1975

AMPÈRE'S LAW

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INTRODUCTION

Everyone has seen a bar magnet in the form of a compass or a door catch. Anyone who has ever casually played with magnets or magnetic toys knows that magnets interact with other magnets; i.e., a magnet experiences a force caused by the presence of an external magnetic field produced by the other magnet. A wire carrying a current experiences a force caused by the presence of a nearby magnet (as you saw in the module Magnetic Forces). We then expect the converse to also hold true, i.e., that the bar magnet will also experience a force from the presence of the current-carrying wire. This expectation can be verified experimentally by putting a compass needle near a current-carrying wire.

Thus both a bar magnet and a current-carrying wire produce a magnetic field. A bar magnet, however, cannot be broken down into a single magnetic pole similar to the electric charge. Even on the atomic scale, there are always two magnetic poles similar to the two magnetic poles produced by a small loop of current-carrying wire. A bar magnet is really just a collection of atomic current loops or charges in motion. In this module (and the module Magnetic Forces), you are actually investigating the interaction at a distance of moving electric charges (e.g., electric currents). The intermediary in this interaction is the magnetic field $\vec{B}(r)$. By introducing $\vec{B}$, you separate the interaction into two parts: (1) creation of the $\vec{B}$ field by given currents (to be treated in this module), and (2) the action of this field on other given currents or moving charges (which you studied in Magnetic Forces).

Along beside the other memorable force relations involving magnetic fields, you will now add to your collection of fond memories the field relations known as Ampère's Law and the Biot-Savart Law. (Please note: The latter is pronounced Bee-oh Sah-var´.)

PREREQUISITES

Before you begin this module, you should be able to:	Location of Prerequisite Content
*Integrate simple polynomials (needed for Objectives 1 through 3 of this module)	Calculus Review
*Evaluate cross products (needed for Objective 3 of this module)	Vector Multiplication Module
*Define the magnetic field, and find the force on a current-carrying wire in a magnetic field (needed for Objectives 1 through 3 of this module)	Magnetic Forces Module

LEARNING OBJECTIVES

After you have mastered the content of this module, you will be able to:

1. Ampère's law - Write Ampère's law and use it to calculate the magnitude and direction of the magnetic field $\vec{B}$ caused by currents flowing in a conductor of cylindrical cross-sectional area with a simple symmetric shape, such as a long, straight wire, a solenoid, or a toroid, or a combination of these (principle of superposition).

2. Forces between currents - Given the currents in parallel conductors, solve for the force on one of the conductors.

3. Biot-Savart law - Write the Biot-Savart law and employ it to find the magnitude and direction of the magnetic field $d\vec{B}$ at a point P_1 caused by a current element at another point P_2; and/or find the magnetic field $\vec{B}$ at the center of a circular or semicircular loop of current-carrying wire.

TEXT: Frederick J. Bueche, <u>Introduction to Physics for Scientists and Engineers</u> (McGraw-Hill, New York, 1975), second edition

SUGGESTED STUDY PROCEDURE

Your readings are from Chapter 24. First read General Comments 1 and 2 and Sections 24.1 and 24.2. Then study Problem A before working Problem F in this study guide and Problem 2 in Chapter 24 of your text. Now read General Comment 3 and Section 24.2. Study Illustration 24.2 and Problem C before working Problems H and 15. Then go back to Objective 1 and read General Comment 4 and Sections 24.3 and 24.4. Study Problem B and Illustrations 24.1 and 24.4 before working Problem G and Problems 7 and 17 in Chapter 24. For Objective 3, read General Comments 5 and 6, Sections 24.5, 24.6, 24.8; study Illustrations 24.5 and 24.6 and Problems D and E; and work Problems I and J.

Try the Practice Test, and work some of the Additional Problems if necessary, before attempting a Mastery Test.

BUECHE

Objective Number	Readings	Problems with Solutions		Assigned Problems		Additional Problems
		Study Guide	Text	Study Guide	Text (Chap. 24)	(Chap. 24)
1	Secs. 24.1, 24.2, 24.3, 24.4, General Comments 1, 2, 4	A, B	Illus.[a] 24.1, 24.4	F, G	2, 7, 17	1, 3, 4, 5, 6, 8
2	Sec. 24.2, General Comment 3	C	Illus. 24.2	H	15	
3	Secs. 24.5, 24.6, 24.8, General Comments 5, 6	D, E	Illus. 24.5, 24.6	I, J		

[a] Illus. = Illustration(s).

TEXT: David Halliday and Robert Resnick, Fundamentals of Physics (Wiley, New York, 1970; revised printing, 1974)

SUGGESTED STUDY PROCEDURE

Your readings are all from Chapter 30. First read General Comments 1 and 2 and Sections 30-1 and 30-2. Then study Problem A and work Problem F in this study guide, and Problems 1 and 23 in Chapter 30. Next read General Comment 3 and Sections 30-3 and 30-4, study Problem C, and work Problems H and 18. Return to Objective 1, reading General Comment 4 and Sections 30-3 and 30-5, studying Problem B and Examples 1 through 4 in Chapter 30, before working Problems G and 27. Then read General Comments 5 and 6, and Section 30-6, excluding Example 5. Study Problems D and E before working Problems I, J, and 33.

Take the Practice Test, and work some Additional Problems if necessary, before attempting a Mastery Test.

HALLIDAY AND RESNICK

Objective Number	Readings	Problems with Solutions		Assigned Problems		Additional Problems
		Study Guide	Text	Study Guide	Text (Chap. 30)	(Chap. 30)
1	Secs. 30-1, 30-2, 30-3, 30-5, General Comments 1, 2, 4	A, B	Ex.[a] 1, 2, 3, 4	F, G	1, 23, 27	4, 5, 15, 19, 21, 22, 23
2	Secs. 30-3, 30-4, General Comment 3	C		H	18	16, 17, 24, 25
3	Sec. 30-6, General Comments 5, 6	D, E		I, J	33	29, 30, 34, 35(a)

[a]Ex. = Example(s).

TEXT: Francis Weston Sears and Mark W. Zemansky, University Physics (Addison-Wesley, Reading, Mass., 1970), fourth edition

SUGGESTED STUDY PROCEDURE

The learning objectives cover the material in the reverse order of your text. You may want to skim Chapter 32 before following this study procedure in detail. (Note: Your text changes notation from $4\pi k'$ to μ_0 in Section 32-6.)

For Objective 1, read General Comments 1 and 2 and Section 32-6. Then study Problem A and work Problems F and 32-3. Next read General Comment 3 and Section 32-4, study Problem C, and work Problems H and 32-11. Return to Objective 1, reading General Comment 4, and Section 32-7, studying Problem B, and working Problems G and 32-17. Next master Objective 3 by reading General Comments 5 and 6 and Section 32-2, studying Problems D and E, and working Problems I, J, 32-1, and 32-15.

Take the Practice Test, and work some Additional Problems if necessary, before attempting a Mastery Test.

SEARS AND ZEMANSKY

Objective Number	Readings	Problems with Solutions	Assigned Problems		Additional Problems
		Study Guide	Study Guide	Text	
1	Secs. 32-6, 32-7, General Comments 1, 2, 4	A, B	F, G	32-3, 32-17	32-2, 32-4, 32-6, 32-18, 32-19
2	Sec. 32-4, General Comment 3	C	H	32-11	32-7, 32-9, 32-10
3	Sec. 32-2, General Comments 5, 6	D, E	I, J	32-1, 32-15	

TEXT: Richard T. Weidner and Robert L. Sells, Elementary Classical Physics (Allyn and Bacon, Boston, 1973), second edition, Vol. 2

SUGGESTED STUDY PROCEDURE

The learning objectives cover the material in the reverse order of your text. You may want to skim through Chapter 30 and then come back and read the particular sections of the text as listed below in the table.

