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The Magnetic Pole in the Formulation of Electricity and Magnetism*

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The use of the magnetic pole in the development of the concepts of electricity and magnetism leads unambiguously to a relativistic formulation of the field vectors which is well within the grasp of the sophomore student. The development is wholly consistent with Maxwell's equations and leads to clear and understandable definitions of the field vectors both in vacuum and in the material medium, as well as to the relations defining the transformations of the field vectors.

In classic electromagnetic field theory there has been considerable discussion of the manner in which the field vectors should be formulated.[1,2] Must the theory be wholly based on charges, or may poles be used? It is generally accepted that the vectors **E** and **B** should have preferred status over **H** and **D**, though the reason is often unclear. It is sometimes difficult to find precise statements of the difference between **B** and **H** and between **E** and **D** which carry conviction to the undergraduate student. There are further questions, such as the shift in meaning of the field vectors on passage from vacuum to the material medium, and the treatment of the motion of the medium with respect to the observer.[3] These questions may be clearly resolved by the use of magnetic poles in the development of the theory of electricity and magnetism. Such a theory must necessarily fall short of a parochial operationalism, at least until the Dirac monopole has been discovered, but it may serve to point the way to the detection of the monopole.[4] Entirely apart from the question of the existence or nonexistence of a free magnetic pole, the use of the pole concept provides elementary electromagnetism with a symmetry and clarity which are otherwise inaccessible (without the full apparatus of the theory of relativity). These are large gains for a modest relaxation in the demands of operationalism. Indeed, similar relaxations in the operational demands on the theory are not unknown in electromagnetism. In speaking of the electric fields within an atom, we are well aware that a probe charge does not exist with which we can probe the field without altering it. Nor does the macroscopic probe charge exist with which we speak of measuring the electric and magnetic fields within a dielectric or a magnetized medium without experiencing the microscopic irregularities in the field.

The present program may be achieved by strict adherence to Maxwell's equations, extended to include magnetic poles and currents of magnetic poles.[5] We write

$$\mathrm{div}\mathbf{D}=\rho,\qquad \mathrm{curl}\mathbf{E}=-(\partial\mathbf{B}/\partial t+\mathbf{j}_m),$$
$$\mathrm{div}\mathbf{B}=\rho_m,\qquad \mathrm{curl}\mathbf{H}=(\partial\mathbf{B}/\partial t+\mathbf{j}).\tag{1}$$

Set up in this way magnetic poles satisfy the continuity equation; we demand the conservation of poles. We note that Eqs. (1) are unaltered if we write **B** for **D**, **H** for **E**, q for q_m, and conversely, and if we replace t by $-t$. Thus **j** must be replaced by $-\mathbf{j}_m$, $\partial/\partial t$ by $-\partial/\partial t$, and **v** by $-\mathbf{v}$. The equation for the Lorentz force on a charged particle may be immediately extended for poles, as

$$\mathbf{F}=q(\mathbf{E}+\mathbf{v}\times\mathbf{B})\ \text{becomes}\ \mathbf{F}=q_m(\mathbf{H}-\mathbf{v}\times\mathbf{D}).\tag{2}$$

In the customary way the effect of the medium upon the field may be ascribed to a distribution of polarization **P** and magnetization **M**, to be measured through the torque **G** on a volume element ΔV in a uniform field, as $\mathbf{G}=(\mathbf{P}\Delta V)\times\mathbf{E}$ and $\mathbf{G}=(\mathbf{M}\Delta V)\times\mathbf{H}$. The influence of the medium may be thought of as an electromagnetic constraint upon the fields, commonly called the constitutive equations

$$\mathbf{D}=\epsilon_0\mathbf{E}+\mathbf{P}\quad\text{and}\quad\mathbf{B}=\mu_0\mathbf{H}+\mathbf{M}.\tag{3}$$

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[1] Am. J. Phys. **18**, 1–25, 69–88 (1950).
[2] *The Teaching of Electricity*, Science Masters' Association (John Murray Ltd., London, 1954).
[3] D. L. Webster, Am. J. Phys. **29**, 262 (1961).
[4] R. Katz and D. R. Parnell, Phys. Rev. **116**, 236 (1959).
[5] *American Institute of Physics Handbook* (McGraw-Hill Book Company, Inc., New York, 1957), p. 5–40.

