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The average response of an infinite thin plate with statistically homogeneous attached random impedances is examined. The added impedances, which represent typical heterogeneities that might occur on complex shells, provide light coupling between the extensional, shear, and flexural waves. The mean plate response is formulated in terms of the Dyson equation which is solved within the assumptions of the first-order smoothing approximation, or Keller approximation, valid when the heterogeneities are weak. Scattering attenuations are derived for each propagation mode. It is shown that the attenuation of one wave type due to coupling to another is proportional to the modal density of the other wave type. Thus the attenuation of extensional and shear waves is predominantly due to mode conversion into flexural waves and is proportional to the large modal density of flexural waves. The flexural degrees of freedom serve as a sink for the energy of the membrane modes and constitute for them an effective fuzzy structure. The specific case of delta-correlated springs is considered for purposes of illustration.

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INTRODUCTION

The propagation of waves on homogeneous submerged shells is a difficult problem because of the spatial variation in the curvature of most realistic structures. The problem is further complicated when concentrated heterogeneities are also included in the analysis. Bulkheads, ribs, welds, rivets, and other heterogeneities may be effective scatterers for waves propagating on the shell. Recent research efforts have approached this problem a number of ways. Guo has examined the sound scattered from submerged cylindrical shells with an attached mass-spring[1] or attached thin plate.[2] The frequency spectrum of the sound scattered from such shells showed distinct differences from that of the homogeneous shell. In particular, certain resonant frequencies appeared at which the energy exciting the shell quickly propagated to the internal attachments. Others have considered a simple model of a plate with enormous numbers of attached mass–spring systems.[3–5] Descriptive information of the attached oscillators is assumed to be known only in some statistical sense. This ''fuzzy structure'' approach is appealing because the heterogeneities need not be described in detail.

In the present article the prototypical fuzzy problem[3] is examined with some major differences. Instead of coupling the fuzzies to the flexural motion only, the attached impedances here will couple the in-plane and out-of-plane modes. The coupling of the flexural and membrane modes is necessary in order to model realistic heterogeneities which would, in general, couple all modes to some extent. Coupling of the plate to a surrounding fluid is also neglected here for simplicity.

The Dyson equation, first used in quantum field theory and discussed by Frisch[6] for applications of classical waves in random media, is used to describe the mean displacements of the plate. It is solved here within the limits of the Keller approximation[7] also called the first-order smoothing approximation.[6] This approximation is valid if the covariance of the heterogeneities is the highest-order statistical quantity describing the scatterers that is of significance.

A similar diagramatic approach was discussed by Rybak for the study of plate wave propagation.[8,9] He examined the much simpler case of one-dimensional steady-state propagation of weakly coupled flexural and longitudinal waves using diagram methods. In this paper we extend his analysis to two dimensions and to the coupling of extensional, shear, and flexural waves, generalize the mathematics, and emphasize certain key features in the results. The paper is also intended to lay a groundwork for development of radiative transfer equations for these plates and to make contact with the recent work on fuzzy structures on plates.

Section I begins with a discussion of the stochastic plate equation. The mean plate response is then formulated in terms of the Dyson equation. In Sec. II, the Dyson equation is solved and expressions for the attenuations derived. In Sec. III, the added impedance is specialized to the case of attached point springs. Thus, any resonant conditions created by a mass and spring will not be present and the attached ''fuzzy'' has no internal degrees of freedom. The springs act as lossless scatterers, and the energy in the plate must stay in the plate. The modal attenuation is thus not due to any true energy loss mechanism acting on the plate nor to a loss to degrees of freedom of an internal fuzzy structure. Rather, it is scattering from the heterogeneities that results in an energy loss out of the propagating beam into other modes and other directions.

The results show that the attenuation of shear and extensional waves is mainly due to mode conversion into flexural

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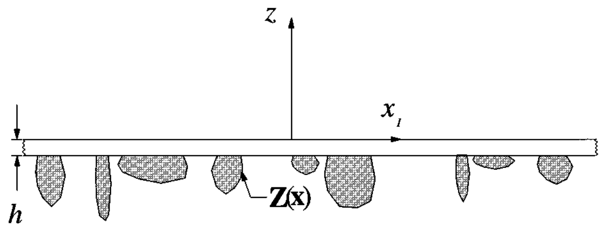

FIG. 1. Infinite flat plate with added spatially varying impedance.

waves. These results are expected to be similar to those one would find for the more complex problem involving submerged thin shells with arbitrary geometry, random heterogeneities, and complex internals.

I. STOCHASTIC PLATE EQUATION

Consider an unwetted, thin, flat plate that extends infinitely in the x_1 and x_2 directions as shown in Fig. 1. The plate has in-plane displacements u_1 and u_2 and out-of-plane displacement w in the z direction. Random spatially varying complex impedances are attached such that extensional, shear, and flexural motion are all coupled. These impedances can be thought of as a means of modeling the heterogeneities that would exist in a real structure, such as ribs, bulkheads, rivets, and attached internals. The added impedances are assumed to have spatial distribution and coupling strengths with known first and second-order statistics. The forces created by the added impedance are proportional to the displacements.

The motion of the plate can be described by the Green's dyadic $G_{ij}(\mathbf{x},\mathbf{x}';t)$ which defines the ith Cartesian displacement response at $\mathbf{x}$ due to an excitation in the jth Cartesian direction applied at $\mathbf{x}'$, where the vector $\mathbf{x}$ is a two-dimensional vector (x_1,x_2) defined within the plane of the plate. The temporal Fourier transform pair of $\mathbf{G}$ is defined as

$$G_{ij}(\mathbf{x},\mathbf{x}';\omega)=\int_0^{+\infty} G_{ij}(\mathbf{x},\mathbf{x}';t)e^{i\omega t}\,dt, \tag{1a}$$

$$G_{ij}(\mathbf{x},\mathbf{x}';t)=\frac{1}{2\pi}\int_{-\infty}^{+\infty} G_{ij}(\mathbf{x},\mathbf{x}';\omega)e^{-i\omega t}\,d\omega. \tag{1b}$$

The temporally Fourier transformed equation of motion for this plate is

$$[Z_{ki}(\mathbf{x},\omega)+\mathscr{L}_{ki}]G_{ij}(\mathbf{x},\mathbf{x}';\omega)=\delta_{kj}\delta^2(\mathbf{x}'-\mathbf{x}). \tag{2}$$

The plate operator, $\mathscr{L}_{ki}$, is given by

$$\mathscr{L}=\begin{bmatrix} -\frac{\partial^2}{\partial x_1^2}-\frac{1-\nu}{2}\frac{\partial^2}{\partial x_2^2}-\frac{(\omega+i\epsilon)^2}{c_p^2} & -\frac{1+\nu}{2}\frac{\partial^2}{\partial x_1\partial x_2} & 0 \\ -\frac{1+\nu}{2}\frac{\partial^2}{\partial x_1\,\partial x_2} & -\frac{\partial^2}{\partial x_2^2}-\frac{1-\nu}{2}\frac{\partial^2}{\partial x_1^2}-\frac{(\omega+i\epsilon)^2}{c_p^2} & 0 \\ 0 & 0 & \frac{h^2}{12}\nabla^4-\frac{(\omega+i\epsilon)^2}{c_p^2} \end{bmatrix}, \tag{3}$$

where $c_p^2=E/(1-\nu^2)\rho$ defines the plate wave speed with E the elastic modulus, ν Poisson's ratio, and ρ the volumetric density of the plate material. The shear wave speed is defined by $c_s^2=(1-\nu)c_p^2/2$. An infinitesimal imaginary part, ϵ, has been added to the frequency, ω, to emphasize that the transform, Eq. (1a), is defined for $\mathrm{Im}\{\omega\}>0$.