For Objective 1, read General Comments 1 and 2 and Section 30-7. Then study Problem A and work Problems F and 30-5. For Objective 2, read General Comment 3 and Section 30-5, study Problem C and Example 30-2, and work Problems H and 30-13. Return to Objective 1, reading General Comment 4 and Section 30-8, studying Problem B and Example 30-3, and working Problems G and 30-22(a), 30-23. Next read General Comments 5 and 6 and Sections 30-1 through 30-3 (pp. 604 and 605 only), for Objective 3. Study Problems D and E before working Problems I, J, and 30-6.

Take the Practice Test, and work some Additional Problems if necessary, before attempting a Mastery Test.

WEIDNER AND SELLS

Objective Number	Readings	Problems with Solutions: Study Guide	Problems with Solutions: Text	Assigned Problems: Study Guide	Assigned Problems: Text	Additional Problems
1	Secs. 30-7, 30-8, General Comments 1, 2, 4	A, B	Ex.[a] 30-3	F, G	30-5, 30-22(a), 30-23	30-1, 30-16, 30-17, 30-19, 30-20, 30-21
2	Sec. 30-5, General Comment 3	C	Ex. 30-2	H	30-13	30-3, 30-11, 30-14, 30-15
3	Secs. 30-1, 30-2, 30-3, General Comments 5, 6	D, E		I, J	30-6	30-4

[a]Ex. = Example(s).

GENERAL COMMENTS

1. Comparison of Ampère's Law and the Biot-Savart Law

The examples of the long wire, long solenoid, and toroid make very good illustrations of the use of Ampère's law. If you learn Ampère's law well, you will not need to memorize the expressions for $\vec{B}$ in these three cases (which are of some importance in themselves) - you will always be able to derive them! (Note that the long solenoid is the limiting case of a toroid as the radius approaches infinity.)

Ampère's law is completely general for steady currents, and can be used to calculate the magnetic field in certain symmetric situations in much the same way that Gauss' law can be used in electrostatics. However, many cases (e.g., the circular loop) cannot be handled with Ampère's law, since one must have a sufficiently simple situation so that $\vec{B}$ is constant and can be removed from the integral sign. For the circular loop this becomes very impractical. Therefore, we go to the Biot-Savart law for a more general approach. The Biot-Savart law, as stated, represents the magnetic field in terms of contributions from infinitesimal elements composing the current circuit. Ampère's law is still true in these cases - it just does not give any calculational help.

Note that Ampère's law and the Biot-Savart law are not independent of one another. Ampère's law can be derived from the Biot-Savart law and therefore does not contain more information than the Biot-Savart law. The Biot-Savart law is just a more general statement.

2. Straight Wires and Paths

The main thrust of Objective 1 is to familiarize oneself with Ampère's law:

$$\oint \vec{B} \cdot d\vec{\ell} = \mu_0 I.$$

On the right side of this equation, I is the net current passing through the area enclosed by the path of integration. In Figure 1 the net current through the area bounded by the curve C is $I_1 + I_2 + I_3$. The positive direction of the current through the area can be obtained by a right-hand rule - the fingers of the right hand curling in the direction of $d\vec{\ell}$ around the curve C and the thumb pointing in the positive direction of the current:

$$\oint \vec{B} \cdot d\vec{\ell} = \mu_0(I_1 + I_2 + I_3).$$

For an area pierced by no current, the line integral around the closed path is zero.

How do we know that the $\vec{B}$ field from a current in a long straight wire is circular around the wire? Since the wire looks the same from one side as from another, the magnitude of $\vec{B}$ at a point near the wire should remain unchanged as the wire is rotated. Thus $|\vec{B}|$ is constant on a circle of radius r centered on the axis of the wire. But what about the direction of $\vec{B}$? If we take a compass and move it around the circle of radius r, the needle will always be tangent to the circle. If the current in the wire is reversed, the needle will reverse end for end. The direction of $\vec{B}$ is taken to be the direction of the compass, and thus we have the right-hand rule. The Biot-Savart law may also be used to find the $\vec{B}$ field from a current-carrying wire and gives the same result.

To utilize Ampère's law to determine $\vec{B}$ for a given situation, we must have a high degree of symmetry just as we did to determine $\vec{E}$ from Gauss' law for the electric field,

$$\int \vec{E} \cdot d\vec{A} = q/\varepsilon_0.$$

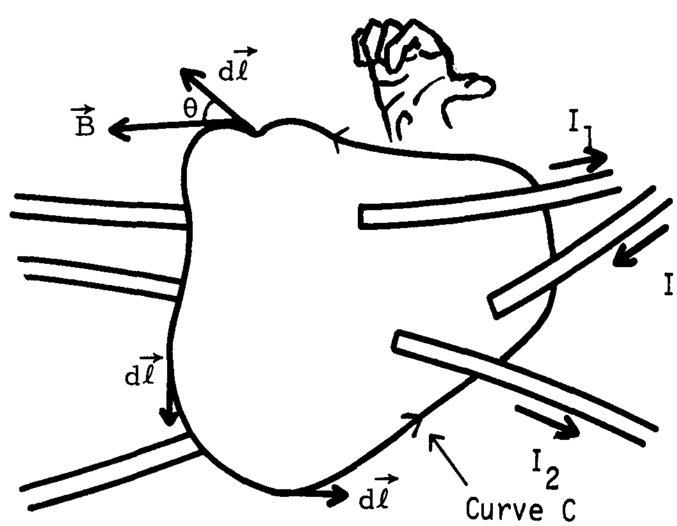


Figure 1

3. <u>Forces</u>

This comment treats problem solving for Objective 2. From the module Magnetic Forces, we know that a current-carrying wire in a magnetic field experiences a force, $\vec{F} = I\vec{\ell} \times \vec{B}$. To find the force, therefore, that one wire carrying a current I exerts on another parallel wire a distance d away and carrying current I_2, we find the $\vec{B}$ field of wire 1 at wire 2:

$$|\vec{B}_1| = \mu_0 I_1/2\pi d.$$

Then use this $\vec{B}$ field to find the force on wire 2:

$$\vec{F}_2 = I_2\vec{\ell} \times \vec{B}_1.$$

Note: This applies to parallel wires only.

4. Uniform Distribution of Current

To find the $\vec{B}$ field at a point inside a cylindrical wire as shown in Figure 2 that carries a current distributed uniformly over the cross section of the wire, we can use Ampère's law for a circular path about the axis of the wire:

$$\oint \vec{B} \cdot d\vec{\ell} = \mu_0 i,$$

where i is the net current inside our circular path of integration (see Figure 2). A total current I is distributed uniformly over the cross section of the wire and into the page. The direction of $\vec{B}$ is determined by the right-hand rule.

Since $d\vec{\ell}$ and $\vec{B}$ are in the same direction all around the dotted circle, $\vec{B} \cdot d\vec{\ell} = B\, d\ell$, and symmetry suggests that $|\vec{B}|$ is constant along the circular path so that

$$\oint \vec{B} \cdot d\vec{\ell} = \oint B\, d\ell = B2\pi r.$$

Thus

$$B = \mu_0 i/2\pi r.$$

But i is the current inside the dotted circle:

$$i = I(\pi r^2/\pi R^2) = Ir^2/R^2.$$

Therefore,

$$|\vec{B}| = \mu_0 Ir/2\pi R^2.$$

See Figure 3. This reduces to the expected expression for $|\vec{B}|$ at r = R.

