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STABILITY OF POISEUILLE-COUETTE FLOW BETWEEN CONCENTRIC CYLINDERS

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SYNOPSIS

The stability of laminar flow between concentric cylinders is analyzed. Combined Poiseuille and Couette flows are considered. It is shown that instabilities exist if the laminar flow is symmetric or near-symmetric on part of the annulus. Instabilities are reported for aspect ratios as small as η = 0.001.

KEYWORDS

Poiseuille-Couette Flow, Stability

SLEUTELWOORDE

Poiseuille-Couette-Vloei, Stabiliteit

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ENCLATURE

complex wave velocity of perturbation, $c = c_R + ic_I$

linear operator defined by equation (4)

wave number of azimuthal operations

density of fluid, kg/m^3

dimensionless constant axial pressure gradients $\frac{dp_0'}{dz'}/(\rho r_2' u_{max}^2)$

- constant axial pressure gradient, Pa/m

dimensionless radial variable, r'/r_2'

outer radius of annulus, m

inner radius of annulus, m

Reynolds number $\rho r_2' u_{max}/\mu$

dimensionless time $t'/(\rho r_2'^2/\mu)$

dimensionless laminar velocity u'/u'_{max}

laminar velocity, m/s

maximum velocity, m/s

dimensionless velocity of inner cylinder, V'/u'_{max}

velocity of inner cylinder, m/s

dimensionless perturbation of radial velocity component, $v_r'/(\mu/\rho r_2')$

perturbation of radial velocity component, m/s

dimensionless axial distance z'/r_2'

axial distance, m

k symbols

wave-number

$V/(Re\frac{dP_0}{dz})$

r_1'/r_2'

dynamic viscosity of fluid, Ns/m^2

INTRODUCTION

Since the time of Osborne Reynolds, the stability of laminar flow in pipes, has been the subject of theoretical and experimental studies. Experimental results indicated that the laminar flow became unstable at Re numbers of 2 000 for rough surfaces, to 100 000 for smooth surfaces[1]. It is clear that the surface finish, which acts as a source for disturbances of finite amplitude, plays a very important role in determining the critical Reynolds number, Re_C.

In this work we are particularly interested in the stability of axial laminar flow in an annulus to axisymmetric perturbations of infinitesimal amplitude. Thus we are considering the case of near-zero roughness, of the walls. Two concentric cylinders of aspect ratio $r_1'/r_2' = \eta < 1$, is considered. In the limit $\eta \to 1$ the flow pattern is similar to flow between two parallel plates. The other limit is $\eta \to 0$; although the transition in this limit to Hagen-Poiseuille flow is not smooth (one must still assume non-slip condition at cylinder surface of infinitesimally small radius) the Re_C goes to infinity, indicating that the laminar flow becomes unconditionally stable to axisymmetric infinitesimal perturbations. Davey and Drazin[2] reported numerical results for Hagen-Poiseuille which are in support of the asymptotic results. Graebel[3] also did a linear stability analysis of Hagen-Poiseuille flow, but he considered azimuthal perturbations and found instability for wavenumbers $n \geq 2$. Furthermore, experimental observations by Fox et al[4] showed that laminar flow became unstable to azimuthal perturbations. All these results are in compliance with the fact that there is no cylindrical equivalent of Squire's theorem, which states that in the planar case, two-dimensional perturbations are most unstable.

The laminar flow in the annulus originates from a constant axial pressure gradient (Poiseuille flow) and by moving the centre cylinder at constant velocity V (Couette flow). If the Couette and Poiseuille flows are in the same direction, the line of maximum velocity gets closer to the centre and for V large enough, the maximum is at the walls of the center cylinder. If the ylinder is pulled anti-parallel to the Poiseuille flow direction, strong boundary-layer develops adjacent to the center cylinder. nder conditions where the Couette flow dominates, the profile is lmost inverted.