The symmetric random operator $\mathbf{Z}$ is not properly termed an impedance. It is an areal density of impedance divided by $-i\omega$ and is a kind of density of generalized dynamic stiffness. It is assumed here to be a local operator. More general, nonlocal, impedances act such that the spatial convolution

$$\int \Lambda_{ki}(\mathbf{x},\mathbf{x}'',\omega)G_{ij}(\mathbf{x}'',\mathbf{x}',\omega)d^2x'', \tag{4}$$

is the impedance related part of Eq. (2). When the attached impedance is presumed, as done here, to be local and couple a point only to itself then $\Lambda_{ki}(\mathbf{x},\mathbf{x}'',\omega)=Z_{ki}(\mathbf{x},\omega)\delta^2(\mathbf{x}-\mathbf{x}'')$ and the equation of motion is as given by Eq. (2).

The random impedance parameter $Z_{ki}(\mathbf{x},\omega)$ is a function of position and frequency and will in general couple all propagation modes. The frequency dependence of $\mathbf{Z}$ is henceforth to be considered implicit. The lowest order statistics of the added impedance are assumed known. The fluctuations of the added impedance are assumed small such that the average added impedance $\mathbf{Z}^0=\langle\mathbf{Z}\rangle$ and covariance $\langle(\mathbf{Z}(\mathbf{x})-\mathbf{Z}^0)(\mathbf{Z}(\mathbf{y})-\mathbf{Z}^0)\rangle$ are the only statistics of importance. The brackets, $\langle\ \rangle$, denote an ensemble average.

Equation (2) is a stochastic partial differential equation because of the random nature of $\mathbf{Z}$. It has deterministic operator, $\mathscr{L}_{ki}$, characteristic of a classical thin plate. It also has the random operator, Z_{ki}. The stochastic nature of the equation of motion precludes us from finding the Green's dyadic itself which will also be a random function of position. However, useful information regarding the plate motion can be found by considering the statistics of the Green's dyadic. In this paper we consider the mean Green's dyadic, $\langle\mathbf{G}\rangle$. Higher-order statistics of $\mathbf{G}$, which are also of interest, will not be discussed in detail here. A brief discussion of the covariance of the Green's dyadic, $\langle\mathbf{G}\mathbf{G}^*\rangle$, with regards to energy propagation is given at the conclusion of this article.

Equation (2) is of the form considered by Frisch[6] for the propagation of waves through random media. Thus, we may immediately write the equation that governs the propagation of the mean Green's dyadic. The Dyson equation is

$$\langle G_{ij}(\mathbf{x},\mathbf{x}')\rangle = G^0_{ij}(\mathbf{x},\mathbf{x}') + \int\int G^0_{i\alpha}(\mathbf{x},\mathbf{y}) m_{\alpha\beta}(\mathbf{y},\mathbf{z}) \times \langle G_{\beta j}(\mathbf{z},\mathbf{x}')\rangle d^2y\ d^2z, \tag{5}$$

where $\mathbf{m}$ is the self-energy or mass operator. The "bare" Green's dyadic, $\mathbf{G}^0$, is the solution of Eq. (2) with the impedance operator set to zero—the classical plate wave solution. The Dyson equation is exact and, if $\mathbf{m}$ is, like $\mathbf{G}^0$, invariant under spatial translations, easily solved in the Fourier domain. The self-energy operator, however, must be approximated. The most common approximation is the first-order smoothing approximation (FOSA)[6] and is equivalent to the Keller approximation.[7] The self-energy operator, $\mathbf{m}$, can be written as an infinite expansion in orders of impedance correlations.[6] The smoothing approximation of $\mathbf{m}$ includes only the first two terms of this expansion[6] and is given as

$$m_{\alpha\beta}(\mathbf{y},\mathbf{z}) = Z^0_{\alpha\beta}\delta^2(\mathbf{y}-\mathbf{z}) + G^0_{\gamma\delta}(\mathbf{y},\mathbf{z})\langle (Z_{\alpha\gamma}(\mathbf{y}) - Z^0_{\alpha\gamma}) \times (Z_{\delta\beta}(\mathbf{z}) - Z^0_{\delta\beta})\rangle. \tag{6}$$

An assumption of statistical homogeneity ensures that the average impedance $\mathbf{Z}^0 = \langle \mathbf{Z}(\mathbf{x})\rangle$ is spatially invariant. We also assume that $\mathbf{Z}^0$ is transversely isotropic—that the average properties do not couple the propagation modes. This form for $\mathbf{m}$ is expected to be valid as long as the impedance fluctuations are small, i.e., such that higher-order terms in $\mathbf{m}$ are unimportant. It is also the simplest level of approximation at which interesting modal coupling and attenuation will occur.

The covariance is assumed to have the following form:

$$\langle (Z_{\alpha\gamma}(\mathbf{x}) - Z^0_{\alpha\gamma})(Z_{\delta\beta}(\mathbf{y}) - Z^0_{\delta\beta})\rangle = {}^{\alpha}_{\beta}\Xi^{\gamma}_{\delta} W(\mathbf{x}-\mathbf{y}), \tag{7}$$

where W is a spatial two-point correlation function. For the form shown in Eq. (7), it has been assumed that the distribution of impedance orientation is independent of the spatial distribution. The assumption of statistical homogeneity is evident in the form of W which is a function of $\mathbf{x}$-$\mathbf{y}$ only. Transverse isotropy requires that $W(\mathbf{x}-\mathbf{y}) = W(|\mathbf{x}-\mathbf{y}|)$. The correlation function is a real-valued dimensionless function that will usually approach zero as $\mathbf{x}-\mathbf{y}\to\infty$. The fourth-rank tensor Ξ and the function W will, in general, also be functions of frequency, ω. Exemplary forms for Ξ and W are presented in Sec. III.

II. MEAN PLATE RESPONSE

The Dyson equation, Eq. (5), governs the mean plate response for the homogeneous plate with added impedance. It is now solved for the effective complex wave numbers of the three plate modes. The attenuation of each mode, given by the imaginary part of the respective wave number, is found in terms of the statistics of the added impedances.

The spatial Fourier transform of the bare Green's dyadic is defined as

$$\begin{aligned} \tilde{G}^0_{ij}(\mathbf{p})\,\delta^2(\mathbf{p}-\mathbf{q}) &= \frac{1}{(2\pi)^2}\int\int e^{-i\mathbf{p}\cdot\mathbf{x}} G^0_{ij}(\mathbf{x},\mathbf{y}) e^{i\mathbf{q}\cdot\mathbf{y}} \times d^2x\ d^2y, \\ G^0_{ij}(\mathbf{x},\mathbf{y}) &= \frac{1}{(2\pi)^2}\int\int e^{i\mathbf{p}\cdot\mathbf{x}} \tilde{G}^0_{ij}(\mathbf{p}) \times \delta^2(\mathbf{p}-\mathbf{q}) e^{-i\mathbf{q}\cdot\mathbf{y}}\ d^2p\ d^2q. \end{aligned} \tag{8}$$

The transform of the bare Green's dyadic is diagonal in wave vector space by virtue of the homogeneity of the plate. The operator $\mathscr{L}$ can also be spatially Fourier transformed allowing the bare Green's dyadic, which is the inverse of $\mathscr{L}$, to be written as

$$\tilde{\mathbf{G}}^0(\mathbf{p}) = \hat{\mathbf{p}}\hat{\mathbf{p}} g^0_e(p) + (\mathbf{I} - \hat{\mathbf{p}}\hat{\mathbf{p}}) g^0_s(p) + \hat{\mathbf{z}}\hat{\mathbf{z}} g^0_f(p), \tag{9}$$

where $\hat{\mathbf{p}}$ is the direction of propagation and lies within the plane of the plate, $\mathbf{I}$ is the two-dimensional identity dyadic $(\mathbf{I} = \hat{\mathbf{x}}_1\hat{\mathbf{x}}_1 + \hat{\mathbf{x}}_2\hat{\mathbf{x}}_2)$, and $\hat{\mathbf{z}}$ is perpendicular to the plate. The extensional, shear, and flexural bare propagators are given by

$$\begin{aligned} g^0_e(p) &= [p^2 - (\omega + i\epsilon)^2/c_p^2]^{-1}, \\ g^0_s(p) &= [(1-\nu)p^2/2 - (\omega+i\epsilon)^2/c_p^2]^{-1}, \\ g^0_f(p) &= [h^2p^4/12 - (\omega+i\epsilon)^2/c_p^2]^{-1}. \end{aligned} \tag{10}$$