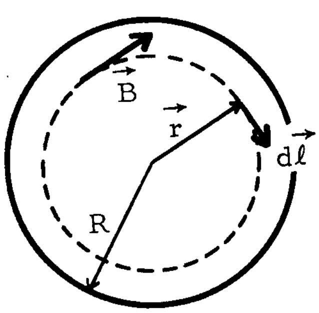


Figure 2

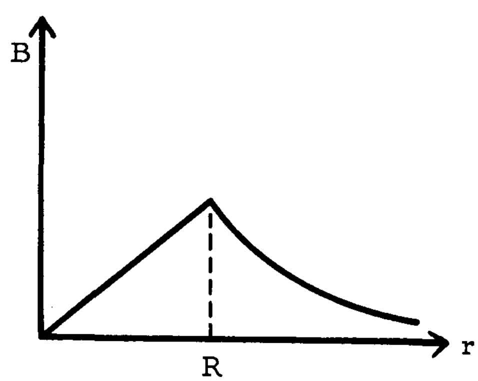


Figure 3

5. Current Elements

In electrostatics the fundamental law telling us how to calculate the electric field caused by an element of charge (a point charge) is Coulomb's law. The corresponding law for magnetic fields is called the Biot-Savart Law. The fundamental source of the magnetic field is a current element, which we can think of as a small length of wire $d\ell$ carrying a current I in the direction of $d\vec{\ell}$. The Biot-Savart law says that the magnetic field $d\vec{B}$ at a distance $\vec{r}$ caused by this incremental length is given by

$$d\vec{B} = \frac{\mu_0 I}{4\pi}\,\frac{d\vec{\ell} \times \hat{r}}{r^2} = \frac{\mu_0 I}{4\pi}\,\frac{d\vec{\ell} \times \vec{r}}{r^3},$$

where μ_0 is a constant ($\mu_0 = 4\pi \times 10^{-7}$ if the units for B, I, and r are teslas, amperes, and meters, respectively), and $\hat{r}$ is a unit vector in the direction of $\vec{r}$. See Figure 4.

Notice that the Biot-Savart law is an inverse-square law. It is more complicated than Coulomb's and the gravitational laws, however, because of the cross product: $d\vec{B}$ points not along $\vec{r}$ but in a direction perpendicular to both $\vec{r}$ and $d\vec{\ell}$.

Observe that the $d\vec{\ell}$ that occurs in Ampère's law has quite a different significance from the $d\vec{\ell}$ in the Biot-Savart law. In Ampère's law, it is along an imaginary path that need bear no relation to the direction of current flow, whereas in the Biot-Savart law it is along the direction of the current flow in the wire.

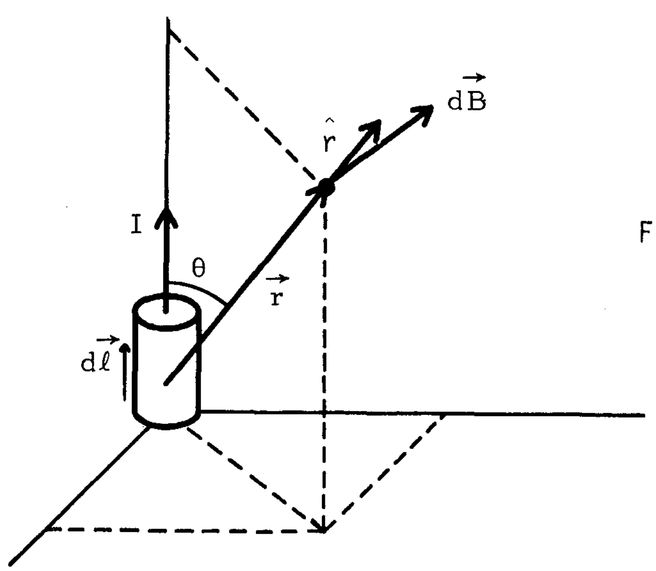


Figure 4

6. Semicircular Current Loops

The Biot-Savart law gives the magnetic field produced by an infinitesimal current, as we shall see in Problems D and I. However, no such isolated current element can exist by itself. There must always be a complete loop of current in any circuit.

The most general case of current-carrying wires involves a detailed integration of the Biot-Savart law, but there is a special case that requires only the interpretation of a line integral. This is the problem of finding the $\vec{B}$ field at the center of a circular or semicircular loop of current. Here, the current element $I\,d\vec{\ell}$ is always perpendicular to the position vector $\vec{r}$:

$I\,d\vec{\ell} \times \vec{r}$ = a vector whose magnitude is $Ir\,d\ell$, in a direction perpendicular to both $\vec{r}$ and $d\vec{\ell}$.

Since we are finding the field at the center of the loop, the distance r is a constant and equal to the radius a. Thus,

$$|d\vec{B}| = (\mu_0 I/4\pi a^2)d\ell.$$

In summing up or integrating the contributions from all the significant current elements, $\int d\ell$ is just the length of the semicircular loop.

PROBLEM SET WITH SOLUTIONS

A(1). Two long straight parallel wires both carry a current of 8.0 A, but in opposite directions as shown in Figure 5. Calculate the magnetic field at the indicated point P caused by these currents.

Solution

Divide this problem into two problems, and use Ampère's law in both. The $\vec{B}$ field caused by the lower wire is found by integrating $\vec{B}$ around a circle of radius 2.00 m that is concentric with the lower wire:

$$\oint \vec{B}_1 \cdot d\vec{\ell} = B_1 2\pi(2) = \mu_0(8), \qquad |\vec{B}_1| = 8.0 \times 10^{-7}\ \mathrm{T}.$$

According to the right-hand rule, $\vec{B}_1$ at the point of interest is directed out of the page.

To find $\vec{B}_2$, the contribution from the upper wire, we integrate $\vec{B}_2$ around a circle of 4.0 m radius that is concentric with the upper wire. Note that we include only the upper current element in the term for enclosed current, because we have divided this problem into two parts, each of which involves only one current-carrying wire. The result is

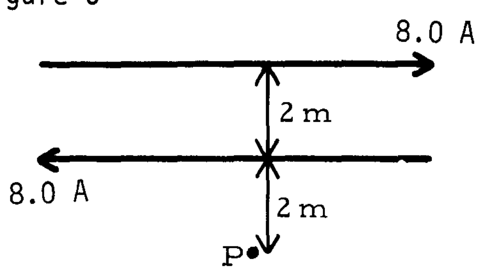
$$\oint \vec{B}_2 \cdot d\vec{\ell} = B_2 2\pi(4) = \mu_0(8), \qquad \vec{B}_2 = 4.0 \times 10^{-7}\ \text{T} \quad \underline{\text{into}} \text{ the page.}$$

Now to find the net field at the point indicated, we superimpose these two partial solutions (principle of superposition), obtaining

$$\vec{B}_{net} = 4.0 \times 10^{-7}\ \text{T} \quad \underline{\text{out}} \text{ of the page.}$$

Note: The presence of a second wire does not destroy the symmetry of the $\vec{B}$ field from the first wire. The separate $\vec{B}$ fields just add together vectorially.

