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Farey Sequences, Ford Circles and Pick's Theorem

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Expository Paper

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One of the ongoing themes through the Math in the Middle coursework has been the idea of identifying patterns. From our first course, Math as a Second Language, patterns have been useful to explain phenomena and determine future values. Some patterns are numerical but can be described using algebra. Some are visual or geometric and also can be described using numbers and symbols. Many of these patterns have resurfaced in different forms and at different times in new and interesting ways. It has been a humbling experience to see the interconnectedness of seemingly unconnected ideas. Pick's Theorem, Farey Sequences and Ford Circles are concepts quite different on the surface but linked in interesting ways.

The Life of Georg Alexander Pick

Georg Alexander Pick was born to a Jewish family on August 10, 1859 in Vienna, Austria. His parents, Josefa Schleisinger and Adolf Josef Pick educated him at home until the age of eleven. He then entered the fourth class of the Leopoldstaedter Communal Gymnasium. He qualified for university entrance in 1875, at age 16.

According to the St. Andrews website, Pick entered the University of Vienna in 1875, and published his first mathematics paper the following year. He studied both math and physics, and graduated with an endorsement to teach the two subjects. An interesting side note is that Leo Konigsberger was his advisor during this period. Pick received his doctorate in 1780.

Pick studied or worked with other notable mathematicians such as Emil Weyr, Felix Klein, Charles Loewner and Albert Einstein. Terms such as the, "Schwarz-Pick lemma, 'Pick matrices' and the, 'Pick-Nevanlinna Interpolation' are still used today. Pick is best remembered for Pick's Theorem. This theorem first appeared in his 1899 paper, *Gepmetrisches zur Zahlenlehr. (Geometrical to the Teaching of Numbers)*

Pick's Theorem is on reticular geometry. A polygon whose edges are reticular lines Pick calls a reticular polygon. Pick's theorem states that the area of a reticular polygon is L + B/2-1 where L is the number of reticular points bordering the polygon and B is the number of reticular points on the edges of the polygon. This theorem can easily be seen on a geoboard. This theorem was largely ignored until 1969 until Hugo Steinhaus included the theorem in his famous book, *Mathematical Snapshots.* From that point on, Pick's Theorem has been recognized for its simplicity and elegance.

Pick's academic and professional career was quite successful. At the German University of Prague he was the dean of the philosophy faculty from 1900-1901. He supervised students for the doctorate program. In 1910, he was on a committee set up by the university to consider appointing Einstein to the university. Pick was the driving force behind Einstein's appointment. He and Einstein were close friends during Einstein's appointment at the university.

After Pick retired in 1927, he was named professor Emeritus and returned to Vienna. Unfortunately, he was unable to live out his life in peace. In 1938, after the Anschluss, he returned to Prague. However, the German government asked the Czech government to give Germany all districts of Bohemia and Moravia with populations that were 50% or more German. Many Czech leaders resigned rather than agree to this, but the new leaders gave in to the request. Hitler's armies invaded in March of 1939 and Hitler installed his representatives to run the country. The Nazis set up the

Theresienstadt concentration camp. The camp was supposed to house the elderly, privileged and famous Jews. The Nazis portrayed a façade that the camp was more of a community of Jewish artists and musicians. In the end, the "Terezin" camp was in fact a transport to Auschwitz, and was extremely overpopulated. Many prisoners died of disease or starvation. Of the 144.000 Jews sent to Terezin, about a quarter died there, including Pick. 60% of other Jews sent there were eventually sent to Auschwitz. Pick was sent to Terezin on July 13, 1942, and died there two weeks later, at age 82. A vulgar death for a gentleman described as, "…a bachelor…uncommonly correct in clothes and attitude."
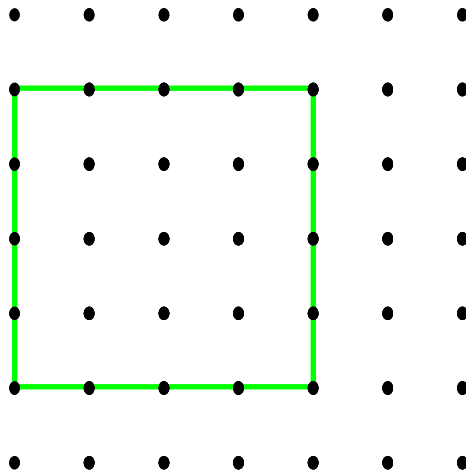
Pick's Theorem

First published in 1899, Pick's Theorem was brought to greater attention in 1969 through the popular book *Mathematical Snapshots* by Hugo Steinhaus. The theorem gives an elegant formula for the area of simple lattice polygons, where "simple" only means the absence of self-intersection. Polygons covered by the theorem have their vertices located at lattice points of a square grid or lattice whose points are spaced at a distance of one unit from their immediate neighbors. The formula doesn't require math proficiency beyond middle grade school and can be easily verified with the help of a geoboard. In fact, the use of a geoboard will make our proof of this theorem simple. Pick's Theorem states:

> *Let P be a simple (i.e., nonintersecting) lattice polygon, containing B lattice points on its boundary, and I lattice points in its interior. Then the area, A (P) of P is given by:*

$$A(P) = \frac{1}{2}B + I - 1$$

In the Euclidean plane, a lattice point is one whose coordinates are (x,y) where (x.y) are both integers. A lattice polygon is one whose vertices are on lattice points. Below we will see a simple example of this theorem:

the square (in green) has 9 interior lattice points, and 16 exterior points. Therefore, the area of the square using Pick's Theorem is:

$$9 + \frac{16}{2} - 1$$
$$= 9 + 8 - 1$$
$$A = 16$$

To illustrate Pick's Theorem for triangles is slightly more complicated. First, let us consider a primitive triangle. A primitive triangle is one that has vertices on exterior lattice points, with no interior points in between. For such a case, the area of the triangle will always be: $A = 0 + (\frac{3}{2}) - 1 = 0.5$

For the purposes of this paper, any polygon with vertices lying on lattice points can be decomposed into primitive triangles. The triangle below has been decomposed into a group of primitive triangles. Using Pick's Formula, its area can be shown:

$$A = 6 + \frac{10}{2} - 1 = 10$$
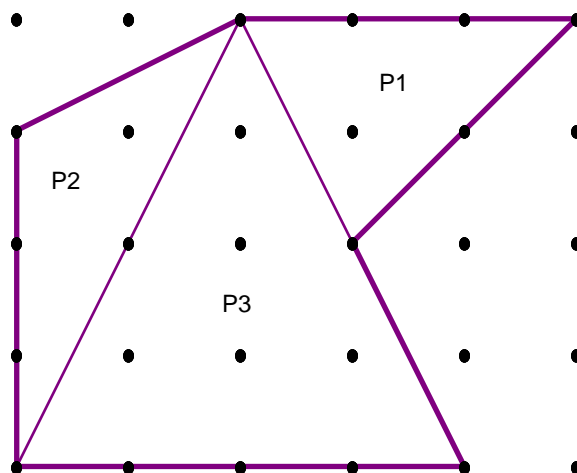
This can be checked for accuracy using the standard area formula for triangles (A=bh)/2

You can visually see the base of the triangle is 5, and the height is 4. A=$\frac{1}{2}(5)(4) = 10$

Pick's Theorem holds true for lattice triangles.

