ADAPT Lessons: Mathematics | Lesson Plans from the ADAPT Program

1996

Windchimes, Hexagons, and Algebra

Melvin C. Thornton
University of Nebraska - Lincoln, melthorn@windstream.net

Windchimes, Hexagons, and Algebra

Melvin Thornton

Department of Mathematics
University of Nebraska-Lincoln
melthorn@windstream.net

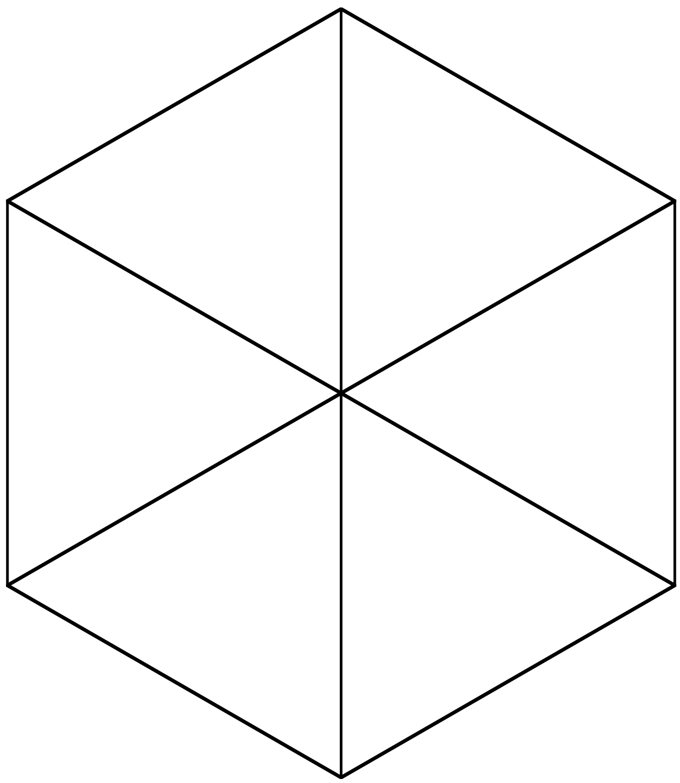
Comments to Instructors:

This lesson begins with work on the three puzzles. They can be done in any order, but all students should work on all puzzles. Working in small groups is usually very effective. A real plus is to have six appropriate sized tubes to hang on a regular hexagon frame for the windchime. Available scissors would allow students to cut a 9-inch equilateral triangle to experiment with the hexagons by folding.

The presentation of the “Invention” materials should be done in Windchime-Hexagon-Algebra order. Possible applications are suggested by the Challenge on the last page of the algebra materials.

The Windchime Puzzle

A windchime is be constructed with six sound tubes hanging from the vertices of a regular hexagon top. The tubes are 9”, 12”, 15”, 18”, 21” and 24” long. Is it possible to hang these tubes from the top so that, when the top is suspended from the center, it will hang perfectly horizontal?



