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Can College Students Reason ?

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> If X teaches Y, then X acts upon Y's environment in such a manner that Y develops in a desired way.
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> How Y develops in a given environment varies not only according to that environment but also according to the way Y perceives that environment.
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> Hence if X teaches Y, then X must consider not only how to act upon Y's environment but also Y's perception of his environment. Equivalently, if X is not sensitive to Y's perception of his environment, then X cannot be teaching Y.

Simplistic as this argument may sound, most of us who try to teach college mathematics have tended to ignore it and its implications. We rarely consider how various students perceive their learning environments--specifically, our classrooms. We also are mostly ignorant of recent research that dramatizes some important consequences of our omissions.

> Item. Towler and Wheatley of Purdue University asked students in an introductory mathematics course whether or not changing the shape of a clay ball affected a) the amount of clay, b) the weight of the clay, or c) the amount of space occupied by the clay. Although most students realized that mass and weight of the clay were invariant, 39% of them believed that the volume changed when the ball was rolled into a sausage shape.[1]

Purdue University, of course, has no monopoly on such thinking; we all have experienced the student who just can't seem to catch on to our mathematics, no matter how hard he tried, the student who can do no more than memorize how to manipulate some formulas. Moreover, in this day of opening admissions and dropping enrollments, it is unlikely that the number

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of these students in our courses will decrease. Not all these students are lazy, or dumb. Some work very hard for us, meeting only frustration; some are quite successful in other courses. Is there anything we can do for them?

I believe we can find at least partial answers to such questions if we consider a psychological theory that makes good use of the concept of the learner's perception of his environmental stimuli. The framework I wish to oversimplify for you today is that developed by the Swiss investigator Jean Piaget.[2] Although Piaget began work in the 1920's, he was largely unknown in this country until the last decade or so. His work by now has influenced elementary and, to some degree, secondary school teaching, but still has not received the attention it deserves from most college educators.

Piaget describes the mental development of a human being in terms of an undefined concept that is roughly translated as "mental structure". An individual organizes environmental stimuli according to his mental structure, and adapts this structure to assimilate such stimuli. Except for inherited reflexes, an infant's mental structure is very narrow; it can assimilate very few of the many stimuli encountered. But, given sufficient numbers of these stimuli, the structure accommodates itself for organizing a broadening range of them. A structure changes when it encounters stimuli that differ only slightly from those it can handle. If there is no incongruity, the stimuli will be assimilated without structural change; if there is too much incongruity, the stimuli will be ignored.

Although Piaget and his followers describe typical mental growth in terms of a refined system of stages and substages, the most important observation for our purposes is that persons encounter the stages in order. They do not backtrack, and, ideally, development does not involve skipping stages, which can lead to problems later in the growing period. (In an extreme case, a special educator might take a "slow" 8- or 9-year-old back to the crawling stage, and then teach him to walk again, and so forth, gradually rebuilding his mental structures to catch up with his physical development.)

With the hope that we can separate the wheat from the chaff, which is abundant in any psychological theory, we shall concentrate here on two of the major stages--those of "concrete operations" and "formal operations". Piaget uses the term "concrete operations" to refer to an extended period between the approximate ages of 7 and 11 in which a child has become able to set up one-to-one correspondences, to count, to recognize that the number of objects in a set is independent of its configuration, and to imagine himself in the position of others. He could perform none of these operations before reaching this stage, and his aptitude improves during this stage. On the other hand, he *cannot* yet operate *on these operations* by designing an experiment that requires holding all but one variable constant, or by formulating hypotheses, or by recognizing that volume is independent of shape or weight, or by responding to the form rather than the content of a logical argument. He will probably not take a fastidious interest in the rules of games. Ability to perform these operations on operations, or "formal operations", is acquired around the age of 11 or 12, according to Piaget.

Item. At the University of Oklahoma, Renner and Lawson found that 58% of the 185 freshmen tested could not isolate variables sufficiently well to determine whether the period of a pendulum is affected by string length, weight of bob, both, or neither. The students were also asked to hypothesize whether a heavier or lighter object of identical volume would displace more water. Twenty-eight percent of the freshmen either predicted incorrectly or reasoned incorrectly in their prediction.[3]

So Oklahoma joins Purdue--and, of course, the rest of us. A growing ody of research indicates that many college students, at least in North merica, do not think at the formal operations stage. This means that a enerous portion of students cannot be expected to "reason" in what we like o think of as a logical way (i.e., as formalized by the "laws of logic"). ince a favored assumption of most college mathematics teachers is that 'reasonable" explanations promote understanding, the indications are that large number of our students are incapable of learning from us if we each in the ways to which we are accustomed.

