Class Notes for Math 953: Algebraic Geometry, Instructor Roger Wiegand

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Topics include: Affine schemes and sheaves, morphisms, dimension theory, projective varieties, graded rings, Artin rings

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1 Section 1.5: Projective Varieties (Ueno)

Definition. *Let $W = k^{n=1} \setminus \{(0,0,...,0)\}$ and define an equivalence relation $\sim$ by*

$$(a_0, ..., a_n) \sim (b_0, ..., b_n) \text{ if and only if } (a_0, ..., a_n) = \alpha(b_0, ..., b_n)$$

*for $\alpha \in k \setminus \{0\}$. Define the $n-$dimensional **projective space** as $\mathbb{P}^n_k = W/\sim$.*

Notation. Say $(a_0 : a_1 : \cdots : a_n)$ is the equivalence class determined by $(a_0, ..., a_n) \in W$ and call $(a_0 : a_1 : \cdots : a_n)$ a point in $\mathbb{P}^n_k$.

We call $\mathbb{P}^1_k$ the **projective line** and $\mathbb{P}^2_k$ the **projective plane**.

$\mathbb{P}^n_k$ can be obtained by gluing together $n+1$ copies of $\mathbb{A}^n_k$.

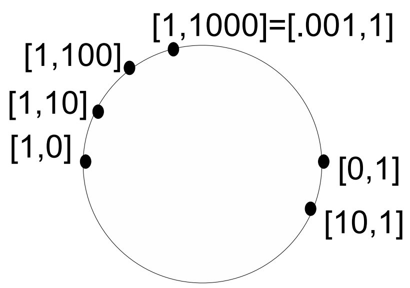
- Define for all $j = 0, ..., n$, $U_j = \{(a_0 : a_1 : \cdots : a_n) \in \mathbb{P}^n_k | a_j \neq 0\}$. Since $(a_0 : a_1 : \cdots : a_n) = (\frac{a_0}{a_j} : \frac{a_1}{a_j} : \cdots : 1 : \cdots : \frac{a_n}{a_j})$, the map $\phi_j : \mathbb{A}^n_k \to U_j$ defined by $(a_1, ..., a_n) \mapsto (a_1 : \cdots : a_j : 1 : a_{j+1} : \cdots : a_n)$ is a bijection. Thus $U_j \cong \mathbb{A}^n_k$.

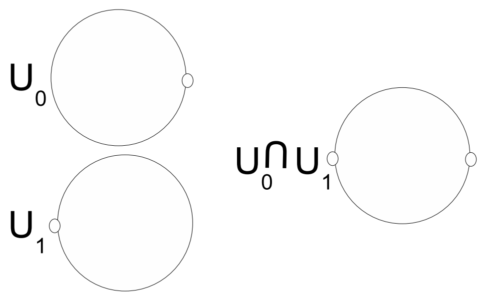
Let's take a closer look at the projective line:

- As we've just seen
$$U_0 = \{[x : y] \in \mathbb{P}^1_k | x \neq 0\} \cong \mathbb{A}^1_k \text{ by } [1 : y] \xrightarrow{\phi} y$$
$$U_1 = \{[x : y] \in \mathbb{P}^1_k | y \neq 0\} \cong \mathbb{A}^1_k \text{ by } [x : 1] \xrightarrow{\psi} x$$

- Consider $U_0 \cap U_1 \subseteq U_0$. Then $U_0 \cap U_1 \xrightarrow{\phi} D(y) := \{y \in \mathbb{A}^1_k | y \neq 0\}$, which has coordinate ring $k[y][\frac{1}{y}] = k[y, \frac{1}{y}]$. Similarly, if we consider $U_0 \cap U_1 \subseteq U_1$, then $U_0 \cap U_1 \xrightarrow{\psi} D(x)$, with coordinate ring $k[x, \frac{1}{x}]$.

- Pictures:





Projective Sets/Projective Varieties

Motivation: Let $f = x_1 - x_0^2$. Then f vanishes at $(1,1,0,...0)$ but not at $(2,2,0,...,0)$. This makes NO sense in $\mathbb{P}^n_k$ as those points lie in the same equivalence class.

Definition. *A **homogenous polynomial** in $k[x_0, ..., x_n]$ is one in which each term has the same degree.*

Let f be a homogenous polynomial of degree d. Then $f(t\alpha) = t^d f(\alpha)$. Thus $f(t\alpha) = 0$ if and only if $f(\alpha) = 0$. So it does make sense to talk about the zero point in $\mathbb{P}^n_k$ of a homogenous polynomial.

Definition. *A **projective set** is $V(f_1, ..., f_t) = \{\alpha \in \mathbb{P}^n_k | f_i(\alpha) = 0 \text{ for all } i\}$ where f_i is homogenous. An ideal I of $k[x_0, ..., x_n]$ is said to be a **homogenous ideal** provided: If $f \in I$ is decomposed into a sum of homogenous polynomials $f_1, ..., f_t$, then $f_i \in I$ for all $i = 1, .., t$.*

Exercise (prob 14, p 32). Let I be an ideal of $k[x_0, ..., x_n]$. Then I is homogenous if and only if I is generated by a finite set of homogenous ideals.

This allows us to define a projective set as $V(I)$, where I is a homogenous ideal of $k[x_0, ..., x_n]$.

Facts.

1. If I is homogenous, so is $\sqrt{I}$ (prob 15, p 32). Moreover, $V(I) = V(\sqrt{I})$.

2. $\{$projective sets $\neq 0\} \leftrightarrow \{$homogenous radical ideals $\neq (x_0, ..., x_n)\}$.

Lets look at some examples of curves in $\mathbb{A}^2_k$ and $\mathbb{P}^2_k$.

The Cuspidal Cubic: $y^2 = x^3$, that is $C = V(y^2 - x^3) \subseteq \mathbb{A}^2_k$. (Assume char $k \neq 2, 3$ and $k = \overline{k}$.)

- We can parameterize the curve by $x = t^2, y = t^3$.
  - Consider the line passing through $(1, -1)$ and $(2, \sqrt{8})$. This is the line $y + 1 = (1 + \sqrt{8})(x - 1)$. A little work shows the third point of intersection is $(6 + 2\sqrt{8}, 20 + 7\sqrt{8})$. Interestingly, $\frac{1}{-1} + \frac{1}{\sqrt{2}} + \frac{1}{2+\sqrt{2}} = 0$.
  - Consider the line passing through $(1, -1)$ and $(4, 8)$, namely $y - 8 = 3(x - 4)$.

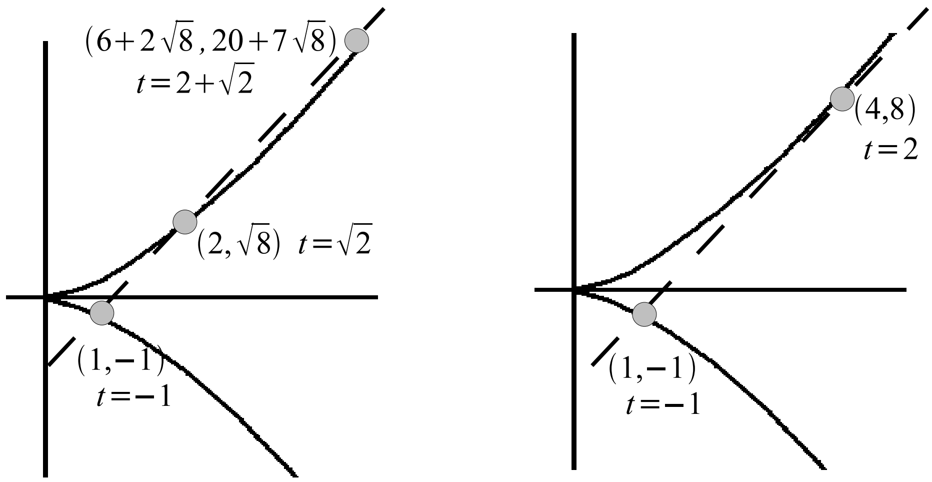
    Claim: $(4, 8)$ has intersection multiplicity 2.

    Proof: Define $x' = x - 4$ and $y' = y - 8$. Then we have the line $y' = 3x'$ and our curve is $y^2 - x^3 = (y')^2 + 16y' - (x')^3 - 12(x')^2 - 48x'$. So the intersection multiplicity is

$$dim_k \frac{k[x, y]_{(x,y)}}{(y - 3x, y^2 + 16y - x^3 - 12x^2 - 48x)} = dim_k \frac{k[x]_{(x)}}{(-x^3 - 3x^2)} = dim_k \frac{k[x]_{(x)}}{(x^2)} = 2$$

    as $-x^2 - 3x^2 = x^2(-x - 3)$ and $-x - 3$ is a unit.

    Note again that $\frac{1}{-1} + \frac{1}{2} + \frac{1}{2} = 0$.



- Lets parameterize with $x = \frac{1}{t^2}, y = \frac{1}{t^3}$. This parameterizes $C - \{(0, 0)\} \cup \{\infty\}$, which is fine as $(0, 0)$ is a singularity.
  - Consider the line $x = 1$. This also crosses the curve at 3 places- one is just the point at ∞. Again, note $\frac{1}{-1} + \frac{1}{1} + \frac{1}{\infty} = 0$.

Exercise. Show P_v, P_u, P_w lie in a straight line if and only if $v + u + w = 0$ (where $P_v = (\frac{1}{v^2}, \frac{1}{v^3})$).

Proof. P_v, P_u, P_w are colinear

$\Leftrightarrow$ There exists a, b (not both 0) such that $ax + by = 1$ has solutions at P_v, P_u, P_w.

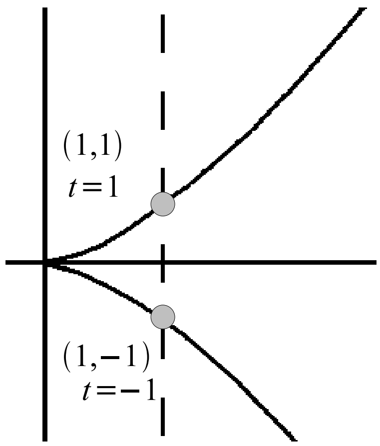
$\Leftrightarrow$ There exists a, b such that $at^{-2} + bt^{-3} = 1$ has solutions at P_v, P_u, P_w.

$\Leftrightarrow$ There exists a, b such that $t^3 - at - b = (t - u)(t - v)(t - w)$

$\Leftrightarrow$ $u + v + w = 0$ as $(t - u)(t - v)(t - w) = t^3 - (u + v + w)t^2 + (uw + uv + vw)t - uvw$.

□

How do we find $\overline{C}$, the closure of C in $\mathbb{P}^2_k$?



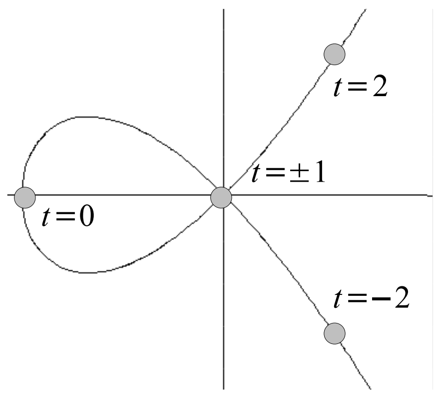
- Homogenize: Introduce the variable z and add it to terms so that the polynomial is homogenous: $y^2z - x^3$.
- If we let $z = 0$, then we see the line $z = 0$ intersects with $y^2z - x^3$ at $(0 : 1 : 0)$. This is a point at ∞.

Exercise. Let D be another plane curve in $\mathbb{A}_k^2$ which meets $C = V(y^2 - x^3)$ at some point $P \neq (0, 0)$. Then D meets C at another point $Q \neq P$.

Proof. Say $D = V(f)$ and $h = f(t^2, t^3)$. Note that $h = 0$ are the points where C and D intersect. Say P corresponds to the point t_0. Then $(t - t_0)|h(t)$. So, suppose for contradiction that $h(t) = a(t - t_0)^n$ (that is, there are no other roots). Then the coefficient of t is $\pm ant_0^{n-1} \neq 0$. This is a contradiction to the fact that $h(t) \in k[t^2, t^3]$ (that is, there are no linear factors). Thus there exists another root, which implies there is another point at which C and D intersect. □

The Nodal Cubic: $C = V(y^2 - x^3 - x^2)$.

- This can be parameterized by $x = t^2 - 1, y = t(t^2 - 1)$.
- This should have a group structure isomorphic to $k^x = k \setminus \{0\}$ (if you through out the singularity at $(0, 0)$.)



2 Section 2.2 (Ueno)

Zariski Topology: To each open set of a prime spectrum $X = SpecR$, we will associate a ring of "regular functions."
Notation. For $f \in R$, denote the open set $D(f) = \{p \in SpecR | f \notin p\}$ by X_f or $(SpecR)_f$.

Proposition (2.8). *For $\{f_\alpha\}_{\alpha \in A} \subseteq R$, $SpecR = \cup_{\alpha \in A}(SpecR)_{f_\alpha}$ if and only if the ideal $(f_\alpha)_{\alpha \in A}$ generated by $\{f_\alpha\}_{\alpha \in A}$ equals R.*

Corollary (2.9). *The topological space $X = SpecR$ is quasicompact, that is, for an open covering $X = \cup_{\lambda \in \Lambda} U_\lambda$, there exist finitely many U_{λ_j} such that $X = \cup_{j=1}^m U_{\lambda_j}$.*

Lemma (2.10). *Let $X = SpecR$, $f, g \in R$. Then*

1. $X_f \cup X_g = X_{fg}$.

2. $X_f \supset X_g$ *if and only if* $g \in \sqrt{(f)}$.

Inductive Limit

Let $X_f \supset X_g$. Then $g \in \sqrt{(f)}$, that is $g^n = af$ for $a \in R$. Thus we can define the homomorphism $\rho_{X_f,x_g} : R_f \to R_g$ by $\frac{r}{f^m} \mapsto \frac{a^m r}{g^{nm}}$. We say ρ_{X_f,X_g} is the **restriction mapping** which restricts a function X_f to a function X_g.

Lemma (2.11). *For $X_f \supset X_g \supset X_h$, $\rho_{X_h,X_g} \circ \rho_{X_g,X_f} = \rho_{X_h,X_f}$.*

Define $\mathcal{U}_p = \{X_f : p \in X_f\}$. (Note: Roger calls this $\mathcal{N}(p)$ and says it is the **neighborhood of** p). Order $\mathcal{U}_p$ by $X_f < X_g$ if and only if $X_f \supset X_g$. Note since $X_{h_1} \cap X_{h_2} = X_{h_1 h_2}$, we have $X_{h_1} X_{h_2} < X_{h_1 h_2}$. Thus $\mathcal{U}_p$ is a **directed set**.

Now, associate $\frac{r}{f_m} \in R_f$ with $(\frac{r}{f^m}, X_f)$ and let $\mathcal{R} = \{(\frac{r}{f^m}, X_f) | X_f \in \mathcal{U}_p\}$. Define an equivalence relation $\sim$ on $\mathcal{R}$ by $(\frac{r}{f^m}, X_f) \equiv (\frac{s}{g^n}, X_g)$ if and only if $X_n \in \mathcal{U}_p$ such that $X_f, X_g < X_h$ and $\rho_{X_h,X_f}(\frac{r}{f^m}) = \rho_{X_h,X_g}(\frac{s}{g^n})$. Then the **inductive limit** is $\varinjlim_{X_f \in \mathcal{U}_p} R_f := \mathcal{R}/\sim$.

Theorem (p 61 - 64). *Let $p \in Spec(R)$. Then $\varinjlim_{f \notin p} R_f = R_p$ (where recall $R_f = R[f^{-1}]$).*

Proof. Define an ordering $R_f \leq R_g$ if and only if $X_f \supseteq X_g$. To get the map $R_f \to R_g$, we need to show $\frac{f}{1}$ is a unit in R_g. Now, $X_f \supseteq X_g$ implies $g^r = af$ for some r, a. Then $\frac{f}{1} \cdot \frac{a}{g^r} = \frac{fa}{g^r} = \frac{1}{1}$. Thus $\frac{f}{1}$ is a unit in R_g. Thus, if $f \leq g$, then we have a corresponding homomorphism $R_f \to R_g$.

By the universal property of direct limits, we get a unique map $\phi : \varinjlim_{f \notin p} R_f \to R_p$. We want to show it is an isomorphism. Its clearly onto. Thus we need only show its 1-1. If $\phi(\alpha) = 0$, since we can write $\alpha = \phi_g(\frac{a}{g^m})$, we have $\frac{a}{g^m} = 0$ in R_p. Thus there exists $h \in R \setminus p$ such that $ha = 0$. Then $\frac{a}{g^m} \mapsto \frac{ah}{g^m h} = 0 \mapsto \alpha$. Since these are homomorphisms, we see $\alpha = 0$. $\square$

Structure Sheaf of the Prime Spectrum

First, we must understand what a sheaf is. Then we can define one for the prime spectrum.

Lemma (2.15). *If $X_f = \cup_{\alpha \in A} X_{f_\alpha}$ and $\rho_{X_{f_\alpha},X_f}(a) = 0$ for $a \in R_f$, then $a = 0$.*

Lemma (2.16). *If $X_f = \cup_{\alpha \in A} X_{f_\alpha}$ and $g_\alpha \in F_{f_\alpha}$ for all α and $\rho_{X_{f_\alpha f_\beta} X_{f_\alpha}}(g_\alpha) = \rho_{X_{f_\alpha f_\beta} X_{f_\beta}}(g_\beta)$ for all α, β, then there exists $g \in R_f$ satisfying $g_\alpha = \rho_{X_{f_\alpha} X_f}(g)$.*

Definition. *Let X be a topological space. A **presheaf** $\mathcal{F}$ on X assigns to each open set U an abelian group $\mathcal{F}(U)$ and to each pair of open sets $V \subseteq U$ a homomorphism (called a "restriction map") $\rho_{v,u} : \mathcal{F}(U) \to \mathcal{F}(V)$ subject to*

1. *If $U \subseteq V \subseteq W$ are open sets, then $\rho_{wu} = \rho_{wv} \circ \rho_{vu}$*

2. $\rho_{uu} = 1_{\mathcal{F}(U)}$

3. $\mathcal{F}(\emptyset) = 0$.

We can similarly define a presheaf of commutative rings where $\mathcal{F}(U)$ is a commutative ring and ρ_{uv} a ring homomorphism. We'll often write $f|_V$ for $\rho_{vu}(f)$.

Example. Let X be a compact Hausdorff space. Then for all open sets U, let

$$\mathcal{F}(U) = \{continuous\ functions\ U \to \mathbb{R}\}.$$

If $U \supseteq V$ and $f \in \mathcal{F}(U)$, let $\rho_{vu}(f) = f|_V$.

Definition. *A presheaf $\mathcal{F}$ is said to be a **sheaf** provided for $U = \cup_{\alpha \in A} U_\alpha$ where U_α are open sets in X we have*

(F1) *If $s \in \mathcal{F}(U)$ with $s|_{U_\alpha} = 0$ for all α, then $s = 0$.*

(F2) *If $s_\alpha \in \mathcal{F}(U_\alpha)$ for all α with $s_\alpha|_{U_\alpha \cap U_\beta} = s_\beta|_{U_\alpha \cap U_\beta}$ for all α, β, then there exists $s \in \mathcal{F}(U)$ such that $s|_{U_\alpha} = s_\alpha$ for all α.*

Note: The s in F2 is unique by F1.

By Property 1 of a presheaf, we can consider the direct limit of $\mathcal{F}(U)$ for open sets U.

Definition. *The **stalk** of $\mathcal{F}$ at p is defined to be $\mathcal{F}_p = \varinjlim \mathcal{F}(U)$.*

Exercise. Let $A = \{\text{cont functions} : X \to \mathbb{R}\}$ where X is a compact Hausdorff Space. For $p \in X$, let $m_p = \{f \in A | f(p) = 0\} = \ker(A \to \mathbb{R}, f \mapsto f(p))$, the "maximal ideal of the point p." Let J_p be the ideal of functions vanishing in a neighborhood of p (Thus fIJ_p if and only if there is an open set U such that $p \in U$ and $f|_U = 0$.) Prove the following:

1. The map $\phi : X \to Spm(A)$ defined by $p \mapsto m_p$ is a homeomorphism.

   Proof. To prove this, we will need to know the following Lemma:

   Urysohn's Lemma: Let Y and Z be disjoint closed subsets of X. Then there exists $f \in A$ such that $f|_Y = 0$ and $f|_Z = 1$.

   Now, we may get on with proving the assertion.

   - ϕ is surjective: We will show any maximal ideal of A is of the form m_c for $c \in X$. Let $I \in Spm(A)$ and suppose $I \not\subseteq m_c$ for all $c \in X$. Then there exists $g_c \in I \setminus m_C$ for all $c \in X$. As $g_c(c) \neq 0$ by continuity of g_c, there exists an open set $U_c \subseteq X$ such that $c \in U_c$ and g_c does not vanish in U_c.

     Why? Fix c. As $g_c = g : X \to \mathbb{R}$ is continuous and $\{0\}$ is closed in $\mathbb{R}$, $g_c^{-1}(\{0\})$ is closed in X. So $g_c(c) \neq 0$ which implies $c \notin g_c^{-1}(\{0\})$. So there exists an open set U_c such that $c \in U_c$ and $U_c \cap g^{-1}(\{0\}) = \emptyset$.

     Then, for all $z \in U_c$, $g(z) \neq 0$. Let $F_c := (g_C)^2$ for $c \in X$. Then $f_c > 0$ in U_c. Since $X \subseteq \cup_{c \in C} U_c$ is compact, there exists $c_1, ..., c_n \in X$ such that $X = U_{c_1} \cup \cdots \cup U_{c_n}$. Now, let $h = f_{c_1} + ... + f_{c_n}$. Note that $h \in I$ and $h > 0$ on X. Then $\frac{1}{h} \in A$ and hence $1 = \frac{1}{h} \cdots f \in I$, a contradiction. Hence $I \subseteq m_c$ for some $c \in X$. Since I is maximal, $I = m_c$.
   - ϕ is injective: Let $p \neq q \in Z$. By Urysohn's Lemma, there exists a continuous $f : X \to [0, 1]$ such that $f(p) = 0$ and $f(q) = 1$. Then $m_p \neq m_q$ which implies ϕ is injective.
   - ϕ maps basis elements of X to basis elements of $Spm(A)$: For $f \in A$, define $U_f = X - f^{-1}(\{0\}) = \{x \in X : f(x) \neq 0\}$ and let $\overline{U_f} = D(f) \cap Spm(A) = \{m \in Spm(A) | f \notin m\}$. Note that this is a basis for $Spm(A)$ as $D(f)$ is a basis for $Spec(A)$. So it is enough to show that $\{U_f\}$ is a basis for the topology on X and $\phi(U_f) = \overline{U_f}$ for $f \in A$.
     - $\phi(U_f) = \{\phi(c) : c \in U_f\} = \{m_c : f(c) \neq 0\}$. Now $m_c \in \phi(U_f)$ if and only if $f(x) \neq 0$ if and only if $f \notin m_c$ if and only if $m_c \in \overline{U_f}$.
     - Let $U \neq \emptyset$ be an open set in X and $x \in U$. By Urysohn's Lemma, there exists $f \in A$ such that $f(c) = 1$ and $f(x) = 0$ for all $x \in X \setminus U$. Thus $c \in U_f \subseteq U$ for some f. So $\{U_f\}$ is a basis for X.

   ∎

2. Note that $\mathcal{F}(U) = \{\text{cont functions} : U \to \mathbb{R}\}$ for all open sets U. Prove that, for each $p \in X$, the canonical localization map $A \to A_{m_p}$ is surjective and J_p is its kernel. (Thus $\mathcal{F}_p \cong A_{m_p} \cong A/J_p$.)

3. If $X = [0, 1] \subseteq \mathbb{R}$, then $Spec(A) \neq Spm(A)$.

   Proof. Suppose, for contradiction, that $Spec(A) = Spm(A)$ and define $S := \{0 \neq p(x) \in \mathbb{R}[x]\}$. Note S is a mcs of A. Let $p \in SpecA$. Then, by assumption, $p = m_c$ for some $c \in [0, 1]$. This says $x - c \in p \cap S$. Thus, for all $p \in SpecA$, we have $p \cap S \neq \emptyset$. Then $Spec(S^{-1}A) = \emptyset$ which implies $S^{-1}A = \{0\}$. Thus $0 \in S$, a contradiction. ∎

Now that we have a handle on sheafs, we can construct one for $Spec(A)$.

Let A be a commutative ring with 1. We want to define a sheaf $\mathcal{O}$ of commutative rings on $X = Spec(A)$ where $\mathcal{O}(D(f)) \cong A[f^{-1}]$ for all $f \in A$ and $\mathcal{O}_p \cong A_p$ for all $p \in X$ (where $\mathcal{O}_p = \varinjlim \mathcal{O}(U)$). Define $\mathcal{O}(U)$ to be a subring of $\prod_{p \in U} A_p$, the ring of functions $s : U \to \dot{\cup}_{p \in U} A_p$ with the property that $s(p) \in A_p$ where $s(p)$ is the pth coordinate of s. Further, we want that $s \in \mathcal{O}(U)$ if and only if for all $p \in U$ there exists an open neighborhood $V \subseteq U$ of p and elements $a, f \in A$ such that $V \subseteq D(f)$ (that is, $f \notin q$ for all $q \in V$) and $s(q) = \frac{a}{f}$ in A_q for all $q \in V$.

1. $\mathcal{O}(U)$ is a subring of $\prod_{p\in U} A_p$.

   Proof. Let $s_1, s_2 \in \mathcal{O}(U)$. Given $p \in U$, choose $V_1, V_2, a_1, a_2, f_1, f_2$ such that $s(q) = \frac{a_i}{f_i}$ in A_q for $q \in V_i$. Now, to simplify things a bit, take $V = V_1 \cap V_2 \cap D(g)$ (this is still an open neighborhood of p). Then, take $a = a_1 f_2 + a_2 f_1$ and $f = f_1 f_2$. Then, for $q \in V$, we define $s(q) := s_1(q) + s_2(q) = \frac{a_1}{f_1} + \frac{a_2}{f_2} = \frac{a}{f}$. Thus, it is closed under addition (and similarly multiplication). □

2. $\mathcal{O}(U)$ is a presheaf.