Figure 5



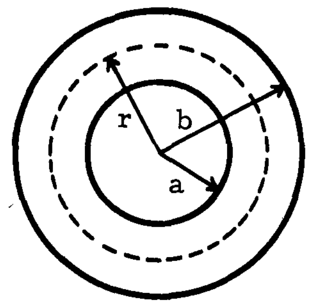


Figure 6

B(1). A hollow cylindrical conductor as shown in Figure 6 of radii a and b ($a < b$) carries a current I (out of the paper) uniformly spread over its cross section.
(a) Show that the magnetic field $\vec{B}$ for points inside the body of the conductor (that is, $a < r < b$) is given by

$$|\vec{B}| = [\mu_0 I/2\pi(b^2 - a^2)][(r^2 - a^2)/r].$$

Check this formula for the limiting case of $a = 0$, r.
(b) Make a rough plot of the general behavior of $\vec{B}(r)$ from $r = 0$ to $r \to \infty$.

Solution

Since the current I is uniformly spread over the cross section between a and b, J (current density) = I/A = const, where A is the area between a and b:

$$A = \pi b^2 - \pi a^2 = \pi(b^2 - a^2).$$

See Figure 7. Apply Ampère's law

$$\oint \vec{B} \cdot d\vec{\ell} = \mu_0 I$$

around a circle of radius r such that $a < r < b$. Since the current is uniform, there is no preferred point on the circle of radius r; therefore $|\vec{B}|$ must be constant on this circle. Since the lines of $\vec{B}$ are concentric circles outside the wire, there is no reason to suspect they are not likewise inside the wire. The direction of $\vec{B}$ is gained by the right-hand rule: the current coming toward you (thumb), fingers curl in the direction of $\vec{B}$.

Integrating around the circle in a counterclockwise direction dictates the current to be positive (right-hand rule).

$$\oint \vec{B} \cdot d\ell = \oint B\, d\ell \cos\theta = \oint B\, d\ell \cos\theta = B \oint d\ell,$$

since $\vec{B}$ and $d\vec{\ell}$ are in the same direction at every point. $\int d\ell = 2\pi r$ = circumference of the circle. $\oint \vec{B} \cdot d\vec{\ell} = B2\pi r$. The I in Ampère's law is the net current that passes through the circular area of radius r. Since $J = I/\pi(b^2 - a^2)$, the current interior to r is $[I/\pi(b^2 - a^2)]\pi(r^2 - a^2)$. Therefore

$$B2\pi r = \mu_0 \pi I(r^2 - a^2)/\pi(b^2 - a^2), \qquad |\vec{B}| = [\mu_0 I/2\pi(b^2 - a^2)][(r^2 - a^2)/r],$$

with the direction as shown in Figure 7. When a = 0,

$$B = \frac{\mu_0 I}{2\pi(b^2)} \frac{r^2}{r} = \frac{\mu_0 I r}{2\pi b^2}.$$

Check this for the limiting case, r = a. Then let b = 2a, $K = \mu_0 I/6\pi a^2$. Then

$$|\vec{B}| = \left(\frac{\mu_0 I}{2\pi(b^2 - a^2)}\right)\left(\frac{r^2 - a^2}{r}\right) = K\left(\frac{r^2 - a^2}{r}\right) \quad \text{for } a < r < 2a.$$

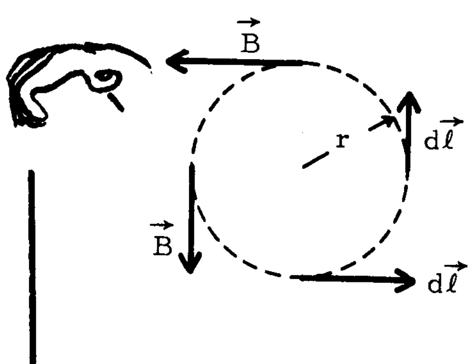


Figure 7

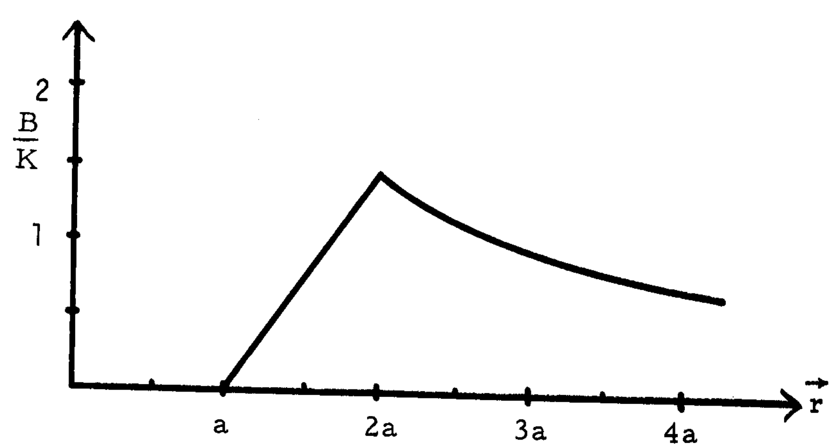


Figure 8

Figure 8 is a graph for $b = 2a$, which is not required for this problem. You could have chosen any other value for b as long as the point b is somewhat near the center of the graph so that the curve of $\vec{B}$ versus $\vec{r}$ is easily seen.

C(2). In Problem A, find the force per unit length on the bottom wire.

Solution

The $\vec{B}$ field caused by the top wire is given by $\oint \vec{B} \cdot d\vec{\ell} = \mu_0 I$, as we found in Problem A:

$$\vec{B}_1 = -(\mu_0 I_1/2\pi d)\hat{k}$$

at the location of the bottom wire. The force on the bottom wire is given by $\vec{F}_2 = I_2\vec{\ell} \times \vec{B}_1$:

$$\vec{F}_2 = [-I_2\ell\hat{i}] \times [(\mu_0 I_1/2\pi d)(-\hat{k})] = -(I_1 I_2 \mu_0/2\pi d)\hat{j}.$$

Thus the force per unit length is

$$\vec{F}/\ell = -(I_1 I_2 \mu_0 \hat{j})/2\pi d = -6.4 \times 10^{-6}\hat{j} \text{ N/m} \quad \text{repulsion.}$$

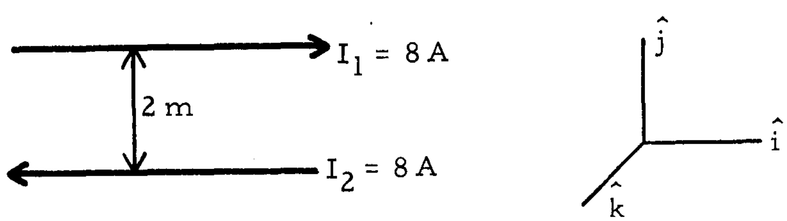


Figure 9

D(3). Imagine a current element $I\,d\vec{\ell}$ in which the current lies in the xy plane, and is directed to the right parallel to the x axis. What is the magnetic field at the origin if the current increment is located:
(a) at the point $(x, y, z) = (0, a, 0)$ m?
(b) at the point $(a, a, 0)$ m?
(c) at the point $(a, 0, 0)$ m?

Solution

(a) See Figure 10. $\hat{r} = -\hat{j}$.

$$d\vec{B} = \left(\frac{\mu_0 I}{4\pi}\right)\left(\frac{|d\vec{\ell}|[\hat{i} \times (-\hat{j})]}{a^2}\right) = \frac{\mu_0 I|d\vec{\ell}|}{4\pi a^2}\,(-\hat{k}).$$

Alternately we could use $|\hat{i} \times (-\hat{j})| = |\hat{i}|\ |-\hat{j}|\ [\sin(90°)]$ and the right-hand rule. From the right-hand rule, we would find, as above, that $d\vec{B}$ points in the -z direction (into the page).

