

2010

Class Notes for Math 921/922: Real Analysis, Instructor Mikil Foss

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Class Notes for Math 921/922: Real Analysis, Instructor Mikil Foss

Topics include: Semicontinuity, equicontinuity, absolute continuity, metric spaces, compact spaces, Ascoli's theorem, Stone Weierstrass theorem, Borel and Lebesgue measures, measurable functions, Lebesgue integration, convergence theorems, L^p spaces, general measure and integration theory, Radon-Nikodym theorem, Fubini theorem, Lebesgue-Stieltjes integration, Semicontinuity, equicontinuity, absolute continuity, metric spaces, compact spaces, Ascoli's theorem, Stone Weierstrass theorem, Borel and Lebesgue measures, measurable functions, Lebesgue integration, convergence theorems, L^p spaces, general measure and integration theory, Radon-Nikodym theorem, Fubini theorem, Lebesgue-Stieltjes integration.

Prepared by Laura Lynch, University of Nebraska-Lincoln

August 2010

Course Information

Office Hours: 11:30-1 MWF

Optional Meeting Time: Thursdays at 5pm

Assessment

Homework	6	34pts
Midterm		33pts
Final		33pts (Dec 13 : 1 – 3pm)

1 Chapter 1

Drawbacks to Riemann Integration

1. Not all bounded functions are Riemann Integrable.
2. All Riemann Integrable functions are bounded.
3. To use the following theorem, we must have $f \in \mathcal{R}[a, b]$.

Theorem 1 (Dominated Convergence Theorem for Riemann Integrals, Arzela). *Let $\{f_n\}_{n=1}^{\infty} \subset \mathcal{R}[a, b]$ and $f \in \mathcal{R}[a, b]$ be given. Suppose there exists $g \in \mathcal{R}[a, b]$ such that $|f_n(x)| < g(x)$ for all $x \in [a, b]$ and $\lim_{n \rightarrow \infty} f_n(x) = f(x)$ for all $x \in [a, b]$, then $\lim_{n \rightarrow \infty} \int_a^b f_n(x) dx = \int_a^b f(x) dx$.*

Example. Define $f_n(x) = \begin{cases} 1 & \text{if } x = \frac{p}{q} \text{ in lowest terms with } 1 \leq q \leq n \text{ on } [0, 1], \\ 0 & \text{otherwise.} \end{cases}$

Then $f_n(x) \rightarrow \chi_{\mathbb{Q} \cap [0, 1]} =: f$. Notice here $|f_n(x)| < 2$ for all $x \in [0, 1]$ but f is not Riemann Integrable. Thus we can not use the theorem.

4. The space $\mathcal{R}[a, b]$ is not complete with respect to many useful metrics.

Good Properties of Riemann Integration

1. $\mathcal{R}[a, b]$ is a vector space
2. The functional $f \mapsto \int_a^b f(x) dx$ is linear on $\mathcal{R}[a, b]$.
3. $\int_a^b f(x) dx \geq 0$ when $f(x) \geq 0$ for all $x \in [a, b]$.
4. Theorem 1 holds.

1.1 Measurable and Topological Spaces

Definition (p 113,119). *Let X be a nonempty set.*

1. A collection \mathcal{T} of subsets of X is called a **topology** of X if it possesses the following properties:

- (a) $\emptyset \in \mathcal{T}$ and $X \in \mathcal{T}$.
- (b) If $\{U_j\}_{j=1}^n \subseteq \mathcal{T}$, then $\bigcap_1^n U_j \in \mathcal{T}$.
- (c) If $\{U_\alpha\}_{\alpha \in A} \subseteq \mathcal{T}$, then $\bigcup_{\alpha \in A} U_\alpha \in \mathcal{T}$.

2. If \mathcal{T} is a topology on X , then (X, \mathcal{T}) is called a **topological space**. If \mathcal{T} is understood, we may just call X itself a topological space. The members of \mathcal{T} are called **open sets** in X . The complements of open sets in X are called **closed sets**.

3. If X and Y are topological spaces and $f : X \rightarrow Y$, then f is called **continuous** if $f^{-1}(V)$ is open in X for all V that are open in Y .

Examples.

1. If $X = \mathbb{R}$, then $\{\emptyset, \mathbb{R}\}$ is a topology on \mathbb{R} .
2. If $X = \mathbb{R}$, then the power set $\{\mathcal{P}(\mathbb{R})\}$ is a topology on \mathbb{R} .
3. $\{\emptyset, \cup_{a>0}\{(-a, a)\}, \mathbb{R}\}$ is a topology on \mathbb{R} .
4. $\{\emptyset, \cup_{a<b\in\mathbb{R}}\{(a, b)\}, \cup_{a\in\mathbb{R}}\{(-\infty, a), (a, \infty)\}, \bar{\mathbb{R}}\}$ is a topology on $\bar{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$.

Definition (pg21,25,43). 1. A collection \mathcal{M} of subsets of X is called a **σ -algebra** if

- (a) $\emptyset \in \mathcal{M}$ and $X \in \mathcal{M}$.
- (b) If $E \in \mathcal{M}$, then $E^C \in \mathcal{M}$.
- (c) If $\{E_j\}_{j=1}^\infty \subseteq \mathcal{M}$, then $\cup_{j=1}^\infty E_j \in \mathcal{M}$.

2. If \mathcal{M} is a σ -algebra on X , then (X, \mathcal{M}) is called a **measurable space**. If \mathcal{M} is understood, then we may just call X itself a measurable space. The members of \mathcal{M} are called **measurable sets**.
3. If (X, \mathcal{M}) and (Y, \mathcal{N}) are measurable spaces, then $f : X \rightarrow Y$ is called **$(\mathcal{M}, \mathcal{N})$ -measurable** or **measurable** if $f^{-1}(E) \in \mathcal{M}$ whenever $E \in \mathcal{N}$.

Examples.

1. $(\mathbb{R}, \{\emptyset, \mathbb{R}\})$ is a σ -algebra
2. $(\mathbb{R}, \{\emptyset, \{1\}, \mathbb{R} \setminus \{1\}, \mathbb{R}\})$ is a σ -algebra
3. $(\mathbb{R}, \{E \subseteq \mathbb{R} | E \text{ or } E^C \text{ is countable}\})$ is a σ -algebra

Lemma 1. If $\{E_j\}_{j=1}^\infty \subseteq \mathcal{P}(X)$, then $\{F_j\}_{j=1}^\infty \subseteq \mathcal{P}(X)$ defined by $F_j = E_j \setminus \cup_{k=1}^{j-1} E_k$ is a sequence of mutually disjoint sets and $\cup_{j=1}^\infty E_j = \cup_{j=1}^\infty F_j$.

Note. As a result of Lemma 1, we can actually modify part (c) of our definition of a σ -algebra to say

(c) If $\{E_j\}_{j=1}^\infty \subseteq \mathcal{M}$ is a sequence of mutually disjoint sets, then $\cup_1^\infty E_j \in \mathcal{M}$.

Remarks.

1. Property (1a) of our definition for σ - algebra could be replaced with “ $\emptyset \in \mathcal{M}$ or $X \in \mathcal{M}$.”
2. If $\{E_j\}_{j=1}^\infty \subseteq \mathcal{M}$, then $\{E_j^C\}_{j=1}^\infty \subseteq \mathcal{M}$ and $\cap_{j=1}^\infty E_j = [\cup_{j=1}^\infty E_j^C]^C \subseteq \mathcal{M}$.
3. If $E, F \in \mathcal{M}$, then $E \setminus F = E \cap F^C \in \mathcal{M}$.

Theorem 2. If \mathcal{E} is a collection of subsets of X , then there exists a unique smallest σ -algebra $\mathcal{M}(\mathcal{E})$ that contains the members of \mathcal{E} . Note: By smallest, we mean any other σ -algebra will contain all the sets in $\mathcal{M}(\mathcal{E})$.

Proof. Let Ω be the family of all σ -algebras containing \mathcal{E} . Note $\Omega \neq \emptyset$ as $\mathcal{P}(X) \in \Omega$. Define $\mathcal{M}(\mathcal{E}) = \cap_{\mathcal{M} \in \Omega} \mathcal{M}$. We want to show $\mathcal{M}(\mathcal{E})$ is a σ -algebra.

1. $\emptyset, X \in \mathcal{M}(\mathcal{E})$ since $\emptyset, X \in \mathcal{M}$ for all $\mathcal{M} \in \Omega$.
2. Let $E \in \mathcal{M}(\mathcal{E})$. Then $E \in \mathcal{M}$ for all $\mathcal{M} \in \Omega$ which implies $E^C \in \mathcal{M}$ for all $\mathcal{M} \in \Omega$ which implies $E^C \in \mathcal{M}(\mathcal{E})$.
3. If $\{E_j\}_{j=1}^\infty \subseteq \mathcal{M}(\mathcal{E})$, then $\{E_j\}_{j=1}^\infty \subseteq \mathcal{M}$ for all $\mathcal{M} \in \Omega$ which implies $\cup_1^\infty E_j \in \mathcal{M}$ for all $\mathcal{M} \in \Omega$ which implies $\cup_1^\infty E_j \in \mathcal{M}(\mathcal{E})$. □

Remark. $\mathcal{M}(\mathcal{E})$ is called the σ -algebra generated by \mathcal{E} .

Definition. Let (X, \mathcal{T}) be a topological space. The σ -algebra generated by \mathcal{T} is called the **Borel σ -algebra** on X and is denoted \mathcal{B}_X . The members of a Borel σ -algebra are called **Borel sets**.

Proposition 1 (p 22). $\mathcal{B}_{\mathbb{R}}$ is generated by each of the following:

1. $\mathcal{E}_1 = \{(a, b) : a < b\}$
2. $\mathcal{E}_2 = \{[a, b] : a < b\}$
3. $\mathcal{E}_3 = \{(a, b) : a < b\}$ or $\mathcal{E}_4 = \{[a, b] : a < b\}$
4. $\mathcal{E}_5 = \{(a, \infty) : a \in \mathbb{R}\}$ or $\mathcal{E}_6 = \{(-\infty, a) : a \in \mathbb{R}\}$
5. $\mathcal{E}_7 = \{[a, \infty) : a \in \mathbb{R}\}$ or $\mathcal{E}_8 = \{(-\infty, a] : a \in \mathbb{R}\}$

Proof. In text. □

Remark. The Borel σ -algebra on $\overline{\mathbb{R}}$ is $\mathcal{B}_{\overline{\mathbb{R}}} = \{E \subseteq \overline{\mathbb{R}} : E \cap \mathbb{R} \in \mathcal{B}_{\mathbb{R}}\}$. It can be generated by $\mathcal{E} = \{(a, \infty] : a \in \mathbb{R}\}$.

Proposition 2. If (X, \mathcal{M}) is a measurable space and $f : X \rightarrow \mathbb{R}$, then TFAE

1. f is measurable
2. $f^{-1}((a, \infty)) \in \mathcal{M}$ for all $a \in \mathbb{R}$
3. $f^{-1}([a, \infty)) \in \mathcal{M}$ for all $a \in \mathbb{R}$
4. $f^{-1}((-\infty, a)) \in \mathcal{M}$ for all $a \in \mathbb{R}$
5. $f^{-1}((-\infty, a]) \in \mathcal{M}$ for all $a \in \mathbb{R}$

Proposition 3 (p 43). Let (X, \mathcal{M}) and (Y, \mathcal{N}) be measurable spaces. If \mathcal{N} is generated by $\mathcal{E} \subseteq \mathcal{P}(Y)$, then $f : X \rightarrow Y$ is $(\mathcal{M}, \mathcal{N})$ -measurable if and only if $f^{-1}(E) \in \mathcal{M}$ for all $E \in \mathcal{E}$.

Proof. (\Rightarrow) Since $\mathcal{E} \subseteq \mathcal{N}$, if f is measurable then $f^{-1}(E) \in \mathcal{M}$ for all $E \in \mathcal{E}$ by definition.

(\Leftarrow) Define $O = \{E \in \mathcal{Y} : f^{-1}(E) \in \mathcal{M}\}$. Want to show O is a σ -algebra. Then since $\mathcal{E} \subseteq O$ and \mathcal{N} is generated by \mathcal{E} , we will be able to conclude $\mathcal{N} \subseteq O$. Recall (p4 of text) $f^{-1}(E^C) = [f^{-1}(E)]^C$ and $f^{-1}(\cup_{j=1}^{\infty} E_j) = \cup_{j=1}^{\infty} f^{-1}(E_j)$.

Claim: O is a σ -algebra.

Proof:

1. Since $f^{-1}(\emptyset) = \emptyset$ and $f^{-1}(Y) = f^{-1}(\emptyset^C) = (f^{-1}(\emptyset))^C = (\emptyset)^C = X \in \mathcal{M}$, we have $\emptyset, Y \in O$.
2. Suppose $F \in O$. Then $f^{-1}(F) \in \mathcal{M}$ by definition of O and $f^{-1}(F^C) = [f^{-1}(F)]^C \in \mathcal{M}$ since \mathcal{M} is a σ -algebra.
3. Suppose $\{F_j\}_{j=1}^{\infty} \in O$. Then $\{f^{-1}(F_j)\}_{j=1}^{\infty} \in \mathcal{M}$ and $f^{-1}(\cup_{j=1}^{\infty} F_j) = \cup_{j=1}^{\infty} f^{-1}(F_j) \in \mathcal{M}$ as \mathcal{M} is a σ -algebra. Thus $\cup_{j=1}^{\infty} F_j \in O$.

Hence O is a σ -algebra on Y , which contains \mathcal{E} . Then $\mathcal{N} \subseteq O$ implies f is measurable. □

Definition. Let X be a nonempty set and let $\{Y_{\alpha}, \mathcal{N}_{\alpha}\}$ be a family of measurable spaces. If $f_{\alpha} : X \rightarrow Y_{\alpha}$ is a map for all $\alpha \in A$ (some index set), then the **σ -algebra on X generated by $\{f_{\alpha}\}_{\alpha \in A}$** is the unique smallest σ -algebra on X that makes each f_{α} measurable. It is generated by $\{f^{-1}(E) : \alpha \in A \text{ and } E \in \mathcal{N}_{\alpha}\}$.

Proposition 4. If (X, \mathcal{M}) is a measurable space, then $f : X \rightarrow \overline{\mathbb{R}}$ is measurable if and only if $f^{-1}((a, \infty]) \in \mathcal{M}$.

Theorem 3. Let (X, \mathcal{M}) be a measurable space, let Y, Z be topological spaces. Let $\phi : Y \rightarrow Z$ be a continuous function. If $f : X \rightarrow Y$ is $(\mathcal{M}, \mathcal{B}_Y)$ -measurable, then $\phi \circ f$ is $(\mathcal{M}, \mathcal{B}_Z)$ -measurable.

Proof. By definition, we need to check $(\phi \circ f)^{-1}(E) \in \mathcal{M}$ for all $E \in \mathcal{B}_Z$. By Proposition 3, we need only to check $(\phi \circ f)^{-1}(E) \in \mathcal{M}$ for all E open in Z . Let E be open in Z . Then $\phi^{-1}(E)$ is open in Y since ϕ is continuous. Since f is $(\mathcal{M}, \mathcal{B}_Y)$ -measurable and $\phi^{-1}(E)$ is open in Y , we have $f^{-1}(\phi^{-1}(E)) \in \mathcal{M}$. \square

Fact. If V is an open set in \mathbb{R}^2 , then there exists a family $\{R_j\}_{j=1}^{\infty}$ of open rectangles in \mathbb{R}^2 satisfying

1. $R_j \subseteq V$ for all $j = 1, 2, 3, \dots$
2. $\cup_{j=1}^{\infty} R_j = V$

Proposition 5. Let (X, \mathcal{M}) be a measurable space. Let $u, v : X \rightarrow \mathbb{R}$ be $(\mathcal{M}, \mathcal{B}_{\mathbb{R}})$ -measurable. If $\phi : \mathbb{R}^2 \rightarrow \mathbb{R}$ is continuous, then $h : X \rightarrow \mathbb{R}$ defined by $h(x) = \phi(u(x), v(x))$ is $(\mathcal{M}, \mathcal{B}_{\mathbb{R}})$ -measurable.

Proof. Define $f : X \rightarrow \mathbb{R}^2$ by $f(x) = (u(x), v(x))$. Since $h = \phi \circ f$ and ϕ is continuous, by Theorem 3 it is enough to show that f is $(\mathcal{M}, \mathcal{B}_{\mathbb{R}^2})$ -measurable. First we show that if $R \subseteq \mathbb{R}^2$ is an open rectangle, then $f^{-1}(R) \in \mathcal{M}$. Let $(a, b), (c, d) \subseteq \mathbb{R}$ be open intervals such that $R = \{(y, z) \in \mathbb{R}^2 \mid a < y < b, c < z < d\}$. If $(u(x), v(x)) \in R$, then $u(x) \in (a, b)$ and $v(x) \in (c, d)$ implies $x \in u^{-1}((a, b)) \cap v^{-1}((c, d))$. Hence $f^{-1}(R) = u^{-1}((a, b)) \cap v^{-1}((c, d))$. Since u, v are $(\mathcal{M}, \mathcal{B}_{\mathbb{R}})$ -measurable, we see $f^{-1}(R) \in \mathcal{M}$ as \mathcal{M} is closed under countable intersections. Now let V be an open set in \mathbb{R}^2 . By the above fact, there is a family $\{R_j\}_{j=1}^{\infty}$ of open rectangles such that $\bigcup_1^{\infty} R_j = V$. So $f^{-1}(V) = f^{-1}\left(\bigcup_1^{\infty} R_j\right) = \bigcup_1^{\infty} f^{-1}(R_j) \in \mathcal{M}$. Thus f is $(\mathcal{M}, \mathcal{B}_{\mathbb{R}^2})$ -measurable. By Theorem 3, $h = \phi \circ f$ is $(\mathcal{M}, \mathcal{B}_{\mathbb{R}})$ -measurable. \square

Proposition 6 (p45). Let (X, \mathcal{M}) be a measurable space. If $c \in \mathbb{R}$ and $f, g : X \rightarrow \mathbb{R}$ are $(\mathcal{M}, \mathcal{B}_{\mathbb{R}})$ -measurable, then

1. cf is $(\mathcal{M}, \mathcal{B}_{\mathbb{R}})$ -measurable
2. $f + g$ is $(\mathcal{M}, \mathcal{B}_{\mathbb{R}})$ -measurable
3. fg is $(\mathcal{M}, \mathcal{B}_{\mathbb{R}})$ -measurable

Proof. 1. Define $\phi : \mathbb{R} \rightarrow \mathbb{R}$ by $\phi(y) = cy$. Then $\phi \circ f = cf$ and by Theorem 3, cf is measurable.

2. Define $\phi : \mathbb{R}^2 \rightarrow \mathbb{R}$ by $\phi(y, z) = y + z$. Then $\phi(f, g) = f + g$ and since ϕ is continuous, by Proposition 5, $f + g$ is measurable.

3. Define $\phi : \mathbb{R}^2 \rightarrow \mathbb{R}$ by $\phi(y, z) = yz$. Then $\phi(f, g) = fg$ and since ϕ is continuous, by Prop 5, fg is measurable. \square

Note. In the above proposition, points (1) and (2) imply its a vector space and adding on point (3) implies it is an algebra. Also, the proposition is true if we consider $f, g : X \rightarrow \overline{\mathbb{R}}$.

Proposition 7. If $\{f_j\}_{j=1}^{\infty}$ is a sequence of $\overline{\mathbb{R}}$ -valued measurable functions on (X, \mathcal{M}) then

$$\begin{aligned} g_1(x) &= \sup_{j \geq 1} f_j(x) & g_3(x) &= \limsup_{j \geq 1} f_j(x) \\ g_2(x) &= \inf_{j \geq 1} f_j(x) & g_4(x) &= \liminf_{j \geq 1} f_j(x) \end{aligned}$$

are all $(\mathcal{M}, \mathcal{B}_{\overline{\mathbb{R}}})$ -measurable. Moreover, if $f(x) = \lim_{j \rightarrow \infty} f_j(x)$ exists for all $x \in X$, then f is measurable.

Proof. Let $a \in \mathbb{R}$ be given. Then $\{x \in X : g_1(x) > a\} = \bigcup_{j=1}^{\infty} \{x \in X : f_j(x) > a\}$ implies $g_1^{-1}((a, \infty]) = \bigcup_1^{\infty} f_j^{-1}((a, \infty]) \in \mathcal{M}$

since f_j is measurable. Thus g_1 is measurable. Also $\{x \in X : g_3(x) > a\} = \bigcap_{j=1}^{\infty} \bigcup_{k=j}^{\infty} \{x \in X : f_k(x) > a\}$ implies $g_3^{-1}((a, \infty]) =$

$\bigcap_{j=1}^{\infty} \bigcup_{k=j}^{\infty} f_k^{-1}((a, \infty]) \in \mathcal{M}$. So g_3 is measurable. Since $g_2(x) = \inf_{j \geq 1} f_j(x) = -\sup_{j \geq 1} -f_j(x)$ and $g_4(x) = -\limsup -f_j(x)$,

we see g_2 and g_4 are measurable. \square

Corollary 1. If $f, g : X \rightarrow \overline{\mathbb{R}}$ are measurable functions, then $\max\{f, g\}$ and $\min\{f, g\}$ are also measurable.

Corollary 2. If $f : X \rightarrow \overline{\mathbb{R}}$ is measurable, then so are $f^+ = \max\{f, 0\}$ and $f^- = -\min\{f, 0\}$.

Corollary 3. If $f : X \rightarrow \overline{\mathbb{R}}$ is measurable, then so is $|f| = f^+ + f^-$.

1.2 Simple Functions (Generalized Step Functions)

Recall that for $E \subseteq X$, the characteristic function of E is

$$\chi_E(x) = \begin{cases} 1 & \text{if } x \in E, \\ 0 & \text{if } x \notin E. \end{cases}$$

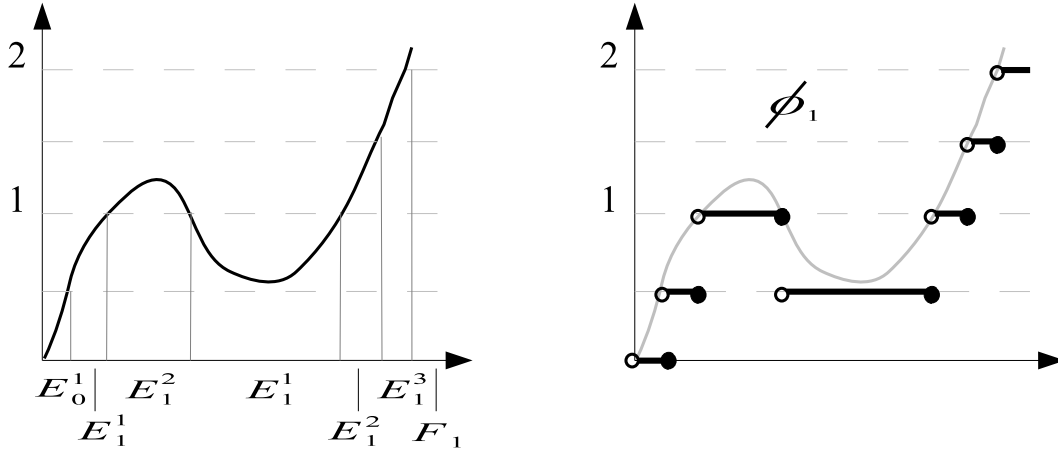
Definition (p46). A **simple function** on X is a measurable function whose range consists of a finite number of values in \mathbb{R} .

If ϕ is a simple function with range $\{a_1, \dots, a_n\}$, then for all $j = 1, 2, \dots, n$, the set $E_j = \phi^{-1}(a_j)$ is measurable. The standard representation for ϕ is $\phi(x) = \sum_{j=1}^n a_j \chi_{E_j}(x)$.

Theorem 4 (p47). Let (X, \mathcal{M}) be a measurable space.

- If $f : X \rightarrow [0, \infty]$ is a measurable function, then there exists a sequence $\{\phi_n\}_{n=0}^{\infty}$ of simple functions such that
 - $0 \leq \phi_0 \leq \phi_1 \leq \dots \leq f$
 - $\phi_n(x) \rightarrow f(x)$ for all $x \in X$
 - ϕ_n converges uniformly to f on the sets where f is uniformly bounded.
- If $f : X \rightarrow \mathbb{C}$ is measurable, there is a sequence $\{\phi_n\}$ of simple functions such that $0 \leq |\phi_1| \leq |\phi_2| \leq \dots \leq |f|$, $\phi_n \rightarrow f$ pointwise and $\phi_n \rightarrow f$ uniformly on any set on which f is bounded.

Proof. (of 1) For all $n = 0, 1, 2, \dots$ and $k = 0, 1, \dots, 2^{2n} - 1$, set $E_n^k = f^{-1}((k2^{-n}, (k+1)2^{-n}])$ and $F_n = f^{-1}((2^n, \infty])$.



Define $\phi_n(x) = \sum_{k=0}^{2^{2n}-1} k2^{-n} \chi_{E_n^k} + 2^n \chi_{F_n}(x)$. We see that

$$E_n^k = f^{-1} \left(\left(k2^{-n}, \left(k + \frac{1}{2} \right) 2^{-n} \right) \right) \cup f^{-1} \left(\left(\left(k + \frac{1}{2} \right) 2^{-n}, (k+1)2^{-n} \right) \right) = E_{n+1}^{2k} \cup E_{n+1}^{2k+1}.$$

On the set E_n^k , we see $\phi_n = k2^{-n} \chi_{E_n^k}$ and $\phi_{n+1} = (2k)2^{-n-1} \chi_{E_{n+1}^{2k}} + (2k+1)2^{-n-1} \chi_{E_{n+1}^{2k+1}} = k2^{-n} \chi_{E_{n+1}^{2k}} + (k + \frac{1}{2})2^{-n} \chi_{E_{n+1}^{2k+1}}$. So $\phi_{n+1} \geq \phi_n$ on each E_n^k . Also, we can see $\phi_{n+1} \geq \phi_n$ on F_n . Therefore $\phi_{n+1} \geq \phi_n$. Since $\phi_n \leq f$, on each E_n^k we see $0 \leq f - \phi_n \leq (k+1)2^{-n} - k2^{-n}$. It follows that $\phi_n \rightarrow f$ and on the sets where f is bounded, it converges uniformly (as these sets fall into some E_n^k .) \square

Definition. Let (X, \mathcal{M}) be a measurable space.

1. A **positive measure** on \mathcal{M} is a function $\mu : \mathcal{M} \rightarrow [0, \infty]$ with the properties $\mu(\emptyset) = 0$ and if $\{E_j\}_{j=1}^{\infty} \subseteq \mathcal{M}$ is a sequence of mutually disjoint sets, then $\mu\left(\bigcup_{j=1}^{\infty} E_k\right) = \sum_{j=1}^{\infty} \mu(E_j)$. To avoid trivialities, we assume $\mu(E) < \infty$ for some $E \subseteq \mathcal{M}$. Usually, we refer to a positive measure as just a **measure**.
2. A **measure space** is a triple (X, \mathcal{M}, μ) where μ is a measure on \mathcal{M} .

Theorem 5. Let (X, \mathcal{M}, μ) be a measure space. Then

1. $\mu(\emptyset) = 0$
2. (monotonicity) If $E, F \in \mathcal{M}$ and $E \subseteq F$, then $\mu(E) \leq \mu(F)$.
3. (subadditivity) If $\{E_j\}_{j=1}^{\infty} \subseteq \mathcal{M}$, then $\mu\left(\bigcup_{j=1}^{\infty} E_j\right) \leq \sum_{j=1}^{\infty} \mu(E_j)$.
4. (continuity from above) If $\{E_j\}_{j=1}^{\infty} \subseteq \mathcal{M}$ and $E_1 \subseteq E_2 \subseteq \dots$, then $\mu(\bigcup E_j) = \lim_{j \rightarrow \infty} \mu(E_j)$.
5. (continuity from below) If $\{E_j\}_{j=1}^{\infty} \subseteq \mathcal{M}$ and $E_1 \supseteq E_2 \supseteq \dots$ and $\mu(E_1) < \infty$, then $\mu(\bigcap E_j) = \lim_{j \rightarrow \infty} \mu(E_j)$.

Proof. 1. Since there exists $E \in \mathcal{M}$ such that $\mu(E) < \infty$, we see $\mu(E) = \mu(E \cup \emptyset) = \mu(E) + \mu(\emptyset)$ since E and \emptyset are disjoint. Now, subtracting $\mu(E)$ from both sides, we see $\mu(\emptyset) = 0$.

2. Let $E, F \in \mathcal{M}$ such that $E \subseteq F$. Then $F = E \cup (F \setminus E)$. Since E and $F \setminus E$ are disjoint, we see

$$\mu(F) = \mu(E \cup (F \setminus E)) = \mu(E) + \mu(F \setminus E) \geq \mu(E).$$

3. Use Lemma 1

4. Let $\{E_j\}_{j=1}^{\infty} \subseteq \mathcal{M}$ satisfying $E_1 \subseteq E_2 \subseteq \dots$ with $E_0 = \emptyset$. Define $F_j = E_j \setminus \bigcup_{k=1}^{j-1} E_k = E_j \setminus E_{j-1}$. By Lemma 1, $\bigcup_{j=1}^{\infty} F_j = \bigcup_{j=1}^{\infty} E_j$. Thus

$$\mu(\bigcup E_j) = \mu(\bigcup F_j) = \sum \mu(F_j) = \sum \mu(E_j \setminus E_{j-1}) = \sum \mu(E_j) - \mu(E_{j-1}) = \lim_{n \rightarrow \infty} \mu(E_n)$$

as $\mu(E_0) = 0$.

5. Similar □

Definition. 1. Let (X, \mathcal{M}, μ) be a measure space. Then a μ -**null set**, or simply **null set**, is a set in \mathcal{M} that has measure 0.

2. If some statement P is true for all points in X except possibly those points in a null set, then we say P holds **almost everywhere (a.e.)** or we may say P holds for almost every $x \in X$ or P holds μ -a.e.

1.3 Integration

Let (X, \mathcal{M}, μ) be a measure space. We set $L^+ = \{f : X \rightarrow [0, \infty] : f \text{ is measurable}\}$.

Definition. Let $\phi \in L^+$ be a simple function. Then there exists $\{a_1, a_2, \dots, a_n\} \subseteq [0, \infty)$ and $\{E_j\}_{j=1}^n \subseteq \mathcal{M}$ such that $\phi = \sum_{j=1}^n a_j \chi_{E_j}$. We define the **Lebesgue Integral of ϕ** with respect to μ by $\int_X \phi d\mu := \sum_{j=1}^n a_j \mu(E_j)$. More generally, if $A \in \mathcal{M}$ is measurable, then we define the **Lebesgue Integral of ϕ over A** with respect to μ as $\int_A \phi d\mu := \int_X \phi \chi_A d\mu = \sum_{j=1}^n a_j \mu(E_j \cap A)$.

Definition. Let $f \in L^+$ be any function. Then the **Lebesgue Integral of f** with respect to μ is

$$\int_X f d\mu = \sup \left\{ \int_X \phi d\mu \mid 0 \leq \phi \leq f, \phi \in L^+, \phi \text{ is simple} \right\}.$$

Also, if $A \in \mathcal{M}$ is measurable, then the **Lebesgue Integral of f over A** with respect to μ is given by $\int_A f d\mu = \int_X f \chi_A d\mu$.

Proposition 8. Let $f, g \in L^+$ and $c \in [0, \infty]$. Then

1. If $f \leq g$, then $\int_X f d\mu \leq \int_X g d\mu$.
2. If $A, B \in \mathcal{M}$ and $A \subseteq B$ then $\int_A f d\mu \leq \int_B f d\mu$.
3. If $A \in \mathcal{M}$, then $\int_A c f d\mu = c \int_A f d\mu$.
4. If $f(x) = 0$ for all $x \in A \subseteq \mathcal{M}$, then $\int_A f d\mu = 0$.
5. If $A \in \mathcal{M}$ and $\mu(A) = 0$, then $\int_A f d\mu = 0$.

Proposition 9. Let $\phi \in L^+$ be a simple function. Define $\lambda : \mathcal{M} \rightarrow [0, \infty]$ by $\lambda(E) = \int_E \phi d\mu$. Then λ is a measure on \mathcal{M} .

Proof. Since ϕ is simple, there exists $\{a_1, a_2, \dots, a_n\} \in [0, \infty)$ and $\{E_j\}_{j=1}^n \subseteq \mathcal{M}$ such that $\phi = \sum_{j=1}^n a_j \chi_{E_j}$. Let $\{A_k\}_{k=1}^\infty \subseteq \mathcal{M}$ be mutually disjoint sets. Then

$$\begin{aligned} \lambda \left(\bigcup_{k=1}^\infty A_k \right) &= \int_{\bigcup_{k=1}^\infty A_k} \phi d\mu \\ &= \int_X \phi \chi_{\bigcup_{k=1}^\infty A_k} d\mu \\ &= \sum_{j=1}^n a_j \mu \left(E_j \cap \left(\bigcup_{k=1}^\infty A_k \right) \right) \\ &= \sum_{j=1}^n a_j \mu \left(\bigcup_{k=1}^\infty (E_j \cap A_k) \right) \\ &= \sum_{j=1}^n a_j \sum_{k=1}^\infty \mu(E_j \cap A_k) \\ &= \sum_{k=1}^\infty \sum_{j=1}^n a_j \mu(E_j \cap A_k) \\ &= \sum_{k=1}^\infty \int_{A_k} \phi d\mu = \sum_{k=1}^\infty \lambda(A_k) \end{aligned}$$

□

Theorem 6 (Monotone Convergence Theorem). Let $\{f_n\}_{n=1}^\infty \subseteq L^+$ be given. Suppose that

1. $f_j \leq f_{j+1}$ for all $j = 1, 2, \dots$
2. $f(x) = \lim_{n \rightarrow \infty} f_n(x)$ for all $x \in X$

Then $f \in L^+$ and $\int_X f d\mu = \lim_{n \rightarrow \infty} \int_X f_n d\mu$.

Proof. Since $f(x) = \sup_{n \geq 1} f_n(x)$, by Prop 7, $f \in L^+$. By Prop 8(1), we see $\{\int_X f_n d\mu\}_{n=1}^\infty \subset [0, \infty]$ is a nondecreasing sequence of real numbers and thus by the MCT for \mathbb{R} , there exists $M \in [0, \infty]$ such that $\lim_{n \rightarrow \infty} \int_X f_n d\mu = M$. Since $f_n \leq f$ for all n , Prop 8(1) also tells us $\int_X f_n d\mu \leq \int_X f d\mu$. Thus $M \leq \int_X f d\mu$. Thus we just need to show $M \geq \int_X f d\mu$.

Let $\alpha \in (0, 1)$ and $\phi \in L^+$ be a simple function such that $0 \leq \phi \leq f$. Set $E_n := \{x \in X \mid f_n(x) \geq \alpha \phi(x)\}$. Since $f_j \leq f_{j+1}$, we see $E_1 \subseteq E_2 \subseteq \dots$. Since $\alpha \phi \leq f$, we also have $\bigcup_{n=1}^\infty E_n = X$. Thus

$$\int_X f_n d\mu \geq \int_{E_n} f_n d\mu \geq \alpha \int_{E_n} \phi d\mu. \tag{1}$$

By Prop 9 and Thm 5(4), $\lim_{n \rightarrow \infty} \int_{E_n} \phi d\mu = \int_X \phi d\mu$. Thus, taking the limit of Equation 1 $M = \lim_{n \rightarrow \infty} \int_X f_n d\mu \geq \alpha \int_X \phi d\mu$. By definition of the Lebesgue Integral for f , taking the sup over ϕ and α gives us $M \geq \int_X f d\mu$. \square

Proposition 10. Let $\phi, \psi \in L^+$ be simple functions. Then $\int_X (\phi + \psi) d\mu = \int_X \phi d\mu + \int_X \psi d\mu$.