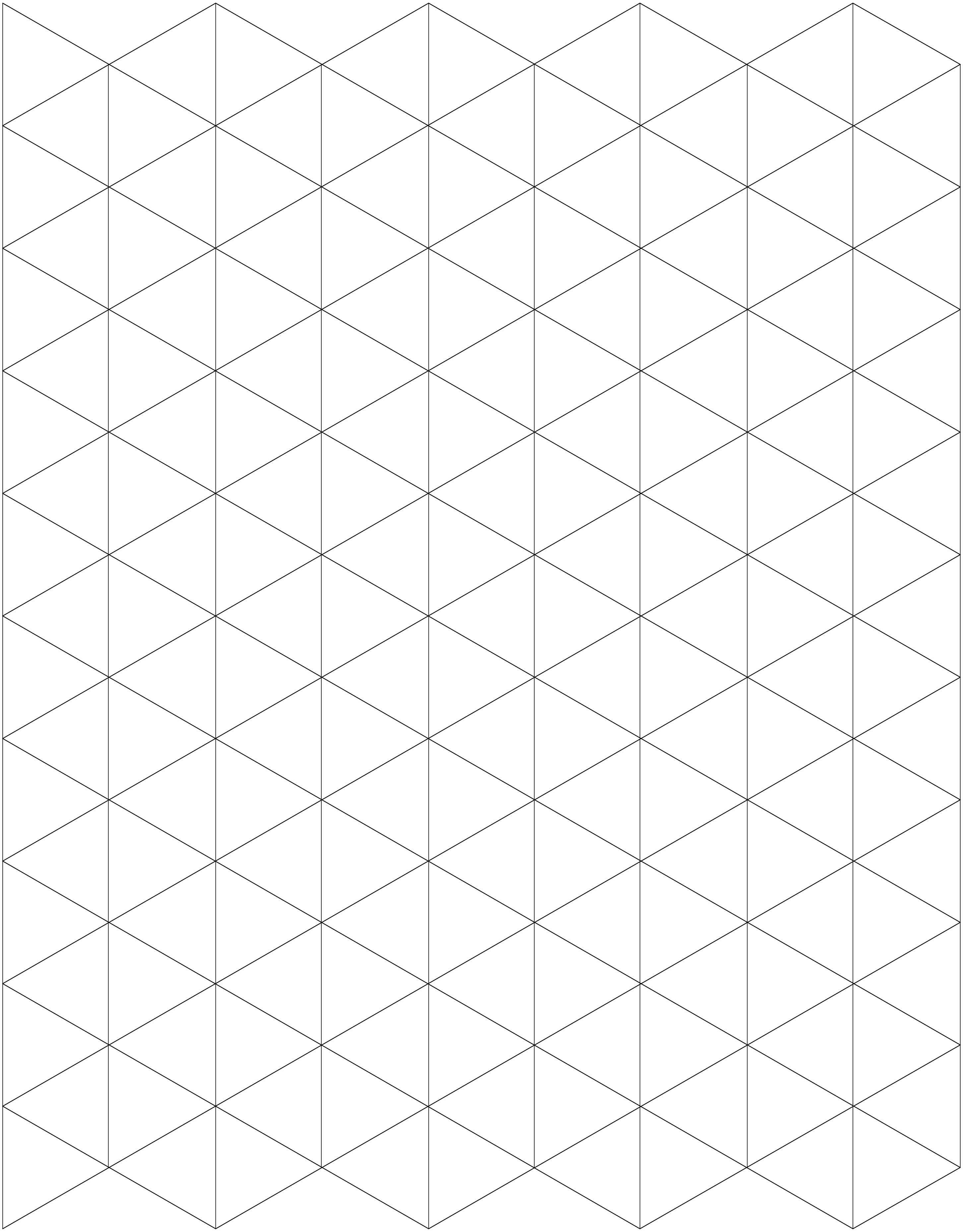
Is there a solution?

How many different solutions are there?

For which sets of six numbers can chimes of that length be hung so the top stays horizontal?

Equiangular Hexagon Puzzle

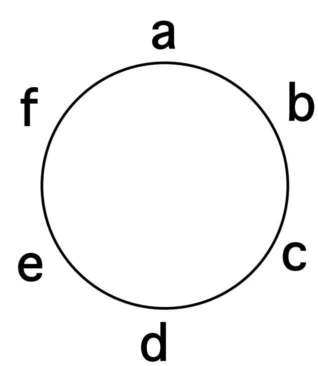
Draw an equiangular hexagon with sides 1, 2, 3, 4, 5, and 6 in some order.



Six in a Circle Algebra Puzzle

Suppose there are six numbers, not necessarily whole numbers. Denote them by a, b, c, d, e, and f. Consider them arranged equally around a circle in the given order. Can definite values for these six numbers be chosen so that any two adjacent ones add to the same sum as the two opposite numbers?

This means, if we have the numbers arranged as shown on the right, that:



$a + b = d + e$

*** $b + c = e + f$

$c + d = f + a$

Thus the problem is to find numerical values for these letters so that all three equations *** are satisfied. It would also be nice if we could find all such choices that would work.