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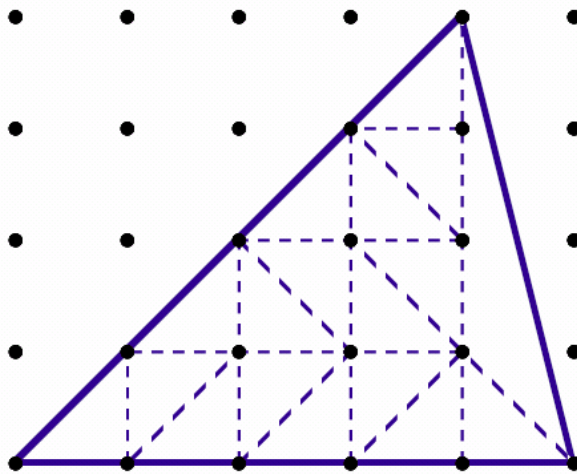
Let's try a more complicated, irregular polygon:

Pick's Theorem can be used to find the area of this shape. This will also show that Pick's Theorem has an additive character. The irregular shape has been divided into six separate polygons, which we will call P1, P2, P3, The interiors of these polygons are separate; however, some edges are shared. Because we have previously shown Pick's Theorem to be true for triangles, you can check this polygon using Pick's Theorem:

$$A = 8 + \frac{14}{2} - 1 = 14$$

Because we want to prove that Pick's Theorem is additive, we will decompose the polygon into triangles.

The interior lattice points will then be:
$$I = I_1 + I_2 + I_3$$
The boundary points will be:
$$B = B_1 + B_2 + B_3 - 3$$
The total area is therefore:
$$A = P_1 + P_2 + P_3$$

$$p_1 = 1 + \frac{6}{2} - 1 = 3$$

$$p_2 = 1 + \frac{6}{2} - 1 = 3$$

$$p_3 = 5 + \frac{8}{2} - 1 = 8$$

Hence, $\left(1 + \dfrac{6}{2} - 1\right) + \left(1 + \dfrac{6}{2} - 1\right) + \left(5 + \dfrac{8}{2} - 1\right) = 14$

In conclusion, Pick's Theorem is a simple way to find the area of a lattice polygon. The theorem is easy for elementary-aged students to understand and apply. There are many classroom applications for Pick's Theorem, especially with the use of a geoboard or Geometer's Sketchpad. This is an alternative way to solve for the area of simple lattice polygons.

The Farey Sequence

The Farey Sequence is a pattern that has its origin in quite common numbers. The Farey fractions can be found in all sorts of different applications. The Farey sequence was so named for British born geologist, John Farey (1766-1826). In 1816 Farey wrote about the "curious nature of vulgar fractions" in the publication *Philosophical Magazine*. Given a sequence F_n where b, d and b + d are all less than n, what Farey noticed is that if two fractions $\frac{a}{b}$ and $\frac{c}{d}$ were combined in this way $\frac{a+c}{b+d}$, the resulting fraction was also in the series. Farey was not able to prove this but prolific French mathematician Augustin Cauchy (1789-1857) was able to provide a proof in 1816 published in *Exercices de mathemátiques*. Despite Farey's inability to prove the "curious nature" of these fractions, Cauchy still attributed the sequence to Farey. Unbeknownst to either Cauchy or Farey, there was a paper with a description of the sequence and proof by Haros fourteen years before.

The Farey Sequence of fractions (F_n) are made up of fractions in lowest terms where the denominator is less than or equal a number *n*. When the fractions of F_1 are added together, incorrectly, $\frac{0+1}{1+1} = \frac{1}{2}$, a new fraction falls between the original two is generated. This fraction is called the *mediant*. The next series is found by adding the first two fractions of F_2 to find the mediant $\frac{0}{1} \oplus \frac{1}{2} = \frac{1}{3}$. One finds the mediant of the last two fractions in F_2, $\frac{1}{2} \oplus \frac{1}{1} = \frac{2}{3}$, and the next Farey sequence is found. This procedure of finding the mediant between each pair of fraction in the previous Farey sequence is repeated to find the next sequence.

$$F_1 = \left\{ \frac{0}{1}, \frac{1}{1} \right\}$$

$$F_2 = \left\{ \frac{0}{1}, \frac{1}{2}, \frac{1}{1} \right\}$$

$$F_3 = \left\{ \frac{0}{1}, \frac{1}{3}, \frac{1}{2,}, \frac{2}{3}, \frac{1}{1} \right\}$$

$$F_4 = \left\{ \frac{0}{1}, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{1}{1} \right\}$$

$$F_5 = \left\{ \frac{0}{1}, \frac{1}{5}, \frac{1}{4}, \frac{1}{3}, \frac{2}{5}, \frac{1}{2}, \frac{3}{5}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{1}{1} \right\}$$

It is interesting to note that the sequence F_5 contains all the fractions from F_4. In fact, only new fractions in F_5 have a denominator of 5. For the purposes of this paper, for all Farey sequences F_n the only fractions that will appear for the first time in F_n will have a denominator of n

One of the properties of the Farey sequence is that given two consecutive Farey fractions $\frac{a}{b}, \frac{c}{d}$ where then $bc - ad = 1$. This can be proved by induction.

Suppose in F_{n+1} there are three consecutive Farey fractions, $\frac{a}{b}, \frac{r}{q}, \frac{c}{d}$ where q = n+1, then $\frac{a}{b}$ and $\frac{c}{d}$ are consecutive fractions in F_n. The middle fraction, $\frac{r}{q}$, is the mediant formed by $\frac{a}{b}$ and $\frac{c}{d}$ and so $\frac{r}{q} = \frac{a+c}{b+d}$.

For F_1 $\left\{\frac{0}{1}, \frac{1}{1}\right\}$, $1 \bullet 1 - 0 \bullet 1 = 1$, so $bc - ad = 1$ is true.

Let us assume that it is true for F_n. Given two consecutive Farey fractions $\frac{a}{b}, \frac{c}{d}$ where, then $bc - ad = 1$.

Let's see what happens at F_{n+1}.

> **Case 1:** Let's randomly select two fractions from F_{n+1}. If the two fractions are both elements of F_n, then we already know $bc - ad = 1$ by the above inductive assumption.

> **Case 2:** If the two randomly selected fractions from F_{n+1} contain one fraction, $\frac{r}{q}$ where q = n+1, we know this is a new Farey fraction. That is, it has not appeared in any previous sequence. This fraction would have one fraction on either side, $\frac{a}{b}$ and $\frac{c}{d}$ where r = a + c and q= b + d. It can be shown the above relationship holds in this situation.

Let's examine $\frac{a}{b}$ and $\frac{r}{q}$, remember $\frac{r}{q} = \frac{a+c}{b+d}$. Then b(a+ c) - a(b + d) which can be simplified to ab + bc – ab – ad or bc – ad which equals 1.

The same argument can be made as we examine $\dfrac{r}{q}$ and $\dfrac{c}{d}$. Using

substitution we get $\dfrac{a+c}{b+d}$ and $\dfrac{c}{d}$ so we need show that c(b + d) – d(a + c)=1. Using some algebra, bc + cd- ad – cd would simplify to bc – ad which is still equal to one.