Proof. Let $\sum_{j=1}^n a_j \chi_{E_j}$ and $\sum_{k=1}^m b_k \chi_{F_k}$ be the standard representations for ϕ and ψ respectively. Clearly, $E_j = \cup_{k=1}^m (E_j \cap F_k)$ for each j and $F_k = \cup_{j=1}^n (E_j \cap F_k)$ for each k . So

$$\begin{aligned} \int_X (\phi + \psi) d\mu &= \sum_{j=1}^n \sum_{k=1}^m (a_j + b_k) \mu(E_j \cap F_k) \\ &= \sum_{j=1}^n a_j \sum_{k=1}^m \mu(F_k \cap E_j) + \sum_{k=1}^m b_k \sum_{j=1}^n \mu(F_k \cap E_j) \\ &= \sum_{j=1}^n a_j \mu\left(\bigcup_{k=1}^m F_k \cap E_j\right) + \sum_{k=1}^m b_k \mu\left(\bigcup_{j=1}^n F_k \cap E_j\right) \\ &= \sum_{j=1}^n a_j \mu(E_j) + \sum_{k=1}^m b_k \mu(F_k) \\ &= \int_X \phi d\mu + \int_X \psi d\mu \end{aligned}$$

\square

Theorem 7. If $\{f_n\}_{n=1}^\infty \subseteq L^+$ and $f(x) = \sum_{n=1}^\infty f_n(x)$ for all $x \in X$, then $\int_X f d\mu = \sum_{n=1}^\infty \int_X f_n d\mu$.

Proof. First we will show for a sum of two functions, then n functions, then an infinite series of functions. So let $f_1, f_2 \in L^+$, then by Theorem 4, there exists $\{\phi_j\}_{j=1}^\infty, \{\psi_j\}_{j=1}^\infty \subseteq L^+$ such that ϕ_j, ψ_j are simple with

- $0 \leq \phi_1 \leq \phi_2 \leq \dots \leq f_1$ and $0 \leq \psi_1 \leq \psi_2 \leq \dots \leq f_2$
- $\lim \phi_j = f_1$ and $\lim \psi_j = f_2$.

From these it follows that

- $0 \leq \phi_1 + \psi_1 \leq \phi_2 + \psi_2 \leq \dots \leq f_1 + f_2$
- $\lim \phi_j + \psi_j = f_1 + f_2$.

By the Monotone Convergence Theorem and Proposition 10,

$$\int_X (f_1 + f_2) d\mu = \lim_{j \rightarrow \infty} \int_X (\phi_j + \psi_j) d\mu = \lim_{j \rightarrow \infty} \int_X \phi_j d\mu + \int_X \psi_j d\mu = \int_X f_1 d\mu + \int_X f_2 d\mu$$

Using Induction, we can show for n functions. To show for an infinite series, note that

- $0 \leq \sum_{n=1}^1 f_n \leq \sum_{n=1}^2 f_n \leq \dots \leq \sum_{n=1}^\infty f_n$
- $\lim_{N \rightarrow \infty} \sum_{n=1}^N f_n(x) = f(x)$

Thus, applying the Monotone Convergence Theorem again, we see

$$\int_X f d\mu = \lim_{N \rightarrow \infty} \int_X \sum_{n=1}^N f_n d\mu = \lim_{N \rightarrow \infty} \sum_{n=1}^N \int_X f_n d\mu = \sum_{n=1}^\infty \int_X f_n d\mu.$$

\square

Lemma (Fatou's Lemma- P.52). If $\{f_n\}_{n=1}^\infty \subseteq L^+$, then $\int_X (\liminf f_n) d\mu \leq \liminf \int_X f_n d\mu$.

Proof. Define $g_k(x) = \inf_{n \geq k} f_n(x)$ for all k and for all $x \in X$. By Proposition 7, $g_k \in L^+$ for all k . Also $(g_k)_{k=1}^\infty$ is a monotone sequence with $g(x) := \lim_{k \rightarrow \infty} g_k(x) = \liminf_{n \rightarrow \infty} f_n(x)$. By the Monotone Convergence Theorem,

$$\lim_{k \rightarrow \infty} \int_X g_k d\mu = \int_X \lim_{k \rightarrow \infty} g_k d\mu = \int_X \liminf_{n \rightarrow \infty} f_n d\mu.$$

We also see

$$\lim_{k \rightarrow \infty} \int_X g_k d\mu = \liminf_{k \rightarrow \infty} \int_X g_k d\mu \leq \liminf_{k \rightarrow \infty} \int_X f_k d\mu$$

since $g_k \leq f_k$ for all k . Combining these two equations, we get what we wanted. \square

Proposition 11 (P 51). *If $f \in L^+$, then $\int_X f d\mu = 0$ if and only if $f = 0$ a.e.*

Proof. First, we will show for simple functions. Let $\phi \in L^+$ be a simple function and say $\phi = \sum_{j=1}^n a_j \chi_{E_j}$. Suppose $\phi = 0$ a.e. Then either $a_j = 0$ or $\mu(E_j) = 0$ for all $j = 1, \dots, n$. Thus $\int_X \phi d\mu = \sum_{j=1}^n a_j \mu(E_j) = 0$. Now suppose $\int_X \phi d\mu = 0$. Then $a_j \mu(E_j) = 0$ for all $j = 1, \dots, n$, which implies either $a_j = 0$ or $\mu(E_j) = 0$ for all j . Thus $\phi = 0$ a.e.

Now let $f \in L^+$. Suppose $f = 0$ a.e. Then for all simple $\phi \in L^+$ such that $0 \leq \phi \leq f$, $\phi = 0$ a.e. Then $\int_X \phi d\mu = 0$ and by the definition of a Lebesgue Integral, $\int_X f d\mu = \sup\{\int_X \phi d\mu : \phi \in L^+, 0 \leq \phi \leq f, \text{ and } \phi \text{ is simple}\} = 0$. Now suppose $f \neq 0$ a.e. Then for sufficiently large n , $\mu(\{x \in X : f(x) > \frac{1}{n}\}) > 0$. Set $E = \{x \in X : f(x) > \frac{1}{n}\}$. Then $\mu(E) > 0$. Consider the simple functions $\frac{1}{n} \chi_E$. We see $0 \leq \frac{1}{n} \chi_E \leq f$. By Proposition 8(1),

$$\int_X f d\mu \geq \int_X \frac{1}{n} \chi_E d\mu = \frac{1}{n} \mu(E) > 0.$$

\square

Remark. This shows for $f \in L^+$, the Lebesgue Integral does not see values of f on the null sets.

Corollary 4. *If $\{f_n\}_{n=1}^\infty \subseteq L^+$ and $\liminf_{n \rightarrow \infty} f_n(x) \geq f(x)$ a.e. with $f \in L^+$, then $\int_X f d\mu \leq \liminf_{n \rightarrow \infty} \int_X f_n d\mu$.*

Proof. Set $E = \{x \in X : \liminf_{n \rightarrow \infty} f_n(x) < f(x)\}$. By hypothesis, $\mu(E) = 0$. Thus we have

$$\liminf_{n \rightarrow \infty} f_n \chi_{E^c} \geq f \chi_{E^c} \text{ for all } x \in X, \text{ and} \quad (*)$$

$$f \chi_E = 0 \text{ a.e. implies } \int_X f \chi_E d\mu = 0. \quad (**)$$

Using these together with Fatou's Lemma, we see

$$\begin{aligned} \liminf \int_X f_n d\mu &\geq \liminf \int_X f_n \chi_{E^c} d\mu \text{ by Proposition 8(1)} \\ &\geq \int_X \liminf f_n \chi_{E^c} d\mu \text{ by Fatou} \\ &\geq \int_X f \chi_{E^c} d\mu \text{ by Proposition 8(1) and } (*) \\ &= \int_X f \chi_{E^c} d\mu + \int_X f \chi_E d\mu \text{ by Prop 11} \\ &= \int_X f \chi_{E^c} + f \chi_E d\mu \text{ by } (**). \\ &= \int_X f d\mu. \end{aligned}$$

\square

Definition. Let (X, \mathcal{M}, μ) be a measure space. Then the measure μ is **complete** if whenever $E \in \mathcal{M}$ is a nullset, we find $F \in \mathcal{M}$ for all $F \subseteq E$.

Note. If μ is complete, then for $E \in \mathcal{M}$ with $\mu(E) = 0$ and $F \subseteq E$, we must have $\mu(F) = 0$.

Theorem 8. *Suppose (X, \mathcal{M}, μ) is a measure space. Set $\mathcal{N} = \{N \in \mathcal{N} | \mu(N) = 0\}$ and $\overline{\mathcal{M}} = \{E \cup F | E \in \mathcal{M}, F \subseteq N \text{ for some } N \in \mathcal{N}\}$. Then $\overline{\mathcal{M}}$ is a σ -algebra and there exists a unique extension of μ to a measure $\overline{\mu}$ on $\overline{\mathcal{M}}$. Say $\overline{\mu}$ is the **completion** of μ .*

Proof. 1. Clearly $\emptyset, X \in \overline{\mathcal{M}}$.

2. Let $G \in \overline{\mathcal{M}}$. Want to show $G^C \in \overline{\mathcal{M}}$. Find $E \in \mathcal{M}$ and $F' \subseteq N' \in \mathcal{N}$ such that $G = E \cup F'$. Define $N = N' \setminus E$ and $F = F' \setminus E$. Then $G = E \cup F$ and $E \cap N = E \cap F = \emptyset$. Also $F \subseteq N \in \mathcal{N}$. Then $E \cup F' = E \cup F = (E \cap N^C) \cup F = ((E \cup N) \cap N^C) \cup ((E \cup N) \cap F) = (E \cup N) \cap (N^C \cup F)$. So $G^C = (E \cup F')^C = [(E \cup N) \cap (N^C \cup F)]^C = (E \cup N)^C \cup (N^C \cup F)^C = (E \cup N)^C \cup (N \cap F^C)$. Now $E \cup N \in \mathcal{M}$ which implies $(E \cup N)^C \in \mathcal{M}$. Also $N \cap F^C \subseteq N$. So $G^C \in \overline{\mathcal{M}}$ by definition.

3. If $\{G_j\}_{j=1}^\infty \subseteq \overline{\mathcal{M}}$, then there exists $\{E_j\}_{j=1}^\infty \subseteq \mathcal{M}$ and $\{N_j\}_{j=1}^\infty \subseteq \mathcal{M}$ and $\{F_j\}_{j=1}^\infty$ such that $F_j \subseteq N_j$ and $G_j = E_j \cup F_j$. Then $\bigcup_{j=1}^\infty G_j = \bigcup_{j=1}^\infty E_j \cup F_j = \left(\bigcup_{j=1}^\infty E_j \right) \cup \left(\bigcup_{j=1}^\infty F_j \right)$. Notice that $\bigcup_{j=1}^\infty F_j \subseteq \bigcup_{j=1}^\infty N_j$ and $\mu(\bigcup_{j=1}^\infty N_j) \leq \sum \mu(N_j) = 0$. So $\bigcup_{j=1}^\infty F_j \subseteq N \in \mathcal{N}$. So $\bigcup_{j=1}^\infty G_j \subseteq \overline{\mathcal{M}}$. □

Definition. Define $\bar{\mu} : \overline{\mathcal{M}} \rightarrow [0, \infty]$ by $\bar{\mu}(E) = \mu(E)$ for all $E \in \mathcal{M}$ and $\bar{\mu}(E \cup F) = \mu(E)$ for all $E \in \mathcal{M}$ and $F \subseteq N \in \mathcal{N}$.

Notes.

1. $\bar{\mu}$ defines a measure. (prove)

2. $\bar{\mu}$ is well-defined and unique.

- Well-defined: Suppose $E_1 \cup F_1 = E_2 \cup F_2$ with $E_1, E_2 \in \mathcal{M}$ and $F_1 \subseteq N_1 \in \mathcal{N}, F_2 \subseteq N_2 \in \mathcal{N}$. Then $\bar{\mu}(E_1 \cup F_1) = \mu(E_1) \leq \mu(E_2 \cup N_2) = \mu(E_2) = \bar{\mu}(E_2 \cup F_2)$. Similarly, \geq . So $\bar{\mu}(E_1 \cup F_1) = \bar{\mu}(E_2 \cup F_2)$.
- Unique: Suppose $\bar{\nu} : \overline{\mathcal{M}} \rightarrow [0, \infty]$ is another completion. Let $E \cup F \in \overline{\mathcal{M}}$. Then $\bar{\nu}(E \cup F) \leq \bar{\nu}(E \cup N) = \mu(E \cup N) = \mu(E) = \bar{\mu}(E \cup F) = \mu(E) = \bar{\nu}(E) \leq \bar{\nu}(E \cup F)$. Thus $\bar{\nu}(E \cup F) = \bar{\mu}(E \cup F)$.

Definition. Let (X, \mathcal{M}, μ) be a measure space. If $E = \bigcup_{j=1}^\infty E_j$ with $\{E_j\}_{j=1}^\infty \subseteq \mathcal{M}$ and $\mu(E_j) < \infty$ for all j , then E is σ -finite.

Proposition 12 (p. 52). If $f \in L^+$ and $\int_X f d\mu < \infty$, then $\{x \in X | f(x) = \infty\}$ is a null set and $\{x \in X | f(x) > 0\}$ is a σ -finite set.

Proof. Set $E = \{x \in X | f(x) = \infty\}$. Then $\infty > \int_X f d\mu \geq \int_E f d\mu = \infty \cdot \mu(E)$, which implies $\mu(E) = 0$. Also for all $j \geq 1$, set $E_j = \{x \in X | f(x) > \frac{1}{j}\}$. Then $\{x \in X | f(x) > 0\} = \bigcup E_j$ and $\infty > \int_X f d\mu \geq \int_{E_j} f d\mu > \frac{1}{j} \mu(E_j)$, which says $\mu(E_j) < \infty$. □

Definition. Let (X, \mathcal{M}, μ) be a measure space. Define $\tilde{L}'(X, \mathcal{M}, \mu)$ to be the collection of all measurable functions $f : X \rightarrow \overline{\mathbb{R}}$ such that $\int_X |f| d\mu < \infty$.

Note. If f is measurable, so is $|f| \in L^+$ and $|f| = f^+ + f^-$. Thus $\int f^\pm d\mu \leq \int |f| d\mu < \infty$ if $f \in \tilde{L}'$.

Definition. If $f \in \tilde{L}'(X, \mathcal{M}, \mu)$, then f is **integrable** and define $\int_X f d\mu = \int_X f^+ d\mu - \int_X f^- d\mu$.

Proposition 13 (p. 53). If $f \in \tilde{L}'(X, \mathcal{M}, \mu)$, then $|\int_X f d\mu| \leq \int_X |f| d\mu$.

Proof. By Theorem 7,

$$\left| \int_X f d\mu \right| = \left| \int_X f^+ d\mu - \int_X f^- d\mu \right| \leq \left| \int_X f^+ d\mu + \int_X f^- d\mu \right| \leq \int_X |f| d\mu = \int_X |f| d\mu. \quad \square$$

Proposition 14 (p. 54). If $f, g \in \tilde{L}'(X, \mathcal{M}, \mu)$, then TFAE

1. $\int_E f d\mu = \int_E g d\mu$ for all $E \in \mathcal{M}$.
2. $\int_X |f - g| d\mu = 0$

3. $f = g$ a.e.

Proof. (2) \Rightarrow (3) By Prop 11

(3) \Rightarrow (2) If $f = g$ a.e., then $f - g = 0$ a.e. which implies $|f - g| = 0$ a.e. and thus $\int_X |f - g|d\mu = 0$ by Prop 11.

(2) \Rightarrow (1) If $\int_X |f - g|d\mu = 0$, then for all $E \in \mathcal{M}$

$$\left| \int_E f d\mu - \int_E g d\mu \right| = \left| \int_X (f - g)\chi_E d\mu \right| \leq \int_X |f - g|\chi_E d\mu \leq \int_X |f - g|d\mu = 0.$$

Thus $\int_E f d\mu = \int_E g d\mu$.

(1) \Rightarrow (3) Contrapositive. Then $\mu(\{x \in X | f(x) - g(x) \neq 0\}) > 0$. Define $E_1 = \{x \in X | f(x) - g(x) > 0\}$ and $E_2 = \{x \in X | f(x) - g(x) < 0\}$. Then either $\mu(E_1), \mu(E_2)$ or both are > 0 . Suppose $\mu(E_1) > 0$. Then $(f - g)\chi_{E_1} \in L^+$ and so $\int_{E_1} (f - g)d\mu > 0$. This implies $\int_{E_1} f d\mu > \int_{E_1} g d\mu$ and so (1) does not hold. Similarly for $\mu(E_2) > 0$. \square

Note. We say f and g are related if $f = g$ μ -a.e. This defines an equivalence relation between functions in $\tilde{L}^1(X, \mathcal{M}, \mu)$.

Definition. Let (X, \mathcal{M}, μ) be a measure space. Define $L^1(X, \mu)$ to be the collection of all equivalence classes of integrable functions with respect to the relation just described.

Notation. If we write $f \in L^1(\mu)$, then we really mean f is a representative for its equivalence class.

Proposition 15 (p.47). Suppose μ is a complete measure. Then

1. If f is measurable and $g = f$ μ -a.e., then g is measurable.
2. If $\{f_n\}_{n=1}^\infty$ is a sequence of measurable functions and $\lim_{n \rightarrow \infty} f_n(x) = f(x)$ a.e., then f is measurable.

Proposition 16 (p.48). Let (X, \mathcal{M}, μ) be a measure space and $(X, \overline{\mathcal{M}}, \overline{\mu})$ be its completion. If $f : X \rightarrow \overline{\mathbb{R}}$ is $(\overline{\mathcal{M}}, \mathcal{B}_{\overline{\mathbb{R}}})$ -measurable, then there exists an $(\mathcal{M}, \mathcal{B}_{\overline{\mathbb{R}}})$ -measurable function g such that $f = g$ $\overline{\mu}$ -a.e.

Note. We identify $L^1(\mu)$ with $L^1(\overline{\mu})$.

Definition. Let (X, \mathcal{M}, μ) be a measure space. Define $\rho_1 : L^1(\mu) \times L^1(\mu) \rightarrow [0, \infty)$ by $\rho_1(f, g) = \int_X |\tilde{f} - \tilde{g}|d\mu$ where $\tilde{f}, \tilde{g} \in \tilde{L}^1(X, \mathcal{M}, \mu)$ are representatives for the equivalence classes f and g .

Proposition 17. The function ρ_1 is a metric on $L^1(\mu)$.

Proof. Clearly, $\rho_1(f, g) = \rho_1(g, f)$. Also if $f, g, h \in L^1(\mu)$, then

$$\rho_1(f, g) = \int_X |f - g|d\mu = \int_X |f - h + h - g|d\mu \leq \int_X |f - h| + |h - g|d\mu = \rho_1(f, h) + \rho_1(h, g).$$

Finally, let $f, g \in L^1(\mu)$ and $\tilde{f}, \tilde{g} \in \tilde{L}^1(X, \mathcal{M}, \mu)$ be representatives for f and g . Then $\rho_1(f, g) = \int_X |\tilde{f} - \tilde{g}|d\mu = 0$ if and only if $\tilde{f} = \tilde{g}$ a.e. which happens if and only if f, g are in the same equivalence class. \square

Definition. If $\{f_n\}_{n=1}^\infty \subseteq L^1(\mu)$ and $f \in L^1(\mu)$ satisfies $\lim_{n \rightarrow \infty} \rho_1(f_n, f) = 0$, then we write $f_n \rightarrow f$ in $L^1(\mu)$ and say f_n converges (**strongly**) to f in $L^1(\mu)$.

Theorem (Lebesgue's Dominated Convergence Theorem). Let $\{f_n\}_{n=1}^\infty \subseteq L^1(\mu)$ be a sequence such that $\lim_{n \rightarrow \infty} f_n(x) = f(x)$ μ -a.e. and there exists $g \in L^1(\mu)$ such that $|f_n(x)| \leq g$ μ -a.e. for all n . Then $f \in L^1(\mu)$ and $\int_X f d\mu = \lim_{n \rightarrow \infty} \int_X f_n d\mu$.

Proof. By Propositions 15 and 16, we may assume f is measurable. By hypothesis, we see $|f(x)| \leq g(x)$ μ -a.e. which implies $\int_X |f|d\mu \leq \int_X g d\mu < \infty$. So $f \in L^1(\mu)$. Since $|f_n(x)| < g(x)$ μ -a.e., we also see that $g + f_n \geq 0$ and $g - f_n \geq 0$ μ -a.e. for all n . Notice that since $\lim_{n \rightarrow \infty} f_n = f$ μ -a.e., $\liminf_{n \rightarrow \infty} (g + f_n)(x) = g(x) + f(x)$ μ -a.e. and $\liminf_{n \rightarrow \infty} (g - f_n)(x) = g(x) - f(x)$ μ -a.e. By the corollary to Fatou's Lemma (Corollary 4), $\int_X (g + f)d\mu \leq \liminf_{n \rightarrow \infty} \int_X (g + f_n)d\mu = \int_X g d\mu + \liminf_{n \rightarrow \infty} \int_X f_n d\mu$ and $\int_X (g - f)d\mu \leq \liminf_{n \rightarrow \infty} \int_X (g - f_n)d\mu = \int_X g d\mu - \limsup_{n \rightarrow \infty} \int_X f_n d\mu$. Thus $\limsup_{n \rightarrow \infty} \int_X f_n d\mu \leq \int_X f d\mu \leq \liminf_{n \rightarrow \infty} \int_X f_n d\mu$. Since \geq is obvious, we see they are all $=$ and thus $\int_X f d\mu = \lim_{n \rightarrow \infty} \int_X f_n d\mu$. \square

Corollary 5. Suppose $\{f_n\}_{n=1}^\infty \subseteq L^1(\mu)$ satisfies the hypotheses of the LDC Theorem. Then $f_n \rightarrow f$ in $L^1(\mu)$, that is, $\lim_{n \rightarrow \infty} \int_X |f_n - f|d\mu = 0$.

Proof. Notice

- $\lim_{n \rightarrow \infty} |f_n(x) - f(x)| = 0$ μ -a.e.
- $|f_n(x) - f(x)| \leq 2g(x)$ μ -a.e. for all n .

Then by the LDC Theorem, $\int_X |f_n - f|d\mu = \int_X 0d\mu = 0$. \square

Theorem 9 (p 55). Suppose $\{f_n\}_{n=1}^\infty \subseteq L^1(\mu)$ satisfies $\sum_{n=1}^\infty \int_X |f_n|d\mu < \infty$. Then $\sum_{n=1}^\infty f_n$ converges μ -a.e. to some function $f \in L^1(\mu)$ and $\int_X \sum_{n=1}^\infty f_n d\mu = \sum_{n=1}^\infty \int_X f_n d\mu$.

Proof. Define $g(x) = \sum_{n=1}^\infty |f_n(x)|$ for all $x \in X$. By Theorem 7 and our hypotheses

$$\int_X g(x)d\mu = \int_X \sum_{n=1}^\infty |f_n|d\mu = \sum_{n=1}^\infty \int_X |f_n|d\mu < \infty.$$

Then $g \in L^1(\mu)$. By Proposition 12, $\sum_{n=1}^\infty |f_n(x)| < \infty$ μ -a.e. Hence $\sum f_n$ convergence absolutely μ -a.e. So we may put $f(x) = \sum_{n=1}^\infty f_n(x)$ for those x where the series converges and $f(x) = 0$ everywhere else (i.e., on a null set). Moreover,

$|\sum_{n=1}^\infty f_n(x)| \leq \sum_{n=1}^\infty |f_n(x)| \leq g(x)$ μ -a.e. By the LDC Theorem, $f \in L^1(\mu)$ and $\int_X f d\mu = \int_X \lim_{N \rightarrow \infty} \sum_{n=1}^N f_n(x) =$

$$\lim_{N \rightarrow \infty} \sum_1^N \int_X f_n(x) = \sum_1^\infty \int_X f_n d\mu. \quad \square$$

Types of Convergence

- $f_n \rightarrow f$ pointwise if $\lim_{n \rightarrow \infty} f_n(x) = f(x)$ for all $x \in X$.
- $f_n \rightarrow f$ a.e. if $\lim_{n \rightarrow \infty} f_n(x) = f(x)$ μ -a.e.
- $f_n \rightarrow f$ uniformly if for all $\epsilon > 0$ there exists N_ϵ such that for all $n > N_\epsilon$ we have $|f_n - f| < \epsilon$ for all $x \in X$.
- $f_n \rightarrow f$ in L^1 if $\lim_{n \rightarrow \infty} \int |f_n - f|d\mu = 0$. (**strong convergence**)
- $f_n \rightarrow f$ in measure if for all $\epsilon > 0$ we have $\lim_{n \rightarrow \infty} \mu(\{x \in X \mid |f_n(x) - f(x)| \geq \epsilon\}) = 0$.

Definition. We say that $\{f_n\}_{n=1}^\infty \subseteq L^1(\mu)$ is **Cauchy in measure** if for all $\epsilon > 0$ we have $\lim_{m, n \rightarrow \infty} \mu(\{x \in X \mid |f_n(x) - f_m(x)| \geq \epsilon\}) = 0$.

Proposition 18. Suppose $\{f_n\}_{n=1}^\infty \subseteq L^1(\mu)$ and $f \in L^1(\mu)$. If $f_n \rightarrow f$ in L^1 , then $f_n \rightarrow f$ in measure.

Proof. Let $\epsilon > 0$ be given. Set $E_n = \{x \in X \mid |f_n(x) - f(x)| \geq \epsilon\}$. Now

$$0 = \frac{1}{\epsilon} \lim_{n \rightarrow \infty} \int_X |f_n - f|d\mu \geq \lim_{n \rightarrow \infty} \frac{1}{\epsilon} \int_{E_n} |f_n - f|d\mu \geq \lim_{n \rightarrow \infty} \frac{1}{\epsilon} \int_{E_n} \epsilon d\mu = \lim_{n \rightarrow \infty} \mu(E_n) \geq 0.$$

Thus $\mu(E_n) = 0$. \square

Theorem 10. Suppose that $\{f_n\}_{n=1}^\infty$ is a sequence of measurable functions that are Cauchy in measure. Then there exists a measurable function f such that $f_n \rightarrow f$ in measure.

Proof. Choose $\{g_j\}_{j=1}^\infty = \{f_{n_j}\}_{j=1}^\infty \subseteq \{f_n\}_{n=1}^\infty$ such that for all j we have

$$\mu(\underbrace{\{x \in X : |g_j(x) - g_{j+1}(x)| \geq 2^{-j}\}}_{E_j}) \leq 2^{-j}.$$

Set $F_k = \bigcup_{j=k}^\infty E_j$. Then $\mu(F_k) \leq \sum_{j=k}^\infty \mu(E_j) \leq \sum_{j=k}^\infty 2^{-j} = 2^{1-k}$. For $x \notin F_k$, we have for all $i \geq j \geq k$ $|g_j(x) - g_i(x)| \leq \sum_{\ell=j}^{i-1} |g_{\ell+1}(x) - g_\ell(x)| \leq \sum_{\ell=j}^{i-1} 2^{-\ell} \leq 2^{1-j}$. It follows that $\{g_j\}_{j=1}^\infty$ is pointwise Cauchy on F_k^C for all k . Then there exists $f : X \rightarrow \mathbb{R}$ such that $g_j \rightarrow f$ on F_k^C for all k , that is, $g_j \rightarrow f$ pointwise on $X \setminus (\bigcap_1^\infty F_k)$ and $f = 0$ on the null set.

Since $\mu(F_1) = \sum \mu(E_j) \leq \sum 2^{-j} = 1$ and $F_1 \supseteq F_2 \supseteq F_3 \supseteq \dots$, we find that $0 \leq \mu(\bigcap_1^\infty F_k) = \lim \mu(F_k) \leq \lim 2^{1-k} = 0$. Thus $\mu(\bigcup_1^\infty F_k) = 0$. Thus $g_j \rightarrow f$ μ -a.e. and by Proposition 15, f is measurable.

For each $x \notin F_j$ we see

$$|g_j(x) - f(x)| \leq \lim_{i \rightarrow \infty} |g_j(x) - g_i(x)| + \lim_{i \rightarrow \infty} |g_i(x) - f(x)| \leq \lim_{i \rightarrow \infty} \sum_{\ell=j}^{i-1} |g_{\ell+1}(x) - g_\ell(x)| \leq \lim_{i \rightarrow \infty} \sum_{\ell=j}^{i-1} 2^{-\ell} \leq 2^{1-j}.$$

(We know $\lim |g_i(x) - f(x)| = 0$ as $x \notin F_j$ implies $x \notin F_i$.) Thus $g_j \rightarrow f$ in measure.

Observe $|f_n(x) - f(x)| \leq |f_n(x) - g_j(x)| + |g_j(x) - f(x)|$. So if $|f_n(x) - f(x)| \geq \epsilon$ then either $|f_n(x) - g_j(x)| \geq \frac{\epsilon}{2}$ or $|g_j(x) - f(x)| \geq \frac{\epsilon}{2}$. Thus

$$\mu(\{x \in X : |f_n(x) - f(x)| \geq \epsilon\}) \leq \mu\left(\left\{x \in X : |f_n(x) - g_j(x)| \geq \frac{\epsilon}{2}\right\}\right) + \mu\left(\left\{x \in X : |g_j(x) - f(x)| \geq \frac{\epsilon}{2}\right\}\right).$$

So taking the limit of both sides as $n, j \rightarrow \infty$, we get

$$\lim_{n \rightarrow \infty} \mu(\{x \in X : |f_n(x) - f(x)| \geq \epsilon\}) = 0$$

since $\lim \mu(\{x \in X : |f_n(x) - g_j(x)| \geq \frac{\epsilon}{2}\}) = 0$ for f_n is Cauchy in measure and $\lim \mu(\{x \in X : |g_j(x) - f(x)| \geq \frac{\epsilon}{2}\}) = 0$ for f_n converges in measure. \square

Theorem 11. Suppose $\{f_n\}_{n=1}^\infty$ is a sequence of measurable functions such that $f_n \rightarrow f$ in measure with f measurable. Then there exists $\{f_{n_j}\}_{j=1}^\infty \subseteq \{f_n\}_{n=1}^\infty$ such that $f_{n_j} \rightarrow f$ μ -a.e.

Proof. Choose a subsequence $\{f_{n_j}\}_{j=1}^\infty$ such that $\mu(\underbrace{\{x \in C : |f_{n_j} - f| \geq 2^{-j}\}}_{E_j}) \leq 2^{-j}$. Setting $F_k = \bigcup_{j=k}^\infty E_j$, $\mu(F_k) \leq 2^{1-k}$.

For $x \notin F_k$ and $j \geq k$ we see that $|f_{n_j}(x) - f(x)| \leq 2^{-j}$. It follows that $f_{n_j} \rightarrow f$ pointwise in $X \setminus \bigcap_{k=1}^\infty F_k$. Thus $f_{n_j} \rightarrow f$ μ -a.e. since $\mu(\bigcup_{k=1}^\infty F_k) = 0$. \square

Theorem 12. Suppose $\{f_n\}_{n=1}^\infty$ is a sequence of measurable functions and $f_n \rightarrow f$ and $f_n \rightarrow g$ in measure for some measurable f and g . Then $f = g$ μ -a.e.

Corollary 6. If $\{f_n\}_{n=1}^\infty \subseteq L^1(\mu)$ and $f \in L^1(\mu)$ with $f_n \rightarrow f$ in L^1 , then there exists a subsequence $\{f_{n_j}\}_{j=1}^\infty$ such that $f_{n_j} \rightarrow f$ μ -a.e.

Examples. Take $X = \mathbb{N}$, $\mathcal{M} = \mathcal{P}(\mathbb{N})$, $\mu(E) =$ the number of elements in E . (that is, the counting measure).

- If $f \in L^+(\mu)$, then $\int_X f d\mu = \sum_1^\infty f(k)$
- If $f \in L^1(\mu)$, then $\sum_{k=1}^\infty |f(k)| = \int_{\mathbb{N}} |f| d\mu < \infty$. So $\sum_{k=1}^\infty f(k)$ is absolutely convergent.
- Suppose that $f_n(k) = \frac{k}{n}$. Then $f_n(k) \rightarrow 0$ pointwise (and thus μ -a.e.), but not uniformly (as for all $\epsilon > 0$, $\frac{k}{n} \geq \epsilon$ when $k \geq n\epsilon$) and not in measure (as $\mu(\{k \in \mathbb{N} : \frac{k}{n} \geq \epsilon\}) = \infty$)
- Suppose that $f_n(k) = \frac{1}{n}$. Then $f_n(k) \rightarrow 0$ pointwise, uniformly, in measure, and μ -a.e., but not in L^1 .

- Suppose that $f_n(k) = \begin{cases} \frac{k}{n^2} & \text{for } k \leq n, \\ 0 & \text{otherwise.} \end{cases}$ Then $f_n(k) \rightarrow 0$ pointwise, uniformly, μ -a.e., and in measure, but not in L^1 .

Theorem (Egoroff's Theorem). Suppose $\mu(X) < \infty$ and f_1, f_2, \dots, f are complex valued and measurable functions on X such that $f_n \rightarrow f$ a.e. Then for all $\epsilon > 0$ there exists $E \subseteq X$ such that $\mu(E) < \epsilon$ and $f_n \rightarrow f$ uniformly on $X \setminus E$.

Proof. Let N be the set of all points where $f_n(x) \not\rightarrow f(x)$. So $\mu(N) = 0$. For each $k, n \in \mathbb{N}$, define $E_{n,k} = \bigcup_{m=n}^{\infty} \{x \in X \setminus N : |f_m(x) - f(x)| \geq \frac{1}{k}\}$. Observe for all k that $E_{n+1,k} \subseteq E_{n,k}$ and $\bigcap_{n=1}^{\infty} E_{n,k} = \emptyset$. Since $\mu(E_{1,k}) \leq \mu(X) < \infty$, we may use Theorem 5 to conclude that $0 = \mu(\bigcap_{n=1}^{\infty} E_{n,k}) = \lim \mu(E_{n,k})$. So for all k there exists n_k such that $\mu(E_{n_k,k}) < 2^{-k}\epsilon$. Set $E = N \cup (\bigcup_{k=1}^{\infty} E_{n_k,k})$. Then $\mu(E) \leq \mu(N) + \mu(\bigcup_{k=1}^{\infty} E_{n_k,k}) \leq \epsilon \sum_{k=1}^{\infty} 2^{-k} < \epsilon$. If $x \notin E$, then for all $n > n_k$, $|f_n(x) - f(x)| < \frac{1}{k}$, that is, $f_n \rightarrow f$ uniformly on $X \setminus E$. \square

1.4 L^p Spaces

Definition. A function $F : (a, b) \rightarrow \mathbb{R}$ is **convex** on $(a, b) \subseteq \mathbb{R}$ if

$$F(\lambda x + (1 - \lambda)y) \leq \lambda F(x) + (1 - \lambda)F(y)$$

for all $\lambda \in [0, 1]$ and $x, y \in (a, b)$.

Theorem 13. If F is convex on $(a, b) \subseteq \mathbb{R}$, then for all $[x, y] \subset (a, b)$ with $x < y$, we find that there exists $M < \infty$ such that $F(s) \geq -M$ for all $s \in [x, y]$.

Proof. Suppose there does not exist $M < \infty$. Then for all $n \in \mathbb{N}$ there exists $s_n \in [x, y]$ such that $F(s_n) < -n$. Since $[x, y]$ is compact, there exists a subsequence (call it s_n for simplicity) such that $s_n \rightarrow s^*$ for some $s^* \in [x, y]$. Let $s \in [x, y] \setminus \{s^*\}$ be given. For each $\lambda \in [0, 1)$, we have

$$F(\lambda s + (1 - \lambda)s_n) \leq \lambda F(s) + (1 - \lambda)F(s_n) < \lambda F(s) + (1 - \lambda)(-n) \rightarrow -\infty.$$

It follows that $F(\lambda s + (1 - \lambda)s^*) = -\infty$ for all $\lambda \in [0, 1)$. So $F(s) = -\infty$ for all $s \in [x, y] \setminus \{s^*\}$, which contradicts the fact that $F : (a, b) \rightarrow \mathbb{R}$. \square

Theorem 14. If F is convex on $(a, b) \subseteq \mathbb{R}$, then F is continuous on (a, b) .

Proof. We will first prove a claim.

Claim: For each $x, y, z \in (a, b)$ satisfying $x < y < z$, we have $\frac{F(y) - F(x)}{y - x} < \frac{F(z) - F(y)}{z - y}$.