   Proof. We can easily define the restriction map $\rho_{VU} : \mathcal{U} \to \mathcal{V}$ as just $\rho_{VU}(s) = s|_V$. (Since s is a function, all of the axioms work out). □

3. $\mathcal{O}(U)$ is a sheaf.

   Proof. (F1) is clear as s is a function and (F2) is almost as equally clear by just defining s by s_i. □

Observation. Currently, our sheaf has a picture similar to Figure 1. However, this can be simplified. Rather than starting with U, shrinking down to a smaller open neighborhood intersected with a $D(f)$, we can simply choose a $D(f) \subseteq U$ and bypass some of the sets all together. How do we do this? With the above notation, choose g such that $p \in D(g) \subseteq V$. (We can do this as open sets of $SpecA$ look like some $D(I) = \cup_\alpha D(f_\alpha)$ where I is generated by f_α). Now, if $q \in D(g)$, then $s(q) = \frac{a}{f} \in A_q$. Since $D(g) \subseteq D(f)$, we know $g^m = rf$ for some m, r. Thus $\frac{a}{f} = \frac{ar}{g^m}$. Now, let $h = g^m$ and $b = ar$. Then $D(h) = D(g)$ and $s(q) = \frac{a}{f} = \frac{b}{h}$.

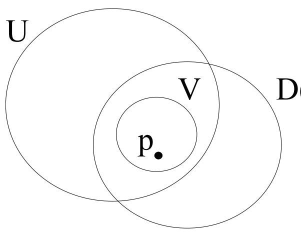


Figure 1

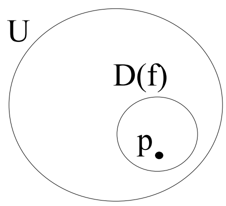


Figure 2

Thus (refreshing notation), we now have $s \in \mathcal{O}(U)$ if and only if for all $p \in U$ there exists f such that $o \in D(f) \subseteq U$ and $s(q) = \frac{a}{f} \in A_q$ for all $q \in U$. This is portrayed in Figure 2.

Note. The following is found on page 71 of Hartshorne.

Proposition. *$\mathcal{O}_p \cong A_p$ for all $p \in Spec(R)$.*

Proof. Define $\psi : \mathcal{O}_p \to A_p$ as follows: Let $\sigma \in \mathcal{O}_p$. Choose an open neighborhood U of p and $s \in \mathcal{O}(U)$ such that s represents σ. Define $\psi(\sigma) = s(p)$.

- ψ is well-defined: Let V be another open neighborhood of p and $t \in \mathcal{O}(V)$ such that t represents σ. Then, there exists $W \in U \cap V$ such that $p \in W$ and $s|_W = t|_W$. Since $p \in W$, we have $s(p) = t(p)$.
- ψ is a homomorphism: This follows from Chapter 2, A&M.
- ψ is surjective: Let $\alpha \in A_p$. Write $\alpha = \frac{a}{f}$ where $a \in A, f \in A \setminus p$. Then $p \in D(f)$. Define $s(q) = \frac{a}{f}$ in A_q for all $q \in D(f)$. Then $s \in \mathcal{O}(D(f))$. Then, let σ be the equivalence class of s in $\mathcal{O}_p$. Then $\psi(\sigma) = \alpha$.
- ψ is injective: Suppose $\psi(\sigma) = 0$. Choose U, s as before such that $s \mapsto \sigma$. Now, there exists $a, f \in A$ such that $p \in D(f) \subseteq U$ and $s(q) = \frac{a}{f}$ for all $q \in U$. Then $0 = \psi(\sigma) = s(p) = \frac{a}{f}$ in A_p. So there exists $h \in A \setminus p$ such that $ha = 0$. For $q \in D(fh) = D(f) \cap D(h)$, we have $s(q) = \frac{a}{f}$ in A_q. Since $h \notin q$, we see $s(q) = 0$. Thus $s|_{\mathcal{O}(D(fh))} = 0$. Take $W = D(fh)$. Then $s = 0$ in $\mathcal{O}(W)$ which implies $\sigma = 0$.

□

Proposition. *$\mathcal{O}(D(f)) = A[f^{-1}]$ for all $f \in A$.*

Proof. Define the map $\psi_f : A[f^{-1}] \to \mathcal{O}(D(f))$ by $\frac{a}{f^m} \mapsto (s : q \mapsto \frac{a}{f^m} \in A_q)$. We will show this is an isomorphism of rings.

- ψ_f is well-defined: Suppose $\frac{a}{f^m} = \frac{b}{f^n}$ in $A[f^{-1}]$. Then $\frac{a}{f^m} = \frac{b}{f^n}$ in A_q whenever $q \in D(f)$.
- ψ_f is a homomorphism: Easy
- ψ_f is injective: Suppose $\frac{a}{f^m} \mapsto 0$. Then $\frac{a}{f^m} = 0$ in A_q whenever $q \in D(f)$. Let $I = (0 : a)$. Then for all $q \in D(f)$, there exists an element $I \setminus q$ such that $I \not\subseteq q$. Thus for all $q \in D(f)$, $q \in D(I)$ which implies $D(f) \subseteq D(I)$ and therefore $f \in \sqrt{I}$. Say $f^n \in I$ for some n. Then $f^n a = 0$ which implies $\frac{a}{f^m} - 0$ in $A[f^{-1}]$.
- ψ_f is surjective: Let $s \in \mathcal{O}(D(f))$. Cover $D(f)$ by open sets of the form $D(f_i)$ such that $s(q) = \frac{a_i}{f_i}$ in A_q. Note that $D(f)$ is quasicompact (in general, $D(I)$ is quasicompact if and only if there exists a finitely generated J such that $\sqrt{I} = \sqrt{J}$). Thus we can extract a finite subcover, say $D(f) = D(f_1) \cup \cdots \cup D(f_n)$. Now, for all $i, j = 1, ..., n$ and for all $q \in D(f_i) \cap D(f_j)$ where $\frac{a_i}{f_i} = s(q) = \frac{a_j}{q_j}$ in A_q, consider $\psi_{f_i f_j} : A[(f_i f_j)^{-1}] \to \mathcal{O}(D(f_i f_j))$ and let $s' = \psi_{f_i f_j}(\frac{a_i f_j - a_j f_i}{f_i f_j})$. Then for all $q \in D(f_i f_j) = D(f_i) \cap D(f_j)$, we have $s'(q) = 0$. Thus $s' = 0$. Since $\psi_{f_i f_j}$ is injective, $\frac{a_i f_j - a_j f_i}{f_i f_j} = 0$. Thus, there exists N such that $(f_i^N f_j^N (a_i f_j - a_j f_i) = 0$. Now, since we have a finite subcover, we can choose N such that the above is true for all $i, j = 1, ..., n$. For simplicity, let $g_i := f_i^{N+1}$ and $b_i := f_i^N a_i$. Then we have $g_j b_i = g_i b_j (*)$. Note $D(g_j) = D(f_j)$, so $D(f) = D(g_1) \cup \cdots \cup D(g_n) = D(ag_1 + ... + ag_n)$. Thus $f \in \sqrt{Ag_1 + ... + Ag_n}$. Write $f^t = c_1 g_1 + ... + c_n g_n$ for $c_i \in A$ and let $c := b_1 c_1 + ... + b_n c_n$. Then, for all j, we have

$$\begin{aligned} g_j c &= b_1 g_j c_1 + ... + b_n g_j c_n \\ &= b_j g_1 c_1 + ... + b_j g_n c_n \text{ by } (*) \\ &= b_j f^t. \end{aligned}$$

  Claim: $\psi(\frac{c}{f^t}) = s$.

  Proof: Let $q \in D(f)$. Choose j such that $q \in D(g_j) = D(f_j)$. In A_q, we have $s(q) = \frac{a_j}{f_j} = \frac{a_j f_j^N}{f_j^{N+1}} = \frac{b_j}{g_j} = \frac{c}{f^t}$.

□

Special Case: Take $f = 1$ in the above proposition. Then, we have $D(f) = X$ and thus $\mathcal{O}(X) \cong A$.

So now, we think of $Spec(A) = (X, \mathcal{O}_X)$, where $X = \{\text{prime ideals}\}$ and $\mathcal{O}_X$ is the sheaf of rings defined above.

Notation. If $\mathcal{F}$ is a sheaf on a topological space X and U is an open set, we often write $\Gamma(U, \mathcal{F}) := \mathcal{F}(U)$. An element $s \in \Gamma(U, \mathcal{F})$ is called a **section** of $\mathcal{F}$ over U.

When are all prime ideals maximal?

Theorem. *Let R be a commutative ring. TFAE*

1. *Every prime ideal is maximal and R is reduced.*
2. *R_m is a field for all maximal ideals m.*
3. *Every $R-$module is flat (i.e., R is absolutely flat).*
4. *$I \cap J = IJ$ for all ideals $I, J \in R$.*
5. *$Rx = Rx^2$ for all $x \in R$ (i.e., R is von Neumann Regular]*

Proof. $1 \Rightarrow 2:$ R_m has only one prime ideal, namely mR_m. Thus $mR_m = \text{Nilrad} = 0$, since R is reduced. Thus R_m is a field.

$2 \Rightarrow 3:$ Fact: Over any ring, M is flat if and only if M_p is flat for all maximal ideals p. Since M_p is a field, M_p is a free R_p modules. Thus M_p is flat.

$3 \Rightarrow 4:$ We know $0 \to I \to R \to R/I \to 0$ is exact. Now, tensor with R/J to get $0 \to I \otimes_R R/J \xrightarrow{1-1} R \otimes R/J \to R/I \otimes R/J \to 0$. This is exact as R/J is flat. Now, $I \otimes_R RJ = I/IJ$ and $R \otimes R/J \cong R/J$. Since injective, we have $I \cap J \subseteq IJ$. Since the other containment is trivial, we have equality.

$4 \Rightarrow 5:$ $Rx^2 = (Rx)^2 = Rx \cap Rx = Rx$.

$5 \Rightarrow 1:$ If $x \neq 0$, then $Rx^2 \neq 0$ and similarly $Rx^n \neq 0$ which says $x^n \neq 0$. Thus R is reduced. Let p be prime. We want to show R/p is a field (and thus p is maximal). Let $D = R/p$, a domain. If $x \in D$, then $x \in Dx^2$. So suppose $x \neq 0$. then $x = dx^2$ which implies $1 = dx$. Thus D is a field.

□

Note that $Spec(A/\sqrt{(0)}) \cong Spec(A)$ by $p/\sqrt{(0)} \leftrightarrow p$ since every prime ideal contains $\sqrt{(0)}$. Thus, we may assume there do not exist nilpotent elements.

Corollary. *$SpecR = SpmR$ if and only if $R/\sqrt{(0)}$ is absolutely flat.*

Definition. *A **ringed space** is a pair $(X, \mathcal{O}_X)$ where X is a topological space and $\mathcal{O}_X$ is a sheaf of commutative rings on X.*

Definition. *Suppose $(X, \mathcal{O}_X)$ is a ringed space. An $\mathcal{O}_S-$**module** (sometimes called a sheaf of $\mathbb{O}_X$ modules) is a sheaf $\mathcal{F}$ of abelian groups on X such that $\mathcal{F}(U)$ is an $\mathcal{O}_X(U)-$module such that the restrictions maps are compatible with scalar multiplication, that is, if $V \subset U$ are opens sets then the diagram below commutes:*

$$\begin{array}{ccc} (r,m) & \longrightarrow & rm \\ \mathcal{O}_X(U) \times \mathcal{F}(U) & \xrightarrow{mult} & \mathcal{F}(U) \\ {\scriptstyle rest\ map \times rest\ map}\Big\downarrow & & \Big\downarrow{\scriptstyle rest\ map \times rest\ map} \\ \mathcal{O}_X(V) \times \mathcal{F}(V) & \xrightarrow{mult} & \mathcal{F}(V) \\ r|_V m|_V & \longrightarrow & rm|_V \end{array}$$

Definition. *Let $(X, \mathcal{O}_X) = SpecR$. Given an $R-$module, define an $\mathcal{O}_X-$module $\widetilde{M}$ as follows: $s \in \Gamma(U, \widetilde{M}) = \widetilde{M}(U)$ if and only if there exists $a_i \in M, f_i \in R$ such that $U = \cup_i D(f_i)$ and $s(p) = \frac{a_i}{f_i}$ in M_p for all $p \in D(f_i)$.*

Exercise. Prove that $\widetilde{M_p} \cong M_p$ as R_p modules for all $p \in X$ (where, recall $\widetilde{M_p} := \varinjlim_{p \in U} \widetilde{M}(U)$ is the stalk and M_p is the localization). Also, prove $\Gamma(D(f), \widetilde{M}) \cong M[\frac{1}{f}]$.

Proof. First, we wish to show $\widetilde{M_p} \cong M_p$. So, define $\phi : \widetilde{M_p} \to M_p$ such that if $\sigma \in \widetilde{M_p}$, then find U and $s \in \widetilde{M}(U)$ such that $\overline{s} = \sigma$. Then $\phi(\sigma) = s(p)$. We see ϕ is well-defined as if there is $s \in \widetilde{M}(U)$ and $t \in \widetilde{M}(V)$ such that $\overline{s} = \sigma = \overline{t}$, then there exists an open $W \subseteq U \cap V$ where $s|_W = t|_W$ which implies $s(p) = t(p)$. To show ϕ is injective, suppose $\sigma \in \widetilde{M_p}$ such that $\phi(\sigma) = 0$. Then if $\overline{S} = \sigma$ for some $s \in \widetilde{M}(U)$ for some U, then $s(p) = \frac{m}{f} = \frac{0}{1}$ in M_p. So there exists $h \in A \setminus p$ such that $hm = 0$ in M. Then $p \in D(f) \cap D(h) = D(fh)$. Now $s \in widetildeM(D(f))$ and $s(q) = \frac{m}{f}$ in M_q for all $q \in D(f)$. So $s|_{D(fh)}(q) = \frac{m}{f}$ for all $q \in D(fh)$. Note that $h \in A \setminus q$ for all $q \in D(fh)$ which implies $hm = 0$. Thus $s|_{D(fh)} = 0$ for all $q \in D(fh)$ which says $\overline{s} = 0$. Thus ϕ is injective. To show ϕ is surjective, let $\frac{m}{f} \in M_p$. Consider $s \in \widetilde{M}(D(f))$ where $s(q) = \frac{m}{f}$ in M_q for all $q \in D(f)$. Then $s(p) = \frac{m}{f}$.

Now, we need to show $\widetilde{M}(D(f)) \cong M[f^{-1}]$. So define $\psi : M_f \to \widetilde{M}(D(f))$ such that $\frac{m}{f^i} \mapsto s(q) = \frac{m}{f^i}$ in M_q for all $q \in D(f)$. Well definedness follows similarly to the previous argument. To ϕ is injective, suppose $\psi(\frac{m}{f_i}) = 0$. Then $\frac{m}{f^i} = \frac{0}{1}$ in M_q for all $q \in D(f)$, that is, $Ann_A(m) \subsetneq q$ for $q \in D(f)$ which says $D(f) \subseteq D(Ann(m))$. Thus $f \in \sqrt{Ann(m)}$ which says $f^n \in Ann(m)$ for some n. Then $f^n m = 0$ in M which implies $\frac{m}{f^i} = \frac{0}{1}$ in $M[f^{-1}]$. Thus we are left with showing ψ is surjective. So suppose $s \in \widetilde{M}(D(f))$. Now, there exists $f_a \in A, m_a \in M$ such that $D(f) = \cup D(f_a)$ and $s(q) = \frac{m_a}{f_a}$ in M_q for all $q \in D(f)$. Now, $D(f)$ is quasicompact (since $D(f) = SpecA[f^-1]$ and the spec of a ring always quasicompact). So there exists a finite subcover $D(f) = \cup_{i=1}^n D(f_i)$. Fix $i, j \in [n]$. Then for all $q \in D(f_i) \cap D(f_j)$, $s(q) = \frac{m_i}{f_i} = \frac{m_j}{f_j}$ in M_q. Define $\psi_{f_i f_j} : M[(f_i f_j)^{-1} \to \widetilde{M}(D(f_i f_j))$. We've shown this is injective. Thus $\psi_{f_i f_j}(\frac{f_j m_i - f_i m_j}{f_i f_j}) = 0$ implies $\frac{f_j m_i - f_i m_j}{f_i f_j} = 0$ in $M[(f_i f_j)^{-1}]$. So there exists N such that $(*)$ $\underbrace{f_j^{N+1}}_{g_j} \underbrace{f_i^N m_i}_{b_i} = \underbrace{f_i^{N+1}}_{g_i} \underbrace{f_j^N m_j}_{b_j}$. Since there finitely many such pairs, we can find an N such that this is true for all pairs i, j. Note that $D(g_) = D(f_j^{n+1}) = D(f_j)$. Thus $D(f) = \cup D(g_i) = D((g_1, ..., g_n))$. Thus $f \in \sqrt{(g_1, ..., g_n)}$ which implies $f^t = c_1 g_1 + ... + c_n g_n$ for some t and some c_i. Define $c = b_1 c_1 + ... + b_n c_n$. Then $g_j c = b_j f^t$ which implies $\frac{m_i}{f_j} = \frac{b_j}{g_j} = \frac{c}{f^t}$. Then $\psi(\frac{c}{f^t}) = s$. □

Exercise. Define the morphisms of ringed spaces and restrictions of a sheaf to open sets $\mathcal{O}_x|_y$. (If $f : X \to Y$ is continuous, $\mathcal{F}$ a sheaf on X, define $f_*\mathcal{F}$ a sheaf on Y.

Definition. *A morphism of ringed spaces $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ consists of a continuous map $f : X \to Y$ and ring homomorphisms $\phi_V : \mathcal{O}_Y(V) \to \mathcal{X}(f^{-1}(V))$ for all open sets $V \subseteq Y$ such that $\{\phi_V\}$ commute with restrictions, that is, for $V_1 \subseteq V_2$, we have the following diagram commutes:*

$$\begin{array}{ccc} \mathcal{O}_Y(V_2) & \xrightarrow[\phi_{V_2}]{} & \mathcal{O}_X(f^{-1}(V_2)) \\ \text{restriction}\downarrow & & \downarrow\text{restriction} \\ \mathcal{O}_Y(V_1) & \xrightarrow[\phi_{V_1}]{} & \mathcal{O}_X(f^{-1}(V_1)) \end{array}$$

Lemma. *Two morphism compose to a morphism*

Proof. Let $(f, \phi) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ and $(g, \psi) : (Y, \mathcal{O}_Y) \to (Z, \mathcal{O}_Z)$. Clearly, $g = g \circ g$ is continuous. Let $W_1^{open} \subseteq W_2^{open} \subseteq Z$. Then, we get the diagram

$$\begin{array}{ccccc} \mathcal{O}_Z(W_2) & \xrightarrow[\psi_{W_2}]{} & \mathcal{O}_Y(g^{-1}(W_2)) & \xrightarrow[\psi_{g^{-1}(W_2)}]{} & \mathcal{O}_X(f^{-1}g^{-1}(W_2)) = \mathcal{O}_X(h^{-1}(W_2)) \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{O}_Z(W_1) & \xrightarrow[\psi_{W_1}]{} & \mathcal{O}_Y(g^{-1}(W_1)) & \xrightarrow[\psi_{g^{-1}(W_1)}]{} & \mathcal{O}_X(f^{-1}g^{-1}(W_1)) = \mathcal{O}_X(h^{-1}(W_1)) \end{array}$$

which commutes. □

Proposition. *A ring homomorphism induces a morphism on the structure sheaves of the spectra.*

Proof. Let $\theta : R \to S$ be a ring homomorphism.

- Then, by AMCH1#21, for $X = SpecS, Y = SpecR$, there exists a continuous map $f : X \to Y$ such that $p \mapsto \theta^{-1}(p)$.
- First define ϕ on basic open sets: Note that $f^{-1}(D(g)) = D(\theta(g))$ as $x \in f^{-1}(D(g))$ if and only if $f(x) \in D(g)$ which is if and only if $g \notin f(x) = \theta^{-1}(x)$ which is if and only $\theta(g) \notin x$ which is if and only if $x \in D(\theta(g))$. So, we can define $\phi_{D(g)} : \underbrace{\mathcal{O}_Y(D(g))}_{R[g^{-1}]} \to \underbrace{\mathcal{O}_X(D(\theta(g)))}_{S[\theta(g)^{-1}]}$ by the universal property of localization applied to $R \to S \to S[\theta(g)^{-1}]$.

  Claim: $\{\phi_{D(g)}\}$ commute with restrictions.

  Proof: Suppose $D(g_2) \subseteq D(g_1)$. Then, we have the diagram where the row are restriction maps

$$\begin{array}{ccccc} R & \longrightarrow & R[g^{-1}] & \longrightarrow & R[g_2^{-1}] \\ \downarrow\theta & & \downarrow\psi_{D(g_1)} & & \downarrow\phi_{D(g_2)} \\ S & \longrightarrow & S[\theta(g_1)^{-1}] & \longrightarrow & S[\theta(g_2)^{-1}] \end{array}$$

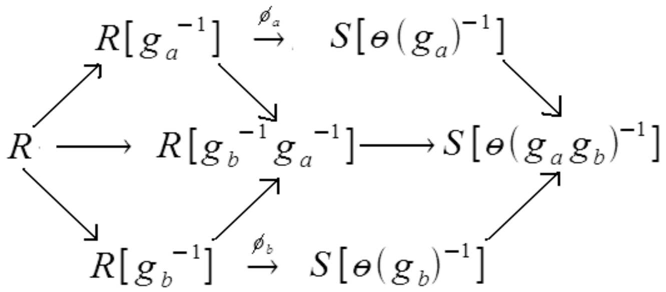
  We know that $R \to R[g_1^{-1}] \to R[g_2^{-1}] \to S[\theta(g_2)^{-1}]$ commutes with $R \to R[g_1^{-1}] \to S[\theta(g_1)^{-1}] \to S[\theta(g_2)^{-1}]$. Since they both go through $R[g_1^{-1}]$, we see the left box of the diagram commutes by uniqueness of the map obtained by the universal property of localization. Thus the diagram commutes.

  Now, we want to extend ϕ to all open sets $V \subseteq Y$. Note that if $V = \cup_a D(g_a)$, then $f^{-1}(V) = \cup_a f^{-1}(D(g_a)) = \cup_a D(\theta(g_a))$. So we can define $\phi_V : \mathcal{O}_Y(V) \to \mathcal{O}_X(f^{-1}(V))$ by extending our definition of $\phi_{D(g)}$. Consider the following diagram

$$\begin{array}{ccc} \mathcal{O}_Y(V) & \longrightarrow & \mathcal{O}_X(f^{-1}(V)) \\ \downarrow & & \downarrow \\ \mathcal{O}_Y(D(g_a)) & \longrightarrow & \mathcal{O}_X(D(\theta(g_a))) \end{array}$$

  For simplicity, let $U_a = D(\theta(g_a))$ so that $f^{-1}(V) = \cup_a U_a$.

  Claim: $\phi_a(s_a)|_{U_a \cap U_b} = \phi_b(s_b)|_{U_a \cap U_b}$.



Proof: First note $U_a \cap U_b = D(\theta(g_a)) \cap D(\theta(g_b)) = D(\theta(g_a g_b))$. Note that $\phi_a(s_a) \mapsto \phi_a(s_a)|_{U_a \cap U_b}$ and $\phi_b(s_b) \mapsto \phi_b(s_b)|_{U_a \cap U_b}$. By commutivity of the diagrams, these must be the same.

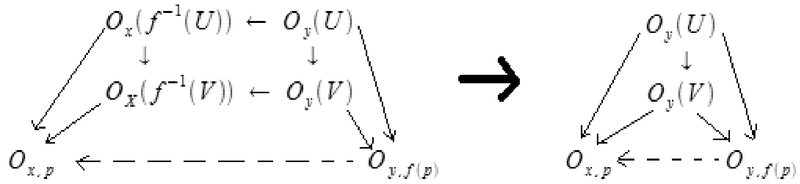
Now, we can use F2 from the definition of sheaf to define a unique ϕ_V. By similar arguments to the above, we can see $\{\phi_V\}$ commute with restrictions.

□

Definition. *A **locally ringed space** $(X, \mathcal{O}_X)$ is a ringed space such that the stalk $\mathcal{O}_X, p$ is a (quasi-)local ring for all $p \in X$.*

Example. If A is a commutative ring, then $Spec(R) = (X, \mathcal{O}_X)$ is a locally ringed space.

Assume $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a morphism of ringed spaces such that X, Y are locally ringed spaces. Then $f : X \to Y$ is continuous and for all open $U \subseteq Y$ we have $\theta_U : \mathcal{O}_Y(U) \to \mathcal{O}_X(f^{-1}(U))$. Let $p \in X$ and $q = f(p) \in Y$. Suppose $f(p) \in V^{open} \subset U^{open} \subset Y$.



Thus we have $\theta_p : \mathcal{O}_{X,p} \leftarrow \mathcal{O}_{Y,f(p)}$ such that the diagrams above commute.

