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# Comments and Corrections

## On the $\eta - \kappa$ Distribution

Saralees Nadarajah and Samuel Kotz

**Index Terms**—Fast fading distribution,  $\eta - \kappa$  distribution.

The recent paper by Yacoub *et al.* [1] introduces what is referred to as the  $\eta - \kappa$  distribution to describe the statistical variation of the envelope in a fast fading environment. The paper discusses several properties of the distribution. Two of the properties discussed are the  $n$ th moment,  $E(P^n)$ , and the cumulative probability function (cdf),  $F_P(\cdot)$ , where  $P$  is a random variable representing the normalized envelope. The expression given for  $E(P^n)$  (see equation (10) in Yacoub *et al.* [1]) is a doubly infinite sum of the Gauss hypergeometric function (which, itself, is an infinite sum). That given for  $F_P(\cdot)$  (see equation (11) in Yacoub *et al.* [1]) is a triple sum of the incomplete gamma function.

We feel that the expressions in equations (10) and (11) of Yacoub *et al.* [1] are too complicated for practical purposes. In the following, we show how one can derive much simpler forms for  $E(P^n)$  and  $F_P(\cdot)$ . Using equations (5)–(8) in Yacoub *et al.* [1], the probability density function (pdf) of  $P$  can be expressed as

$$f_P(p) = \frac{\sqrt{h(1+\kappa)}}{\pi \exp(\kappa)} \int_0^{2\pi} p \exp(Ap - Bp^2) d\theta, \quad (1)$$

where  $A = 2\sqrt{h\kappa(1+\kappa)} \cos \theta$  and  $B = (1+\kappa)h + H(1+\kappa) \cos(2(\theta + \phi))$ . Thus, the  $n$ th moment,  $E(P^n)$ , can be expressed as

$$\begin{aligned} E(P^n) &= \frac{\sqrt{h(1+\kappa)}}{\pi \exp(\kappa)} \int_0^{2\pi} \int_0^\infty p^{n+1} \exp(Ap - Bp^2) dp d\theta \\ &= \frac{\sqrt{h(1+\kappa)}}{\pi \exp(\kappa)} \int_0^{2\pi} \int_0^\infty p^{n+1} \exp(Ap - Bp^2) dp d\theta \\ &= \frac{\sqrt{h(1+\kappa)}}{\pi \exp(\kappa)} \int_0^{2\pi} I(\theta) d\theta. \end{aligned} \quad (2)$$

By equation (2.3.15.3) in Prudnikov *et al.* [2],  $I(\theta)$  can be calculated as

$$I(\theta) = \Gamma(n+2)(2B)^{-(n/2+1)} \exp\left(\frac{A^2}{8B}\right) D_{-n-2}\left(-\frac{A}{\sqrt{2B}}\right), \quad (3)$$

where  $D_p(\cdot)$  denotes the parabolic cylinder function defined by

$$D_p(x) = \frac{\exp(-x^2/4)}{\Gamma(-p)} \int_0^\infty \frac{\exp\{-t(x+t^2/2)\}}{t^{p+1}} dt.$$

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Combining (2) and (3) yields the formula

$$\begin{aligned} E(P^n) &= \frac{\Gamma(n+2)\sqrt{h(1+\kappa)}}{2^{n/2+1}\pi \exp(\kappa)} \int_0^{2\pi} B^{-(n/2+1)} \\ &\quad \times \exp\left(\frac{A^2}{8B}\right) D_{-n-2}\left(-\frac{A}{\sqrt{2B}}\right) d\theta. \end{aligned} \quad (4)$$

This formula applies for any real number  $n > -2$ . If  $n$  is a positive integer then, using equation (2.3.15.7) in Prudnikov *et al.* [2],  $I(\theta)$  can be calculated as

$$I(\theta) = \frac{(-1)^{n+1}\sqrt{\pi}}{2\sqrt{B}} \frac{\partial^{n+1}}{\partial q^{n+1}} \times \left[ \exp\left(\frac{q^2}{4B}\right) \operatorname{erfc}\left(\frac{q}{2\sqrt{B}}\right) \right] \Big|_{q=-A}, \quad (5)$$

where  $\operatorname{erfc}(\cdot)$  denotes the complementary error function defined by