Proof: Let $y = \lambda x + (1 - \lambda)z$ with $\lambda = \frac{z - y}{z - x}$. Then

$$F(y) \leq \frac{z - y}{z - x} F(x) + \frac{y - x}{z - x} F(z)$$

as F is convex. This implies

$$F(x) \geq \frac{z - x}{z - y} F(y) - \frac{y - x}{z - y} F(z)$$

and thus

$$\frac{-1}{y - x} F(x) \leq \frac{1}{z - y} F(z) - \frac{z - x}{(y - x)(z - y)} F(y).$$

Thus

$$\frac{F(y) - F(x)}{y - x} \leq \frac{1}{y - x} F(y) + \frac{1}{z - y} F(z) - \frac{z - x}{(y - x)(z - y)} F(y) = \frac{1}{y - z} F(y) + \frac{1}{z - y} F(z) = \frac{F(z) - F(y)}{z - y}.$$

Let $[x, y] \subset (a, b)$ with $x < y$ be given. Then F is uniformly bounded from below by Theorem 13. Let $s \in (x, y)$ and $t \in (s, y)$. So $x < s < t < y$. Then

$$\frac{F(s) - F(x)}{s - x} \leq \frac{F(t) - F(s)}{t - s} \leq \frac{F(y) - F(t)}{y - t}$$

which implies

$$\frac{t-s}{s-x}[F(s) - F(x)] + F(s) \leq F(t) \leq \frac{t-s}{y-t}[F(y) - F(t)] + F(s).$$

Since F is uniformly bounded, the RHS does not blow up, so as $t \rightarrow s$ we see $F(t) \rightarrow F(s)$. Similarly for $t \in (x, s)$. Thus $\lim_{t \rightarrow s} F(t) = F(s)$. \square

Theorem 15 (Jensen's Inequality). *Suppose that (X, \mathcal{M}, μ) is a measure space with $\mu(X) < \infty$. If F is a convex function on \mathbb{R} and $f \in L^1(\mu)$, then*

$$F\left(\frac{1}{\mu(X)} \int_X f d\mu\right) \leq \frac{1}{\mu(X)} \int_X F \circ f d\mu.$$

Proof. Since $f \in L^1(\mu)$, $\int_X |f| d\mu < \infty$. By Proposition 12, $\{x \in X : |f| = +\infty\}$ is a nullset. So WLOG we may assume f is \mathbb{R} -valued (just redefine it to be 0 on the nullset). Put $t = \frac{1}{\mu(X)} \int_X f d\mu$. For each $s \in (-\infty, t)$ and $u \in (t, \infty)$, the claim above gives us

$$\frac{F(t) - F(s)}{t - s} \leq \frac{F(u) - F(t)}{u - t}.$$

Let $\beta = \sup_s \frac{F(t) - F(s)}{t - s}$. Then $\beta \leq \frac{F(u) - F(t)}{u - t}$ which implies $F(u) \geq F(t) + \beta(u - t)$ for $u \in (t, \infty)$. If $u \in (-\infty, t)$, then $\frac{F(t) - F(u)}{t - u} \leq \beta$ by definition of supremum. Thus $F(t) \leq F(u) + \beta(t - u)$ which implies $F(u) \geq F(t) + \beta(u - t)$. Let $u = f(x)$. Then $F(f(x))$ is measurable and $F(f(x)) \geq F(t) + \beta(f(x) - t)$ which implies

$$\int_X F(f(x)) d\mu \geq \int_X F(t) d\mu + \beta \left(\int_X f(x) d\mu - \int_X t d\mu \right) = F(t)\mu(X) + \beta \left(\int_X f(x) d\mu - t\mu(X) \right).$$

Note that if $F(f(x))$ is not in L^1 then it integrates to ∞ , in which case this inequality is still true. Substituting the value for t , we see

$$\int_X F(f(x)) d\mu \geq F\left(\frac{1}{\mu(X)} \int f d\mu\right) \mu(X).$$

\square

Let $X = [n]$, $\mathcal{M} = \mathcal{P}(X)$, $\mu(k) = a_k$ where $\sum_{k=1}^n a_k = 1$ and $a_k > 0$. So $\int_X f d\mu = \sum f(k)a_k$. Put $F = e^t$, which is convex on \mathbb{R} . Then by Jensen's Inequality, since $\mu(X) = 1$, we have

$$\exp\left(\sum f(k)a_k\right) = \exp\left(\int_X f d\mu\right) \leq \int_X e^f d\mu = \sum a_k \exp(f(k)).$$

Put $y_k = e^{f(k)}$, that is, $f(k) = \ln y_k$. Then $\exp(\sum \ln y_k^{a_k}) \leq \sum a_k \exp(\ln y_k)$ which implies

$$\prod_{k=1}^n y_k^{a_k} = \sum_{k=1}^n a_k y_k.$$

Theorem (Young's Inequality). *Let $\frac{1}{p} + \frac{1}{q} = 1$ with $p, q > 1$. Then $|ab| \leq \frac{1}{p}|a|^p + \frac{1}{q}|b|^q$.*

Proof. Use the above with $\alpha_1 = \frac{1}{p}$, $\alpha_2 = \frac{1}{q}$, $y_1 = |a|^p$, $y_2 = |b|^q$. \square

Theorem (Hölder's Inequality). *Let $\frac{1}{p} + \frac{1}{q} = 1$ with $p, q > 1$. Let $f, g \in L^+$. Then*

$$\int fg d\mu \leq \left(\int f^p d\mu\right)^{1/p} \left(\int g^q d\mu\right)^{1/q}.$$

Proof. If $\int f^p d\mu = 0$, then $f = 0$ a.e. which implies $fg = 0$ a.e. and thus $\int fg d\mu = 0$. Similarly if $\int g^q d\mu = 0$. So assume $\int f^p d\mu, \int g^q d\mu > 0$. If $\int f^p d\mu = \infty$ or $\int g^q d\mu = \infty$, the inequality is clear. So assume $0 < \int f^p d\mu, \int g^q d\mu < \infty$. Put

$$F = \frac{f}{\left(\int f^p d\mu\right)^{1/p}} \text{ and } G = \frac{g}{\left(\int g^q d\mu\right)^{1/q}}.$$

Observe $\int F^p d\mu = \frac{1}{\int f^p d\mu} \int f^p d\mu = 1$. Similarly, $\int G^q d\mu = 1$. Using Young's Inequality,

$$\int_{\mathbb{X}} FG d\mu \leq \int \frac{1}{p} F^p d\mu + \frac{1}{q} G^q d\mu = \frac{1}{p} \int F^p d\mu + \frac{1}{q} \int G^q d\mu = \frac{1}{p} + \frac{1}{q} = 1.$$

So $\int_{\mathbb{X}} \frac{fg}{(\int f^p d\mu)^{1/p} (\int g^q d\mu)^{1/q}} d\mu \leq 1$ which implies

$$\int_{\mathbb{X}} fg d\mu \leq \left(\int f^p d\mu \right)^{1/p} \left(\int g^q d\mu \right)^{1/q}.$$

□

Theorem (Minkowski's Inequality). Suppose $p \geq 1$. Let $f, g \in L^+$ be given. Then

$$\left(\int (f+g)^p d\mu \right)^{1/p} \leq \left(\int f^p d\mu \right)^{1/p} + \left(\int g^p d\mu \right)^{1/p}.$$

Proof. If $p = 1$, then it is trivial. So assume $p > 1$. Then

$$\begin{aligned} \int (f+g)^p d\mu &= \int f(f+g)^{p-1} d\mu + \int g(f+g)^{p-1} d\mu \\ &\leq \left(\int f^p d\mu \right)^{1/p} \left(\int (f+g)^p d\mu \right)^{p-1/p} + \left(\int g^p d\mu \right)^{1/p} \left(\int (f+g)^p d\mu \right)^{p-1/p}. \end{aligned}$$

If $\int (f+g)^p d\mu = 0$, clear. If it is ∞ then $(f+g)^p = 2^p (\frac{1}{2}f + \frac{1}{2}g)^p \leq 2^{p-1} f^p + 2^{p-1} g^p = 2^{p-1} (f^p + g^p)$ (since x^p is convex) which implies one of $\int f^p$ and $\int g^p$ is ∞ . Thus we can divide by $(\int (f+g)^p d\mu)^{p-1/p}$ to get Minkowski's Inequality. □

Definition. For each $p \in [1, \infty)$ and each measurable function f , define $\|f\|_p = (\int_{\mathbb{X}} |f|^p d\mu)^{1/p}$ and $\|f\|_{\infty} = \text{ess sup}_{x \in \mathbb{X}} |f(x)| = \inf \{a \geq 0 : \mu(\{x \in \mathbb{X} : |f(x)| > a\}) = 0\}$ (where $\inf \emptyset = \infty$). This is called the **essential supremum**.

Definition. For each $p \in [1, \infty]$ define $L^p(\mathbb{X}, \mu) = \{f \in L^1(\mu) : \|f\|_p < \infty\}$.

1.5 Normed Vector Spaces

Let K denote \mathbb{R} or \mathbb{C} . Recall that a **vector space** \mathfrak{X} is a set of elements with addition and scalar multiplication. By a **subspace**, we mean a vector subspace of \mathfrak{X} . If $x \in \mathfrak{X}$, denote by Kx the subspace $\{kx \in \mathfrak{X} : k \in K\}$. If \mathcal{M} and \mathcal{N} are subspaces of \mathfrak{X} , then $\mathcal{M} \oplus \mathcal{N} = \{x + y \in \mathfrak{X} : x \in \mathcal{M}, y \in \mathcal{N}\}$.

Definition. A **seminorm** on \mathfrak{X} is a function $\|\cdot\| : \mathfrak{X} \rightarrow [0, \infty)$ such that

- $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in \mathfrak{X}$.
- $\|\lambda x\| = |\lambda| \|x\|$ for all $x \in \mathfrak{X}$ and $\lambda \in K$.

If $\|\cdot\|$ also satisfies

- $\|x\| = 0$ if and only if $x = 0$

then $\|\cdot\|$ is called a **norm** on \mathfrak{X} . A pair $(\mathfrak{X}, \|\cdot\|)$ is called a **normed vector space**.

Examples.

- \mathbb{R}^n is a VS and the function $\|x\|_p = (\sum_{k=1}^n |x_k|^p)^{1/p}$ for $p \in [1, \infty)$ is a norm. So is $\|x\|_{\infty} = \max\{|x_1|, |x_2|, \dots, |x_n|\}$.
- For each $p \in [1, \infty]$, the space $L^p(\mu)$ is a VS and the function $\|\cdot\|_p$ is a norm on L^p .

Fact. If $(\mathfrak{X}, \|\cdot\|)$ is a NVS, then $\rho_{\|\cdot\|}(x, y) = \|x - y\|$ for $x, y \in \mathfrak{X}$ is a metric on \mathfrak{X} . The topology induced by this metric is called the **norm (or strong) topology**.

Definition. If ρ is a metric on a set \mathfrak{X} , the **topology induced by ρ** is generated by $\mathcal{E} = \{U \in \mathfrak{X} : \text{there exists } \epsilon > 0, x \in \mathfrak{X} \text{ such that } \rho(y, x) < \epsilon \text{ for all } y \in U\}$. (In Euclidean Space, these are the open balls of radius ϵ .) If $\mathcal{E} \subseteq \mathcal{P}(\mathfrak{X})$, then the smallest topology $\mathcal{T}(\mathcal{E})$ containing \mathcal{E} is called the **topology generated by \mathcal{E}** .

Note. Each set in \mathcal{E} is open in the topology generated by \mathcal{E} by definition.

Definition. Two norms $\|\cdot\|$ and $\|\cdot\|_1$ are **equivalent** if there exists constants $0 < c_1, c_2 < \infty$ such that $c_1\|x\| \leq \|x\|_1 \leq c_2\|x\|$ for all $x \in \mathfrak{X}$.

Examples.

- If $\mathfrak{X} = \mathbb{R}^n$, then for all $p, q \in [1, \infty]$, the norms $\|\cdot\|_p$ and $\|\cdot\|_q$ are equivalent.
- If $\mathfrak{X} = \mathbb{R}^{\mathbb{N}}$ (that is, the space of infinite sequences of real numbers), then for each $p \neq q \in [1, \infty)$, the norms $\|\cdot\|_p$ and $\|\cdot\|_q$ are not equivalent.

Definition. If $(\mathfrak{X}, \|\cdot\|)$ is a NVS that is complete with respect to $\rho_{\|\cdot\|}$, then we say that $(\mathfrak{X}, \|\cdot\|)$, or just \mathfrak{X} , is a **Banach Space**.

Definition. Let $\{x_n\}_{n=1}^{\infty} \subseteq \mathfrak{X}$ be given. The series $\sum_{n=1}^{\infty} x_n$ **converges** to $x \in \mathfrak{X}$ if $\lim_{N \rightarrow \infty} \sum_{n=1}^N x_n = x$ (i.e., $\lim_{N \rightarrow \infty} \|\sum_{n=1}^N x_n - x\| = 0$). The series $\sum_{n=1}^{\infty} x_n$ is **absolutely convergent** if $\sum_{n=1}^{\infty} \|x_n\| < \infty$.

Theorem 16 (p. 152). A NVS $(\mathfrak{X}, \|\cdot\|)$ is complete if and only if every absolutely convergent series is convergent.

Proof. (\Rightarrow) Suppose $(\mathfrak{X}, \|\cdot\|)$ is complete. Let $\{x_n\}_{n=1}^{\infty} \subseteq \mathfrak{X}$ such that $\sum_{n=1}^{\infty} \|x_n\| < \infty$. Then for all N we can define $S_N = \sum_{n=1}^N x_n \in \mathfrak{X}$. Want to show S_N is Cauchy. Let $M > N$ be given. Then $\|S_M - S_N\| = \|\sum_{n=N+1}^M x_n\| \leq \sum_{n=N+1}^M \|x_n\| \rightarrow 0$ as $M, N \rightarrow \infty$. Thus $\{S_N\}_{N=1}^{\infty}$ is a Cauchy Sequence in \mathfrak{X} and since \mathfrak{X} is complete there exists $x \in \mathfrak{X}$ such that $\lim_{N \rightarrow \infty} \|S_N - x\| = 0$. Thus $\sum_{n=1}^{\infty} x_n$ converges to $x \in \mathfrak{X}$.

(\Leftarrow) Suppose every absolutely convergent series converges. Let $\{x_n\}_{n=1}^{\infty} \subseteq \mathfrak{X}$ be a Cauchy Sequence. Select a subsequence $\{x_{n_j}\}_{j=1}^{\infty}$ such that for all j and $n, m \geq n_j$, we have $\|x_n - x_m\| < 2^{-j}$. Put $y_1 = x_{n_1}$ and $y_j = x_{n_j} - x_{n_{j-1}}$ for all $j > 1$. Then $x_{n_k} = \sum_{j=1}^k y_j$. Also,

$$\sum_{j=1}^{\infty} \|y_j\| = \|y_1\| + \sum_{j=2}^{\infty} \|y_j\| = \|x_{n_1}\| + \sum_{j=2}^{\infty} \|x_{n_j} - x_{n_{j-1}}\| \leq \|x_{n_1}\| + \sum_{j=2}^{\infty} 2^{-j} \leq \|x_{n_1}\| + 1 < \infty.$$

Then, by hypothesis, there exists $x \in \mathfrak{X}$ such that $\lim_{J \rightarrow \infty} \sum_{j=1}^J y_j = x$. Then $\lim_{J \rightarrow \infty} x_{n_J} = x$. Of course $\|x_n - x\| \leq \|x_n - x_{n_j}\| + \|x_{n_j} - x\|$ and for $n \geq n_j$, $\|x_n - x_{n_j}\| \leq 2^{-j}$ and $\|x_{n_j} - x\| \rightarrow 0$. Thus $\|x_n - x\| \rightarrow 0$. \square

Corollary 7. If (X, \mathcal{M}, μ) is a measure space, then $(L^1(\mu), \|\cdot\|)$ is a Banach Space.

Theorem 17 (p. 183). For $p \in [1, \infty]$, the space $(L^p(\mu), \|\cdot\|_p)$ is a Banach Space.

Proof. Case 1: $p \in [1, \infty)$. Suppose that $\{f_k\}_{k=1}^{\infty} \subseteq L^p(\mu)$ satisfying $\sum_{k=1}^{\infty} \|f_k\|_p < \infty$. Put $A^p = \sum_{k=1}^{\infty} \|f_k\|_p^p$ and $G_n = \sum_{k=1}^n |f_k|$, with $G = \sum_{k=1}^{\infty} |f_k|$. Clearly, $G_1^p < G_2^p < \dots < G^p$ and by Minkowski's inequality

$$\|G_n\|_p^p = \int_X \left(\sum_{k=1}^n |f_k| \right)^p d\mu \leq \sum_{k=1}^n \int_X |f_k|^p d\mu = \sum_{k=1}^n \|f_k\|_p^p \leq A^p.$$

By the Monotone Convergence Theorem,

$$\lim_{n \rightarrow \infty} \|G_n\|_p^p = \lim_{n \rightarrow \infty} \int_X \left(\sum_{k=1}^n |f_k| \right)^p d\mu = \int_X \lim_{n \rightarrow \infty} \left(\sum_{k=1}^n |f_k| \right)^p d\mu = \int_X \left(\sum_{k=1}^{\infty} |f_k| \right)^p d\mu = \|G\|_p^p \leq A^p < \infty$$

since $\|G_n\|_p^p \leq A^p$ for all n . This implies that $(\sum_1^\infty |f_k|)^p < \infty$ a.e. and so $\sum_1^\infty |f_k| < \infty$ a.e. Thus for almost every x there exists $F(x) < \infty$ such that $F(x) = \lim_{N \rightarrow \infty} \sum_1^N f_k(x)$. Define $F(x) = 0$ for those x where the sum is infinite. We need to show $F \in L^p(\mu)$ and $\lim \|F - \sum_1^N f_k\| = 0$. We have

$$\int_X |F|^p d\mu = \int \left| \sum_1^\infty f_k(x) \right|^p d\mu \leq \int_X \left(\sum_1^\infty |f_k| \right)^p d\mu = \|G\|_p^p \leq A^p < \infty.$$

Thus $F \in L^p(\mu)$. Finally, for all n , $|F - \sum_1^n f_k|^p = |\sum_{n+1}^\infty f_k|^p \leq (\sum_1^\infty |f_k|)^p = G \in L^1$. Also $\lim_{n \rightarrow \infty} F(x) - \sum_1^n f_k(x) = 0$ a.e. So by the Dominated Convergence Theorem,

$$\lim_{n \rightarrow \infty} \|F - \sum_1^n f_k\|_p^p = \lim_{n \rightarrow \infty} \int |F - \sum_1^n f_k|^p d\mu = 0.$$

Thus by Theorem 16, $L^p(\mu)$ is complete.

Case 2: $p = \infty$. Let $\{f_k\} \subset L^\infty(\mu)$ satisfying $\sum_1^\infty \|f_k\|_\infty < \infty$. For each k , set $A_k = \{x \in X : |f_k(x)| > \|f_k\|_\infty\}$. By the definition of $\|\cdot\|_\infty$, each A_k is a null set. Also $A = \bigcup_1^\infty A_k$ is a null set. For each $x \in X \setminus A$, $\sum_1^\infty |f_k(x)| \leq \sum_1^\infty \|f_k\|_\infty < \infty$. So there exists F such that $F(x) = \sum_1^\infty f_k(x) < \infty$ for all $x \in X \setminus A$. Put $F(x) = 0$ for all $x \in A$. So $F = \lim_{N \rightarrow \infty} \sum_1^N f_k(x)$ μ -a.e. Now for $x \in X \setminus A$, $|F(x)| = |\sum_1^\infty f_k(x)| \leq \sum_1^\infty |f_k(x)| \leq \sum_1^\infty \|f_k\|_\infty < \infty$. So $F \in L^\infty$ as $\mu(A) = 0$. Finally

$$\lim_{n \rightarrow \infty} \|F - \sum_1^n f_k\|_\infty = \lim_{n \rightarrow \infty} \left\| \sum_{n+1}^\infty f_k \right\|_\infty \leq \lim_{n \rightarrow \infty} \sum_{n+1}^\infty \|f_k\|_\infty = 0$$

since $\sum_1^\infty \|f_k\|_\infty < \infty$. □

Proposition 19. *Let $S = \{\text{simple functions on } X\}$. For each $p \in [1, \infty]$, the set $S \cap L^p$ is dense in L^p .*

Proof. The case $p = \infty$ is covered by Theorem 4. Suppose $p \in [1, \infty)$ and let $f \in L^p(\mu)$ be given. Want to find a sequence $\{f_n\}_{n=1}^\infty \subseteq S \cap L^p$ such that $\lim_{n \rightarrow \infty} \|f_n - f\|_p = 0$. By Theorem 4 (applied to f^+ and f^-), there exists a sequence $\{f_n\}_{n=1}^\infty \subseteq S$ such that $|f_n| \leq |f|$ and $\lim_{n \rightarrow \infty} f_n(x) = f(x)$ for all $x \in X$. Since $f \in L^p$, we find that $\int_X |f_n|^p d\mu \leq \int_X |f|^p d\mu < \infty$ which implies $f_n \in L^p$ for all n . So $\{f_n\}_{n=1}^\infty \subseteq S \cap L^p$. Moreover, $|f_n - f|^p \leq (|f_n| + |f|)^p \leq 2^p |f|^p \in L^1$ and $|f_n - f|^p \rightarrow 0$ for all $x \in X$. By the LDC, $\lim_{n \rightarrow \infty} \int_X |f_n - f|^p d\mu = 0$ which implies $\lim \|f_n - f\|_p = 0$. □

Proposition 20. *If $1 \leq p \leq q \leq r \leq \infty$, then $L^p \cap L^r \subseteq L^q$ and $\|f\|_q \leq \|f\|_p^\lambda \|f\|_r^{1-\lambda}$ where $\frac{1}{q} = \frac{\lambda}{p} + \frac{1-\lambda}{r}$.*

Proof. If $p = q = r = \infty$, trivial. If $p < \infty$ and $q = r = \infty$ then clearly $L^p \cap L^\infty \subseteq L^\infty$ and if we take $\lambda = 0$, then we see $\|f\|_q \leq \|f\|_r = \|f\|_p^0 \|f\|_r^1$. So suppose $p, q < \infty$. If $r = \infty$, then

$$\begin{aligned} \left(\int_X |f|^q d\mu \right)^{1/q} &= \left(\int_X |f|^p |f|^{q-p} d\mu \right)^{1/q} \\ &\leq \left(\int_X |f|^p \|f\|_\infty^{q-p} d\mu \right)^{1/q} \\ &= \|f\|_\infty^{\frac{q-p}{q}} \left(\int_X |f|^p d\mu \right)^{1/q} \\ &= \|f\|_p^{p/q} \|f\|_\infty^{1-p/q}. \end{aligned}$$

Let $\lambda = \frac{p}{q}$. If $f \in L^p \cap L^\infty$, then $\|f\|_p, \|f\|_\infty < \infty$, so $\|f\|_q \leq \|f\|_p^\lambda \|f\|_\infty^{1-\lambda} < \infty$. Thus $f \in L^q$. Now suppose $r < \infty$. Note

that $\lambda q < p$.

$$\begin{aligned}
\left(\int_X |f|^q d\mu\right)^{1/q} &= \left(\int_X |f|^{\lambda q} |f|^{(1-\lambda)q} d\mu\right)^{1/q} \\
&\leq \left(\int_X (|f|^{\lambda q})^{p/\lambda q} d\mu\right)^{\frac{1}{q}(\frac{\lambda q}{p})} \left(\int_X (|f|^{(1-\lambda)q})^{\frac{p}{p-\lambda q}} d\mu\right)^{\frac{1}{q}(\frac{p-\lambda q}{p})} \quad \text{by Holder's Inequality} \\
&= \left(\int_X |f|^p d\mu\right)^{\lambda/p} \left(\int_X |f|^r d\mu\right)^{\frac{1-\lambda}{r}} \\
&= \|f\|_p^\lambda \|f\|_r^{1-\lambda}.
\end{aligned}$$

By the same argument as above, $f \in L^p \cap L^\infty$ implies $f \in L^q$. □

Proposition 21. If $\mu(X) < \infty$, then for all $1 \leq p \leq q \leq \infty$, we have $L^q \subseteq L^p$ and $\|f\|_p \leq \|f\|_q \mu(X)^{\frac{1}{p} - \frac{1}{q}}$.

Proof. If $q < \infty$, then

$$\begin{aligned}
\left(\int_X |f|^p d\mu\right)^{1/p} &= \left(\int_X 1 \cdot |f|^p d\mu\right)^{1/p} \\
&\leq \left(\int_X (|f|^p)^{q/p} d\mu\right)^{\frac{p}{q}(\frac{1}{p})} \left(\int_X |f|^{\frac{q}{q-p}} d\mu\right)^{\frac{1}{p}(\frac{q-p}{q})} \\
&= \|f\|_q \mu(X)^{\frac{1}{p} - \frac{1}{q}}.
\end{aligned}$$
□

2 Measure Theory

The Lebesgue Measure on \mathbb{R}^n . Suppose $(\mathbb{R}^n, \mathcal{M}, m^n)$ is a measure space where the measure $m^n : \mathcal{M} \rightarrow [0, \infty]$ with $\mathcal{M} \subseteq \mathcal{P}(\mathbb{R}^n)$ is the unique measure such that

$$m^n \left(\prod_{k=1}^n (a_k, b_k) \right) = \prod (b_k - a_k)$$

where $(a_k, b_k) \subseteq \mathbb{R}$ for all k . What can we say about \mathcal{M} if we want to measure all the open boxes $\prod (a_k, b_k)$?

- Since any open set is a countable union of open boxes, all open sets in the usual topology must be in \mathcal{M} .
- The smallest σ -algebra \mathcal{M} must be the Borel σ -algebra.

So we want to somehow extend m^n from the boxes to all of $\mathcal{B}_{\mathbb{R}^n}$.

Definition. Let $X \neq \emptyset$. A family of sets $\mathcal{C} \subseteq \mathcal{P}(X)$ is a **semialgebra** if

1. $\emptyset, X \in \mathcal{C}$
2. If $E_1, E_2 \in \mathcal{C}$, then $E_1 \cap E_2 \in \mathcal{C}$ (and thus all finite intersections are in \mathcal{C}).
3. If $E \in \mathcal{C}$, then there exists a finite sequence $\{E_i\}_{i=1}^k \in \mathcal{C}$ with $E_i \cap E_j = \emptyset$ for all $i \neq j$ such that $E^C = \bigcup_{i=1}^k E_i$.

Examples. The following are semialgebras:

- $I = \{ \text{open, half-open, closed intervals on } \mathbb{R} \}$.
- $I^n = \{ \text{crossproduct of any } n \text{ elements of } I \}$.

Notation. Denote any interval with endpoints a and b by $I(a, b)$.

Definition. Let $X \neq \emptyset$. A family of sets $\mathcal{F} \subseteq \mathcal{P}(X)$ is an **algebra** if

1. $\emptyset, X \in \mathcal{F}$
2. If $E_1, E_2 \in \mathcal{F}$, then $E_1 \cap E_2 \in \mathcal{F}$ (and thus all finite intersections are in \mathcal{F}).

3. If $E \in \mathcal{F}$, then $E^C \in \mathcal{F}$.

Note that by 2 and 3, we are only allowing finite unions to be in \mathcal{F} , unlike in a σ -algebra.

Examples. The following are algebras

- $\mathcal{F}(I) = \{E \subseteq \mathbb{R} \mid E = \bigcup_{k=1}^{\ell} I_k, I_k \in I, I_j \cap I_k = \emptyset \text{ for } j \neq k\}$.
- $\mathcal{F}(I)^n = \{E \subseteq \mathbb{R}^n \mid E = \bigcup_{k=1}^{\ell} I_k, I_k \in I^n, I_j \cap I_k = \emptyset \text{ for } j \neq k\}$.

In general, if \mathcal{C} is a semialgebra, then

$$\mathcal{F}(\mathcal{C}) = \left\{ E \subseteq X \mid E = \bigcup_{k=1}^{\ell} E_k, E_k \in \mathcal{C}, E_j \cap E_k = \emptyset \text{ for } j \neq k \right\}$$

is an algebra.

Definition. Let $\mathcal{C} \subseteq \mathcal{P}(X)$. A set function $\mu : \mathcal{C} \rightarrow [0, \infty]$ is called

- **monotone** if for all $A, B \in \mathcal{C}$ satisfying $A \subseteq B$, we have $\mu(A) \leq \mu(B)$.
- **finite additive** if $\{E_k\}_{k=1}^{\ell} \subseteq \mathcal{C}$ such that $E_j \cap E_k = \emptyset$ and $\bigcup_{k=1}^{\ell} E_k \in \mathcal{C}$ implies $\mu(\bigcup_{k=1}^{\ell} E_k) = \sum_{k=1}^{\ell} \mu(E_k)$.
- **countably additive** if $\{E_k\}_{k=1}^{\infty} \subseteq \mathcal{C}$ such that $E_j \cap E_k = \emptyset$ and $\bigcup_{k=1}^{\infty} E_k \in \mathcal{C}$ implies $\mu(\bigcup_{k=1}^{\infty} E_k) = \sum_{k=1}^{\infty} \mu(E_k)$.
- **countably subadditive** if $\{E_k\}_{k=1}^{\infty} \subseteq \mathcal{C}$ such that $\bigcup_{k=1}^{\infty} E_k \in \mathcal{C}$ implies $\mu(\bigcup_{k=1}^{\infty} E_k) \leq \sum_{k=1}^{\infty} \mu(E_k)$.

1st Goal: Given a monotone countably additive set function μ defined on a semialgebra \mathcal{C} , we want to extend μ to a monotone countably additive function $\tilde{\mu}$ defined on an algebra $\mathcal{F}(\mathcal{C})$ generated by \mathcal{C} .

Proposition 22. Let $\mathcal{C} \subseteq \mathcal{P}(X)$. Then there exists a unique algebra $\mathcal{F}(\mathcal{C}) \subseteq \mathcal{P}(X)$ such that $\mathcal{C} \subseteq \mathcal{F}(\mathcal{C})$ and if $\mathcal{A} \subseteq \mathcal{P}(X)$ is an algebra such that $\mathcal{C} \subseteq \mathcal{A}$, then $\mathcal{F}(\mathcal{C}) \subseteq \mathcal{A}$. So $\mathcal{F}(\mathcal{C})$ is the “smallest” algebra containing \mathcal{C} .

Proof. Define $\mathcal{F}(\mathcal{C}) = \bigcap \{A \mid \mathcal{C} \subseteq A \subseteq \mathcal{P}(X), A \text{ is an algebra}\}$. □

Definition. Given $\mathcal{C} \subseteq \mathcal{P}(X)$, the algebra $\mathcal{F}(\mathcal{C})$ provided by Prop 22 is called the **algebra generated by \mathcal{C}** .

Proposition 23. If \mathcal{C} is a semialgebra, then the algebra generated by \mathcal{C} is $\mathcal{F}(\mathcal{C}) := \{E \subseteq X : E = \bigcup_{k=1}^{\ell} E_k, E_j \cap E_k = \emptyset, j \neq k, E_k \in \mathcal{C}\}$.

Example. Recall I was a semialgebra. What kind of properties does $m : I \rightarrow [0, \infty]$ defined by $m(I(a, b)) = b - a$ have? It is monotone, finitely additive, countably additive (2 cases: if the union is an interval which is finite or infinite), countably subadditive (by monotonicity, countable additivity and Lemma 1).

Theorem 18. Suppose μ is a finitely additive and countable subadditive set function on a semialgebra \mathcal{C} such that $\mu(\emptyset) = 0$. Then there exists a unique countably additive set function $\tilde{\mu}$ on $\mathcal{F}(\mathcal{C})$ such that $\tilde{\mu}(E) = \mu(E)$ for all $E \in \mathcal{C}$.

Proof. For all $E \in \mathcal{F}(\mathcal{C})$, by Prop 23, there exists $\{E_k\}_{k=1}^n \subseteq \mathcal{C}$ such that $E = \bigcup_{k=1}^n E_k$ and $E_j \cap E_k = \emptyset$ if $j \neq k$. Define $\tilde{\mu}(E) = \sum_{k=1}^n \mu(E_k)$.

Claim 1: $\tilde{\mu}$ is well-defined.

Proof: Let $E \in \mathcal{F}(\mathcal{C})$ and suppose there exists $\{E_k\}_{k=1}^n$ and $\{F_{\ell}\}_{\ell=1}^m \subseteq \mathcal{C}$ such that $E_j \cap E_k = \emptyset$ for $j \neq k$ and $F_j \cap F_{\ell} = \emptyset$ for $j \neq \ell$ and $\bigcup_{k=1}^n E_k = E = \bigcup_{\ell=1}^m F_{\ell}$. Then for all $\ell = 1, 2, \dots, m$, $F_{\ell} = F_{\ell} \cap E = F_{\ell} \cap (\bigcup_{k=1}^n E_k) = \bigcup_{k=1}^n (F_{\ell} \cap E_k)$ and for all $k = 1, 2, \dots, n$, $E_k = E_k \cap E = E_k \cap (\bigcup_{\ell=1}^m F_{\ell}) = \bigcup_{\ell=1}^m (E_k \cap F_{\ell})$. So

$$\begin{aligned} \tilde{\mu}(E) &= \sum_{k=1}^n \mu(E_k) &= \sum_{k=1}^n \mu(\bigcup_{\ell=1}^m (E_k \cap F_{\ell})) \\ & &= \sum_{k=1}^n \sum_{\ell=1}^m \mu(E_k \cap F_{\ell}) \\ & &= \sum_{\ell=1}^m \sum_{k=1}^n \mu(E_k \cap F_{\ell}) \\ & &= \sum_{\ell=1}^m \mu(\bigcup_{k=1}^n (E_k \cap F_{\ell})) \\ & &= \sum_{\ell=1}^m \mu(F_{\ell}). \end{aligned}$$

Claim 2: $\tilde{\mu}$ is finitely additive on $\mathcal{F}(\mathcal{C})$.

Proof: Suppose $\{E_k\}_{k=1}^n \subseteq \mathcal{F}(\mathcal{C})$ with $E_k \cap E_j = \emptyset$ for $j \neq k$, then $\cup_{k=1}^n E_k \subseteq \mathcal{F}(\mathcal{C})$ since $\mathcal{F}(\mathcal{C})$ is an algebra. By Prop 23, there exists $\{G_r\}_{r=1}^s \subseteq \mathcal{C}$ such that $\cup_{r=1}^s G_r = \cup_{k=1}^n E_k$. Also, for all $k = 1, 2, \dots, n$, there exist mutually disjoint $\{F_{k,\ell}\}_{\ell=1}^{m_k} \subseteq \mathcal{C}$ such that $E_k = \cup_{\ell=1}^{m_k} F_{k,\ell}$. Then for all $k = 1, \dots, n$

$$E_k = E_k \cap \left(\bigcup_{k=1}^n E_k \right) = E_k \cap \left(\bigcup_{r=1}^s G_r \right) = \left(\bigcup_{\ell=1}^{m_k} F_{k,\ell} \right) \cap \left(\bigcup_{r=1}^s G_r \right) = \bigcup_{\ell=1}^{m_k} \bigcup_{r=1}^s (F_{k,\ell} \cap G_r).$$

Also for all $r = 1, \dots, s$, $G_r = G_r \cap \left(\bigcup_{k=1}^n E_k \right) = G_r \cap \left(\bigcup_{k=1}^n \bigcup_{\ell=1}^{m_k} F_{k,\ell} \right) = \bigcup_{k=1}^n \bigcup_{\ell=1}^{m_k} G_r \cap F_{k,\ell}$. Now

$$\begin{aligned} \tilde{\mu} \left(\bigcup_{k=1}^n E_k \right) &= \tilde{\mu} \left(\bigcup_{r=1}^s G_r \right) = \sum_{r=1}^s \mu(G_r) = \sum_{r=1}^s \mu \left(\bigcup_{k=1}^n \bigcup_{\ell=1}^{m_k} G_r \cap F_{k,\ell} \right) \\ &=^* \sum_{r=1}^s \sum_{k=1}^n \sum_{\ell=1}^{m_k} \mu(G_r \cap F_{k,\ell}) = \sum_{k=1}^n \tilde{\mu} \left(\bigcup_{r=1}^s \bigcup_{\ell=1}^{m_k} G_r \cap F_{k,\ell} \right) = \sum_{k=1}^n \tilde{\mu}(E_k). \end{aligned}$$

Claim 3: $\tilde{\mu}$ is countably subadditive.