By the principal of mathematical induction, if given two consecutive fractions, $\dfrac{a}{b}$ and $\dfrac{c}{d}$ in the sequence F_{n+1}, then bc – ad = 1.

Another interesting relationship of two consecutive Farey fractions, $\dfrac{a}{b}$ and $\dfrac{c}{d}$ is that the mediant, $\dfrac{a+c}{b+d}$ determined by these two fractions will lie between $\dfrac{a}{b}$ and $\dfrac{c}{d}$. We already know that bc – ad = 1 because the value is a positive number, then $\dfrac{a}{b} < \dfrac{a+c}{b+d}$.

The same argument can be made for the last two fractions, $\dfrac{a+c}{b+d}$ and $\dfrac{c}{d}$ so therefore,

$$\dfrac{a}{b} < \dfrac{a+c}{b+d} < \dfrac{c}{d}.$$

Through the course of my work with the Farey Sequence, I discovered other aspects I would like to explore more thoroughly at a later time. Many sites referenced the idea that all rational numbers between 0 and 1 can be generated by the Farey tree.
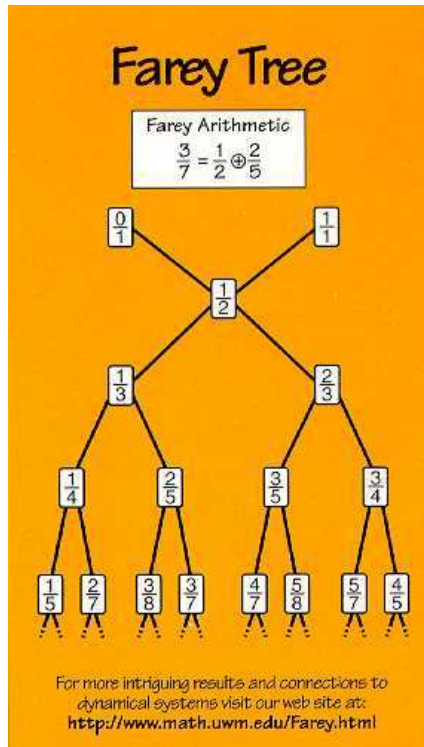
When I first started manipulating the fractions, I discovered Fibonacci numbers can be found in the Farey Fractions in the examples shown below.

$$\frac{0}{1} + \frac{1}{1} = \frac{1}{2}$$

$$\frac{1}{2} + \frac{1}{1} = \frac{2}{3}$$

$$\frac{1}{2} + \frac{2}{3} = \frac{3}{5}$$

$$\frac{2}{3} + \frac{3}{5} = \frac{5}{8}$$

$$\frac{3}{5} + \frac{5}{8} = \frac{8}{13}$$

Farey fractions seem to be very evident in many different areas of mathematics. It is interesting that such a simple idea would generate so many discussion topics through a myriad of mathematical topics.

FORD CIRCLES

Lester Randolph Ford, Sr. was an American mathematician born in 1886. Ford received a PhD in mathematics from Harvard University in 1917. Ford circles are named after Ford, who introduced the concept in a 1938 article called "Fractions" (American Mathematical Monthly, volume 45, number 9, pages 586-601).

Ford was the editor of the American Mathematical Monthly magazine from 1942 to1946, and President of the Mathematical Association of America from1947 to 1948. In 1964 the Mathematical Association of America recognized his contribution to mathematics by establishing the 'Lester R. Ford Award' for authors of published mathematics in The American Mathematical Monthly. Ford's son, Lester Randolph Ford, Jr., (who was born in 1927), is also a famous mathematician.

Ford Circles are a geometric representation of fractions. Ford wanted to illustrate fractions, like $\frac{a}{b}$ and $\frac{c}{d}$, as circles. Ford showed that you could find a fraction in between the fractions $\frac{a}{b}$ and $\frac{c}{d}$ by finding their *mediant,* a new fraction $\frac{(a+c)}{(b+d)}$, as shown with the diagram below:
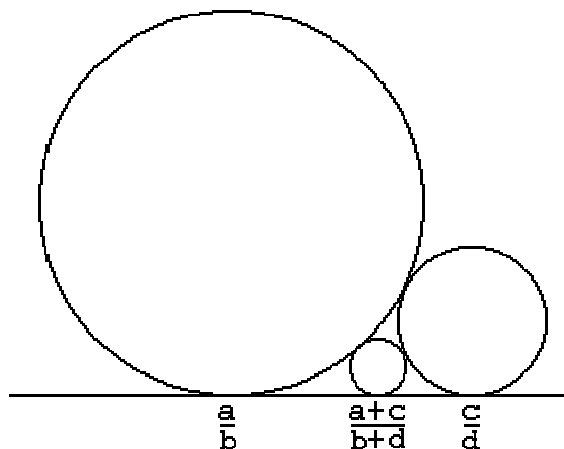

To further this geometric representation of fractions, we will plot points on a line that is the x-axis on an '*x, y*' coordinate plane. Here $x=\frac{a}{b}$ where *a* and *b* are integers and the fraction is in its lowest terms. Given $x=\frac{a}{b}$, we construct a circle with a *radius equal*

to $\dfrac{1}{(2b^2)}$. Now we have our circle at $\left(\dfrac{a}{b}\right), \left(\dfrac{1}{(2b^2)}\right)$, tangent to the x-axis and found in quadrant I on the coordinate plane.

For example:

	$x = \left(\dfrac{a}{b}\right)$	$y = \dfrac{1}{(2b^2)}$
Circle L	$\dfrac{1}{2}$	$\dfrac{1}{8}$
Circle M	$\dfrac{2}{3}$	$\dfrac{1}{18}$
Circle N	$\dfrac{3}{4}$	$\dfrac{1}{32}$
Circle O	$\dfrac{4}{3}$	$\dfrac{1}{18}$
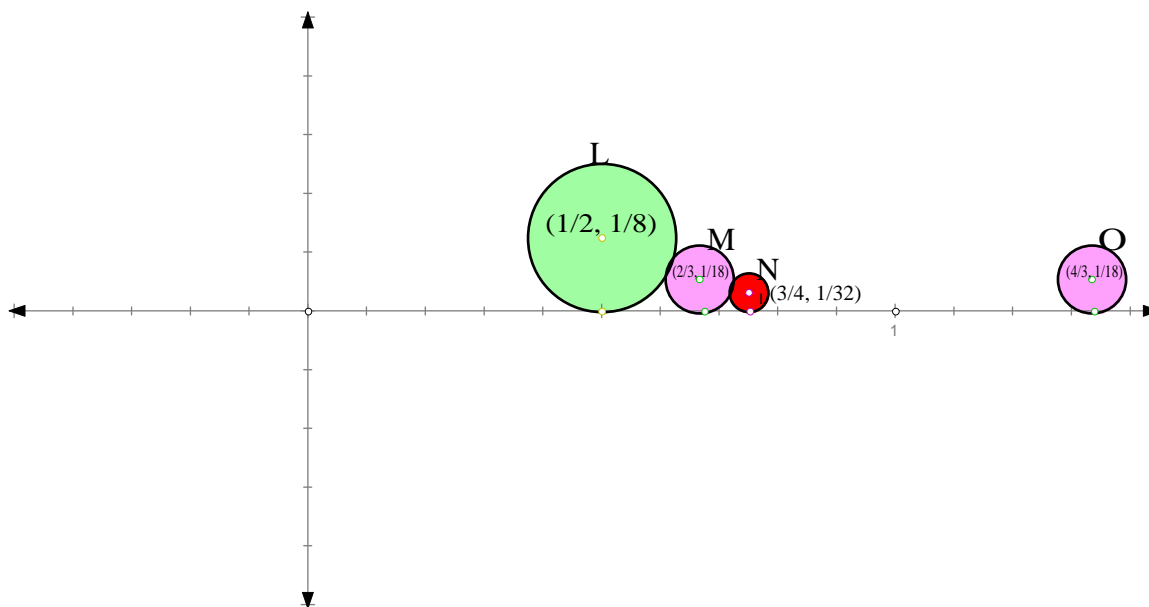
Plotting fractions as circles allowed Ford to state his first theorem related to these fractions:

> THEOREM 1. The representative circles of two distinct fractions are either tangent or wholly external to one another (Ford, 1938).