$$\operatorname{erfc}(x) = -\frac{2}{\sqrt{\pi}} \int_0^x \exp(-t^2) dt.$$

Combining (2) and (5) yields the simpler formula

$$\begin{aligned} E(P^n) &= \frac{(-1)^{n+1}\sqrt{h(1+\kappa)}}{2\sqrt{\pi} \exp(\kappa)} \int_0^{2\pi} B^{-1/2} \frac{\partial^{n+1}}{\partial q^{n+1}} \\ &\quad \times \left[ \exp\left(\frac{q^2}{4B}\right) \operatorname{erfc}\left(\frac{q}{2\sqrt{B}}\right) \right] \Big|_{q=-A} d\theta. \end{aligned} \quad (6)$$

Various simple expressions can be obtained from (6) by setting specific values for  $n$ . For instance, if  $n = 1$ ,  $n = 3$  and  $n = 5$  then (6) can be reduced to the simple forms

$$\begin{aligned} E(P) &= \frac{\sqrt{h(1+\kappa)}}{8\pi \exp(\kappa)} \int_0^{2\pi} B^{-3} \exp\left(\frac{A^2}{4B}\right) \\ &\quad \times \left[ 2B^{3/2}\sqrt{\pi} + A^2\sqrt{B}\sqrt{\pi} + 2B^{3/2}\sqrt{\pi} \operatorname{erf}\left(\frac{A}{2\sqrt{B}}\right) \right. \\ &\quad \left. + A^2\sqrt{B}\sqrt{\pi} \operatorname{erf}\left(\frac{A}{2\sqrt{B}}\right) \right. \\ &\quad \left. + 2AB \exp\left(-\frac{A^2}{4B}\right) \right] d\theta, \end{aligned} \quad (7)$$

$$\begin{aligned} E(P^3) &= \frac{\sqrt{h(1+\kappa)}}{32\pi \exp(\kappa)} \int_0^{2\pi} B^{-5} \exp\left(\frac{A^2}{4B}\right) \\ &\quad \times \left[ 12B^{5/2}\sqrt{\pi} + 12A^2B^{3/2}\sqrt{\pi} + A^4\sqrt{B}\sqrt{\pi} \right. \\ &\quad \left. + 12B^{5/2}\sqrt{\pi} \operatorname{erf}\left(\frac{A}{2\sqrt{B}}\right) + 12A^2B^{3/2} \right. \\ &\quad \left. \times \sqrt{\pi} \operatorname{erf}\left(\frac{A}{2\sqrt{B}}\right) + 20AB^2 \exp\left(-\frac{A^2}{4B}\right) \right. \\ &\quad \left. + A^4\sqrt{B}\sqrt{\pi} \operatorname{erf}\left(\frac{A}{2\sqrt{B}}\right) \right. \\ &\quad \left. + 2A^3B \exp\left(-\frac{A^2}{4B}\right) \right] d\theta \end{aligned} \quad (8)$$

and

$$\begin{aligned}
E(P^5) &= \frac{\sqrt{h}(1+\kappa)}{128\pi \exp(\kappa)} \int_0^{2\pi} B^{-7} \exp\left(\frac{A^2}{4B}\right) \\
&\times \left[ 120B^{7/2} \sqrt{\pi} + 180A^2 B^{5/2} \sqrt{\pi} + 30A^4 B^{3/2} \sqrt{\pi} \right. \\
&\quad + A^6 \sqrt{B} \sqrt{\pi} + 120B^{7/2} \sqrt{\pi} \operatorname{erf}\left(\frac{A}{2\sqrt{B}}\right) \\
&\quad + 180A^2 B^{5/2} \sqrt{\pi} \operatorname{erf}\left(\frac{A}{2\sqrt{B}}\right) \\
&\quad + 264AB^3 \exp\left(-\frac{A^2}{4B}\right) + 30A^4 B^{3/2} \sqrt{\pi} \left(\frac{A}{2\sqrt{B}}\right) \\
&\quad + 56A^3 B^2 \exp\left(-\frac{A^2}{4B}\right) + A^6 \sqrt{B} \sqrt{\pi} \operatorname{erf}\left(\frac{A}{2\sqrt{B}}\right) \\
&\quad \left. + 2A^5 B \exp\left(-\frac{A^2}{4B}\right) \right] d\theta, \quad (9)
\end{aligned}$$

respectively, where  $\operatorname{erf}(\cdot)$  denotes the error function defined by

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x \exp(-t^2) dt.$$

If  $n$  is an even number then one does not need to use (6) since  $E(P^n) = E((X^2 + Y^2)^{n/2}) / \{E(X^2 + Y^2)\}^{n/2}$ , where  $X$  and  $Y$  are independent Gaussian random variables (see equation (15) in Yacoub *et al.* [1]).