Proof: Same as above, except replace n with ∞ and change the $=^*$ to \leq .

Note that the countable additivity of $\tilde{\mu}$ follows from the next theorem (Theorem 19) □

Theorem 19. Let \mathcal{F} be an algebra of sets on X and $\tilde{\mu} : \mathcal{F} \rightarrow [0, \infty]$ be a set function such that $\tilde{\mu}(\emptyset) = 0$. Then $\tilde{\mu}$ is countably additive if and only if it is both finitely additive and countably subadditive.

Proof. First note that if $\tilde{\mu}$ is finitely additive, then (since \mathcal{F} is an algebra) for $A, B \in \mathcal{F}$ with $A \subseteq B$, we see $\tilde{\mu}(B) = \tilde{\mu}(A \cup B \setminus A) = \tilde{\mu}(A) + \tilde{\mu}(B \setminus A) \geq \tilde{\mu}(A)$. Thus $\tilde{\mu}$ is monotone.

(\Rightarrow): Suppose $\tilde{\mu}$ is countably additive. Clearly $\tilde{\mu}$ is finitely additive as $\tilde{\mu}(\emptyset) = 0$. To show subadditive, let $\{E_k\}_{k=1}^{\infty} \subseteq \mathcal{F}$ such that $\cup_{k=1}^{\infty} E_k \in \mathcal{F}$. By Lemma 1, there exists a sequence $\{F_k\}_{k=1}^{\infty}$ of mutually disjoint sets such that $\cup_{k=1}^{\infty} F_k = \cup_{k=1}^{\infty} E_k$. Using countable additivity and monotonicity, we see $\tilde{\mu}(\cup E_k) = \tilde{\mu}(\cup F_k) = \sum \tilde{\mu}(F_k) \leq \sum \tilde{\mu}(E_k)$.

(\Leftarrow): Suppose $\tilde{\mu}$ is finitely additive and countably subadditive. Let $\{E_k\}_{k=1}^{\infty} \subseteq \mathcal{F}$ be mutually disjoint sets such that $\cup_{k=1}^{\infty} E_k \in \mathcal{F}$. Since $\tilde{\mu}$ is countably subadditive, $\tilde{\mu}(\cup_{k=1}^{\infty} E_k) \leq \sum_{k=1}^{\infty} \tilde{\mu}(E_k)$. To show the opposite inequality, we use finite additivity and monotonicity to conclude $\tilde{\mu}(\cup_{k=1}^n E_k) \geq \tilde{\mu}(\cup_{k=1}^n E_k) = \sum_{k=1}^n \tilde{\mu}(E_k)$ for all n . Taking the limit as $n \rightarrow \infty$, we get $\tilde{\mu}(\cup_{k=1}^{\infty} E_k) \geq \sum_{k=1}^{\infty} \tilde{\mu}(E_k)$. Thus $\tilde{\mu}(\cup_{k=1}^{\infty} E_k) = \sum_{k=1}^{\infty} \tilde{\mu}(E_k)$. □

Definition. Suppose $\mathcal{A} \subseteq \mathcal{P}(X)$ is an algebra. A function $\tilde{\mu} : \mathcal{A} \rightarrow [0, \infty]$ is called a **premeasure** if $\tilde{\mu}(\emptyset) = 0$ and $\tilde{\mu}$ is countably additive.

Theorem 18 shows how to construct a premeasure on an algebra, generated from a semialgebra, from a finitely additive countably subadditive function on that semialgebra.

Notation. Define $\tilde{I} := \{(a, b] : a < b \in \mathbb{R}\} \cup \{(-\infty, b] : b \in \mathbb{R}\} \cup \{(a, \infty) : a \in \mathbb{R}\} \cup \{(-\infty, \infty)\} \cup \{\emptyset\}$. Note the σ -algebra generated by \tilde{I} is $\mathcal{B}_{\mathbb{R}}$. Also \tilde{I} is a semialgebra.

Proposition 24. Let $F : \mathbb{R} \rightarrow \mathbb{R}$ be an increasing function. Define $\mu_F : \tilde{I} \rightarrow [0, \infty]$ by $\mu_F((a, b]) = F(b) - F(a)$, $\mu_F((-\infty, b]) = F(b) - \lim_{x \rightarrow -\infty} F(x)$, $\mu_F((a, \infty)) = \lim_{x \rightarrow \infty} F(x) - F(a)$, $\mu_F((-\infty, \infty)) = \lim_{x \rightarrow \infty} F(x) - \lim_{x \rightarrow -\infty} F(x)$, $\mu_F(\emptyset) = 0$. Then μ_F is well-defined, finitely additive and monotone. Moreover, if F is right continuous, then μ_F is countably subadditive.

Proof. It is clear that μ_F is well-defined. Suppose $\{I_k\}_{k=1}^n \subseteq \tilde{I}$ are disjoint. First suppose each I_k is of the form $(a_k, b_k]$ and $\cup_{k=1}^n (a_k, b_k] = (a, b) \in \tilde{I}$. Then WLOG, assume $a = a_1 < b_1 = a_2 < b_2 = \dots = a_n < b_n = b$. So

$$\sum_{k=1}^n \mu_F(I_k) = \sum_{k=1}^n \mu_F((a_k, b_k]) = \sum_{k=1}^n (F(b_k) - F(a_k)) = F(b) - F(a) = \mu_F((a, b]).$$

Now suppose $\cup_{k=1}^n I_k = (-\infty, b] \in \tilde{I}$. WLOG, assume $I_1 = (-\infty, b_1]$ and $I_k = (a_k, b_k]$ with $b_1 = a_2 < b_2 = \dots = a_n < b_n = b$. So

$$\begin{aligned} \sum_1^n \mu_F(I_k) &= \mu_F(I_1) + \sum_2^n \mu_F((a_k, b_k]) &= F(b_1) - \lim_{x \rightarrow -\infty} F(x) + F(b_n) - F(a_2) \\ &= F(b_n) - \lim_{x \rightarrow -\infty} F(x) &= \mu_F((-\infty, b]). \end{aligned}$$

Similarly, the other cases hold. Thus μ_F is finitely additive. Monotonicity follows. Thus we need only to show countable subadditivity in the case that F is right continuous. Suppose $I = (a, b] \subseteq \cup_{k=1}^\infty I_k$ with $\{I_k\}_{k=1}^\infty \subseteq \tilde{I}$. Let $\epsilon \in (0, b - a)$. For each k , define

$$I'_k = \begin{cases} (a_k, b_k + \delta_k) & \text{if } I_k = (a_k, b_k], \\ (-\infty, b_k + \delta_k) & \text{if } I_k = (-\infty, b_k], \\ I_k & \text{otherwise,} \end{cases}$$

where δ_k satisfies $F(b_k + \delta_k) - F(b_k) < \epsilon 2^{-k}$. Now $\{I'_k\}_{k=1}^\infty$ is an open cover for $[a + \epsilon, b]$. Since compact, Heine Borel says there exists a finite subcover, call it $\{I'_k\}_{k=1}^m$ for simplicity. WLOG, assume $a + \epsilon \in I'_1$. If $b \notin I'_1$, then $b_1 + \delta_1 < b$ and thus $[b_1 + \delta_1, b] \subseteq \cup_{k=2}^m I'_k$. WLOG, assume $b_1 + \delta_1 \in I'_2$. Then $a_2 < b_1 + \delta_1$. If $b \notin I'_2$, then $b_2 + \delta_2 < b$ and so $[b_2 + \delta_2, b] \subseteq \cup_{k=3}^m I'_k$. Continue to find $m < n$ such that $a < a + \epsilon < b_1 + \delta_1 < b_2 + \delta_2 < \dots < b < b_m + \delta_m$, that is, $b \in I'_m$. Note that this also says $a_{i+1} < b_i + \delta_i$. Now

$$\begin{aligned} F(b) - F(a) &= F(b) - F(a + \epsilon) + F(a + \epsilon) - F(a) \\ &\leq F(b_m + \delta_m) - F(a_1) + F(a + \epsilon) - F(a) \\ &= \sum_{k=1}^{m-1} ((F(b_{k+1} + \delta_{k+1}) - F(b_k - \delta_k)) + F(b_1 + \delta_1) - F(a_1) + F(a + \epsilon) - F(a)) \\ &\leq \sum_{k=1}^{m-1} ((F(b_{k+1} + \delta_{k+1}) - F(a_{k+1})) + F(b_1 + \delta_1) - F(a_1) + F(a + \epsilon) - F(a)) \\ &= \sum_{k=1}^m ((F(b_k + \delta_k) - F(a_k)) + F(a + \epsilon) - F(a)) \\ &= \sum_{k=1}^m ((F(b_k + \delta_k) - F(b_k) + F(b_k) - F(a_k)) + F(a + \epsilon) - F(a)) \\ &\leq \sum_{k=1}^m \epsilon 2^{-k} + \sum_{k=1}^m \mu_F(I_k) + F(a + \epsilon) - F(a) \\ &\leq \epsilon + F(a + \epsilon) - F(a) + \sum_{k=1}^m \mu_F(I_k) \end{aligned}$$

Letting $\epsilon \rightarrow 0^+$, the right continuity of F yields

$$\mu_F(I) = F(b) - F(a) \leq \sum_1^\infty \mu_F(I_k).$$

Now suppose I is an infinite interval. If $I = (-\infty, b]$, then for each $M > -\infty$, the same argument shows that $\mu((M, b]) = F(b) - F(M) \leq \sum_{k=1}^\infty \mu_F(I_k)$. Now letting $M \rightarrow -\infty$, we see $\mu_F((-\infty, b]) = F(b) - \lim_{M \rightarrow -\infty} F(M) \leq \sum_{k=1}^\infty \mu_F(I_k)$. Similarly for the other cases. \square

Note. It is also the case that μ_F is countably additive, but we don't prove that here. For reference, Folland refers to this as μ_0 . This is similar to Prop 1.15 in Folland.

Proposition 25. *Let $F : \mathbb{R} \rightarrow \mathbb{R}$ be increasing and right continuous. Define $\mu_F : \tilde{I} \rightarrow [0, \infty]$ as in Proposition 24. Then $\tilde{\mu}_F : \mathcal{F}(\tilde{I}) \rightarrow [0, \infty]$ defined by $\tilde{\mu}_F(\cup_1^n I_k) = \sum_1^n \mu_F(I_k)$ whenever $\{I_j\}_{k=1}^n \subseteq \tilde{I}$ satisfies $I_j \cap I_k = \emptyset$ for $j \neq k$ is a premeasure on $\mathcal{F}(\tilde{I})$.*

Proof. Follows from Prop 23, Thm 18, and Prop 24. \square

Remark. If $F = x$, then μ_F is the usual length of an interval.

Proposition 26. *Suppose $\mu : \tilde{I} \rightarrow [0, \infty]$ is finitely additive and $\mu((a, b]) < \infty$ for each $a, b \in \mathbb{R}$. Then there exists an increasing function $F : \mathbb{R} \rightarrow \mathbb{R}$ such that $\mu((a, b]) = F(b) - F(a)$ for all $(a, b] \subseteq \mathbb{R}$. If μ is also countably additive on \tilde{I} , then F is right continuous and $\mu_F = \mu$.*

Proof. For all $x \in \mathbb{R}$, define $F(x) = \begin{cases} \mu((0, x]) & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -\mu((x, 0]) & \text{if } x < 0 \end{cases}$. We want to verify that $\mu((a, b]) = F(b) - F(a)$ for all $(a, b] \in \mathbb{R}$. Since μ is finitely additive, if $0 < a < b$

$$\mu((a, b]) = \mu((0, b] \setminus (0, a]) = \mu((0, b]) - \mu((0, a]) = F(b) - F(a).$$

Similarly for $a \leq 0 < b$ and $a < b \leq 0$. To show F is increasing, note that for $0 < a < b$, $F(b) - F(a) = \mu((a, b]) \geq 0$. Similarly for the other two cases. Now, suppose μ is countably additive. We want to show F is right continuous. Let $x \in \mathbb{R}$ and $\{x_k\}_{k=1}^{\infty} \subset (x, \infty)$ such that $x_k \rightarrow x$ as $k \rightarrow \infty$ and $\{x_k\}_{k=1}^{\infty}$ is decreasing. Notice $\{F(x_k)\}_{k=1}^{\infty}$ is decreasing and bounded below by $F(x)$, so it converges.

Case 1: Let $x > 0$. Then

$$\begin{aligned} F(x_1) = \mu((0, x_1]) &= \mu((0, x]) + \mu(\cup_{k=1}^{\infty} (x_{k+1}, x_k]) \\ &= F(x) + \lim_{N \rightarrow \infty} \sum_{k=1}^N \mu((x_{k+1}, x_k]) \\ &= F(x) + \lim_{N \rightarrow \infty} \sum_{k=1}^N F(x_k) - F(x_{k+1}) \\ &= F(x) + \lim_{N \rightarrow \infty} F(x_1) - F(x_{N+1}). \end{aligned}$$

Thus $F(x) = \lim_{N \rightarrow \infty} F(x_{N+1})$. Similarly if $x = 0$.

Case 2: Let $x < 0$. Then for some $m \in \mathbb{N}$ we find $x_m < 0$. Then

$$\begin{aligned} F(x) = -\mu((x, 0]) &= -(\mu((x_m, 0]) + \mu((x, x_m])) \\ &= F(x_m) - \mu(\cup_{k=n}^{\infty} (x_{k+1}, x_k]) \\ &= F(x_m) - \lim_{n \rightarrow \infty} \sum_{k=n}^{\infty} F(x_k) - F(x_{k+1}) \\ &= F(x_m) - F(x_m) + \lim F(x_{n+1}). \end{aligned}$$

Thus $F(x) = \lim_{n \rightarrow \infty} F(x_{n+1})$.

To show $\mu = \mu_F$, we need only to compare $\mu(I)$ and $\mu_F(I)$ on infinite intervals. Suppose $I = (-\infty, \infty)$. Then $I = \cup_{k=1}^{\infty} ((-k, -k+1] \cup (k-1, k])$. Since $\mu(I) \geq 0$ we have

$$\begin{aligned} \mu(I) &= \lim_{n \rightarrow \infty} \sum_{k=1}^n \mu((-k, -k+1]) + \mu((k-1, k]) \\ &= \lim_{n \rightarrow \infty} \sum_{k=1}^n F(-k+1) - F(-k) + \lim_{n \rightarrow \infty} \sum_{k=1}^n F(k) - F(k-1) \\ &= \lim_{n \rightarrow \infty} F(-n) + \lim_{n \rightarrow \infty} F(n) = \mu_F(I). \end{aligned}$$

□

2nd Goal: Given a premeasure on an algebra A , we want to extend a measure on the σ -algebra generated by A .

Intermediate Goal: Approximate “measure” of any subset of a nonempty X using the premeasure on an algebra A .

Idea: Recall $m : I \rightarrow [0, \infty]$ was given by $m(I(a, b)) = b - a$ (where $I(a, b)$ is any interval with endpoints a and b) for $a, b \in \overline{\mathbb{R}}$. The algebra generated by I is $\mathcal{F}(I)$, which is the collection of all finite unions of disjoint intervals in I . The extension of m to \tilde{m} on $\mathcal{F}(I)$ is $\tilde{m}(E) = \sum_{k=1}^n (b_k - a_k)$ for $E = \cup_{k=1}^n I(a_k, b_k)$, with $\{I(a_k, b_k)\}_{k=1}^n$ mutually disjoint. Now, we want to extend \tilde{m} to a set function that “measures” any subset of \mathbb{R} . Suppose $E \subseteq \mathbb{R}$. Then we can find at least one countable family $\{I_k\}_{k=1}^{\infty} \subseteq \mathcal{F}(I)$ such that $E \subseteq \cup_{k=1}^{\infty} I_k$ (take $I_k = \mathbb{R}$ for all k). Since $E \subseteq \cup_{k=1}^{\infty} I_k$, we expect the “measure” of E to be $\leq \tilde{m}(\cup_{k=1}^{\infty} I_k) \leq \sum_{k=1}^{\infty} \tilde{m}(I_k)$. So, in general, we want

$$\text{“measure of” } E \leq \sum_{k=1}^{\infty} \tilde{m}(I_k).$$

So we should define it as

$$\text{“measure” of } E = \inf \left\{ \sum \tilde{m}(I_k) : \{I_k\} \subseteq \mathcal{F}(I), E \subseteq \cup I_k \right\}.$$

Note that we do not get anything new by trying to approximate the “measure” from the inside, since if $E \subseteq \mathbb{R} \in \mathcal{F}(I)$, then the “inner measure of” E is $\tilde{m}(\mathbb{R}) -$ “the outer measure of” $\mathbb{R} \setminus E$.

Definition. Let $\mathcal{E} \subseteq \mathcal{P}(X)$ be such that $\emptyset, X \in \mathcal{E}$ and $\rho : \mathcal{E} \rightarrow [0, \infty]$ satisfy $\rho(\emptyset) = 0$. For each $E \in \mathcal{E}$, define $\mu^*(E) = \inf \{ \sum \rho(E_k) : \{E_k\}_{k=1}^\infty \subseteq \mathcal{E}, E \subseteq \cup E_k \}$ to be the **outermeasure of E induced by ρ** .

In general,

Definition. If $\mu^* : \mathcal{P}(X) \rightarrow [0, \infty]$ is monotone, countably subadditive, and satisfies $\mu^*(\emptyset) = 0$, call μ^* an **outermeasure on X** .

Proposition 27. The set function in the former definition is an outer measure in the sense of the latter definition.

Proof. Clearly, $\mu^*(\emptyset) = 0$. Suppose $A, B \subseteq X$ with $A \subseteq B$. Observe that there is at least one collection $\{E_k\}_{k=1}^\infty \subseteq \mathcal{E}$ such that $B \subseteq \cup_{k=1}^\infty E_k$. Then $A \subseteq \cup_{k=1}^\infty E_k$. This is true for all covers of B . So $\mu^*(A) \leq \mu^*(B)$. To show subadditivity, let $\epsilon > 0$ and $\{A_k\}_{k=1}^\infty \subseteq \mathcal{P}(X)$. Then for all j we can find a sequence $\{E_{k,j}\}_{j=1}^\infty \subseteq \mathcal{E}$ such that $A_k \subseteq \cup E_{k,j}$. Then $\mu^*(A_k) \geq \sum_{j=1}^\infty \rho(E_{k,j}) - \epsilon 2^{-k}$. Now $\cup_{k=1}^\infty A_k \subseteq \cup_{k=1}^\infty \cup_{j=1}^\infty E_{k,j}$. By the definition of μ^* ,

$$\mu^*(\cup A_k) \leq \sum_{k=1}^\infty \sum_{j=1}^\infty \rho(E_{k,j}) \leq \sum_{k=1}^\infty (\mu^*(A_k) + \epsilon 2^{-k}) \leq \sum_{k=1}^\infty \mu^*(A_k) + \epsilon.$$

Since true for all ϵ , we get $\mu^*(\cup A_k) \leq \sum \mu^*(A_k)$. □

Roughly speaking, if μ^* were a measure and $A \subset E$, then $\mu^*(A) + \mu^*(E \setminus A) = \mu^*(E)$ if A, E are measurable. Then $\mu^*(E \cap A) + \mu^*(E \cap A^C) = \mu^*(E)$.

Definition. If μ^* is an outer measure on X , then a set $A \subseteq X$ is called **μ^* -measurable** if and only if $\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^C)$ for all $E \subseteq X$.

Remark. By countable subadditivity, we already have the \leq direction as $E = (E \cap A) \cup (E \cap A^C)$.

Theorem (Carathéodory’s Theorem). If μ^* is an outermeasure on X , then the collection of all μ^* -measurable sets, call it \mathcal{M} is a σ -algebra. Moreover, μ^* is a complete positive measure on \mathcal{M} .

Proof. We prove that \mathcal{M} is a σ -algebra:

1. Let $E \subseteq X$ be given. Then

$$\mu^*(E) = \mu^*(\emptyset \cap E) + \mu^*(X \cap E).$$

Thus \emptyset, X are μ^* measurable sets.

2. Suppose A is μ^* measurable and $E \subseteq X$. Then

$$\mu^*(E) = \mu^*(A \cap E) + \mu^*(A^C \cap E) = \mu^*(A^C \cap E) + \mu^*((A^C)^C \cap E).$$

Thus A^C is μ^* -measurable.

3. First, we will show μ^* is finitely additive on \mathcal{M} . Let $A, B \in \mathcal{M}$ with $A \cap B = \emptyset$ and let $E \subseteq X$. Notice

$$A \cup B = (A \cap X) \cup (A^C \cap B) = (A \cup (B \cap B^C)) \cup (A^C \cap B) = (A \cap B) \cup (A \cap B^C) \cup (A^C \cap B).$$

Now

$$\begin{aligned}
\mu^*(E) &= \mu^*(E \cap A) + \mu^*(E \cap A^C) \\
&= \mu^*(E \cap A \cap B) + \mu^*(E \cap A \cap B^C) + \mu^*(E \cap A^C \cap B) + \mu^*(E \cap A^C \cap B^C) \\
&\geq \mu^*(E \cap (A \cap B) \cup (A \cap B^C) \cup (A^C \cap B)) + \mu^*(E \cap A^C \cap B^C) \\
&= \mu^*(E \cap (A \cup B)) + \mu^*(E \cap (A \cup B)^C).
\end{aligned}$$

By the remark, we know \leq is true, thus we have equality and $A \cup B$ is μ^* measurable. Now, let $E = A \cup B$. Then, as A is μ^* measurable, we see

$$\mu^*(A \cup B) = \mu^*(A \cap (A \cup B)) + \mu^*(A^C \cap (A \cup B)) = \mu^*(A) + \mu^*(B).$$

Thus μ^* is finitely additive. Now we want to show that \mathcal{M} is closed under countable unions. Let $\{A_k\}_{k=1}^\infty \subset \mathcal{M}$ be mutually disjoint. For all n , set $B_n = \cup_{k=1}^n A_k$ and set $B = \cup_{k=1}^\infty A_k$. Let $E \subseteq X$. Notice that $B_n \cap A_n = A_n$ and $B_n \cap A_n^C = B_{n-1}$. Thus

$$\mu^*(E \cap B_n) = \mu^*(E \cap B_n \cap A_n) + \mu^*(E \cap B_n \cap A_n^C) = \mu^*(E \cap A_n) + \mu^*(E \cap B_{n-1}) = \sum_{k=1}^n \mu^*(E \cap A_k)$$

by iterative applications. Since $B_n^C \supset B^C$, we see

$$\mu^*(E) = \mu^*(E \cap B_n) + \mu^*(E \cap B_n^C) \geq \sum_{k=1}^n \mu^*(E \cap A_k) + \mu^*(E \cap B^C).$$

Since this is true for all n , we get

$$\mu^*(E) \geq \sum_{k=1}^\infty \mu^*(E \cap A_k) + \mu^*(E \cap B^C) \geq \mu^*(E \cap B) + \mu^*(E \cap B^C)$$

by countable subadditivity. Thus $B = \cup_{k=1}^\infty A_k \subseteq \mathcal{M}$.

Thus \mathcal{M} is a σ -algebra. By Thm 19, we see that μ^* is countable on \mathcal{M} (as σ -algebra implies algebra).

To show μ^* is complete, let $N \in \mathcal{M}$ with $\mu^*(N) = 0$. Let $A \subseteq N$. Notice $0 \leq \mu^*(A) \leq \mu^*(N) = 0$. So $\mu^*(A) = 0$. Let $E \subseteq X$. Then

$$\mu^*(E \cap A) + \mu^*(E \cap A^C) = \mu^*(E \cap A^C) \leq \mu^*(E)$$

as $\mu^*(E \cap A) \leq \mu^*(N) = 0$. Thus $A \in \mathcal{M}$. □

Example. Let $\mathcal{E} = \{\emptyset, \{x\}, X\}$ with $x \in X$ and $X \setminus \{x\} \neq \emptyset$. Consider $\rho : \mathcal{E} \rightarrow [0, \infty]$ defined by $\rho(\emptyset) = 0, \rho(X) = 1, \rho(\{x\}) = 2$. Then, by definition $\mu^*(\emptyset) = 0, \mu^*(X) = \inf\{\sum \rho(A_j) : \{A_j\} \subseteq \mathcal{E} \text{ and } X \subseteq \cup A_j\} = 1$. Let $A \subseteq X$ with $A \neq \emptyset$. Then $\mu^*(A) = 1$ (as X covers A). What sets are μ^* -measurable?

- Clearly \emptyset, X are.
- Let $A \subsetneq X$ such that $A \neq \emptyset$. Note that $\mu^*(X \cap A) + \mu^*(X \cap A^C) = 1 + 1 \neq 1 = \mu^*(X)$. Thus A is not μ^* -measurable.

This example shows that \mathcal{M} is *not* generated by \mathcal{E} as $\mathcal{E} \subsetneq \mathcal{M}$. As we shall see, if \mathcal{E} is an algebra, then \mathcal{M} is the σ -algebra generated by \mathcal{E} .

Proposition 28. *If $\tilde{\mu}$ is a premeasure on an algebra $\mathcal{A} \subseteq \mathcal{P}(X)$ and μ^* is the outer measure induced by $\tilde{\mu}$, then*

1. $\mu^*|_{\mathcal{A}} = \tilde{\mu}$
2. Every set in \mathcal{A} is μ^* -measurable.

Proof. 1. Let $E \in \mathcal{A}$. We will show $\mu^*(E) = \tilde{\mu}(E)$. Since $\mu^*(E) = \inf\{\sum_{j=1}^{\infty} \tilde{\mu}(A_j) : \{A_j\} \subseteq \mathcal{A}, E \subseteq \cup A_j\}$, we see $\mu^*(E) \leq \tilde{\mu}(E)$ (take $A_1 = E$ and $A_j = \emptyset$ for $j > 1$). Now, let $\{A_j\}_1^{\infty} \subseteq \mathcal{A}$ such that $E \subseteq \cup A_j$. By Lemma 1, the sequence $\{B_j\} \subseteq \mathcal{A}$ defined by $B_j = A_j \setminus \cup_{k=1}^{j-1} A_k$ is such that B_j 's are mutually disjoint and $\cup B_j = \cup A_j$. We see

$$\bigcup_{j=1}^{\infty} (B_j \cap E) = \left(\bigcup_{j=1}^{\infty} B_j \right) \cap E = \left(\bigcup_{j=1}^{\infty} A_j \right) \cap E = E.$$

Since $\tilde{\mu}$ is a premeasure and $B_j \cap E \subseteq \mathcal{A}$ for all j ,

$$\tilde{\mu}(E) = \tilde{\mu} \left(\bigcup_{j=1}^{\infty} (B_j \cap E) \right) = \sum_{j=1}^{\infty} \tilde{\mu}(B_j \cap E) \leq \sum_{j=1}^{\infty} \tilde{\mu}(A_j).$$

Now, since this holds for all $\{A_j\}$, taking the infimum gives us $\tilde{\mu}(E) \leq \mu^*(E)$.

2. Let $A \in \mathcal{A}$ and $E \subseteq X$. By definition of μ^* , for all $\epsilon > 0$ there exists a sequence of A_j such that $E \subseteq \cup A_j$ and $\mu^*(E) \geq \sum_{j=1}^{\infty} \tilde{\mu}(A_j) - \epsilon$. Since $\tilde{\mu}$ is additive and $A \cap A_j, A^C \cap A_j \in \mathcal{A}$, we see

$$\begin{aligned} \mu^*(E) &\geq \sum \tilde{\mu}(A_j) - \epsilon \\ &= \sum \tilde{\mu}((A \cap A_j) \cup (A^C \cap A_j)) - \epsilon \\ &= \sum \tilde{\mu}(A \cap A_j) + \sum \tilde{\mu}(A^C \cap A_j) - \epsilon \\ &= \sum \tilde{\mu}(A \cap A_j) + \sum \tilde{\mu}(A^C \cap A_j) - \epsilon \\ &= \sum \mu^*(A \cap A_j) + \sum \mu^*(A^C \cap A_j) - \epsilon \\ &\geq \mu^*(A \cap (\cup A_j)) + \mu^*(A^C \cap (\cup A_j)) - \epsilon \\ &\geq \mu^*(A \cap E) + \mu^*(A^C \cap E) - \epsilon \end{aligned}$$

Since this is true for all ϵ , we see $\mu^*(E) \geq \mu^*(A \cap E) + \mu^*(A^C \cap E)$. Thus $\mathcal{A} \in \mathcal{M}$. \square

Definition. Let $\mathcal{A} \subseteq \mathcal{P}(X)$ be an algebra and $\tilde{\mu}$ a premeasure on \mathcal{A} . Then $\tilde{\mu}$ is called

- **finite** if $\tilde{\mu}(X) < \infty$.
- **σ -finite** if there exists $\{A_j\}_{j=1}^{\infty} \subseteq \mathcal{A}$ such that $X = \cup_1^{\infty} A_j$ and $\tilde{\mu}(A_j) < \infty$.
- **semifinite** if for all $E \in \mathcal{A}$ with $\tilde{\mu}(E) = \infty$, then there exists $A \subseteq E$ such that $0 < \tilde{\mu}(A) < \infty$.

Theorem 20 (p 31). Let $\mathcal{A} \in \mathcal{P}(X)$ be an algebra. Let $\tilde{\mu}$ be a premeasure on \mathcal{A} and \mathcal{M} the σ -algebra generated by \mathcal{A} .

1. Then there exists a measure μ on \mathcal{M} such that $\mu|_{\mathcal{A}} = \tilde{\mu}$.
2. If there exists another measure ν such that $\nu|_{\mathcal{A}} = \tilde{\mu}$ then $\nu(E) \leq \mu(E)$ for all $E \in \mathcal{M}$. If $\mu(E) < \infty$, then $\nu(E) = \mu(E)$.
3. If $\tilde{\mu}$ is σ -finite, then μ is the unique extension of $\tilde{\mu}$ to \mathcal{M} .

Proof. 1. Follows from Caratheodory's Theorem and Prop 28 if we take $\mu = \mu^*$ (the outer measure induced by $\tilde{\mu}$.)

2. Suppose ν is another measure on \mathcal{M} which extends $\tilde{\mu}$. If $\mu(E) = \infty$, then $\nu(E) \leq \mu(E)$. So assume $\mu(E) < \infty$. Then for all $\epsilon > 0$, there exists $\{A_j\}_{j=1}^{\infty} \subseteq \mathcal{A}$ such that $\mu(E) \geq \sum_{j=1}^{\infty} \tilde{\mu}(A_j) - \epsilon$. Since ν is a measure, $\nu(E) \leq \nu(\cup A_j) \leq \sum \nu(A_j) = \sum \mu(A_j) \leq \mu(E) + \epsilon$. Since ϵ was arbitrary, $\nu(E) \leq \mu(E)$. To show $\nu(E) = \mu(E)$ if $\mu(E) < \infty$, let $\epsilon > 0$ and take $\{A_j\}$ as above. By continuity of measures from above (Theorem 5),

$$\nu(\cup_1^{\infty} A_j) = \lim_{n \rightarrow \infty} \nu(\cup_1^n A_j) = \lim_{n \rightarrow \infty} \mu(\cup_1^n A_j) = \mu(\cup_1^{\infty} A_j).$$

Now $\mu(\cup_1^{\infty} A_j \setminus E) = \mu(\cup_1^{\infty} A_j) - \mu(E)$ and $\mu(\cup_1^{\infty} A_j) \leq \sum \mu(A_j) = \sum \tilde{\mu}(A_j) < \mu(E) + \epsilon$. Thus $\mu(\cup_1^{\infty} A_j \setminus E) < \epsilon$. So $\mu(E) \leq \mu(\cup_1^{\infty} A_j) = \nu(\cup_1^{\infty} A_j) = \nu(E) + \nu(\cup_1^{\infty} A_j \setminus E) \leq \nu(E) + \mu(\cup_1^{\infty} A_j \setminus E) < \nu(E) + \epsilon$. Since ϵ was arbitrary, we see $\nu(E) > \mu(E)$. Thus $\nu(E) = \mu(E)$.

3. If $\tilde{\mu}$ is σ -finite, then $X = \cup_1^\infty A_j$ with $\tilde{\mu}(A_j) = \mu(A_j) = \nu(A_j) < \infty$. Let $E \in \mathcal{M}$. Consider $E \cap \cup_1^n A_j$. By (2), we see $\nu(E \cap \cup_1^n A_j) = \mu(E \cap \cup_1^n A_j)$. Thus

$$\nu(E) = \nu(E \cap \cup_1^\infty A_j) = \lim \nu(E \cap \cup_1^n A_j) = \lim \mu(E \cap \cup_1^n A_j) = \mu(E \cap \cup_1^\infty A_j) = \mu(E). \quad \square$$

Recall the premeasure μ_F obtained in Prop 25 where $\tilde{\mu}_F|_{\tilde{I}} = \mu_F$. By Caratheodory's Theorem, the μ_F^* -measurable sets form a σ -algebra where $\mu_F^*(E) = \inf\{\sum \mu_F((a_j, b_j]) : E \subseteq \cup(a_j, b_j]\}$. We denote this σ -algebra by \mathcal{M}_{μ_F} and $\mu_F^*|_{\mathcal{M}_{\mu_F}}$ by μ_F . This is the extension of μ_F on \tilde{I} to all of \mathcal{M}_{μ_F} . It follows that $\mathcal{B}_{\mathbb{R}} \subseteq \mathcal{M}_{\mu_F}$ (note that in general this is a strict containment).

Notes.

1. F is called the distribution function for μ_F .
2. μ_F is a complete measure on \mathcal{M}_{μ_F} (by Caratheodory's Theorem).
3. The measure μ_F is called the **Lebesgue-Stieltjes Measure** associated with F . If $F = x$, then μ_F is called the **Lebesgue Measure** and \mathcal{M}_{μ_F} is called the Lebesgue Measurable sets.

Note that for any $E \subseteq \mathcal{M}_{\mu_F}$, we define $\mu_F(E)$ to be $\mu_F^*(E)$ as we defined it above.

Fix an F which is right continuous and increasing.

Lemma 2. For any $E \in \mathcal{M}_{\mu_F}$, $\mu_F(E) = \inf\{\sum_1^\infty \mu_F((a_j, b_j]) : E \subseteq \cup(a_j, b_j]\}$.

Proof. Define $\nu(E) = \inf\{\sum \mu_F((a_j, b_j]) : E \subseteq \cup(a_j, b_j]\}$. We want to show $\nu(E) \geq \mu_F(E) \geq \nu(E)$.

1. Let $\{(a_j, b_j]\}_1^\infty$ be such that $E \subseteq \cup(a_j, b_j]$. For all j , let $\{c_{k,j}\}_{k=1}^\infty$ be a sequence in (a_j, b_j) such that $c_{k,j}$ increases up to b_j . Then $(a_j, b_j) = (a_j, c_{1,j}] \cup \cup_1^\infty (c_{k,j}, c_{k,j+1}]$. So $\mu_F((a_j, b_j)) = \mu_F((a_j, c_{1,j}]) + \sum_1^\infty \mu_F((c_{k,j}, c_{k,j+1}])$. It follows that $\sum_1^\infty \mu_F((a_j, b_j)) \geq \sum_1^\infty \mu_F((a_j, c_{1,j}]) + \sum_{j,k} \mu_F((c_{k,j}, c_{k,j+1}]) \geq \mu_F(E)$. Since this holds for all intervals $\{(a_j, b_j]\}$ such that $E \subseteq \cup(a_j, b_j]$, we see $\nu(E) \geq \mu_F(E)$.
2. Let $\epsilon > 0$. By definition of μ_F , there exists $\{a_j, b_j\}$ such that $\mu_F(E) \geq \sum \mu_F((a_j, b_j]) - \epsilon$. Since F is right continuous, for all j there exists δ_j such that $F(b_j + \delta_j) - F(b_j) \leq \epsilon 2^{-j}$. Since $E \subseteq \cup_1^\infty (a_j, b_j + \delta_j)$, we see

$$\begin{aligned} \nu(E) &\leq \sum \mu_F((a_j, b_j + \delta_j)) \\ &= \sum \mu_F((a_j, b_j]) + \mu_F((b_j, b_j + \delta_j)) \\ &\leq \mu_F(E) + \epsilon + \sum F(b_j + \delta_j) - F(b_j) \\ &\leq \mu_F(E) + 2\epsilon. \end{aligned}$$

Of course, ϵ is arbitrary. Thus $\nu(E) \leq \mu_F(E)$. □

Theorem 21. If $E \subseteq \mathcal{M}_{\mu_F}$, then

1. $\mu_F(E) = \inf\{\mu_F(U) : U \text{ is open, } E \subseteq U\}$ (that is, μ_F is outer-regular)
2. $\mu_F(E) = \sup\{\mu_F(K) : K \text{ is compact, } E \supseteq K\}$ (that is, μ_F is inner-regular)

Proof. 1. Let $E \in \mathcal{M}_{\mu_F}$. By Lemma 2, for all ϵ there exists $\{(a_j, b_j]\}_1^\infty \subset \mathcal{P}(\mathbb{R})$ such that $E \subset \cup(a_j, b_j]$ and $\mu_F(E) \geq \sum \mu_F(a_j, b_j) - \epsilon$. Since μ_F is subadditive, $\mu_F(E) \geq \mu_F(\cup(a_j, b_j]) - \epsilon$. Let $U = \cup(a_j, b_j)$, an open set. Then $\mu_F(U) \leq \mu_F(E) + \epsilon$. Now ϵ is arbitrary and since all open sets are the union of open intervals, we see $\inf\{\mu_F(U)\} \leq \mu_F(E)$. Of course, \geq is true by monotonicity, so they are equal.