To prove this theorem we will be looking at the distance between the centers of the two circles formed by the fractions $\dfrac{a}{b}$ and $\dfrac{c}{d}$ (both in lowest terms).
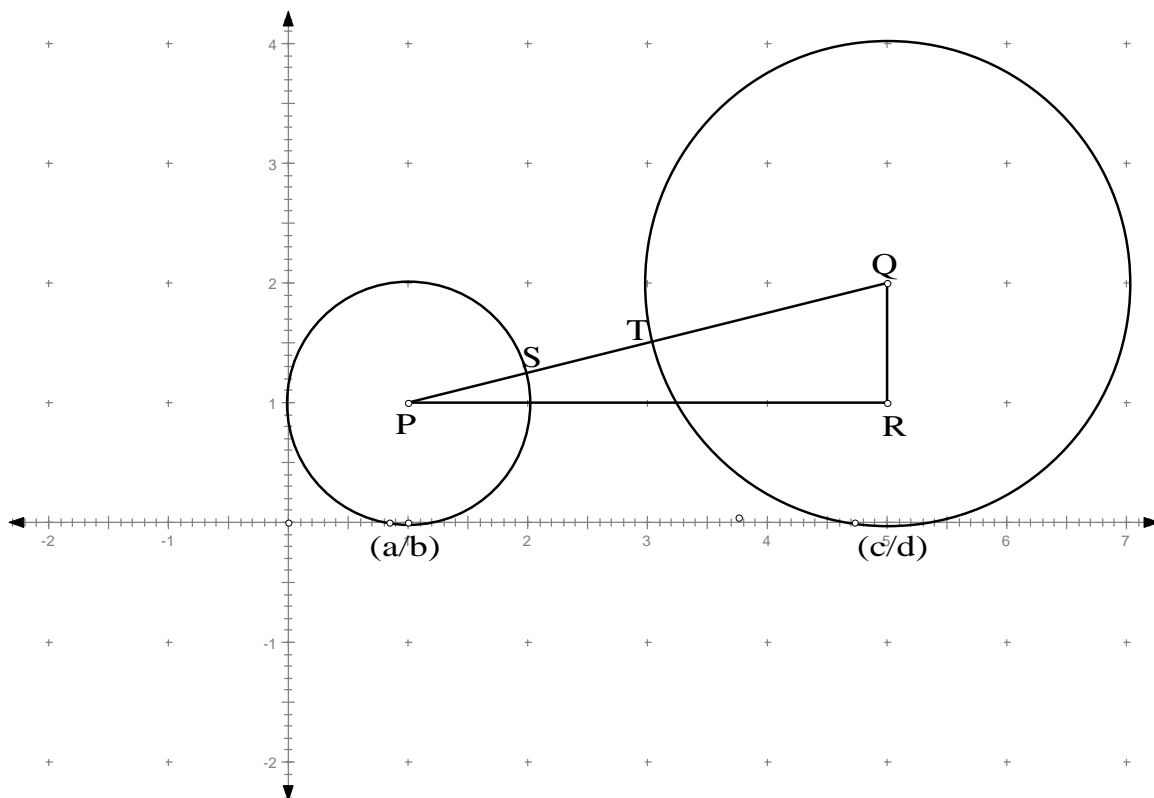
The distance between these circles' radii will create a line ($\overline{PQ}$) from $\dfrac{1}{(2b^2)}$ to $\dfrac{1}{(2d^2)}$.

The distance between the circles, $\left|\left(\dfrac{c}{d}\right)-\left(\dfrac{a}{b}\right)\right|$, will create another line ($\overline{PR}$) parallel to the *x-axis*. The two lines $\overline{PQ}$ and $\overline{PR}$ can be made into a right triangle with a vertical line ($\overline{QR}$) that has a distance of $\left|\left(\dfrac{1}{(2d^2)} - \dfrac{1}{(2b^2)}\right)\right|$.

Using the Pythagorean Theorem ($a^2 + b^2 = c^2$), we can then state that:

$$\overline{PQ}^2 = \overline{PR}^2 + \overline{QR}^2$$

$$\overline{PQ}^2 = \left(\frac{c}{d} - \frac{a}{b}\right)^2 + \left(\frac{1}{2d^2} - \frac{1}{2b^2}\right)^2$$

$$\overline{PQ}^2 = \left(\frac{1}{2d^2} + \frac{1}{2b^2}\right)^2 + \frac{(cb - ad)^2 - 1}{d^2 b^2}$$

$$\overline{PQ}^2 = \left(\overline{PS} + \overline{TQ}\right)^2 + \frac{(cb - ad)^2 - 1}{d^2 b^2}$$

With this equation, we can say that if |bc-ad| > 1, then $\overline{PQ} > \overline{PS} + \overline{TQ}$, and the two

circles are external to one another. If |bc-ad| = 1, then $\overline{PQ} = \overline{PS} + \overline{TQ}$, and the two circles

are tangent. If, however, |bc-ad| < 1, then the fractions are not different fractions; |bc-ad|

< 1 is not possible.

When |bc-ad| = 1, then $\overline{PQ} = \overline{PS} + \overline{TQ}$, and the two circles are tangent, we can

then look at the relationship between the two tangent circles to construct a smaller circle,

tangent to both original circles, that lies on the x axis.