Definition. *Let $(A, m), (B, n)$ be local rings. A **local homomorphism** from A to B is a ring homomorphism $\phi : A \to B$ such that $\phi(m) \subseteq n$. Equivalently $\phi^{-1}(n) = m$. Now $(f, \theta) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a **morphism of locally ringed spaces** provided both are locally ringed spaces and for all $p \in X$, we have $\theta_p : \mathcal{O}_{X,p} \leftarrow \mathcal{O}_{Y,f(p)}$ is a local homomorphism.*

Suppose $\phi : R \leftarrow S$ is a ring homomorphism, $X = SpecR, Y = SpecS$. This yields the maps $(X, \mathcal{O}_X \xrightarrow{f,\theta} (Y, \mathcal{O}_Y)$ where $f : X \to Y$ is defined by $p \mapsto \phi^{-1}(p)$.

$$\begin{array}{ccc} O_x(D(\phi(S))) = O_x(f^{-1}(D(S))) = R\left[\frac{1}{\phi(s)}\right] & \leftarrow & O_y(D(S)) = S\left[\frac{1}{s}\right] \\ \downarrow & & \downarrow \\ R_p = O_{x,p} & \leftarrow & O_{y,f(p)} = S_{f(p)} \end{array}$$

Now θ_p is a local homomorphism (If $\frac{a}{s} \in qS_q$, then $a \in q = \phi^{-1}(p)$ implies $\phi(a) \in p$ and thus $\frac{\phi(a)}{\phi(s)} \in pR_p$).

Recall. If (X, R) is a ringed space and $U \subseteq X$ is open, then we have a ringed space $(U, R|_U)$ where $R|_U(V) = R(V)$ for all $V \subseteq U$.

Definition. *A **scheme** is a locally ringed space $(X, \mathcal{O}_X)$ such that for each point p there exists an open neighborhood U of p and a ring R such that $(U, \mathcal{O}_X|_U)$ is isomorphic to $SpecR$ as locally ringed spaces. An **affine scheme** is a locally ringed space $(X, \mathcal{O}_X)$ that is isomorphic as locally ringed spaces to a scheme of the form $SpecR$ for some ring R.*

Theorem. *The category of affine schemes is dual to the category of rings.*

Proof. Define functions $R \rightsquigarrow SpecR$ and $(X, \mathcal{O}_X) \rightsquigarrow \mathcal{O}_X(X)$. Check that $\rightsquigarrow$ is faithful (i.e., distinct maps go to distinct maps), full (i.e., every arrow comes from an arrow), and representative (i.s., every affine scheme is isomorphic to one in $SpecR$). □

2.1 Graded Rings

Definition. *A **graded ring** is a commutative ring R with a direct sum decomposition as abelian additive groups ($R = R_0 \oplus R_1 \oplus \cdots$) such that $R_m R_n \subseteq R_{m+n}$.*

Example. $R = k[x_0, ..., x_n]$. Let $R_i = \{cx_0^{r_0} \cdots x_n^{r_n} | x \in k, r_0 + ... + r_n = i\}$. Then $R = \oplus R_i$ is clearly a graded ring.

Definition. *An element of R_d is said to be homogenous of degree d.*

Definition. *If R is a graded ring, a **graded $R-$module** is an $R-$module M such that M decomposes ($M = \oplus M_i$) and $R_i M_j = M_{i+j}$. An element of M_i is homogenous of degree i.*

Lemma. *Suppose R is a graded ring, M a graded $R-$module. Let $N \subseteq M$ be a submodule. Then TFAE*

- *N is generated by homogenous elements of M (that is, N is generated by $\cup_d (N \cap M_d)$).*
- *$N = \oplus_d (N \cap M_d)$.*
- *If $a \in N$ and $a = a_0 + ... + a_m$ with $a_i \in M_i$, then $a_i \in N$ for all i.*

Proof. Exercise. □

Definition. *A **homogenous submodule** of M is a submodule N satisfying the properties of the Lemma. In this case, N is a graded module and so is $M/N = \oplus_d \frac{M_d}{N \cap M_d}$.*

Let $S = \oplus_{d>0} S_d$ be a graded ring. Note that S_0 is a ring (as two element of degree 0 multiply to an element of degree 0). Let $S_+ = \oplus_{d>1} S_d$. This is an ideal of S, called the **irrelevant ideal** (as $V(S_+) = 0$). If $S = k[x_0, ..., x_n]$, then $S_+ = (x_0, ..., x_n)$.

We are going to build a (non-affine) scheme $Proj(S) = (X, \mathcal{O}_X)$.

- Define $X = \{P \in SpecS | P \not\supseteq S_+$ is a homogenous ideal$\}$. (Note: This says that if $P \in Proj(S)$, then there exists a homogenous element $f \in S_+$ such that $f \notin P$).
- If f is a homogenous element of positive degree, let $D_+(f) = \{P \in Proj(S) | f \notin P\}$. (By the note, this says $\{D_+(f)\}$ covers $Proj(S)$).
- Closed sets in $Proj(S)$ are $V(I) = \{P \in Proj(S) | P \supseteq I\}$ where I is an homogenous ideals. This defines our topology.
- Let $P \in SpecS$. Define $S_{(P)} = \{\frac{a}{t} \in T^{-1}S | a, t$ are homog of same degree$\}$ where $T = \{$ homog elements of $S \setminus T\}$. This is a local ring and will be the stalks for our scheme.
- Define the Sheaf: If $U^{open} \subseteq X$, an element of $\mathcal{O}_X(U)$ is a function s such that
  1. $s(P) \subseteq S_{(P)}$ for all $P \in U$.
  2. For all $P \in U$, there exists a neighborhood V such that $p \in V \subseteq U$ and there exists $a, f \in S_d$ for some d such that for all $Q \in V$ $f \notin Q$ and $s(Q) = \frac{a}{f}$ in $S_{(Q)}$.
- $\mathcal{O}_X(D_+(f)) = S_{(f)} = \{$degree 0 elements of $S[\frac{1}{f}]\}$.

2.2 Regular Functions and Projective Varieties

Suppose Y is an affine variety in $\mathcal{A}_k^n$ with $k = \overline{k}$. Then $Y = V(P)$ where $P \in Spec(k[x_1, ..., x_n])$. The affine coordinate ring $A(Y)$ is $k[x_1, ..., x_n]/P$. We can think of elements of $A(Y)$ as functions $U \to k$.

Definition. *If U is an open subset of Y, a* ***regular function*** *of U is a function $s : U \to k$ that is locally a quotient of elements of $A = A(Y)$. Denote $\mathcal{O}_Y(U)$ to be the ring of regular functions on U. Then a function $s : U \to k$ is in $\mathcal{O}_Y(U)$ if and only if for all $p \in U$ there exists an open neighborhood $V \subseteq U$ with $p \in V$ and elements $a, f \in A$ such that $V \subseteq D(f)$ and $s(q) = \frac{a(q)}{f(q)}$ for all $q \in V$.*

Note. $(Y, \mathcal{O}_Y)$ is a ringed space. Call $\mathcal{O}_Y$ the **sheaf of regular functions** on Y.

Now, we have an embedding $\beta : Y \xrightarrow{homeomorphism} Spm(A) \hookrightarrow Spec(A) = X$. We can define a morphism of ringed spaces from this:

- If $U^{open} \subseteq X$, we want a ring homomorphism $\mathcal{O}_X(U) \to \mathcal{O}_Y(\beta^{-1}(U))$ defined by $s \mapsto s'$ where $s'(y) = s(m_Y) \in A_{m_y} \twoheadrightarrow \frac{A_{m_Y}}{m_Y A_{m_Y}} = \frac{A}{m_Y} = k$ (i.e., $s'(y) = s(m_Y) + m_Y A_{m_Y} = \overline{s(m_Y)} \in k$.)

  Claim: $\mathcal{O}_X(U) \xrightarrow{\theta_U} \mathcal{P}_Y(\beta^{-1}(U))$ is an isomorphism of rings.

  Proof: Onto: Let $\sigma \in \mathcal{O}_Y(\beta^{-1}(D(f)))$ (we may assume $U = D(f)$ by the sheaf axioms). Shrink $D(f)$ to a smaller set, if necessary, such that $\sigma(y) = \frac{a(y)}{f(y)}$ for all $y \in D(f)$. Let $s \in \mathcal{O}_X(D(f))$ be defined by $s(p) = \frac{a}{f} \in A_p$ for all $p \in D(f)$. Obviously, $\theta_{D(f)}(s) = \sigma$.

  1-1: Suppose $s \in \mathcal{O}_X(U)$ and $s \mapsto 0$ By the sheaf axioms, it is enough to show s is locally 0. Shrink down to a set $D(f)$ where $s = \frac{a}{f}$. Then $\overline{a(p)} = 0$ for all $p \in D(f)$ (i.e., when $f(p) \neq 0$). Then $(p) \in D(f)$ implies $a \in m_p$ and so $p \notin D(a)$. So $D(fa) = D(f) \cap D(a) = \emptyset$ which says $fa = 0$. Thus $s|_{D(f)} = 0$.

Definition. *Suppose $\gamma : Y \to X$ is a continuous map of topological spaces. Let $\mathcal{F}$ be a sheaf on Y. Then $\gamma_*\mathcal{F}$, a sheaf on X, is defined by $(\gamma_*\mathcal{F})(U) = \mathcal{F}(\gamma^{-1}(U))$. This is called the* ***direct image sheaf****.*

We've shown if an affine variety $Y \hookrightarrow X = Spec(A)$ for $A = A(Y)$, then $(X, \mathcal{O}_X \cong (X, \beta_*\mathcal{O}_Y)$ as ringed spaces.

Let Y be a **projective variety**, that is, a closed irreducible subset of $\mathbb{P}_k^n$. Then Y is the vanishing set of some homogenous prime ideal P in $k[x_0, ..., x_n]$. Let $S = k[x_0, ..., x_n]/P$ be the homogenous coordinate ring. Note that in this case (the projective case), the coordinate ring is not unique. We have $Proj(S) = (X, \mathcal{O}_X)$. Now, again we have

$$\beta : Y \xrightarrow{homeomorphism} \{\text{closed points of } X\} \hookrightarrow X$$

where $(X, \mathcal{O}_X) \cong (X, \beta_*(\mathcal{O}_Y))$. Here, the regular functions are quotients of homogenous polynomials of the same degree. Why can't we think of an arbitrary homogenous element σ of S as a function $Y \to k$? Say $\deg \sigma = d > 0$. Then $t^d\sigma(a_0, ..., a_n) = \sigma(ta_0, ..., ta_n)$.

Recall. Suppose $Y \subset \mathcal{A}_k^n$ is closed. Then $I(Y) = \{f \in k[x_1, ..., x_m] | f(p) = 0 \text{ for all } p \in Y\}$ and $A = A(Y) = k[x_1, ..., x_n]/I(Y)$. Then $\mathcal{O}_Y$ is the sheaf of regular functions (i.e., $\sigma : Y \to L$ which are locally defined as fractions) on Y and $\Gamma(Y, \mathcal{O}_Y) = A$.

Now, let $Y = \mathbb{P}_k^n$ (so that $Y = U_0 \cup \cdots \cup U_n$) where $U_i = \{$ elements with nonzero entry in the i^{th} spot$\})$ and let $S = k[x_0, ..., x_n]$. Then $\mathcal{O}_y$ is the sheaf of regular functions(i.e., $\sigma : Y \to k$ which are locally defined as fractions of degree 0) on Y.

Claim: $\Gamma(Y, \mathcal{O}_Y) = k$.

Proof: Let $\sigma \in \Gamma(Y, \mathcal{O}_Y)$. Consider $\sigma_i := \sigma|_{U_i}$. This is an element of degree 0 in the ring $S[\frac{1}{x_i}]$. Write $\sigma_i = \frac{1}{x_i^{a_i}} f_i$ where $a_i \geq 0$ and $f_i \in S$ is homogenous of degree a_i. We can assume (by cancelling out factors of x_i) that $x_i \nmid f_i$. Choose $j \neq i$. Then $\sigma_j = \frac{1}{x_j^{a_j}} f_j$ is such that $x_j \nmid f_j$. Of course, by properties of sheafs, we see $\sigma_i = \sigma_j$ implies $x_j^{a_j} f_i = x_i^{a_i} f_j$. Now, if $a_i > 0$, $x_i | x_j^{a_j} f_i$ and since $x_i \nmid x_j^{a_j}$, we have $x_i \mid f_i$, a contradiction. Thus $a_i = 0$ for all i which implies $\sigma_i = f_i$ for all i and $deg f_i = 0$ for all i. Thus $\sigma_i \in k$.

2.3 Artin Rings - Ch 8 A&M

Proposition (8.1)**.** *In Artinian Rings, prime ideals are maximal.*

Proof. Let $p \in SpecA$ where A is an Artinian ring. Then $D = A/p$ is Artin and a domain. Let $x \in D \setminus \{0\}$. Then $(x^n) = (x^{n+1})$ for some n. So there exists $y \in D$ such that $x^n = x^{n+1}y$. Cancelling, this says $1 = xy$ which implies D is a field and thus p is maximal. □

Corollary (8.2)**.** *The Jacobson Radical and Nilradical of an Artinian ring are the same.*

Proposition (8.3)**.** *An Artin ring A has finitely many maximal ideals.*

Proof. Let $S = \{m_1 \cap \cdots \cap m_r | m_i \in Spm(A)\}$. Then $S \neq \emptyset$ and has a minimal element by definition of Artinian. Say $m_1 \cap \cdots \cap m_n$ is the minimal element. Then for all $m \in Spm(A)$, we see $m \cap m_1 \cap \cdots \cap m_n = m_1 \cap \cdots \cap m_n$. By Prop 1.11, this says $m \supseteq m_i$ for some i, but since both are maximal, we have $m = m_i$. □

Proposition (8.4)**.** *In an Artin ring A, the nilradical $\mathcal{R}$ is nilpotent.*

Proof. By DCC, we know $\mathcal{R}^k = \mathcal{R}^{k+1} = ... =: a$ for some $k > 0$. Suppose $a \neq 0$. Let $\Sigma = \{b | b \text{ is an ideal of } A, ab \neq 0\}$. Then $\Sigma \neq \emptyset$ as $a \in \Sigma$. Now let c be a minimal element of Σ. Then there exists $x \in c$ such that $xa \neq 0$. So $(x) \subseteq c$ which implies $(x) = c$ by minimality. But $(xa)(a) = xa^2 = xa \neq 0$ and $(xa) \subseteq (x)$ which implies $(xa) = (c)$ by minimality. Hence $x = xy$ for some $y \in a$. Then $x = xy = (xy)y - xy^2 = xy^3 = \cdots$. Since $y \in a = \mathcal{R}^k \subseteq \mathcal{R}$, y is nilpotent. So $x = xy^n = 0$, a contradiction. So $a = 0 = \mathcal{R}^k$. □

Definition. *A chain of prime ideals in A is a sequence of prime ideals $p_0 \subsetneq p_1 \subsetneq \cdots \subsetneq p_n$. Then*

$$dim(A) = \sup\{ \textit{ lengths of chains of primes}\}.$$

Proposition (8.5)**.** *A ring A is Artin if and only if A is Noetherian and $dim(A) = 0$.*

Proof. (⇒:) By Proposition 8.1, $dim(A) = 0$. Let $m_1, ..., m_n$ be the maximal ideals of A (by Proposition 8.3). Now $\prod_{i=1}^n m_i^k = (\cap_{i=1}^n m_i)^k = \mathcal{R}^k = 0$ for some k. Thus A is Noetherian by Proposition 6.11.

(⇐:) By Proposition 7.13, the zero ideal has a primary decomposition, which implies A has only a finite number of minimal prime ideals. Since $dim(A) = 0$, those are all maximal. Hence $\mathcal{R} = \cap_1^n m_i$ where m_i are the minimal primes. By Proposition 7.15, $\mathcal{R}^k = 0$ which implies $\prod_1^n m_i^k = 0$ for some k. Again, by Proposition 6.1, A is Artinian. □

Remarks. If A is a local Artin ring with maximal ideal m, then $m = \mathcal{R}(A)$. Hence m is nilpotent by Proposition 8.4 and $x \in m$ is nilpotent. Thus in a local Artinian ring, each element is either a unit or a nilpotent.

Proposition (8.6)**.** *Let A be a Noetherian local ring with maximal ideal m. Then either*

- $m^n \neq m^{n+1}$ *for all n or*
- $m^n = 0$ *for some n, in which case A with be Artinian.*

Proof. Suppose $m^n = m^{n+1}$ for some n. By Nakayama's Lemma, $m^n = 0$. Let $p \in SpecA$. Then $m^n = 0 \subseteq p$ implies $m = p$. Then every prime is maximal and by 8.4, A is Artinian. □

Theorem (8.7 Structure Theorem for Artinian Rings)**.** *Let A be Artinian. Then $A \cong \prod_{m \in Spm(A)} A_m =: A'$. (Note A_m are local Artinian rings).*

Proof. Note that $|Spm(A)| < \infty$, so say $A' = \prod_{i=1}^n A_{m_i}$. Define $\phi : A \to A'$ by $a \mapsto (\frac{a}{1}, ..., \frac{a}{1})$. This is clearly a well-defined homomorphism. It's also injective as for $a \in A$, we see $\phi(a) = (\frac{0}{1}, ..., \frac{0}{1})$ implies $Anna = A$, and thus $a = 0$. So it remains only to show ϕ is surjective.

Claim 1: If $i \neq j$, then $0 \in (A \setminus m_i)(A \setminus m_j) =: S$.

Proof: If not, then there exists $p \in SpecA$ such that $p \cap S = \emptyset$ which implies $p \subseteq m_i \cap m_j$. Since A is Artinian, p is maximal and thus $p = m_i \cap m_j$ which implies $m_i = m_j$, a contradiction.

Claim 2: If $i \neq j$, then $(A_{m_i})_{m_j} = (0)$.

Proof: Let $\alpha = \frac{\frac{a}{s}}{t} \in (A_{m_i})_{m_j}$. If $\alpha = \frac{0}{1}$, then we need that there exists $b \in A \setminus m_j$ (since A_{m_i} is an $A-$module) such that $b(\frac{a}{s}) = 0$ in A_{m_i} which implies there exists $c \in A \setminus m_i$ such that $cba = 0$. Of course, by Claim 1, cb exists. Thus $\alpha = \frac{0}{1}$.

Now consider $\phi_{m_j} : A_{m_j} \to (\prod_1^n A_{m_i})_{m_j}$. By the claim, we see that $(\prod_1^n A_{m_i})_{m_j} = \prod_1^n (A_{m_i})_{m_j} \cong (A_{m_i})_{m_j} = A_{m_j}$. Thus ϕ_{m_j} is clearly onto. Since ϕ_{m_j} is onto for all m_j, we have that ϕ is onto. □

The following is a random result by Roger:

Theorem. *Let $R \neq 0$ be a commutative ring. Then there does not exist an injective $R-$homomorphism such that $R^{n+1} \rightarrowtail R^n$.*

Proof. Suppose first that R is Artinian. Let $\lambda(R) = r$. WE have the short exact sequence $0\ toR^m \to R^{m+1} \to R \to 0$ which inductively by the additive of λ gives us $\lambda(R^{m+1}) = (m+1)r$. So $\lambda(R^n) = nr$ and $\lambda(R^{n+1}) = (n+1)r$. Thus if $R^{n+1} \rightarrowtail R^n$, then we get the short exact sequence $0 \to R^{n+1} \to R^n \to coker \to 0$ and so $\lambda(R^n) = \lambda(R^{n+1} + \lambda(coker)$, a contradiction as $\lambda > 0$. Now, assume R is Noetherian. Suppose $\alpha : R^{n+1}R^n$. Let p be a minimal prime. Then $\alpha_p : R_p^{n+1} \rightarrowtail R_p^n$. Note $dim R_p = 0$ and R_p is Noetherian. Thus R_p is Artinian, a contradiction. Lastly, assume R is an arbitrary commutative ring. Suppose β is an $n \times n+1$ matrix of β over R. We want to show $\ker \beta \neq 0$. Let $\beta = [b_{ij}]$. Let R_0 be the smallest subring of R. Then R_0 is either $\mathbb{Z}$ or $\mathbb{Z}/(k)$. Let $S = R_0[b_{ij}] \hookrightarrow R$. Note S is Noetherian by the Hilbert Basis Theorem. Therefore, there exists $s = \begin{bmatrix} s_1 \\ \vdots \\ s_n \end{bmatrix} \in S^{n+1}$ such that $\beta s = 0$, where $s \neq 0$. But $s \in R^{n+1}$ too. Thus $\ker \neq 0$ and thus there are no injections. □

2.4 Graded Rings - Ch 10 A&M

Recall. We say A is a **graded ring** if $A = \oplus_{n=0}^{\infty} A_n$ where $A_m A_n \subseteq A_{m+n}$. Say M is a **graded** $A-$ **module** if $M = \oplus_{n=0}^{\infty}$ and $A_m M_n \subseteq M_{m+n}$.

Proposition (10.7 and more)**.** *Let $R = \oplus_{i \geq 0} R_j$ be a graded ring and $R^+ = \oplus_{i \geq 1} R_i$ an ideal of R. TFAE*

1. *R is Noetherian.*
2. *R_0 is Noetherian and R^+ is a finitely generated ideal of R.*
3. *$_0$ is Noetherian and R a finitely generated R_0-algebra.*

Proof. (3) ⇒ (1) Any finitely generated algebra over a Noetherian ring is Noetherian by the Hilbert Basis Theorem.

(1) ⇒ (2) Note that $R = R_0 \oplus R^+$ implees $R/R^+ \cong R_0$. Thus R_0 is Noetherian and R^+ is finitely generated as it is an ideal of a Noetherian ring.

(2) ⇒ (3) As R^+ is finitely generated, it has a finite generating set where each generator is a sum of homogenous elements of positive degree. Say $R^+ = (f_1, ..., f_n)R$. We will show $R = R_0[f_1, ..., f_n]$. To do so, assume not, that is, suppose $R_0[f_1, ..., f_n] \subsetneq R$. Then there exists an element of $R \setminus R_0[f_1, ..., f_n]$ and thus a homogenous element $g \in R \setminus R_0[f_1, ..., f_n]$ of least degree. Then $g \notin R_0$ which implies $g \in R^+$. So $g = \sum r_i f_i$ for $r_i \in R$ where r_i is a sum of homogenous elements of R. Then $g \in \sum (R_{\deg g - \deg f_i}) f) i$. Say $g = \sum h_i f_i$ where h_i is homogenous of degree $\deg g - \deg f_i$. Then $\deg h_i < \deg g$. Since g was chosen to have least degree, each $h_i \in R_0[f_1, ..., f_n]$ and thus so is g.

□

Note that this says if $R = \oplus R_i$ is a graded Noetherian ring, then we can write $R = R_0[\alpha_1, ..., \alpha_s]$ where α_i is homogenous of degree $\alpha_i > 0$. Note this says R_n is the R_0 span of monomials of degree n. Since there are finitely many such monomials, R_n is a finitely generated R_0-module.

Now let $M = \oplus M_i$ be a finitely generated $R-$module. As M is finitely generated, it can be generated by finitely many homogenous elements, say $u_1, ..., u_r$ such that $\deg u_j = r_j$. If $a = \inf\{r_j\} \in \mathbb{Z}$, then any homogenous element of M has degree $\geq a$. So $M_n = 0$ for $n < a$. Now, $M_n = \sum_{j=1}^{r} (R_{n-r_j}) u_j$ and since each R_{n-r_j} is finitely generated over R_0, we see M_n is a finitely generated R_0-module.

Thus, if R_0 is an Artinian ring, then the length of M_n over R_0 is finite.

Definition. *Let R be a ring, α an ideal of R and M an $R-$module. An (infinite) chain $M = M_0 \supseteq M_1 \supseteq \cdots \supseteq M_n \supseteq \cdots$, where M_n are submodules of M, is called a **filtration** of M, denoted (M_n). It is an $\alpha-$**filtration** if $\alpha M_n \subseteq M_{n+1}$ for all n and a **stable** $\alpha-$**filtration** if $\alpha M_n = M_{n+1}$ for $n >> 0$.*

Example. $(\alpha^n M)$ is a stable $\alpha-$filtration.

Let A be a ring, α an ideal of A. Then we can form a graded ring $A^* = \oplus_{n\geq 0}\alpha^n$. Similarly, if M is an $A-$module and (M_n) is an $\alpha-$filtration of M, then $M^* = \oplus_{n\geq 0}M_n$ is a graded A^*-module as $a^m M_n \subseteq M_{m+n}$.

Lemma (10.8)**.** *Let A be a Noetherian ring, M a finitely generated $A-$module, (M_n) an $\alpha-$filtration of M. TFAE*

1. *M^* is a finitely generated A^* module.*
2. *The filtration (M_n) is stable.*

Proof. Define $M_n^* = M_0 \oplus \ldots \oplus M_n \oplus \alpha M_n \oplus \alpha^2 M_n \oplus \ldots$. This is an A^*-module. Note that M_n is finitely generated as an $A-$module for all which implies $M_0 \oplus \cdots \oplus M_n$ is finitely generated as an $A-$module and thus M_n^* is finitely generated as an A^*-module. Note that $\{M_n^*\}$ form an ascending chain and $\cup M_n^* = M^*$. Now, since A is Noetherian, α is finitely generated. Say $\alpha = (x_1, ..., x_n)$. Then $A^* = A[x_1, ..., x_n]$ which is Noetherian by the Hilbert Basis Theorem. So M^* is finitely generated as an A^*-module if and only if the chain $\{M_n^*\}$ stops (that is $M^* = M_n^*$ for some n) which is if and only if $M_{n+r} = \alpha^r M_n$ for all $r >> 0$ which is if and only if (M_n) is a stable filtration. □

Proposition (10.9 **Artin-Rees Lemma**)**.** *Let A be a Noetherian ring, α an ideal of A, M a finitely generated $A-$module, (M_n) a stable $\alpha-$filtration of M. If M' is a submodule of M, then $(M' \cap M_n)$ is a stable $\alpha-$filtration of M'.*

Proof. Certainly $M' \cap M_0 \subseteq M' \cap M_1 \subseteq \cdots$. Furthermore, $\alpha(M' \cap M_n) \subseteq \alpha M' \cap \alpha M_n \subseteq M' \cap M_{n+1}$. Thus $(M' \cap M_n)$ is an $\alpha-$filtration. Hence it defines a graded A^*-module $M'^* = \oplus_n (M' \cap M_n)$ which is a submodule of M^*. Since A^* is Noetherian, M'^* is finitely generated. Thus the filtration $(M' \cap M_n)$ is stable by Lemma 10.8. □

Corollary (10.10)**.** *There exists $k \in \mathbb{Z}$ such that $\alpha^n M) \cap M' = \alpha^{n-k}((\alpha^k M) \cap M')$ for all $n \geq k$.*

Proof. Take $M_n = \alpha^n M$ in 10.9. Since $(\alpha^n M \cap M')$ is stable, we have $a^r(M' \cap M_k) = M' \cap M_{k+r}$. Now, take $r = n - k$. □

Note. Corollary 10.10 is often called the Artin-Rees Lemma.