2. If E is compact, clearly $\mu_F(E) = \sup\{\mu_F(K) : K \subseteq E, K \text{ is compact}\}$. If E is bounded, then the closure of E , \bar{E} , is compact. Note that it is also measurable as it is a Borel Set. Thus by (1) for all $\epsilon > 0$ we can find an open U such

that $\overline{E} \setminus E \subseteq U$ and $\mu_F(\overline{E} \setminus E) \geq \mu_F(U) - \epsilon$. Note that $\overline{E} \setminus U$ is compact and $\overline{E} \setminus U \subseteq E$. Since $\overline{E} \setminus U = E \setminus (E \cap U)$, we have

$$\begin{aligned} \mu_F(\overline{E} \setminus U) &= \mu_F(E \setminus E \cap U) \\ &= \mu_F(E) - \mu_F(E \cap U) \text{ since } E \text{ is bounded, } \mu_F(E) < \infty \\ &= \mu_F(E) - \mu_F(U \setminus (U \setminus E)) \\ &= \mu_F(E) - \mu_F(U) + \mu_F(U \setminus E) \text{ since } \mu_F(U) \leq \mu_F(\overline{E} \setminus E) + \epsilon < \infty \\ &\geq \mu_F(E) - \mu_F(\overline{E} \setminus E) - \epsilon + \mu_F(U \setminus E) \\ &\geq \mu_F(E) - \mu_F(\overline{E} \setminus E) - \epsilon + \mu_F(\overline{E} \setminus E) = \mu_F(E) - \epsilon \end{aligned}$$

So for all compact sets $\overline{E} \setminus U$ we have $\mu_F(E) \leq \mu_F(\overline{E} \setminus U) + \epsilon \leq \mu_F(E) + \epsilon$. Since this is true for all ϵ , we see $\mu_F(E) = \sup\{\mu_F(K) : K \subseteq E, K \text{ is compact}\}$. If E is not necessarily bounded or closed, consider $E_j = E \cap (j, j+1]$. Clearly $\cup_{j=-\infty}^{\infty} E_j = E$ and E_j is bounded for all j . Let $\epsilon > 0$. By previous argument, for all j there exists K_j such that K_j is compact, $K_j \subseteq E_j$, and $\mu_F(K_j) \leq \mu_F(E_j) \leq \mu_F(K_j) + \epsilon 2^{-|j|}$. Put $H_n = \cup_{j=-n}^n K_j$. Then H_n is compact and $H_n \subseteq E$. So $\mu_F(H_n) \leq \mu_F(\cup_{j=-n}^n E_j) = \sum_{j=-n}^n \mu_F(E_j) \leq \sum_{j=-n}^n \mu_F(K_j) + 3\epsilon = \mu_F(H_n) + 3\epsilon$. If $\mu_F(E) = +\infty$, then $\lim_{n \rightarrow \infty} \mu_F(\cup_{j=-n}^n E_j) = \infty$ which implies $\mu_F(H_n) \rightarrow \infty$. Then $\sup\{\mu_F(K) : K \subseteq E, K \text{ is compact}\} = \infty$. If $\mu_F(E) < \infty$, then $\lim_{n \rightarrow \infty} \mu_F(\cup_{j=-n}^n E_j) = \mu_F(E)$. Then there exists $N \in \mathbb{N}$ such that $|\mu_F(E) - \mu_F(\cup_{j=-N}^N E_j)| < \epsilon$. Hence $\mu_F(H_N) \leq \mu_F(E) \leq \mu_F(H_N) + 4\epsilon$. It follows, since ϵ was arbitrary, that $\mu_F(H_N) \rightarrow \mu_F(E)$ and $\mu_F(E) = \sup\{\mu_F(K) : K \subseteq E, K \text{ is compact}\}$. \square

Theorem 22. If $A \subseteq \mathbb{R}$, then there exists $E \in \mathcal{B}_{\mathbb{R}}$ such that $\mu_F(E) = \mu_F^*(A)$ and $A \subseteq E$.

Proof. If $\mu_F(A) = \infty$, let $E = \mathbb{R}$. Otherwise, assume $\mu_F(A) < \infty$. For all j , we may select $\{(a_{j,k}, b_{j,k})\}_{k=1}^{\infty}$ such that $A \subseteq \cup_{k=1}^{\infty} (a_{j,k}, b_{j,k}]$ and $\mu^*(A) \geq \sum_{k=1}^{\infty} \mu_F((a_{j,k}, b_{j,k}]) - \frac{1}{j}$. Put $B_j = (a_{j,k}, b_{j,k}]$. Then $B_j \in \mathcal{B}_{\mathbb{R}}$ and we may assume $\mu_F(B_j) < \infty$. Then $\mu_F^*(A) \geq \mu_F(B_j) - \frac{1}{j}$ and $\mu_F^*(A) \leq \mu_F(B_j)$ as $A \subseteq B_j$. Let $B = \cap_{j=1}^{\infty} B_j \in \mathcal{B}_{\mathbb{R}}$. Then $\mu_F(B) = \lim_{\ell \rightarrow \infty} \mu_F(\cap_{j=1}^{\ell} B_j) \leq \lim_{\ell \rightarrow \infty} \mu_F(B_{\ell}) \leq \lim_{\ell \rightarrow \infty} \mu_F^*(A) + \frac{1}{\ell} = \mu_F^*(A)$. Since $A \subseteq B_j$ for all j , we know $A \subseteq B$ and so $\mu_F^*(A) \leq \mu_F(B)$. Combining these two equations, we get equality. \square

Definition. Suppose g is an $(\mathcal{M}_{\mu_F}, \mathcal{B}_{\mathbb{R}})$ -measurable function, with μ_F a Lebesgue-Stieltjes measure. Then $\int_{\mathbb{R}} g d\mu_F$ is called the **Lebesgue-Stieltjes Integral**.

Theorem 23. Suppose F is increasing and differentiable on \mathbb{R} . Then

$$\int_{\mathbb{R}} g \chi_{(a,b]} d\mu_F = \int_a^b g F' dx.$$

Note. If $F = x$, then $\mu_F = m$ is the Lebesgue measure and Thm 23 reduces to $\int_{\mathbb{R}} g \chi_{(a,b]} d\mu_F = \int_a^b g dx$.

Theorem 24. If $E \subseteq \mathbb{R}$, then TFAE

1. $E \in \mathcal{M}_{\mu_F}$
2. $E = V \setminus N_1$, where $V \in G_{\delta} = \{\cap_{j=1}^{\infty} U_j | U_j \text{ is open}\}$ and $\mu_F(N_1) = 0$.
3. $E = H \cup N_2$, where $H \in F_{\sigma} = \{\cup_{j=1}^{\infty} K_j | K_j \text{ is closed}\}$ and $\mu_F(N_2) = 0$.

Proof. Since $\mathcal{B}_{\mathbb{R}} \subseteq \mathcal{M}_{\mu_F}$, we see that (b) \Rightarrow (a) and (c) \Rightarrow (a). So suppose $E \in \mathcal{M}_{\mu_F}$. First, suppose $\mu_F(E) < \infty$. Then by Theorem 21, there exists $\{U_j\}_{j=1}^{\infty} \subseteq \mathcal{P}(X)$ of open sets and $\{K_j\}_{j=1}^{\infty} \subseteq \mathcal{P}(X)$ of compact sets such that $E \subseteq U_j$ and $K_j \subseteq E$ and $\mu(U_j) - 2^{-j} \leq \mu_F(E) \leq \mu_F(K_j) + 2^{-j}$. Put $V = \cap_{j=1}^{\infty} U_j$ and $H = \cup_{j=1}^{\infty} K_j$. So $H \subseteq E \subseteq V$. Then $\mu_F(H) \leq \mu_F(E) \leq \mu_F(V)$ and

$$\mu_F(V) = \lim_{\ell \rightarrow \infty} \mu_F(\cap_{j=1}^{\ell} U_j) \leq \mu_F(E) \leq \lim_{\ell \rightarrow \infty} \mu_F(\cup_{j=1}^{\ell} K_j) = \mu_F(H).$$

Thus $\mu_F(H) = \mu_F(E) = \mu_F(V)$. It follows that $\mu_F(V \setminus E) = 0$ and $\mu_F(E \setminus H) = 0$. Let $N_1 = V \setminus E$ and $N_2 = E \setminus H$. Note that $V \in G_{\delta}$ and $H \in F_{\sigma}$. Now suppose $\mu_F(E) = \infty$. Consider $E_j = E \cap (j, j+1]$ for all $j \in \mathbb{Z}$. Follow the argument in Theorem 21. \square

The Lebesgue Measure is the most commonly used measure on \mathbb{R}^n . Let \mathcal{L} denote the set of Lebesgue Measurable sets. Note that \mathcal{L} is complete and is Borel, that is $\mathcal{B}_{\mathbb{R}} \subseteq \mathcal{L}$.

Theorem 25. *If $E \in \mathcal{L}$, then $E + s = \{x + s \in \mathbb{R} | x \in E\}$, $rE = \{rx \in \mathbb{R} | x \in E\} \in \mathcal{L}$ for all $r, s \in \mathbb{R}$. Moreover, $m(E + s) = m(E)$ and $m(rE) = |r|m(E)$.*

Proof. If E is open, then so is $E + s$ and rE (for $r \neq 0$). It follows if $E \in \mathcal{B}_{\mathbb{R}}$, then so is $E + s$ and rE for all $r, s \in \mathbb{R}$. Denote $m(E + s)$ by $m_s(E)$ and $m(rE)$ by $m^r(E)$ for all $E \in \mathcal{L}$. Clearly

$$m_s(I(a, b)) = m(I(a + s, b + s)) = (b + s) - (a + s) = b - a = m(I(a, b))$$

and

$$m^r(I(a, b)) = m(I(|r|a, |r|b)) = |r|b - |r|a = |r|m(I(a, b))$$

for all intervals $I(a, b)$. Since \mathbb{R} is σ -finite with respect to m_s, m^r, m , our earlier propositions imply that the extension of m_s, m^r, m from the left open, right closed intervals to $\mathcal{B}_{\mathbb{R}}$ is unique. Thus we see $m_s(E) = m(E)$ and $m^r(E) = |r|m(E)$ for all $E \in \mathcal{B}_{\mathbb{R}}$. Suppose $E \in \mathcal{L}$ and $m(E) = 0$. By Theorem 24, there exists $V \in \mathcal{B}_{\mathbb{R}}$ and a null set N such that $E = V \setminus N$. Now $0 = m(E) = m(V \setminus N) = m(V) - m(N) = m(V)$. So V is a Lebesgue null set. Since $V \in \mathcal{B}_{\mathbb{R}}$, we have $m_s(V) = m(V) = 0$ and $m^r(V) = |r|m(V) = 0$. By monotonicity and completeness, as $E \subseteq V$, we see $m_s(E) = 0 = m^r(E)$. In general, if $E \in \mathcal{L}$, then use Theorem 24(3) to conclude $E + s, rE \in \mathcal{L}$ and $m_s(E) = m(E)$ and $m^r(E) = |r|m(E)$. \square

Example. Let $E = \mathbb{Q} \cap [0, 1]$. Then there exists an enumeration $\{r_j\}_1^\infty$ of E (since \mathbb{Q} is countable). Observe that $m(\{r_j\}) = 0$. So $m(E) = m(\cap\{r_j\}) = \sum m(\{r_j\}) = 0$. So this is measure-theoretically small. On the other hand, $\overline{E} = [0, 1]$, so this is topologically large.

Now set $U_j = (r_j - \epsilon 2^{-j}, r_j + \epsilon 2^{-j}) \cap [0, 1]$ for all $j \in \mathbb{N}$. So $m(\cup U_j) \leq \sum m(U_j) = \sum \epsilon 2^{-j+1} = 2\epsilon$. So again, this is set is measure-theoretically very small. However, we still see $\overline{\cup_1^\infty U_j} = [0, 1]$, and so the set is topologically large.

Define $F = [0, 1] \setminus (\cup_1^\infty U_j \cap [0, 1])$. Notice F is compact (and thus $F = \overline{F}$). Clearly, $F \neq [0, 1]$, in fact F is nowhere dense (that is, there does not exist an open interval contained in F). So this set is topologically small. However, $m(F) = m([0, 1] \setminus (\cup U_j \cap [0, 1])) = m([0, 1]) - m(\cup U_j \cap [0, 1]) \geq 1 - 2\epsilon$. Thus this is measure-theoretically large.

Proposition 29. *Let C be the Cantor Set.*

1. C is compact, nowhere dense, and totally disconnected. However, C has no isolated points.
2. $m(C) = 0$.
3. C is uncountable.

Proof. 1. That C is compact follows from the fact that it is a countable intersection of closed sets (and therefore closed) and clearly bounded (as $C \subseteq [0, 1]$). For the other properties, we will use decimal expansions base 3, that is for $x \in [0, 1]$, we will find $a_j \in \mathbb{Z}_3$ such that $x = \sum_1^\infty \frac{a_j}{3^j}$ and write $x = .a_1a_2a_3 \dots_3$. Note that $\sum_{j=2}^\infty \frac{2}{3^j} = 2(\frac{1}{1-\frac{1}{3}} - 1 - \frac{1}{3}) = \frac{1}{3}$, so $.0\overline{2}_3 = .1_3$. Now we will apply this to our Cantor Set:

$$\begin{aligned} S_0 &= [0_3, 1_3] \\ S_1 &= [0_3, .1_3] \cup [.2_3, 1_3] = [0_3, .0\overline{2}_3] \cup [.2_3, .2\overline{2}_3] \\ S_2 &= [0_3, .00\overline{2}_3] \cup [.02_3, .02\overline{2}_3] \cup [.2_3, .20\overline{2}_3] \cup [.22_3, .22\overline{2}_3] \end{aligned}$$

It follows that C contains all the points that have only 0's and 2's in its base 3 expansion. It seems all be in C , however this is not the case:

$$\frac{1}{4} = \sum \frac{2}{9^j} = .\overline{02}_3 \text{ which implies } \frac{1}{4} \in C$$

yet it is clear that $\frac{1}{4}$ is not an endpoint. Now C has no open intervals as we can always choose a number with a 1 in its base 3 expansion inside any open interval of $[0, 1]$. This says C is totally disconnected. To show it has no isolated points, we will use a particular example (as all other points will follow from there). Consider $\frac{1}{4} = .02\overline{02}_3$ and $x_1 = .000\overline{2}_3$.

These are in C and differ by $.02_3 = \frac{2}{9}$. Now consider $x_2 = .0200\overline{02}_3$. This is also in C and $\frac{1}{4} - x_2 = .0002 = \frac{2}{9^2}$. We can continually do this, finding a sequence $\{x_i\} \subseteq C$ such that $\frac{1}{4} - x_i = \frac{2}{9^i}$. Then $x_i \rightarrow \frac{1}{4}$, which says it is not an isolated point.

2. Now $m(C) = m(\cap S_n) = \lim m(S_n)$ (we can do that as $m(S_0) = 1 < \infty$). Note that $m(S_0) = 1, m(S_1) = 1 - \frac{1}{3} = \frac{2}{3}, m(S_2) = \frac{2}{3} - \frac{2}{9} = \frac{4}{9}, m(S_3) = \frac{4}{9} - \frac{4}{27} = \frac{8}{27}$, etc. Thus $m(S_n) = \frac{2^n}{3^n} \rightarrow 0$. Thus $m(C) = 0$.
3. To show C is uncountable, we will show there exists a surjective map $C \rightarrow [0, 1]$, as we know $[0, 1]$ is uncountable. For all $x = \sum \frac{a_j}{3^j}$ with $a_j = 0, 2$ (i.e., for all $x \in C$), define $f(x) = \sum \frac{a_j}{2^{2j}}$, a binary expansion. Now let $y \in [0, 1]$. Then $y = \sum \frac{b_j}{2^j}$ for some $b_j \in \mathbb{Z}_2$. Let $x = \sum \frac{2b_j}{3^j}$. Then $x \mapsto y$. □

Generalized Cantor Set

Let I be a bounded interval and call J the open α^{th} middle of I . If J is open, then $m(J) = \alpha m(I)$ and the midpoint of J is the same as I . (Here, we take $\alpha \in [0, 1]$.) Inductively define $S_0 = [0, 1]$ and S_n to be S_{n-1} with the α_n^{th} middle of each interval in S_{n-1} removed. The generalized Cantor set is then $C = \cap S_n$. If $\{\alpha_n\}_{n=1}^\infty \subseteq (0, 1)$, then C is compact, totally disconnected, and uncountable. For n , we see $m(S_n) = (1 - \alpha_n)m(S_{n-1}) = \prod_1^n (1 - \alpha_i)$. So $m(C) = \prod_1^\infty (1 - \alpha_i)$. If $\{\alpha_n\}$ are bounded away from 0 uniformly, then $m(C) = 0$. If $\{\alpha_n\}$ go to 0 slowly enough, then $m(C) = 0$. If $\{\alpha_n\}$ go to 0 too fast, then $m(C) > 0$.

Vitali Function, AKA Cantor-Lebesgue Function

Definition (1). *The complement of the Cantor Set C in $[0, 1]$ is*

$$O = [0, 1] \setminus C = [0, 1] \setminus \cap_{n=0}^\infty S_n = \cup_{n=1}^\infty [0, 1] \setminus S_n = \cup_{j=1}^\infty \cup_{\alpha \in \{0, 2\}^j} O_\alpha$$

where $\{0, 2\}^0 = \emptyset, \{0, 2\}^1 = \{(0), (2)\}, \{0, 2\}^2 = \{(0, 0), (0, 2), (2, 0), (2, 2)\}$, etc and $O_\emptyset = (\frac{1}{3}, \frac{2}{3}), O_{(0)} = (\frac{1}{9}, \frac{2}{9}), O_{(2)} = (\frac{7}{9}, \frac{8}{9}), O_{(0,0)} = (\frac{1}{27}, \frac{2}{27})$, and for $\alpha = (a_1, \dots, a_n)$, we see $O_\alpha = (\sum_{i=1}^n a_i 3^i + 3^{-(n+1)}, \sum_{i=1}^n a_i 3^i + 2 \cdot 3^{-(n+1)})$.

Now, for $x \in O_\emptyset$, define $f(x) = \frac{1}{2}$ and for $x \in O_\alpha$ for $\alpha \neq \emptyset$, define $f(x) = \sum_{i=1}^n 6n \frac{a_i}{2} 2^{-i} + 2^{-(n+1)}$. Note that f is uniformly continuous on O . Thus let F be the unique extension of f to $[0, 1]$ that is continuous. Note $F(0) = 0, F(1) = 1$, and F is non-decreasing.

Definition (2). Define $F(x) = \begin{cases} \frac{1}{2}F(3x) & \text{if } 0 \leq x \leq \frac{1}{3} \\ \frac{1}{2} & \text{if } \frac{1}{3} < x < \frac{2}{3} \\ \frac{1}{2} + \frac{1}{2}F(3x - 2) & \text{if } \frac{2}{3} \leq x \leq 1 \end{cases}$. Then, for example, $F(\frac{1}{4}) = \frac{1}{2}F(\frac{3}{4}) = \frac{1}{2}(\frac{1}{2} + \frac{1}{2}F(\frac{1}{4}))$, and

solving we get $F(\frac{1}{4}) = \frac{1}{3}$.

Fact. F is piecewise constant on O , so F is constant a.e. Also, F is differentiable a.e. and $F'(x) = 0$ a.e. In fact, $F'(x) = 0$ for all $x \in O$. However, F is not constant as $F(0) = 0$ and $F(1) = 1$.

Another Amazing Function

Define $g : [0, 1] \rightarrow [0, 2]$ by $g(x) = F(x) + x$. Then g is continuous and strictly increasing. Also $g(0) = 0, g(1) = 2$. So $m(g([0, 1])) = 2$. What is $m(g(O))$? Since $O = \cup_{j=1}^\infty \cup_{\alpha \in \{0, 2\}^j} O_\alpha$ with O_α mutually disjoint and g is strictly increasing, $g(O) = \cup g(O_\alpha)$. Thus $m(g(O)) = \sum m(g(O_\alpha)) = \sum m(O_\alpha) = 1$ (since $O_\alpha, g(O_\alpha) = O_\alpha + c$ and m is translation invariant). So $m(g(C)) = m(g([0, 1] \setminus O)) = m(g([0, 1])) - m(g(O)) = 1$. So g maps the Cantor Set to a set of measure 1 (in a continuous way!).

2.1 Hausdorff Measure (p 350)

For all $\delta > 0$, define $\mathcal{E}_\delta = \{E \subseteq \mathbb{R}^n : diam(E) < \delta\}$, where $diam(E) = \sup\{\|x - y\| : x, y \in E\}$. Then, for all $p \geq 0$, define $H_{p, \delta}(A) = \inf\{\sum_{j=1}^\infty (diam(B_j))^p : \{B_j\}_{j=1}^\infty \subseteq \mathcal{E}_\delta \text{ and } A \subseteq \cup_1^\infty B_j\}$. Note that this is an outer measure. Define the **Hausdorff Outer-Measure** H_p by $H_p(A) = \lim_{\delta \rightarrow 0^+} H_{p, \delta}(A)$.

Notes.

1. $\mathcal{E}_{\delta_1} \subset \mathcal{E}_{\delta_2}$ whenever $\delta_1 < \delta_2$. So $H_{p,\delta_1}(A) \geq H_{p,\delta_2}(A)$. Thus $\{H_{p,\delta}\}$ is increasing and thus the limit exists.

2. You may restrict \mathcal{E}_δ (and still get the same results) to $E \subseteq \mathbb{R}^n$ where E is open or closed (see p 350).

Proposition 30. H_p is an outermeasure on \mathbb{R}^n .

Proof. We see that H_p is nonnegative and $H_p(\emptyset) = 0$. If $A_1 \subseteq A_2$, then since $H_{p,\delta}$ is an outermeasure, $H_{p,\delta}(A_1) \leq H_{p,\delta}(A_2) \leq H_p(A_2)$. Since this is true for all δ , take the limit to get $H_p(A_1) \leq H_p(A_2)$. To show subadditivity, let $\{A_j\}_{j=1}^\infty \subset \mathbb{R}^n$. As $H_{p,\delta}$ are outermeasures,

$$H_{p,\delta}(\cup A_j) \leq \sum H_{p,\delta}(A_j) \leq \sum H_p(A_j).$$

Since true for all δ , take the limit to get $H_p(\cup A_j) \leq \sum H_p(A_j)$. □

Let (X, ρ) be a metric space. We define $\rho(A, B) = \inf\{\rho(x, y) : x \in A, y \in B\}$. If $\rho(A, B) > 0$, then $A \cap B = \emptyset$. If $\rho(A, B) = 0$, anything may happen.

Definition. Suppose μ^* is an outermeasure on X . We say that μ^* is a metric outer measure if $\mu^*(A \cup B) = \mu^*(A) + \mu^*(B)$ whenever $\rho(A, B) > 0$.

Proposition 31. If μ^* is a metric outer measure, then every Borel Set is μ^* measurable.

Proof. Recall that the Borel Sets are generated by the closed sets. Thus it suffices to show all closed sets are μ^* measurable. Suppose $F \subseteq X$ is closed. We need to show $\mu^*(A) = \mu^*(A \cap F) + \mu^*(A \setminus F)$ for all $A \subseteq X$. Recall that “ \leq ” follows from subadditivity. If $\mu^*(A) = \infty$, clear. So suppose $\mu^*(A) < \infty$. If $\rho(A \cap F, A \setminus F) > 0$, done by the definition. Define $B_n = \{x \in A \setminus F | \rho(x, F) \geq \frac{1}{n}\}$. Now $\rho(B_n, F) \geq \frac{1}{n}$. Since μ^* is a metric outermeasure, $\mu^*(A) = \mu^*((A \cap F) \cup (A \setminus F)) \geq \mu^*((A \cap F) \cup B_n) = \mu^*(A \cap F) + \mu^*(B_n)$. So we need only show $\lim_{n \rightarrow \infty} \mu^*(B_n) = \mu^*(A \setminus F)$.

Claim: Let $C_{n+1} = B_{n+1} \setminus B_n$. Then $\rho(C_{n+1}, B_n) \geq \frac{1}{n(n+1)}$.

Proof: Note that the distance between points in B_n and F is $\geq \frac{1}{n}$ and the distance between points in C_{n+1} and F is $\leq \frac{1}{n+1}$. So the distance between C_{n+1} and B_n is $\geq \frac{1}{n} - \frac{1}{n+1} = \frac{1}{n(n+1)}$.

Now $B_{2k+1} = C_{2k} \cup B_{2k} \supseteq C_{2k} \cup B_{2k-1}$. So

$$\begin{aligned} \mu^*(B_{2k+1}) &\geq \mu^*(C_{2k} \cup B_{2k-1}) \\ &= \mu^*(C_{2k}) + \mu^*(B_{2k-1}) \\ &\geq \mu^*(C_{2k}) + \mu^*(C_{2k-2} \cup B_{2k-3}) \\ &\geq \sum_{j=1}^k \mu^*(C_{2j}) + \mu^*(B_1) \\ &\geq \sum_{j=1}^k \mu^*(C_{2j}). \end{aligned}$$

Thus we have $\mu^*(B_{2k+1}) \geq \sum_{j=1}^k \mu^*(C_{2j})$ and it follows that $\mu^*(B_{2k}) \geq \sum_{j=1}^k \mu^*(C_{2j-1})$. Since $\mu^*(A) < \infty, \mu^*(B_n) < \mu^*(A) < \infty$. Thus $\sum_1^\infty \mu^*(C_{2j}) < \infty$ and $\sum_{j=1}^k \mu^*(C_{2j-1}) < \infty$. Since these sums both converge absolutely, $\sum_{j=n}^\infty \mu^*(C_j) \rightarrow 0$. By subadditivity of μ^* , $\mu^*(A \setminus F) = \mu^*(B_n \cup (\cup_{j=n}^\infty C_j)) \leq \mu^*(B_n) + \sum_{j=n}^\infty \mu^*(C_j)$. Taking this limit as $n \rightarrow \infty$, we see

$$\mu^*(A \setminus F) \leq \liminf \mu^*(B_n) \leq \limsup \mu^*(B_n) \leq \mu^*(A \setminus F).$$

Thus $\mu^*(A \setminus F) = \lim \mu^*(B_n)$. □

Note. For the above proof, we assumed F was closed in order to deduce that $\cup_1^\infty B_n = A \setminus F$ as if $x \in A \setminus F$, then as F is closed, we know $\rho(x, F) \geq \epsilon > 0$ which implies $x \in B_n$ for $n \geq \frac{1}{\epsilon}$.

Proposition 32. H_p is a metric outer measure on \mathbb{R}^n .

Proof. By Proposition 30, we know that H_p is an outermeasure. Thus we need only show $H_p(A \cup B) = H_p(A) + H_p(B)$ whenever $\rho(A, B) > 0$. Let $A, B \subseteq \mathbb{R}^n$ with $\rho(A, B) > 0$. Since H_p is an outermeasure, we already have “ \leq ”. To show “ \geq ”, we select $\delta \in (0, \rho(A, B))$ and $\{E_j\}_{j=1}^\infty \subset \mathcal{E}_\delta$ such that $A \cup B \subseteq \cup E_j$ and $H_{p,\delta}(A \cup B) \geq \sum_{j=1}^\infty (\text{diam}(E_j))^p - \epsilon$ for a given ϵ .

Since $\delta < \rho(A, B)$, no E_j can intersect both A and B . So we split our covering for $A \cup B$ into 2 families: $\{C_j\}_{j=1}^{\infty}$ (which are the E_j such that $E_j \cap A \neq \emptyset$) and $\{D_j\}_{j=1}^{\infty}$ (which are all the other sets). Then $\{C_j\}$ is a cover for A and $\{D_j\}$ is a cover for B . Now,

$$H_{p,\delta}(A) + H_{p,\delta}(B) \leq \sum_{j=1}^{\infty} (\text{diam}(C_j))^p + \sum_{j=1}^{\infty} (\text{diam}(D_j))^p = \sum_{j=1}^{\infty} (\text{diam}(E_j))^p \leq H_{p,\delta}(A \cup B) + \epsilon.$$

Now, since ϵ is arbitrary, we have $H_{p,\delta}(A) + H_{p,\delta}(B) \leq H_{p,\delta}(A \cup B) \leq H_p(A \cup B)$. Again, this is true for all δ , so letting $\delta \rightarrow 0$, we see $H_p(A) + H_p(B) \leq H_p(A \cup B)$. \square

Corollary 8. *All the Borel Sets of \mathbb{R}^n are H_p -measurable.*

Example. The Cantor Set. Define it as $S_0 = C_0, S_1 = C_{1,1} \cup C_{1,2}, S_2 = C_{2,1} \cup C_{2,2} \cup C_{2,3} \cup C_{2,4}$ and in general $S_k = \cup_1^{2^k} C_{k,j}$ where $\text{diam}(C_{j,k}) = (\frac{1}{3})^k$. Since $C = \cap S_k$, each S_k covers C . It follows that $C \subseteq \cup_1^{2^j} C_{j,k}$. So if $\delta = (\frac{1}{3})^k$, we see that

$$H_p(C) = \lim_{\delta \rightarrow 0^+} H_{p,\delta}(C) \leq \lim_{k \rightarrow 0^+} \left(\frac{2}{3^p}\right)^k = \begin{cases} 0 & \text{if } p > \frac{\ln 2}{\ln 3} \\ 1 & \text{if } p = \frac{\ln 2}{\ln 3} \\ \infty & \text{if } p < \frac{\ln 2}{\ln 3} \end{cases}$$

Interestingly, it can be show the inequality is actually equality.

Proposition 33 (p 351). *If $H_p(A) < \infty$, then $H_q(A) = 0$ for all $q > p$. If $H_p(A) > 0$, then $H_q(A) = \infty$ for all $q < p$. It follows that*

$$\inf\{p | H_p(A) = 0\} = \sup\{p \geq 0 | H_p(A) = \infty\}.$$

The **Hausdorff Dimension** of A is the above number.

2.2 Product Measures

Goal: Given measure spaces (X, \mathcal{M}, μ) and (Y, \mathcal{N}, ν) , we want to define a measure on $X \times Y$ (the cartesian product) such that the measure of $E \times F$, with $E \in \mathcal{M}$ and $F \in \mathcal{N}$ is $\mu(E)\nu(F)$.

Definition. Let $\{(X_\alpha, \mathcal{M}_\alpha)\}_{\alpha \in A}$ be measurable spaces. The **product σ -algebra** on $X = \prod_{\alpha \in A} X_\alpha$ is the σ -algebra generated by $\{\pi_\alpha^{-1}(E_\alpha) : E_\alpha \in \mathcal{M}_\alpha, \alpha \in A\}$ where $\pi_\alpha : X \rightarrow X_\alpha$ is the α^{th} coordinate map.

Notation. The product σ -algebra is denoted by $\otimes_{\alpha \in A} \mathcal{M}_\alpha$.

Proposition 34. *If A is countable, then $\otimes_{\alpha \in A} \mathcal{M}_\alpha$ is the σ -algebra generated by $\{\prod_{\alpha \in A} E_\alpha : E_\alpha \in \mathcal{M}_\alpha\}$.*

Proof. See page 22-23. \square

Proposition 35. *Let X_1, \dots, X_n be metric spaces and let $X = \prod_1^n X_j$ be equipped with the product measure (p 13). Then $\otimes_1^n \mathcal{B}_{X_j} \subseteq \mathcal{B}_X$. If each X_j is separable, then $\otimes_1^n \mathcal{B}_{X_j} = \mathcal{B}_X$.*

Corollary 9. $\mathcal{B}_{\mathbb{R}^n} = \otimes_1^n \mathcal{B}_{\mathbb{R}}$.

Definition. If $E \in \mathcal{M}$ and $F \in \mathcal{N}$, we call $E \times F$ a **measurable rectangle**. We denote the set of all measurable rectangles by \mathcal{R} .

By HW3 #3, since \mathcal{M} and \mathcal{N} and semialgebras, we see $\mathcal{R} = \mathcal{M} \times \mathcal{N}$ is a semialgebra. So we can use the Carathéodory construction to extend it to an algebra:

Theorem 26. *Let $\Pi : \mathcal{R} \rightarrow [0, \infty]$ be defined by $\Pi(A \times B) = \mu(A)\nu(B)$. Then Π is well-defined, countably additive, and $\Pi(\emptyset) = 0$.*

Proof. Clearly $\Pi(\emptyset) = 0$ and Π is well-defined. To show countably additive, suppose we have $\{A_n\}_1^\infty \subseteq \mathcal{M}$ and $\{B_n\}_1^\infty \subseteq \mathcal{N}$ which satisfy

- $\cup_1^\infty (A_n \times B_n) = A \times B$ with $A \in \mathcal{M}$ and $B \in \mathcal{N}$.
- $(A_m \times B_m) \cap (A_n \times B_n) = \emptyset$ if $n \neq m$.

Then we need to show $\Pi(A \times B) = \sum \Pi(A_n \times B_n)$. Note that

$$\chi_A(x)\chi_B(y) = \chi_{A \times B}((x, y)) = \sum_1^\infty \chi_{A_j \times B_j}((x, y)) = \sum_1^\infty \chi_{A_j}(x)\chi_{B_j}(y).$$

By Theorem 7, we see

$$\begin{aligned} \mu(A)\chi_B(y) &= \int_X \chi_A(x)\chi_B(y)d\mu(x) \\ &= \int_X \sum_1^\infty \chi_{A_j}(x)\chi_{B_j}(y)d\mu(x) \\ &= \sum_1^\infty \int_X \chi_{A_j}(x)\chi_{B_j}(y)d\mu(x) \\ &= \sum_1^\infty \mu(A_j)\chi_{B_j}(y). \end{aligned}$$

Now, using Theorem 7 again, we have

$$\begin{aligned} \mu(A)\nu(B) &= \int_Y \mu(A)\chi_B(y)d\nu(y) \\ &= \int_Y \sum_1^\infty \mu(A_j)\chi_{B_j}(y)d\nu(y) \\ &= \sum_1^\infty \int_Y \mu(A_j)\chi_{B_j}(y)d\nu(y) \\ &= \sum_1^\infty \mu(A_j)\nu(B_j). \end{aligned}$$

Thus, by our definition of Π , we see Π is countably additive. □

Theorem 27. *With Π defined as in Theorem 26, there exists a unique extension of Π to a premeasure $\tilde{\Pi}$ on $\mathcal{F}(\mathcal{R})$, the algebra generated by \mathcal{R} .*

Proof. Done, by Theorems 19 and 18. □

Theorem 28. *The premeasure $\tilde{\Pi}$ generates an outer measure Π^* on $X \times Y$ whose restriction to $\mathcal{M} \otimes \mathcal{N}$ is a measure extending Π . Moreover, if μ and ν are σ -finite, then so is $\Pi^*|_{\mathcal{M} \otimes \mathcal{N}}$ and Π^* is unique.*

Proof. Done, by Proposition 27, Carathéodory's Theorem, and Theorem 20. □

Notation. We denote $\Pi^*|_{\mathcal{M} \otimes \mathcal{N}}$ by $\mu \times \nu$.