SPECIFIC FORD CIRCLES

Given the following picture of two tangent circles, we can now look for a relationship between the larger circle $\left(\dfrac{1}{1},\dfrac{1}{2}\right)$ and the small circle $\left(\dfrac{1}{2},\dfrac{1}{8}\right)$ in the diagram:
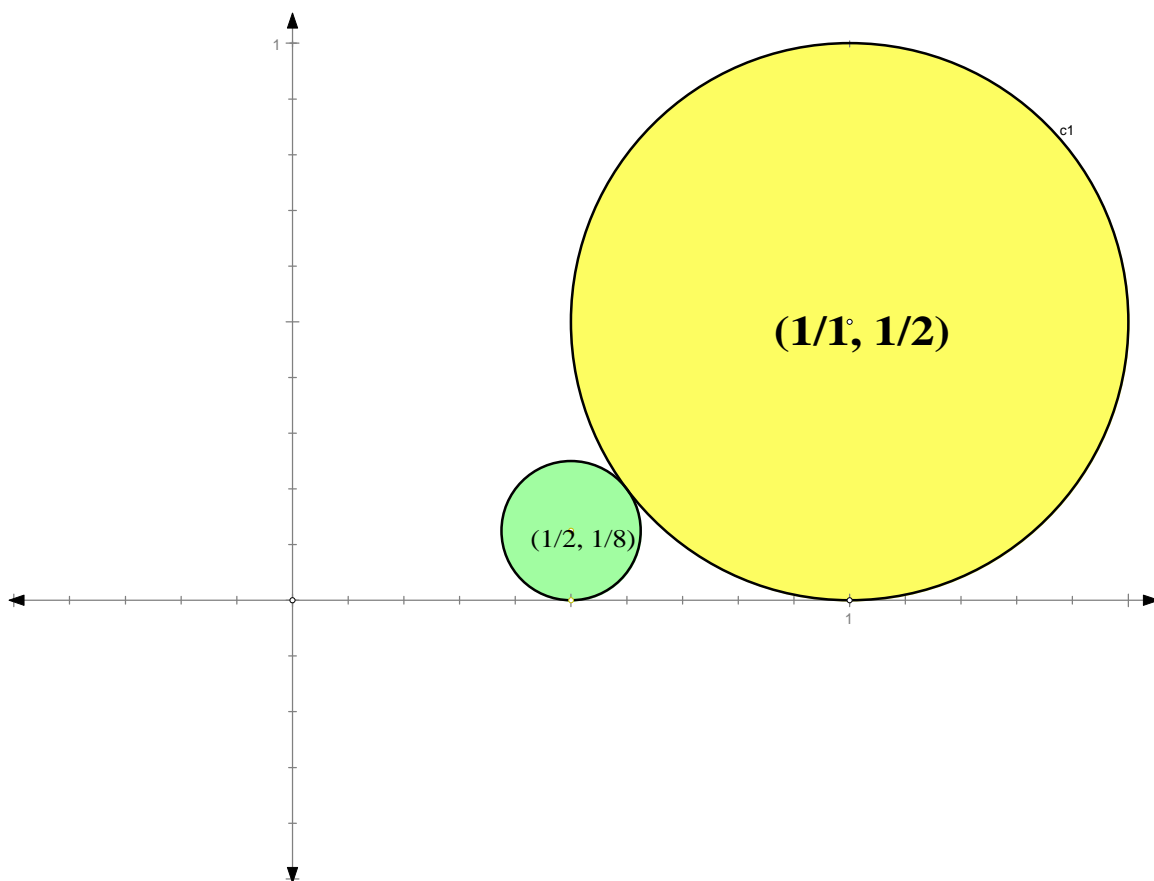
The radius of the larger circle $\left(\dfrac{1}{2}\right)$ is four times the radius of the small circle $\left(\dfrac{1}{8}\right)$. Knowing that the smaller radius is $\dfrac{1}{4}$ that of the larger radius, we might wonder if the ratio of other tangent circles' radii will also be $\dfrac{1}{4}$. To see, we'll find another fraction circle tangent to the existing circles.

We can find a smaller circle, tangent to both original circles, looking at the x-axis values: $\dfrac{1}{2}$ and $\dfrac{1}{1}$. By finding the *mediant,* a new fraction in between the existing radii, $\dfrac{(1+1)}{(2+1)}$, we are given $\dfrac{2}{3}$ as the x-axis point for our new circle.   To find the y value of our new circle, we use the formula $\dfrac{1}{(2b^2)}$ and see that the new circle's radius will be at $\dfrac{1}{2(3^2)}$, or $\dfrac{1}{18}$.

(1/1, 1/2)

(1/2, 1/8)

(2/3, 1/18)

c1

1
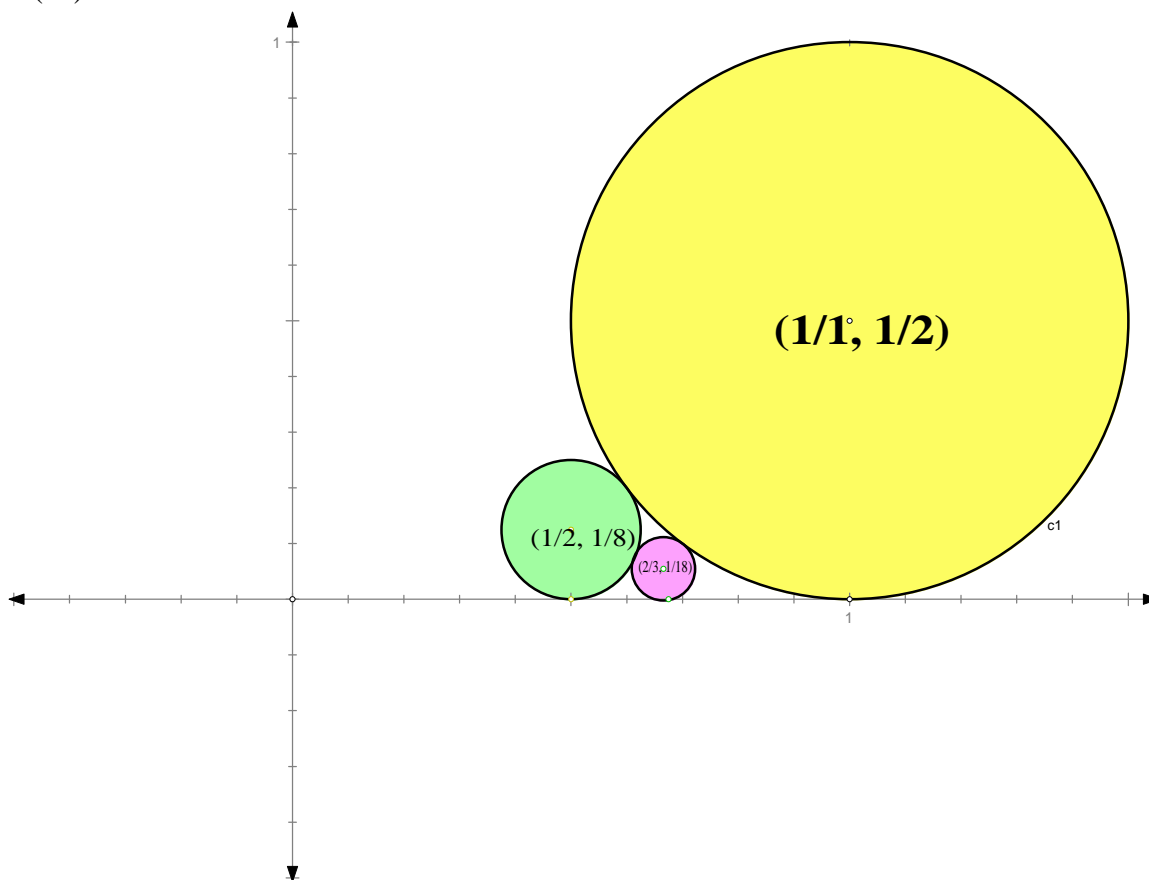
With our three circles, we can look to see if the new circle's radius is $\dfrac{1}{4}$ the value either of the existing circles. The new circle, at $\left(\dfrac{2}{3}, \dfrac{1}{18}\right)$, has a radius that is $\dfrac{1}{9}$ of the largest circle and is $\dfrac{4}{9}$ the radius of the second circle. Therefore, $\dfrac{1}{4}$ is not a constant ratio.