The imaginary parts of these expressions are used later and are, in the limit $\epsilon\to 0$,

$$\begin{aligned} \operatorname{Im}\{g^0_e(p)\} &= \pi\ \operatorname{sgn}(\omega)\,\delta(p^2 - \omega^2/c_p^2), \\ \operatorname{Im}\{g^0_s(p)\} &= \pi\ \operatorname{sgn}(\omega)\,\delta((1-\nu)p^2/2 - \omega^2/c_p^2), \\ \operatorname{Im}\{g^0_f(p)\} &= \pi\ \operatorname{sgn}(\omega)\,\delta(h^2p^4/12 - \omega^2/c_p^2). \end{aligned} \tag{11}$$

The spatial Fourier transform pairs of $\langle \mathbf{G}\rangle$ and $\mathbf{m}$ are similarly defined as

$$\begin{aligned} \langle \tilde{G}_{ij}(\mathbf{p})\rangle\,\delta^2(\mathbf{p}-\mathbf{q}) &= \frac{1}{(2\pi)^2}\int\int e^{-i\mathbf{p}\cdot\mathbf{x}} \langle G_{ij}(\mathbf{x},\mathbf{y})\rangle \times e^{i\mathbf{q}\cdot\mathbf{y}}\ d^2x\ d^2y, \\ \langle G_{ij}(\mathbf{x},\mathbf{y})\rangle &= \frac{1}{(2\pi)^2}\int\int e^{i\mathbf{p}\cdot\mathbf{x}} \langle \tilde{G}_{ij}(\mathbf{p})\rangle \times \delta^2(\mathbf{p}-\mathbf{q}) e^{-i\mathbf{q}\cdot\mathbf{y}}\ d^2p\ d^2q, \end{aligned} \tag{12}$$

and

$$\begin{aligned} \tilde{m}_{\alpha\beta}(\mathbf{p})\,\delta^2(\mathbf{p}-\mathbf{q}) &= \frac{1}{(2\pi)^2}\int\int e^{-i\mathbf{p}\cdot\mathbf{x}} m_{\alpha\beta}(\mathbf{x},\mathbf{y}) \times e^{i\mathbf{q}\cdot\mathbf{y}}\ d^2x\ d^2y, \\ m_{\alpha\beta}(\mathbf{x},\mathbf{y}) &= \frac{1}{(2\pi)^2}\int\int e^{i\mathbf{p}\cdot\mathbf{x}} \tilde{m}_{\alpha\beta}(\mathbf{p})\,\delta^2(\mathbf{p}-\mathbf{q}) \times e^{-i\mathbf{q}\cdot\mathbf{y}}\ d^2p\ d^2q. \end{aligned} \tag{13}$$

The transforms of both $\langle \mathbf{G}\rangle$ and $\mathbf{m}$ are, like $\tilde{\mathbf{G}}^0$, also diagonal due to the statistical homogeneity of the scattering medium.

With these definitions, the Dyson equation, Eq. (5), can be spatially Fourier transformed and solved algebraically for the mean field as

$$\langle \tilde{\mathbf{G}}(\mathbf{p})\rangle = [\mathbf{I} - \tilde{\mathbf{G}}^0(\mathbf{p})\tilde{\mathbf{m}}(\mathbf{p})]^{-1}\tilde{\mathbf{G}}^0(\mathbf{p}). \tag{14}$$

The assumed transverse isotropy allows $\tilde{\mathbf{m}}$ and therefore $\langle\tilde{\mathbf{G}}\rangle$ to be written in the form

$$\begin{aligned} \tilde{\mathbf{m}}(\mathbf{p}) &= \hat{\mathbf{p}}\hat{\mathbf{p}}m_e(p) + (\mathbf{I} - \hat{\mathbf{p}}\hat{\mathbf{p}})m_s(p) + \hat{\mathbf{z}}\hat{\mathbf{z}}m_f(p), \\ \langle\tilde{\mathbf{G}}(\mathbf{p})\rangle &= \hat{\mathbf{p}}\hat{\mathbf{p}}g_e(p) + (\mathbf{I} - \hat{\mathbf{p}}\hat{\mathbf{p}})g_s(p) + \hat{\mathbf{z}}\hat{\mathbf{z}}g_f(p). \end{aligned} \tag{15}$$

The solution of the Dyson equation given by Eq. (14) is then used to give the mean-field propagators of $\langle\tilde{\mathbf{G}}\rangle$,

$$\begin{aligned} g_e(p) &= [g_e^0(p)^{-1} - m_e(p)]^{-1}, \\ g_s(p) &= [g_s^0(p)^{-1} - m_s(p)]^{-1}, \\ g_f(p) &= [g_f^0(p)^{-1} - m_f(p)]^{-1}. \end{aligned} \tag{16}$$

Equations (15) and (16) define $\langle\tilde{\mathbf{G}}\rangle$ in terms of the components of $\tilde{\mathbf{G}}^0$ and $\tilde{\mathbf{m}}$. The inverse Fourier transform of $\langle\tilde{\mathbf{G}}\rangle$ will be dominated by the behavior near the poles of the mean field propagators given in Eqs. (16). Thus the following set of dispersion equations must be solved for the mean-field wave numbers, p^{eff}:

$$\begin{aligned} &(p_e^{\text{eff}})^2 - (\omega + i\epsilon)^2/c_p^2 - m_e(p) = 0, \\ &(1-\nu)(p_s^{\text{eff}})^2/2 - (\omega + i\epsilon)^2/c_p^2 - m_s(p) = 0, \\ &h^2(p_f^{\text{eff}})^4/12 - (\omega + i\epsilon)^2/c_p^2 - m_f(p) = 0. \end{aligned} \tag{17}$$

Equations (17) for the effective wave number must, in general, be solved numerically for the entire frequency range. However, a simplifying assumption (sometimes called a Born approximation[10]) allows for reduction of Eqs. (17). We have assumed that the impedance fluctuations are small which implies that $\tilde{\mathbf{m}}$ is also small. Therefore, the effective wave numbers are close to the wave numbers, $p^0 = \omega/c$, of the plate without the attached random impedances. Each factor $m(p)$ is then approximated by $m(p^0)$. This approximation allows Eqs. (17) to be solved for the three effective wave numbers as

$$\begin{aligned} p_e^{\text{eff}} &= [(p_e^0)^2 + m_e(p_e^0)]^{1/2} \cong p_e^0 + m_e(p_e^0)c_p/2\omega, \\ p_s^{\text{eff}} &= [(p_s^0)^2 + 2m_s(p_s^0)/(1-\nu)]^{1/2} \\ &\cong p_s^0 + m_s(p_s^0)c_s/(1-\nu)\omega, \\ p_f^{\text{eff}} &= [(p_f^0)^4 + 12\tilde{m}_f(p_f^0)/h^2]^{1/4} \\ &\cong p_f^0 + m_f(p_f^0)\sqrt[4]{12}(c_p h/\omega)^{3/2}/4h^2, \end{aligned} \tag{18}$$

where the wave numbers of the homogeneous plate are

$$p_e^0 = \omega/c_p, \quad p_s^0 = \omega/c_s, \quad p_f^0 = \sqrt[4]{12}\sqrt{\omega/hc_p}. \tag{19}$$

The effective wave numbers given by Eqs. (18) reduce to the wave numbers of the bare plate when the heterogeneities, and thus the m's, go to zero.[11] The approximation used to obtain Eqs. (18) fails when m becomes comparable to $(p^0)^2$, but will allow much simplification to be made in the subsequent analysis.