2.5 Dimension Theory - Ch 11 A&M

Definition. *Let R be a $\mathbb{N}-$graded, Noetherian ring and M a finitely generated $\mathbb{Z}-$graded $R-$module. Let ℓ be an additive function with values in $\mathbb{Z}$ on the class of finitely generated R_0 modules. The numerical function $t_M : \mathbb{Z} \to \mathbb{Z}$ defined by $n \mapsto \ell(M_n)$ is called the **Hilbert function** of M with resect to the function ℓ. The **Hilbert (Poincaré) Series** for M (with respect to ℓ) is defined as $P(M,t) = \sum_{n\in\mathbb{Z}} P(M,n)t^n \in \mathbb{Z}((t))$ where $\mathbb{Z}((t))$ is the set of all formal Laurent series with integer coefficients.*

Remarks. Let $R = \oplus_{i\geq 0}R_i$ be a graded righ.

1. Let M be a graded module over R. Then for all $d \in \mathbb{Z}$, define $M(d)$ such that $M(d)_n = M_{n+d}$. Then $M(d) = \oplus_{n\in\mathbb{Z}}M(d)_n$.

   Example. $R(d)$ is the ring R itself whose basis $\{1\}$ is homogenous of degree $-d$, that is $1 \in R(d)_{-d} = R_0$.

   Note. $R(d)$ is a graded module but not a graded ring as if $x, y \in R(d)_0$, then $xy \in R(2d) = R(d)_d$ and hence if $d \neq 0$, then $R(d)$ is not graded.

2. If M, N are graded $R-$modules, then an $R-$linear map $f : M \to N$ is called a **graded homomorphism of degree** r if $f(M_i) \subseteq N_{i+r}$ for all i.

3. If $f : M \to M'$ is a graded homomorphism of degree 0, then $\ker f$ and imf are homogenous submodules (that is, they are graded). In this case

$$0 \to \ker f \to M \xrightarrow{f} M' \to cokerf \to 0$$

   is an exact sequence of graded modules (that is, the maps have degree 0).

   Recall.

(a) A submodule N of a graded module M is homogenous if it is generated by homogenous elements. Equivalently, N is homogenous if $u \in N$, then all homogenous components are in N.

(b) If N is homogenous, then it is graded, that is $N = \oplus N_i$ where $N_i = N \cap M_i$. So $N \to M$ is a degree zero map. Moreover, $M/N = \oplus M_i/N_i$ where $M_i/N_i = M_i/M_i \cap N \cong M_i + N/N$. So $M \twoheadrightarrow M/N$ is a degree 0 map.

Proof. For $y \in imf$, we know $y = \sum f(x_i)$ for some $x_i \in M$. As each $f(x_i) \in M_i'$, imf is generated by homogenous elements of M' belonging to imf. So imf is graded. For $y \in \ker f$, we have $\sum f(x_i) = 0$ where $y = \sum x_i$ with $x_i \in M_i$. As $f(x_i) \in M_i'$, we have $f(x_i) = 0$ for all i. So each $x_i \in \ker f$ and so $\ker f$ is graded. □

4. If M, N, L, M_i are finitely generated $\mathbb{Z}-$graded modules, then

(a) $H_{M(r)}(t) = t^{-r}P(M,t)$ for all $r \in \mathbb{Z}$.

(b) If $0 \to L \to M \to N \to 0$ is an exact sequence of graded modules, then $P(M,t) = H_L(T) + H_N(T)$. So if $0 \to M_s \to M_{s-1} \to \cdots \to M_0 \to 0$ is exact, then $\sum_{i=1}^{n}(i1)^i H_{M_i}(t) = 0$.

Proof. (a) Now $M(r)_n = M_{n+r}$ which implies $\ell(M(r)_n)t^n = \ell(M_{r+n})t^n$. Summing, this says $\sum \ell(M(r)_n)t^n = \sum \ell(M_{n+r}t^n = (\sum \ell(M_{n+r})t^{n+r})t^{-r}$. Thus $H_{M(r)}(t) = P(M,t)t^{-r}$.

(b) Note we can say $0 \to L_n \to M_n \to 0$ is an exact sequence of R_0 modules for all n. So $\ell(M_n) = \ell(L_n) + \ell(N_n)$ by the additivity of ℓ. So $\sum \ell(M_n)t^n = \sum \ell(L_n)t^n + \sum \ell(N_n)t^n$ which implies $P(M,t) = H_L(t) + H_N(t)$.

□

Theorem (11.1 **Hilbert Serre**)**.** *Let $R = \oplus_{n \geq 0} R_n$ be a graded Noetherian ring and $M = \oplus_{n \in \mathbb{Z}} M_n$ a finitely generated graded $R-$module. Write $R = R_0[f_1, ..., f_s]$ where $\deg f_i = k_i > 0$. Then $P(M,t) = \frac{f(t)}{\prod_{i=1}^{s}(1-t^{k_i})}$ for some $f(t) \in \mathbb{Z}[t^{\pm 1}]$. If M is $\mathbb{N}-$generated, then $f(t) \in \mathbb{Z}[t]$.*

Proof. Induct on s. If $s = 0$, then $R = R_0$ and M finitely generated over R_0 implies M can be generated by a finite number of homogenous elements $\{m_i\}$. If $e := \max\{\deg(m_i)\}$, then $M_j = 0$ for $j > e$. In this case, $P(M,t)$ is a Laurent polynomial. Now, suppose $s < 0$. Multiplication by f_s defines a homomorphism of graded modules $f_s : M(-k_s) \to M$. This yields the exact sequence of graded modules

$$0 \to K = \ker_{f_S} \to M(-k_S) \xrightarrow{f_S} M \to C = coker f_s \to 0.$$

As M is finitely generated over R, we have K, C are finitely generated over R. So $(*) = 0 = P(K,t) - t^{k_S}P(M,t) + P(M,t) - P(C,t)$. Note that $f_sK = f_sC = 0$ implies K, C are finitely generated over $B := R/(f_s)$ where B is a Noetherian graded R_0-algebra generated by $s-1$ elements. By induction,

$$P(K,t) = \frac{g(t)}{\prod_{i=1}^{s-1}(1-t^{k_i})}, P(C,t) = \frac{h(t)}{\prod_{i=1}^{s-1}(1-t^{k_i})}$$

for some $g(t), h(t) \in \mathbb{Z}[t^{\pm 1}]$. By $(*)$, we have

$$(1-t^{k_s})P(M,t) = P(C,t) - P(K,t) = \frac{f(t)}{\prod_{i=1}^{s-1}(1-t^{k_i})}$$

where $f \in \mathbb{Z}[t^{\pm 1}]$. □

Now, suppose $k_i = 1$ for all i. Then $\sum_{n \geq 0} P(M,t)t^n = \frac{f(t)}{(1-t)^d}$ and using the binomial expansionsion of $\frac{1}{(1-t)^d}$, we get $P(M,n) = \sum_{k=0}^{N} a_k \begin{pmatrix} d+n-k-1 \\ d-1 \end{pmatrix}$ for $n \geq N$ where d is the number of integers such that $1 - t^{k_i} \nmid f(t)$.

Corollary (11.2)**.** *Let $R = \oplus_{i \geq 0} R_i$ be a Noetherian ring, $M = \oplus_{i \geq 0} M_i$ finitely generated over R. If R is generated over R_0 by degree one elements, then there exists ϕ_M of degree $d(M) - 1 =: d - 1$ such that $P(M,n) = \phi_M(n)$ for $n >> 0$. Here ϕ_M is called the* ***Hilbert polynomial****.*

Notation. Define $d(M)$ to be the order of the pole of $P(M,t)$ at $t = 1$.

Proposition (11.3). *If $x \in A_k$ is not a zero divisor of M, then $d(M/xM) = d(M) - 1$.*

Proof. Let $x \in A_k$, not a zero divisor. Then $0 \to M_n \xrightarrow{f_x} M_{n+k} \to L_{n+k} := M_{n+k}/xM_n \to 0$ is a short exact sequence where f_x is multiplication by x. Applying λ, we see $\lambda(M_n) - \lambda(M_{n+k}) + \lambda(L_{n+k}) = 0$. Multiplying by t^{n+k} and summing, we see $t^k P(M,t) - (g(t) + P(M,t)) + P(L,t) = 0$ for $g(t) = \sum_1^k \lambda(M_n)t^n$. Then $(1-t^k)P(M,t) = P(L,t) + g(t)$ which says $d(M/xM) = d(L) = d(M) - 1$ as we can factor $1-t$ from the left hand side. □

Proposition (11.4). *Let A be a Noetherian local ring with maximal ideal m, M a finitely generated $A-$module, q a $m-$primary ideal (that is, $m = r(q)$) and (M_n) a stable $q-$filtration of M. Then*

1. *M/M_n has finite length for $n \geq 0$.*
2. *For $n >> 0$, the length is a polynomial $g(n)$ of degree $\leq s$ where s is the least number of generators of q.*
3. *The degree and leading coefficient of $g(n)$ depend only on M and q, not on (M_n).*

Proof.

1. Let $G(A) = \oplus_{n \geq 0} q^n/q^{n+1}$, $G(M) = \oplus_{n\geq 0} M_n/M_{n+1}$. Now, $G_0(A) = q^0/q^1 = A/q$, an Artinian local ring by 8.5. Also, $G(A)$ is Noetherian and $G(M)$ is a finitely generated graded $G(A)-$module by 10.22. Each M_n/M_{n+1} is a Noetherian $A-$module. Since (M_n) is a $q-$filtration, $qM_n \subseteq M_{n+1}$ which implies M_n/M_{n+1} is annihilated by q. So it is a Noetherian $A/q-$module and thus $\lambda(M_n/M_{n+1}) < \infty$ as A/q is Artinian. This implies $\lambda(M/M_n) < \infty$ and $\ell_n := \lambda(M/M_n) = \sum_{r=1}^n \lambda(M_{r-1}/M_r)$ (induct with the short exact sequence $0 \to M_{n-1}/M_n \to M/M_n \to M/M_{n-1} \to 0$).

2. Say $x_1, ..., x_s$ generate q. Then $\overline{x_i} \in q/q^2$ generate $G(A)$ as an $A/q-$ algebra. Note $\overline{x_i}$ have degree 1. So by Corollary 11.2, $\lambda(M_n/M_{n+1}) = f(n)$ for some polynomial $f(n)$ of degree $\leq s-1$ for $n >> 0$ (say $n > N$). By above, $\ell_{n+1} - \ell_n = \lambda(M_n/M_{n+1}) = f(n)$.

   Fact: Let f be a polynomial in $\mathbb{Q}[x]$ of degree d. Then there exists unique $r_i \in \mathbb{Q}$ such that $f(n) = \sum_{i=0}^d r_i \begin{pmatrix} n+i \\ i \end{pmatrix}$.

   Set $d := \deg f(n) \leq s-1$. Then for $n > N$, we have $\ell_n = (\ell_n - \ell_{n-1}) + (\ell_{n-1} - \ell_{n-2}) + ... + (\ell_{N+1} - \ell_N) + \ell_N$. Thus $\ell_n - \ell_N = f(n-1) + ... + f(N)$. Set $p(n) = f(n+N)$. Then $\ell_n - \ell_N = p(n-1-N) + ... + p(0)$. Set $m = n-1-N$. Then $\ell_{m+1+N} - \ell_N = \sum_{j=0}^m p(j) = \sum_{j=1}^m (\sum_{i=0}^d r_i \begin{pmatrix} j+i \\ i \end{pmatrix}$ by the fact. Switching the sums, $\ell_{m+1+N} - \ell_N = \sum_{i=i}^d r_i \sum_{j=0} \begin{pmatrix} j+i \\ i \end{pmatrix} = \sum_{i=0}^d r_i \begin{pmatrix} m+i+1 \\ i+1 \end{pmatrix}$, a polynomial of degree $d+1$. Then $g = \ell_N + \sum_{i=0}^p (i)$. Let $D = 1+N$ and $h(n) = g(n-D)$. Then $\ell_n = h(N)$ for $n \geq N$. Thus ℓ_n is a polynomial of degree $d+1 \leq s$.

3. Let $(\hat{M}_n)$ be another stable $q-$filtration of M. By 10.6, the two filtrations have a bounded difference. So there exists $n_0 \in \mathbb{N}$ such that $\hat{M}_{n+n_0} \subseteq M_n$ and $\hat{M}_{n+n_0} \supseteq M_n$ for all $n \geq 0$ and $g(n+n_0) \geq \hat{g}(n)$ and $\hat{g}(n+n_0) \geq g(n)$. By (2), $g, \hat{g}$ are polynomials for $n >> 0$. So $\lim_{n\to\infty} \frac{g(n)}{\hat{g}(n)} = 1$ which implies $\deg g(n) = \deg \hat{g}(n)$ and they have the same leading coefficient.

□

Notation. The polynomial $g(n)$ corresponding to the filtration $(q^n <)$ is denoted by $\chi_q^M(n) = \ell(M/q^nM)$. If $M = A$, write $\chi_q(n)$ and call it the **characteristic polynomial** of the $m-$primary ideal q.

Corollary (11.5). *For $n >> 0$, $\ell(A/q^n)$ is a polynomial $\chi_q(n)$ of degree $\leq s$, were s is the least number of generators for q.*

Proposition (11.6). *If A, m, q are as above, then $\deg \chi_q(n) = \deg \chi_m(n)$.*

Proof. Note $m \supseteq q \supseteq m^r$ for some r by 7.16. Then $m^n \supseteq q^n \supseteq m^{rn}$ which implies $\chi_m(n) \leq \chi_q(n) \leq \chi_m(rn)$ for all $n >> 0$. Take the limit as $n \to \infty$. □

Assume (A, m) is a local Noetherian ring, $M \neq (0)$ a finitely generated $A-$module. Let q be an $m-$primary ideal.
Notation. Define $\delta(A)$ to be the least number of generators of an $m-$primary ideal. For example, if $A = k[[t^2, t^3]$, then $m = (t^2, t^3)$. Take $q = At^2$. Then $\sqrt{q} = m$ and thus $\delta(A) = q$.

Note. If (A, m) is Artinian, then $m^t = (0)$ for some t. Then $m = \sqrt{(0)} = \sqrt{\emptyset}$ and thus $\delta(A) = 0$.

Proposition (11.7). $d(A) \leq \delta(A)$.

Proof. Theorem 11.4 and 11.6 give $d(A) = \deg \chi_m^M = \deg \chi_q^M \leq \delta(A)$. □

In fact, we will show $d(a) = \delta(A)$.

Proposition (11.8). *Let $x \in m$ and assume x is a non-zero-divisor of M. Then* $\deg \chi_q^{M/xM} \leq \deg \chi_q^M - 1$.

Proof. Let $N = xM \subseteq M$ (observe that $N \cong M$ as $0 \to M \xrightarrow{x} M \to M/N \to 0$ is exact). Let $N_n = N \cap q^n M$. By Artin-Rees (10.9), (N_n) is a stable $q-$filtration. Now for all n there exists a short exact sequence $0 \to \frac{N}{N_n} \to \frac{M}{q^n M} \to \frac{N}{N+q^n M} \to 0$. Note that $\frac{M/xM}{q^n(M/xM)} \cong \frac{M/xM}{q^n M + xM/xM} \cong \frac{M}{q^n M + xM} \cong \frac{M}{N+q^n M}$. Let $g(n) = \ell(N/N_n)$. Then $g(n) - \chi_q^M(n) + \chi_q^{M/xM}(n) = 0$ for all $n >> 0$. By 11.4(3), χ_q^M and the polynomial which represents $g(n)$ for all $n >> 0$ have the same degree and leading coefficient (as $M \cong N$). Thus $\deg \chi_q^{M/xM} < \deg \chi_q^M$. □

Note. That if $x \notin m$, then x is a unit and $M/xM = 0$. Thus the proposition is still true, it is just trivial.

Corollary (11.9). *Let $x \in m$ and assume x is a non-zero-divisor of A. Then $d(A/(x)) \leq d(A) - 1$.*

Proposition (11.10). $d(A) \geq \dim A =$ *supremum of lengths of chains in A.*

Proof. Induct on $d = d(A)$. Suppose $d = 0$. Then $\chi_m^A(n)$ is a polynomial of degree 0 (that is, a constant). Then $\ell_A(A/m^{n+1}) = \ell_A(A/m^n)$ for all $n >> 0$. Thus $m^n = m^{n+1}$ for $n >> 0$ which implies $m^n = 0$ by Nakayama's Lemma. Thus A is Artinian and m is the only prime ideal. Thus $\dim A = 0$. Now suppose $d > 0$. Let $p_0 \subset p_1 \subset \cdots \subset p_r$ be primes in A. Want to show $r \leq d$. Let $\overline{A} = A/p_0$, an integral domain. Then we have the chain $(0) = \overline{p_0} \subset \overline{p_1} \subset \cdots \subset \overline{p_r}$ where $\overline{p_i} = p_i/p_0$. Choose $x \in \overline{p_1} \setminus \overline{p_0}$. Then x is a nonzero divisor of $\overline{A}$ and so $d(\overline{A}/(x)) \leq d(\overline{A}) - 1$. Let $\overline{m} = m/p_0$, the maximal ideal of $\overline{A}$. Then $A/m^n \twoheadrightarrow A/m^n + p_0 = \overline{A}/\overline{m}^n$. Thus $\lambda(A/m^n) \geq \lambda(\overline{A}/\overline{m}^n)$. Thus $d(\overline{A}/(x)) \leq d(A) - 1$ and by induction $\dim(\overline{A}/(x)) \leq d(\overline{A}/(x)) \leq d - 1$. Note we have $\overline{p}_1/(x) \subset \cdots \subset \overline{p}_r/(x)$, a chain of length $r - 1$. Thus $r - 1 \leq d - 1$ which implies $r \leq d$. □

Proposition (11.13). $\delta(A) \leq \dim(A)$ *(and thus they are equivalent)*

Proof. Induct on $\dim(A)$. If $\dim A = 0$, then m is the only prime and thus $m = \sqrt{(0)} = \sqrt{\emptyset}$ and so $\delta(A) = 0$. Assume $\dim A > 0$. Let $p_1, ..., p_s$ be the minimal primes (there are finitely many as A is Noetherian). Then $p_i \subsetneq m$ for all i. Thus $m \not\subset p_1 \cup \cdots \cup p_s$. So choose $x \in m \setminus p_1 \cup \cdots \cup p_s$. Then there is a bijective correspondence between the set of primes of $A/(x)$ and the primes of A which contain (x). Then $p_i/(x)$ are note primes in $A/(x)$ which implies $dim A/(x) < \dim(A)$ (as a chain will be at least 1 prime shorter). So there exists an $m/(x)-$primary ideal $Q/(x)$ of $A/(x)$ such that $Q/(x)$ is generated by $\leq \dim A/(x)$ elements by induction. Of course, the number of generators for x is at most the number of generators for $Q/(x) + 1$ which is at most $\dim A$. Clearly, $\sqrt{Q} = m$ as $\sqrt{Q/(x)} = m/(x)$. □

Corollary (11.18). *Let (A, m) be local Noetherian and let $x \in m$. Then*

1. $\dim A/(x) =$ *either* $\dim A$ *or* $\dim A - 1$.
2. *If x is a non-zero-divisor, then* $\dim A/(x) = \dim A - 1$.

Proof. Say $\dim A/(x) =: d$. Choose an $m/(x)-$primary ideal $Q/(x)$ generated by d elements. Then Q is $m-$primary and generated by $\leq d + 1$ elements. Thus $\dim A \leq d + 1$. Since $\dim A/(x) \leq \dim A$, 1 is proved. Use Corollary 11.9 for 2. □

Examples. If $k = \overline{k}$, then $\dim k[x_1, ..., x_n] = n$.

Proof. Show $\dim k[x_1, ..., x_n]_m \leq n$ for all maximal m. (By the correspondence for primes of rings and primes of localizations, the dimension is the sup of length of chains of the localization). But $m = (x_1 - c_1, ..., x_n - c_n)$ for $c_i \in k$ by the Nullstellensatz. Thus mA_m is generated by n elements which says $\dim A_m \leq n$. To see $\dim A_m \geq n$, consider the chain $(0) \subsetneq (x_1) \subsetneq (x_1, x_2) \subsetneq \cdots \subsetneq (x_1, ..., x_n)$. □

A similar proof shows that $\dim k[[x_1, ..., x_n]] = n$.

Example. Let $R = k[[x, y]]/(x^2, xy)$. (This is known as the "Emmy Ring"). This ring shows that Cor 11.18 2 is not an if and only if statment. Note that everything in the maximal ideal (x, y) is a zero divisor. We claim $\dim R = 1$. To prove this, note that $(x) \subsetneq (x, y)$ is a maximal chain of primes. You can't add anymore as $\dim k[[x, y]] = 2$ which implies $(0) \subsetneq (x) \subseteq (x, y)$ is maximal in $k[[x, y]]$. Note that $\dim R/(y) = \dim k[[x, y]]/(x^2, xy, y) = \dim k[[x]]/(x^2) = 0$.

Corollary (11.16)**.** *Let I be an ideal of a Noetherian ring R and assume I is generated by n elements. Let P be a prime minimal over I (that is, its minimal among all primes $p \supseteq I$). Then $\dim R_p = n$.*

Proof. I_p is a pR_p−primary ideal of R_p generated by $\leq n$ elements. Thus $\dim R_p \leq n$. But $height(p) = \dim R_p$. □

Corollary (11.17: Krull's Principal Ideal Theorem)**.** *Let A be a Noetherian ring and $x \in A$ a nonunit non-zero-divisor. Then every minimal prime of (x) has height 1.*

Definition. *Let (A, m) be a local Noetherian ring of $\dim d$. A* ***system of parameters*** *for R (**SOP**) is a list $f_1, ..., f_d \in R$ such that $\sqrt{(f_1, ..., f_d)} = m$.*

Definition. *A* ***regular local ring (RLR)*** *is a Noetherian local ring (A, m) such that m is generated by $\dim A$ elements.*

Nonexample. Consider $k[[x, y]]/(y^2, x^3) \cong k[[t^2, t^3]]$. Here, the dimension is 1, but m needs two generators. More generally, if $0 \neq f \in m^2$ where $m = (x_1, ..., x_n)$ is the maximal ideal of $k[[x_1, ..., x_n]]$, then $k[[x_1, ..., x_n]]/(f)$ is not a RLR.

Proof. $\dim k[[x_1, ..., x_n]]/(f) = n - 1$ as f is a non zero divisor. But $(x_1, ..., x_n)/(f)$ still needs n generators as since $f \in m^2$, it does not help generate by Nakayama. □

Theorem (Krull's Intersection Theorem)**.** *Let M be a finitely generated module over a Noetherian ring R and I an ideal of R. Let $E = \cap_{n=1}^{\infty} I^n M$. Then $IE = E$.*

Proof. Note that $(I^n M)$ is a stable I−filtration of M. So by Artin-Rees, $(I^n M \cap E)$ is a stable I−filtration of E. Choose $n >> 0$ such that $I(I^n M \cap E) = (I^{n+1} M) \cap E$. Then $IE = E$. □

Ties with Algebraic Geometry Let k be an algebraically closed field. Then the set of varieties (irreducuble closed sets) in $\mathbb{A}^n_k$ are in bijective correspondence with the set of prime ideals of $k[x_1, ..., x_n]$. Then $\dim Y = \dim A(Y) = \dim k[x_1, ..., x_n]/I(Y)$ (where $I(Y)$ is prime as we are only looking at the irreducible closed sets).

Definition. *A* ***hypersurface*** *in $\mathbb{A}^n_k$ is a variety of dimension $n - 1$.*

Suppose Y is a hypersurface in A^n_k. Then $\dim k[x_1, ..., x_n]/I(Y) = n - 1$. Thus *height* $I(Y) = 1$.

Lemma. *Let R be a Noetherian UFD. Then the height 1 primes are exactly the ideals (f) where f is irreducible.*

Proof. Let p be a height 1 prime. Choose $g \in P \setminus \{0\}$. Factor $g = f_1 \cdots f_t$ where f_i is irreducible. Choose i such that $f := f_i \in P$. Then (f) is a nonzero prime ideal. So $0 \neq (f) \subseteq p$ which implies $(f) = p$ has p has height 1. Conversely, let f be irreducible. Then (f) is a nonzero prime ideal. Since R is Noetherian, $ht(f) \leq 1$ as it is principal. Since R is a UFD, its an integral domain and thus (0) is a prime ideal. Therefore $ht(f) = 1$. □

If Y is a hypersurface in $\mathbb{A}^n_k$, then $I(Y) = (f)$ where f is an irreducible polynomial. Conversely, if $f \in k[x_1, ..., x_n]$ is irreducible, then $V((f))$ is a hyperspace.

- Choose a maximal ideal $m \supseteq f$. We know $ht(m) = n$. Then $\dim Rm = n$ and so $\dim Rm/(f) = n - 1$, as f is a non-zero-divisor. Therefore, $\dim R/(f) \geq n - 1$. If $\dim R/(f) = n$, then we could add the ideal (0) to the longest chain in R to get $\dim R/(f) > n$, a contradiction. Thus $\dim R/(f) = n - 1$.

Consider a curve $C \subseteq \mathbb{A}^3_k$, that is, C is a one-dimensional variety. Let $I(C) \subseteq k[x, y, z]$ be the ideal of functions vanishing on C. Then $I(C)$ is a prime ideal of $k[x, y, z]$ with $\dim k[x, y, z]/I(C) = 1$. Once could show $I(C)$ has height 2.

Exercises.