By using finding the mediant fractions for the next few terms, we create new adjacent fractions, and we can try to find a pattern between the ratios of the circles. This table shows the circles created by the adjacent fractions and compares the ratio of their radii:

Term	x, y $\left(\dfrac{a}{b}, \dfrac{1}{2b^2}\right)$ of New Circle	Large Circle Radius	New Circle Radius	Ratio of Radii
1	$\dfrac{1}{1}, \dfrac{1}{2}$	$\dfrac{1}{2}$	$\dfrac{1}{2}$	$\dfrac{1}{1}$
2	$\dfrac{1}{2}, \dfrac{1}{8}$	$\dfrac{1}{2}$	$\dfrac{1}{8}$	$\dfrac{4}{1}$
3	$\dfrac{2}{3}, \dfrac{1}{18}$	$\dfrac{1}{2}$	$\dfrac{1}{18}$	$\dfrac{9}{1}$
4	$\dfrac{3}{4}, \dfrac{1}{32}$	$\dfrac{1}{2}$	$\dfrac{1}{32}$	$\dfrac{16}{1}$
5	$\dfrac{4}{5}, \dfrac{1}{50}$	$\dfrac{1}{2}$	$\dfrac{1}{50}$	$\dfrac{25}{1}$
n	$\dfrac{n-1}{n}, \dfrac{1}{2(n^2)}$	$\dfrac{1}{2}$	$\dfrac{1}{2(n^2)}$	$\dfrac{n^2}{1}$

In fact, by looking at a table that shows the radii of mediant fractions staring with circles at $\left(\dfrac{1}{1}, \dfrac{1}{2}\right)$ and $\left(\dfrac{1}{2}, \dfrac{1}{8}\right)$, we do begin to see a pattern in the ratio of their radii:

$$\frac{1}{2} \div \frac{1}{8} = 4, \ \frac{1}{2} \div \frac{1}{18} = 9, \ \frac{1}{2} \div \frac{1}{32} = 16, \ \frac{1}{2} \div \frac{1}{50} = 25.$$

The ratio of the largest circle's radius divided by the newest circle's radius is always a perfect square. In setting up a table that labels the circles as terms, we can see that the consecutive squares are related to the term (or number of the new circle). Given the 'n[th]' circle, the ratio of the original large circle that circle, compared to the ratio of the 'n[th]' circle will be $\dfrac{n^2}{1}$.

We can even write a formula for plotting this 'nth' circle, knowing that it will be located at $\left(\dfrac{n-1}{n}, \dfrac{1}{2(n^2)} \right)$.
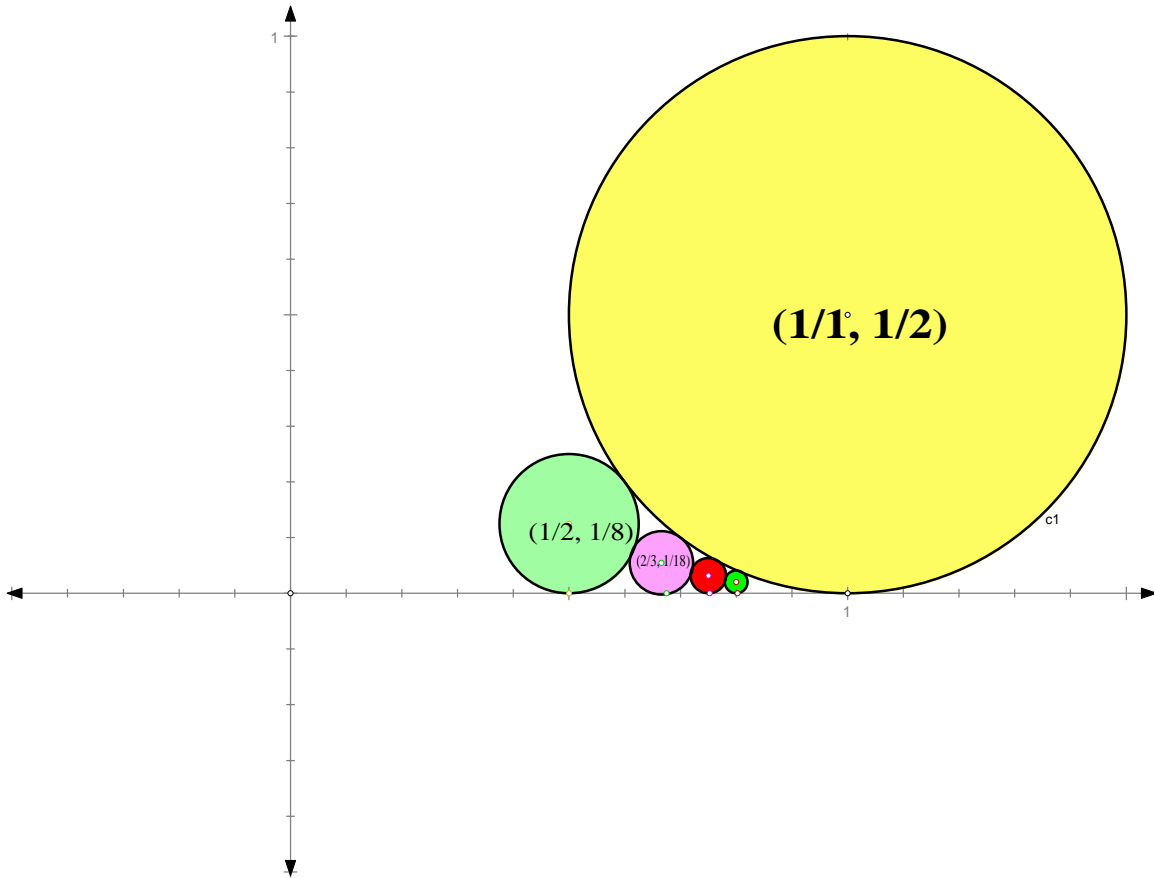
There is a Ford circle associated with every rational number. In addition, the line y=1 is considered a Ford circle (as it can be on the coordinate plane at 1, 0, or thought of as $\dfrac{1}{0}$, associated with infinity).

L.R. Ford was able to take fractions, sometimes considered part of number sense or arithmetic, and create a geometric presentation of how fractions and their radii are related. Ford Circles help to visually represent the concept of mediant and the patterns associated with Farey fractions.