The attenuation α is the imaginary part of the effective wave number. These attenuations are

$$\begin{aligned} \alpha_e &= \text{Im}\{m_e(p_e^0)\}c_p/2\omega, \\ \alpha_s &= \text{Im}\{m_s(p_s^0)\}c_s/(1-\nu)\omega, \\ \alpha_f &= \text{Im}\{m_f(p_f^0)\}\sqrt[4]{12}(c_p h/\omega)^{3/2}/4h^2, \end{aligned} \tag{20}$$

to the same approximation. Thus the attenuations due to scattering from the heterogeneities are described by the imaginary part of the spatial Fourier transform of the self-energy operator, $\mathbf{m}$.

In Eq. (6) it was shown that the FOSA expression for $\mathbf{m}$ is related to the covariance of the random impedance operator. In Sec. I the assumed form for the covariance was discussed. Using the definition for the covariance given by Eq. (7), the spatial Fourier transform of $\mathbf{m}$ becomes

$$\tilde{m}_{\alpha\beta}(\mathbf{p}) = Z_{\alpha\beta}^0 + \int e^{-i\mathbf{p}\cdot\mathbf{r}} G_{\gamma\delta}^0(\mathbf{r})\, {}^{\alpha}_{\beta}\Xi^{\gamma}_{\delta} W(\mathbf{r})\, d^2r. \tag{21}$$

Defining the spatial Fourier transform of the two-point correlation function as

$$\tilde{W}(\mathbf{p}) = \frac{1}{(2\pi)^2}\int W(\mathbf{r})e^{-i\mathbf{p}\cdot\mathbf{r}}\, d^2r, \tag{22}$$

gives the integral term of $\tilde{\mathbf{m}}$ as a convolution between the bare Green's dyadic and the two-point correlation function. Thus

$$\tilde{m}_{\alpha\beta}(\mathbf{p}) = Z_{\alpha\beta}^0 + \int {}^{\alpha}_{\beta}\Xi^{\gamma}_{\delta}\tilde{G}_{\gamma\delta}^0(\mathbf{s})\tilde{W}(\mathbf{p}-\mathbf{s})\, d^2s, \tag{23}$$

with a sum over γ and δ. The imaginary part of $\tilde{\mathbf{m}}$ now becomes

$$\begin{aligned} \text{Im}\{\tilde{m}_{\alpha\beta}(\mathbf{p})\} = \text{Im}\{Z_{\alpha\beta}^0\} &+ \int \text{Im}\{{}^{\alpha}_{\beta}\boldsymbol{\Xi}^{\gamma}_{\delta}\}\text{Re}\{\tilde{G}_{\gamma\delta}^0(\mathbf{s})\} \\ &\times \tilde{W}(\mathbf{p}-\mathbf{s})\, d^2s + \int \text{Re}\{{}^{\alpha}_{\beta}\boldsymbol{\Xi}^{\gamma}_{\delta}\}\text{Im}\{\tilde{G}_{\gamma\delta}^0(\mathbf{s})\} \\ &\times \tilde{W}(\mathbf{p}-\mathbf{s})\, d^2s. \end{aligned} \tag{24}$$

For a purely reactive impedance $\text{Im}\{\mathbf{Z}^0\} = 0$ and $\text{Im}\{\boldsymbol{\Xi}\} = 0$.[12] Each component of Eq. (24) now becomes, in direct notation,

$$\begin{aligned} \text{Im}\{m_e(\mathbf{p})\} &= \int {}^{\hat{\mathbf{p}}\cdot}_{\hat{\mathbf{p}}\cdot}\boldsymbol{\Xi}{}^{\cdot}_{\cdot}[{}^{\hat{\mathbf{s}}}_{\hat{\mathbf{s}}}\text{Im}\{g_e^0(s)\} + (\mathbf{I} - {}^{\hat{\mathbf{s}}}_{\hat{\mathbf{s}}})\text{Im}\{g_s^0(s)\} \\ &\quad + {}^{\hat{\mathbf{z}}}_{\hat{\mathbf{z}}}\text{Im}\{g_f^0(s)\}]\tilde{W}(\mathbf{p}-\mathbf{s})\, d^2s, \\ \text{Im}\{m_s(\mathbf{p})\} &= \int (\mathbf{I} - {}^{\hat{\mathbf{p}}}_{\hat{\mathbf{p}}}){}^{\cdot}_{\cdot}\boldsymbol{\Xi}{}^{\cdot}_{\cdot}[{}^{\hat{\mathbf{s}}}_{\hat{\mathbf{s}}}\text{Im}\{g_e^0(s)\} \\ &\quad + (\mathbf{I} - {}^{\hat{\mathbf{s}}}_{\hat{\mathbf{s}}})\text{Im}\{g_s^0(s)\} \\ &\quad + {}^{\hat{\mathbf{z}}}_{\hat{\mathbf{z}}}\text{Im}\{g_f^0(s)\}]\tilde{W}(\mathbf{p}-\mathbf{s})\, d^2s, \end{aligned} \tag{25}$$

$$\begin{aligned} \text{Im}\{m_f(\mathbf{p})\} &= \int {}^{\hat{\mathbf{z}}\cdot}_{\hat{\mathbf{z}}\cdot}\boldsymbol{\Xi}{}^{\cdot}_{\cdot}[{}^{\hat{\mathbf{s}}}_{\hat{\mathbf{s}}}\text{Im}\{g_e^0(s)\} + (\mathbf{I} - {}^{\hat{\mathbf{s}}}_{\hat{\mathbf{s}}})\text{Im}\{g_s^0(s)\} \\ &\quad + {}^{\hat{\mathbf{z}}}_{\hat{\mathbf{z}}}\text{Im}\{g_f^0(s)\}]\tilde{W}(\mathbf{p}-\mathbf{s})\, d^2s, \end{aligned}$$

where the factors such as ${}^{\hat{\mathbf{p}}\cdot}_{\hat{\mathbf{p}}\cdot}\Xi^{\cdot\hat{\mathbf{s}}}_{\cdot\hat{\mathbf{s}}}$ are inner products of unit vectors ($\hat{\mathbf{p}}$, $\hat{\mathbf{s}}$, and $\hat{\mathbf{z}}$) on the fourth-rank tensor Ξ. These inner products correspond to the interaction of each propagation mode with the impedances. The contraction on Ξ with different unit vectors can be thought of as a standard scattering problem. The energy of one wave type interacts with the heterogeneities and scatters into all directions and all modes. The inner products of Eqs. (25) are defined as