1. Let $C := \{(t^3, t^4, t^5) \mid t \in k\} \subset \mathbb{A}^3_k$, where k is an algebraically closed field, and let $I(C) \subseteq k[x, y, z]$ be the ideal of functions vanishing on C.

   (a) Prove that $I(C)$ is a prime ideal of $k[x, y, z]$.

   Proof. Put a grading on $k[x, y, z]$ by setting $\deg x = 3, \deg y = 4, \deg z = 5$. Let $k[t]$ have the standard grading. Consider $\phi : S \twoheadrightarrow R$ by $x \mapsto t^3, y \mapsto t^4, z \mapsto t^5$. This is a graded homomorphism of $\deg 0$ and thus $\ker \phi$ is homogenous. Now $S/\ker \phi \cong R$, a domain. Thus $\ker \phi$ is prime. Of course, $\ker \phi = I(C)$ and so $I(C)$ is prime. □

(b) Prove that $I(C) = (x^3 - yz, y^2 - xz, z^2 - x^2y)$, the ideal generated by the three polynomials listed. Interestingly, these relations are the determinants of the 2x2 minors in the matrix $\begin{bmatrix} x^2 & y & z \\ z & x & y \end{bmatrix}$.

Proof. Let $f = x^3 - yz, g = y^2 - xz, h = z^2 - x^2y$. Note that f, g, h are homogenous of degree 9, 8, and 10 respectively. Since $I(C)$ is homogenous, it is generated by homogenous elements. So it is enough to show that any homogenous $F \in I(C)$ can be written as a linear combination of f, g, and h. Since h is monic in z, the division algorithm says there exists $q_1, H \in k[x,y][z]$ such that

$$F = hq_1 + H \text{ where } H \text{ is at most linear in } z.$$

Note that since $G, h \in I(C)$, we have $H \in I(C)$. Also, H is homogenous of the same degree as F. Now, since g is monic in y, the division algorithm says there exists $q_2, G \in k[x,z][y]$ such that

$$H = gq_2 + G \text{ where } G \text{ is at most linear in } y.$$

As H was at most linear in z and g has a term with z, we see G is at most quadratic in z. Also, as $H, g \in I(C)$ we see $G \in I(C)$. Applying the division algorithm one more time, we find $q_3, R \in k[y,z][x]$ such that

$$G = fq_3 + R \text{ where } R \text{ is at most quadratic in } x \text{ and } y \text{ and linear in } z.$$

Note we have $F = hq_1 + gq_2 + fq_3 + R$. We wish to show $R = 0$. Using degree arguments (similar to the following proof of 3b), we see it is impossible for R to be nonzero. Thus any homogenous element of $I(C)$ is a linear combination of f, g, and h. □

(c) Prove that $C = V(I(C))$ (that is, C is a variety).

Proof. Note that $\subseteq$ is clear, so we need only to prove $\supseteq$. Let $(a,b,c) \in V(I(C))$. Then $a^3 = bc, b^2 = ac, c^2 = a^2b$. If one of a, b, c are 0, then clearly all are. So assume a, b, c are all nonzero. Let $t_0 = \frac{b}{a}$. Then $t_0^3 = \frac{b^3}{a^3} = \frac{b^3}{bc} = \frac{b^2}{c} = a$. Similarly, $t_0^4 = \frac{b^4}{a^4} = \frac{a^2c^2}{a^4} = \frac{c^2}{a^2} = b$ and $t_0^5 = \frac{b^5}{a^5} = \frac{a^2c^2b}{a^2bc} = c$. Thus $(a,b,c) = (t_0^3, t_0^4, t_0^5)$. □

2. With C and $I(C)$ as above, prove that $I(C) = \sqrt{(x^4 - 2xyz + y^3, z^2 - x^2y)}$.

Proof. **$\supseteq$:** Using the definition of $I(C)$ as in (b) above, first note that $(d,c) \in I(C)$ as $x^4 - 2xyz + y^3 = x(x^3 - yz) - y(y^2 - xz) \in I(C)$ and clearly $(z^2 - x^2y) \in I(C)$. Let $f \in \sqrt{(x^4 - 2xyz + y^3, z^2 - x^2y)}$. Then $f^n = g(x^4 - 2xyz + y^3) + h(z^2 - x^2y) \in I(C)$. Thus $f^n(t^3, t^4, t^5) = 0$ which implies $f(t^3, t^4, t^5 = 0$ and thus $f \in I(C)$.

$\subseteq$: With a little work, we see

$$(x^3 - yz)^2 = x^2(x^4 - 2xyz + y^3) + y^2(z^2 - x^2y)$$

which says $(x^2 - yz)$ is in the RHS,

$$(y^2 - xz)^2 = y(x^4 - 2xyz + y^3) + x^2(z^2 - x^2y)$$

which says $y^2 - xz$ is in the RHS and of course its easy to see $z^2 - x^2y$ is in the RHS. Thus each term is separately in the RHS. Using this, it can be shown that a general element $(f(x^3 - yz) + g(y^2 - xz) + h(z^2 - x^2y))$ is in the RHS, proving our claim.

□

3. Let $D = \{(t^4, t^5, t^6) \mid t \in k\} \subset \mathbb{A}^3_k$, where k is an algebraically closed field. Let $I(D) \subseteq k[x,y,z]$ be the ideal of functions vanishing on D.

(a) Prove that $I(D)$ is a prime ideal of $k[x,y,z]$.

Proof. Put a grading on $k[x,y,z]$ by setting $\deg x = 4, \deg y = 5, \deg z = 6$. Define $\phi : k[x,y,z] \to k[t]$ by $x \mapsto t^4, y \mapsto t^5, z \mapsto t^6$ so that ϕ is a graded homomorphism of degree 0. Then $\ker\phi$ is a homogenous ideal of $k[x,y,z]$. Clearly, $\ker\phi = I(D)$ and thus $k[x,y,z]/I(D) \cong im\phi \subseteq k[t]$, a domain. Thus $k[x,y,z]/I(D)$ is a domain which implies $I(D)$ is prime. □

(b) Prove that $I(D) = (x^3 - z^2, y^2 - xz)$.

Proof. Note that $\supseteq$ is clear. So we need only show $\subseteq$. By (a), $I(D)$ is homogenous which implies it is generated by homogenous elements. So it is enough to show that any homogenous element f of $I(D)$ is such that $f \in (x - z^2, y^2 - xz)$. Let $f \in I(D)$ be homogenous of degree d. Since the leading coefficient of $y^2 - xz$ is a unit in $k[x,z][y]$, the division algorithm applies to say there exists $q_1, r_1 \in k[x,z][y]$ such that $f = (y^2 - xz)q_1 + r_1$ where r_1 is at most linear in y. Since $f, y^2 - xz \in I(D)$, we see $r_1 \in I(D)$. Also, r_1 is homogenous of the same degree as f. Applying the division algorithm again, we see there exists $q_2, r_2 \in k[y,z][x]$ such that $r_1 = (x^3 - z^2)q_2 + r_2$ where r_2 is at most quadratic in x. Note that we can also say r_2 is at most linear in y and again, since $r_1, x^3 - z_2 \in I(D)$ we see $r_2 \in I(D)$. Combining these equations, we have $f = (y^2 - xz)q_1 + (x^3 - z^2)q_2 + r_2$ with r_2 at most linear in y, at most quadratic in x and $r_2 \in I(D)$. Then

- Any term of r_2 containing x^2 will have degree $8 + n6$ (from x^2z^n) or $13 + n6$ (from x^2yz^n) (here n may be 0).
- Any term containing an x, but not x^2 will have degree $4 + n6$ (from xz^n) or $9 + n6$ (from xyz^n).
- Any term containing no x will have degree $5 + n6$ (from yz^n) or 0 (from a constant).

Of course, $8, 13, 4, 9, 5, 0$ are all different modulo 6. Thus this list consists of all distinct degrees. Now, r_2 homogenous implies r_2 has at most 1 nonzero term. By $r_2(t^4, t^5, t^6) = 0$ implies $r_1 = 0$. Thus $f \in (x^3 - z^2, y^2 - xz)$. □

(c) Prove that $D = V(I(D))$ (so that D is a variety).

Proof. Now $\subseteq$ is clear, so we need only to prove $\supseteq$. Suppose $(a,b,c) \in V(I(D))$. Then (a,b,c) is a solution to $x^3 - z^2$ and $y^2 = xz$. Thus $a^3 = c^2$ and $b^2 = ac$. If $a = 0$, then $b = c = 0$ and we know $(0,0,0) \in D$. Otherwise, let $t = \frac{b}{a}$. Then $t^4 = \frac{b^4}{a^4} = \frac{a^2c^2}{ac^2} = a$. Also $t = \frac{b}{a} = \frac{b}{t^4}$ implies $t^5 = b$. Finally, $c = \frac{b^2}{a} = \frac{t^{10}}{t^4} = t^6$. Thus $(a,b,c) \in D$. □

4. (Unsolved) Show if C is an irreducible curve in $\mathbb{A}^3_k$, then $I(C) = \sqrt{(f,g)}$ for $f, g \in k[x,y,z]$. (This is known for *char* $k = p > 0$.)

These examples are indicative of a general phenomenon for "monomial curves" in $\mathbb{A}^3_k$, that is, curves of the form $C = \{(t^a, t^b, t^c)\}$, where $a < b < c$ are positive integers with greatest common divisor 1: The ideal $I(C)$ is always the radical of a two-generated ideal, and $I(C)$ is actually two-generated if and only if the submonoid S of $\mathbb{Z}$ generated by a, b, c is symmetric. (A submonoid S of $\mathbb{Z}$ is symmetric provided (i) there is a largest integer $f \notin S$ and (ii) for each $n \in \mathbb{Z}$, $n \in S \Leftrightarrow f - n \notin S$.) The situation with monomial curves in $\mathbb{A}^n_k$ is much more mysterious when $n > 3$.

2.6 Singularities

Consider $C = (y^2 - x^3) \subseteq \mathbb{A}^2_k$. (Assume *char* $k \neq 2, 3$).

- Let $f = y^2 - x^3$. Then $\frac{\partial f}{\partial x} = 3x^2$ and $\frac{\partial f}{\partial y} = 2y$. To define the equation for the tangent line, we need one of the partials to not be 0. So, we can find a tangent line everywhere but at $(0,0)$.

Consider $D = V(y^2 - x^3 - x^2) \subseteq \mathbb{A}^2_k$. Again, assume *char* $k \neq 2, 3$.

- Let $f = y^2 - x^3 - x^2$. Then $\frac{\partial f}{\partial x} = -3x^2 - 2x$ and $\frac{\partial f}{\partial y} = 2y$. Let $p \in D$. Then $\frac{\partial f}{\partial y}(p) = 0$ implies $p = (x,0)$. This says for $p = (x,y)$ that $x^3 + x^2 = 0$ as $p \in D$. So $p = (0,0)$ or $p = (-1,0)$. Doing the same with $\frac{\partial f}{\partial y}$, we see $p = (0,0)$ or $p = (-\frac{2}{3}, \pm\sqrt{\frac{4}{27}})$. Thus the tangent line exists everywhere but at $(0,0)$.

Note. In these examples, the origin is known as a **singular point**.

Let $Y \subseteq \mathbb{A}^n_k$ be an affine variety of dimension r. Then $I(Y)$ is a prime ideal of $k[x_1, ..., x_n]$ and $\dim \frac{k[x_1,...,x_n]}{I(Y)} = r$. Let $I(Y) = (f_1, ..., f_m)$ (it is finitely generated as the ring is Noetherian). Now, form the Jacobian matrix $J = [\frac{\partial f_i}{\partial x_j}]$ (an $m \times n$

matrix. We can think of $J(p) : k^n \to k^m$ where $p \in Y$ and we would like $\dim \ker J(p) = r$ (where $\ker J(p)$ is the tangent space).

Definition. *The point $p \in Y$ is a **smooth / non-singular** point of Y provided rank of $J(p)$ is $n - r$.*

Fact. This definition is independent of the choice of generators for $I(Y)$ (that is, independent of m).

Exercise. Let Y be a variety in $\mathbb{A}_k^n$ (that is, an irreducible closed subset of $\mathbb{A}_k^n$), and let $p \in Y$. Prove that p is a smooth point of Y if and only if $\mathcal{O}_{Y,p}$ is a regular local ring. (Recall that $\mathcal{O}_{Y,p}$ is nothing scary: Let $A = k[x_1, \ldots, x_n]/I(Y)$, the affine coordinate ring of Y, and let m be the maximal ideal of A corresponding to the point p. Then $\mathcal{O}_{Y,p} = A_m$.)

- Recall $\mathcal{O}_{p,Y} \cong (A(Y))_{\mathcal{M}}$ where $A(Y) = k[x_1, ..., x_n]/I(Y)$. Here m is the maximal ideal of $A(Y)$ corresponding to p (so $m = m_p/I(Y)$ where $m_p = (x_1 - a_1, ..., x_n - a_n)$ for $p = (a_1, ..., a_n)$).

- **Claim.** $\dim \mathcal{O}_{p,Y} = \dim Y$.
  Proof. Recall $\dim \mathcal{O}_{p,Y} = \dim(A(Y))_m = ht\ m$. Also $ht\ m + \dim A(Y)/m = \dim A(Y)$ (we will prove this later). Since $A(Y)/m$ is a field, it has dimension 0. Thus $ht\ m = \dim A(Y) = \dim Y$. Therefore, $\dim \mathcal{O}_{p,Y} = ht\ m = \dim Y$. ☐

Proof. (Of exercise) Say Y has dimension r and $p = (a_1, ..., a_n)$. Define $\theta : m_p \to k^n$ by $f \mapsto (\frac{\partial f}{\partial x_1}(p), ..., \frac{\partial f}{\partial x_n}(p))$. Notice $\theta(x_i - a_i) = (0, ..., 0, 1, 0, ..., 0)$ (where the 1 is in the i^{th} spot). Thus θ is surjective. Also, m_p^2, which is generated by $(x_i - a_i)(x_j - a_j)$, is in the kernel as $\frac{\partial}{\partial x_i}(x_i - a_i)(x_j - a_j)(p) = (x_2 - a_2)(p) = 0$. If $f \in \ker \theta$, then $\frac{\partial f}{\partial x_i}(p) = 0$ for all i. So $f(x_1, ..., x_n) = q_1(x_1, ..., x_n)(x_1 - a_1) + r(x_2, ..., x_n)$. Since $\frac{\partial f}{\partial x_1} = \frac{\partial r}{\partial x_1} = 0$, we see $q_1(p) = 0$ which implies $q = \sum b_i(x_i - a_i)$. So $f(x_1, ..., x_n) = \sum b_i(x_1 - a_1)(x_i - a_i) + r(x_2, ..., x_n)$. Of course, now, $f(x_1, ..., x_n), \sum b_i(x_1 - a_1)(x_i - a_i) \in \ker \theta$ and so $r(x_2, ..., x_n) \in \ker \theta$. So $\frac{\partial r}{\partial x_i} = 0$ for all i. Breaking up r as we did f, we can proceed inductively to get $f(x_1, ..., x_n) = \sum b_{i,j}(x_i - a_i)(x_j - a_j) \in m_p^2$. Thus $\ker \theta = m_p^2$ which implies $m_p/m_p^2 \cong k^n$. Then $\dim_k m_p/m_p^2 = n$. Let $I(Y) = (f_1, ..., f_s)$. Then $(p) \subseteq Y$ which implies $I(Y) \subseteq m_p$. So $\theta(I(Y))$ is a subspace of k^n of dimension $rank(\frac{\partial f_j}{\partial x_i}(p))$. Note that $\theta(I(Y) + m_p^2) = \theta(I(Y))$ and so $\frac{I(I)+m_p^2}{m_p} \cong \theta(I(Y))$. Therefore $\dim \frac{I(Y)+m_p^2}{m_p^2} = rankJ$. One can show $\frac{m(A(Y))_m}{m^2(A(Y))_m} \cong \frac{m}{m^2} = \frac{m_p/I(Y)}{m_p^2+I(Y)/I(Y)} \cong \frac{m_p}{m_p^2+I(Y)}$. Note $V = \frac{m_p}{m_p^2} \supseteq \frac{m_p^2+I(Y)}{m_p^2} = W$ which implies $\dim V = \dim W + \dim \frac{V}{W}$. So $n = rank\ J + \dim \frac{m(A(Y))_m}{m^2(A(Y))_m}$. Now, $\mathcal{O}_{p,y}$ is a regular local ring if and only if $\dim \frac{m(A(Y))_m}{m^2(A(Y))_m} = \dim \mathcal{O}_{p,y} = \dim Y$ which is if and only if $rankJ = n - \dim Y$ which is if and only if p is a smooth point for Y. ☐

Recall. If (R, m) is a Noetherian local ring of dimension d, then there exists a polynomial $\chi_m \in \mathbb{Q}[t]$ such that $\chi_m(n) = \ell_R(R/m^n)$ for all $n >> 0$. Note that $\deg \chi_m = d$.

Examples.

1. $R = k[[x, y]]$. Then $m = (x, y)$ and m^n is minimal generated by $\{x^n, x^{n-1}y, ..., xy^{n-1}, y^n\}$. Then R/m^n has a $k-$basis consisting of monomials of degree $\leq n-1$. How many? There is/are 1 monomial of degree 0, 2 of degree 1, ..., n of degree $n-1$. So $\chi_m(n) = \ell(R/m^n) = \dim_k R/m^n = 1+2+...+n = \frac{n(n+1)}{2}$. This is a polynomial of degree 2 nd leading coefficient $\frac{1}{2}$.

2. $R = k[[x_1, ..., x_d]]$. There are $\begin{pmatrix} r+d-1 \\ d-1 \end{pmatrix}$ monomials of degree r. Thus $\dim(R/m^n) = \sum_{r=0}^{n-1} \begin{pmatrix} r+d-1 \\ d-1 \end{pmatrix} = \begin{pmatrix} n+d-1 \\ d-1 \end{pmatrix}$. So $\chi_m(n)$ has degree d with leading coefficient $\frac{1}{d!}$.

Exercise. Let $f \in \mathbb{Q}[t]$ be a polynomial of degree d such that $f(n) \in \mathbb{Z}$ for all sufficiently large integers n. Prove that the leading coefficient is of the form $\frac{e}{d!}$ for some integer e. Note that, in particular, this says $\chi_m = \frac{e}{d!}t^d +$ lower terms for some $e \geq 1$ in $\mathbb{Z}$. We say $e = e(r)$ is the **multiplicity** of R.

Proof. Induct on d. If $d = 0$, then $f(n) \in \mathbb{Z}$ for all n (as there are no variable terms). Thus $f(x) = \frac{a}{b} = \frac{a}{1} = \frac{a}{0!}$. So let $d > 0$. Consider $g(x) = f(x+1) - f(x)$. Then

$$\begin{aligned} g(x) &= \tfrac{a_d}{b_d}(x+1)^d + \tfrac{a_{d-1}}{b_{d-1}}(x+1)^{d-1} - \tfrac{a_d}{b_d}x^d - \tfrac{a_{d-1}}{b_{d-1}}x^{d-1} + (lower\ terms) \\ &= \tfrac{a_d}{b_d}(x^d + dx^{d-1} + (lower\ terms) - x^d) + \tfrac{a_{d-1}}{b_{d-1}}(x^{d-1} + (d-1)x^{d-2} + (lower\ terms) - x^{d-1}) + (lower\ terms) \\ &= d\tfrac{a_d}{b_d}x^{d-1} + (lower\ terms) \end{aligned}$$

By induction, as $\deg g = d-1$, we see $d\frac{a_d}{b_d} = \frac{e}{(d-1)!}$ for some e which implies $\frac{a_d}{b_d} = \frac{e}{d!}$. □

Exercise. Let (R, m) be a regular local ring of dimension d. Prove that $e(R) = 1$. (Recall that $e(R)$ is the multiplicity of R, that is, $d!$ times the leading coefficient of χ_m.)

Exercise. If (S, n) is a regular local ring and $f \in m^e - m^{e+1}$, prove that $e(S/(f)) = e$.

Example. Let $R = k[[t^4, t^{11}]]$, where $m = t^4, t^{11})$. This is a hypersurface. (Note that 29 is the last exponent not in R). Let $S = k[[x, y]]$. Then $R \cong S/(y^4 - x^{11})$. Notice

$$\begin{aligned} &m^2 = (t^8, t^{15}, t^{22}) \\ &m^3 = (t^{12}, t^{19}, t^{26}, t^{33}) \\ &m^4 = (t^{16}, t^{23}, t^{30}, t^{37}, t^{44}) = (t^{16}, t^{23}, t^{30}, t^{37}) \text{ as } t^{44} = t^{28}t^{16} \in (t^{16}, t^{23}, t^{30}, t^{37}). \end{aligned}$$

In a similar manner, m^k requires 4 generators for all $k \geq 5$. Recall that the number of generators of m^t is the number of generators of m^t/m^{t+1} (by NAK as the maximal ideal does not help generate) which is just $\dim_k m^t/m^{t+1}$ (as m^t/m^{t+1} is an R/m vector space for m kills it) which is exactly $\ell_R m^t/m^{t+1}$. Thus $\ell(R/m^n) = \sum_0^{n-1} \ell(m^t/m^{t+1}) = 4n - 6$ if $n \geq 3$. Note that dim $= 1$ so we have the leading coefficient of χ_m as $\frac{4}{1!} = 4$ as the fact above says. Thus $e(R) = 4$.

Note. For 1 dimension, the number of generators for m^n for $n >> 0$ is exactly $e(R)$.

Example. $R = k[[x, y]]/(x^2, xy)$. Here, we see

$$\begin{aligned} &m = (x, y) \\ &m^2 = (x^2, xy, y^2) = (y^2) \\ &m^3 = (x^3, x^2y, xy^2, y^3) = (y^3) \\ &\vdots \\ &m^n = (y^n) \end{aligned}$$

Thus the number of generators for m^n for $n \geq 2$ is 1 and so $e(R) = 1$.

2.7 Back to Affine Varieties

Suppose X, Y are affine varieties. Say $X \subseteq \mathbb{A}_k^m \leftrightarrow k[x_1, ..., x_m]$ and $Y \subseteq \mathbb{A}_k^n \leftrightarrow k[y_1, ..., y_n]$. Then $X \times Y \subseteq \mathbb{A}_k^{m+n} \leftrightarrow k[x_1, ..., x_m, y_1, ..., y_n]$. Now, we just want to show $X \times Y$ is a variety. Let $I(X) = P = (f_1, ..., f_s) \subseteq k[x_1, ..., x_m]$ and

$I(Y) = Q = (g_1, ..., g_t) \subseteq k[y_1, ..., y_n]$. Then $X \times Y = V(f_1, ..., f_s, g_1, ..., g_t)$.

Theorem. *$X \times Y$ is irreducible.*

Proof. Suppose $X \times Y = V_1 \cap V_2$ for V_i closed. Let $H_i = \{p \in X | \{p\} \times Y \subseteq V_i\}$. Note $\{p\} \times Y \cong Y$. Thus $\{p\} \times Y$ is irreducible which implies, for all p, either $\{p\} \times Y \subseteq V_1$ or $\{p\} \times Y \subseteq V_2$. Thus $X = H_1 \cap H_2$. Want to show H_1, H_2 are closed. As V_1 is closed, say $V_1 = V(G_1, ..., G_t)$. Fix $q \in Y$. Then for all p we see $(p, q) \subseteq V_1$ if and only if $G_j(p, q) = 0$ for all j. Let $G_{j,q}(x_1, ..., x_m) = G_j(x_1, ..., x_m, q)$. Then $(p, q) \in V_1$ if and only if $G_{j,p}(p) = 0$ for all j. Letting q vary, we see $p \in H_!$ if and only if $(p, q) \in V_1$ for all $q \in Y$ which if and only if $G_{j,q}(p) = 0$ for all $q \in Y$ and all j. Thus $H_! = \cap_{j,q} V(G_{j,q}(p))$. Thus H_1 is closed and similarly for H_2. So X irreducible implies $X = H_1$ or $X = H_2$. WLOG, say $X = H_!$. Thus $X \times Y \subseteq V_1$ which implies $X \times Y = V_1$. □

Let $A = A(X)$ and $B = A(Y)$. Then $A \otimes_k B = A(X \times Y)$.

Corollary. *Let $k = \overline{k}$ be a field and A, B affine domains over k. Then $A \otimes_k B$ is a domain.*

Example. Note that $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} \cong \mathbb{C} \times \mathbb{C}$ is not a domain. Thus we need the fact that $k = \overline{k}$ in the above corollary.

2.8 A Crash Course in Transcendence Bases

Start with a fixed field extension K/F. A list $\alpha_1, ..., \alpha_n$ in K is **algebraically independent** over F provided the $F-$algebra homomorphisms $F[x_1, ..., x_n] \to F[\alpha_1, ..., \alpha_n]$ defined by $x_i \mapsto \alpha_i$ is an isomorphism. A subset $\mathcal{B}$ of K is algebraically independent over F provided each list $x_1, ..., x_n$ of distinct elements of $\mathcal{B}$ is algebraically independent over F. A **transcendence basis** for K over F is an $F-$algebraically independent subset $\mathcal{B}$ of K such that K is algebraic over $F(\mathcal{B})$.
Facts.

1. If $\mathcal{B} \subseteq K$ and K is algebraic over $F(\mathcal{B})$, then $\mathcal{B}$ contains a transcendence basis.
2. If $\mathcal{C} \subseteq K$ is an algebraically independent set, then $\mathcal{C}$ can be extended to a transcendence basis for K over F.
3. Let $\mathcal{C} \subseteq K$ be algebraically independent and $\mathcal{B} \subseteq K$ such that K is algebraic over $F(\mathcal{B})$. Then there exists $\mathcal{B}' \subseteq \mathcal{B}$ such that $\mathcal{B}' \cap \mathcal{C} = \emptyset$ and $\mathcal{C} \cup \mathcal{B}'$ is a transcendence basis.
4. Any two transcendence bases for K/F have the same cardinality.

