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The Exact Distribution of the Multilook Magnitude

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***Abstract*—Gierull provides a statistical analysis of multilook synthetic aperture radar interferograms. Various expressions for the probability density function, cumulative distribution function, and the moments of associated statistics are derived. It appears, however, that most of these expressions are based on some approximation. In this letter, the corresponding expressions are derived in their exact form, including some elementary representations for certain expressions given by Gierull. A numerical comparison of the exact and approximate expressions is provided.**

***Index Terms*—Appell function, Gauss hypergeometric function, interferogram's phase, multilook magnitude, synthetic aperture radar (SAR) interferograms.**

I. Introduction

GIERULL [1] examines the statistics of the phase and magnitude of multilook synthetic aperture radar (SAR) interferograms toward deployment of along-track interferometry (ATI) for slow ground moving-target indication (GMTI) and derives various expressions for the probability density function (pdf), cumulative distribution function (cdf), and the moments of the statistics. Most of these expressions are based on certain approximations, whereas others involve nonstandard functions such as the Gauss hypergeometric function, which is defined as

$$ {}_2F_1(a,b;c;x) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{x^k}{k!} \tag{1} $$

where $(e)_k = e(e+1)\cdots(e+k-1)$ denotes the ascending factorial. The parameters a, b, c, and x control the nature of the infinite series in (1). The series terminates if a or b is equal to a negative integer or to zero. For $c=-n$ $(n=0,1,\ldots)$, the series is indeterminate if neither a nor b is equal to $-m$ (where $m<n$ and m is a natural number). The series in (1) converges in the unit circle $|x|<1$ and has a branch point at $x=1$. The following conditions apply for convergence on the unit circle. If $0 \le a+b-c<1$, then the series converges throughout the entire unit circle except at the point $x=1$. If $a+b-c<0$, then the series converges absolutely throughout the entire circle. If $a+b-c \ge 1$, then the series diverges on the entire unit circle. Numerical routines for the computation of (1) are widely available in packages such as Matlab, Maple, and Mathematica.

In this letter, we show that one can actually derive exact expressions and that some of the expressions given in terms of nonstandard functions can be reduced to elementary forms. The outline is given as follows. In Section II, we derive an elementary expression for the marginal pdf of interferogram's phase (compare with [1, eq. (4)], which involves the Gauss hypergeometric function). In Section III, an exact expression for the pdf of the multilook magnitude is derived (compare with [1, eq. (11)], which provides an approximate form). In Section IV, an exact expression for the cdf of the multilook magnitude is derived (compare with [1, eq. (21)], which provides an approximate form involving the Gauss hypergeometric function). In Section V, we derive exact formulas for the moments of the multilook magnitude (compare with [1, eqs. (12) and (19)], which provide some approximate forms). Finally, in Section VI, we illustrate a numerical comparison of the exact and approximate formulas for the moments of the multilook magnitude.

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II. Marginal PDF of Interferogram's Phase

The joint pdf of interferogram's magnitude ($\mathcal{E}$) and phase (Ψ) is given as [1, eq. (3)]

$$ f_{\varepsilon,\Psi}(\eta,\psi) = \frac{2n^{n+1}\eta^n}{\pi\Gamma(n)(1-\rho^2)} \exp(-p\eta) K_{n-1}(c\eta) \tag{2} $$

where $p=-2n\rho\cos\psi/(1-\rho^2)$, $c=2n/(1-\rho^2)$, ρ is the complex correlation coefficient, n is the number of looks, $\Gamma(\cdot)$ denotes the gamma function, and $K_{n-1}(\cdot)$ denotes the modified Bessel function of the third kind of order $n-1$. Usually, one defines

$$ K_\nu(x) = \frac{\pi\{I_{-\nu}(x) - I_\nu(x)\}}{2\sin(\nu\pi)} \tag{3} $$

(when ν is an integer, the right-hand side should be interpreted as a limit), where $I_\nu(\cdot)$ denotes the modified Bessel function of the first kind defined as

$$ I_\nu(x) = \frac{x^\nu}{2^\nu\Gamma(\nu+1)} \sum_{k=0}^{\infty} \frac{1}{(\nu+1)_k k!}\left(\frac{x^2}{4}\right)^k. $$