$$
\begin{aligned}
\xi_{ee}(\Phi) &\equiv {}^{\hat{\mathbf{p}}\cdot}_{\hat{\mathbf{p}}\cdot}\Xi^{\cdot\hat{\mathbf{s}}}_{\cdot\hat{\mathbf{s}}} = {}^{\alpha}_{\beta}\Xi^{\gamma}_{\delta}\hat{p}_{\alpha}\hat{p}_{\beta}\hat{s}_{\gamma}\hat{s}_{\delta},\\
\xi_{es}(\Phi) &\equiv {}^{\hat{\mathbf{p}}\cdot}_{\hat{\mathbf{p}}\cdot}\Xi^{\cdot}_{\cdot}(\mathbf{I}-{}^{\hat{\mathbf{s}}}_{\hat{\mathbf{s}}}) = {}^{\alpha}_{\beta}\Xi^{\gamma}_{\delta}\hat{p}_{\alpha}\hat{p}_{\beta}(\delta_{\gamma\delta}-\hat{s}_{\gamma}\hat{s}_{\delta}),\\
\xi_{ef}(\Phi) &\equiv {}^{\hat{\mathbf{p}}\cdot}_{\hat{\mathbf{p}}\cdot}\Xi^{\cdot\hat{\mathbf{z}}}_{\cdot\hat{\mathbf{z}}} = {}^{\alpha}_{\beta}\Xi^{3}_{3}\hat{p}_{\alpha}\hat{p}_{\beta},\\
\xi_{se}(\Phi) &\equiv (\mathbf{I}-{}^{\hat{\mathbf{p}}}_{\hat{\mathbf{p}}})^{\cdot}_{\cdot}\Xi^{\cdot\hat{\mathbf{s}}}_{\cdot\hat{\mathbf{s}}} = {}^{\alpha}_{\beta}\Xi^{\gamma}_{\delta}(\delta_{\alpha\beta}-\hat{p}_{\alpha}\hat{p}_{\beta})\hat{s}_{\gamma}\hat{s}_{\delta},\\
\xi_{ss}(\Phi) &\equiv (\mathbf{I}-{}^{\hat{\mathbf{p}}}_{\hat{\mathbf{p}}})^{\cdot}_{\cdot}\Xi^{\cdot}_{\cdot}(\mathbf{I}-{}^{\hat{\mathbf{s}}}_{\hat{\mathbf{s}}}) = {}^{\alpha}_{\beta}\Xi^{\gamma}_{\delta}(\delta_{\alpha\beta}-\hat{p}_{\alpha}\hat{p}_{\beta})\\
&\qquad\times(\delta_{\gamma\delta}-\hat{s}_{\gamma}\hat{s}_{\delta}),\\
\xi_{sf}(\Phi) &\equiv (\mathbf{I}-{}^{\hat{\mathbf{p}}}_{\hat{\mathbf{p}}})^{\cdot}_{\cdot}\Xi^{\cdot\hat{\mathbf{z}}}_{\cdot\hat{\mathbf{z}}} = {}^{\alpha}_{\beta}\Xi^{3}_{3}(\delta_{\alpha\beta}-\hat{p}_{\alpha}\hat{p}_{\beta}),\\
\xi_{fs}(\Phi) &\equiv {}^{\hat{\mathbf{z}}\cdot}_{\hat{\mathbf{z}}\cdot}\Xi^{\cdot}_{\cdot}(\mathbf{I}-{}^{\hat{\mathbf{s}}}_{\hat{\mathbf{s}}}) = {}^{3}_{3}\Xi^{\gamma}_{\delta}(\delta_{\gamma\delta}-\hat{s}_{\gamma}\hat{s}_{\delta}),\\
\xi_{fe}(\Phi) &\equiv {}^{\hat{\mathbf{z}}\cdot}_{\hat{\mathbf{z}}\cdot}\Xi^{\cdot\hat{\mathbf{s}}}_{\cdot\hat{\mathbf{s}}} = {}^{3}_{3}\Xi^{\gamma}_{\delta}\hat{s}_{\gamma}\hat{s}_{\delta},\quad \xi_{ff}(\Phi) \equiv {}^{\hat{\mathbf{z}}\cdot}_{\hat{\mathbf{z}}\cdot}\Xi^{\cdot\hat{\mathbf{z}}}_{\cdot\hat{\mathbf{z}}} = {}^{3}_{3}\Xi^{3}_{3},
\end{aligned}
\tag{26}
$$

with a sum over all indices $\alpha,\beta,\gamma,\delta=1,2$. The angle Φ is the angle between the unit vectors $\hat{\mathbf{p}}$ and $\hat{\mathbf{s}}$, which define the incident and scattered propagation directions.

From Eqs. (25), it can be seen that the imaginary part of the self-energy operator, and thus the attenuations, are related to a combination of the imaginary parts of the bare Green's dyadic propagators. An alternate definition of the bare Green's dyadic, given by Eq. (9), can also be imagined in terms of an expansion in the normal modes of the plate. The imaginary part of such a modal expansion can be shown to be proportional to the modal density.[13] It is then evident that these attenuations develop due to a loss of energy from a propagating wave into other available modes and directions through an interaction with the heterogeneities. This result is now derived explicitly.

The definitions of the bare propagators, Eqs. (10), and use of the approximation discussed above allows for simplification of Eqs. (25). The integration over the magnitude of $\mathbf{s}$ is easily done due to the delta function behavior of the imaginary parts of the bare propagators. The imaginary part of $\tilde{\mathbf{m}}$ is then found in terms of an integration over the unit circle

$$
\begin{aligned}
\mathrm{Im}\{m_e(\omega/c_p)\} &= \frac{\pi}{2}\int_0^{2\pi}[\xi_{ee}(\Phi)\tilde{W}_{ee}(\Phi)\\
&\quad+2\xi_{es}(\Phi)\tilde{W}_{es}(\Phi)/(1-\nu)\\
&\quad+\xi_{ef}(\Phi)\tilde{W}_{ef}(\Phi)\sqrt{12}c_p/2h\omega]d\Phi,\\
\mathrm{Im}\{m_s(\omega/c_s)\} &= \frac{\pi}{2}\int_0^{2\pi}[\xi_{se}(\Phi)\tilde{W}_{se}(\Phi)\\
&\quad+2\xi_{ss}(\Phi)\tilde{W}_{ss}(\Phi)/(1-\nu)\\
&\quad+\xi_{sf}(\Phi)\tilde{W}_{sf}(\Phi)\sqrt{12}c_p/2h\omega]d\Phi,
\end{aligned}
\tag{27}
$$

$$
\begin{aligned}
\mathrm{Im}\{m_f(p_f^0)\} &= \frac{\pi}{2}\int_0^{2\pi}[\xi_{fe}(\Phi)\tilde{W}_{fe}(\Phi)\\
&\quad+2\xi_{fs}(\Phi)\tilde{W}_{fs}(\Phi)/(1-\nu)\\
&\quad+\xi_{ff}(\Phi)\tilde{W}_{ff}(\Phi)\sqrt{12}c_p/2h\omega]d\Phi.
\end{aligned}
$$

The scattering functions, $\xi(\Phi)$, are the inner products defined in Eqs. (26). The specific correlation functions are defined as

$$
\begin{aligned}
&\tilde{W}_{ee}(\Phi)=\tilde{W}(\hat{\mathbf{p}}p_e^0-\hat{\mathbf{s}}p_e^0),\quad \tilde{W}_{es}(\Phi)=\tilde{W}(\hat{\mathbf{p}}p_e^0-\hat{\mathbf{s}}p_s^0),\\
&\tilde{W}_{ef}(\Phi)=\tilde{W}(\hat{\mathbf{p}}p_e^0-\hat{\mathbf{s}}p_f^0),\\
&\tilde{W}_{se}(\Phi)=\tilde{W}(\hat{\mathbf{p}}p_s^0-\hat{\mathbf{s}}p_e^0),\quad \tilde{W}_{ss}(\Phi)=\tilde{W}(\hat{\mathbf{p}}p_s^0-\hat{\mathbf{s}}p_s^0),\\
&\tilde{W}_{sf}(\Phi)=\tilde{W}(\hat{\mathbf{p}}p_s^0-\hat{\mathbf{s}}p_f^0),\\
&\tilde{W}_{fe}(\Phi)=\tilde{W}(\hat{\mathbf{p}}p_f^0-\hat{\mathbf{s}}p_e^0),\quad \tilde{W}_{fs}(\Phi)=\tilde{W}(\hat{\mathbf{p}}p_f^0-\hat{\mathbf{s}}p_s^0),\\
&\tilde{W}_{ff}(\Phi)=\tilde{W}(\hat{\mathbf{p}}p_f^0-\hat{\mathbf{s}}p_f^0).
\end{aligned}
\tag{28}
$$