Specifically, what can they not learn from us that we would like them o understand? Borrowing heavily from a recent paper[4] concerning the im-lications of Piaget's theory for teaching chemistry, I have stuck my neck ut and prepared a list of concepts which I suspect most students who are ot at the formal operations stage cannot really understand. (Figure 1.) f I am at all correct, it follows that they are not able to follow a ormal argument, much less to come up with a proof of their own. They annot grasp the concept of a function, because the concept of variable is ot clear. And, in terms of attitude toward our field of study, they cer-ainly cannot appreciate playing mathematics, seen as a rule-oriented game.

Concrete-operational students

can	but can't
make routine measurements and observations	**measure** "indirectly" quantities such as speed and acceleration, perhaps even area and volume
answer acceptably the question, "Are there more squares or rectangles in the diagram"? if they realize that all squares are rectangles	**respond** correctly to the choice, "If all squares are rectangles, then: 1. all rectangles are squares; 2. some rectangles are squares; 3. no rectangles are squares."
order a collection of sticks according to length	**decide** who is tallest if told that Bill is taller than Tammy and shorter than Sheila
count and perform elementary arithmetic operations	**systematize** counting procedures well enough to understand permutations and combinations
manipulate algebraic expressions, including fractions	**given** the equation $y=3x^2$ or $y=1/x$, decide what happens to y as x increases
generalize simply from given data: All quadratic equations (in x) represent parabolas	**perform** "once-removed" generalization Since quadratic equations in x represent parabolas, so do quadratic equations in y.

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Figure 1.

Untested conjectures

<u>Item</u>. As long ago as 1944, a study presented college students with an argument for which they were to choose a correct conclusion. "Some ruthless men deserve a violent death; since one of the most ruthless of men was Heydrich, the Nazi hangman:

1. Heydrich, the Nazi hangman, deserved a violent death.
2. Heydrich may have deserved a violent death.
3. Heydrich did not deserve a violent death.
4. None of these conclusions logically follows."

More than 37% of the students chose number one.[5]

<u>Item</u>. Recent experimentation reported in the journal <u>Science</u> indicates not only that 50% of the freshman women at Penn State are unaware of the general principle that the surface of still water is invariantly horizontal, but also that they do not learn this principle by correcting their own errors.[6] My own informal experimentation verifies that a large number of college students, by no means limited to freshman women, do not understand this principle. (Figure 2.)

Sketch in the water:

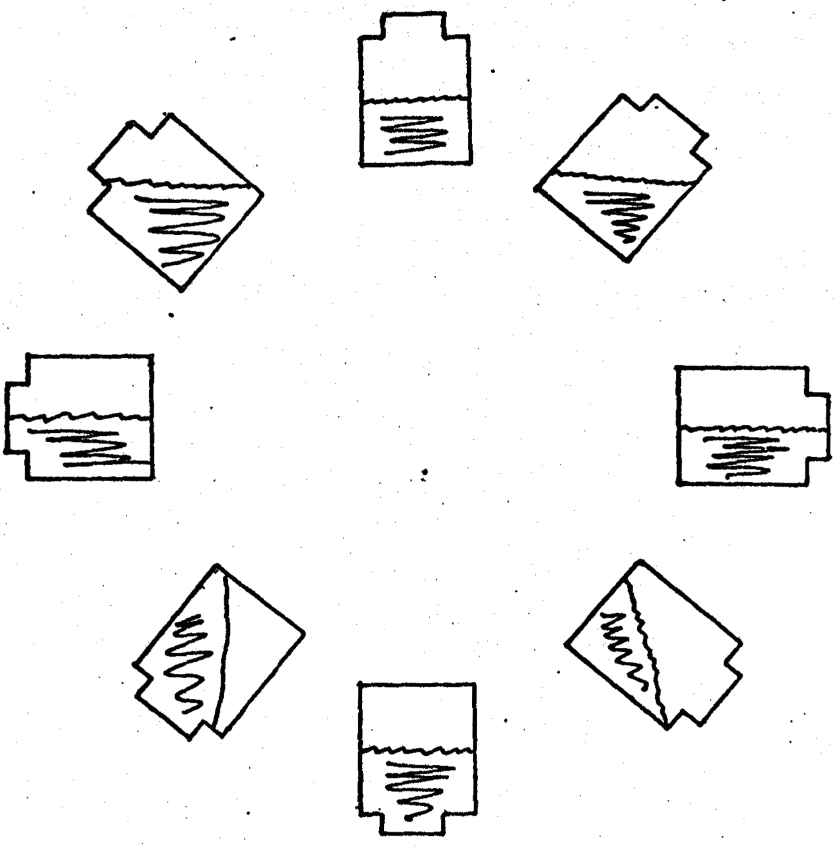


Figure 2.
Actual result of my own testing.