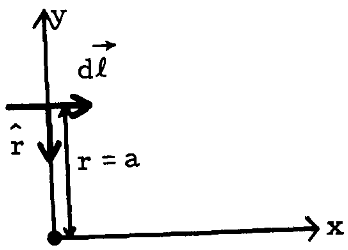


Figure 10

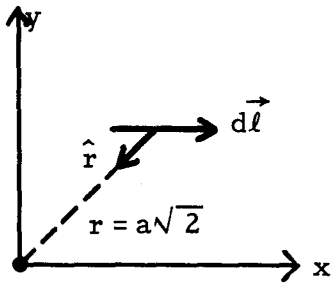


Figure 11

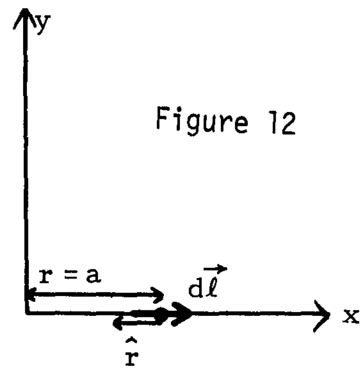


Figure 12

(b) See Figure 11, where $\hat{r} = (\sqrt{2}/2)(-\hat{i} - \hat{j})$. The $\sqrt{2}/2$ makes $\hat{r}$ a unit vector.

$$d\vec{B} = \left(\frac{\mu_0 I}{4\pi}\right)\left(\frac{|d\ell|[\hat{i} \times (-\hat{i} - \hat{j})(\sqrt{2}/2)]}{2a^2}\right) = \frac{\sqrt{2}\mu_0 I|d\vec{\ell}|}{16\pi a^2}(-\hat{k}).$$

(c) See Figure 12, where $\hat{r} = -\hat{i}$.

$$d\vec{B} = (\mu_0 I/4\pi)\{|d\vec{\ell}|[\hat{i} \times (-\hat{i})]/a^2\} = 0.$$

E(3). The wire shown in Figure 13 carries a current I. What is the magnetic field at the center C of the semicircle arising from:
(a) each infinite straight segment,
(b) the semicircular segment of R,
(c) the entire wire?

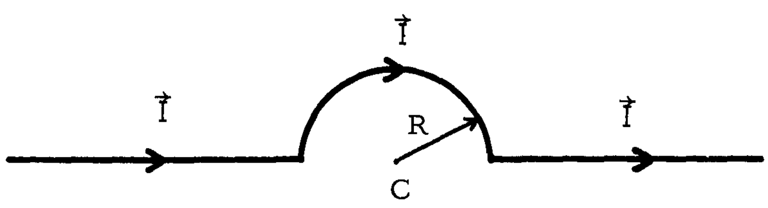


Figure 13

Solution

(a) For the left-hand segment $I\,d\vec{\ell}$ is in the same direction as r, thus $I d\vec{\ell} \times r = I\,d\ell \sin 0° = 0$. See Figure 14. For the right-hand segment $I\,d\ell$ is directly opposite of r, thus $I\,d\vec{\ell} \times \hat{r} = I\,d\ell \sin 180° = 0$. See Figure 15. Thus the straight segments contribute zero to the field at C, since C is in a direct line with each segment.

Figure 14

Figure 15

(b) For the circular portion, $I\,d\vec{\ell} \times \hat{r}$ = a vector whose magnitude is $I\,d\ell$ and whose direction is into the page. Therefore,

$$|d\vec{B}| = \left|\left(\frac{\mu_0 I}{4\pi}\right)\left(\frac{d\vec{\ell} \times \hat{r}}{r^2}\right)\right| = \frac{\mu_0 I\,d\ell}{4\pi R^2}.$$

Now since μ_0, I, and R are constants,

$$|\vec{B}| = \int dB = (\mu_0 I/4\pi R^2)\int d\ell.$$

The $\int d\ell$ is just the length of the circular segment, $(1/2)(2\pi R) = \pi R$. Thus $|\vec{B}|$ is $\mu_0 I/4R$ and the direction of $\vec{B}$ is into the page.

(c) The total $\vec{B}$ field is just the vector sum of parts (a) and (b), or $\mu_0 I/4R$, into the page.

Problems

F(1). Two infinitely long wires carry currents of 3.00 A and 10.0 A as shown in Figure 16. At what point(s) in the plane of the paper is $\vec{B} = 0$?

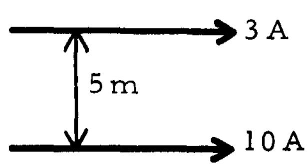
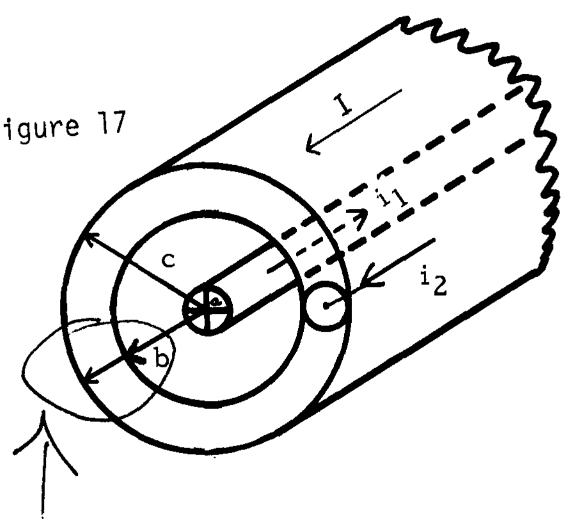


Figure 16

Figure 17



G(1). A long coaxial cable consists of two concentric conductors with the dimensions as shown in Figure 17 (a = radius of inner conductor). The conductors carry equal currents but in the opposite directions. Find the $\vec{B}$ field at a distance r from the center, where (a) $b < r < c$; and (b) $r > c$.

H(2). In Problem F, find the force per unit length on the top wire, carrying current $I = 3.00$ A, caused by the bottom wire, carrying current $I = 10.0$ A.

I(3). Imagine a current element $I\,d\vec{\ell}$ in which the current lies in the xy plane and is directed to the left parallel to the x axis. What is the magnetic field $d\vec{B}$ (magnitude and direction), at the point P = (0, 1, 0), if the current element is located at (a) (4, -1, 0) m? (b) (-2, 1, 0) m? See Figure 18.

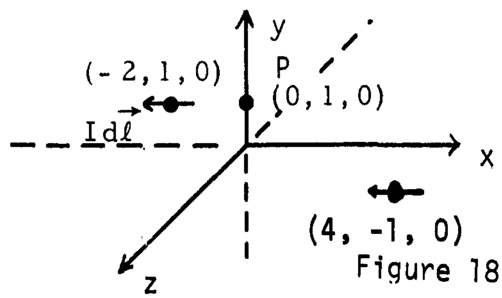


Figure 18

J(3). (a) A straight conductor is split in identical semicircular turns as shown in Figure 19. What is the magnetic field at the center of the circular loop? (Hint: This follows immediately from Problem E. Can you guess the answer?)