Note that by iterative applications of the above we can define a product measure for any finite number of measure spaces. In that case, we denote the product measure on $\mathcal{M}_1 \otimes \cdots \otimes \mathcal{M}_n$ by $\prod_1^n \mu_j$.

How do we find $\mu \times \nu(E)$ for $E \in \mathcal{M} \otimes \mathcal{N}$?

Simple Case: Let $E \subseteq \mathbb{R}^2$ with $E = \{(x, y) \in \mathbb{R}^2 | a \leq x \leq b, f(x) \leq y \leq g(x)\}$. Then the measure of E is $\int_a^b g - f(x)dx$.
Now $g - f(x) = \text{meas}(E_x)$ where $E_x = \{y \in \mathbb{R} | (x, y) \in E\}$. So we see

$$\text{meas}(E) = \int_{\mathbb{R}} m(E_x)dm.$$

Questions

1. Given $E \in \mathcal{M} \otimes \mathcal{N}$, $x \in X$, is $E_x \in \mathcal{N}$?
2. Is the function $x \mapsto \nu(E_x)$ μ -measurable?
3. Is $\mu \times \nu(E) = \int_X \nu(E_x)d\mu$?
4. Can we interchange μ and ν ?

Definition. If $E \subseteq X \times Y$, then for all $x \in X$ define the x -**section** $E_x = \{y \in Y \mid (x, y) \in E\}$ and for all $y \in Y$ define the y -**section** $E^y = \{x \in X \mid (x, y) \in E\}$. If $f : X \times Y \rightarrow \overline{\mathbb{R}}$, we define the x -**section** f_x and the y -**section** f^y as $f_x(y) = f^y(x) = f(x, y)$.

Example. If $E = A \times B$ with $A \in \mathcal{M}$ and $B \in \mathcal{N}$ then $E_x = \emptyset$ if $x \notin A$ and $E_x = B$ if $x \in A$.

Proposition 36 (p 65). 1. If $E \in \mathcal{M} \otimes \mathcal{N}$, then $E_x \in \mathcal{N}$ for all $x \in X$ and $E^y \in \mathcal{M}$ for all $y \in Y$.

2. If f is a $(\mathcal{M} \otimes \mathcal{N})$ -measurable function, then f_x is \mathcal{N} -measurable and f^y is \mathcal{M} -measurable.

Proof. 1. Let \mathcal{O} be the collection of all $E \subseteq X \times Y$ satisfying $E_x \in \mathcal{N}$ for all $x \in X$ and $E^y \in \mathcal{M}$ for all $y \in Y$. We show \mathcal{O} is a σ -algebra:

- Clearly, $\emptyset, X \times Y \in \mathcal{O}$.
- Let $E \in \mathcal{O}$. Then $E_x \in \mathcal{N}$ which implies $(E^C)_x = E_x^C \in \mathcal{N}$ for all $x \in X$ as \mathcal{N} is a σ -algebra and $E^y \in \mathcal{M}$ which implies $(E^C)^y = (E^y)^C \in \mathcal{M}$ for all $y \in Y$ as \mathcal{M} is a σ -algebra. Thus $E^C \in \mathcal{O}$.
- Suppose $\{E_j\}_{j=1}^\infty \subseteq \mathcal{O}$. Then $(\cup E_j)_x = \cup (E_j)_x \in \mathcal{N}$ for all $x \in X$ and similarly $(\cup E_j)^y \in \mathcal{M}$ for all $y \in Y$. This $\cup E_j \in \mathcal{O}$.

Now we observe that all measurable rectangles are in \mathcal{O} and since $\mathcal{M} \otimes \mathcal{N}$ is defined to be the smallest σ -algebra which contains the measurable rectangles, we see $\mathcal{M} \otimes \mathcal{N} \subseteq \mathcal{O}$.

2. Suppose $E \subseteq \mathcal{B}_{\overline{\mathbb{R}}}$. Then $f^{-1}(E) \in \mathcal{M} \otimes \mathcal{N}$. By part 1, $(f^{-1}(E))_x \in \mathcal{N}$ and $(f^{-1}(E))^y \in \mathcal{M}$. Notice $(f^{-1}(E))_x = \{y \in Y : f(x, y) \in E\} = \{y \in Y : f_x(y) \in E\} = f_x^{-1}(E)$. Thus $f_x^{-1}(E) \in \mathcal{N}$ which implies f_x is measurable. Similarly, f^y is measurable. \square

Definition. A subset $\mathcal{C} \subseteq \mathcal{P}(X)$ is a **monotone class** if it possesses the following properties:

- If $\{E_j\}_{j=1}^\infty \subseteq \mathcal{C}$ and $E_1 \subseteq E_2 \subseteq \dots$, then $\cup_1^\infty E_j \in \mathcal{C}$.
- If $\{E_j\}_{j=1}^\infty \subseteq \mathcal{C}$ and $E_1 \supseteq E_2 \supseteq \dots$, then $\cap_1^\infty E_j \in \mathcal{C}$.

Note. A σ -algebra is a monotone class.

Given a subset $\mathcal{E} \subseteq \mathcal{P}(X)$, there exists a smallest monotone class $\mathcal{C}(\mathcal{E})$ containing \mathcal{E} . We say $\mathcal{C}(\mathcal{E})$ is the **monotone class generated by \mathcal{E}** .

Theorem 29. $\mathcal{M} \subseteq \mathcal{P}(X)$ is a σ -algebra if and only if \mathcal{M} is a monotone class and an algebra.

Proof. (\Rightarrow): Clear

(\Leftarrow): Suppose \mathcal{M} is an algebra and a monotone class. Then

1. $\emptyset, X \in \mathcal{M}$ as \mathcal{M} is an algebra.
2. \mathcal{M} is closed under complements as \mathcal{M} is an algebra.
3. As \mathcal{M} is an algebra, it is closed under finite unions. Let $\{E_j\}_{j=1}^\infty \subseteq \mathcal{M}$. Define $A_k := \cup_{j=1}^k E_j$. Then $A_k \in \mathcal{M}$ for all k and $A_1 \subseteq A_2 \subseteq \dots$. Since \mathcal{M} is a monotone class, we see $\cup_{j=1}^\infty E_j = \cup_{k=1}^\infty A_k \in \mathcal{M}$. \square

Lemma (Monotone Class Lemma (p66)). If $\mathcal{A} \subseteq \mathcal{P}(X)$ is an algebra, then the monotone class $\mathcal{C}(\mathcal{A})$ generated by \mathcal{A} and the σ -algebra $\mathcal{M}(\mathcal{A})$ generated by \mathcal{A} are equal.

Proof. Since $\mathcal{M}(\mathcal{A})$ is a monotone class, we see $\mathcal{C}(\mathcal{A}) \subseteq \mathcal{M}(\mathcal{A})$. So it is enough to show $\mathcal{C}(\mathcal{A})$ is a σ -algebra. By Theorem 29, it is enough to show $\mathcal{C}(\mathcal{A})$ is an algebra.

1. Since \mathcal{A} is an algebra, $\emptyset, X \in \mathcal{A} \subseteq \mathcal{C}(\mathcal{A})$.

2. Define $\mathcal{E} := \{E \subseteq X \mid E^C \in \mathcal{C}(\mathcal{A})\}$. We show \mathcal{E} is a monotone class.

- If $\{E_j\}_{j=1}^\infty \subseteq \mathcal{E}$ such that $E_1 \subseteq E_2 \subseteq \dots$, then $\{E_j^C\}_{j=1}^\infty \in \mathcal{C}(\mathcal{A})$ and $E_1^C \supseteq E_2^C \supseteq \dots$ which implies $(\cup E_j)^C = \cap E_j^C \in \mathcal{C}(\mathcal{A})$ which implies $\cup E_j \in \mathcal{E}$.
- Similar

So $\mathcal{C}(\mathcal{A}) \subseteq \mathcal{E}$ which implies $\mathcal{C}(\mathcal{A})$ is closed under complements.

3. We want to show $\mathcal{C}(\mathcal{A})$ is closed under finite unions. Define $\mathcal{E}(F) := \{E \subseteq X \mid E \cup F \in \mathcal{C}(\mathcal{A})\}$ for all $F \in \mathcal{C}(\mathcal{A})$. Now, suppose $F \in \mathcal{A}$. Then $\mathcal{A} \subseteq \mathcal{E}(F)$ as \mathcal{A} is an algebra. Continuing under the assumption that $F \in \mathcal{A}$, we want to show $\mathcal{E}(F)$ is a monotone class (as then $\mathcal{C}(\mathcal{A}) \subseteq \mathcal{E}(F)$). So

- Let $\{A_j\}_{j=1}^\infty \subseteq \mathcal{E}(F)$ with $A_1 \subseteq A_2 \subseteq \dots$. Then $\{A_j \cup F\}_{j=1}^\infty \subseteq \mathcal{C}(\mathcal{A})$ and of course $A_1 \cup F \subseteq A_2 \cup F \subseteq \dots$ which implies $(\cup A_j) \cup F = \cup(A_j \cup F) \in \mathcal{C}(\mathcal{A})$. Thus $\cup A_j \in \mathcal{E}(F)$.
- Similar

Now, suppose $E \in \mathcal{C}(\mathcal{A})$. Then $E \in \mathcal{E}(F)$ if and only if $E \cup F \in \mathcal{C}(\mathcal{A})$ if and only if $F \in \mathcal{E}(E)$. Thus for all $E \in \mathcal{C}(\mathcal{A})$, we have $\mathcal{A} \in \mathcal{E}(E)$. By the above, $\mathcal{E}(E)$ is a monotone class. So $\mathcal{C}(\mathcal{A}) \subseteq \mathcal{E}(E)$ for all $E \in \mathcal{C}(\mathcal{A})$. Thus $\mathcal{C}(\mathcal{A})$ is closed under finite unions.

Thus $\mathcal{C}(\mathcal{A})$ is an algebra, which implies it is a σ -algebra and thus $\mathcal{C}(\mathcal{A}) = \mathcal{M}(\mathcal{A})$. □

Theorem 30 (p 66). *Suppose (X, \mathcal{M}, μ) and (Y, \mathcal{N}, ν) are σ -finite measure spaces. Let $E \in \mathcal{M} \otimes \mathcal{N}$. Then*

1. $x \mapsto \nu(E_x)$ and $y \mapsto \mu(E^y)$ are measurable in X and Y , respectively.
2. $\mu \times \nu(E) = \int_X \nu(E_x) d\mu(x) = \int_Y \mu(E^y) d\nu(y)$.

Proof. Let $\mathcal{E} := \{E \in \mathcal{M} \otimes \mathcal{N} : (1), (2) \text{ hold}\}$. We want to show $\mathcal{E} = \mathcal{M} \otimes \mathcal{N}$, that is $\mathcal{E} \supseteq \mathcal{M} \otimes \mathcal{N}$. For this, we show \mathcal{E} is a monotone class that contains the algebra $\mathcal{F}(\mathcal{R})$. Since $\mathcal{M} \otimes \mathcal{N}$ is the σ -algebra generated by \mathcal{R} , it is also generated by $\mathcal{F}(\mathcal{R})$. So it is the monotone class generated by $\mathcal{F}(\mathcal{R})$ by the Monotone Class lemma. Thus, if we show $\mathcal{F}(\mathcal{R}) \subseteq \mathcal{E}$ and \mathcal{E} is a monotone class, then $\mathcal{M} \otimes \mathcal{N} \subseteq \mathcal{E}$.

First, we assume μ, ν are finite measures. Now, if $A \times B \in \mathcal{R}$, then $x \mapsto \nu((A \times B)_x) = \nu(B)\chi_A(x)$ and $y \mapsto \mu((A \times B)^y) = \mu(A)\chi_B(y)$, which are measurable. Thus property (1) holds for \mathcal{R} . For property (2),

$$\int_X \nu((A \times B)_x) d\mu(x) = \int_X \nu(B)\chi_A(x) d\mu(x) = \nu(B)\mu(A) = \mu \times \nu(A \times B).$$

and similarly $\int_Y \mu((A \times B)^y) d\nu(y) = \mu \times \nu(A \times B)$. Thus $\mathcal{R} \in \mathcal{E}$. Since \mathcal{R} is a semialgebra, it is enough to show \mathcal{E} is closed under finite disjoint unions (by Proposition 23) to show $\mathcal{F}(\mathcal{R}) \subseteq \mathcal{E}$. Let $E_1, E_2 \in \mathcal{E}$ with $E_1 \cap E_2 = \emptyset$. Observe $(E_1 \cup E_2)_x = (E_1)_x \cup (E_2)_x$ with $(E_1)_x \cap (E_2)_x = \emptyset$. Similarly for the y -sections. Thus

1. $x \mapsto \nu((E_1 \cup E_2)_x) = \nu((E_1)_x \cup (E_2)_x) = \nu((E_1)_x) + \nu((E_2)_x)$ and similarly $y \mapsto \mu((E_1 \cup E_2)^y) = \mu((E_1)^y) + \mu((E_2)^y)$, which are measurable.
2. Since $\mu \times \nu$ is a measure, we see $\mu \times \nu(E_1 \cup E_2) = \mu \times \nu(E_1) + \mu \times \nu(E_2) = \int \nu((E_1)_x) d\mu + \int \nu((E_2)_x) d\mu = \int \nu((E_1)_x) + \nu((E_2)_x) d\mu = \int \nu((E_1 \cup E_2)_x) d\mu$ and similarly $\mu \times \nu(E_1 \cup E_2) = \int \mu((E_1 \cup E_2)^y) d\nu$.

Thus \mathcal{E} is closed under disjoint unions of two sets and (by induction) thus is closed under finite disjoint unions. Thus $\mathcal{F}(\mathcal{R}) \subseteq \mathcal{E}$. Now, we need to show \mathcal{E} is a monotone class.

- Suppose $\{E_n\}_{n=1}^\infty \subseteq \mathcal{E}$ with $E_1 \subseteq E_2 \subseteq \dots$. Now, $(E_1)_x \subseteq (E_2)_x \subseteq \dots$ and $(E_1)^y \subseteq (E_2)^y \subseteq \dots$. So $\{x \mapsto \nu((E_n)_x)\}_n$ and $\{y \mapsto \mu((E_n)^y)\}_n$ are increasing sequences of measurable functions. By Theorem 5,

$$\lim_{n \rightarrow \infty} \nu((E_n)_x) = \nu\left(\bigcup_{n=1}^{\infty} (E_n)_x\right) = \nu\left(\left(\bigcup_{n=1}^{\infty} E_n\right)_x\right) \text{ and } \lim_{n \rightarrow \infty} \mu((E_n)^y) = \mu\left(\left(\bigcup_{n=1}^{\infty} E_n\right)^y\right).$$

So $x \mapsto \nu((\cup E_n)_x)$ and $y \mapsto \mu((\cup E_n)^y)$ are measurable functions (as the supremum of measurable functions is measurable). For property 2, by the MCT and Theorem 5,

$$\int_y \mu((\cup_{n=1}^{\infty} E_n)^y) d\nu(y) = \lim_{n \rightarrow \infty} \int \mu((E_n)^y) d\nu = \lim_{n \rightarrow \infty} \mu \times \nu(E_n) = \mu \times \nu(\cup_{n=1}^{\infty} E_n)$$

and similarly for the x -sections. Thus $\cup_{n=1}^{\infty} E_n \subseteq \mathcal{E}$.

- Similar

Thus, if μ and ν are finite, we see \mathcal{E} is a monotone class containing $\mathcal{F}(\mathcal{R})$ which implies $\mathcal{M} \otimes \mathcal{N} \subseteq \mathcal{E}$. If μ and ν are σ -finite, then X and Y are the unions of finite increasing sets, in which case we can use the above with the MCT to get the limit. \square

Theorem (Fubini-Tonelli Theorem p.67). *Suppose (X, \mathcal{M}, μ) and (Y, \mathcal{N}, ν) are σ -finite measure spaces. Then*

1. (Tonelli) *If $f \in L^+(X \times Y)$, then the functions $g(x) = \int_Y f_x d\nu$ and $h(y) = \int_X f^y d\mu$ are in $L^+(X), L^+(Y)$, respectively and $(*) \int_{X \times Y} f(x, y) d(\mu \times \nu) = \int_X [\int_Y f(x, y) d\nu] d\mu = \int_Y [\int_X f(x, y) d\mu] d\nu$.*
2. (Fubini) *If $f \in L^1(\mu \times \nu)$, then $g \in L^1(X)$ and $h \in L^1(Y)$ and $(*)$ holds almost everywhere.*

Proof. 1. If $f = \chi_E$ for $E \in \mathcal{M} \otimes \mathcal{N}$, then $f_x = \chi_{E_x}$. So $g(x) = \int_Y f_x d\nu = \int_Y \chi_{E_x} d\nu = \nu(E_x)$, which is measurable. Thus $g \in L^+$ by Theorem 30. Similarly, $h(y) = \mu(E^y)$ and $h \in L^+$. Also by Theorem 30,

$$\int f d(\mu \times \nu) = \int \chi_E d(\mu \times \nu) = \mu \times \nu(E) = \int_X \nu(E_x) d\mu = \int_X \int_Y f_x(y) d\nu d\mu$$

and similarly $\int f d(\mu \times \nu) = \int_Y \int_X f^y(x) d\mu d\nu$. Since $f_x(y) = f^y(x) = f(x, y)$, the theorem holds for all characteristic functions of measurable sets. If f is a simple function in L^+ , then it is a finite linear combination of characteristic functions of measurable sets. Thus, the theorem holds for all simple functions.

If $f \in L^+$, not necessarily simple, then we may select $\{\phi_n\}_{n=1}^{\infty} \subseteq L^+$ such that ϕ_n is simple, $0 \leq \phi_1 \leq \phi_2 \leq \dots \leq f$, and $\phi_n \rightarrow f$ pointwise everywhere. Clearly, $(\phi_n)_x \rightarrow f_x$, $(\phi_n)^y \rightarrow f^y$, and $(\phi_n)_x, (\phi_n)^y$ are increasing sequences. Define $g_n(x) = \int_Y (\phi_n)_x d\nu$ and $h_n(x) = \int_X (\phi_n)^y d\mu$. By the MCT,

$$\lim_{n \rightarrow \infty} g_n(x) = \int_Y \lim_{n \rightarrow \infty} (\phi_n)_x d\nu = \int_Y (f)_x d\nu = g(x)$$

and similarly $\lim_{n \rightarrow \infty} h_n(x) = h(x)$. We also see that $0 \leq g_1 \leq g_2 \leq \dots \leq g$ and $0 \leq h_1 \leq h_2 \leq \dots \leq h$. Thus again by the MCT

$$\int_X \int_Y f(x, y) d\nu d\mu = \int_X g d\mu = \lim_{n \rightarrow \infty} \int_X g_n d\mu = \lim_{n \rightarrow \infty} \int_X \int_Y (\phi_n)_x d\nu d\mu = \lim_{n \rightarrow \infty} \int_{X \times Y} \phi_n d(\mu \times \nu) = \int_{X \times Y} f d(\mu \times \nu).$$

Similarly $\int_Y \int_X f(x, y) d\mu d\nu = \int_{X \times Y} f d(\mu \times \nu)$.

2. Follows from applying part (a) to f^+ and f^- separately (as if $f \in L^1$, then $\int f^+, \int f^-$ are finite). \square

Note. A common way to use this theorem is to use part a in order to use part b. That is, if f is measurable, then $|f| \in L^+$. Then we have $\int_{X \times Y} |f(x, y)| d(\mu \times \nu) = \int_X \int_Y |f(x, y)| d\nu d\mu = \int_Y \int_X |f(x, y)| d\mu d\nu$ and we can show that one of those integrals is finite to conclude that $f \in L^1$. Then, we can use part b.

Definition. *The n -dimensional Lebesgue measure m^n is the completion of $(\mathbb{R}^n, \mathcal{L} \otimes \dots \otimes \mathcal{L}, m \times \dots \times m)$. The domain of m^n is \mathcal{L}^n , the class of n -dimensional Lebesgue measurable sets.*

Remarks.

1. Often, the superscript n is dropped. For example, just write $(\mathbb{R}, \mathcal{L}, m)$ for $(\mathbb{R}^n, \mathcal{L}^n, m^n)$. Integrals with respect to the Lebesgue measure are usually written as $\int_{\mathbb{R}^n} f dx$ instead of $\int_{\mathbb{R}^n} f dm$.

2. By Theorem 8 (1.9 in Folland), if $\mathcal{N} = \{N \in \mathcal{B}_{\mathbb{R}^n} : m(N) = 0\}$, then $\mathcal{L}^n = \{E \cup F : E \in \mathcal{B}_{\mathbb{R}^n}, F \subseteq N \text{ for some } N \in \mathcal{N}\}$.

3. If $\{E_j\}_{j=1}^n \subseteq \mathcal{L} \subseteq \mathcal{P}(\mathbb{R})$, then $m^n(\prod_{j=1}^n E_j) = \prod_{j=1}^n m(E_j)$.

If $E = \prod_{j=1}^n E_j$, then we will refer to each E_j as a **side/edge** of E . Recall that $E \triangle F = (E \setminus F) \cup (F \setminus E)$. Let $\mathcal{R} = \{\prod_{k=1}^n E_k : \{E_k\}_{k=1}^n \subseteq \mathcal{L} \subseteq \mathcal{P}(\mathbb{R})\}$.

Theorem 31 (2.41 in Folland). *Suppose $E \in \mathcal{L}^n$. Then*

1. $m(E) = \inf\{m(U) : E \subseteq U \text{ and } U \text{ is open}\}$.
2. $m(E) = \sup\{m(K) : K \subseteq E \text{ and } K \text{ is compact}\}$.
3. $E = V \setminus N_1$, where V is a G_δ set and $m(N_1) = 0$, where $G_\delta = \{\cap_{j=1}^\infty U_j : U_j \text{ is open}\}$.
4. $E = H \cup N_2$, where H is an F_σ set and $m(N_2) = 0$, where $F_\sigma = \{\cup_{j=1}^\infty D_j : D_j \text{ is closed}\}$.

Note that this is just the n -dimensional version of Theorem 21 (1.18 in Folland) and Theorem 24 (1.19 in Folland).

Proof. 1. Recall that m is the restriction to \mathcal{L}^n of the outer measure m^* which is induced by $\prod_{k=1}^n A_k \mapsto \prod_{k=1}^n m(A_k)$. Thus for a given $E \in \mathcal{L}^n$, we have $m(E) = m^*(E) = \inf\{\sum_{j=1}^\infty m(E_j) : \{E_j\}_{j=1}^\infty \subseteq \mathcal{R}, E \subseteq \cup_{j=1}^\infty E_j\}$. Let $\epsilon \in (0, 1)$. Then there exists $\{T_j\}_{j=1}^\infty \subseteq \mathcal{R}$ such that $E \subseteq \cup_{j=1}^\infty T_j$, and $\sum_{j=1}^\infty m(T_j) \leq m(E) + \frac{1}{2}\epsilon$. Set $\mathcal{Q}_0 = \{\prod_{k=1}^n [a_k, a_k + 1) : a_k \in \mathbb{Z}\}$. Notice that this is a countable collection of mutually disjoint sets such that $\cup_{Q \in \mathcal{Q}_0} Q = \mathbb{R}^n$. Let $\{Q_r\}_{r=1}^\infty$ be an enumeration of \mathcal{Q}_0 . Let $j \in \mathcal{N}$ be given. We have that $T_j = \cup_{r=1}^\infty Q_r \cap T_j = \cup_{r=1}^\infty Q_r \cap \prod_{k=1}^n E_{j,k}$ where $\{E_{j,k}\}_{k=1}^n \subseteq \mathcal{L} \subseteq \mathcal{P}(\mathbb{R})$ (that is, they are the one dimensional edges of T_j). Let $r \in \mathbb{N}$ be given. Since $Q_r = \prod_{k=1}^n [a_k, a_k + 1)$ for some $\{a_k\}_{k=1}^n \subseteq \mathbb{Z}$, we conclude $Q_r \cap T_j = \prod_{k=1}^n [a_k, a_k + 1) \cap E_{j,k}$. Now $[a_k, a_k + 1) \cap E_{j,k} \in \mathcal{L} \subseteq \mathcal{P}(\mathbb{R})$, so by Theorem 21, for all $k = 1, \dots, n$, there exists $F_{r,j,k} \subseteq \mathbb{R}$ that is open and satisfies $F_{r,j,k} \supseteq [a_k, a_k + 1) \cap E_{j,k}$ and $m(F_{r,j,k}) \leq m([a_k, a_k + 1) \cap E_{j,k}) + \frac{1}{2^n n} \epsilon 2^{-r-j}$. It follows that $Q_r \cap T_j \subseteq \prod_{k=1}^n F_{r,j,k}$, which is open and

$$\begin{aligned}
m\left(\prod_{k=1}^n F_{r,j,k}\right) &= \prod_{k=1}^n m(F_{r,j,k}) \leq \prod_{k=1}^n \left(m([a_k, a_k + 1) \cap E_{j,k}) + \frac{1}{2^n n} \epsilon 2^{-r-j}\right) \\
&= \prod_{k=1}^n \left(m([a_k, a_k + 1) \cap E_{j,k}) + \frac{1}{2^n n} \epsilon 2^{-r-j}\right) \prod_{k=2}^n \left(m([a_k, a_k + 1) \cap E_{j,k}) + \frac{1}{2^n n} \epsilon 2^{-r-j}\right) \\
&= \prod_{k=1}^n \{m([a_k, a_k + 1) \cap E_{j,k})\} \prod_{k=2}^n \left\{m([a_k, a_k + 1) \cap E_{j,k}) + \frac{1}{2^n n} \epsilon 2^{-r-j}\right\} \\
&\quad + \frac{1}{2^n n} \epsilon 2^{-r-j} \prod_{k=2}^n \left\{m([a_k, a_k + 1) \cap E_{j,k}) + \frac{1}{2^n n} \epsilon 2^{-r-j}\right\} \\
&\leq \prod_{k=1}^n \{m([a_k, a_k + 1) \cap E_{j,k})\} \prod_{k=2}^n \left\{m([a_k, a_k + 1) \cap E_{j,k}) + \frac{1}{2^n n} \epsilon 2^{-r-j}\right\} \\
&\quad + \frac{1}{2^n n} \epsilon 2^{-r-j} \left(1 + \frac{1}{2^n n} \epsilon 2^{-r-j}\right)^{n-1} \\
&\leq \prod_{k=1}^n \{m([a_k, a_k + 1) \cap E_{j,k})\} \prod_{k=3}^n \left\{m([a_k, a_k + 1) \cap E_{j,k}) + \frac{1}{2^n n} \epsilon 2^{-r-j}\right\} \\
&\quad + \frac{1}{2^n n} \epsilon 2^{-r-j} m([a_1, a_1 + 1) \cap E_{j,1}) \prod_{k=3}^n \left\{m([a_k, a_k + 1) \cap E_{j,k}) + \frac{1}{2^n n} \epsilon 2^{-r-j}\right\} \\
&\quad + \frac{1}{2^n n} \epsilon 2^{-r-j} \left(1 + \frac{1}{2^n n} \epsilon 2^{-r-j}\right)^{n-1}
\end{aligned}$$

$$\begin{aligned}
&\leq \prod_{k=1}^2 \{m([a_k, a_k + 1] \cap E_{j,k})\} \prod_{k=3}^n \left\{ m([a_k, a_k + 1] \cap E_{j,k}) + \frac{1}{2^n n} \epsilon 2^{-r-j} \right\} \\
&\quad + 2 \frac{1}{2^n n} \epsilon 2^{-r-j} \left(1 + \frac{1}{2^n n} \epsilon 2^{-r-j} \right)^{n-1} \\
&\leq \prod_{k=1}^n m([a_k, a_k + 1] \cap E_{j,k}) + n \left(\frac{1}{2^n n} \epsilon 2^{-r-j} \right) \left(1 + \frac{1}{2^n n} \epsilon 2^{-r-j} \right)^{n-1} \\
&\leq \prod_{k=1}^n m([a_k, a_k + 1] \cap E_{j,k}) + \frac{1}{2} \epsilon 2^{-r-j} \text{ as } \left(1 + \frac{1}{2^n n} \epsilon 2^{-r-j} \right)^{n-1} \leq 2^{n-1} \\
&= m(Q_r \cap T_j) + \frac{1}{2} \epsilon 2^{-r-j}
\end{aligned}$$

Thus for all $r, j \in \mathbb{N}$, there exists an open set of the form $\prod F_{r,j,k}$ such that $Q \cap T_j \subseteq \prod F_{r,j,k}$ and $m(\prod F_{r,j,k}) \leq m(Q_r \cap T_j) + \frac{1}{2} \epsilon 2^{-r-j}$. Now, for all j set $U_j = \cup_{r=1}^{\infty} \prod_{k=1}^{\infty} F_{r,j,k}$. So $T_j \subseteq U_j$, with U_j open and

$$\begin{aligned}
m(U_j) &\leq \sum_{r=1}^{\infty} m(\prod_{k=1}^n F_{r,j,k}) \\
&\leq \sum_{r=1}^{\infty} (m(Q_r \cap T_j) + \frac{1}{2} \epsilon 2^{-r-j}) \\
&\leq m(\cup_{r=1}^{\infty} Q_r \cap T_j) + \frac{1}{2} \epsilon 2^{-j} \sum_{r=1}^{\infty} 2^{-r} \\
&= m(T_j) + \frac{1}{2} \epsilon 2^{-j}
\end{aligned}$$

Set $U = \cup_{j=1}^{\infty} U_j$, so U is open and $E \subseteq \cup_{j=1}^{\infty} T_j \subseteq U$ and $m(U) \leq \sum_{j=1}^{\infty} m(U_j) \leq \sum_{j=1}^{\infty} (m(T_j) + \frac{1}{2} \epsilon 2^{-j}) = \sum_{j=1}^{\infty} m(T_j) + \frac{1}{2} \epsilon \leq m(E) + \epsilon$. Since $\epsilon \in (0, 1)$ was arbitrary, we're done.

2. Follows exactly from Theorem 21b (1.18 in Folland)

3. Follows exactly from Theorem 24 (1.19 in Folland)

4. Follows exactly from Theorem 24. □

For each $k \in \mathbb{Z}$, define $\mathcal{Q}_k^n = \{\prod_{j=1}^n [a_j 2^{-k}, (a_j + 1) 2^{-k}] : a_j \in \mathbb{Z}\}$, the set of n dimensional dyadic cubes.

Remarks.

- For each $k \in \mathbb{Z}$, $\mathbb{R}^n = \cup_{Q \in \mathcal{Q}_k^n} Q$.
- If $Q_1 \in \mathcal{Q}_k^n$ and $Q_2 \in \mathcal{Q}_\ell^n$ with $k < \ell$, then either $Q_2 \subset Q_1$ or $Q_2 \cap Q_1 = \emptyset$.
- If $Q \in \mathcal{Q}_k^n$, then $m(Q) = 2^{-kn}$.
- If $Q \in \mathcal{Q}_\ell^n$, then there are exactly $2^{(k-\ell)n}$ elements of \mathcal{Q}_k^n contained in Q .

Lemma 3. *Let $U \subseteq \mathbb{R}^n$ be an open set, then there exists a countable collection of disjoint dyadic cubes $\{Q_r\}_{r=1}^{\infty} \subseteq \cup_{k=0}^{\infty} \mathcal{Q}_k^n$ such that $U = \cup_{r=1}^{\infty} Q_r$.*

Proof. See Rudin. □

Theorem 32 (2.40c in Folland). *Suppose $E \in \mathcal{L}^n$ and $m(E) < \infty$. Then for all $\epsilon > 0$, there exists a finite collection $\{Q_r\}_{r=1}^N$ of disjoint dyadic cubes such that $m(E \Delta \cup_{r=1}^N Q_r) < \epsilon$.*

Proof. By Theorem 31a (2.40a), there exists an open set $U \subseteq \mathbb{R}^n$ such that $m(U) < m(E) + \frac{1}{2} \epsilon$. By Lemma 3, there exists a collection $\{Q_r\}_{r=1}^{\infty}$ of disjoint dyadic cubes such that $U = \cup_{r=1}^{\infty} Q_r$. Then $\sum_{r=1}^{\infty} m(Q_r) = m(U) < m(E) + \frac{1}{2} \epsilon < \infty$. Since $\sum m(Q_r)$ is absolutely convergent, there exists an $N \in \mathbb{N}$ such that $\sum_{r=N+1}^{\infty} m(Q_r) < \frac{1}{2} \epsilon$. Thus

$$\begin{aligned}
m(E \Delta \cup_{r=1}^N Q_r) &= m((E \setminus \cup_{r=1}^N Q_r) \cup (\cup_{r=1}^N Q_r \setminus E)) \\
&= m(E \setminus \cup_{r=1}^N Q_r) + m(\cup_{r=1}^N Q_r \setminus E) \\
&\leq m(U \setminus \cup_{r=1}^N Q_r) + m(U \setminus E) \\
&= m(U) - \sum_{r=1}^N m(Q_r) + m(U) - m(E) \\
&< m(E) + \frac{1}{2} \epsilon - \sum_{r=1}^{\infty} m(Q_r) + \sum_{r=N+1}^{\infty} m(Q_r) + m(U) - m(E) = \epsilon.
\end{aligned}$$
□

Theorem 33 (2.42 in Folland). *The n -dimensional Lebesgue measure is translation invariant. To be more precise, for all $a \in \mathbb{R}^n$, define $\tau_a : \mathbb{R}^n \rightarrow \mathbb{R}^n$ by $\tau_a(x) = x + a$. Then*

1. *If $E \in \mathcal{L}^n$, then $\tau_a(E) \in \mathcal{L}^n$ and $m(\tau_a(E)) = m(E)$.*

2. *If $f : \mathbb{R}^n \rightarrow \mathbb{C}$ is Lebesgue measurable, then so is $f \circ \tau_a$. Moreover, if either $f \geq 0$ is real valued or $f \in L^1(m)$, then $\int_{\mathbb{R}^n} f \circ \tau_a dm = \int_{\mathbb{R}^n} f dm$.*

Proof. Key Observation: Suppose λ is a Borel measure on \mathbb{R}^n and there exists a constant c such that $\lambda(Q) = cm(Q)$ for all dyadic cubes. Then, by Lemma 3, $\lambda(U) = \sum_{r=1}^{\infty} \lambda(Q_r) = \sum_{r=1}^{\infty} cm(Q_r) = cm(U)$ for all open sets $U \subseteq \mathbb{R}^n$. Thus $\lambda(E) = cm(E)$ for all $E \in \mathcal{B}_{\mathbb{R}^n}$.