Integrating (2) with respect to the magnitude η, one obtains

$$ \begin{aligned} f_\Psi(\psi) &= \frac{2n^{n+1}}{\pi\Gamma(n)(1-\rho^2)} \int_0^\infty \eta^n \exp(-p\eta) K_{n-1}(c\eta)\, d\eta \\ &= \frac{n^{n+1}c^{1-n}\Gamma(2n)}{\sqrt{\pi}\Gamma(n)\rho^2(1-\rho^2)2^n\Gamma(n+3/2)} \\ &\quad \times {}_2F_1\left(1,\frac{3}{2};n+\frac{3}{2};1-\frac{c^2}{p^2}\right) \end{aligned} \tag{4} $$

for $|1-c^2/p^2|<1$, which follows by using [2, eq. (2.16.6.3)]. Using [3, eqs. (7.3.1.1), (7.3.1.9), (7.3.1.127), (7.3.2.83)],

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the hypergeometric term in (4) can be simplified to the elementary form

$$\begin{aligned} &{}_2F_1\left(1, \frac{3}{2}; n+\frac{3}{2}; x\right) \\ &= \frac{(-n-1/2)_n (x-1)^{n-1}}{(1/2-n)_n x^n} \\ &\quad \times \Bigg[-1 + n x^{-1/2} \operatorname{arc\,tanh}(\sqrt{x}) + \sum_{k=2}^{n} (-1)^{k-1} \binom{n}{k} \\ &\quad \times \left\{ \frac{(2k-3)!! x^{-1/2}}{2(2k-2)!!} \ln \frac{1+\sqrt{x}}{1-\sqrt{x}} + \sum_{l=1}^{k-1} a_l x^{l-1} (x-1)^{-l} \right\} \Bigg] \end{aligned} \tag{5}$$

where $x = 1 - c^2/p^2 (|x| < 1)$ and

$$a_l = \frac{(2k-3)!!(l-1)}{2^{k-l}(k-1)!} \sum_{j=l}^{k-1} (-1)^{l-j} \binom{k-1}{j} \frac{(2j-2l-1)!!}{(2j-1)!!}.$$

Here, $(2m)!! = 2^m m!$ and $(2m+1)!! = 2^{m+1}\pi^{-1/2}\Gamma(m+3/2)$ with the convention that $(0)!! = 1$. Combining (4) and (5) yields an elementary expression for the marginal pdf of interferogram's phase (compare with [1, eq. (4)]).

III. PDF of Multilook Magnitude

Multilook magnitude is defined by the random variable $Z = w\mathcal{E}$, where w and $\mathcal{E}$ are independent random variables specified by the pdfs

$$f_W(w) = \frac{\nu^\mu}{\Gamma(\mu)} w^{-\mu-1} \exp\left(-\frac{\nu}{w}\right) \tag{6}$$

(for $w > 0$) and

$$f_\varepsilon(\eta) = \frac{4n^{n+1}\eta^n}{\Gamma(n)(1-\rho^2)} I_0(b\eta) K_{n-1}(c\eta) \tag{7}$$

(for $\eta > 0$), respectively, where $b = 2n\rho/(1-\rho^2)$, $\mu > 0$ denotes the degrees of freedom and $\nu > 0$ is a shape parameter. Gierull [1] provides an approximate formula for the pdf of Z [1, eq. (11)]. Here, we derive an expression for the pdf of Z. First, we write

$$\begin{aligned} f_Z(z) &= \int_0^\infty \frac{1}{w} f_W(w) f_\mathcal{E}\left(\frac{z}{w}\right) dw \\ &= \frac{4n^{n+1} z^n \nu^\mu}{\Gamma(\mu)\Gamma(n)(1-\rho^2)} \\ &\quad \times \int_0^\infty w^{-n-\mu-2} \exp\left(-\frac{\nu}{w}\right) I_0\left(\frac{bz}{w}\right) K_{n-1}\left(\frac{cz}{w}\right) dw \\ &= \frac{4n^{n+1} z^n \nu^\mu}{\Gamma(\mu)\Gamma(n)(1-\rho^2)} \\ &\quad \times \int_0^\infty x^{n+\mu} \exp(-\nu x) I_0(bzx) K_{n-1}(czx) dx \end{aligned} \tag{8}$$

which follows after substituting $x = 1/w$. Using the definition in (3), one can reexpress (8) as

$$\begin{aligned} &f_Z(z) \\ &= \frac{2\pi n^{n+1} z^n \nu^\mu \{I(n+\mu+1, \nu, bz, cz) - J(n+\mu+1, \nu, bz, cz)\}}{\Gamma(\mu)\Gamma(n)(1-\rho^2)\sin((n-1)\pi)} \end{aligned} \tag{9}$$

where

$$I(\alpha, p, b, c) = \int_0^\infty x^{\alpha-1} \exp(-px) I_0(bx) I_{1-n}(cx) dx$$

$$J(\alpha, p, b, c) = \int_0^\infty x^{\alpha-1} \exp(-px) I_0(bx) I_{n-1}(cx) dx.$$