The attenuations of each propagation mode can now be written in terms of the bare plate modal densities[13,14] discussed above as

$$\alpha_e=\frac{\pi^2}{2}\frac{c_p^3}{\omega^2}\left[n_e(\omega)\int_0^{2\pi}\xi_{ee}(\Phi)\tilde{W}_{ee}(\Phi)d\Phi\right.$$
$$+n_s(\omega)\int_0^{2\pi}\xi_{es}(\Phi)\tilde{W}_{es}(\Phi)d\Phi$$
$$\left.+n_f(\omega)\int_0^{2\pi}\xi_{ef}(\Phi)\tilde{W}_{ef}(\Phi)d\Phi\right],$$

$$\alpha_s=\frac{\pi^2}{2}\frac{c_p^4}{c_s\omega^2}\left[n_e(\omega)\int_0^{2\pi}\xi_{se}(\Phi)\tilde{W}_{se}(\Phi)d\Phi\right.$$
$$+n_s(\omega)\int_0^{2\pi}\xi_{ss}(\Phi)\tilde{W}_{ss}(\Phi)d\Phi$$
$$\left.+n_f(\omega)\int_0^{2\pi}\xi_{sf}(\Phi)\tilde{W}_{sf}(\Phi)d\Phi\right], \tag{29}$$

$$\alpha_f=\frac{\pi^2}{4}\frac{\sqrt[4]{12}c_p}{h^3}\left(\frac{c_p h}{\omega}\right)^{5/2}\left[n_e(\omega)\right.$$
$$\times\int_0^{2\pi}\xi_{fe}(\Phi)\tilde{W}_{fe}(\Phi)d\Phi$$
$$+n_s(\omega)\int_0^{2\pi}\xi_{fs}(\Phi)\tilde{W}_{fs}(\Phi)d\Phi$$
$$\left.+n_f(\omega)\int_0^{2\pi}\xi_{ff}(\Phi)\tilde{W}_{ff}(\Phi)d\Phi\right],$$

where $n_e(\omega)$, $n_s(\omega)$, and $n_f(\omega)$ are the bare plate modal densities of the extensional, shear, and flexural waves, respectively. These modal densities are defined as the number of modes per frequency interval per unit area and are given by[14]

$$n_e(\omega)=\omega/2\pi c_p^2,\quad n_s(\omega)=\omega/2\pi c_s^2,$$
$$n_f(\omega)=\sqrt{12}/4\pi h c_p. \tag{30}$$

We note that for the frequencies of our chief interest, where $\omega h/c_p \ll 1$, that $n_f \gg n_s > n_e$. Thus the third term in each of Eqs. (29) is dominant. Losses into flexural degrees of freedom are most important. The use of the bare plate modal densities is valid to the extent that the heterogeneities remain small, i.e., within the assumptions of the Keller approximation.

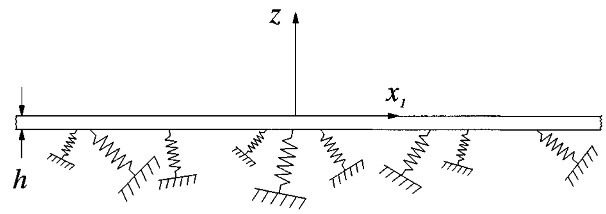

FIG. 2. Infinite flat plate with attached point stiffeners.

The general result given by Eqs. (29) defines the scattering attenuations for the extensional, shear, and flexural propagation modes in terms of known quantities defining the heterogeneities attached to the plate and the modal densities. Equations (29) show that the attenuation of each mode is equal to a weighted sum of the modal densities of all propagation modes. The weighting is determined by the strength of the scattering from the heterogeneities. It is then apparent that most of the extensional and shear wave energy will be lost into flexural energy because of the high flexural modal density. The flexural modes act as energy sinks for the energy of the extensional and shear modes. In the following section, attached impedances in the form of springs are used to illustrate the above result.

III. RESULTS FOR ISOTROPICALLY DISTRIBUTED DELTA-CORRELATED SPRINGS

The above derivation for the attenuations of the three plate modes concluded with the result that the attenuations are related to the modal densities, the spatial correlation function W, and the scattering functions $\xi(\Phi)$. We now consider the specific case of added impedances in the form of N discretely located springs as shown in Fig. 2. Each spring has dimensionless stiffness $k_n=\zeta_n(1-\nu^2)/Eh$, where ζ_n is the dimensional spring stiffness of the nth spring. Each spring also has an orientation angle defined by θ_n, measured from the negative z axis, and ψ_n, measured in the x_1-x_2 plane from the x_1 axis. The springs are attached at locations $\mathbf{x}_n$. In this case

$$Z_{ki}(\mathbf{x})=\sum_{n=1}^{N}k_n M_{ki}(\theta_n,\psi_n)\,\delta^2(\mathbf{x}-\mathbf{x}_n). \tag{31}$$

The matrix $\mathbf{M}$ couples the different modes and is a continuous function of the solid angle of orientation of the spring. With the definitions of θ and ψ given above, $\mathbf{M}$ is given by

$$\mathbf{M}(\theta_n,\psi_n)=\begin{bmatrix}\sin^2\theta_n\cos^2\psi_n & \sin^2\theta_n\sin\psi_n\cos\psi_n & -\sin\theta_n\cos\theta_n\cos\psi_n\\ \sin^2\theta_n\sin\psi_n\cos\psi_n & \sin^2\theta_n\sin^2\psi_n & -\sin\theta_n\cos\theta_n\sin\psi_n\\ -\sin\theta_n\cos\theta_n\cos\psi_n & -\sin\theta_n\cos\theta_n\sin\psi_n & \cos^2\theta_n\end{bmatrix}. \tag{32}$$

The average of the stiffness coupling is then

$$Z_{ki}^0=\sigma\bar{k}\bar{M}_{ki}, \tag{33}$$

where σ is the areal number density, N/A of the springs, $\bar{k}$ is the average spring stiffness, and $\bar{\mathbf{M}}$ is the average of the orientation matrix. The spring stiffness and orientation angle have been assumed uncorrelated. The quantities $\bar{k}$ and $\bar{\mathbf{M}}$ are defined as

$$\bar{k} = \frac{1}{N} \sum_{n=1}^{N} k_n ,$$
$$\bar{M}_{ki} = \frac{1}{2\pi} \int_0^{\pi/2} \int_0^{2\pi} M_{ki}(\theta,\psi) \sin \theta \, d\theta \, d\psi, \tag{34}$$

where it has been assumed that the springs are isotropically distributed over a hemisphere of solid angle.

It is easily found that

$$\bar{\mathbf{M}} = \begin{bmatrix} 1/3 & 0 & 0 \\ 0 & 1/3 & 0 \\ 0 & 0 & 1/3 \end{bmatrix}. \tag{35}$$

Thus $\mathbf{Z}^0$ is diagonal and isotropic.

The covariance given by Eq. (6) can be specialized for this case as well. The covariance is explicitly calculated as

$$\langle Z_{\alpha\gamma}(\mathbf{x}) Z_{\delta\beta}(\mathbf{y}) \rangle - Z^0_{\alpha\gamma} Z^0_{\delta\beta}$$
$$= \sigma \bar{k}^2 [\overline{M_{\alpha\gamma} M_{\delta\beta}} \delta^2(\mathbf{x}-\mathbf{y}) - \bar{M}_{\alpha\gamma} \bar{M}_{\delta\beta}/A]. \tag{36}$$

The second term above is ineffectual [it has no $\tilde{\mathbf{m}}(\mathbf{p} \neq 0)$] and will be neglected.[15] Recalling the assumed form for the covariance [Eq. (6)], we find

$${}^{\alpha}_{\beta}\Xi^{\gamma}_{\delta} = \sigma \bar{k}^2 \overline{M_{\alpha\gamma} M_{\delta\beta}}/A, \quad W(\mathbf{r}) = A\,\delta^2(\mathbf{r}). \tag{37}$$

The factor of plate area A is included in the definition of W in order that W remain dimensionless.

