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# Comments and Corrections

## On the $\eta - \kappa$ Distribution

Saralees Nadarajah and Samuel Kotz

***Index Terms*—Fast fading distribution, $\eta - \kappa$ distribution.**

The recent paper by Yacoub *et al.* [1] introduces what is referred to as the $\eta - \kappa$ *distribution* to describe the statistical variation of the envelope in a fast fading environment. The paper discusses several properties of the distribution. Two of the properties discussed are the $n$th moment, $E(P^n)$, and the cumulative probability function (cpf), $F_P(\cdot)$, where $P$ is a random variable representing the normalized envelope. The expression given for $E(P^n)$ (see equation (10) in Yacoub *et al.* [1]) is a doubly infinite sum of the Gauss hypergeometric function (which, itself, is an infinite sum). That given for $F_P(\cdot)$ (see equation (11) in Yacoub *et al.* [1]) is a triple sum of the incomplete gamma function.

We feel that the expressions in equations (10) and (11) of Yacoub *et al.* [1] are too complicated for practical purposes. In the following, we show how one can derive much simpler forms for $E(P^n)$ and $F_P(\cdot)$. Using equations (5)–(8) in Yacoub *et al.* [1], the probability density function (pdf) of $P$ can be expressed as

$$f_P(p)\frac{\sqrt{h}(1+\kappa)}{\pi\exp(\kappa)}\int_0^{2\pi} p\exp(Ap-Bp^2)d\theta, \tag{1}$$

where $A = 2\sqrt{h\kappa(1+\kappa)}\cos\theta$ and $B = (1+\kappa)h + H(1+\kappa)\cos(2(\theta+\phi))$. Thus, the $n$th moment, $E(P^n)$, can be expressed as

$$\begin{aligned} E(P^n) &= \frac{\sqrt{h}(1+\kappa)}{\pi\exp(\kappa)}\int_0^{\infty}\int_0^{2\pi} p^{n+1}\exp(Ap-Bp^2)d\theta dp \\ &= \frac{\sqrt{h}(1+\kappa)}{\pi\exp(\kappa)}\int_0^{2\pi}\int_0^{\infty} p^{n+1}\exp(Ap-Bp^2)dp d\theta \\ &= \frac{\sqrt{h}(1+\kappa)}{\pi\exp(\kappa)}\int_0^{2\pi} I(\theta)d\theta. \end{aligned} \tag{2}$$

By equation (2.3.15.3) in Prudnikov *et al.* [2], $I(\theta)$ can be calculated as

$$I(\theta) = \Gamma(n+2)(2B)^{-(n/2+1)}\exp\left(\frac{A^2}{8B}\right)D_{-n-2}\left(-\frac{A}{\sqrt{2B}}\right), \tag{3}$$

where $D_p(\cdot)$ denotes the parabolic cylinder function defined by

$$D_p(x) = \frac{\exp(-x^2/4)}{\Gamma(-p)}\int_0^{\infty}\frac{\exp\left\{-(tx+t^2/2)\right\}}{t^{p+1}}dt.$$

Combining (2) and (3) yields the formula

$$E(P^n) = \frac{\Gamma(n+2)\sqrt{h}(1+\kappa)}{2^{n/2+1}\pi\exp(\kappa)}\int_0^{2\pi} B^{-(n/2+1)} \times \exp\left(\frac{A^2}{8B}\right)D_{-n-2}\left(-\frac{A}{\sqrt{2B}}\right)d\theta. \tag{4}$$

This formula applies for any real number $n > -2$. If $n$ is a positive integer then, using equation (2.3.15.7) in Prudnikov *et al.* [2], $I(\theta)$ can be calculated as

$$I(\theta) = \frac{(-1)^{n+1}\sqrt{\pi}}{2\sqrt{B}}\frac{\partial^{n+1}}{\partial q^{n+1}} \times \left[\exp\left(\frac{q^2}{4B}\right)\mathrm{erfc}\left(\frac{q}{2\sqrt{B}}\right)\right]|_{q=-A}, \tag{5}$$

where $\mathrm{erfc}(\cdot)$ denotes the complementary error function defined by

$$\mathrm{erfc}(x) = -\frac{2}{\sqrt{\pi}}\int_0^x \exp(-t^2)dt.$$

Combining (2) and (5) yields the simpler formula

$$E(P^n) = \frac{(-1)^{n+1}\sqrt{h}(1+\kappa)}{2\sqrt{\pi}\exp(\kappa)}\int_0^{2\pi} B^{-1/2}\frac{\partial^{n+1}}{\partial q^{n+1}} \times \left[\exp\left(\frac{q^2}{4B}\right)\mathrm{erfc}\left(\frac{q}{2\sqrt{B}}\right)\right]\Bigg|_{q=-A} d\theta. \tag{6}$$