Let D be an $F-$algebra with quotient field K. Then $F(D) = K$. So by 1, D contains a transcendence basis for K over F. Such a transcendence basis will be called a transcendence basis for D over F. Note that tr $\deg_F K =$ tr $\deg_F D$ which is the cardinality of some transcendence basis.

Definition. *Let K be a field. An **affine domain** over K is an integral domain that is finitely generated as a $K-$algebra.*

Say $D = k[\alpha_1, ..., \alpha_n]$. Let K be the quotient field of D. Then $K = k(\alpha_1, ..., \alpha_n)$. Thus *tr* $\deg_k D \leq n$. In particular, it is finite.

Lemma. *Let k be a field, and let A, B be affine domains over k. Let $\phi : A \to B$ be a surjective $k-$algebra homomorphism. Then tr $deg_k A \geq$ tr $deg_k B$ with equality if and only if ϕ is an isomorphism.*

Proof. Let $\{b_1, ..., b_n\}$ be a transcendence basis for B over k where $n =$ tr $\deg_k B$. Choose $a_i \in A$ such that $\phi(a_i) = b_i$ for all i. This induces the homomorphism ϕ' as seen in the diagram.

$$\begin{array}{ccc} k[x_1, ..., x_n] & \xrightarrow[\alpha]{} & k[a_1, ..., a_n] \subseteq A \\ & & \downarrow \phi' \\ & \xrightarrow[\beta]{} & k[b_1, ..., b_n] \subseteq B \end{array}$$

Note $\alpha : x_i \mapsto a_i$ and $\beta : x_i \mapsto b_i$ where x_i are indeterminants. If α is not 1-1, neither is $\phi'\alpha = \beta$, a contradiction as β is an isomorphism as $\{b_1, ..., b_n\}$ is algebraically independent. Thus $\{a_1, ..., a_n\}$ is algebraically independent. Thus tr $\deg_k A \geq$ tr $\deg_k B$.

For the if and only if statement, note that if ϕ is injective, then $A \cong B$ and thus tr deg$A =$ tr deg$B$. For the other direction, assume tr deg$_k A = n =$ tr deg$_k B$. Then the a_i's form a transcendence basis for A over k and thus $qf(A)$ (the quotient field) is algebraic over $k(a_1, ..., a_n)$. Thus every element of A is algebraic over $k[a_1, ..., a_n]$ (just clear the denominators). Suppose there exists $c \in A \setminus \{0\}$ such that $c \in \ker\phi$. Choose an expression with d minimal such that $f_d c^d + ... + f_1 c + f_0 = 0$ where $f_i \in k[a_1, ..., a_n]$. Then $f_0 \neq 0$ (as otherwise, we could cancel c and get a contradiction to the minimality of d). Thus $0 = \phi(0) = \phi(f_d c^d + ... + f_0) = \phi(f_d)\phi(c)^d + ... + \phi(f_0) = 0$ which implies $\phi(f_0) = 0$. But $f_9 \in k[a_1, ..., a_n]$ and ϕ' is an isomorphism. Thus $\phi'(f_0) = \phi(f_0) = 0$, a contradiction as $\phi^{-1} = \beta\alpha^{-1}$, an isomorphism. □

Theorem. $\dim k[x_1, ..., x_n] = n$.

Proof. Note that $0 \subsetneq (x_1) \subsetneq (x_1, x_2) \subsetneq \cdots \subsetneq (x_1, ..., x_n)$. Thus dim $\geq n$. To show dim $\leq n$, suppose $(0) \subsetneq P_1 \subsetneq \cdots \subsetneq P_{n+1}$. Then $n =$ tr deg$_k S$ where $S = k[x_1, ..., x_n$ and $S \to S/P_1$ is not 1-1. Thus $n =$ tr deg$_k S >$ tr deg$_k S/P_1 > \cdots >$ tr deg$S/P_{n+1} \geq 0$, a contradiction. □

We've seen $\dim k[x_1, ..., x_d] = d =$ tr deg$_k k[x_1, ..., x_d]$. This is in fact true for all affine domains.

Lemma (**Noether Normalization Lemma**). *Let k be an infinite field and $\neq 0$ be a finitely generated $k-$algebra. Then there exist algebraically independent elements $y_1, ..., y_r \in A$ over k such that A is integral over $k[y_1, ..., y_r]$.*

Proof. First, we must prove a claim:

Claim: Let k be an infinite field, $g \in k[z_1, ..., z_n] \setminus \{0\}$ be homogenous. Then there exists $\lambda_1, ..., \lambda_{n-1} \in k$ such that $g(\lambda_1, ..., \lambda_{n-1}, 1) \neq 0$.

Proof: Let $g = \sum a_j z_1^{e_{j_1}} \cdots z_n^{e_{j_n}}$ where $z_1^{e_{j_1}} \cdots z_n^{e_{j_n}}$ are distinct for all j. Let $g' = g(z_1, ..., z_{n-1}, 1) = \sum a_j z_1^{e_{j_1}} \cdots z_{n_1}^{e_{j_{n-1}}} \in k[z_1, ..., z_{n-1}] \setminus \{0\}$. (If $g' = 0$, then there would exist j, k such that $a_j z_1^{e_{j_1}} \cdots z_{n-1}^{e_{j_{n-1}}} = a_k z_1^{e_{k_1}} \cdots z_{n-1}^{e_{k_{n-1}}}$. But homogeneity implies then that $e_{j_n} = e_{k_n}$, a contradiction as we just stipulated that the terms of g were distinct.) Since k is infinite, there exists $\lambda_{n-1} \in k$ such that $g'(z_1, ..., z_{n-2}, \lambda_{n-1}) \neq 0$. Now, iteratively, since k is infinite we can find $\lambda_{n-2}, ..., \lambda_1$ such that $g(\lambda_1, ..., \lambda_{n-1}, 1) = g'(\lambda_1, ..., \lambda_{n-1}) \neq 0$.

Let $x_1, ..., x_n$ generate A over k. Induct on n. If $n = 0$, done. So let $n \geq 1$. Reorder $x_1, ..., x_n$ such that $\{x_1, ..., x_r\}$ are algebraically independent over k and $x_{r+1}, ..., x_n$ are all algebraically dependent over $k[x_1, ..., x_r]$ (that is $\{x_1, ..., x_r\}$ is a transcendence base for A over k). If $n = r$, done. So suppose $n > r$. Then x_n is algebraic over $k[x_1, ..., x_{n-1}]$ which implies $f \in k[z_1, ..., z_n] \setminus \{0\}$ such that $f(x_1, ..., x_n) = 0$. Let $F = \sum a_j z_1 e^{j_1} \cdots z_n^{e_{jn}}$ be the homogenous part of f of largest degree. By the claim, there exists $\lambda_1, ..., \lambda_{n-1} \in k$ such that $F(\lambda_1, ..., \lambda_{n-1}, 1) \neq 0$. Set $x_i' = x_i - \lambda_i x_n$ for all $i \in [n-1]$. Then for $g := f(z_1 + \lambda_1 z_n, ..., z_{n-1} + \lambda_{n-1} z_n, z_n)$, we see $g(x_1', ..., x_{n-1}', x_n) = 0$. Consider $G := F(z_1 + \lambda_1 z_n, ... z_{n-1} + \lambda_{n-1} z_n, z_n)$. Note $G(z_1, ..., z_n) = \sum a_j (z_1 + \lambda_1 z_n)^{e_{j_1}} \cdots (z_{n-1} + \lambda_{n-1} z_n)^{e_{j_{n-1}}} z_n^{e_{jn}}$. If we expand on G, we find that the z_n^d term has a coefficient of $c = F(\lambda_1, ..., \lambda_{n-1}, 1) \neq 0$. Further, any other terms in g involving z_n^e must have $e < d$. Thus $g(z_1, ..., z_n) \in k[z_1, ..., z_n] \setminus \{0\}$. Furthermore, the leading coefficient (that of the z_n^d term) is a unit and $g(x_1', ..., x_{n-1}', x_n) = 0$. Thus x_n is integral over $k[x_1', ..., x_n']$. By induction, we see $k[x_1', ..., x_{n-1}']$ is integral over $k[y_1, ..., y_r]$ and so $k[x_1, ..., x_n]$ is integral over $k[y_1, ..., y_r]$. □

Theorem. *Let A be an affine domain over a field k. Then* $\dim A =$ *tr deg$_k A$.*

Proof. Let $d =$ tr deg$_k A$. By the Noether Normalization Lemma, let $a_1, ..., a_d \in A$ be algebraically independent such that A is integral over $k[a_1, ..., a_d]$. By Ch 5 of A& M, $\dim A = \dim k[a_1, ..., a_d] = d$ (using the Lying Over Theorem, Going Up Theorem, and incomprability). □

Lemma. *Let k be a field and p a height 1 prime in $k[x_1, ..., x_d]$. Then* $\dim \frac{k[x_1, ..., x_d]}{p} = d - 1$.

Proof. Recall that height 1 primes in a UFD are principal. Let $S = k[x_1, ..., x_d]$ and $p = (f)$. Note that f is not a unit and thus some variable, say x_d, occurs in f, that is $f \notin k[x_1, ..., x_{d-1}]$. Let $\overline{x_1}, ..., \overline{x_{d-1}}$ be the images of $x_1, ..., x_{d-1}$ in S/p.

Claim: $\overline{x_1}, ..., \overline{x_{d-1}}$ are algebraically independent over k.

Proof: Suppose not. Then the map $\phi : k[x_1, ..., x_{d-1}] \to k[\overline{x_1}, ..., \overline{x_{d-1}}]$ is not injective (where ϕ is the map induced by $S \twoheadrightarrow S/p$). Let $0 \neq g \in \ker\phi$. Then $g \in P = (f)$. So $g = hf$ for some $h \in S$. Since $h \neq 0$, x_d occurs in g, a contradiction as $g \in k[x_1, ..., x_{d-1}]$.

Thus tr deg$S/p \geq d-1$. Of course, $S/p \subseteq k[\overline{x_1}, ..., \overline{x_{d-1}}]$ which has dimension $\leq d-1$. Thus $\dim S/p = d-1$. □

Note. This proof does not yield itself to height 2 primes as S/p would not necessarily be a UFD.

Theorem. *Let A be an affine domain over k, a field. Let P be a height 1 prime ideal. Then* $\dim A/P = \dim A - 1$.

Proof. The Noether Normalization Lemma gives a "polynomial ring" $S = k[a_1, ..., a_d] \subseteq A$ such that A is integral over S. Let $p = P \cap S$. We claim that p has height 1. Now, we know $S/p \hookrightarrow A/P$ is an integral extension (as S a UFD implies S is integrally closed). Thus $\dim S/p = \dim A/P < \dim A = d$ as $P \neq 0$. Thus $p \neq (0)$. Now, suppose $p \supsetneq q \neq 0$ for a prime ideal q. By the Going Down Theorem, there exists a prime Q such that $P \subsetneq Q \neq 0$, a contradiction as P is a height 1 prime. Thus p has height 1. Thus $\dim S/p = \dim S - 1 = d-1$ and therefore $\dim A/P = d-1$. □

Corollary. *Let P be a prime ideal in an affine domain over k. Then $htP + \dim A/p = \dim A$.*

Proof. Induct on the height of P. □

Corollary. *Let A be an affine domain over k and P, Q primes with $P \subsetneq Q$ and no primes strictly between them. Then $ht(Q) = 1 + ht(P)$.*

Proof. Note that Q/P is a height 1 prime in A/P. □

Note. This says that we never have $P \subsetneq Q$ where there is a chain consisting of only one prime in between and also a chain of more than one prime in between. This is called the Catenary Chain Condition.

Examples.

1. Let $R = k[x,y]_{(x,y)\cup(x+1)}$. Note that $\dim R = 2$, (x,y) has height 2, $(x+1)$ has height 1, and $ht(x+1) + \dim R/(x+1) = 1 + 0 = 1 < 2$.
2. Let $R = \frac{k[x,y,z]}{(x,y)\cap(z)}$. Here $(x,y) \subsetneq (x,y,z)$ has no primes in between and $(z) \subseteq (x,y,z)$ has many.

2.9 Back to Affine Varieties

We are now aiming to prove the following:

Theorem. *Let X, Y be affine algebraic varieties in $\mathbb{A}^n_k$. Let $\dim X = r, \dim Y = s$. Then each irreducible component of $X \cap Y$ has $\dim \geq r + s - n$.*

Example. $X = V(xy - uv) \subseteq \mathbb{A}^4_k$. Then $\dim Z = 3$. Let $X = V(xy - uv, x, u) = V(x, y)$ and $Y = V(xy - uv, y, v) = V(y, v)$. Then $\dim X = \dim Y = 2$ and $X \cap Y = V(x, u, y, v) = \{(0, 0, 0, 0)\}$. [The issue: This has a singularity at the origin]

First, we need some results on morphisms and products of affine algebraic sets. Unless otherwise stated, k is an algebraically closed field. Let X be a closed subset of $\mathbb{A}^m_k$, with affine coordinate ring $A(X) = k[x_1, \ldots, x_m]/I(X)$. We can think of $A(X)$ as the k-algebra of polynomial functions: $X \to k$.

Suppose now that Y is a closed subset of $\mathbb{A}^n_k$. Recall that a map $f : X \to Y$ is a *morphism* of affine algebraic sets provided $\varphi \circ f : X \to k$ belongs to $A(X)$ for each $\varphi \in A(Y)$. Equivalently, f corresponds to a k-algebra homomorphism $A(Y) \to A(X)$ as follows: A morphism $f : X \to Y$ yields the k-algebra homomorphism $f^* : A(Y) \to A(X)$ taking φ to $\varphi \circ f$. On the other hand, if $F : A(Y) \to A(X)$ is a k-algebra homomorphism, we get a morphism $F^* : X \to Y$ in the following way: Given a point $p \in X$, let m_p be the corresponding maximal ideal of $A(X)$ consisting of polynomial functions vanishing at p. Then $F^{-1}(m_p)$ is a maximal ideal n of $A(Y)$. (Obviously it's a prime ideal; the fact that it's a *maximal* ideal is far from obvious, but it follows easily from the Nullstellensatz.) We know (again by the Nullstellensatz) that n corresponds to a point $q \in Y$. We put $F^*(p) = q$. This map $F^* : X \to Y$ is a morphism. We showed that these correspondences give an antiequivalence between the category of affine algebraic sets and the category of finitely generated reduced k-algebras. ("reduced" means 0 is the only nilpotent element.) In particular, $f^{**} = f$ and $F^{**} = F$. We will say that f is the morphism *corresponding* to the k-algebra homomorphism F.

Now consider $X \times Y \subseteq \mathbb{A}^{m+n}_k$. Then $X \times Y$ is a closed subset of $\mathbb{A}^{m+n}_k$, since $X \times Y = V(I(X)k[x_1, \ldots, x_m, y_1 \ldots, y_n] + I(Y)k[x_1, \ldots, x_m, y_1 \ldots, y_n])$.

Lemma. *Suppose we have closed $X \subseteq \mathbb{A}_k^m$ and closed $Y \subseteq \mathbb{A}_k^n$ are affine algebraic sets. Then the projection maps $\pi_X : X \times Y \to X$ and $\pi_Y : X \times Y \to Y$ where $X \times Y \subseteq \mathbb{A}_k^{m+n}$ are morphisms.*

Proof. Consider the following commutative diagram:

$$\begin{array}{ccc} X \times Y & \xrightarrow{\ \ell\ } & \mathbb{A}_k^{m+n} \\ {\scriptstyle \pi_X}\big\downarrow & & {\scriptstyle \pi}\big\downarrow \\ X & \xrightarrow{\ i\ } & \mathbb{A}_k^m \end{array}$$

The maps ℓ, π and i are morphisms, since they come from the corresponding k-algebra homomorphisms in the following diagram:

$$\begin{array}{ccc} \frac{k[x_1,\ldots,x_m,y_1,\ldots,y_n]}{I(X\times Y)} & \longleftarrow & k[x_1,\ldots,x_m,y_1,\ldots,y_n] \\ & & \uparrow \\ k[x_1,\ldots,x_m]/I(X) & \longleftarrow & k[x_1,\ldots,x_m] \end{array}$$

Since the kernel of the map $k[x_1,\ldots,x_m] \to \frac{k[x_1,\ldots,x_m,y_1,\ldots,y_n]}{I(X\times Y)}$ contains $I(X)$, we can fill in another $\uparrow$ on the left side of the bottom diagram. This is a k-algebra homomorphism, so it corresponds to a morphism $\sigma : X \times Y \to X$ making the first diagram commute. Since i is one-to-one and π_X makes the diagram commute, it follows that $\sigma = \pi_X$. □

Proposition. *For all affine algebraic sets Z and all pairs of morphisms $f : Z \to X$ and $g : Z \to Y$, there exists a unique morphism $h : Z \to X \times Y$ such that $\pi_X h = f$ and $\pi_Y h = g$.*

Proof. Define $h : Z \to X \times Y$ by $h(z) = (f(z), g(z))$. Let $i : X \to \mathbb{A}_k^M$, $j : X \to \mathbb{A}_k^n$ and $\ell : X \times Y \to \mathbb{A}_k^{m+n}$ be the inclusion maps. Since i and j are morphisms, so are $if : Z \to \mathbb{A}_k^m$ and $jg : Z \to \mathbb{A}_k^n$. We define $F : k[x_1,\ldots,x_m,y_1,\ldots,y_n] \to A(Z)$ by $x_i \mapsto (if)^*(x_i)$ and $y_j \mapsto (jg)^*(y_j)$. The morphisms (actual and purported) are pictured here:

$$\begin{array}{ccccc} \mathbb{A}_k^m & \twoheadleftarrow & \mathbb{A}_k^{m+n} & \twoheadrightarrow & \mathbb{A}^n(Y) \\ {\scriptstyle i}\big\uparrow & & {\scriptstyle \ell}\big\uparrow & & {\scriptstyle j}\big\uparrow \\ X & \xleftarrow{\ \pi_X\ } & X \times Y & \xrightarrow{\ \pi_Y\ } & Y \\ & \nwarrow f & \uparrow h & g \nearrow & \\ & & Z & & \end{array}$$

One checks that $F^* = \ell \circ h$, so $\ell \circ h$ is a morphism. The next lemma, with $\ell : X \times Y \to \mathbb{A}_k^{m+n}$ in place of $i : X \to \mathbb{A}_k^m$, shows that h is a morphism. Clearly h is the unique morphism with the property that $\pi_X \circ h = f$ and $\pi_Y \circ h = g$. This proves the following: □

Proposition. *The diagram $X \xleftarrow{\pi_X} X \times Y \xrightarrow{\pi_Y} Y$ is the product in the category of affine algebraic sets.*

Lemma. *Let $q : Z \to \mathbb{A}_k^m$ be a morphism of affine algebraic sets, let X be a closed subset of $\mathbb{A}_k^m$, and let $i : X \to \mathbb{A}_k^m$ be the inclusion morphism. Suppose $q(z) \in X$ for each $z \in Z$, and let $h : Z \to X$ be defined by $i \circ h = q$. Then h is a morphism.*

Proof. Let $\varphi : X \to k$ be an element of $A(X)$. We want to show that $\varphi \circ h \in A(Z)$. Since $k[x_1,\ldots,x_m] \twoheadrightarrow A(X)$ is surjective, we can choose $\psi \in k[x_1,\ldots,x_m]$ mapping to φ. Then $\psi : \mathbb{A}_k^m \to k$, and $\psi \circ i = \phi$. Therefore $\phi \circ h = \psi \circ i \circ h = \psi \circ q \in A(Z)$. □

Lemma. *Let k be any field (not necessarily algebraically closed), and let A and B be finitely generated k-algebras. Then the diagram $A \xleftarrow{u} A \otimes_k B \xrightarrow{v} B$ (where $u(a) = a \otimes 1$ and $v(b) = 1 \otimes b$) is the coproduct in the category of finitely generated k-algebras.*

Proof. Clearly $A \otimes_k B$ is finitely generated as a k-algebra. Suppose C is another finitely generated k-algebra and $A \xrightarrow{F} C \xleftarrow{G} B$ are k-algebra homomorphisms. We want to show that there is a unique k-algebra homomorphism $H : A \otimes_k B \to C$ such that $H \circ u = F$ and $H \circ v = G$. For uniqueness, we note that $A \otimes_k B$ is generated, as a k-algebra (even as an abelian group), by simple tensors $a \otimes b$, and any H that does the job must satisfy $H(a \otimes b) = H((a \otimes 1)(1 \otimes b)) = H(u(a))H(v(b)) = F(a)G(b)$. To prove existence, we observe that the mapping $(a, b) \mapsto F(a)G(b)$ is k-bilinear and therefore induces a map $H : A \otimes_k B \to C$ that does all the right things. □

Now we know that the category of affine algebraic sets over an algebraically closed field k is dual to the category of finitely generated reduced k-algebras, so the product in the first category must correspond to the coproduct in the second category (that is, $A(X \times Y)$ is the coproduct in the category of finite generated reduced $k-$algebras). What is the coproduct in the category of finitely generated reduced k-algebras? Well, it's easy to see, just by looking at boring diagrams, that it's $(A \otimes_k B)_{\text{red}}$, where for any commutative ring R, $R_{\text{red}} = R/\{\text{nilpotents}\}$. (The point is that any ring homomorphism from R to a reduced ring has to factor through the nilradical.) This proves the following:

Theorem. *Let k be an algebraically closed field, and let X and Y be affine algebraic sets over k. Then $A(X \times Y) = (A(X) \otimes_k A(Y))_{red}$. (But see Corollary 2 below.)*

As it turns out, the red subscript doesn't need to be there, but its removal requires some hard field theory (which we refer to below). Since redness has no effect on the main result we are after, we'll leave the subscript there for now.

Corollary. *Let X and Y be affine varieties, of dimensions r and s, respectively. Then $X \times Y$ is an affine variety of dimension $r + s$.*

Proof. We have already shown that $X \times Y$ is a variety. By the Nullstellensatz, $\dim A(X) = \dim X$ and $\dim A(Y) = \dim Y$. By the Noether Normalization Lemma there are transcendence bases $\{x_1, \ldots, x_r\} \subseteq A(X)$ and $\{y_1, \ldots, y_s\} \subseteq A(Y)$ such that $A(X)$ and $A(Y)$ are integral over $k[x_1, \ldots, x_r]$ and $k[y_1, \ldots, y_s]$, respectively. It is easy to see that $A(X) \otimes_k A(Y)$ is integral over $k[x_1, \ldots, x_r] \otimes_k k[y_1, \ldots, y_s]$ (as $A(x) \otimes 1$ is integral over $k[x_1, ..., x_r]$ and $1 \otimes A(Y)$ is integral over $k[y_1, ..., y_s]$) and that $k[x_1, \ldots, x_r] \otimes_k k[y_1, \ldots, y_s] \cong k[x_1, \ldots, x_r, y_1, \ldots, y_s]$. By LO-GU-INC, $\dim_k(A(X) \otimes_k A(Y)) = \dim(k[x_1, \ldots, x_r, y_1, \ldots, y_s]) = r + s$. Since killing nilpotents has no effect on Krull dimension, $\dim(A(X) \otimes_k A(Y)) = r + s$. By the Theorem, $X \times Y$ has dimension $r + s$. □

Getting the red out. We want to show that the tensor product of reduced k-algebras is reduced. In fact, all we need is that the field k is perfect. Note that we can't get around this assumption:

Example. Let k be an imperfect field of characteristic p, and let α be an element of k with no p^{th} root. Let $F = k(\beta)$, where $\beta^p = \alpha$. Thus $F \cong k[x]/(x^p - \alpha)$, and F is amazingly reduced since it's a field. On the other hand, $F \otimes_k F \cong F[x]/(x^p - \alpha) = F[x](x^p - \beta^p) = F[x]/(x - \beta)^p$, and the coset of $x - \beta$ is a non-zero nilpotent element.

Proposition. *Let k be a perfect field, and let A and B be reduced Noetherian k-algebras. Then $A \otimes_k B$ is reduced.*

Note. We don't need Noetherian here. We can just reduce to the finitely generated/Noetherian case as each element is in a finitely generated subalgebra.

Proof. (Sketch) Let $P_1, \ldots, P_r$ be the minimal prime ideals of A. Then $A \hookrightarrow \frac{A}{P_1} \times \cdots \times \frac{A}{P_r}$, so $R \hookrightarrow K_1 \times \cdots \times K_r$, where K_i is the quotient field of $\frac{A}{P_i}$. Similarly, $B \hookrightarrow L_1 \times \cdots \times L_s$, where the L_j are field extensions of k. Then $A \otimes_k B$ embeds in the direct product of the k-algebras $K_i \otimes_k L_j$. Therefore it will suffice to show that $K \otimes_k L$ is reduced, whenever K and L are field extensions of k. Given any element $\alpha \in K \otimes_k L$, there are finitely generated subfields K_1 and L_1 of K and L, respectively, such that $\alpha \in K_1 \otimes_k L_1$. Changing notation, we may assume that K and L are finitely generated over k.

Now we need some serious field theory. See Matsumura's *Commutative Ring Theory*, pp. 199–200. Since k is perfect, L has a separating transcendence basis $S = \{x_1, \ldots, x_n\}$ over k. This means that L is a finite separable extension of $k(S)$. We can regard the x_i as indeterminates, so that $K \otimes_k k[S] \cong K[x_1, \ldots, x_n]$, an integral domain. Therefore $D := K \otimes_k k(S)$ is a domain too, being a localization of a domain. Now $K \otimes_k L = D \otimes_{k(S)} L$, so it will suffice to show that $F \otimes_{k(S)} L$ is reduced, where F is an algebraic closure of the quotient field of D. For notational bliss, put $E = k(S)$. By the Primitive Element Theorem, $L = E(\beta)$ for some element $\beta \in L$. Let $f(T)$ be the minimum polynomial of β over E. Then $L = E[T]/(f)$, so $F \otimes_k L = F[T]/(f)$. Since f is separable over E and $E \subseteq F$, f splits into distinct linear factors over F. By the Chinese Remainder Theorem, $F[T]/(f) = K \times \cdots \times K$, which is reduced. □

Corollary. *Let X and Y be affine algebraic sets over an algebraically closed field k. Then $A(X \times Y) = A(X) \otimes_k A(Y)$.*

Corollary. *Let A and B be affine domains over an algebraically closed field k. Then $A \otimes_k B$ is a domain.*

Over any field k that is not algebraically closed there are counterexamples to the corollary: Let $f(T)$ be an irreducible polynomial of degree bigger than 1, and put $A = B = k[T]/(f)$. Then $A \otimes_k B = B[T]/(f)$. But f has a root in B, so f is not irreducible in $B[T]$. Therefore $A \otimes_k B$ is not a domain.