If it's not clear even how to teach these forms of abstract thinking, then, what should we do? Should we give up all hope of bringing about understanding and retention, and fall back on conditioning and drill? But then aren't we building our houses on sand? Or should we abandon altogether the notion of teaching mathematics to these students? But then, where do we find our new jobs? Or should we perhaps take our students back to a "crawl" stage, in some sense?

Although I don't pretend to have any final answers to these questions, I am growing increasingly in favor of the last alternative, of concentration on rebuilding mental structures—but only if we keep in mind a few caveats. For example, we should be aware that responses to "why" questions can be just as automatic as responses to "how" questions, as I believe some of the "new math" programs demonstrated. Thus we must be very careful in assessing progress. Also, we should probably at least consider Piaget's personal opinion that we should not unnaturally accelerate a person's development--in what he calls, of course, the "American Way"--although some limited experimentation indicates that it can be done.

Since I'm generally coming down on the side of optimism, though, I should probably go even farther out on our limb and speculate about techniques we can use for teaching mathematics to these college students. It seems clear by now that we must find ways to bridge the gap between concrete and formal operations—at least to the point of giving our students intuitive feelings for whatever mathematical concepts can be communicated this way. To do so would require at least paying a great deal more attention to concrete materials in the college classroom than we're accustomed to—and by "concrete materials" I mean paper and scissors and compass and measuring tape, not overhead projector and programmed text and teaching machine (although the extent to which these aids can provide relatively concrete operational experiences is a fascinating and unexplored question). The student we're discussing needs to "mess around" with basic mathematical concepts--independently of our telling him how to mess around--before he can begin to formalize them, or appreciate anyone's desire to formalize them.

While I'm at it, I should say that I suspect we only impede development toward more abstract ways of thought if we continue to think of these students as dumb, or slow. To ask "What if . . ." kinds of questions, after all, requires an openness to new ideas that presupposes some degree of ease with one's current view of the world.[7] Our labeling a student as "slow" cannot help in building this self-confidence and thus will probably become a self-fulfilling prophesy. Moreover, it is a prophesy that ignores the fact that all of us are concrete-operational in some areas of thought.

So we need to provide learning environments that give a student concrete experiences yet don't insult his dignity. We need to find materials for this that are conducive to mental development. And we would like to find ways of evaluating the success of such a program. I have gleaned some ideas about these problems from student experiences in a few courses I have taught using concrete materials in a college mathematics laboratory setting.[8] I'd like to share a few of them with you before turning you loose to do your own experimentation.

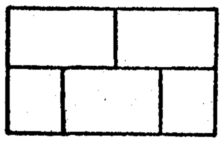
1. The Tower of Hanoi is an old puzzle, consisting of three spindles and a stack of punctured disks, decreasing in size, which fit over the spindle: The goal is to transfer the pile of disks from one spindle to another, moving only one disk at a time and never putting a larger disk on top of a smaller one.

Ideas for such concrete materials can come from many sources: articles in mathematics education journals such as <u>The Mathematics Teacher</u>, NCTM yearbooks, catalogues of educational materials, browsing through toy stores, and so on. In the case of the Tower, I believe I was originally inspired by some Madison Project material.[9] I usually let students play with it for awhile, devising strategies for transferring the disks if possible. Some never get beyond this point, although most do so in the course of a semester of periodic attempts. Then I ask them to vary the number of disks and to keep track of the minimum number of moves required to transfer the piles. Eventually many of them actually derive an expression for the function involved; some go on to explore deeper mathematical relationships exemplified by the Tower. Those who cannot generalize this way need more practice with concrete materials, so I suggest that they play with other puzzles and games that give similar experiences.

Incidentally, I have never encountered a student who was not intrigued by the Tower, no matter what his mathematical ability. One student last fall went on to derive a new method for moving disks, which, while not the most efficient, required little thought. She thus discovered what another student had once proudly proclaimed to me--that mathematics is the process of working very hard to find easier ways of doing things!