Application of [2, eq. (2.15.20.2)] shows that one can calculate

$$\begin{aligned} I(\alpha, p, b, c) &= \frac{c^{1-n}\Gamma(\alpha+1-n)}{2^{1-n} p^{\alpha+1-n}\Gamma(2-n)} F_4 \\ &\quad \times \left(\frac{\alpha+1-n}{2}, \frac{\alpha+2-n}{2}; 1, 2-n; \frac{b^2}{p^2}, \frac{c^2}{p^2}\right) \end{aligned} \tag{10}$$

$$\begin{aligned} J(\alpha, p, b, c) &= \frac{c^{n-1}\Gamma(\alpha+n-1)}{2^{n-1} p^{\alpha+n-1}\Gamma(n)} F_4 \\ &\quad \times \left(\frac{\alpha+n-1}{2}, \frac{\alpha+n}{2}; 1, n; \frac{b^2}{p^2}, \frac{c^2}{p^2}\right) \end{aligned} \tag{11}$$

where F_4 denotes the Appell function of the fourth kind defined as

$$F_4(a, b; c, c'; z, \xi) = \sum_{k=0}^\infty \sum_{l=0}^\infty \frac{(a)_{k+l}(b)_{k+l} z^k \xi^l}{(c)_k (c')_l k! l!}.$$

Combining (9), (10), and (11), one obtains an exact expression for the pdf of Z in terms of the Appell function. The Appell functions are well known, and numerical routines for their exact computation are available in packages such as Mathematica.

IV. CDF of Multilook Magnitude

Here, we derive an exact formula for the cdf of Z (compare with the approximate formula given by [1, eq. (21)]). Noting that $Z = w\mathcal{E}$, where w and $\mathcal{E}$ are independent random variables specified by (6) and (7), respectively, one can write

$$F_Z(z) = \int_0^\infty F_W\left(\frac{z}{\eta}\right) f_\varepsilon(\eta) d\eta. \tag{12}$$

Because W is a reciprocal of a chi-square random variable, one has [4, Ch. 18]

$$F_W(w) = \exp\left(-\frac{\nu}{w}\right) \sum_{k=0}^{\mu-1} \frac{1}{k!} \left(\frac{\nu}{w}\right)^k. \tag{13}$$

Substituting (13) into (12), one obtains

$$F_Z(z) = \frac{4n^{n+1}}{\Gamma(n)\,(1-\rho^2)} \sum_{k=0}^{\mu-1} \frac{1}{k!} \left(\frac{\nu}{z}\right)^k \times \int_0^\infty \eta^{n+k} \exp\left(-\frac{\nu\eta}{z}\right) I_0(b\eta) K_{n-1}(c\eta) d\eta. \quad (14)$$

The integral in (14) is of the same form as that in (8). Thus, (14) can be simplified to

$$F_Z(z) = \frac{2\pi n^{n+1}}{\Gamma(n)(1-\rho^2)\sin((n-1)\pi)} \sum_{k=0}^{\mu-1} \frac{1}{k!} \left(\frac{\nu}{z}\right)^k \times \left\{ I\left(n+k+1, \frac{\nu}{z}, b, c\right) - J\left(n+k+1, \frac{\nu}{z}, b, c\right) \right\}$$

where $I(\cdot)$ and $J(\cdot)$ are given by (10) and (11), respectively. Hence, one obtains an exact expression for the cdf of Z in terms of the Appell function of the fourth kind.

V. Moments of Multilook Magnitude

Because $Z = w\mathcal{E}$ and w and $\mathcal{E}$ are independent random variables, the rth moment of Z is simply $E(Z^r) = E(w^r)E(\mathcal{E}^r)$. It is well known that the rth moment of a reciprocal of a chi-square random variable is given as [4, Ch. 18]

$$E(W^r) = \frac{\nu^r \Gamma(\mu - r)}{\Gamma(\mu)}. \quad (15)$$

The rth moment of $\mathcal{E}$ can be calculated as

$$E(\varepsilon^r) = \frac{4n^{n+1}}{\Gamma(n)(1-\rho^2)} \int_0^\infty \eta^{n+r} I_0\left(\frac{2n\rho\eta}{1-\rho^2}\right) K_{n-1}\left(\frac{2n\eta}{1-\rho^2}\right) d\eta$$
$$= \frac{(1-\rho^2)^{n+r}}{n^r \Gamma(n)} \Gamma\left(n+\frac{r}{2}\right) \Gamma\left(1+\frac{r}{2}\right) \times {}_2F_1\left(n+\frac{r}{2}, 1+\frac{r}{2}; 1; \rho^2\right) \quad (16)$$