We will prove 1. Fix $a \in \mathbb{R}^n$ and define $\lambda : \mathcal{B}_{\mathbb{R}^n} \rightarrow [0, \infty]$ by $\lambda(E) = m(\tau_a(E))$. It is easy to verify λ is a Borel measure. Let Q be a dyadic cube. Then $\tau_a(Q)$ is still a dyadic cube and has the same volume. Thus $\lambda(Q) = m(Q)$. By the Key Observation, (*) $m(\tau_a(E)) = \lambda(E) = m(E)$ for all $E \in \mathcal{B}_{\mathbb{R}^n}$. Of course, we want to show this for a general $E \in \mathcal{L}^n$. If $N \in \mathcal{L}^n$ and $m(N) = 0$, then by Theorem 31(3) (2.40b), there exists $V \in \mathcal{B}_{\mathbb{R}^n}$ such that $N \subseteq V$ and $m(V) = 0$ by (*). Since m is complete, it follows that $\tau_a(N) \in \mathcal{L}^n$ and $m(\tau_a(N)) = 0$. In general, if $E \in \mathcal{L}^n$, then by Theorem 31(4), there is an $H \in \mathcal{B}_{\mathbb{R}^n}$ and a null set $N \in \mathcal{L}^n$ such that $E = H \cup N$, so $\tau_a(E) = \tau_a(H) \cup \tau_a(N) \in \mathcal{L}^n$. Thus the translation of a Lebesgue measurable set is still Lebesgue measurable. Furthermore,

$$\begin{aligned} m(E) &= \inf\{m(U) \mid U \text{ is open and } E \subseteq U\} \\ &= \inf\{m(\tau_a(U)) \mid U \text{ is open and } E \subseteq U\} \\ &= \inf\{m(U) \mid U \text{ is open and } \tau_a(E) \subseteq U\} \\ &= m(\tau_a(E)) \end{aligned}$$

Thus, we conclude $m(\tau_a(E)) = m(E)$ for all $E \in \mathcal{L}^n$. □

Theorem 34. *Suppose μ is a Borel measure satisfying $\mu(\tau_a(E)) = \mu(E)$ for all $E \in \mathcal{B}_{\mathbb{R}^n}$ and $a \in \mathbb{R}^n$. Suppose also $\mu(Q_0) < \infty$ for some unit dyadic cube. Then $\mu(E) = \mu(Q_0)m(E)$ for all $E \in \mathcal{B}_{\mathbb{R}^n}$.*

Proof. Since $\mu(\tau_a(Q)) = \mu(Q)$ for all $a \in \mathbb{R}^n$ and dyadic cubes Q , we may assume that $Q_0 \in \mathcal{Q}_0^n$. Let $Q \in \mathcal{Q}_k^n$ for some $k \in \mathbb{N}$. Now $Q_0 = \cup_{r=1}^{2^{nk}} Q_r$ for some family $\{Q_r\}_{r=1}^{2^{nk}} \subseteq \mathcal{Q}_k^n$, where $\mu(Q_r) = \mu(Q_s)$ for all $r, s = 1, \dots, 2^{nk}$. Thus $\mu(Q_0) = \mu(\cup_{r=1}^{2^{nk}} Q_r) = \sum_{r=1}^{2^{nk}} \mu(Q) = 2^{nk} \mu(Q)$. Thus $\mu(Q) = 2^{-nk} \mu(Q_0) = m(Q) \mu(Q_0)$. By the Key Observation, $\mu(E) = \mu(Q_0)m(E)$ for all $E \in \mathcal{B}_{\mathbb{R}^n}$. □

Corollary 10 (11.20 in Folland). *If H_p is the p -dimensional Hausdorff measure on \mathbb{R}^n , then there is a constant $j_{p,n} \geq 0$ such that $H_p(E) = j_{p,n}m(E)$ for each $E \in \mathcal{B}_{\mathbb{R}^n}$ (we assume $p \geq n$).*

- *If $p = n$, then $j_{p,n} = H_n(Q_0) = \frac{1}{m(B)}$ where B is a ball of radius 1.*
- *If $p > n$, then $j_{p,n} = 0$.*

Theorem 35. *Suppose that $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a linear transformation (that is, $T(ax + by) = aT(x) + bT(y)$ for all $x, y \in \mathbb{R}^n, a, b \in \mathbb{R}$). Then there exists a number $\delta < \infty$ such that $m(T(E)) = \delta m(E)$ for all $E \in \mathcal{L}^n$.*

Proof. If the dimension of the range of T is less than n then $m(T(\mathbb{R}^n)) = 0$ which implies $m(T(E)) = 0$ for all $E \in \mathcal{L}^n$, so we have $\delta = 0$. If the dimension of the range of T is n , then T can be represented by an invertible matrix. In particular, T^{-1} exists and is also linear (and thus continuous). It follows that T^{-1} is a Borel measurable mapping. Thus $T(E) \in \mathcal{B}_{\mathbb{R}^n}$ whenever $E \in \mathcal{B}_{\mathbb{R}^n}$. Define $\mu : \mathcal{B}_{\mathbb{R}^n} \rightarrow [0, \infty]$ by $\mu(E) = m(T(E))$. Since T is linear, it is easy to verify that μ is a measure. Let $a \in \mathbb{R}^n$ be given. Then $\mu(\tau_a(E)) = \mu(a + E) = m(T(a + E)) = m(T(a) + T(E)) = m(T(E)) = \mu(E)$ (as m is translation invariant) for all $E \in \mathcal{B}_{\mathbb{R}^n}$. By Theorem 34, $\mu(E) = \mu(Q_0)m(E)$ for all $E \in \mathcal{B}_{\mathbb{R}^n}$ with Q_0 a unit cube.

For the general case where $E \in \mathcal{L}^n$, use essentially the same argument used at the end of the proof for Thm 33. □

2.3 Signed Measures and Differentiation

Major Goal: Develop a theory of differentiation for measures.

Suppose that $g \in C^1(\mathbb{R})$ with $g(0) = 0$. Then by the Fundamental Theorem of Calculus, there exists $f \in C(\mathbb{R})$ such that $g(x) = \int_0^x f(s)ds$. (Here, of course, $f = g'$.) We want to do something similar for measures:

- Suppose that μ, ν are measures on a σ -algebra \mathcal{M} . When is it true that there is a \mathcal{M} -measurable function such that for each $A \in \mathcal{M}$, $\nu(A) = \int_A f d\mu$? In some sense, f is the derivative of ν with respect to μ .

To develop an answer to this question, we extend our notion of measures to signed measures.

Definition. Let (X, \mathcal{M}) be a measurable space. A **signed measure** on (X, \mathcal{M}) is a function $\nu : \mathcal{M} \rightarrow [-\infty, \infty]$ such that

- $\nu(\emptyset) = 0$.
- ν assumes at most one of the values $\pm\infty$, that is, if there exists $A \in \mathcal{M}$ such that $\nu(A) = \infty$, then there does not exist $B \in \mathcal{M}$ such that $\nu(B) = -\infty$.
- if $\{E_j\}_{j=1}^\infty \subseteq \mathcal{M}$ are mutually disjoint sets then $\nu(\cup_{j=1}^\infty E_j) = \sum_{j=1}^\infty \nu(E_j)$ and $\sum_{j=1}^\infty |\nu(E_j)| < \infty$ whenever $|\sum_{j=1}^\infty \nu(E_j)| < \infty$.

Remark. Since countable unions are invariant under rearrangement, if $|\sum_{j=1}^\infty \nu(E_j)| < \infty$, then one can show $\sum_{j=1}^\infty |\nu(E_j)| < \infty$.

Examples.

1. Suppose $\alpha, \beta \in \mathbb{R}$ and μ_1, μ_2 are positive measures on \mathcal{M} such that either $\mu_1(X) < \infty$ or $\mu_2(X) < \infty$. Then $\alpha\mu_1 + \beta\mu_2$ is a signed measure. [The condition that one must be finite is to prevent $\alpha\mu_1 + \beta\mu_2$ from taking on values of both $\pm\infty$.]
2. If $f \in L^1(\mu)$ where μ is a positive measure on \mathcal{M} , then the function $\nu : \mathcal{M} \rightarrow (-\infty, \infty)$ defined by $\nu(A) = \int_A f d\mu$ is a signed measure.

Proposition 37 (3.1). Let ν be a signed measure on (X, \mathcal{M}) .

1. If $\{E_j\}_{j=1}^\infty \subseteq \mathcal{M}$ and $E_1 \subseteq E_2 \subseteq \dots$, then $\nu(\cup_{j=1}^\infty E_j) = \lim_{j \rightarrow \infty} \nu(E_j)$.
2. If $\{E_j\}_{j=1}^\infty \subseteq \mathcal{M}$ and $E_1 \supseteq E_2 \supseteq \dots$ and $|\nu(E_1)| < \infty$, then $\nu(\cap_{j=1}^\infty E_j) = \lim_{j \rightarrow \infty} \nu(E_j)$.

Proof. This is similar to the proof for Theorem 5 (1.8 in Folland). Thus we will prove only (1). If there exists $N \in \mathbb{N}$ such that $|\nu(E_N)| = \infty$, then for all $j \geq N$ $\nu(E_j) = \nu(E_j \setminus E_N) + \nu(E_N) = \nu(E_N) = \pm\infty$ by property 3 of the definition of signed measures. Thus $|\nu(E_j)| = \infty$ and $\lim_{j \rightarrow \infty} \nu(E_j) = \pm\infty$. Also, $\nu(\cup_{j=1}^\infty E_j) = \nu(\cup_{j=1}^\infty E_j \setminus E_N) + \nu(E_N) = \infty = \nu(E_N) = \lim_{j \rightarrow \infty} \nu(E_j)$. So we may assume $|\nu(E_j)| < \infty$ for all $j \in \mathbb{N}$. Define $\{F_j\}_{j=1}^\infty \subseteq \mathcal{M}$ by $F_1 := E_1$ and $F_j := E_j \setminus \cup_{k=1}^{j-1} E_k$ for all $j \geq 2$. Then by Lemma 1, $\nu(\cup_{j=1}^\infty E_j) = \nu(\cup_{j=1}^\infty F_j) = \sum_{j=1}^\infty \nu(F_j)$. For each $j \geq 2$, we see $\nu(E_j) = \nu(F_j) + \nu(\cup_{k=1}^{j-1} E_k) = \nu(F_j) + \nu(E_{j-1})$. Since $|\nu(E_k)| < \infty$, this says $\nu(F_j) = \nu(E_j) - \nu(E_{j-1})$. Thus we have

$$\nu(\cup_{j=1}^\infty E_j) = \sum_{j=1}^\infty \nu(F_j) = \nu(E_1) + \sum_{j=1}^\infty \nu(E_j) - \nu(E_{j-1}) = \lim_{j \rightarrow \infty} \nu(E_j). \quad \square$$

Definition. Suppose ν is a signed measure. A set $E \in \mathcal{M}$ is called

1. **positive** if $\nu(F) \geq 0$ for all $F \subseteq E$ such that $F \in \mathcal{M}$,
2. **negative** if $\nu(F) \leq 0$ for all $F \subseteq E$ such that $F \in \mathcal{M}$,
3. **null** if $\nu(F) = 0$ for all $F \subseteq E$ such that $F \in \mathcal{M}$,

Lemma 4. Suppose $\{P_j\}_{j=1}^\infty \subseteq \mathcal{M}$ are positive sets with respect to ν . Then $\cup_{j=1}^\infty P_j$ is also positive.

Theorem (Hahn Decomposition Theorem- p.86). Suppose ν is a signed measure. Then there exists a positive set P and a negative set N such that $X = P \cup N$ and $P \cap N = \emptyset$. Moreover, if P', N' are another such pair, then $P \Delta P' = N \Delta N'$ are null sets.

Proof. WLOG, assume $\nu(A) < \infty$ for all $A \in \mathcal{M}$ (if not, work with $-\nu$). Put $M = \sup\{\nu(P) : P \text{ is positive}\}$. Then there exists $\{P_j\}_{j=1}^\infty \subseteq \mathcal{M}$ such that each P_j is a positive set and $\lim_{j \rightarrow \infty} \nu(P_j) = M$. WLOG, assume $P_1 \subseteq P_2 \subseteq \dots$ as otherwise we can just use $s_j = \cup_{k=1}^j P_k$ where still $\nu(s_j) \rightarrow M$. Set $P = \cup_{j=1}^\infty P_j$. Then by Proposition 37, $\nu(P) = \lim_{j \rightarrow \infty} \nu(P_j) = M < \infty$. Set $N = X \setminus P$.

Claim: N is a negative set.

Proof: Suppose not. Then there exists $A \in \mathcal{M}$ such that $A \subseteq N$ and $\nu(A) > 0$.

Subclaim: There exists a positive set E such that $E \subseteq A$ and $\nu(E) > 0$.

Proof: If A is a positive set, done. Otherwise, there exists $C \in \mathcal{M}$ such that $C \subseteq A$ and $\nu(C) < 0$. Put $L_1 = \inf\{\nu(C) : C \in \mathcal{M}, C \subseteq A\} < 0$. Let $n_1 \in \mathbb{N}$ be the smallest such integer such that $L_1 < -\frac{1}{n_1}$. Then there exists $C_1 \in \mathcal{M}$ such that $C_1 \subseteq A$ and $\nu(C_1) < -\frac{1}{n_1}$. Set $A_1 = A \setminus C_1$. If A_1 is a positive set, done. Otherwise, there exists $C \in \mathcal{M}$ such that $C \subseteq A$ and $\nu(C) < 0$. Put $L_2 = \inf\{\nu(C) : C \in \mathcal{M}, C \subseteq A_1\} < 0$. Let $n_2 \in \mathbb{N}$ be the least such integer such that $L_2 < -\frac{1}{n_2}$. Then there exists $C_2 \in \mathcal{M}$ such that $C_2 \subseteq A_1$ and $\nu(C_2) < -\frac{1}{n_2}$. Set $A_2 = A_1 \setminus C_2$ and continue inductively to get sequences of sets $\{A_j\}_{j=1}^\infty, \{C_j\}_{j=1}^\infty$, and positive integers $\{n_j\}_{j=1}^\infty$ such that for $j \geq 2$ we have $C_j \subseteq A_{j-1}$ and for all $j \in \mathbb{N}$, $\nu(C_j) < -\frac{1}{n_j}$. Notice $\nu(A_j) > \nu(A) + \sum_{k=1}^j \frac{1}{n_k}$. Put $E = \cap_{j=1}^\infty A_j$. Since $A_1 \supseteq A_2 \supseteq \dots$ and $\nu(A_1) < \infty$, by Proposition 37, we have $\nu(E) = \lim \nu(A_k) > \nu(A) + \sum_{k=1}^\infty \frac{1}{n_k} > 0$. Since $\nu(E) < \infty$, we have $\sum \frac{1}{n_k} < \infty$ and thus $n_k \rightarrow \infty$ as $k \rightarrow \infty$. Now, suppose E is not a positive set. Then there exists $C \in \mathcal{M}$ such that $C \subseteq E$ and $\nu(C) < 0$. Since $n_k \rightarrow \infty$, there exists k_0 such that $\nu(C) < -\frac{1}{n_{k_0-1}}$. Since $C \subseteq E \subseteq \cap_{j=1}^{n_{k_0-1}} A_j \subseteq A_{k_0-1}$ and $L_{k_0} = \inf\{\nu(\tilde{C}) : \tilde{C} \in \mathcal{M}, \tilde{C} \subseteq A_{k_0-1}\}$, we see $L_{k_0} < -\frac{1}{n_{k_0-1}}$. But $n_{k_0} - 1 < n_{k_0}$ and n_{k_0} was chosen to be the smallest integer, a contradiction. Thus E is a positive set and $\nu(E) > \nu(A) > 0$.

By the subclaim, if N was not a negative set, then there exists a positive set E such that $\nu(E) > 0$. But this contradicts the fact that $\nu(P) = \sup\{\nu(\tilde{P}) : \tilde{P} \text{ is a positive set}\}$ as $P \cup E$ is a positive set with $\nu(P \cup E) > \nu(P)$. Thus N is a negative set.

If P', N' is another such decomposition, then $P \setminus P' \subseteq P$ and $P \setminus P' \subseteq N'$. Thus $\nu(P \setminus P') = 0$. Similarly for $P' \setminus P$ and thus $P \Delta P'$ is a null set. \square

Definition (p 87). Any decomposition of X into a positive set P and a negative set N (that is, $P \cup N = X$ and $P \cap N = \emptyset$) is called a **Hahn Decomposition**.

Definition (p 87). Suppose that μ and ν are signed measures on (X, \mathcal{M}) . We say μ and ν are **mutually singular**, denoted $\mu \perp \nu$, if there exists a set $E \in \mathcal{M}$ such that E is a null set for μ and $X \setminus E$ is a null set for ν . We also say μ is **singular with respect to ν** and vice versa.

Example. Suppose m is the Lebesgue measure and ν any discrete signed measure, that is, there exists a countable set $K \subset \mathbb{R}^n$ such that $\mathbb{R}^n \setminus K$ is a ν -null set (for example, the counting measure on \mathbb{Z}). Then $m \perp \nu$. (since $m(K) = 0$)

Example. Put $D = \{(x, y) \in \mathbb{R}^2 | x = y\}$. Define $\nu : \mathcal{B}_{\mathbb{R}^2} \rightarrow [0, \infty]$ by $\nu(E) = m(\{x \in \mathbb{R} : (x, x) \in E\})$. Then $m|_{\mathcal{B}_{\mathbb{R}^2}}(D) = 0$ and $\nu(D^c) = 0$.

Theorem (Jordan Decomposition Theorem p.87). If ν is a signed measure on (X, \mathcal{M}) , then there exist unique positive measures ν^+ and ν^- such that $\nu = \nu^+ - \nu^-$ and $\nu^+ \perp \nu^-$.

Proof. Let P, N be a Hahn Decomposition for ν . Define $\nu^+, \nu^- : \mathcal{M} \rightarrow [0, \infty]$ by $\nu^+(E) = \nu(E \cap P)$ and $\nu^- = -\nu(E \cap N)$. Its easy to check ν^+, ν^- are positive measures. Also $\nu^+ \perp \nu^-$ as $P \cap N = \emptyset$. Finally,

$$\nu(E) = \nu((E \cap P) \cup (E \cap N)) = \nu(E \cap P) + \nu(E \cap N) = \nu^+(E) - \nu^-(E).$$

To show uniqueness, suppose there exist mutually singular positive μ^+, μ^- such that $\nu = \mu^+ - \mu^-$. Since $\mu^+ \perp \mu^-$, there exists $E \in \mathcal{M}$ such that $\mu^-(E) = \mu^+(E^C) = 0$. Now, for any $A \in \mathcal{M}$ such that $A \subseteq X \setminus E$, $\nu(A) = \mu^+(A) - \mu^-(A) = -\mu^-(A)$ by monotonicity. So $X \setminus E$ is a negative set and similarly E is a positive set. Thus $E, X \setminus E$ is another Hahn Decomposition of ν which implies $\nu(E \triangle P) = \nu(E^C \triangle N) = 0$. Let $A \in \mathcal{M}$ be given. Then

$$\begin{aligned}\mu^+(A) &= \mu^+(A \cap E) = \nu(A \cap E) &= \nu(A \cap ((E \setminus P) \cup (P \cap E))) \\ & &= \nu(A \cap E \setminus P) + \nu(A \cap (P \cap E)) \\ & &= \nu((A \cap E) \cap P) = \nu^+(A \cap E).\end{aligned}$$

Also

$$\begin{aligned}\nu^+(A) &= \nu^+((A \cap E) \cup (A \setminus E)) &= \nu^+(A \cap E) + \nu^+(A \setminus E) \\ & &= \nu^+(A \cap E) + \nu((A \setminus E) \cap P) \\ & &= \nu^+(A \cap E) \text{ as } (A \setminus E) \cap P \subseteq E \triangle P.\end{aligned} \quad \square$$

Definition. The decomposition of a signed measure ν into a difference of two positive mutually singular measures ν^+, ν^- is called a **Jordan Decomposition**. The positive measure ν^+ is called the **positive variation** of ν and ν^- is called the **negative variation** of ν . The **total variation** of ν is defined by $|\nu|(E) = \nu^+(E) + \nu^-(E)$ for $E \in \mathcal{M}$.

Note. This is a generalization of bounded variation.

Remarks.

1. $|\nu|$ is a positive measure on \mathcal{M} .
2. $A \in \mathcal{M}$ is a null set for ν if and only if it is for $|\nu|$.
3. If ν is a signed measure on \mathcal{M} , then $\nu \perp \mu$ if and only if $|\nu| \perp \mu$ if and only if $\nu^+ \perp \mu$ and $\nu^- \perp \mu$.
4. If P, N is a Hahn Decomposition for ν , then $\nu(A) = \nu^+(A) - \nu^-(A) = |\nu|(A \cap P) - |\nu|(A \cap N) = \int_{A \cap P} 1d|\nu| - \int_{A \cap N} 1d|\nu| = \int_A |\chi_P - \chi_N|d|\nu|$.

Definition. Suppose ν is a signed measure on (X, \mathcal{M}) . We set $L^1(\nu) = L^1(\nu^+) \cap L^1(\nu^-)$ and define $\int f d\nu = \int f d\nu^+ - \int f d\nu^-$ for $f \in L^1(\nu)$.

Example. Define $f \in C^\infty(\mathbb{R})$ by $f(x) = x^2 + 2$. Define $\delta_0, \delta_1 : \mathcal{B}_{\mathbb{R}} \rightarrow [0, \infty]$ by $\delta_i(A) = 0$ if $i \notin A$ and 1 if $i \in A$. Put $\nu = m - \delta_0 - \delta_1$. This is a signed measure on $\mathcal{B}_{\mathbb{R}}$ as $\delta_0 + \delta_1$ is finite. Notice

$$\int_{[0,1)} f d\nu = \int_{[0,1)} f dm - \int_{[0,1)} f d\delta_0 - \int_{[0,1)} f d\delta_1 = \frac{1}{3}x^3 + 2|_0^1 - f(0) - 0 = \frac{1}{3}$$

(as $1 \notin [0, 1)$) but

$$\int_{[0,1]} f d\nu = \frac{1}{3} - f(0) - f(1) = -\frac{8}{3}.$$

2.4 The Lebesgue-Radon-Nikodym Theorem

Definition. Suppose ν is a signed measure and μ a positive measure on (X, \mathcal{M}) . We say ν is **absolutely continuous** with respect to μ , denoted $\nu \ll \mu$, if $\nu(E) = 0$ whenever $\mu(E) = 0$.

Remarks.

- If $\nu \ll \mu$, then each null set for μ is a null set for ν .
- $\nu \ll \mu$ if and only if $|\nu| \ll \mu$ if and only if $\nu^+ \ll \mu$ and $\nu^- \ll \mu$.
- If $\nu \ll \mu$ and $\nu \perp \mu$, then $\nu = 0$.

Examples.

1. Suppose $f \in L^1(\mu)$ and define $\nu : \mathcal{M} \rightarrow (-\infty, \infty)$ by $\nu(E) = \int_E f d\mu$ for all $E \in \mathcal{M}$. Then $\nu \ll \mu$.
2. Recall the dirac measures δ_0, δ_1 . Then $\delta_0 \ll \delta_0 + \delta_1$ and $\delta_1 \ll \delta_0 + \delta_1$, but $\delta_0 + \delta_1$ is not absolutely continuous with respect to δ_0, δ_1 .

Theorem 36 (3.5). *Let ν be a finite signed measure and let μ be a positive measure. Then $\nu \ll \mu$ if and only if for all $\epsilon > 0$ there exists $\delta > 0$ such that $|\nu(E)| < \epsilon$ whenever $\mu(E) < \delta$.*

Proof. If the $\epsilon - \delta$ condition holds and $\mu(E) = 0$, then for all $\epsilon > 0$, we have $|\nu(E)| < \epsilon$. Thus $\nu(E) = 0$. Now suppose $\nu \ll \mu$, but there exists $\epsilon > 0$ such that for all $\delta > 0$ there exists $E \in \mathcal{M}$ such that $|\nu(E)| \geq \epsilon$ and $\mu(E) < \delta$. Then for all $n \in \mathbb{N}$, find $E_n \in \mathcal{M}$ such that $\mu(E_n) < \frac{1}{2^n}$ yet $|\nu(E_n)| \geq \epsilon$. Set $F = \liminf_{j \rightarrow \infty} E_j = \bigcap_{j=1}^{\infty} \bigcup_{k=j}^{\infty} E_k$. So for $j \in \mathbb{N}$, we have $0 \leq \mu(F) \leq \mu(\bigcup_{k=j}^{\infty} E_k) \leq \sum_{k=j}^{\infty} \frac{1}{2^k} = \frac{1}{2^{j-1}}$. Since this holds for all $j \in \mathbb{N}$, we have $\mu(F) = 0$. Since $\nu \ll \mu$, $|\nu| \ll \mu$ and thus $|\nu|(F) = 0$. Observe $|\nu|(\bigcup_{k=1}^{\infty} E_k) < \infty$ and $\bigcup_{k=j}^{\infty} E_k \supseteq \bigcup_{k=j+1}^{\infty} E_k$. Since $|\nu|$ is a positive measure, Theorem 5 gives $0 = |\nu|(F) = \lim_{j \rightarrow \infty} |\nu|(\bigcup_{k=j}^{\infty} E_k) \geq \lim_{j \rightarrow \infty} |\nu|(E_j) \geq \lim_{j \rightarrow \infty} |\nu(E_j)| \geq \epsilon$, a contradiction. \square

Corollary 11. *If $f \in L^1(\mu)$, then for all $\epsilon > 0$ there exists $\delta > 0$ such that $|\int_A f d\mu| < \epsilon$ whenever $\mu(A) < \delta$.*

Notation. If $\nu(E) = \int_E f d\mu$ for $E \in \mathcal{M}$ we write $\frac{d\nu}{d\mu}$ for f . Also, write $d\nu$ for $f d\mu$.

Lemma 5 (3.7). *Suppose ν, μ are finite positive measures. Either $\nu \perp \mu$ or there exists $\epsilon > 0$ and $E \in \mathcal{M}$ such that $\mu(E) > 0$ and E is a positive set for $\nu - \epsilon\mu$.*

Proof. For all $n \in \mathbb{N}$, let λ_n be the signed measure $\nu - \frac{1}{n}\mu$ and P_n, N_n be a Hahn Decomposition for λ_n . Set $P = \bigcup_{n=1}^{\infty} P_n, N = \bigcap_{n=1}^{\infty} N_n$. Note $N = X \setminus P$. We see N is a negative set for all λ_n . Thus $0 \geq \lambda_n(N) = (\nu - \frac{1}{n}\mu)(N)$ which implies $\frac{1}{n}\mu(N) \geq \nu(N)$. Taking the limit as $n \rightarrow \infty$, since ν is a positive measure, $\nu(N) = 0$. If P is a null set for μ , then $\nu \perp \mu$. So suppose P is not a null set for μ , that is, $\mu(P) > 0$ (since μ is a positive measure). Then there exists $n_0 \in \mathbb{N}$ such that $\mu(P_{n_0}) > 0$ and since P_{n_0} is a positive set for λ_{n_0} , the lemma is proved (that is, take $E = P_{n_0}$). \square

Theorem (Lebesgue Radon Nikodym Theorem- p.90). *Let ν be a σ -finite signed measure and μ be a σ -finite positive measure. There are unique σ -finite signed measures λ, ρ on (X, \mathcal{M}) such that $\lambda \perp \mu, \rho \ll \mu$ and $\lambda + \rho = \nu$ (this is called the **Lebesgue Decomposition** of ν and μ). Moreover, there exists an $\overline{\mathbb{R}}$ -valued μ -integrable function f such that $d\rho = f d\mu$. Any other such function is equal to f μ -a.e.*

Note. By μ -integrable, we mean either $\int f^+ d\mu$ or $\int f^- d\mu$ is finite.

Convention: If ν is a signed measure, we may refer to $d\nu$ as a signed measure, but we are actually referring to $E \mapsto \int_E d\nu$.

Proof. Step 1: First, we will assume μ, ν are finite positive measures. Set $\mathcal{F} = \{f \in L^1(\mu) : \int_E f d\mu \leq \nu(E) \text{ for all } E \in \mathcal{M}\}$. Note that $0 \in \mathcal{F}$, thus it is non-empty. Also, if $f, g \in \mathcal{F}$, so is the function $x \mapsto \max\{f(x), g(x)\}$ as if $A = \{x \in X | f(x) > g(x)\}$, then for $E \in \mathcal{M}$ $\int_E \max\{f(x), g(x)\} d\mu = \int_{E \cap A} f d\mu + \int_{E \setminus A} g d\mu \leq \nu(E \cap A) + \nu(E \setminus A) = \nu(E)$. Put $a := \sup\{\int_X f d\mu : f \in \mathcal{F}\}$ so that $a \leq \nu(X) < \infty$. We may select $\{f_n\}_{n=1}^{\infty} \subseteq \mathcal{F}$ such that $\lim_{n \rightarrow \infty} \int f_n d\mu = a$. For each $n \in \mathbb{N}$, define $g_n = \max\{f_1, \dots, f_n\}$. Then $\{g_n\}_{n=1}^{\infty} \subseteq \mathcal{F}$ is an increasing sequence. Define $f : X \rightarrow \overline{\mathbb{R}}$ by $f(x) := \sup_{n \in \mathbb{N}} f_n(x)$.

Claim: $f \in \mathcal{F}$.

Proof: Observe $g_n \rightarrow f$ pointwise and $g_1 \leq g_2 \leq \dots \leq f$. By the Monotone Convergence Theorem, for all $E \in \mathcal{M}$, $\int_E f d\mu = \lim_{n \rightarrow \infty} \int_E g_n d\mu \leq \nu(E)$. Then $f \in \mathcal{F}$.

Note that $\int_X f d\mu = a$.

Claim: The measure $d\lambda = d\nu - f d\mu$ is singular with respect to μ .

Proof: Note $d\lambda$ is a positive measure as $\nu(E) - \int f d\mu \geq 0$ for all $E \in \mathcal{M}$. Suppose λ was not singular with respect to μ . By Lemma 5(3.7), there exists $\epsilon > 0$ and $E \in \mathcal{M}$ such that $\mu(E) > 0$ and $\lambda(A) - \epsilon\mu(A) \geq 0$ for all $A \in \mathcal{M}$ with $A \subseteq E$. This implies for all $A \in \mathcal{M}$ that $\epsilon\mu(A \cap E) \leq \lambda(A \cap E) = \nu(A \cap E) - \int_{A \cap E} f d\mu$. Thus $\int_A \{f + \epsilon\chi_E\} d\mu = \int_A f d\mu + \epsilon\mu(A \cap E) \leq \int_A f d\mu + \nu(A \cap E) - \int_{A \cap E} f d\mu = \int_{A \setminus E} f d\mu + \nu(A \cap E) \leq \nu(A)$. Therefore, $f + \epsilon\chi_E \in \mathcal{F}$ but $\int_X f + \epsilon\chi_E d\mu = \int f d\mu + \epsilon\mu(E) > \int f d\mu$, a contradiction. Thus $\lambda \perp \mu$.

For uniqueness, suppose there exists λ', ρ', f' satisfying the conclusion of the theorem. Then $d\nu = d\lambda + fd\mu = d\lambda' + f'd\mu$ which implies $d\lambda - d\lambda' = (f' - f)d\mu$. Since $\lambda \perp \mu$ and $\lambda' \perp \mu$, we see $\lambda - \lambda' \perp \mu$. Also, since $f'd\mu \ll d\mu$ and $fd\mu \ll d\mu$ we have $(f' - f)d\mu \ll d\mu$. This implies that $\lambda - \lambda'$ is singular and absolutely continuous with respect to μ which says $\lambda - \lambda' = 0$. Now, $\int_X |f - f'|d\mu = 0$ and thus by Proposition 14 (2.23), $f' = f$ μ -a.e. Thus, the theorem is proved when μ, ν are finite positive measures.

Step 2: Now, assume μ, ν are positive σ -finite measures. We may find a sequence $\{A_j\}_{j=1}^\infty \subseteq \mathcal{M}$ of mutually disjoint sets such that $\cup_{j=1}^\infty A_j = X$ and $\nu(A_j), \mu(A_j) < \infty$. For each $j \in \mathbb{N}$, define the positive finite measures ν_j and μ_j by $\mu_j(E) = \mu(E \cap A_j)$ and $\nu_j(E) = \nu(E \cap A_j)$ for all $E \in \mathcal{M}$. Apply Step 1 to each pair (μ_j, ν_j) to obtain $\{\lambda_j\}_{j=1}^\infty$ of signed measures and $\{f_j\}_{j=1}^\infty$ of μ_j -integrable functions (in fact, in $L^1(\mu_j)$, since $\mu_j(X) < \infty$) such that for all $j \in \mathbb{N}$, $\lambda_j \perp \mu_j$ and $d\nu_j = d\lambda_j + f_j d\mu_j$. Since $\mu_j(X \setminus A_j) = 0$, WLOG, assume $f_j = 0$ on $X \setminus A_j$. Also, observe for all $E \in \mathcal{M}$ such that $E \subseteq X \setminus A_j$ we have $\lambda_j(E) = \nu_j(E) - \int_E f_j d\mu_j = 0$. So $X \setminus A_j$ is a null set for λ_j . Put $\lambda := \sum_{j=1}^\infty \lambda_j, f := \sum_{j=1}^\infty f_j$. Then it can be shown $\lambda \perp \nu$ and $d\nu = d\lambda + fd\mu$. Also, λ and $fd\mu$ are σ -finite.

Step 3: If ν is a σ -finite signed measure, then $\nu = \nu^+ - \nu^-$ where ν^+, ν^- are σ -finite positive measures. Apply Step 2 to ν^+ and ν^- separately and take the difference of the results. \square

Definition. The function f in the Lebesgue-Radon-Nikodym Theorem is called the **Radon-Nikodym derivative** of ν with respect to μ . It is traditionally denoted by $\frac{d\nu}{d\mu}$ and if $\nu \ll \mu$ then $d\nu = \frac{d\nu}{d\mu} d\mu$.

Example. Let ν be a σ -finite signed measure on (X, \mathcal{M}) . We see $\nu \ll |\nu|$. Also, we observed $\nu(E) = \int_E [\chi_P - \chi_N]d|\nu|$ where P, N is a Hahn Decomposition of ν . Thus $\chi_P - \chi_N$ is the Radon-Nikodym derivative of ν with respect to $|\nu|$.

Examples. Let $F : \mathbb{R} \rightarrow \mathbb{R}$ be given by $F(x) = \begin{cases} 0 & \text{if } x < 0, \\ 3 - e^{-x} & \text{if } 0 \leq x < 1 \\ 4 - e^{-x} & \text{if } 1 \leq x < \infty. \end{cases}$ Thus F is nondecreasing, right continuous. So

there exists a Lebesgue-Stieltjes measure μ_F with F as its distribution. What is the Lebesgue Decomposition of μ_F with respect to m ? Note that $m(\{0\}) = 0$ but $\mu_F(\{0\}) = \mu_F((-\infty, 0] \setminus (-\infty, 0)) = 2$. Also, $m(\{1\}) = 0$ but $\mu_F(\{1\}) = 1$.

Then there exists a singular measure on $\{0, 1\}$. Define $G : \mathbb{R} \rightarrow \mathbb{R}$ by $G(x) = \begin{cases} 0 & \text{if } x < 0, \\ e^{-x} & \text{if } 0 \leq x < \infty. \end{cases}$ Notice $\int_0^x G(s)ds$ is 0

if $x < 0$ and $-e^{-x} + 1$ if $x \geq 0$. So define $\rho : \mathbb{R} \rightarrow [0, \infty]$ by $\rho(E) = \int_E G(x)dx$. We see that $\mu_F = \rho + 2\delta_0 + \delta_1$ and $\frac{d\rho}{dm} = G(x)$.

Expansion of discussion on p 106

Definition. Let (X, \mathcal{M}) be a measurable space such that $\{x\} \in \mathcal{M}$ for all $x \in X$. Let ν be a signed measure on (X, \mathcal{M}) . We say $x \in X$ is an **atom** of ν if $\nu(\{x\}) \neq 0$.

Definition. Let (X, \mathcal{M}) be a measurable space such that $\{x\} \in \mathcal{M}$ for all $x \in X$. Let ν be a signed measure. Then,

- We say ν is **continuous** if $\nu(\{x\}) = 0$ for all $x \in X$.
- We say ν is **discrete** if there exists a countable set $k \subseteq \mathcal{M}$ such that $|\nu|(k^c) = 0$.

Definition. Let (X, \mathcal{M}) be a measure space such that $\{x\} \in \mathcal{M}$ for all $x \in X$. For each $x \in X$ define the **dirac measure** concentrated at x by $\delta_x = \begin{cases} 1 & \text{if } x \in E, \\ 0 & \text{if } x \notin E \end{cases}$, for all $E \in \mathcal{M}$.

Examples.

- The Lebesgue measure, the 0 measure, and all Lebesgue Stieltjes measures with continuous distribution functions are continuous.
- The 0 measure and the dirac measures are discrete.
- There exist measures which are neither continuous nor discrete. For example $m + \delta_0$.

Proposition 38. Let (X, \mathcal{M}) be a measurable space such that $\{x\} \in \mathcal{M}$ for all $x \in X$. Let ν be a σ -finite positive measure on (X, \mathcal{M}) . Then there exist σ -finite positive measures ν_c and ν_d such that $\nu_c \perp \nu_d$, $\nu = \nu_c + \nu_d$, ν_c is continuous, and ν_d is discrete.

Proof. Step 1: Assume ν is finite.

Claim: ν has only a countable number of atoms.

Proof: Let $F \in \mathcal{M}$ be a set consisting of a countable number of atoms. Then $\nu(F) = \sum_{x \in F} \nu(\{x\}) \leq \nu(X) < \infty$.