2. While studying polygons, one freshman was asked to cut some geometric figures out of construction paper--apparently the first time in her life she had applied scissors to paper! She enjoyed this, and went on to construct polygons out of popsickle sticks by weaving the sticks to make stable figures. (Interestingly, she was not satisifed that her early figures were stable until they had remained together for several days.) She also constructed polyhedra with the help of Superstructures, a modern plastic version of Tinkertoys. By the end of the course she was making fairly accurate predictions about two-dimensional patterns required for paper polyhedra, although she had a long way yet to go.

Many students are delighted with problems involving cutting and tracing graphs. They are usually attracted by the dual challenges of cutting each line segment of a given figure exactly once with a single continuous curve



and of tracing various figures without repeating line segments or lifting thei pencils. Some students derive conditions for traceability fairly quickly, and either continue to another project or expand into relationships demonstrating Euler's formula. On the other hand, one music major persisted with the tracing project for several weeks before finally coming across a relationship between order of vertices and traceability. The "discovery" came only after he had physically traced literally hundreds of graphs, mostly of his own making, and had constructed several charts. Even then the dawn was almost accidental--he was not yet comfortable with designing his experimentation so as to eliminate variables methodically.

4. As I hinted before, we all seem to experience a need to work in the concrete-operational mode upon first approaching an area of investigation that is new to us. Of course, this is a primary justification for laboratories in the natural sciences and even in the social sciences, but it can be true for us as we intially confront an area of mathematics. We are like the person who first encounters Piet Hein's Soma puzzle. I have never seen anyone pick up the blocks for the first time and immediately form them into a cube. However, most of my acquaintances who have "messed around" with it for awhile eventually have come up with a solution--and, as time goes on, they have become quite conscious of combinations that will or won't be suc-

cessful, even without a logical analysis. Developing this intuitive feeling by "messing around" is the concrete operational work which, I suspect, must precede formal operational learning in any field.

It should be clear by now that the students we are discussing are not stupid or lazy. Perhaps they are not "reasoning", in our logical sense of the term, but we need to consider the possibility that this is due to gaps in the development of their mental structures rather than to inherent lack of growth potential. If we channel our impatience toward providing concrete, "hands-on" learning environments, I believe we may teach more effectively in the long run.

Therefore I encourage you to take Piaget's work seriously--if not for its specifics, at least for its metaphorical value--for it presents a very compelling model for describing the growth of our students. "Mess around" with the theory, be sensitive to the grain of students' mental structures, and experiment with some concrete teaching materials. And, please, let me hear from you.

References

1. J. O. Towler and G. Wheatley, Conservation concepts in college students: a replication and critique, *Journal of Genetic Psychology* 118: 265-70, 1971.

2. For background reading about Piaget's theory, see J. L. Phillips, *The Origins of Intellect: Piaget's Theory* (Freeman, 1969); H. Ginsburg and S. Opper, *Piaget's Theory of Intellectual Development: an Introduction* (Prentice-Hall, 1969); and R. W. Copeland, *How Children Learn Mathematics: Teaching Implications of Piaget's Research* (Macmillan, 1970).

3. J. W. Renner and A. E. Lawson, Promoting intellectual development through science teaching, *Physics Teacher* 11: 273-6, 1973.

4. D. Heron, Piaget for chemists, Journal of Chemical Education 52: 146-50, 1975. See also D. W. Beistel, A Piagetian approach to general chemistry, in the same issue.

5. J. J. B. Morgan and J. T. Morton, The distortion of syllogistic reasoning produced by convictions, Journal of Social Psychology 20: 39-59, 1944. Although the authors attribute such false reasoning to personal bias on the parts of the subjects, they admit that "even when a subject is presented with a syllogism in which the terms. . . have little or no personal significance, he has difficulty in selecting the correct conclusion. . . . The only circumstance under which we can be relatively sure that the inferences of a person will be logical is when they lead to a conclusion which he has already accepted." (p. 39)

6. H. Thomas, W. Jamison, D. D. Hummel, Observation is insufficient for discovering that the surface of still water is invariantly horizontal, Science 181: 173-4, 1973.

7. For some fascinating ideas about the "What if not . . . "nature of mathematical exploration, see articles by M. Walter and S. I. Brown in Mathematics Teaching, Nos. 46 (Spring, 1969, pp. 38-45) and 51 (Summer, 1970, pp. 9-17).

8. A talk I gave last spring on some of these courses has been expanded into a paper, Multi-structured college courses in general mathematics and calculus, available from me. For an attempt to fit them into philosophical and psychological contexts, see my doctoral dissertation, Teaching Models for College Mathematics, available from University Microfilms, Ann Arbor, Michigan.

9. For example, Robert B. Davis, Explorations in Mathematics: A Text for Teachers (Addison-Wesley, 1967), pp. 257-63.

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