for $|\rho| < 1$, which follows by using [2, eq. (2.16.28.1)]. Combining (15) and (16), one obtains the rth moment of Z as

$$E(Z^r) = \frac{\nu^r (1-\rho^2)^{n+r} \Gamma(\mu - r)}{n^r \Gamma(n) \Gamma(\mu)} \Gamma\left(n+\frac{r}{2}\right) \Gamma\left(1+\frac{r}{2}\right) \times {}_2F_1\left(n+\frac{r}{2}, 1+\frac{r}{2}; 1; \rho^2\right)$$

for $|\rho| < 1$ and $r \geq 1$ (compare with [1, eq. (12)]). In particular, the first two moments of Z are

$$E(Z) = \frac{\sqrt{\pi}\nu(1-\rho^2)^{n+1}}{2n\Gamma(n)(\mu-1)} \Gamma\left(n+\frac{1}{2}\right) {}_2F_1\left(n+\frac{1}{2}, \frac{3}{2}; 1; \rho^2\right) \quad (17)$$

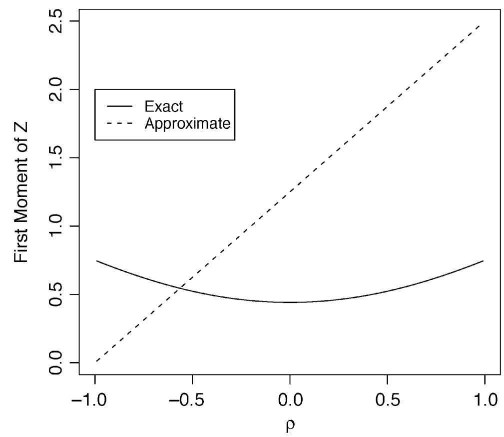

Fig. 1. Comparison of the exact (17) and the approximate (19) expressions for $E(Z)$. It is assumed that $n = 2$, $\mu = 5$, $\nu = 3$, and $\rho = -0.99, -0.98, \ldots, 0.99$.

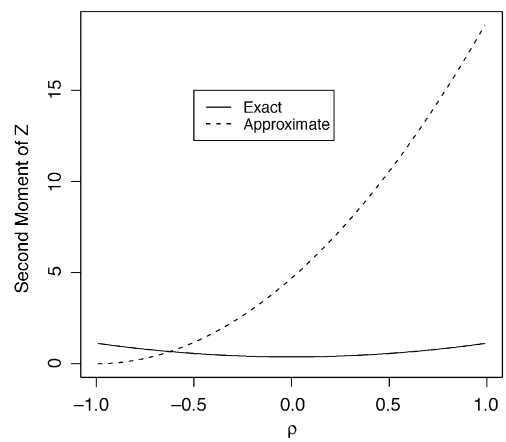

Fig. 2. Comparison of the exact (18) and the approximate (20) expressions for $E(Z^2)$. It is assumed that $n = 2$, $\mu = 5$, $\nu = 3$, and $\rho = -0.99, -0.98, \ldots, 0.99$.

and

$$E(Z^2) = \frac{\nu^2(1+n\rho^2)}{n(\mu-1)(\mu-2)} \quad (18)$$

respectively (compare with [1, eq. (19)]), for $|\rho| < 1$.

VI. Numerical Comparison

Here, we provide a numerical comparison of the exact expressions previously derived with the approximate ones discussed in [1]. Specifically, we compare the moment expressions in (17) and (18) with

$$m_1 = \frac{B(n+1, \nu-1)}{\gamma B(n, \nu)} \quad (19)$$

$$m_2 = \frac{B(n+2, \nu-2)}{\gamma^2 B(n, \nu)} \quad (20)$$

where $\gamma = 2n/(\mu(1+\rho))$ and $B(a,b) = \Gamma(a)\Gamma(b)/\Gamma(a+b)$, which are the corresponding approximate expressions given in [1]. The numerical comparisons of (17) versus (19) and of

(18) versus (20) are shown in Figs. 1 and 2, respectively. It is clear that there is a substantial difference between the exact and approximate expressions.

VII. Conclusion

We have derived explicit expressions for the pdf, cdf, and the moments of multilook magnitude as well as an elementary expression for the marginal pdf of interferogram's phase. We have also provided a numerical comparison of these expressions with the approximate ones suggested by Gierull [1]. We expect that these new results will be of use with respect to modeling SAR interferograms.

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