Work below:

Windchimes Puzzle (Physics)

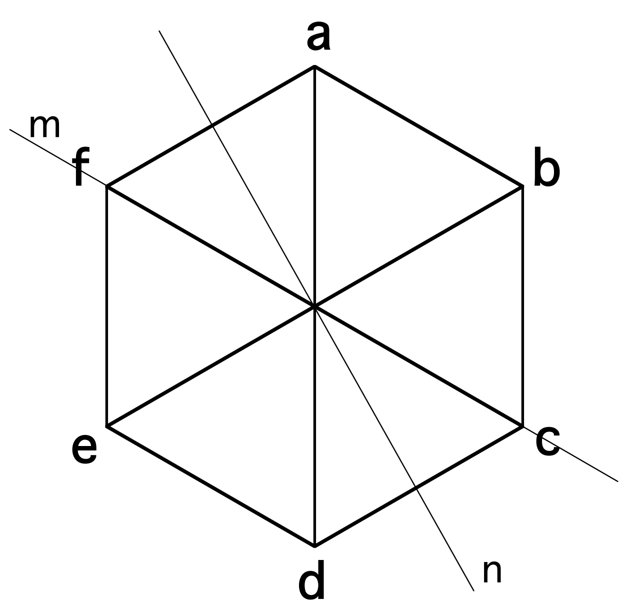
The puzzle as given has the lengths of the pipes as 9, 12, 15, 18, 21 and 24 inches. A reasonable assumption is that each of the pipes is made from the same material and that the weight of the pipes is proportional to their length. Thus we can take 9, 12, 15, 18, 21 and 24 as weights.

These weights are in what units? Clearly the units don't make any difference. Whether they are 9 grams, 9 pounds or 9 whatever, it doesn't matter. So why not take them in the units of the weight of 3 inches of pipe, so that the weights are 3, 4, 5, 6, 7, and 8?

But we can do better. Suppose the weights of 3, 4, 5, 6, 7, and 8 are balanced around a regular hexagon. Then if each weight were reduced (or increased) by the same amount the windchime would still be balanced. So let us reduce all weight by 2. Hence if we can balance the windchime with weights of 1, 2, 3, 4, 5, and 6 then we can balance with lengths of 9, 12, 15, 18, 21, and 24 and conversely.

What conditions must be true if weights of a, b, c, d, e, and f balance when placed on the vertices of a regular hexagon? Here balanced will mean the hexagon is horizontal when suspended (or supported) from the center of the hexagon.

Consider line m as a knife edge supporting the hexagon with the weights attached. This line is a line of symmetry and goes through the center. So if the windchime is balanced, we must have a + b equal to d + e since these give opposite moments to twist around line m. Considering the two other lines of symmetry through vertices in the hexagon, we get these equations:



P1)
$$a + b = d + e$$
$$b + c = e + f$$
$$c + d = f + a$$

Also:

P2)
$$a + 2b + c = d + 2e + f$$
$$b + 2c + d = e + 2f + a$$
$$c + 2d + e = f + 2a + b$$