ott and Joseph[5] studied the stability of laminar flow in annuli, nder the action of Poiseuille flow. An interesting result is the mportance of the symmetry of the basic laminar profile. If the rofile is symmetric around the centre-line of the annulus ($r' = r'_1 + r'_2)/2$), they found that the laminar flow is the least table. As the basic profile becomes more asymmetric, the tability increases ($Re_c \to \infty$). Returning to our problem of the ombined Poiseuille-Couette flow, it is possible for certain alues of η, V and Re, that the laminar profile is near-symmetric n a part of the annulus, not necessarily for all $r' \epsilon [r'_1, r'_2]$. f we now seek infinitesimal perturbations which are non-zero on nly this symmetric part of the flow, satisfying the boundary onditions already well-away from the two solid boundaries, it is ossible to find Re_c which are much smaller than the values for erturbations which are non-zero for $r' \epsilon (r'_1, r'_2)$. This approach as used with good success by Gatica et al[6] to analyze the tability of non-linearly stratified fluids to thermal nstabilities.

part from academic interest in this problem, several practical ituations exist where this configuration is used. Of specific nterest is the coating of yarn or filaments with a protective ayer by pulling it through a fluid. If a homogeneous coating is equired, it is important to know the maximum stability onditions.

INEAR ANALYSIS

e laminar flow is given by:

$$= 1/4\ Re \frac{dP_0}{dz} [r^2 - \eta^2 + (1 - \eta^2)\ln(r/\eta)/\ln \eta] + V \ln r/\ln \eta \quad (1)$$

ere Re is the Reynolds number.

Let

$$\Omega = V/Re \frac{dP_0}{dz}$$

When $\Omega = 0$ (pure Poiseuille flow), the flow is almost symmetric for $\eta \to 1$ and becomes more skew as $\eta \to 0$. In the case of $\Omega = 0{,}005$, the annulus flow is distorted, but the important effect is the restoring of symmetry on part of the domain (η, 1). For axisymmetric perturbations of the form $v_r(r)\exp(i(\alpha z - Rec\,\alpha t)$, the Orr-Sommerfeld equation for cylindrical coordinates takes the form:

$$L^2 v_r - i\alpha Re[(u-c)Lv_r - v_r(u'' - u'/r)] = 0 \qquad (2)$$

$$v'_r(r) = v_r(r) = 0, \quad r = \eta, 1 \qquad (3)$$

where

$$L \equiv \frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{1}{r^2} - \alpha^2 \qquad (4)$$

A prime denotes differentiation with respect to r. α denotes the axial wave number and c is the wave velocity. Instability sets in when the complex part of c becomes positive. The associated Re value will be referred to as the critical Reynolds number.

An inverse shifted version of the Lanczos algorithm is used to solve the eigenvalue problem. This algorithm produces rapidly and efficiently a cluster of eigenvalues in the vicinity of the one with the smallest real part.

A. $\Omega = 0$

In Figure 1 the critical values of Re is shown for different aspect ratios η. These values are in very good agreement with the results of Mott and Joseph (1968) but for comparison reasons it should be remembered that they used Re = $u'_{max}(r'_2 - r'_1)/2\mu$. These results also served as a check for our numerical code.

. $\Omega = 0{,}005$

he trough-like behaviour of Re vs η becomes more pronounced and he minimum value of Re is less than the minimum value for the revious case. When η approaches 0 and 1, the axial movement of he inner cylinder plays a stabilizing role and values of Re are igher than the case of a stagnant cylinder, i.e. $\Omega = 0$.

o illustrate the concept of perturbations which are more stable n only a part of the annulus-domain, consider the case of $= 0{,}6$. The critical value was Re = 47 750 and the associated igenfunction was numerically found to be nonzero only on r = 0,6235, 1]. This implies that the perturbation did not survive n the part [0,6; 0,6235]. When the domain [0,6; 1] was used, the tabilizing role of asymmetric basic velocity profiles gave rise o a critical value of Re = 114 300. The sensitivity of Re owards the form of the basic flow is clearly illustrated. Since he eigenfunction satisfies both Neumann and Dirichlet conditions t both ends, a jump in a higher derivative can be expected at $= 0{,}6235$ for the form:

$$*_r = \begin{cases} v_r, & 0{,}6235 \le r \le 1 \\ 0, & 0{,}6 \le r \le 0{,}6235 \end{cases}$$

ne can conclude that the most critical perturbation for the nnulus $r = (r_1, 1)$ is not necessarily associated with this domain, ut Re (r_2) can reach a local minimum for $(r_2, 1)$, $r_2 > r_1$. In rinciple, the same argument holds for the upper limit of the omain, although numerical experiments never showed a decrease in e for a decrease in this value. Since the Reynolds stress istribution and energy transfer from the basic flow usually shows harp changes at the fixed walls, it is expected that at least one olid boundary should remain in this domain.