Connections

Pick's Theorem as Related to Euler's Formula

Pick's Theorem also has other properties as well. Pick's formula is a two-dimensional equivalent to Euler's formula. Recall Euler's Formula as *V-E+F=2*. It is important to note that Pick's Theorem as stated above is only valid for simple polygons. For example, polygons similar to the ones drawn above that consist of a single piece and do not contain "holes." According to W.W. Funkenbusch (1974), the derivation of Pick's formula is seen as the following:

> *Let the polygon have lattice points for vertices, and let Euler's Formula be applied to the related connected planar graph. We then obtain:*

$$V=I + B$$
$$E=3I +2B -3$$
$$\frac{F-1}{2} = area\, of\; polygon$$

> *which when substituted into Euler's Formula gives Pick's Formula.*

The equation V= I + B is the sum of the interior points and all boundary points. The equation E= 3I +2B is generated by the fact that when moving from Pick to Euler, the interior points are three times (3I) the original number when you have any interior point and connect an edge to it. This creates a triangulation effect. Likewise, as you connect the interior lattice points to the exterior vertices, you create twice the number of edges. Therefore, you now have (2B) edges. See below for an example of this effect.
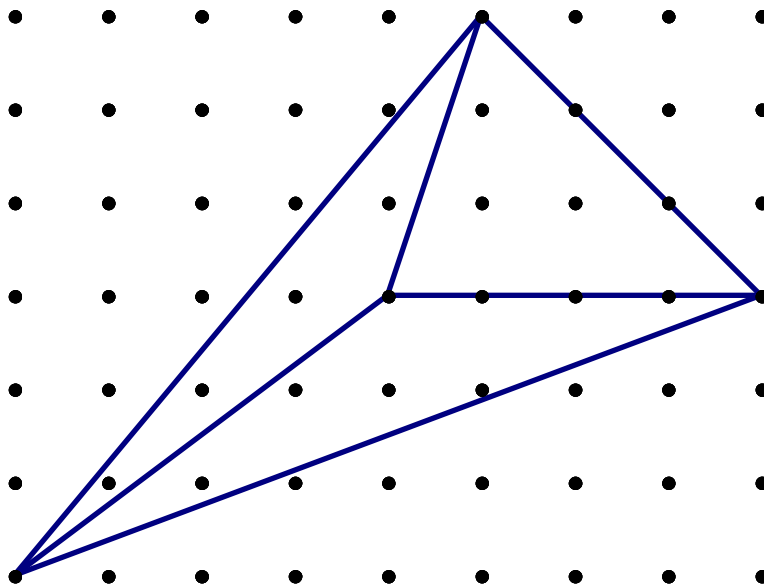
This simplified example shows why when you connect to interior lattice points from exterior vertices, the number of edges is doubled. Similarly, the number of interior lattice points is now tripled. For proof of this we will substitute the above components developed by Funkenbusch for Euler and Pick.

$$To\ solve\ for\ F: \quad \frac{F-1}{2} = area\ of\ polygon$$

$$2\left(\frac{F-1}{2}\right) = A(2)$$

$$F - 1 = 2A$$

$$\underline{-1 \qquad -1}$$

$$*F = 2A + 1$$

$$*V - E + F = 2$$

$$*By\ substitution, we\ now\ generate\ a\ new\ equation:$$

$$I + B - (3I + 2B - 3) + 2A + 1 = 2$$

$$I + B - 3I - 2B + 3 + 2A + 1 = 2$$

$$-2I - B + 4 + 2A = 2$$

$$-2I - B + 2 + 2A = 0$$

$$\frac{-2I - B + 2 = -2A}{-2}$$

$$I + \frac{B}{2} - 1 = A$$

This proof shows the relation of the faces, edges and vertices of Euler's Formula, and the interior and boundary lattice points of Pick's Theorem.

Pick's Theorem and the Farey Series

There exists another relationship between Pick's Theorem and a mathematical idea the Farey Series. The Farey Series F_n of order n a is the ascending series of irreducible fractions $\frac{m}{n}$ between 0 and 1 whose denominators do not exceed N. A fraction $\frac{m}{n}$ belongs to F_N if and only if: $0 \le m \le n \le N$, $\gcd(m,n) = 1$
The relationship between Pick's Theorem and the Farey Sequence is simple: when you plot two consecutive pairs of fractions from a Farey Series on a grid, using the denominator and numerator as an ordered pair, (m,n) and connect to the origin point (0,0), the resulting area will always be $\left(\frac{1}{2}\right)$. The area is always $\left(\frac{1}{2}\right)$ because the points

plotted will never contain interior lattice points. Therefore, the equation created will always be:

$$0(I) + \frac{3}{2}(B) = .5$$

This can be used as an alternative proof of the connection between Pick's Theorem and the Farey Sequence. Several examples are depicted below.
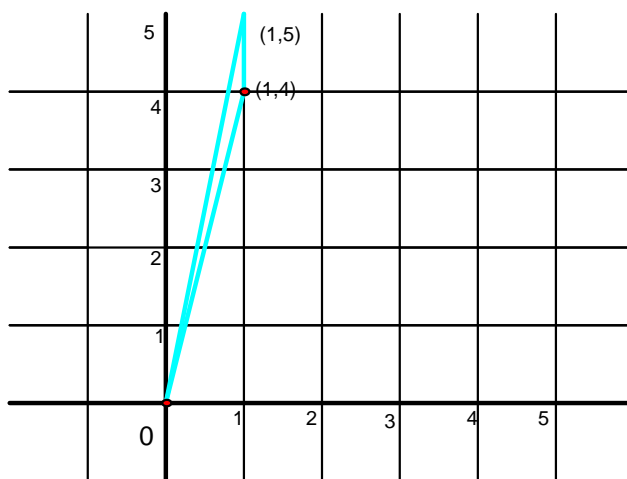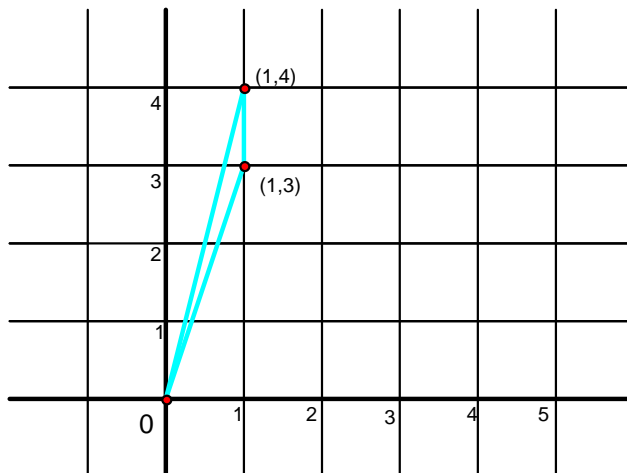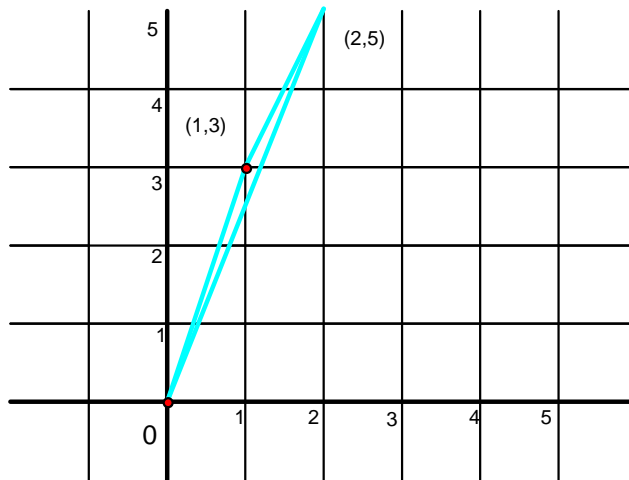
Farey Fractions and Ford Circles