The inner products on Ξ defined in Eqs. (26) can now be calculated explicitly by carrying out the sums over the indices. For example,

$$\hat{\mathbf{p}}\cdot \hat{\mathbf{p}}\cdot \Xi \,\dot{:}\, \mathbf{I} = {}^{\alpha}_{\beta}\Xi^{\gamma}_{\delta} \hat{p}_\alpha \hat{p}_\beta \delta_{\gamma\delta} = \frac{\sigma \bar{k}^2}{15A} [3(\hat{p}_1\hat{p}_1 + \hat{p}_2\hat{p}_2)$$
$$+ (\hat{p}_1\hat{p}_1 + \hat{p}_2\hat{p}_2)] = 4\sigma \bar{k}^2/15A, \tag{38}$$

where $\hat{p}_1\hat{p}_1 + \hat{p}_2\hat{p}_2 = \hat{\mathbf{p}}\cdot\hat{\mathbf{p}} = 1$.

The necessary inner products are similarly calculated and give the required scattering functions

$$\xi_{ee}(\Phi) = \sigma \bar{k}^2 (1 + 2\cos^2 \Phi)/15A,$$
$$\xi_{es}(\Phi) = \xi_{se}(\Phi) = \sigma \bar{k}^2 (1 + 2\sin^2 \Phi)/15A,$$
$$\xi_{ef}(\Phi) = \xi_{fe}(\Phi) = \sigma \bar{k}^2/15A, \tag{39}$$
$$\xi_{ss}(\Phi) = \sigma \bar{k}^2 (1 + 2\cos^2 \Phi)/15A,$$
$$\xi_{sf}(\Phi) = \xi_{fs}(\Phi) = \sigma \bar{k}^2/15A, \quad \xi_{ff}(\Phi) = 3\sigma \bar{k}^2/15A.$$

These functions have the symmetries one might expect for the isotropic distribution assumed.

The delta function correlation implies that the transform $\tilde{W}(\mathbf{p}) = A/(2\pi)^2$ as defined by Eq. (22). Substituting the expressions from Eqs. (39) and the definition of $\tilde{W}$ into Eqs. (29) gives the attenuations[16] as

$$\alpha_e = \frac{\sigma \bar{k}^2 \pi}{60} \frac{c_p^3}{\omega^2} [2n_e(\omega) + 2n_s(\omega) + n_f(\omega)],$$
$$\alpha_s = \frac{\sigma \bar{k}^2 \pi}{60} \frac{c_p^4}{c_s \omega^2} [2n_e(\omega) + 2n_s(\omega) + n_f(\omega)], \tag{40}$$
$$\alpha_f = \frac{\sigma \bar{k}^2 \pi}{120} \frac{\sqrt[4]{12} c_p}{h^3} \left(\frac{c_p h}{\omega} \right)^{5/2} [n_e(\omega) + n_s(\omega) + 3n_f(\omega)].$$

Equations (40) show that the attenuation of each mode is proportional to a weighted sum of the modal densities of all modes. All waves will attenuate predominantly into flexural waves because of the high flexural modal density ($n_f \gg n_s > n_e$) at frequencies of chief interest.

The decay of each wave type compared to the others is now calculated. The ratio of shear attenuation to extensional attenuation is simply the inverse of the wave speed ratio

$$\alpha_s/\alpha_e = c_p/c_s . \tag{41}$$

The ratio of the flexural attenuation to the extensional attenuation is more complicated. It is frequency dependent and given by

$$\frac{\alpha_f}{\alpha_e} = \frac{1}{\sqrt{\omega}} \frac{\sqrt[4]{12}}{2} \left[\frac{(1 + (c_p/c_s)^2) + 3\sqrt{12}/2\varpi}{2(1 + (c_p/c_s)^2) + \sqrt{12}/2\varpi} \right], \tag{42}$$

where the modal densities given by Eqs. (30) have been used. The dimensionless frequency, ϖ, has been defined by $\varpi = \omega h/c_p$. In the limit that ϖ is very small, the ratio of attenuations is approximated by

$$\alpha_f/\alpha_e \cong 3\sqrt[4]{12}/2\sqrt{\varpi}. \tag{43}$$

Thus for low frequencies, the flexural waves decay much faster than both the extensional and shear waves. However, this flexural decay is dominated by flexural–flexural scattering and not mode conversion into the membrane modes.

Also of interest is the attenuation per wave number as a function of the spring stiffness, number of springs, and excitation frequency. These are

$$\frac{\alpha_e}{p_e^0} = \frac{\alpha_s}{p_s^0} = \frac{(\sigma \bar{k})^2 h^2}{120\sigma} \left(\frac{1}{\varpi^2} \right) \left[2\left(1 + \frac{c_p^2}{c_s^2} \right) + \frac{\sqrt{12}}{2\varpi} \right],$$
$$\frac{\alpha_f}{p_f^0} = \frac{(\sigma \bar{k})^2 h^2}{240\sigma} \left(\frac{1}{\varpi^2} \right) \left[\left(1 + \frac{c_p^2}{c_s^2} \right) + \frac{3\sqrt{12}}{2\varpi} \right]. \tag{44}$$

The attenuations are seen to scale with the square of $\sigma \bar{k}$ which is the total added stiffness per area. Equations (44) also vary inversely with the spring density. Thus for fixed total added stiffness $(\sigma \bar{k})$, attenuations will be enhanced by having a small density, σ, of strong springs $\bar{k}$ rather than a large density of weak springs. We also see that for low frequencies the ratio of flexural to extensional attenuation per wave number is 1.5. This result is of course highly dependent on the assumed isotropic distribution of spring orientations. The above expressions for the attenuations are now evaluated using parameters of interest for this problem.

IV. ESTIMATES OF REALISTIC ATTENUATIONS

We now turn to the issue of estimating the required spring stiffnesses and densities to obtain significant values of extensional and shear wave attenuation. The calculation of the required added stiffness will provide some insight into the potential merit of this approach for the modeling of anomalously high attenuations in realistic structures such as complex thin-walled submerged cylinders.

In such a system, $\varpi = \omega h/c_p$ is unity for frequencies of the order of the coincidence frequency. At the "ring" frequency, where the cylinder's circumference fits one extensional wavelength, $\omega_{\text{ring}} = c_p/R$ with R the cylinder radius. This assumption implies that, at the ring frequency, $\varpi = 0.01$ if $R/h = 100$. We take the average total added stiffness to be of the order of, and bounded by, this breathing mode stiffness (so that the mean flexural dispersion relation is not significantly altered by the attached springs),[17]

$$\sigma\bar{\zeta}/\rho_{\text{areal}} \leq \omega_{\text{ring}}^2, \tag{45}$$

where $\rho_{\text{areal}} = \rho h$ is the areal plate density and $\bar{\zeta}$ is the average dimensional spring stiffness. Using the definition of the dimensionless spring stiffness defined as $\bar{k} = \bar{\zeta}(1-\nu^2)/Eh$ we find that

$$\sigma\bar{k} \leq \omega_{\text{ring}}^2/c_p^2. \tag{46}$$

For the assumed radius to thickness ratio of 100, $\omega_{\text{ring}}/c_p = 1/100h$, which gives

$$\sigma\bar{k} \leq 10^{-4}/h^2. \tag{47}$$

The upper bound of this inequality is now used in the expression for the shear attenuation to wave number ratio [Eqs. (44)]

$$\alpha_s/p_s^0 = \frac{10^{-8}}{120\sigma h^2}\left(\frac{1}{\varpi^3}\right)\frac{\sqrt{12}}{2}. \tag{48}$$

At the ring frequency $\varpi = 0.01$, and

$$\alpha_s/p_s^0 = 144(10)^{-6}/\sigma h^2. \tag{49}$$

A significant amount of shear wave attenuation, $\alpha_s/p_s^0 = \alpha_s\lambda_s/2\pi = 0.05$, is obtained when the spring density multiplied by the square of the thickness is

$$\sigma h^2 = 0.00289. \tag{50}$$

Finally, assuming a shell thickness of 5 cm, we find the density of these springs to be on the order of or less than 1.15 springs per square meter. We conclude that the required density of springs is not large.