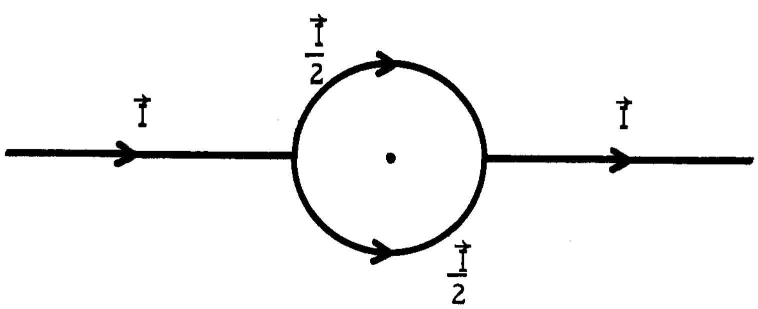


Figure 19

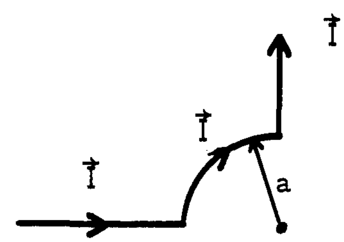


Figure 20

(b) A conductor as shown in Figure 20 consists of a circular arc of 90° and two long wires leading to this arc. Find the $\vec{B}$ field at the center of the arc.

Solutions

F(1). 1.15 m from the 3.00-A wire and toward the 10.0-A wire.

G(1). (a) $|\vec{B}| = (\mu_0 I/2\pi r)[(c^2 - r^2)/(c^2 - b^2)]$; direction is circular, clockwise. (b) $\vec{B} = 0$.

H(2). $\vec{F} = 1.20 \times 10^{-6}$ N/m, down (attraction).

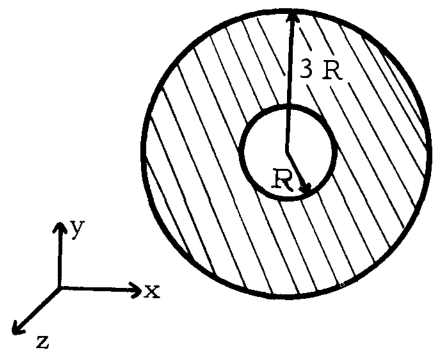
I(3). (a) $d\vec{B} = -\mu_0 I|d\vec{\ell}|\hat{k}/80\pi\sqrt{5}$. (b) $d\vec{B} = 0$.

J(3). (a) $\vec{B} = 0$. (b) $\vec{B} = \mu_0 I/8R$ into the page.

PRACTICE TEST

1. A long, straight hollow (inner radius R, outer radius 3R) conductor in Figure 21 carries a current I (out of paper) that is uniformly distributed over the cross section. If $\vec{r}$ measures the position from the axis of the conductor, determine $\vec{B}$ (magnitude and direction) for (a) $\vec{r} = -(R/2)\hat{i}$, (b) $\vec{r} = 2R\hat{j}$, and (c) $\vec{r} = 5R\hat{i}$.

Figure 21



2. A thin wire of length L carries a current 3I parallel to the conductor of Problem 1. The wire is positioned at $\vec{r} = 5R\hat{i}$ from the axis of the hollow conductor. Determine the force (magnitude and direction) on this wire.

3. Imagine a current element I $d\vec{\ell}$ in which the current lies in the xy plane and is directed downward and to the right at an angle of 45° to the x axis; i.e.,

   $d\vec{\ell} = (\sqrt{2}/2)|d\vec{\ell}|(\hat{i} - \hat{j})$.

   What is the magnetic field $d\vec{B}$ at the origin if the current element is located: (a) at the point (2, 0, 0)? (b) at (-4, 0, 0)?

Practice Test Answers

1. (a) 0. (b) $-(3\mu_0 I/32\pi R)\hat{i}$. (c) $(\mu_0 I/10\pi R)\hat{j}$.
2. $-(3\mu_0 I^2 L/10\pi R)\hat{i}$.
3. Let $C = (\mu_0 I/4\pi)|d\vec{\ell}|$: (a) $d\vec{B} = C(-\sqrt{2}\hat{k})/8$. (b) $d\vec{B} = C(\sqrt{2}\hat{k})/32$.

AMPÈRE'S LAW

Date ______________________

Mastery Test   Form A

pass   recycle

1   2   3

Name ______________________ Tutor ______________________

1. A current I is distributed uniformly over the cylindrical region $2R < r < 5R$ in Figure 1. The current is in the $+\hat{k}$ direction.
   (a) Determine the magnetic field at $\vec{r} = 3R\hat{i}$. Answer in terms of μ_0, I, R, $\hat{i}$, $\hat{j}$, $\hat{k}$.
   (b) Determine the value of r for which $|\vec{B}|$ is a maximum, and calculate the magnitude of $\vec{B}$ at that point.

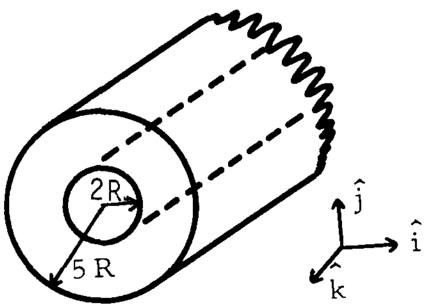


Figure 1

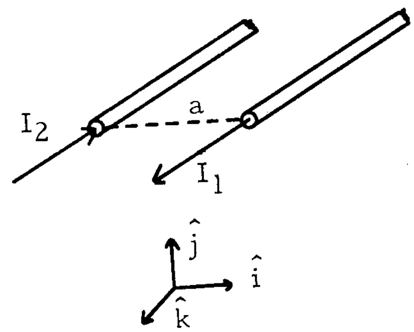


Figure 2

2. Two long parallel wires carry currents as shown in Figure 2. Determine the force on a 20.0-cm length of the wire carrying the I_1 current. $I_1 = I_2 = 60$ A, $a = 2.00$ cm.

3. A current element $I\,d\vec{\ell} = I\,d\ell(\sqrt{3}\hat{i} + \hat{j})/2$ lies in the xy plane and is directed upward and to the right at an angle of 30° with respect to the x axis. What is the magnetic field dB (magnitude and direction) at the origin if the current element is located at the point (2, 0, 0) m?

AMPERE'S LAW Date ____________________

Mastery Test Form B

pass recycle

1 2 3

Name ______________________________ Tutor ______________________________

1. (a) Currents of 6I and 8I are distributed uniformly in the cylindrical conductor as shown in Figure 1. Determine the magnetic field at point P in terms of I, R, μ_0, $\hat{i}$, $\hat{j}$, $\hat{k}$.

   (b) A thin wire of length L carries a current 3I into the paper at point P. It is "parallel" to the cylindrical conductors. Determine the force on this wire in terms of L, I, R, μ_0, $\hat{i}$, $\hat{j}$, $\hat{k}$.

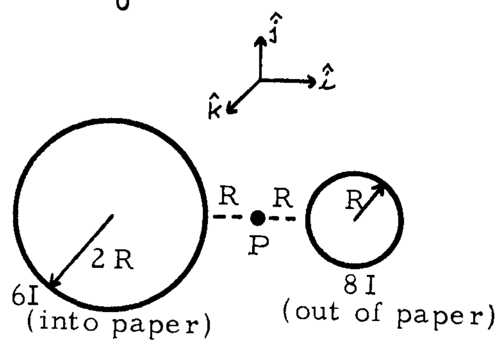


Figure 1

2. A current-carrying wire is bent in the shape shown in Figure 2.
   (a) Which of the four segments A, B, C, D contribute to the field at the point P?
   (b) What is the direction and magnitude of the field at P?