Various simple expressions can be obtained from (6) by setting specific values for $n$. For instance, if $n=1$, $n=3$ and $n=5$ then (6) can be reduced to the simple forms

$$\begin{aligned} E(P) = \frac{\sqrt{h}(1+\kappa)}{8\pi\exp(\kappa)}\int_0^{2\pi} B^{-3}\exp\left(\frac{A^2}{4B}\right) \\ \times\Bigg[2B^{3/2}\sqrt{\pi} + A^2\sqrt{B}\sqrt{\pi} + 2B^{3/2}\sqrt{\pi}\mathrm{erf}\left(\frac{A}{2\sqrt{B}}\right) \\ + A^2\sqrt{B}\sqrt{\pi}\mathrm{erf}\left(\frac{A}{2\sqrt{B}}\right) \\ + 2AB\exp\left(-\frac{A^2}{4B}\right)\Bigg]d\theta, \end{aligned} \tag{7}$$

$$\begin{aligned} E(P^3) = \frac{\sqrt{h}(1+\kappa)}{32\pi\exp(\kappa)}\int_0^{2\pi} B^{-5}\exp\left(\frac{A^2}{4B}\right) \\ \times\Bigg[12B^{5/2}\sqrt{\pi} + 12A^2B^{3/2}\sqrt{\pi} + A^4\sqrt{B}\sqrt{\pi} \\ + 12B^{5/2}\sqrt{\pi}\mathrm{erf}\left(\frac{A}{2\sqrt{B}}\right) + 12A^2B^{3/2} \\ \times\sqrt{\pi}\mathrm{erf}\left(\frac{A}{2\sqrt{B}}\right) + 20AB^2\exp\left(-\frac{A^2}{4B}\right) \\ + A^4\sqrt{B}\sqrt{\pi}\mathrm{erf}\left(\frac{A}{2\sqrt{B}}\right) \\ + 2A^3B\exp\left(-\frac{A^2}{4B}\right)\Bigg]d\theta \end{aligned} \tag{8}$$

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and

$$E(P^5)=\frac{\sqrt{h}(1+\kappa)}{128\pi\exp(\kappa)}\int_0^{2\pi}B^{-7}\exp\left(\frac{A^2}{4B}\right)\times\Big[120B^{7/2}\sqrt{\pi}+180A^2B^{5/2}\sqrt{\pi}+30A^4B^{3/2}\sqrt{\pi}+A^6\sqrt{B}\sqrt{\pi}+120B^{7/2}\sqrt{\pi}\mathrm{erf}\left(\frac{A}{2\sqrt{B}}\right)+180A^2B^{5/2}\sqrt{\pi}\mathrm{erf}\left(\frac{A}{2\sqrt{B}}\right)+264AB^3\exp\left(-\frac{A^2}{4B}\right)+30A^4B^{3/2}\sqrt{\pi}\left(\frac{A}{2\sqrt{B}}\right)+56A^3B^2\exp\left(-\frac{A^2}{4B}\right)+A^6\sqrt{B}\sqrt{\pi}\mathrm{erf}\left(\frac{A}{2\sqrt{B}}\right)+2A^5B\exp\left(-\frac{A^2}{4B}\right)\Big]d\theta, \quad (9)$$

respectively, where $\mathrm{erf}(\cdot)$ denotes the error function defined by

$$\mathrm{erf}(x)=\frac{2}{\sqrt{\pi}}\int_0^x\exp(-t^2)dt.$$

If $n$ is an even number then one does not need to use (6) since $E(P^n)=E((X^2+Y^2)^{n/2})/\{E(X^2+Y^2)\}^{n/2}$, where $X$ and $Y$ are independent Gaussian random variables (see equation (15) in Yacoub *et al.* [1]).

The cpf of $P$, $F_P(\cdot)$, can be expressed as

$$\begin{aligned}F_P(p)&=1-\frac{\sqrt{h}(1+\kappa)}{\pi\exp(\kappa)}\int_p^\infty\int_0^{2\pi}x\exp(Ax-Bx^2)d\theta dx\\&=1-\frac{\sqrt{h}(1+\kappa)}{\pi\exp(\kappa)}\int_0^{2\pi}\int_p^\infty x\exp(Ax-Bx^2)dxd\theta\\&=1-\frac{\sqrt{h}(1+\kappa)}{\pi\exp(\kappa)}\times\int_0^{2\pi}\left[\int_p^\infty(x-p)\exp(Ax-Bx^2)dx+p\int_p^\infty\exp(Ax-Bx^2)dx\right]d\theta\\&=1-\frac{\sqrt{h}(1+\kappa)}{\pi\exp(\kappa)}\int_0^{2\pi}[I_1(\theta)+pI_2(\theta)]\,d\theta. \quad (10)\end{aligned}$$

By equation (2.3.15.1) in Prudnikov *et al.* [2], $I_1(\theta)$ and $I_2(\theta)$ can be calculated as

$$I_1(\theta)=(2B)^{-1}\exp\left\{\frac{A^2}{8B}+\frac{p(A-pB)}{2}\right\}D_{-2}\left(\frac{2pB-A}{\sqrt{2B}}\right) \quad (11)$$

and

$$I_2(\theta)=(2B)^{-1/2}\exp\left\{\frac{A^2}{8B}+\frac{p(A-pB)}{2}\right\}D_{-1}\left(\frac{2pB-A}{\sqrt{2B}}\right). \quad (12)$$

Combining (10), (11) and (12) yields a formula for the cpf $F_P(\cdot)$.

Note that all of the formulas in (4), (6), (7), (8), (9) and (10) involve just one integral (with respect to $\theta$) and are much simpler than those in equations (10) and (11) of Yacoub *et al.* [1]. We feel that the formulas given can help the readers and authors of this journal with respect to modeling the statistical variation of the envelope in a fast fading environment.

## REFERENCES

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