From Eqs. (1) describing the sources of the fields, Eqs. (2) describing the experiences of the fields, and Eqs. (3) describing constraining relations among the field vectors, we are ready to proceed to a pedagogy of electricity and magnetism to be expressed in strict conformity to these equations.

The electric field in a particular inertial frame may be defined through the force per unit charge on a probe charge q at rest in that frame. The field is not simply a point function of space but depends both on the space point and on the coordinate frame. The field in vacuum is to be described through the force on a microscopic probe, while the field in the material medium is experienced through the force on a diffuse macroscopic probe which averages over the discontinuities in the medium. Nevertheless **E** always remains the same in basic concept; it is always the force on a probe charge at rest in the particular inertial frame in which **E** is being determined. Similarly **H** is always the force on a probe pole at rest. Note that the force on a probe pole is given by **H** and not by **B**. Further, we note from Eqs. (2) that **B** is to be defined through the force on a moving charge, while **D** is to be defined through the force on a moving probe pole. All four field vectors retain the same meaning whether in vacuum or in the medium. In all cases we require that the probe be altered from a microscopic to a macroscopic one when we enter the medium from vacuum, but the conceptual sense of the field vectors is retained. We have no need for special definitions of **D** and **E** in the medium, like the Kelvin cavity definitions. It is now clear why the vectors **E** and **B** must be regarded as basic, while the vectors **H** and **D** are to be regarded as auxiliary—we have probe charges, and as yet we can only imagine probe poles.

Just as definitions of the experiences of the field are to be based on the Lorentz force equations, so statements as to the sources of the field must be based on Maxwell's equations, though these must be supplemented by the requirements of the constitutive equations. Free charge is the source of the lamellar part of the electric displacement **D**, while free poles are the source of the lamellar part of the magnetic induction **B**. Similarly, electric currents are the source of the solenoidal part of the magnetic field intensity **H** and currents of magnetic poles are the source of the solenoidal part of the electric field intensity **E**, according to the usual relations given here for slowly moving point charges and poles.

$$\mathbf{D}=q\hat{r}/4\pi r^2, \qquad \mathbf{B}=q_m\hat{r}/4\pi r^2,$$
$$\mathbf{H}=(q\mathbf{v}\times\hat{r})/4\pi r^2, \quad \mathbf{E}=-(q_m\mathbf{v}\times\hat{r})/4\pi r^2. \qquad (4)$$

Note that a moving pole generates an electric field having circular symmetry but directed according to a left-hand rule corresponding to the minus sign in the equation for curl **E**. The familiar demonstration experiment in which an electric current is established in a coil by displacing a bar magnet along the axis of the coil may be interpreted in terms of the electric field due to moving poles rather than through the change of flux in the coil. The calculation is simpler and more direct through the use of this technique. We note that $\partial\mathbf{B}/\partial t$ may be interpreted as a displacement current of poles, just as $\partial\mathbf{D}/\partial t$ is customarily interpreted as a displacement current of charges.

There is here an interesting symmetry of the field quantities. The electric field **E** is generated by poles in motion and is experienced by charge at rest. The magnetic induction **B** is generated by poles at rest and is experienced by charges in motion. Again, the magnetic field **H** is generated by charges in motion but is experienced by poles at rest, while the electric displacement **D** is generated by charges at rest and is experienced by poles in motion.

Statements of the sources and experiences of the fields have always been muddied by the constitutive equations. In engineering the relations between **D** and **E**, and between **B** and **H**, have been treated as cause-effect relationships. In physics some writers have chosen to suppress a pair of the field vectors in vacuum, the suppressed pair being **D** and **B**, or **D** and **H**, the choice associated with the choice of a unit system. In the present development electric (and magnetic) charges and currents are taken as field sources, while the constitutive equations are to be treated as constraining relationships among the field quantities, akin to boundary value conditions. Thus the nonexistence of magnetic poles demands that there be no lamellar component of **B** but permits a nonzero solenoidal **B** through sources of **H** and the existence of **M**.