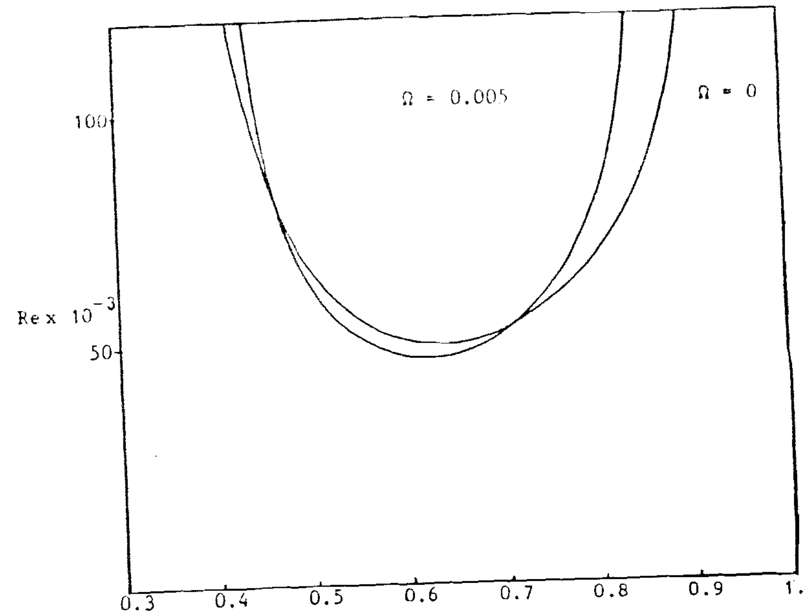

Figure 1: Critical Reynolds Number vs. Aspect Ratio η

C. $\Omega = 1{,}5$ $(\eta = 0{,}001)$

The basic flow profile associated with this case, is shown in Figure 2. Although the inner cylinder is very thin (and often it is assumed that $Re_c \to \infty$ for cylinders this thin), the flow is almost symmetric on the interval (0,4596; 1) and for this domain $c_1 = 0{,}00000$, at Re = 67 134. When considering the whole domain (0,001; 1) the code was unable to find instability.

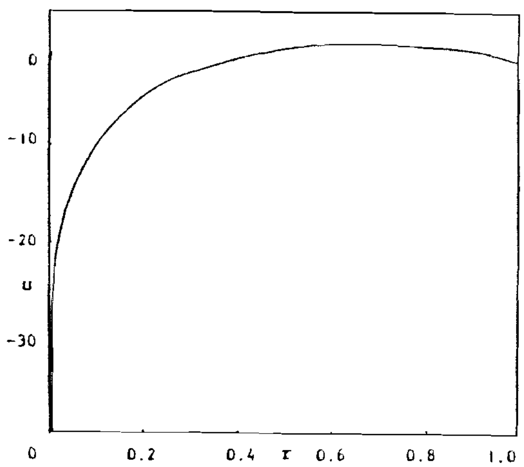

Figure 2: Basic Flow Profile for Ω = 1,5 (η = 0,001)

NCLUSION

e most important points which became apparent from this study e:

laminar flow in an annulus between stagnant cylinders is stabilized when $\eta \to 0$. (In this limit the Reynolds numbers based on gap width and outer radius become identical.)

including Couette flow by axially moving the inner cylinder, the most critical perturbation can be asymptotically zero on a part of the annulus

for very small values of η, one will expect that the laminar flow will become unconditionally stable, but for η = 0,001 pulling rates of the centre cylinder exist that will lead to instability at Re = 67 134

4. an infinite perturbation analysis still leads to critical Re numbers which are considerably higher than experimental values of 2 100 and the key lies in a finite amplitude analysis.

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