Now we can get to the dimension inequality discussed at the beginning of this section. Let X be a closed subset of $\mathbb{A}^m_k$. Define the diagonal $\Delta_{\mathbb{A}^m_k} := \{(p,p)|p \in \mathbb{A}^m\} \subseteq \mathbb{A}^{2m}_k = \mathbb{A}^m_k \times \mathbb{A}^m_k$ and $\Delta_X = \Delta_{\mathbb{A}^m_k} \cap (X \times X) \subseteq X \times X$. Now $\Delta_{\mathbb{A}^m_k} = V(x_1 - y_1, ..., x_m - y_m)$ which says it is closed in $\mathbb{A}^{2m}_k$. Thus Δ_X is closed in $X \times X$.

Theorem (Affine Dimension Inequality)**.** *Let X and Y be closed subvarieties of $\mathbb{A}^m_k$, where k is an algebraically closed field, and let Z be an irreducible component of $X \cap Y$. Then $\dim(Z) \geq \dim(X) + \dim(Y) - m$.*

Proof. (Version 1) Let $I = I(X)$ and $J = I(Y)$, ideals of $k[x_1, \dots, x_m]$. Let Δ be the diagonal of $\mathbb{A}^m_k$. Note that the obvious k-algebra isomorphism $F : \frac{k[x_1,...,x_m,y_1,...,y_m]}{(x_1-y_1,...,x_m-y_m)} \xrightarrow{f} k[x_1, \dots, x_m]$ induces an isomorphism of algebraic sets $\Delta \xleftarrow{F^*} \mathbb{A}^m_k$ taking a point p to (p,p). In particular F^* is a homeomorphism. Since F^* carries $X \cap Y$ onto $(X \times Y) \cap \Delta$, we see that $X \cap Y$ and $(X \times Y) \cap \Delta$ are homeomorphic. (Yeah, they are isomorphic algebraic sets, but we don't need that since irreducibility and dimension are topological things.) We know that $X \times Y$ is an affine variety whose dimension is $\dim(X) + \dim(Y)$. The desired inequality now follows by setting $V := X \times Y$ in the next lemma.

(Version 2) Let $r = \dim X, s = \dim Y$. Let $I = I(X), J = I(Y)$. Then $A(X \cap Y) = \frac{k[x_1,...,x_m]}{\sqrt{I+J}}$. Now, $A(X \times Y \cap \Delta_{\mathbb{A}^m_k}) = \frac{k[x_1,...,x_m,y_1,...,y_m]=:S}{\sqrt{IS+J_1S+(x_1-y_1,...,x_m-y_m)}}$ where $J_1 = \text{im} J$ under the isomorphism $k[x_1, ..., x_m] \to k[y_1, ..., y_m]$ which sends $x_i \mapsto y_i$. Clearly, $A(X \times Y \cap \Delta_{\mathbb{A}^m_k}) = \frac{k[x_1,...,x_m]}{\sqrt{I+J}} = A(X \cap Y)$. Thus $X \cap Y \cong (X \times Y) \cap \Delta_{\mathbb{A}^m_k}$. Let $P = I(X \times Y) = k[x_1, ..., x_m, y_1, ..., y_m]$. Then $p = \sqrt{IS + J_1S}$. An irreducible component of $(X \times Y) \cap \Delta_{\mathbb{A}^m_k}$ is a minimal prime of $A((X \times Y) \cap \Delta)$. We want to show that for all minimal primes Q of $P + (x_1 - y_1, ...x_m - y_m)$, that $\dim \frac{k[x_1,...,x_m,y_1,...,y_m]}{Q} \geq r + s - m$.

Claim: Let A be an affine domain over k and let $f_1, ..., f_m \in A$. Then for all primes Q minimal over $(f_1, ..., f_m)$, we have $\dim A/Q \geq \dim A - m$.

Proof: For affine domains over a field, we have $\dim A/p + ht(p) = \dim A$. Of course, $ht(p) \leq m$.

Let $A = k[x_1, ..., x_m, y_1, ..., y_m]/p$. Then $\dim A = r + s$. By the Claim, we're done. □

Remark. We can't replace "variety" with "affine open set."

Proposition. $\dim \mathbb{P}^n_k = n$.

Proof. To show $\geq$, we need a chain of homogenous prime ideals in $S = k[x_0, ..., x_n]$ that does not include $S_+ = (x_0, ..., x_n)$. Of course, $(0) \subsetneq (x_0) \subsetneq \cdots \subsetneq (x_0, ..., x_{n-1})$ does the job. To show $\leq$, every homogenous prime ideal $\neq S_+$ is properly contained in S_+. Any chain of length $n + 1$ would give a chain of length $n + 2$ when we include S_+. But $\dim k[x_0, ..., x_n] = n + 1$, a contradiction. □

Lemma. *Let V be a closed subvariety of $\mathbb{A}^m_k \times \mathbb{A}^m_k$, and let Z be an irreducible component of $V \cap \Delta$. Then $\dim(Z) \geq \dim(V) - m$.*

Proof. Let $I(V) = P$, a prime ideal of $S := k[x_1, \dots, x_m, y_1, \dots, y_m]$. Then $V \cap \Delta$ is defined by the ideal $J := P + (x_1 - y_1, \dots, x_n - y_n)$, so $A(V \cap \Delta) = S/\sqrt{J}$ by the Nullstellensatz. Let $B = S/P$, and let f_i be the residue of $x_i - y_i$ modulo P. Then $A(V \cap \Delta) = B/\sqrt{(f_1, \dots, f_m)}$. Our mission is to show that for every minimal prime Q of the ideal $\sqrt{(f_1, \dots, f_m)}$ in B, $\dim(B/Q) \geq \dim(V) - m$. Such a prime Q is also minimal over $(f_1, \dots, f_m)$, so by Krull's theorem we have $\text{ht}(Q) \leq m$. Moreover, from the dimension theory of affine domains we know that $\dim(B/Q) + \text{ht}(Q) = \dim(B)$. Since $\dim(B) = \dim(V)$, we're done. □

Example. By the affine dimension inequality, two affine surfaces in $\mathbb{A}^3$ cannot intersect in a point. Let's see what happens if we replace the ambient space $\mathbb{A}^3$ by a more perverse three-dimensional variety C, namely, the zero-set of $x_1x_2 - x_3x_4$ in $\mathbb{A}^4$. Then C is a three-dimensional variety containing two planes $V(x_1, x_3)$ and $V(x_2, x_4)$, whose only point of intersection is the origin.

In order to discuss the dimension theory of projective varieties (cf. Hartshorne, pp. 11, 12, 48), we state a trivial lemma on irreducibility, etc. (cf. Hartshorne, p. 8).

Lemma (TL)**.** *Let X be a non-empty topological space, and let Z be a non-empty subspace of X.*

1. *X is irreducible if and only every non-empty open subset of X is dense in X.*

2. *Z is irreducible if and only if $\overline{Z}$ (closure in X) is irreducible.*

3. $\dim(Z) \leq \dim(X)$

4. *If X is irreducible and Z is open in X, then Z is irreducible.*

5. *If X is irreducible and Z is closed in X and $\dim(Z) = \dim(X) < \infty$, then $Z = X$.*

6. *If $X = \bigcup_i U_i$ and the U_i are open in X, then $\dim(X) = \sup_i \dim(U_i)$.*

Proof. Only (6) seems at all tricky. By (3), we just have to prove "$\leq$", and we may assume that $\sup_i \dim(U_i) = d < \infty$. Suppose we have a chain of irreducible closed sets $Z_0 \subsetneq \cdots \subsetneq Z_t$ in X. Choose $p \in Z_1$, and choose i such that $p \in U_i$ (since the U_i's cover X). We get a chain of non-empty closed subsets of U_i: $U_i \cap Z_0 \subseteq \cdots \subseteq U_i \cap Z_t$, and each of these sets is irreducible by (4). If these were not proper inclusions, say $U_i \cap Z_m = U_i \cap Z_{m+1}$, by (1) we would have $Z_m = \overline{U_i \cap Z_m} = \overline{U_i \cap Z_{m+1}} = Z_{m+1}$, a contradiction. This shows that $t \leq d$ as desired. □

Exercise. The following are useful exercises from Hartshorne (#2.6, 2.6 p12)

1. $U_{x_i} \cap Y \neq \emptyset$ implies $\dim(U_{x_i} \cap Y) = \dim Y$ (Recall $U_{x_i} = \{p \in \mathbb{P}^n_k | x_i(p) \neq 0\}$ and $U_{x_i} = \mathbb{A}^n_k$).

2. Let $Y = U \cap V$, where U is open in $\mathbb{P}^n$ and V is closed in $\mathbb{P}^n$. Then $\dim Y = \dim \overline{Y}$.

Theorem. *Let X be a projective or affine variety. Then $\dim(X) = \dim(\mathcal{O}_X, p)$ for every point $p \in X$.*

[We're not in Schemeland anymore, so points are closed.]

Proof. Suppose first that X is affine, with affine coordinate ring A. We have to show that $\dim(A_m) = \dim(A)$ for each maximal ideal m of A. Since A is an affine domain, we have $\dim A = \dim(A/m) + \mathrm{ht}(m) = 0 + \dim(A_m)$. Now suppose X is a projective variety. Then X has an open cover by non-empty affine open sets $U_1, \ldots, U_m$. Then $U_i \cap U_j \neq \emptyset$ by TL(1). Choose $p \in U_i \cap U_j$. By the affine case above, $\dim(U_i) = \dim(\mathcal{O}_{U_i,p}) = \dim(\mathcal{O}_{U_i,p}) = \dim(U_j)$. Now TL(6) implies that $\dim(X) = \dim(U_i)$ for each i, and another application of the affine case gives $\dim X = \dim U_I = \dim \mathcal{O}_{\S,\checkmark}$. □

Lemma. *Let Y be a projective variety, and let S be the homogeneous coordinate ring of Y corresponding to a particular embedding of Y in some $\mathbb{P}^m_k$. Then $\dim(S) \geq \dim(Y) + 1$.*

[Actually, $\dim(S) = \dim(Y) + 1$, but the other inequality seems a little harder to prove, and we don't really need it at this point.]

Proof. Let $\dim(Y) = d$, and let $Y = Z_0 \supset \cdots \supset Z_d$ be a chain of irreducible closed subsets of Y. By the homogeneous Nullstellensatz, we have a corresponding chain $(0) = P_0 \subset \cdots \subset P_d$ of homogeneous prime ideals of S, with $P_d \neq S_+$. Tossing in the prime ideal S_+, we get a chain of primes of length $d + 1$ in S. □

Theorem (Projective Dimension Theorem)**.** *Let X and Y be projective varieties in $\mathbb{P}^m_k$, with dimensions r and s, respectively. Then each irreducible component of $X \cap Y$ has dimension at least $r + s - m$. Moreover, if $r + s - m \geq 0$, then $X \cap Y \neq \emptyset$.*

Proof. Let Z be an irreducible component of $X \cap Y$, and let $\dim(Z) = d$. We know $\mathbb{P}^m_k$ is covered by open sets isomorphic to $\mathbb{A}^m_k$, so choose such an open set U such that $Z \cap U \neq \emptyset$. By TL(4), $Z \cap U$ is irreducible; by the previous theorem, $\dim(Z \cap U) = d$. To see that $Z \cap U$ is an irreducible component of $X \cap Y \cap U$, suppose $Z \cap U \subsetneq C$, where C is an irreducible closed subset of $X \cap Y \cap U$. Taking closures, we have $Z \subseteq \overline{Z \cap U} \subseteq \overline{C} \subseteq X \cap Y$. Since C is irreducible by TL(2), and since Z is an irreducible component of $X \cap Y$, we must have $Z = \overline{C}$. But then $C \subseteq \overline{C} \cap U = Z \cap U$, contradiction.

Next, we apply the previous theorem to the projective variety Z and the affine variety $Z \cap U$ to conclude that $\dim(Z \cap U) = \dim(Z) = d$. Similarly, $\dim(X \cap U) = r$ and $\dim(Y \cap U) = s$. Since all three of these varieties are in the affine space $U \cong \mathbb{A}^m_k$, affine dimension inequality implies that $d \geq r + s - m$, as desired.

Now we assume that $r + s - m \geq 0$. We want to show that $X \cap Y \neq \emptyset$. Let $S = k[x_0, \ldots x_m]$, the homogeneous coordinate ring of $\mathbb{P}^m_k$. Put $P = I(X)$ and $Q = I(Y)$, the ideals generated by the homogeneous polynomials in S that vanish on X and Y respectively. Thus S/P and S/Q are the homogeneous coordinate rings for X and Y respectively (for the given embeddings in $\mathbb{P}^m_k$). Let's temporarily forget about the grading and think of P and Q as plain old boring prime ideals in the ungraded polynomial ring S. We have the corresponding affine varieties $C(X) := V(P) \subseteq \mathbb{A}^{m+1}_k$ and $C(Y) := V(Q) \subseteq \mathbb{A}^{m+1}_k$. (These are called the *affine cones* over the projective varieties X and Y respectively. Note that they are indeed cones: For each $q \in C(X)$ the line through q and the origin is contained in $C(X)$.) We know $\dim(C(X)) = \dim(S/P) \geq \dim S(x) \geq r + 1$

(by the previous Lemma), and similarly $\dim(C(Y)) \geq s+1$. Also, $C(X)$ and $C(Y)$ are irreducible because P and Q are prime. Since P and Q are homogeneous (remember, I said *temporarily* forget), the origin $p := (0, \ldots, 0)$ is in $C(X) \cap C(Y)$. Thus $C(X) \cap C(Y)$ has an irreducible component, and by the affine dimension inequality its dimension must be at least $(r+1)+(s+1)-(m+1) = r+s-m+1$, which by assumption is positive. Therefore $C(X) \cap C(Y)$ has to contain another point $q \neq p$. (In fact, $C(X) \cap C(Y)$ must be infinite.) Write $q = (a_0, \ldots, a_m)$, and observe that every homogeneous polynomial in $P+Q$ vanishes at the point $(a_0 : \cdots : a_m) \in \mathbb{P}^m_k$. Therefore $(a_0 : \cdots : a_m) \in \mathrm{Z}(P+Q) = \mathrm{Z}(P) \cap \mathrm{Z}(Q) = X \cap Y$. □

Theorem (Graded Prime Filtration (Hartshorne p 50))**.** *Let S be a non-negative graded Noetherian ring and let M be a finitely generated $\mathbb{Z}-$graded $S-$module. Then M has a filtration of graded homogenous submodules $(0) = M^0 \subset M^1 \subseteq \cdots \subseteq M^r = M$ such that for i we have $M^i/M^{i-1} \cong (S/p^i)(\ell_i)$ (where ℓ_i is a shift operator so that we have an isomorphism of* graded *rings). Given any such filtration, we have*

1. *For all homogenous prime ideals P of S, $P \supseteq Ann_S M = (0 :_S M)$ if and only if $P \supseteq P_i$ for some i. Therefore, the minimal elements of $\{P_1, ..., P_r\}$ are exactly the minimal homogenous primes of $(0 :_S M)$.*

2. *For all primes P minimal over $(0 :_S M)$, P occurs exactly $lambda_{S_p}(M_p)$ times on the list $\{P_1, ..., P_r\}$.*

Remark. $P \supseteq (0 : M)$ if and only if $M_p \neq 0$. So we define the **support** of a module as $Supp(M) = \{p \in SpecS | M_p \neq 0\}$. Then P is minimal over $(0 : M)$ if and only if P is minimal over $Supp(M)$ if and only if M_p is a nonzero S_p-module of finite length.

Proof. (Existence of filtrations). We can assume $M \neq 0$. Let $\mathfrak{a} := \{(0 : M) | m$ is a nonzero homogenous element of $M\}$. As S is Noetherian, $\mathfrak{a}$ has a maximal element, call it P. Then P is a homogenous ideal. To show it is prime, let a, b be homogenous elements of S with $ab \in P$. Suppose $b \in P$. Note $P = (0 : m)$ for some homogenous element $m \in M \setminus \{0\}$. So $bm \neq 0$ and $P \subseteq (0 : bm) \in \mathfrak{A}$. Since P is maximal in $\mathfrak{A}$, we see $P = (0 : bm)$. Of course, $a \in (0 : bm)$ as $(ab)m = 0$ which implies $a \in P$. Thus P is prime. Let $M^0 = 0$ and $M^1 = Sm$. Then $P_1 = P$. Note we have the exact sequence $0 \to P \to S \xrightarrow{f} Sm \to 0$ where $f(1) = m$. If $\deg m = \ell$, then $0 \to P(-\ell) \to S(-\ell) \to Sm \to 0$ is a short exact sequence of graded modules. Therefore $M' = Sm \cong S/p(-\ell)$. Take $\ell_1 = -\ell$. Repeat on M/M' to get M^2. Continue in this way and note we must stop as M is Noetherian.

Now, suppose we are given such a filtration as above. Then

1. $P \supseteq (0 : M)$ if and only if $M_p \neq 0$ if and only if $(M^i/M^{i-1})_p \neq 0$ for some i if and only if $(S/p_i)_p \neq 0$ for some i which is if and only if $P_i \subseteq P$ for some i.

2. Let P be minimal over $(0 : M)$, that is P is a minimal element over $\{P_1, ..., P_r\}$. Fix i and look at $M_p^{i-1} \subseteq M_p^i$. If $P \neq P_i$, then $(M^i/M^{i-1})_p \cong (S/P_i)_P = 0$ as $P_i \not\subseteq P$. If $P = P_i$, then $(M^i/M^{i-1})_P \cong (S/P)_P = S_p/pS_p$, which has length 1 as an R_p-module. Since lengths are additive, the length is the number of i such that $P = P_i$, that is, the number of occurrences of P.

□

Theorem (Hartshorne, p 51)**.** *Let $S = k[x_0, ..., x_n]$. For a homogenous ideal J of S, $Y = Z(J) := \{p \in \mathbb{P}^n_k | F(p) = 0$ for all homogenous $F \in J\}$. Suppose that $M \neq 0$ is a finitely generated $\mathbb{Z}-$graded $S-$module. Let $Y = Z(0 : M) \subseteq \mathbb{P}^n$. Define $P_M(t) =$ Hilbert polynomial of M (Recall $P_M(n) = \dim_k M_n$ for all $n >> 0$). Then $\deg P_M = \dim Y = \dim S/(0 : M) - 1(*)$.*

Proof. We will induct on $d = \deg P_M$. By convention, we say the 0 polynomial has degree -1 and also $\emptyset$ has dim $= -1$. Now, if $d = -1$, then M has finite length. Then $\sqrt{(0 : M)} = (x_0, ..., x_n)$ which implies $Z((0 : M)) = Z((x_0, ..., x_n)) = \emptyset$. Also, $\dim S/(0 : M) = 0$. So suppose we have a graded exact sequence $0 \to M' \to M \to M'' \to 0$.

Claim: If $(*)$ holds for M', M'', then it holds for M.

Proof: Note that $P_M = P_{M'} + P_{M''}$. Clearly, $\deg P_M = \max\{\deg P_{M'}, \deg P_{M''}\}$ as the leading coefficients of $P_{M'}$ and $P_{M''}$ are positive. Also, $(**)\sqrt{(0 : M)} = \sqrt{(0 : M') \cap (0 : M'')}$. To see this, since $M'' = M/M'$, we have $aM = 0$ implies $aM' = 0$ and also $aM'' = 0$ and $aM' = 0$ and $aM'' = 0$ implies $a^2 M = 0$. Thus $Y = Z(0 : M) = Z(0 : M') \cup Z(0 : M'')$ and so $\dim Y = \max\{\dim Z(0 : M'), \dim Z(0 : M'')\}$ (as any irreducible component of the union is an irreducible component of one of the pieces). Lastly, if $p \in SpecS$, then $p \supseteq (0 : M)$ if and only if $p \supseteq (0 : M')$ or $p \supseteq (0 : M'')$ by $(**)$. Thus $\dim S/(0 : M) = \max\{\dim S/(0 : M'), \dim S/(0 : M'')\}$.

By the claim and the homogenous prime filtration theorem, we may assume $M = S/Q(\ell)$ where Q is a homogenous prime ideal. Since $d \geq 0$, M does not have finite length. Thus $Q = (0 : M) \subsetneq (x_0, ..., x_n)$. Then choose i such that $x_i \notin Q$. Let $H = Z(x_i) \cong \mathbb{P}^{n-1}$. Thus $\dim H = n-1$. Note x_i is a non-zero divisor on S/Q and thus $S/Q(\ell) = M$. Then we have the short exact sequence of graded modules $- \to M(-1) \xrightarrow{x_i} M \to M'' \to 0$ (where M'' is the cokernel). Now $P_{M(-1)}(t) = P_M(t-1)$. Thus $P_{M''}(t) = P_M(t) - P_M(t-1) = (\Delta P_m)(t-1)$ (the difference operator). Thus $\deg P_{M''}(t) = d-1$ and we can apply the inductive hypothesis. To complete the proof, we want to show $(a) \dim((0 : M'')) = \dim Y - 1$ and $(b) \dim(S/(0 : M'')) = \dim(S/Q) - 1$.

Proof of b: Let $A = S/Q$. Then $M' = A/fA$ where $f = x_i + Q \in A$. We have $(0 : M'') = (Q + Sx_i)$ and so $\dim S/(0 : M'') = \dim A/fA \geq \dim A - 1$. But $f \neq 0$ is in a domain. So $\dim A/fA < \dim A$ as A is a domain. Thus $\dim(S/(0 : M'')) = \dim S/Q - 1$.

Proof of a: Note $Z(0 : M'') = Z(Q + sx_i) = Z(Q) \cap Z(x_i) = Y \cap H$ (where $\dim Y = r, \dim H = n-1$). If $\dim Y = 0$, then $Y = \{point\}$. Recall $Y \not\subseteq H$ as $x_i \notin Q$. Thus $Y \cap H = \emptyset$ and $\dim Z(0 : M'') = -1$. Thus we may assume $r > 0$. Then $\dim Y + \dim H - \dim \mathbb{P}^n = r + (n-1) + n \geq 0$. Thus $Y \cap H \neq \emptyset$. Therefore each irreducible component Z of $Y \cap H$ has $\dim \geq r - 1 \geq 0$. Since $Y \cap H \subsetneq Y$, we see $\dim Z < \dim Y = r$ which says $\dim Z \leq r - 1$. Thus $\dim Z = r - 1$.

□

Corollary. *If $Y = Z(F)$, where F is a homogenous polynomial of positive degree in $S(\mathbb{P}^n) = k[x_0, ..., x_n]$, then $\dim Y = n-1$.*

Definition. *A **hypersurface** in $\mathbb{P}^n$ is a closed subset of the form $Z(F)$, where F is a homogenous polynomial of positive degree.*

Definition. *Define $P_Y = P_{S/I(Y)}$ where $S = k[x_0, ..., x_n]$. We know $P_Y(n) \in \mathbb{Z}$ for $n >> 0$. Say $P_Y(t) = \frac{e}{r!}t^r + ...$, where e is a positive integer. The integer e is called the **degree** of Y and is written $\deg Y = e$. (Recall $\dim Y = r$.)*

Warning. Leading coefficients of $P_{S/I}$ and $P_{S/\sqrt{I}}$ may be different.

Proposition (Hartshorne, p52)**.** *1. Let $Y = Y_1 \cup Y_2$, where each Y_i is closed in $\mathbb{P}^n$. Assume $\dim Y_1 \cap Y_2 < \dim Y_1 = \dim Y_2 =: r$. Then $\deg Y = \deg Y_1 + \deg Y_2$.*

2. $\deg \mathbb{P}^n = 1$.

3. Let $F \in S$ be homogenous of degree $d > 0$. Assume F is square free. Then $\deg Z(F) = d$.

Proof. 1. Note $Y_i \neq \emptyset$. Let $I_{Y_i} = J_i$ for $i = 1, 2$. Then $J_i \neq S_+ = (x_0, ..., x_n)$. So $Z(J_1 + J_2) = Y_1 \cap Y_2$. Thus $P_{Y_1 \cap Y_2}$ has degree $< r$, so P_{S/J_1+J_2} has deg $< r$. We have the short exact sequence $0 \to S/J_1 \cap J_2 \to (S/J_1) \oplus (S/J_2) \to S/(J_1 + J_2) \to 0$. Now $Y_1 \cup Y_2 = Z(J_1 \cap J_2), P_{Y_1} + P_{Y_2} = P_{Y_1 \cup Y_2} + P_{S/(J_1+J_2)}$ where the leading coefficient of the RHS is the leading coefficient of $P_{Y_1 \cup Y_2}$ as $\deg P_{S/(J_1+J_2)} < \deg P_{Y_1 \cup Y_2}$ and $\deg P_{Y_1} = \deg P_{Y_2} = \deg P_{Y_1 \cup Y_2}$. Thus we see lc of $P_{Y_1}+$ lc of $P_{Y_2} =$ lc of $P_{Y_1 \cup Y_2}$.

2. Count Monomials: $P_{\mathbb{P}^n}(t) = (\binom{t+n}{n}) = \frac{1}{n!}t^n + ...$

3. Note $0 \to S(-d) \xrightarrow{F} X \to S/(F) \to 0$ is a short exact sequence. Let $Y = Z(F)$, so $P_Y(t) = P_{\mathbb{P}^n}(t) - P_{\mathbb{P}^n}(t-d) = (\frac{t^n}{n!} + a_{n-1}t^{n-1} + ...) - (\frac{(t-d)^n}{n!} + a_{n-1}(t-d)^{n-1} + ...)$. The t^n terms will cancel, leaving $\deg = d$.