In many problems the availability of two points of view, that of charge and of pole, itself may justify the use of a pole-based pedagogy. As a second illustration, the problem of finding the force between two solenoids carrying current is much more easily solved by replacing each turn of wire by a dipole layer, and then treating the assembly as a pair of permanent magnets. This is somewhat simpler than integrating the force between current elements. But far more significant is the simplicity with which relative motion may be treated through the pole concept.

In order to develop the relativistic transformations between electromagnetic field vectors without approximation, we call on a result from relativistic mechanics. If the force $\mathbf{F}$ on a particle, as measured in the laboratory frame, is compared to the force $\mathbf{F}'$ which would be measured in the rest frame of the particle, their components parallel and perpendicular to $\mathbf{v}$ are given by

$$\mathbf{F}_{\parallel}=\mathbf{F}_{\parallel}', \quad \text{and} \quad \mathbf{F}_{\perp}=\mathbf{F}_{\perp}'/\gamma, \tag{5}$$

where $\gamma=(1-v^2/c^2)^{-\frac{1}{2}}$. In the event that we do not utilize Eq. (5) but rather make the nonrelativistic assertion that $\mathbf{F}=\mathbf{F}'$, then the resulting transformation equations for the field vectors are correct to first order in v/c.

We examine the forces on a probe charge q and on a probe pole q_m at rest in the primed frame, but moving with velocity $\mathbf{v}$ in the $+x$ direction in the unprimed (laboratory) frame. From the conservation of charge and pole we note that all observers will see the same value of the charge and pole strength. As seen in the proper frame

$$\mathbf{F}'=q\mathbf{E}' \quad \text{and} \quad \mathbf{F}'=q_m\mathbf{H}'. \tag{6}$$

The laboratory observer finds the forces to be

$$\mathbf{F}=q(\mathbf{E}+\mathbf{v}\times\mathbf{B}) \quad \text{and} \quad \mathbf{F}=q_m(\mathbf{H}-\mathbf{v}\times\mathbf{D}). \tag{7}$$

From Eqs. (5), (6), and (7) we find that

$$\mathbf{E}_{\parallel}'=\mathbf{E}_{\parallel}, \qquad \mathbf{E}_{\perp}'=\gamma(\mathbf{E}_{\perp}+\mathbf{v}\times\mathbf{B}),$$
$$\mathbf{H}_{\parallel}'=\mathbf{H}_{\parallel}, \quad \text{and} \quad \mathbf{H}_{\perp}'=\gamma(\mathbf{H}_{\perp}-\mathbf{v}\times\mathbf{D}). \tag{8}$$

To find the transformation equations appropriate to the other two field vectors in free space we make use of Eqs. (3) with $\mathbf{P}=\mathbf{M}=0$ to find

$$\mathbf{D}_{\parallel}'=\mathbf{D}_{\parallel}, \qquad \mathbf{D}_{\perp}'=\gamma(\mathbf{D}_{\perp}+\epsilon_0\mu_0\mathbf{v}\times\mathbf{H}),$$
$$\mathbf{B}_{\parallel}'=\mathbf{B}_{\parallel}, \quad \text{and} \quad \mathbf{B}_{\perp}'=\gamma(\mathbf{B}_{\perp}-\epsilon_0\mu_0\mathbf{v}\times\mathbf{D}). \tag{9}$$

In order to retain the same physical meaning for the field vectors in vacuum and in the medium, and noting that $\epsilon_0\mu_0=1/c^2$, an invariant quantity, we demand that Eqs. (9) hold for the material medium as well as for vacuum.