Figure 3 is a plot of the attenuations per wave number, α/p^0, versus dimensionless frequency, $\varpi = \omega/hc_p$, given by Eqs. (44). The added stiffness and spring density used for Fig. 3 are $\sigma\bar{k} = 10^{-4}/h^2$ and $\sigma h^2 = 0.00289$, respectively, as discussed above. The membrane and flexural contributions of the attenuations have been separated to highlight the dominance of the losses into the flexural degrees of freedom. Thus the upper two curves represent the last term of Eqs. (44), while the lower two curves represent the rest of Eqs. (44).

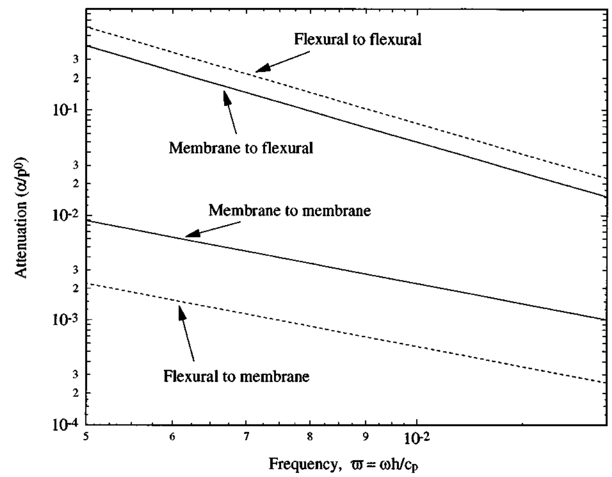

FIG. 3. Attenuation per wave number versus dimensionless frequency as given by Eqs. (44). The added stiffness and spring density used for these plots are $\sigma\bar{k} = 10^{-4}/h^2$ and $\sigma h^2 = 0.00289$, respectively. The losses into flexural and membrane modes have been separated to highlight the large losses into the flexural degrees of freedom expected for this frequency range.

The flexural part of the attenuations is at least ten times the membrane part of the attenuations for this frequency range.

One might also ask to what value of stiffness these parameters correspond. Consider a shell made of steel ($\rho = 7900$ kg/m^3) with a thickness of 5 cm and a ring frequency of 171 Hz. Using the spring density of $\sigma = 1.15/\text{m}^2$ we find a dimensional spring stiffness of about 400 MN/m. The required $\bar{k}$ can be achieved by letting the attached springs be steel rods of 10-cm length with cross-sectional areas, $a_{\text{rod}} = (14\text{ mm})^2$ (with stiffness given by $a_{\text{rod}}E/L$, with L their length). The stiffness and density of springs required to obtain significant values of attenuation are not unlike what one might anticipate finding on real structures. One could model significant attenuation at higher frequencies by invoking lower number densities of springs and higher stiffnesses.

V. DISCUSSION

The model considered here is a highly simplified version of an actual complex submerged shell. The attached impedances have been taken to be local, statistically homogeneous, and connected to a rigid ground. The results are nevertheless important. The method of smoothing has been applied to show that significant amounts of attenuation can be achieved by means of membrane wave/flexural wave mode conversions without the need to appeal to internal fuzzy structures. Inasmuch as the chief sink for membrane wave energy is into flexural waves, there are also important implications for the radiation problem. The loss of energy from membrane waves will lead to weaker acoustic radiation into a surrounding fluid, but the scatterers will also enhance the otherwise weak radiation from the flexural waves. The precise implications for radiation are therefore not yet obvious.

The present results do not comprise a complete description of the statistics of wave propagation on inhomogeneous plates. In order to track the energy propagation across the plate as a function of space and time, one must also derive expressions for the covariance of the Green's dyadic, $\langle \mathbf{GG}^* \rangle$, which is proportional to acoustic energy density. The Bethe–Salpeter equation governs this quantity and can be reduced to an equation of radiative transfer in the limit that the attenuation per wavelength is sufficiently weak.[6,18,19]

ACKNOWLEDGMENT

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[1] Y. P. Guo, ''Sound scattering from an internally loaded cylindrical shell,'' J. Acoust. Soc. Am. **91**, 926–938 (1992).

[2] Y. P. Guo, ''Sound scattering from cylindrical shells with internal elastic plates,'' J. Acoust. Soc. Am. **93**, 1936–1946 (1993).

[3] A. D. Pierce, V. W. Sparrow, and D. A. Russell, ''Fundamental structural-acoustic idealizations for structures with fuzzy internal,'' proceedings of the ASME Winter Annual Meeting, 1993, paper No. 93-WA/NCA-18.

[4] A. D. Pierce, ''Mass per unit natural frequency as a descriptor of internal fuzzy structure,'' Boston University Tech. Rep. No. AM-94-003, June 1994. ''Influence of structural wave dispersion on scattering patterns, with application to scattering from structures with fuzzy internal,'' Boston University Tech. Rep. No. AM-94-006, July 1994. ''Mechanical foundations for a theory of fuzzy structures,'' Boston University Tech. Rep. No. Am-95-003, January 1995.

[5] C. Soize, ''Modeling and numerical method in the medium frequency range for vibroacoustic predictions using theory of structural fuzzy,'' J. Acoust. Soc. Am. **92**, 2365 (1992).

[6] U. Frisch, ''Wave propagation in random media,'' *Probabilistic Methods in Applied Mathematics*, edited by A. T. Bharucha-Reid (Academic, New York, 1968), Vol. 1, pp. 75–198.

[7] F. C. Karal and J. B. Keller, ''Elastic, electromagnetic and other waves in random media,'' J. Math. Phys. **5**, 537–547 (1964).

[8] S. A. Rybak, ''Waves in a plate containing random inhomogeneities,'' Sov. Phys. Acoust. **17**, 345–349 (1972).

[9] S. A. Rybak, ''Randomly coupled flexural and longitudinal vibrations of plates,'' Sov. Phys. Acoust. **18**, 76–79 (1972).

[10] F. E. Stanke and G. S. Kino, ''A unified theory for elastic wave propagation in polycrystalline materials,'' J. Acoust. Soc. Am. **75**, 665–681 (1984).

[11] The other pole of the flexural dispersion equation defines the evanescent flexural wave and is not of interest here. However, this pole would also be shifted slightly due to the presence of impedance heterogeneities.

[12] If the attached impedances have internal resonant degrees of freedom then $\mathrm{Im}\{\mathbf{Z}^0\}\neq 0$, and the attenuation will be found to be augmented by losses into these degrees of freedom. This is presumably equivalent to the currently widely studied case of wave energy loss into internal structural fuzzies.[3,4]

[13] E. N. Economou, *Green's Functions in Quantum Physics* (Springer-Verlag, Berlin, 1979).

[14] R. L. Weaver, ''Diffuse waves on submerged thin shells,'' J. Acoust. Soc. Am. **95**, 857–865 (1994).

[15] The second term of Eq. (36) is identically zero if the number of springs is also allowed to be stochastic with a Poisson distribution.

[16] It may be observed that, for delta-correlated impedances, $\mathrm{Re}\{p_e^{\mathrm{eff}}\}$ and $\mathrm{Re}\{p_s^{\mathrm{eff}}\}$ are undefined. The effective wave speeds are logarithmically divergent as the correlation range vanishes. This means that these quantities depend on the actual correlation range. Fortunately, however, the attenuations are largely independent of this extra parameter.

[17] The change in natural frequency expected for the breathing mode is about 40% and is less than this for the propagating modes.

[18] R. L. Weaver, ''Diffusivity of ultrasound in polycrystals,'' J. Mech. Phys. Solids **38**, 55–86 (1990).

[19] J. A. Turner and R. L. Weaver, ''Radiative transfer and multiple scattering of diffuse ultrasound in polycrystalline media,'' J. Acoust. Soc. Am. **96**, 3675–3683 (1994).