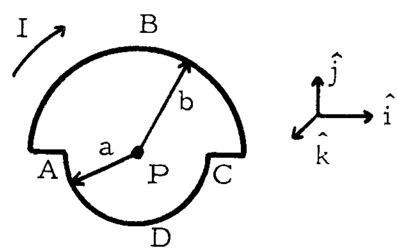


Figure 2

AMPERE'S LAW Date ____________

Mastery Test Form C

pass recycle

1 2 3

Name ____________ Tutor ____________

1. (a) A current of 3I is distributed uniformly in the cylindrical region $R < r < 2R$ as shown in Figure 1. A current of 2I is uniformly distributed in a cylindrical conductor of radius R as shown. Determine the magnetic field at points O and P in terms of I, R, and constants.
(b) A thin wire of length L carries a current 2I <u>into the paper</u> at point P. It is "parallel" to the two cylindrical conductors. Determine the force on this wire in terms of I, R, L, $\hat{i}$, $\hat{j}$, $\hat{k}$, and other constants.

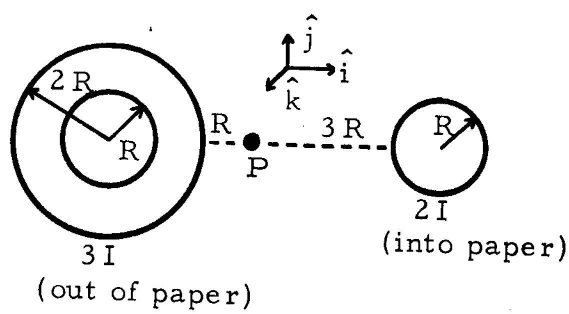


Figure 1

2. Imagine a current element $I\,d\vec{\ell}$ in which the current lies in the xy plane and is directed upward and to the right at an angle of 53° to the x axis, i.e.,

$d\vec{\ell} = d\ell(0.6\hat{i} + 0.8\hat{j})$.

What is the magnetic field $d\vec{B}$ (magnitude and direction) at the origin if the current element is located at the point (2, 0, 0) m?

AMPÈRE'S LAW

Date ____________________

Mastery Test   Form D

pass   recycle

1   2   3

Name ____________________ Tutor ____________________

1. A solenoid consists of 500 turns of wire wound around a cylindrical shell 1.00 cm in radius and 10.0 cm long. What is the magnetic field at the center of the solenoid when a current of 6.0 A is sent through the solenoid?

2. Two wires carrying the same current I are bent into semicircles as shown in Figure 1. The ends are then overlapped closely but not touching to form a circle of radius R, as in Figure 2. What is the magnetic field at the center of the circle (direction and magnitude)?

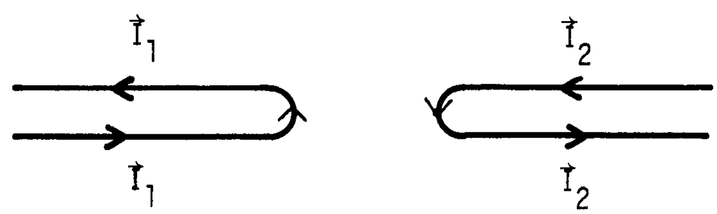


Figure 1

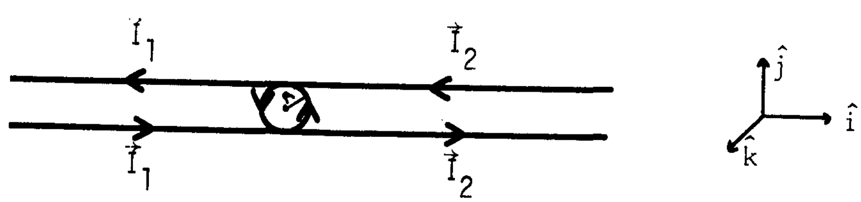


Figure 2

3. Two long wires carry currents of 3.00 A and 10.0 A, as in Figure 3. What is the force per unit length on the bottom wire carrying 10.0 A? (The wires are separated a distance of 5.0 m.)

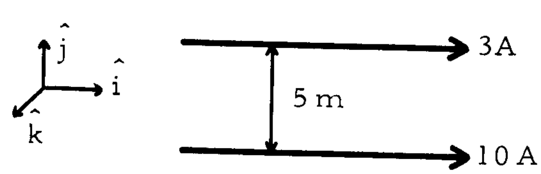


Figure 3

MASTERY TEST GRADING KEY - Form A

1. What To Look For: (a) Do they start off from Ampère's law? What is i? Fraction of I enclosed by circle of radius 3R. How do you determine direction? (Apply right-hand rule, have student do it.) (b) How do you know $\vec{B}$ is maximum at r = 5R? (See Figure 30.)

Solution: (a) $\vec{B}(\vec{r} = 3R\hat{i}) = ?$ $\oint \vec{B} \cdot d\vec{\ell} = \mu_0 i$,

$$|\vec{B}| = \frac{\mu_0 i}{2\pi r}, \qquad i = \left(\frac{I\pi}{\pi}\right)\left(\frac{(3R)^2 - (2R)^2}{(5R)^2 - (2R)^2}\right) = I\left(\frac{5R^2}{21R^2}\right) = \frac{5I}{21},$$

$$\vec{B} = [5\mu_0 I/42\pi(3R)]\hat{j} = [5\mu_0 I/126\pi R]\hat{j}.$$

(b) See Figure 30. $\vec{B}$ is maximum at r = 5R. For (2R < r < 5R), $|\vec{B}| = \mu_0 I(r^2 - 4R^2)/2\pi r(21R^2)$. For 5R < r, $|\vec{B}| = \mu_0 I/2\pi r$. At r = 5R, $|\vec{B}| = \mu_0 I/10\pi R$.

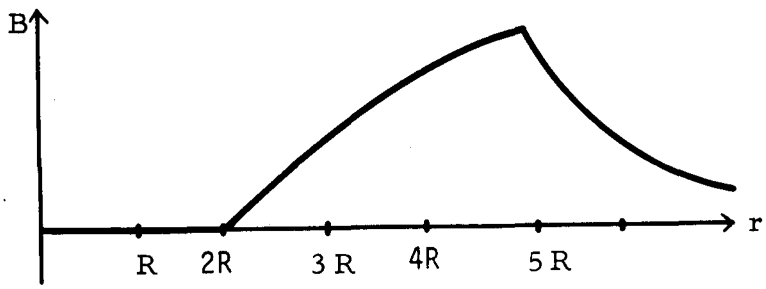


Figure 30

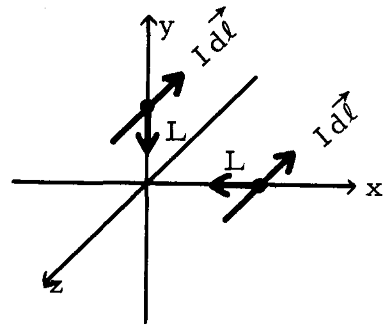


Figure 31

2. What To Look For: Are the directions for $\vec{B}$ from I_2 correct?

Solution: From $\oint \vec{B} \cdot d\vec{\ell} = \mu_0 I$, $\vec{B}$ at I_1 caused by I_2 equals $-(\mu_0 I_2/2\pi a)\hat{j}$.

$$\vec{F} = i\vec{\ell} \times \vec{B}, \qquad I_1 = I_2 = (I_1 \ell\hat{k}) \times \left(-\frac{\hat{j}}{a}\frac{\mu_0 I}{2\pi}\right) = \left(\frac{\mu_0 I^2 \ell}{2\pi}\right)\left(\frac{\hat{i}}{a}\right) = \frac{(4\pi \times 10^{-7})(60)^2(0.200)\hat{i}}{2\pi(0.0200)},$$

$\vec{F} = (7.2 \times 10^{-3})\hat{i}$ N.