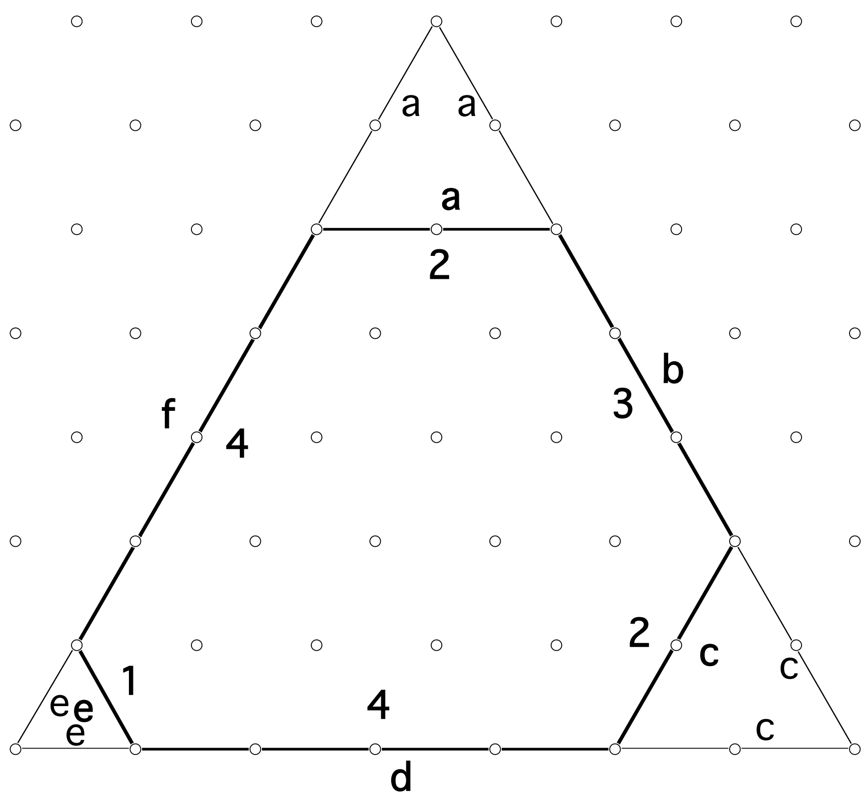
The equations P2) come from looking at the line n of symmetry. Since b is twice as far from that line as a and c are, a+2b+c represents a turning moment on one side and d+2e+f is the turning moment on the other side. These must be equal if the hexagon is balanced. Considering the other two lines of symmetry perpendicular to the sides gives the equations P2).

Living in the physical world we know that the hexagon would be quite stable if it were supported by any two of the knife edges represented by any two of the six lines of symmetry. Note therefore that given any two of the equations in P1) and P2) together, you can derive all the other equations.

Equiangular Hexagon Puzzle (Geometry)

A hexagon (six sided polygon) with all six interior angles of the same size, must have what size angles? Drawing interior diagonals from one vertex to the three non-adjacent vertices will divide the hexagon into four triangles each of whose angles sum to 180°. Thus there are 4×180 or a total of 720° of interior angles. So each of the six equal interior angles must be 120°. This can also be understood by remembering the six exterior angles must add to 360°. The exterior and interior angles at each vertex must be supplementary, so again, the interior angles must all be 120°. Hence to draw an equiangular hexagon, we might just as well draw it on equiangular triangle paper as shown below.

In this example we do have an equiangular hexagon since all of the interior angles are 120°. Note that this means that opposite sides are necessarily parallel. Here the sides are of lengths 2, 3, 2, 4, 1, and 4. How can we draw a hexagon with sides 1, 2, 3, 4, 5, and 6, but not necessarily in that order?



Suppose we have an equiangular triangle with sides a, b, c, d, e, and f. Since the top and bottom are parallel, the lengths of sides b and c must add to those of e and f. This is because the line of the top side is a fixed number (in this case 5) of heights of the triangles from the bottom side. Both b+c and e+f must be this same number. Since each pair of opposite sides are parallel, these equations must be true (we start with sides f and c parallel):

$a + b = d + e$

G1) $b + c = e + f$

$c + d = f + a$

From the geometry of the hexagon with sides restricted to be on the lines of the equilateral triangle paper you can note several other facts:

If opposite sides are increased (or decreased) by the same amount, then the hexagon remains equiangular.

If all sides are increased (or decreased) by the same amount, then the hexagon remains equiangular.

If three non-adjacent sides are selected and extended to meet, the hexagon can be viewed as an equilateral triangle with smaller equilateral triangles snipped from the corners. Since equiangular triangles are equilateral, we get the equations:

G2) $a + b + c = c + d + e = e + f + a$.

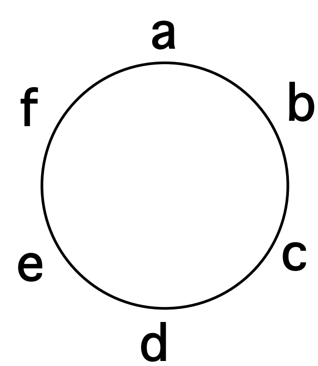
Choosing the other three sides to extend would give these equations:

G3) $f + a + b = b + c + d = d + e + f$.