The cpf of  $P$ ,  $F_P(\cdot)$ , can be expressed as

$$\begin{aligned}
F_P(p) &= 1 - \frac{\sqrt{h}(1+\kappa)}{\pi \exp(\kappa)} \int_p^\infty \int_0^{2\pi} x \exp(Ax - Bx^2) d\theta dx \\
&= 1 - \frac{\sqrt{h}(1+\kappa)}{\pi \exp(\kappa)} \int_0^{2\pi} \int_p^\infty x \exp(Ax - Bx^2) dx d\theta \\
&= 1 - \frac{\sqrt{h}(1+\kappa)}{\pi \exp(\kappa)} \\
&\quad \times \int_0^{2\pi} \left[ \int_p^\infty (x-p) \exp(Ax - Bx^2) dx \right. \\
&\quad \left. + p \int_p^\infty \exp(Ax - Bx^2) dx \right] d\theta \\
&= 1 - \frac{\sqrt{h}(1+\kappa)}{\pi \exp(\kappa)} \int_0^{2\pi} [I_1(\theta) + pI_2(\theta)] d\theta. \quad (10)
\end{aligned}$$

By equation (2.3.15.1) in Prudnikov *et al.* [2],  $I_1(\theta)$  and  $I_2(\theta)$  can be calculated as

$$I_1(\theta) = (2B)^{-1} \exp\left\{\frac{A^2}{8B} + \frac{p(A-pB)}{2}\right\} D_{-2}\left(\frac{2pB-A}{\sqrt{2B}}\right) \quad (11)$$

and

$$I_2(\theta) = (2B)^{-1/2} \exp\left\{\frac{A^2}{8B} + \frac{p(A-pB)}{2}\right\} D_{-1}\left(\frac{2pB-A}{\sqrt{2B}}\right). \quad (12)$$

Combining (10), (11) and (12) yields a formula for the cpf  $F_P(\cdot)$ .

Note that all of the formulas in (4), (6), (7), (8), (9) and (10) involve just one integral (with respect to  $\theta$ ) and are much simpler than those in equations (10) and (11) of Yacoub *et al.* [1]. We feel that the formulas given can help the readers and authors of this journal with respect to modeling the statistical variation of the envelope in a fast fading environment.

#### REFERENCES

- [1] M. D. Yacoub, G. Fraidenraich, H. B. Tercius, and F. C. Martins, "The symmetrical  $\eta - \kappa$  distribution: A general fading distribution," *IEEE Transactions on Broadcasting*, vol. 51, pp. 504–511, December 2005.
- [2] A. P. Prudnikov, Y. A. Brychkov, and O. I. Marichev, *Integrals and Series*. Amsterdam: Gordon and Breach Science Publishers, 1986, vol. 1.

#### Corrections to "A General SFN Structure With Transmit Diversity for TDS-OFDM System"

Jin-Tao Wang, Jian Song, Jun Wang, Chang-Yong Pan,  
Zhi-Xing Yang, and Lin Yang

In the above paper [1], the first author's name was misspelled in the byline: "Jian-Tao Wang" should have read: "Jin-Tao Wang".

The corrected byline should read:

Jin-Tao Wang, Jian Song, Jun Wang, Chang-Yopng Pan,  
Zhi-Xing Yang, and Lin Yang

#### REFERENCES

- [1] J.-T. Wang, J. Song, J. Wang, C.-Y. Pan, Z.-X. Yang, and L. Yang, "A general SFN structure with transmit diversity for TDS-OFDM system," *IEEE Trans. Broadcasting*, vol. 52, no. 2, pp. 245–251, Jun. 2006.

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