□

2.10 Intersection Multiplicity in $\mathbb{P}^n$

Let Y be a closed subset of $\mathbb{P}^n$. Assume Y has *pure* dimension r (that is, every irreducible component of Y has dimension r). Let H be a hypersurface (not necessarily irreducible) so that no irreducible component of Y is contained in H. Let $Z_1, ..., Z_s$ be the irreducible components of $Y \cap H$. Then $\dim Z_i = r - 1$. The **intersection multiplicity** of $Y \cap H$ along Z_j is defined as $i(Y, H; Z_j) := \lambda_{S_{P_j}}((S/(I_Y + I_H))_{P_j})$, where $P_j = I(Z_j)$. Note that P_j is a minimal prime ideal of $\sqrt{I_Y + I_H}$ and thus of $I_Y + I_H$.

Theorem. *With the above setup and $Y \neq \emptyset$, $\sum_{j=1}^{s} i(Y, H; Z_j) \deg Z_j = (\deg Y)(\deg H)$.*

Proof. Let $d := \deg H, e := \deg Y$. Consider the graded short exact sequence $0 \to S/I_Y(-d) \xrightarrow{f} S/I_y \to M \to 0$ where $Sf = I_H$ and $M = S/(I_Y + I_H)$. Then

$$P_M(t) = P_Y(t) - P_Y(t-d) = [\frac{e}{r!} + lower] - [\frac{e}{r!}(t-d)^r + lower] = \frac{ed}{(r-1)!}t^{r-1} + lower(*).$$

Filter M as $(0) = M^0 \subset M^1 \subset \cdots \subset M^q = M$, where $M^i/M^{i-1} \cong (S/Q_i)(\ell_i)$ as graded modules. Let $e_i = \deg Z(Q_i)$ and $r_i = \dim Z(Q_i) = \dim S/Q_i - 1 \leq \dim M - 1 = \dim(Y \cap H) = r-1$. Notice $P_M(T) = \sum_{i=1}^q P_{S/Q_i(\ell_i)}(t) = \sum_{i=1}^q \frac{e_i}{r_i!}(t+\ell_i)^{r_i} + ... = \sum_{i=1}^q \frac{e_i}{r_i!}t^{r_i} + ...$ (as the ℓ_i's get absorbed into the lower degree terms). Want to find the coefficient of t^{r-1} and compare it with that of $(*)$. This is $\sum_{\{i:r_i=r-1\}} \frac{e_i}{(r-1)!}$. These i are the ones for which Q_i is minimal over $I_Y + I_H$ (as they correspond to those of maximal dimension). Thus each Q_i is a P_j as $Z_j = Z(P_j)$. So the coefficient of T^{r-1} in $P_M(t)$ is $\sum_{j=1}^s \frac{\deg Z_j}{(r-1)!}\cdot$(the number of times P_j occurs in the list of $Q_1, ..., Q_q$). Since the number of times is $\lambda_{S_{P_j}} M_{P_j}$, we're done. □

Corollary (Bezout's Theorem). *Let C and D be curves in $\mathbb{P}^2$ (where by curve we mean an algebraic set of pure dimension 1). Assume C and D have no common irreducible components (that is, f and g are relatively prime where $Sf = I_C$ and $Sg = I_D$). Let $C \cap D = \{p_1, ..., p_s\}$. Then $\sum_{j=1}^s i(C, D; p_j) = \deg C \deg D = (\deg f)(\deg g)$.*

Proof. Note that the degree of a point is 1. □

Example. Consider the curves $C := y^2 - x = 0$ and $D := y^2 - x^3 = 0$. Now, five points of intersection are clear: $(0,0), (1, \pm 1), (-1, \pm i)$. However, Bezout's Theorem say there are 6. What's the catch? $(0,0)$ is a double point. To see this, note that the ideal $(y^2 - x, y^2 - x^3)$ contains $x^3 - x = x(x^2 - 1)$. As $x^2 - 1$ is a unit in $k[x,y]_{(x,y)}$, the ideal contains x. Thus $\frac{k[x,y]_{(x,y)}}{(y^2-x,y^2-x^3)}] = \frac{k[x,y]_{(x,y)}}{(x,y^2)} = \frac{k[y]}{(y^2)}$. Thus $i(C, D; (0,0)) = 2$. Thus, we have all six points.

Notice that Bezout's Theorem deals with $\mathbb{P}^2$ and we found all of the points lying in the affine plane. Consider the next example for which we must consider $\mathbb{P}^2$.

Example. Consider the curves $C := y - x^2 = 0$ and $D := y^2 - x^3 = 0$. Here we have two obvious points $(0,0)$ and $(1,1)$. Its easy to see that $(1,1)$ has multiplicity 1 as the curves have different tangent lines at that point.

Claim: $i(C, D; (0,0)) = 3$.

Proof: We want to consider the ring $\frac{k[x,y]_{(x,y)}}{(y-x^2,y^2-x^2)}$. Note that $k[x,y] \to k[x]$ defined by $y \mapsto x^2$ is a surjective homomorphism and thus $\frac{k[x,y]}{(y-x^2)} \cong k[x]$. Notice that $y^2 - x^3 \mapsto x^4 - x^3 = x^3(x-1)$, where $x - 1$ is a unit in $k[x,y]_{(x,y)}$. Thus $\frac{k[x,y]_{(x,y)}}{(y-x^2,y^2-x^2)} \cong \frac{k[x]}{(x^3)}$, and so $(0,0)$ has multiplicity 3.

Of course, this only gives us 4, and Bezout's Theorem says we should have 6. So we must go to the projective plane. In this case, we must homogenize our curves to $\overline{C} := yz - x^2 = 0$ and $\overline{D} := y^2z - x^3 = 0$ (just add factors of z to each term to make the degrees the same, but also keep the degree minimal). Then, we see $(0,1,0)$ is a point and by considering $\frac{k[x,z]_{(x,z)}}{(z-x^2,z-x^3)}$, we see $i(C, D; (0:1:0)) = 2$ (as we see $x^3 - x^2 = x^2(x-1)$ is in the ideal and as $x - 1$ is a unit, we get x^2 in the ideal).

(If we wanted to find the intersection multiplicity of a point that didn't lie on an axis, we'd simply do a change of variables so that it did).

Observation. Let C and D be plane curves and p a point of $C \cap D$. Then $i(C, D; p) = 1$ if and only if p is a smooth point of C and of D and also C and D are not tangent to each other at p.

Proof. Say $I(C) = f, I(D) = g$, which are ideals of $k[x,y]$. We can assume $p = (0,0)$ by translations of the axes. Write $f = a + bx + cy + (higher)$, $a, b, c \in k$. Then $p \in V(f)$ implies $a = 0$. Now, $\frac{\partial f}{\partial x}(p) = b$ and $\frac{\partial f}{\partial y}(p) = c$. Recall that p is a smooth point of C if and only if $[\frac{\partial f}{\partial x}(p) \; \frac{\partial f}{\partial y}(p)]$ has rank $n - r = 1$ which is if and only if one of the partial derivatives at p is nonzero. Note that the tangent line of f is $bx + cy = 0$. If p is a singularity, then $b = c = 0$ which says $f \in m^2$ (where $m = (x,y)k[x,y]_{(x,y)}$). Since m needs two generated and $f \in m^2$, NAK implies $m \neq (f,g)$. Thus $\dim_k \frac{\mathcal{O}_p}{(f,g)} > 1$ and so $i(C, D; p) > 1$.

Claim: $i(C, D; p) = 1$ if and only if $(f,g) = m$.

Proof: As higher degree terms are in m^2, NAK says that $(f,g) = m$ if and only if $(bx + cy, dx + ey) = m$ where $g = dx + ey + (higher)$. Now $\overline{x}, \overline{y}$ form a vector space for m/m^2. So $b\overline{x} + c\overline{y}$ and $d\overline{x} + e\overline{y}$ form a basis if and only if $be - cd \neq 0$ which is if and only if $bx + cy = 0$ and $dx + ey = 0$ define different lines.

□

3 Sheaf Cohomology

Recall. Suppose X is a topological space and suppose $\mathcal{F}$ is a presheaf on X. This assigns to each open set U an abelian group $\mathcal{F}(U)$ and to each pair of open sets $U \subseteq V$ a "restriction map" $\rho_{VU} : \mathcal{F}(U) \to \mathcal{F}(V)$ such that $\rho_{WV} \circ \rho_{VU} = \rho_{WU}$ if $U \subseteq V \subseteq W$ and $\rho_{UU} = 1_{\mathcal{F}(U)}$. $\mathcal{F}$ is a sheaf if it satisfies the following axioms:

If U is open and $U = \cup U_i$ for all U_i open, then

F1 $\sigma \in \mathcal{F}(U)$ and $\sigma|_{U_i} = \rho_{U_i U}(\sigma) = 0$ for all i implies $\sigma = 0$.

F2 If $\sigma_i \in \mathcal{F}(U_i)$ and $\sigma_i|_{U_i \cap U_j} = \sigma_j|_{U_i \cap U_j}$ for all i, j then there exists $\sigma \in \mathcal{F}(U)$ such that $\sigma|_{U_i} = \sigma_i$ for all i (and σ is unique by F1.

Note. Every presheaf $\mathcal{F}$ has an associated sheaf $\mathcal{F}^+$.

Let $\mathcal{F}_p = \lim_{\to U, p \in U} \mathcal{F}(U)$. Let $\mathcal{F}^+(U)$ consist of all functions $\sigma : U \to \cup_p \mathcal{F}_p$ such that for all $p \in U$ there exists open $V \subseteq U$ with $p \in V$ and there exists $\tau \in \mathcal{F}(V)$ for all $q \in V$ such that $\sigma(q) = \operatorname{im}(\tau)$ under the limit map from $\mathcal{F}(V) \to \mathcal{F}_p$. It is easy to see that $\mathcal{F}^+$ is a sheaf and the obvious map $\theta : \mathcal{F}(U) \to \mathcal{F}^+(U)$ is a morphism of presheafs. Moreover, if $\phi : \mathcal{F} \to \mathcal{G}$ is a morphism of presheafs, then there exists a unique $\overline{\phi} : \mathcal{F}^+ \to \mathcal{G}$ such that $\overline{\phi}\theta = \phi$.

Let X be a topological space. Suppose $\mathcal{F}$ and $\mathcal{G}$ are sheaves of abelian groups (or modules) and suppose $\alpha : \mathcal{F} \to \mathcal{G}$ is a morphism. Then the **kernel** of α is the sheaf defined by $(\ker \alpha)(U) := \ker(\alpha_U)$ (where $\alpha_u : \mathcal{F}(U) \to \mathcal{G}(U)$). The **image** of α is the sheafification ($\mathcal{F} \to \mathcal{F}^+$) of the presheaf $U \mapsto \operatorname{im}\alpha_U$ (this need not be a sheaf like the kernel is). A sequence of sheaves (of abelian groups or modules) and sheaf maps $\mathcal{F}' \xrightarrow{\alpha} \mathcal{F} \xrightarrow{\beta}\to \mathcal{F}''$ is **exact** proved $im\alpha = \ker \beta$.

Theorem. *$\mathcal{F}' \xrightarrow{\alpha} \mathcal{F} \xrightarrow{\beta} \mathcal{F}''$ is exact if and only if the induced sequence $\mathcal{F}'_p \xrightarrow{\alpha_p} \mathcal{F}_p \xrightarrow{\beta_p} \mathcal{F}''_p$ is exact for all $p \in X$.*

Theorem. *Let $0 \to \mathcal{F}' \xrightarrow{\alpha} \mathcal{F} \xrightarrow{\beta} \mathcal{F}''$ be an exact sequence of sheaves. Then for all open sets U we have $0 \to \mathcal{F}'(U) \xrightarrow{\alpha_U} \mathcal{F}(U) \xrightarrow{\beta_U} \mathcal{F}''(U)$ is an exact sequence of abelian groups (or modules).*

Proof. We need to show α_U is injective and $im\alpha_U = \ker \beta_U$.

- α_U is injective: Suppose $s \in \mathcal{F}'(U)$ and $\alpha_U(s) = 0$. Let $p \in U$. Let s_p be the image of s in $\mathcal{F}'_p$. Then $\alpha_p(s_p) = 0$ by commutivity of the below diagram. But α_p is 1-1 by the above theorem. Thus $s_p = 0$. Choose an open neighborhood V with $p \in V \subseteq U$ such that $s|_V = 0$. Now, let p vary. The open sets V of U cover U. Since $S|_V = 0$ for all V, we see $s = 0$ by F1.

$$\begin{array}{ccc} \mathcal{F}'(U) & \xrightarrow[\alpha_U]{} & \mathcal{F}(U) \\ \downarrow & & \downarrow \\ \mathcal{F}'_p & \xrightarrow[\alpha_p]{} & \mathcal{F}_p \end{array}$$

- Similarly, $\beta_U \alpha_U = 0$.

- $\ker \beta_U \subseteq \operatorname{im}\alpha_U$. Suppose $s \in \mathcal{F}(U)$ and $\beta_U(s) = 0$.

$$\begin{array}{ccccc} & & \mathcal{F}(U) & \xrightarrow[\beta_U]{} & \mathcal{F}''(U) \\ & & \downarrow & & \downarrow \\ \mathcal{F}'_p & \xrightarrow[\alpha_p]{} & \mathcal{F}_p & \xrightarrow[\beta_p]{} & \mathcal{F}''_p \end{array}$$

There exists $\sigma \in \mathcal{F}'_p$ such that $\alpha_p(\sigma) = s_p$, which is the image of s in $\mathcal{F}_p$. Choose an open neighborhood V with $p \in V$ and an element $\tau \in \mathcal{F}'(V)$ such that $\tau \mapsto \sigma$ in the limit. Now $\alpha_V(\tau) - s|_V \mapsto 0$ in $\mathcal{F}_p$. Thus there exists an open neighborhood W with $p \in W \subseteq V$ such that $\alpha_W(\tau|_W) = s|_W$. Now, let p vary. Define $U_i := W$ and $t_i = \tau|_{U_i}$ for each p. Then $t_i \in \mathcal{F}'(U_i)$ and $\alpha_{U_i}(t_i) = s|_{U_i}$. Now, $\alpha_{U_i \cap U_j}(t_i|_{U_i \cap U_j}) = s|_{U_i \cap U_j} = \alpha_{U_i \cap U_j}(t_j|_{U_i \cap U_j})$. Since $\alpha_{U_i \cap U_j}$ is 1-1, we see $t_i|_{U_i \cap U_j} = t_j|_{U_i \cap U_j}$. By (F2), there exists $t \in \mathcal{F}'(U)$ such that $t|_{U_i} = t_i$ for all i. Now, to show $\alpha_U(t) = s$, notice $\alpha_U(t) - s|_{U_i} = 0$ for all i and so we are done by (F1).

$$\begin{array}{ccc} \mathcal{F}'(U_i) & \xrightarrow[\alpha_{U_i}]{} & \mathcal{F}(U_i) \\ {\scriptstyle restriction}\downarrow & & \downarrow{\scriptstyle restriction} \\ \mathcal{F}'(U_i \cap U_j) & \xrightarrow[\alpha_{U_i \cap U_j}]{} & \mathcal{F}(U_i \cap U_j) \end{array}$$

□

Corollary. *Let $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}''$ be an exact sequence of sheaves on X. Then $0 \to \Gamma(X, \mathcal{F}') \to \Gamma(X, \mathcal{F}) \to \Gamma(X, \mathcal{F}'')$ is exact.*

Example. Consider the origin p in $\mathbb{A}^1_k$ and $U = \mathbb{A}^1_k \backslash \{p\}$. If V is any open subset of $\mathbb{A}^1_k$, then $\mathcal{O}_X(V) = \{\text{rational functions } \frac{f(x)}{g(x)} : g|_V \neq 0\}$. Then $\mathcal{O}_x(U) = \{\frac{f}{g} | g(c) \neq 0 \text{ for all } c \neq 0\} = \{\frac{f}{x^n}\} = k[x, x^{-1}]$. Notice $\frac{1}{x} \in \mathcal{O}_x(U)$ can not be extended to a section $\mathcal{O}_X(X) = k[x]$.

Definition. *A sheaf $\mathcal{F}$ is **flabby** provided every section extends on X, that is, $\Gamma(X, \mathcal{F}) \to \Gamma(U, \mathcal{F})$ is surjective for each open set U of X. (It also often called "flasque")*

Theorem. *Suppose $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ is an exact sequence of sheaves on a topological space X. Assume $\mathcal{F}'$ is flabby. Then $0 \to \Gamma(X, \mathcal{F}') \xrightarrow{\alpha_X} \Gamma(X, \mathcal{F}) \xrightarrow{\beta_X} \Gamma(X, \mathcal{F}'') \to 0$ is exact.*

Proof. It is enough to show β_X is onto. Let $s'' \in \Gamma(X, \mathcal{F}'')$. Look at all pairs (U, s) where U is open in X and $s \in \Gamma(U, \mathcal{F})$ with $\beta_U(s) = s''|_U$. By Zorns Lemma, there exists a maximal pair (U, s). If $U \neq X$, let $p \in X \setminus U$. Choose an open neighborhood V of p and $t \in \mathcal{F}(V)$ such that $\beta_V(t) = s''|_V$. Look at $t|_{U \cap V} - s|_{U \cap V} : \beta_{U \cap B}(t|_{U \cap V} - s|_{U \cap V}) = s''|_{U \cap V} - s''|_{U \cap V} = 0$. Thus $t|_{U \cap V} - s|_{U \cap V} \in \ker \beta_{U \cap V} = \text{im} \alpha_{U \cap V}$. So there exists $f \in \Gamma(U \cap V, \mathcal{F}')$ such that $\alpha_{U \cap V}(f) = t|_{U \cap V} - s|_{U \cap V}$. As $\mathcal{F}'$ is flabby, extend f to a section $g \in \Gamma(V, \mathcal{F}')$. Now consider $(t - \alpha_V(g))|_{U \cap V} = t|_{U \cap V} - \alpha_{U \cap V}(g) = T|_{U \cap V} - \alpha_{U \cap v}(f) = s|_{U \cap V}$. Use F2 to get a unique $q \in \Gamma(U \cap V, \mathcal{F})$ such that $q|_U = s$ and $q|_V = t$. Then $\beta_{U \cup V}(q) = s''|_{U \cup V}$ which implies $(U \cap V, q) \in \{(U, s)\}$, a contradiction to the maximality of (U, s). □

3.1 Products of $\mathcal{O}_X-$modules.

Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}_i$ be a $\mathcal{O}_X-$module. Define $(\prod_i \mathcal{F}_i)(U) = \prod_i \mathcal{F}_i(U)$, with the obvious restriction maps (defined coordinated wise). This is a sheaf of $\mathcal{O}_X-$modules. Now $\pi_j : \Pi \mathcal{F}_i(U) \to \mathcal{F}_j(U)$ gives us the projection maps $\pi_j : \Pi \mathcal{F}_i \to \mathcal{F}_j$. Also, we get $Hom_{\mathcal{O}_X}(\mathcal{G}, \Pi_i \mathcal{F}_i) = \Pi_i Hom_{\mathcal{O}_x}(\mathcal{G}, \mathcal{F}_i)$ for all $\mathcal{G}$. Let $p \in X$ and J an $\mathcal{O}_p-$module. Define an $\mathcal{O}_x-$ module $\mathcal{J}$ by $\mathcal{J}(U) = \begin{cases} J & \text{if } p \in U \\ 0 & \text{if } p \notin U \end{cases}$. If $V \subseteq U$, let $\rho_{VU} = \begin{cases} 1_J & \text{if } p \in V \\ 0 & \text{if } p \notin V. \end{cases}$ This is a sheaf!

Lemma. *Let $p \in X$, J an $\mathcal{O}_p-$module, $\mathcal{J}$ the $\mathcal{O}_X-$module defined above. Let G be any $\mathcal{O}_X-$module. Then $Hom_{\mathcal{O}_x}(\mathcal{G}, \mathcal{G}) = Hom_{\mathcal{O}_x}(\mathcal{G}_p, J)$.*

Idea: Let $\mathcal{F}$ be any $\mathcal{O}_X-$module. For all $p \in X$, choose an injective $\mathcal{O}_p-$module $J(p)$ and an injection $\mathcal{F}_p \hookrightarrow J(p)$. Let $\mathcal{J}(p)$ be the sheaf of $\mathcal{O}_X-$modules defined as above and $\mathcal{J} = \prod_{p \in X} \mathcal{J}(p)$.

Definition. *Let $(X, \mathcal{O}_X)$ be a ringed space. A $\mathcal{O}_X-$module is **injective** provided for all exact sequences $0 \to \mathcal{F}' \xrightarrow{i} \mathcal{F}$ of $\mathcal{O}_X-$modules and for all homomorphisms $f : \mathcal{F}' \to \mathcal{J}$, there exists g such that $gi = f$.*

$$\begin{array}{ccccc} & & \mathcal{J} & & \\ & & f\uparrow & \nwarrow^{\exists g} & \\ 0 & \longrightarrow & \mathcal{F}' & \xrightarrow{i} & \mathcal{F} \end{array}$$

In other words, $Hom_{\mathcal{O}_X}(-, \mathcal{J})$ preserves exactness.

Proposition. *Each $\mathcal{O}_X-$modules $\mathcal{F}$ can be embedded in an injective one.*

Proof. Fix $p \in X$. We can embed $\mathcal{F}_p$ into an injective $\mathcal{O}_p-$module, say $0 \to \mathcal{F}_p \xrightarrow{u(p)} J(p)$ is exact. Let $\mathcal{G}$ be any sheaf of $\mathcal{O}_X-$modules. Define $i : \{p\} \hookrightarrow X$. Think of $J(p)$ as a sheaf over $\{p\}$. Then $\mathcal{J}(p) := I_* J(p)$ is a sheaf of $\mathcal{O}_X-$modules. Now $\mathcal{G} \rightsquigarrow \mathcal{G}_p \rightsquigarrow Hom_{\mathcal{O}_p}(\mathcal{G}_p, J(p)) = Hom_{\mathcal{O}_X}(\mathcal{G}, \mathcal{J})$ ($\rightsquigarrow$ means exact sequences go to exact sequences). Thus $\mathcal{J}(p)$ is an injective $\mathcal{O}_X-$modules. Now let p vary and define $\mathcal{J} = \prod_{p \in X} \mathcal{J}(p)$. Then $\mathcal{J}$ is an injective $\mathcal{O}_X-$modules. Now, we can show $0 \to \mathcal{F} \xrightarrow{u} \mathcal{J}$. □

Thus every sheaf of $\mathcal{O}_X-$modules can be embedded in an injective sheaf.

Given an $\mathcal{O}_X-$module $\mathcal{F}$, we can build an "injective resolution," that is, an exact sequence $0 \to \mathcal{F} \to \mathcal{J}^0 \to \mathcal{J}^1 \to \cdots$, where each $\mathcal{J}^i$ is injective. How?: We have $0 \to \mathcal{F} \to \mathcal{J}^0 \to \mathcal{C} \to 0$ where $\mathcal{C}$ is the cokernel. Now, $0 \to \mathcal{C} \to \mathcal{J}'$ is exact for some injective $\mathcal{J}'$. So look at $0 \to \mathcal{F} \to \mathcal{J} \to \mathcal{J}'$ and continue.

Now, drop the $\mathcal{F}$ term and apply $\Gamma : 0 \xrightarrow{d^{-1}} \Gamma(X, \mathcal{J}^0) \xrightarrow{d^0} \Gamma(X, \mathcal{J}^1) \xrightarrow{d^1} \Gamma(X, \mathcal{J}^2) \xrightarrow{d^2} \cdots$. This gives us a complex of which we can take homology. Note that we can choose any injective resolution, but will always get the same $H^i(X, \mathcal{F})$. Since $\Gamma(X, -)$ is left exact, we see $H^0(X, \mathcal{F}) = \Gamma(X, \mathcal{F})$. Now, a short exact sequence of sheaves

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$$

gives a long exact sequence on homology:

$$0 \to H^0(X, \mathcal{F}') \to H^0(X, \mathcal{F}) \to \cdots \to H^i(X, \mathcal{F}') \to H^i(X, \mathcal{F}) \to H^i(X, \mathcal{F}'') \to H^{i+1}(X, \mathcal{F}') \to \cdots.$$

Theorem. *1. $H^i(X, \mathcal{J}) = 0$ for all $i > 0$ if $\mathcal{J}$ is injective.*

2. $H^i(X, \mathcal{F}) = 0$ for all $i > 0$ if $\mathcal{F}$ is flabby.

As a result, we can compute H^i using flabby resolutions. (Note: injectives are flabby).

Theorem. *If X is Noetherian of dimension d, then for all $\mathcal{O}_X-$modules $\mathcal{F}$ we have $H^i(X, \mathcal{F}) = 0$ for all $i \geq d$.*

Example. Let p be a closed point of X. Define an $\mathcal{O}_X-$module $\mathcal{F}$ by $\mathcal{F}(u) = \begin{cases} \mathcal{O}_p & \text{if } p \in U \\ 0 & \text{if } p \notin U. \end{cases}$ Let $\phi : \mathcal{O}_X \to \mathcal{F}$ be defined by $\phi_u = \begin{cases} \lim(\mathcal{O}_X(U) \to \mathcal{O}_p) & \text{if } p \in U \\ 0 & \text{if } p \notin U. \end{cases}$ Then $\mathcal{F}_q = \begin{cases} \mathcal{O}_p & \text{if } q = p \\ 0 & \text{if } q \neq p. \end{cases}$ Let $k = \ker \phi$. Then $\phi_q : \mathcal{O}_q \to \mathcal{F}_q$ is surjective for all q by the definition of q. So $0 \to k \to \mathcal{O}_X \to \mathcal{F} \to 0$ is exact. By $\Gamma(X, \mathcal{O}_X) \to \Gamma(X, \mathcal{F})$ needs not be surjective.