Six in a Circle Puzzle (Algebra)

Suppose there are six numbers, not necessarily whole numbers. Call them a, b, c, d, e, and f. Consider them arranged equally around a circle in the given order. Can definite values for these numbers be chosen so that any two adjacent ones add to the same sum as the two opposite numbers?

This means, if we have the numbers arranged as shown on the right, that:



A1)
$$a + b = d + e$$
$$b + c = e + f$$
$$c + d = f + a$$

Thus the problem is to find numerical values for these letters so that all three equations in A1) are satisfied. It would also be nice if we could find all such choices that would work.

One Solution

Without any motivation let us rearrange the equations in A1) to get:

A2)
$$b - e = d - a$$
$$b - e = f - c$$
$$d - a = f - c$$

With this rearrangement of A1) into A2) it is even clearer that only two of the equations are needed and in fact,

A3) $b - e = d - a = f - c$. Let us call this common value something else, say x.

Thus $b - e = x$, $d - a = x$, $f - c = x$. We can rewrite this as:

A4)
$$b = e + x$$
$$d = a + x$$
$$f = c + x.$$

From the equations A4) it should be clear that if values of a, c, e and x are arbitrarily assigned, then the values of b, d, and f are determined. And, in fact, all of the equations A1) will be satisfied. Thus we have all possible solutions for the Six in a Circle Algebra puzzle.

So What? and Two Solutions

Without any further comment, the Six in a Circle Algebra puzzle and the solution should seem pointless. But note that equations A1) are identical to equations P1) from the Windchime Puzzle and equations G1) from the Equiangular Hexagon Puzzle.

Thus we are led to ask about solutions to A1) where a, b, c, d, e, and f are equal to 1, 2, 3, 4, 5, and 6 in some order. Then we will have ALL possible solutions to the Windchime and Equiangular Hexagon Puzzles. In fact we should see they are not distinct puzzles, but really the same problem expressed in different words and in different contexts.

Since we can rotate and flip over both the equiangular hexagon and the windchime, it will not make any difference where we start. Thus we may as well assume that a = 1. With that assumption, what are the choices for x? Clearly, x must be a positive integer.

If x = 1, then d = 2. Choose c = 3 and then f = 4. This only leaves e = 5 and hence b = 6. So the windchime and equiangular hexagon puzzles are solved by the numbers in this order: 1, 6, 3, 2, 5, 4.

If x = 2, then d = 3. Whatever letter we choose to equal 2 we get another letter equal to 4. This leaves the numbers 5 and 6. But choosing 5 and adding x = 2 we can't get 6. So there is no solution with x = 2.

If x = 3, then d = 4. Choose c = 2 and then f = 5. This leaves the choice e = 3 so b = 6 and we get the solution 1, 6, 2, 4, 3, 5. This solution is different than the one obtained when x was 1.

If x = 4, then d = 5. Now one of c or e must be 3. But this means that f or b will have to be 7 which can't happen. Thus x cannot be 4. The same reasoning shows that any larger x will not work either.

Thus there are exactly two solutions (up to a rotation or reflection) to the Windchime and the Equiangular Hexagon Puzzle.

Challenge

A rewarding exercise is to take a condition in one of the puzzles and interpret it in the other two contexts. For example in the Algebra A4) equation $b = e + x$ we see that increasing e by a unit will mean increasing b by a unit. In Physics this means that if you add a weight to corner e, the same weight must be added to corner b to retain balance. In Geometry this means lengthening side e means that side b must be lengthened by the same amount.

Also note if x is increased by a unit, b, d and f all increase by the same unit. In Physics this is reflected in the fact that if the windchime is balanced, putting a additional equal weight on three every-other-one vertices would maintain balance.

In Geometry this suggests the surprising (at least non-obvious) fact that increasing or decreasing three every-other sides of a equiangular hexagon by the same amount will produce another equiangular hexagon.