With the above results we are ready to examine the transformation of $\mathbf{P}$ and $\mathbf{M}$. We apply Eqs. (8) and (9) to Eqs. (3) to find

$$\epsilon_0\gamma(\mathbf{E}_{\perp}+\mathbf{v}\times\mathbf{B})+\mathbf{P}_{\perp}'=\gamma(\epsilon_0\mathbf{E}_{\perp}+\mathbf{P}_{\perp}+\epsilon_0\mu_0\mathbf{v}\times\mathbf{H}),$$
$$\mathbf{P}_{\perp}'=\gamma[\mathbf{P}_{\perp}-\epsilon_0\mathbf{v}\times(\mathbf{B}-\mu_0\mathbf{H})],$$

whence

$$\mathbf{P}_{\perp}'=\gamma(\mathbf{P}_{\perp}-\epsilon_0\mathbf{v}\times\mathbf{M}); \tag{10}$$

similarly

$$\mathbf{M}_{\perp}'=\gamma(\mathbf{M}_{\perp}+\mu_0\mathbf{v}\times\mathbf{P}),$$

and

$$\mathbf{P}_{\parallel}'=\mathbf{P}_{\parallel}, \quad \mathbf{M}_{\parallel}'=\mathbf{M}_{\parallel}.$$

Equations (10) are the generally accepted equations for the transformation of polarization and magnetization, though to bring them into accord with some published forms, as where the relationship $\mathbf{B}=\mu_0(\mathbf{H}+\mathbf{M})$ is used, it is necessary to replace $\mathbf{M}$ above by $\mu_0\mathbf{M}$.[6]

These results are consistent with the customary first-order illustration of the observation in the laboratory of a moving ribbon of electret (magnet), displaced parallel to itself and polarized (magnetized) perpendicular to its direction of motion. We imagine the polarization (magnetization) to be replaced by a distribution of polarization charge (magnetization poles) along its edges. A ribbon moving to the right appears to have a current of positive charges (poles) along one edge, and a current of negative charges (poles) along the other edge, moving in the same direction. This may be conceived as a collection of current (pole current) loops in the plane of the ribbon which generates a transverse magnetic (electric) dipole moment according to a right (left) hand rule. The magnitude of the dipole moment generated by a current (pole current) loop of area A is $\mu_0 iA$ ($\epsilon_0 i_m A$). In first order we seen the effect as a transverse magnetization (polarization) given by $\mathbf{M}'=-\mu_0\mathbf{v}\times\mathbf{P}$ ($\mathbf{P}'=\epsilon_0\mathbf{v}\times\mathbf{M}$), which we may calculate by sopho-

[6] W. K. H. Panofsky and M. Phillips, *Classical Electricity and Magnetism* (Addison-Wesley Publishing Company, Inc., Reading, Massachusetts, 1955), Eqs. 17–40 to 17–43; 22–15 to 22–18.

more methods for the case where $\mathbf{P}(\mathbf{M})$ is at rest in the laboratory and an observer is moving in the $-x$ direction so that he sees $\mathbf{M}'(\mathbf{P}')$. The form is chosen to correspond to Eqs. (10), or to the case where the observer is at rest in the laboratory and the ribbon moves to the right.

It is generally accepted that relativity should be introduced to undergraduates, particularly in electricity and magnetism, but the question has been how to do so. Some recent treatments of the problem are those of Rosser[7] and Webster,[3] who deal with charges alone, and of Cullwick[8] who uses magnetic poles, though his treatment is somewhat different from that presented here.

Finally, though no pole has ever been detected, it should be realized that there is no known theoretical argument which excludes the possibility of free magnetic poles, while there are sound theoretical reasons for supposing that they may exist.[9,10]

[7] W. G. V. Rosser, Contemporary Phys. **1**, 134, 453 (1959–1960).

[8] E. G. Cullwick, *Electromagnetism and Relativity* (Longmans Green and Company, Inc., New York, 1957).

[9] P. A. M. Dirac, Phys. Rev. **74**, 817 (1948); Proc. Roy. Soc. (London) **A133**, 60 (1931).

[10] N. F. Ramsey, Phys. Rev. **109**, 225 (1958).