3. What To Look For: Do they start with the Biot-Savart law?

Solution: See Figure 31.

$$d\vec{B} = \frac{\mu_0 I\, d\vec{\ell} \times \hat{r}}{4\pi r^2} = \frac{\mu_0 I\, d\ell(\sqrt{3}\hat{i} + \hat{j}) \times (-\hat{i})}{4\pi 2(2^2)}, \quad \hat{r} = -\hat{i}, \quad |\vec{r}| = 2, \quad d\vec{B} = (\mu_0 I\, d\ell/32\pi)\hat{k}.$$

MASTERY TEST GRADING KEY - Form B

1. What To Look For: (a) Did they start from Ampère's law and get $|\vec{B}| = \mu_0 i/2\pi r$? What is i? What is r? (b) Can they evaluate the cross product correctly?

   Solution: (a) Add $\vec{B}$ fields from each wire at P:

   $\int \vec{B} \cdot d\vec{\ell} = \mu_0 i$, $\vec{B}_{6I} = [\mu_0(6I)/2\pi(3R)](-\hat{j}) = -(\mu_0 I/\pi R)\hat{j}$,

   $\vec{B}_{8I} = [\mu_0(8I)/2\pi(2R)](-\hat{j}) = -(2\mu_0 I/\pi R)\hat{j}$, $\vec{B}_{total} = -(3\mu_0 I/\pi R)\hat{j}$.

   (b) $\vec{F} = I\vec{\ell} \times \vec{B} = 3IL(-\hat{k}) \times (-3\mu_0 I/\pi R)\hat{j} = -(9LI^2\mu_0/\pi R)\hat{i}$.

2. What To Look For: (b) Do they start with Biot-Savart law or know where $|\vec{B}| = \mu_0 I/4R$ comes from?

   Solution: (a) Only B, D contribute. A and C have $d\vec{\ell} \times \hat{r} = 0$.
   (b) For a circle $d\vec{B} = (\mu_0 I\ d\vec{\ell} \times \hat{r})/4\pi r^2$.

   $|d\vec{B}| = (\mu_0 I\ d\ell)/4\pi r^2$ (r = radius).

   For a half-circle:

   $$\vec{B} = \left(\frac{\mu_0 I}{4\pi r^2}\right)\frac{1}{2}(2\pi r) = -\left(\frac{\mu_0 I}{4r}\right)\hat{k}.$$

   Therefore

   $$\vec{B} = -\frac{\mu_0 I\hat{k}}{4}\left(\frac{1}{a} + \frac{1}{b}\right).$$

MASTERY TEST GRADING KEY - Form C

1. What To Look For: (a) Do they start from Ampère's law? How do you know direction? (right-hand rule) Why can you add $\vec{B}_{2I}$ and $\vec{B}_{3I}$? (principle of superposition.)
(b) Did they evaluate cross product correctly?

Solution: (a) Field at 0 = ?

$\oint \vec{B} \cdot d\vec{\ell} = \mu_0 I$ or $|\vec{B}| = \mu_0 I/2\pi r$.

From the 3I current, $\vec{B} = 0$, zero radius. From the 2I current,

$\vec{B} = [\mu_0(2I)/2\pi 7R]\hat{j} = (\mu_0 I/7\pi R)\hat{j}$.

Field at P = ? From 3I current,

$\vec{B} = [\mu_0(3I)/2\pi 3R]\hat{j} = (\mu_0 I/2\pi R)\hat{j}$.

From 2I current,

$\vec{B} = [\mu_0(2I)/2\pi 4R]\hat{j} = (\mu_0 I/4\pi R)\hat{j}$, $\vec{B}_{total} = (3\mu_0 I/4\pi R)\hat{j}$.

(b) $\vec{F} = I\vec{\ell} \times \vec{B} = 2IL(-\hat{k}) \times \left(\frac{3\mu_0 I}{4\pi R}\right)\hat{j} = \left(\frac{3\mu_0 I^2 L}{2\pi R}\right)\hat{i}$.

2. What To Look For: What is the Biot-Savart law? Is the cross product evaluated correctly?

Solution: See Figure 32.

$d\vec{B} = \frac{\mu_0}{4\pi}\frac{I\,d\vec{\ell} \times \hat{r}}{r^2}$, $\hat{r} = -\hat{i}$, $r^2 = 4$,

$d\vec{B} = \frac{\mu_0 I\,d\ell(0.6\hat{i} + 0.8\hat{j}) \times (-\hat{i})}{4\pi(4)} = \left(\frac{0.8\mu_0 I\,d\ell}{16\pi}\right)\hat{k} = \left(\frac{\mu_0 I\,d\ell}{20\pi}\right)\hat{k}$.

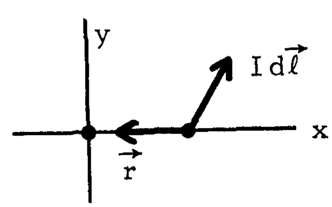


Figure 32

MASTERY TEST GRADING KEY - Form D

1. What To Look For: How do you know the direction of $\vec{B}$? (right-hand rule on one wire) Can they derive $\vec{B} = \mu_0 In\hat{i}$ from Ampère's law?

   Solution: See Figure 33. $\oint \vec{B} \cdot d\ell = \mu_0 I$ for dotted rectangle on sides D, C and $\vec{B} \cdot d\vec{\ell} = 0$. On side B, $\vec{B} = 0$. So for side A, $\oint \vec{B} \cdot d\vec{\ell} = B\ell$. Total I inside $= In\ell$, where n = turns/length and I = current in one turn. Therefore $B\ell = \mu_0 In\ell$.

   $\vec{B} = \mu_0 In\hat{i}$, n = 500 turns/0.100 m = 5000 turns/m.

   $\vec{B} = (4\pi \times 10^{-7})(6)(5000) = (3.77 \times 10^{-2})\hat{i}$ T.

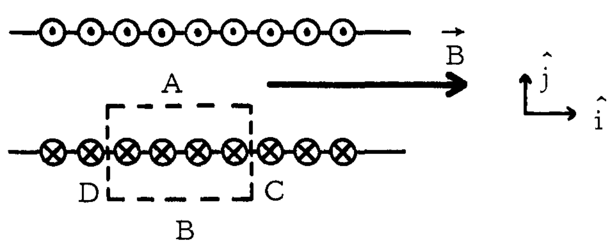


Figure 33

2. What To Look For: How can you determine direction? (right-hand rule) Can they derive each of these equations? (For the Biot-Savart law: r = const = R. $\int d\ell = 2\pi R$.)

   Solution: This is equivalent to two wires plus a current loop. From Ampère's law:

   $\vec{B} = (\mu_0 I/2\pi R)\hat{k}$, each straight wire.

   Circle (from Biot-Savart law):

   $\vec{B} = (\mu_0 I/2R)\hat{k}$. $\vec{B}_{total} = [(\mu_0 I/2R)(1/\pi + 1)]\hat{k}$.

3. What To Look For: How do you know direction of $\vec{B}$? (right-hand rule) Is the cross product evaluated correctly?

   Solution: $\vec{B}$ at bottom wire caused by top wire $= (\mu_0 I/2\pi r)(-\hat{k})$.

   $\vec{B} = (\mu_0 3/2\pi 5)(-\hat{k})$, $\vec{F} = I\vec{\ell} \times \vec{B} = (10\ell\hat{i}) \times [-(\mu_0 3/10\pi)\hat{k}]$,

   $\vec{F} = (\frac{3\mu_0 \ell}{\pi})\hat{j}$, $\frac{\vec{F}}{\ell} = (\frac{3\mu_0}{\pi})\hat{j} = (\frac{3(4\pi \times 10^{-7})}{\pi})\hat{j} = (1.20 \times 10^{-6})\hat{j}$ N/m.