Given two Ford circles C_1 and C_2 with centers consecutive Farey fractions, the circles are tangent to each other. In order for the two circles to be tangent, we need to show that the sum Since the centers are consecutive Farey fractions with $\dfrac{c}{d} > \dfrac{a}{b}$ and $bc - ad = 1$ The center of C_1 is at $\dfrac{a}{b}$ with a radius of $\dfrac{1}{2b^2}$. The center of C_2 is $\dfrac{c}{d}$ with a radius of $\dfrac{1}{2d^2}$. It can be shown that C_1 is tangent to C_2.
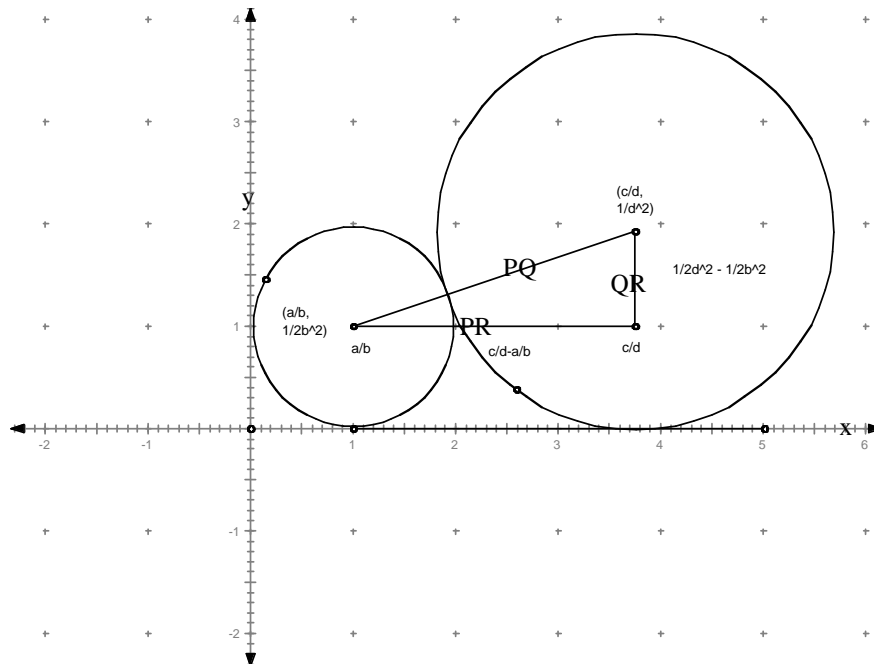
The Pythagorean Theorem can be used to show the length p is equal to the sum of the radius of C_1 and C_2.

$$\left(\frac{c}{d} - \frac{a}{b}\right)^2 + \left(\frac{1}{2d^2} - \frac{1}{2b^2}\right)^2 = \left(\frac{1}{2d^2} + \frac{1}{2b^2}\right)^2$$

$$\frac{c^2}{d^2} - \frac{2ac}{bd} + \frac{a^2}{b^2} + \frac{1}{4d^4} - \frac{2}{4b^2d^2} + \frac{1}{4b^4} = \frac{1}{4d^4} + \frac{2}{4b^2\,d^2} + \frac{1}{4b^4}$$

After some simplification, this translates to

$$\frac{c^2}{d^2} - \frac{2ac}{bd} + \frac{a^2}{b^2} = \frac{4}{4b^2d^2}$$

To work with a simpler problem, the fractions can be eliminated by multiplying by b^2d^2

$$a^2d^2 - 2abcd + b^2c^2 = 1$$

The trinomial on the left hand side can be factored as a binomial square.

$$(ad - bc)^2 = 1$$

Since $\frac{a}{b}$ and $\frac{c}{d}$ are consecutive Farey fractions then $ad - bc = 1$. So Ford circles with centers indicated by consecutive Farey fractions are tangent.

Summary

Through our research we found references to Farey fractions, Pick's theorem and Ford's circles in all kinds of situations. In fractal and chaos theory mathematics for example, Farey fractions are even used in designing stereo equipment. The impression made by this research question emphasized the interconnectedness of mathematical occurrences. The diagram below shows the position of Farey fractions in the Mandelbrot Set.
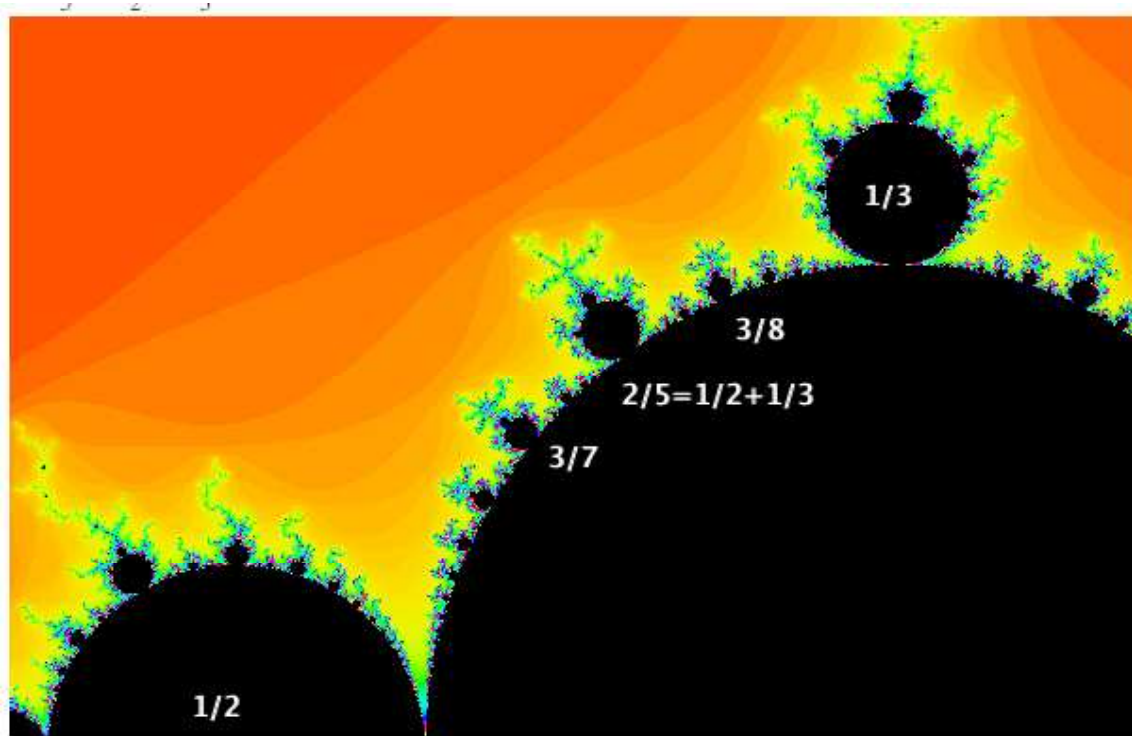
In our classrooms, we believe these topics can be used to connect the ideas from the visual nature of geometry to the abstract nature of algebra. The use of Pick's Theorem takes this idea at an elementary level, and moves towards higher-level mathematical reasoning through the use of Farey Sequences and Ford Circles. In many curriculums, topics are explored in isolation. Through our research, we have discovered the many connections between concepts. These connections are important to the deep understand of mathematics we hope our students will achieve,

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