Put $\alpha := \sup\{\sum_{x \in F} \nu(\{x\}) : F \in \mathcal{M} \text{ and } F \text{ is countable}\}$. Then $\alpha < \infty$ and there exists a sequence $\{F_n\}_{n=1}^\infty \subseteq \mathcal{M}$ such that each F_n is countable and $\alpha = \lim_{n \rightarrow \infty} \sum_{x \in F_n} \nu(\{x\})$. Set $F = \cup_{n=1}^\infty F_n$. Then F is countable and $\alpha = \sum_{x \in F} \nu(\{x\})$. If there were an uncountable number of atoms for ν , then there would exist $x_0 \in X$ such that x_0 is an atom but $x_0 \notin F$. But then $F \cup \{x_0\}$ would be a countable set where $\sum_{x \in F \cup \{x_0\}} \nu(\{x\}) > \alpha$, a contradiction. Thus there exists a set $k = \{\text{atoms}\}$ which is countable.

Define $\nu_d : \mathcal{M} \rightarrow [0, \infty)$ by $\nu_d(E) = \sum_{x \in k} \nu(\{x\}) \delta_x(E)$ for all $E \in \mathcal{M}$. Put $\nu_c = \nu - \nu_d$. Clearly, ν_c is countably additive as ν and ν_d are and $\nu_c(\emptyset) = \nu(\emptyset) - \nu_d(\emptyset) = 0$. To show ν_c is non-negative, let $E \in \mathcal{M}$. Then $\nu_c(E) = \nu(E) - \nu_d(E) = \nu(E \cap k^C) + \nu(E \cap k) - \nu_d(E \cap k^C) - \nu_d(E \cap k) = \nu(E \cap k^C) + \sum_{x \in E \cap k} \nu(\{x\}) - \sum_{x \in E \cap k} \nu(\{x\}) = \nu(E \cap k^C) \geq 0$ (since $\delta_x = 1$ for all $x \in E \cap k$). Thus ν_c is a positive finite measure. Need to show ν_c is continuous. Let $x \in X$ be given. Then $\nu_c(\{x\}) = \nu(\{x\} \cap k^C) = 0$. Since $\nu_c = \nu(E \cap k^C)$ and $\nu_d = \nu(E \cap k)$, clearly $\nu_c \perp \nu_d$. We leave uniqueness as an exercise.

Step 2: Extend to σ -finite measures. (Again, an exercise). □

Theorem 37. Let (X, \mathcal{M}) be a measurable space such that $\{x\} \in \mathcal{M}$ for all $x \in X$. Let μ be a σ -finite positive measure and ν a σ -finite signed measure. Then there exist unique σ -finite signed measures ν_{ac}, ν_{sc} , and ν_d such that

1. $\nu_{ac} \ll \mu, \nu_{sc} \perp \mu$ and $\nu_d \perp \mu$.
2. ν_{sc} is continuous, ν_d is discrete and $\nu_{sc} \perp \nu_d$.
3. $\nu = \nu_{ac} + \nu_{sc} + \nu_d$.

Proof. Use Lebesgue Decomposition for part (1), then use Jordan Decomposition and Proposition 38 for the rest. □

Example. Let m be the Lebesgue measure on \mathbb{R} . Define $\nu : \mathcal{L} \rightarrow [0, \infty]$ by $\nu(E) = m(E) + \delta_0(E) + \delta_1(E)$. Define $\mu : \mathcal{L} \rightarrow [0, \infty]$ by $\mu(E) = \sum_{x \in E \cap \mathbb{N}} 1$. Then $\nu_{ac} = \delta_1, \nu_{sc} = m, \nu_d = \delta_0$ and $\frac{d\nu_{ac}}{d\mu} = \frac{d\delta_1}{d\mu} = \chi_{\{1\}}$.

2.5 Differentiation on Euclidean Space

We consider the setting where $(X, \mathcal{M}) = (\mathbb{R}^n, \mathcal{B}_{\mathbb{R}^n})$. In this setting, we will look at computing $\frac{d\nu}{dm}$ more explicitly.

Consider a positive Borel measure μ on \mathbb{R} such that $\mu \ll m$. By the Radon-Nikodym Theorem there exists $f \in L^+(\mathbb{R})$ such that $\mu(E) = \int_E f dx$ for all $E \in \mathcal{B}_{\mathbb{R}}$. Can we find a formula for f in terms of μ ?

Special Case: Suppose f is continuous. Then for all $x_0 \in \mathbb{R}$ and $h > 0$, we have $\mu((x_0 - h, x_0 + h)) = \int_{x_0-h}^{x_0+h} f dx$ and

$$\frac{\mu((x_0 - h, x_0 + h))}{m((x_0 - h, x_0 + h))} = \frac{1}{2h} \int_{x_0-h}^{x_0+h} f dx \rightarrow f(x_0)$$

as $h \rightarrow 0^+$ by the Fundamental Theorem of Calculus. Thus $\frac{d\mu}{dm} = \lim_{h \rightarrow 0^+} \frac{\mu((x_0-h, x_0+h))}{m((x_0-h, x_0+h))} = f(x_0)$.

Let $B(r, x) \subset \mathbb{R}^n$ be the open ball of radius r centered at $x \in \mathbb{R}^n$. Want to show that if ν is a signed measure on $\mathcal{B}_{\mathbb{R}^n}$ such that $\nu \ll m$, then $d\nu = f dm$ for some m -integrable function and for m -a.e. $x \in \mathbb{R}^n$ we have $\frac{d\nu}{dm} = f(x) = \lim_{r \rightarrow 0^+} \frac{\nu(B(r, x))}{m(B(r, x))}$.

Definition. A measurable function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is **locally integrable** with respect to the Lebesgue measure if $\int_K |f| dx < \infty$ for all bounded measurable sets $K \in \mathcal{L}$. We denote the space of locally integrable functions by $L^1_{loc}(\mathbb{R}^n)$ or just by L^1_{loc} .

Definition. Let $f \in L^1_{loc}$. Then for all bounded measurable sets $K \in \mathcal{L}$ with $m(K) > 0$, we define the **mean (average) value** of f over K by $\frac{1}{m(K)} \int_K f dx$. We denote the mean value of f over K by $\int_K f dx$. (Note that this notation is different, but more standard, from Folland's notation).

Definition. Let $f \in L^1_{loc}$. The **Hardy-Littlewood Maximal Function** $Hf : \mathbb{R}^n \rightarrow \mathbb{R}$ (also notated as Mf) is given by $Hf := \sup_{r>0} \int_{B(r,x)} |f| dx$.

Recall that a function $h : \mathbb{R}^n \rightarrow \mathbb{R}$ is **lower semi-continuous** if the set $\{x \in \mathbb{R}^n : h(x) > a\}$ is open for all $a \in \mathbb{R}$. Also, we say h is lower semicontinuous if $\liminf_{y \rightarrow x} h(y) \geq h(x)$.

Proposition 39. Let $f, g \in L^1_{loc}$ be given. Then

1. $0 \leq Hf \leq +\infty$.
2. $H(f + g) \leq Hf + Hg$
3. $H(cf) = |c|Hf$ for all $c \in \mathbb{R}$.
4. Hf is a lower semicontinuous function in \mathbb{R}^n .
5. Hf is a Borel measurable function.

Proof. (1-3) obvious

- (4) Let $a \in \mathbb{R}$. Consider the set $U_a := \{x \in \mathbb{R}^n : Hf > a\}$. We want to show U_a is open. Let $x_0 \in U_a$. Then there exists a sequence $\{r_n\}_{n=1}^\infty \subseteq (0, \infty)$ such that $\lim_{n \rightarrow \infty} \int_{B(r_n, x_0)} |f| dx > a$. Thus there exists $r_0, \epsilon > 0$ such that $\int_{B(r_0, x_0)} |f| dx \geq a + \epsilon$. Now, the measure $E \mapsto \int_E |f| dx$ is absolutely continuous with respect to m . So by Theorem 36, there exists $\delta > 0$ such that whenever $m(E) < \delta$ we find $\int_E |f| dx < \frac{1}{2}\epsilon m(B(r_0, x_0))$. Consider the set $D = \{x \in \mathbb{R}^n : m(B(x_0, r_0) \Delta B(x, r_0)) < \delta\}$. This is an open set containing x_0 . Let $x \in D$. Then by Theorem 33

$$\begin{aligned} \int_{B(r_0, x_0)} |f| dy &= \frac{1}{m(B(r_0, x_0))} \int_{B(r_0, x_0)} |f| dy \\ &= \frac{1}{m(B(r_0, x_0))} \left[\int_{B(r_0, x_0)} |f| dy - \int_{B(r_0, x_0)} |f| dy \right] + \int_{B(r_0, x_0)} |f| dy \\ &\geq \frac{1}{m(B(r_0, x_0))} \left[- \int_{B(r_0, x) \Delta B(r_0, x_0)} |f| dy \right] + \int_{B(r_0, x_0)} |f| dy \\ &\geq \frac{1}{m(B(r_0, x_0))} \left[- \frac{1}{2} \epsilon m(B(r_0, x_0)) \right] + a + \epsilon \\ &= a + \frac{1}{2} \epsilon \end{aligned}$$

- (5) Since for all $a \in \mathbb{R}$ the set $(Hf)^{-1}((a, \infty]) \in \mathcal{B}_{\mathbb{R}^n}$ by (4), we see Hf is Borel measurable. □

Example. Consider the function $\chi_{[0,1]} \in L^1_{loc}$. We see $H\chi_{[0,1]} = \sup_{r>0} \frac{1}{2r} m(B(r, x) \cap [0, 1]) = \begin{cases} 1 & \text{if } x \in (0, 1), \\ \frac{1}{2x} & \text{if } x \geq 1 \\ \frac{1}{2(1-x)} & \text{if } x \leq 0. \end{cases}$ Note

that even though $\chi_{[0,1]} \in L^1 \cap L^\infty$, the maximal function $H\chi_{[0,1]} \notin L^1$. In general, $Hf \notin L^1(\mathbb{R}^n)$ unless $f = 0$ a.e.

Theorem (Chebyshev's Inequality- p.193). Let (X, \mathcal{M}, μ) be a measure space. If $f \in L^p(\mu)$ for some $p \in [1, \infty)$, then for each $\alpha > 0$ we have $\mu(\{x \in X : |f(x)| > \alpha\}) \leq \frac{1}{\alpha^p} \|f\|_{L^p}^p$.

Proof. Set $E_\alpha := \{x \in X : |f(x)| > \alpha\}$. Then $\int_X |f|^p d\mu \geq \int_{E_\alpha} |f|^p d\mu \geq \alpha^p \int_{E_\alpha} 1 d\mu = \alpha^p \mu(E_\alpha)$. □

Definition. Let (X, \mathcal{M}, μ) be a measure space. For each measurable function $f : X \rightarrow \overline{\mathbb{R}}$, define $[f]_p := \sup_{\alpha>0} [\alpha^p \mu(\{x \in X : |f(x)| > \alpha\})]$. We say f is in **weak- L^p** if and only if $[f]_p < \infty$.

Remarks.

- $[f]_p$ is not a norm (it does not satisfy the triangle inequality).
- By Chebyshev's Inequality, for all $p \in [1, \infty)$ we find $L^p(\mu) \subsetneq weak - L^p(\mu)$. (This is strict as $\frac{1}{x^{1/p}} \notin L^p$, but is in $weak - L^p$.)

Lemma 6 (Simple Vitali Covering Lemma 3.15). *Let \mathcal{C} be a collection of open balls in \mathbb{R}^n and set $U = \cup_{B \in \mathcal{C}} B$. If $c < m(U)$, then there exists disjoint balls $\{B_j\}_{j=1}^k \subseteq \mathcal{C}$ such that $\sum_{j=1}^k m(B_j) > \frac{c}{3^n}$.*

Proof. Let $c < m(U)$. By Theorem 31(b), there exists a compact set $K \subseteq U$ such that $c < m(K) < m(U)$. Since K is compact, there exists a finite subcover $\{A_j\}_{j=1}^m \subseteq \mathcal{C}$ of K . WLOG, assume $m(A_1) \geq m(A_2) \geq \dots \geq m(A_m)$. Set $B_1 = A_1$. Then, pick B_2 to be the next ball in the list A_2, \dots, A_m such that $B_2 \cap B_1 = \emptyset$. Then pick B_3 to be the next ball after B_2 in the collection of $\{A_j\}$ such that $B_3 \cap (\cup_{j=1}^2 B_j) = \emptyset$. Continue until the list is exhausted. So we end up with a disjoint collection of $\{B_j\}_{j=1}^k$ such that $A_i \cap (\cup_{j=1}^k B_j) \neq \emptyset$ for all $i = 1, \dots, m$. Let $1 \leq i \leq m$ be given. Then there exists at least one j such that $A_i \cap B_j \neq \emptyset$. Pick the smallest such j . Then $m(B_j) \geq m(A_i)$. Hence the radius of B_j is greater than the radius of A_i . Let B_j^* be the ball that is concentric to B_j but has 3 times the radius. Then $A_i \subset B_j^*$. It follows that $\cup_{j=1}^k B_j^* \supseteq \cup_{j=1}^m A_j \supseteq K$. Thus $c < m(K) \leq m(\cup_{j=1}^k B_j^*) \leq \sum_{j=1}^k m(B_j^*) = 3^n \sum_{j=1}^k m(B_j)$. \square

Theorem (The Maximal Theorem, AKA The Hardy-Littlewood Theorem). *There exists a constant $c > 0$ such that for all $f \in L^1$ and all $\alpha > 0$ we have $m(\{x \in \mathbb{R}^n : Hf(x) > \alpha\}) \leq \frac{c}{\alpha} \|f\|_{L^1}$, that is, $Hf \in weak - L^1$.*

Proof. Set $E_\alpha = \{x \in \mathbb{R}^n : Hf(x) > \alpha\}$. Then for all $x \in E_\alpha$, we must have $\sup_{r>0} \int_{B(r,x)} |f| dy > \alpha$ which implies $\int_{B(r_x,x)} |f| dy > \alpha$ for some $r_x > 0$. Define $\mathcal{C} = \{B(r_x, x) : x \in E_\alpha\}$. Then $E_\alpha \subseteq \cup_{B \in \mathcal{C}} B$. For each $c < m(E_\alpha)$, we may select (by Lemma 6) a finite number of points $\{x_j\}_{j=1}^k \subseteq E_\alpha$ such that $\{B(r_{x_j}, x_j)\}_{j=1}^k$ are disjoint and $\sum_{j=1}^k m(B(r_{x_j}, x_j)) > \frac{c}{3^n}$. So

$$\begin{aligned} c < 3^n \sum_{j=1}^k m(B(r_{x_j}, x_j)) &= \frac{3^n}{\alpha} (\sum_{j=1}^k m(B(r_{x_j}, x_j)) \alpha) \\ &\leq \frac{3^n}{\alpha} (\sum_{j=1}^k m(B(r_{x_j}, x_j)) \int_{B(r_{x_j}, x_j)} |f| dy) \\ &= \frac{3^n}{\alpha} (\sum_{j=1}^k \int_{B(r_{x_j}, x_j)} |f| dy) \\ &= \frac{3^n}{\alpha} (\int_{\cup_{j=1}^k B(r_{x_j}, x_j)} |f| dy) \\ &\leq \frac{3^n}{\alpha} \int_{\mathbb{R}^n} |f| dy. \end{aligned}$$

Since true for all $c < m(E_\alpha)$, it follows that $m(E_\alpha) \leq \frac{3^n}{\alpha} \int_{\mathbb{R}^n} |f| dx$. \square

Theorem 38 (2.41). *If $f \in L^1(m)$ and $\epsilon > 0$, then there exists a continuous function $g : \mathbb{R}^n \rightarrow \mathbb{R}$ such that $\int_{\mathbb{R}^n} |f - g| dx < \epsilon$.*

Recall that $\lim_{r \rightarrow R} \phi(r) = c$ if and only if $\limsup_{r \rightarrow R} |\phi(r) - c| = 0$.

Theorem 39 (3.18). *Let $f \in L^1_{loc}$. Then for almost every $x \in \mathbb{R}^n$ we have $\lim_{r \rightarrow 0} \int_{B(r,x)} f(y) dy = f(x)$.*

Proof. Let $N \in \mathbb{N}$. Since $f \in L^1_{loc}$, we find $f_N := f \chi_{B(N,0)} \in L^1$. Let $\epsilon > 0$. By Theorem 38, there exists a continuous function $g : \mathbb{R}^n \rightarrow \mathbb{R}$ such that $\int_{\mathbb{R}^n} |f_N - g| dx < \epsilon$.

Claim: $\lim_{r \rightarrow 0^+} \int_{B(r,x)} g dy = g(x)$ for all $x \in \mathbb{R}^n$.

Proof: Let $x \in \mathbb{R}^n$ and $\delta > 0$. Since g is continuous, there exists $r > 0$ such that $|g(y) - g(x)| < \delta$ for all $y \in B(r, x)$.

Thus $\int_{B(r,x)} |g(y) - g(x)| dy < \delta$. Since $\delta > 0$ was arbitrary, we see

$$\begin{aligned} 0 = \lim_{r \rightarrow 0} \int_{B(r,x)} |g(y) - g(x)| dy &\geq \lim_{r \rightarrow 0} |\int_{B(r,x)} g(y) dy - g(x)| \\ &= \lim_{r \rightarrow 0} |\int_{B(r,x)} g(y) dy - g(x)| \geq 0. \end{aligned}$$

Hence, $\lim_{r \rightarrow 0^+} \int_{B(r,x)} g(y) dy = g(x)$.

Now, estimate $\limsup_{r \rightarrow 0^+} |\int_{B(r,x)} f_N(y)dy - f_N(x)|$ by comparing f_N and g . We have

$$\begin{aligned} \limsup_{r \rightarrow 0^+} |\int_{B(r,x)} f_N(y)dy - f_N(x)| &= \limsup_{r \rightarrow 0^+} |\int_{B(r,x)} f_N(y) - g(y)dy + \int_{B(r,x)} g(y) - g(x)dy + g(x) - f_N(x)| \\ &\leq \limsup_{r \rightarrow 0^+} [\int_{B(r,x)} |f_N(y) - g(y)|dy] + |g(x) - f_N(x)| \\ &\leq H(f_N - g)(x) + |f_N(x) - g(x)|. \end{aligned}$$

It follows that for all $\alpha > 0$, if $\limsup_{r \rightarrow 0^+} |\int_{B(r,x)} f_N(y)dy - f_N(x)| > \alpha$, then either $H(f_N - g)(x) > \frac{\alpha}{2}$ or $|f_N(x) - g(x)| > \frac{\alpha}{2}$. Set $E_\alpha = \{x \in \mathbb{R}^n : \limsup_{r \rightarrow 0^+} |\int_{B(r,x)} f_N(y)dy - f_N(x)| > \alpha\}$, $F_\alpha = \{x \in \mathbb{R}^n : H(f_N - g)(x) > \alpha\}$, and $G_\alpha = \{x \in \mathbb{R}^n : |f_N(x) - g(x)| > \alpha\}$. Note that $E_\alpha \subseteq F_{\alpha/2} \cup G_{\alpha/2}$. By the Maximal theorem and Chebyshev's inequality,

$$m(E_\alpha) \leq m(F_{\alpha/2}) + m(G_{\alpha/2}) \leq \frac{2c}{\alpha} \int_{\mathbb{R}^n} |f_N - g|dx + \frac{2}{\alpha} \int_{\mathbb{R}^n} |f_N - g|dx \leq \epsilon \left(\frac{2c}{\alpha} + \frac{2}{\alpha} \right).$$

Since $\epsilon > 0$ was arbitrary, we deduce $m(E_\alpha) = 0$ for all $\alpha < \infty$. Set $E = \cup_{k=1}^{\infty} E_{1/k}$. Then $m(E) = 0$ and for all $x \in E^C$ we see $0 \leq \limsup_{r \rightarrow 0^+} |\int_{B(r,x)} f_N(y)dy - f_N(x)| = 0$. Thus $\lim_{r \rightarrow 0^+} \int_{B(r,x)} f_N(y)dy = f_N(x)$. Since $N \in \mathbb{N}$ was arbitrary and $f_N = f$ on $B(N, 0)$, we conclude $\lim_{r \rightarrow 0^+} \int_{B(r,x)} f(y)dy = f(x)$ for almost every $x \in \mathbb{R}^n$. \square

Definition. For each $f \in L^1_{loc}$, set $L_f := \{x \in \mathbb{R}^n : \lim_{r \rightarrow 0^+} \int_{B(r,x)} |f(y) - f(x)|dy = 0\}$. The set L_f is called the **Lebesgue set** for f . The points in L_f are called the **Lebesgue points** for f .

Theorem 40 (3.20). If $f \in L^1_{loc}$, then $m(\mathbb{R}^n \setminus L_f) = 0$.

Proof. For each $\alpha \in \mathbb{R}$, define $g_\alpha := |f - \alpha|$. Since $f \in L^1_{loc}$ and $\alpha < \infty$, $g_\alpha \in L^1_{loc}$. Then, by Theorem 39, $E_\alpha = \{x \in \mathbb{R}^n : \lim_{r \rightarrow 0^+} \int_{B(r,x)} |f - \alpha|dy \text{ DNE or } \neq |f(x) - \alpha|\}$ is a null set for all $\alpha \in \mathbb{R}$. Put $E := \cup_{\alpha \in \mathbb{Q}} E_\alpha$ so $m(E) = 0$. Let $x \notin E$ and $\epsilon > 0$. Since \mathbb{Q} is dense in \mathbb{R} , select $\alpha \in \mathbb{Q}$ such that $|f(x) - \alpha| < \epsilon$. Since $x \notin E$,

$$0 \leq \limsup_{r \rightarrow 0^+} \int_{B(r,x)} |f(y) - f(x)|dy \leq \underbrace{\limsup_{r \rightarrow 0^+} \int_{B(r,x)} |f(y) - \alpha|dy + |\alpha - f(x)|}_{\text{by Theorem 39}} = 2|f(x) - \alpha| \leq 2\epsilon.$$

Since ϵ was arbitrary, $\lim_{r \rightarrow 0^+} \int_{B(r,x)} |f(y) - f(x)|dy = 0$. \square

Definition. A family of sets $\{E_r\}_{r>0} \subset \mathcal{B}_{\mathbb{R}^n}$ is said to **shrink nicely** to x if

- $E_r \subseteq B(r, x)$ for all $r > 0$.
- There exists a constant $\alpha > 0$ such that for all $r > 0$ we see $m(E_r) > \alpha m(B(r, x))$.

Theorem (Lebesgue's Differentiation Theorem). Suppose $f \in L^1_{loc}$. For each $x \in L_f$ we have $\lim_{r \rightarrow 0^+} \int_{E_r} |f(y) - f(x)|dy = 0$ and $\lim_{r \rightarrow 0^+} \int_{E_r} f(y)dy = f(x)$ for all families $\{E_r\}_{r>0} \subseteq \mathcal{B}_{\mathbb{R}^n}$ that shrink nicely to x .

Proof. By definition, there exists $\alpha > 0$ such that $m(E_r) > \alpha m(B(r, x))$ for all $r > 0$. Thus

$$\int_{E_r} |f(y) - f(x)|dy \leq \frac{1}{m(E_r)} \int_{B(r,x)} |f(y) - f(x)|dy \leq \frac{1}{\alpha m(B(r, x))} \int_{B(r,x)} |f(y) - f(x)|dy = \frac{1}{\alpha} \int_{B(r,x)} |f(y) - f(x)|dy.$$

If $x \in L_f$, then it follows that $\lim_{r \rightarrow 0^+} \int_{E_r} |f(y) - f(x)|dy = 0$. Also

$$0 \leq \lim_{r \rightarrow 0^+} \left| \int_{E_r} f(y)dy - f(x) \right| \leq \lim_{r \rightarrow 0^+} \int_{E_r} |f(y) - f(x)|dy = 0. \quad \square$$

Remark. Recall that the members of L^1 (and L^1_{loc}) are actually equivalence classes. By Theorem 3p, for all $f \in L^1_{loc}$ we have $\lim_{r \rightarrow 0^+} \int_{B(r,x)} f(y)dy$ exists for almost every $x \in \mathbb{R}^n$. The function $f^*(x) = \begin{cases} \int_{B(r,x)} f(y)dy & \text{if it exists} \\ 0 & \text{otherwise} \end{cases}$ is called

the **precise representative** for the equivalence class $f \in L^1_{loc}$. Note that if $f, g \in L^1_{loc}$ and $f = g$ a.e., then $f^* = g^*$ for all $x \in \mathbb{R}^n$.

Definition. Let ν be a signed measure on $(\mathbb{R}^n, \mathcal{B}_{\mathbb{R}^n})$. Then we say ν is **regular** if

1. $|\nu(K)| < \infty$ for all compact $K \subseteq \mathbb{R}^n$.
2. For all $E \in \mathcal{B}_{\mathbb{R}^n}$, we find $|\nu|(E) = \inf\{|\nu|(U) : U \text{ is open, } E \subseteq U\}$.

Remarks.

1. If ν is regular, then ν is σ -finite (by 1).
2. Property 1 implies Property 2.
3. If $d\nu = f dm$, then ν is regular if and only if $f \in L^1_{loc}$.

Proof. By the Jordan Decomposition, $d\nu^+ = f^+ dm$ and $d\nu^- = f^- dm$. So $d|\nu| = |f| dm$. If ν is regular, then (1) implies $f \in L^1_{loc}$. So assume $f \in L^1_{loc}$. Then clearly (1) is satisfied. For (2), let $E \in \mathcal{B}_{\mathbb{R}^n}$. Put $E_1 := E \cap B(1, 0)$ and for $j = 2, 3, \dots$ define $E_j := E \cap (B(j, 0) \setminus B(j-1, 0))$. Note $f \chi_{\overline{B(j+1, 0)}} \in L^1$ for all $j \in \mathbb{N}$. Let $\epsilon > 0$. By Corollary 10 (3.6), for all $j \in \mathbb{N}$ there exists δ_j such that $\int_F |f \chi_{B(j+1, 0)}| dx < \epsilon 2^{-j}$ whenever $m(F) < \delta_j$. By Theorem 31a(2.40), for all $j \in \mathbb{N}$ there exists $U_j \subseteq B(j+1, 0)$ such that U_j is open, $E_j \subseteq U_j$ and $m(U_j) < m(E_j) + \delta_j$. This implies $m(U_j \setminus E_j) = m(U_j) - m(E_j) < \delta_j$. Put $U = \cup_{j=1}^{\infty} U_j$. Then

$$\begin{aligned} |\nu|(U) = \int_U |f| dx &\leq \sum_{j=1}^{\infty} \int_{\mathbb{R}^n} |f \chi_{U_j}| dx \\ &= \sum_{j=1}^{\infty} \int_{U_j} |f| dx \\ &= \sum_{j=1}^{\infty} \int_{E_j} |f| dx + \sum_{j=1}^{\infty} \int_{U_j \setminus E_j} |f| dx \\ &= \int_E |f| dx + \sum_{j=1}^{\infty} \epsilon 2^{-j} \\ &= |\nu|(E) + \epsilon \leq |\nu|(U) + \epsilon \end{aligned}$$

Since $\epsilon > 0$ is arbitrary, $|\nu|(E) = \inf\{|\nu|(U) : U \text{ is open, } E \subseteq U\}$. □

Theorem 41 (3.22). Let ν be a regular signed measure on $(\mathbb{R}^n, \mathcal{B}_{\mathbb{R}^n})$. Let $d\nu = d\lambda + f dm$ be the Lebesgue-Radon-Nikodym decomposition of ν with respect to m . Then for almost every $x \in \mathbb{R}^n$, we have $\lim_{r \rightarrow 0^+} \frac{\nu(E_r)}{m(E_r)} = f(x)$ for every family $\{E_r\}_{r>0}$ that shrinks nicely to x .

Proof. As in the proof of the remark, the Jordan Decomposition implies $d|\nu| = d|\lambda| + |f| dm$.

Claim: λ and $f dm$ are regular.

Proof: If K is compact, then $|\lambda|(K) + \int_K |f| dm = |\nu|(K) < \infty$ by property 1 of regularity. Thus $|\nu(K)|, \int_K |f| dm < \infty$.

Thus $f \in L^1_{loc}$ and by the previous remarks, $|f| dm$ satisfies property 2 of regularity. For $|\lambda|$, note that $d|\lambda| = d|\nu| - |f| dm$ where ν and $f dm$ are regular. So we can use the same exact argument to show if $f \in L^1_{loc}$ then $f dm$ satisfies property 2.

Since $f \in L^1_{loc}$, the Lebesgue Differentiation Theorem implies $\lim_{r \rightarrow 0^+} \int_{E_r} f dy = f(x)$ whenever x is a Lebesgue point for f . It only remains to show $\lim_{r \rightarrow 0^+} \frac{\lambda(E_r)}{m(E_r)} = 0$ for m -almost every $x \in \mathbb{R}^n$ and all $\{E_r\}_{r>0}$ which shrink nicely to x (as then $\lim_{r \rightarrow 0^+} \frac{\nu(E_r)}{m(E_r)} = \lim_{r \rightarrow 0^+} \frac{\int_{E_r} f dm}{m(E_r)} = f(x)$). Since $\lambda \perp m$, there exists $A \in \mathcal{B}_{\mathbb{R}^n}$ such that $\lambda(A) = 0 = m(A^C)$. Since A^C is an m -null set, we need only consider $x \in A$. For each $k \in \mathbb{N}$, define $F_k := \{x \in A : \limsup_{r \rightarrow 0^+} \frac{|\lambda|(B(r, x))}{m(B(r, x))} > \frac{1}{k}\}$. We claim $m(F_k) = 0$ for all $k \in \mathbb{N}$. Let $\epsilon > 0$. Then there exists $U \in \mathcal{B}_{\mathbb{R}^n}$ that is open such that $A \subseteq U$ and $|\lambda|(U) < \lambda(A) + \epsilon = \epsilon$ by property 2 of regularity. Let $x \in F_k$. Then there exists $r_x < 0$ such that $B(r_x, x) \subseteq U$ and $\frac{1}{k} \leq \frac{|\lambda|(B(r_x, x))}{m(B(r_x, x))}$. Set $V = \cup_{x \in F_k} B(r_x, x) \subseteq U$. For each $c < m(V)$, by Lemma 6 there exists a finite collection of disjoint balls $\{B(r_{x_j}, x_j)\}_{j=1}^{\ell}$ such that $c < 3^n \sum_{j=1}^{\ell} m(B(r_{x_j}, x_j)) \leq 3^n k \sum_{j=1}^{\ell} |\lambda|(B(r_{x_j}, x_j)) \leq 3^n k |\lambda|(V) \leq 3^n k |\lambda|(U) \leq 3^n k \epsilon$. Since $c < m(V)$ and $\epsilon > 0$ were arbitrary, we conclude $m(F_k) = 0$. Setting $F = \cup_{k=1}^{\infty} F_k$, we see $m(F) = 0$. Suppose $x \in A$ and there exists $\{E_r\}_{r>0} \subset \mathcal{B}_{\mathbb{R}^n}$

that shrink nicely to x such that $\limsup_{r \rightarrow 0^+} \frac{|\nu(E_r)|}{m(E_r)} = \beta > 0$. Then there exists $\alpha > 0$ such that $m(E_r) > \alpha m(B(r, x))$ for all $r > 0$ and $K > \frac{1}{\alpha\beta}$, we have $\limsup_{r \rightarrow 0^+} \frac{|\lambda|(B(r, x))}{m(B(r, x))} \geq \limsup_{r \rightarrow 0^+} \frac{|\lambda|(E_r)}{m(B(r, x))} \geq \limsup_{r \rightarrow 0} \frac{\alpha|\lambda|(E_r)}{m(E_r)} \geq \alpha\beta > \frac{1}{k}$. So $x \in F$. Thus $\limsup_{r \rightarrow 0^+} \frac{|\lambda|(E_r)}{m(E_r)} = 0$ for almost every $x \in \mathbb{R}^n$. \square

Example. Consider the earlier example $F : \mathbb{R} \rightarrow \mathbb{R}$ defined by $F(x) = \begin{cases} 0 & x < 0, \\ 3 - e^{-x} & 0 \leq x < 1, \\ 4 - e^{-x} & 1 \leq x. \end{cases}$ Let μ_F be the Lebesgue-Stieltjes measure with distribution function F . So $\mu_F((-\infty, x]) = F(x)$. Previously, we found $d\mu_F = d\lambda + f dm$ where $\lambda = 2\delta_0 + \delta_1$ and $f(x) = \begin{cases} 0 & x < 0, \\ e^{-x} & x \geq 0. \end{cases}$ By Theorem 41, for m -almost every x , we see

$$f(x) = \lim_{r \rightarrow 0} \frac{\mu_F((x-r, x+r))}{m((x-r, x+r))} = \lim_{r \rightarrow 0} \frac{F(x+r) - F(x-r)}{2r} = F'(x)$$

if F is differentiable. Note that since $F'(x) = \begin{cases} 0 & x < 0, \\ e^{-x} & x > 0, \end{cases}$ we have a formula for $f(x)$ almost everywhere. Put $d\rho = f dm$. So then if $x \leq 0$, we see $\rho((-\infty, x]) = 0$ and if $x > 0$, we see $\rho((-\infty, x]) = \int_{(-\infty, x]} f ds = \int_{[0, x]} e^{-s} ds = 1 - e^{-x}$. Now, $\lambda = \mu_F - \rho$. In particular, $\lambda(\{0\}) = \mu_F(\{0\}) - \rho(\{0\}) = \mu_F((-\infty, 0]) - \mu_F((-\infty, 0)) - 0 = 2$ and $\lambda(\{1\}) = \mu_F(\{1\}) = 1$. Thus $\lambda = 2\delta_0 + \delta_1$.

Theorem (Lusin's Theorem). Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be a Borel measurable function. Suppose there exists $A \in \mathcal{B}_{\mathbb{R}^n}$ such that $m(A) < \infty$ and $f(x) = 0$ for all $x \in A^C$. Then for all $\epsilon > 0$ there exists a continuous function $g : \mathbb{R}^n \rightarrow \mathbb{R}$ such that

- $g(x) = 0$ for all $x \in \mathbb{R}^n \setminus B(R, 0)$ for some $R > 0$.
- $\sup_{x \in \mathbb{R}^n} |g(x)| \leq \sup_{x \in \mathbb{R}^n} |f(x)|$.
- $m(\{x \in \mathbb{R}^n | f(x) \neq g(x)\}) < \epsilon$.

Proof. (Of Theorem 38) WLOG, assume $f \geq 0$. By Proposition 19 (6.7), the set of simple functions in L^1 is dense in L^1 . Thus we may select a simple function $\phi \in L^1$ such that $\int_{\mathbb{R}^n} |f - \phi| dx < \frac{\epsilon}{2}$. Since ϕ is simple and in L^1 , there is an $A \in \mathcal{B}_{\mathbb{R}^n}$ such that $m(A) < \infty$ and $\phi(x) = 0$ for all $x \in A^C$. By Lusin's Theorem, there exists a continuous $g : \mathbb{R}^n \rightarrow \mathbb{R}$ such that $\sup_{x \in \mathbb{R}^n} |g(x)| < \infty$ and $m(\{x \in \mathbb{R}^n | g(x) \neq \phi(x)\}) < \frac{\epsilon}{4 \sup_{x \in \mathbb{R}^n} |\phi(x)| + 1}$. Thus

$$\int_{\mathbb{R}^n} |\phi - g| dx \leq \int_{\{x \in \mathbb{R}^n | g(x) \neq \phi(x)\}} |\phi - g| dx \leq \sup_{x \in \mathbb{R}^n} |\phi - g| m(\{x \in \mathbb{R}^n : \phi(x) \neq g(x)\}) \leq \frac{(2 \sup_{x \in \mathbb{R}^n} |\phi|) \epsilon}{4 \sup_{x \in \mathbb{R}^n} |\phi(x)| + 1} < \frac{\epsilon}{2}$$

by Holder's Inequality. Hence $\int_{\mathbb{R}^n} |f - g| dx \leq \int_{\mathbb{R}^n} |f - \phi| + |\phi - g| dx < \epsilon$. \square

Theorem 42. Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be a Borel measurable function. Let $A \in \mathcal{B}_{\mathbb{R}^n}$ be such that $m(A) < \infty$. For all $\epsilon > 0$, there exists a compact set $K \subseteq \mathbb{R}^n$ such that $m(A \setminus K) < \epsilon$ and $f|_K$ is continuous.

Proof. Recall that \mathcal{Q}_k^1 is the collection of dyadic intervals in \mathbb{R} with length 2^{-k} . Fix $k \in \mathbb{N}$. Let $\{Q_i\}_{i=1}^\infty$ be an enumeration of \mathcal{Q}_k^1 . For each $i \in \mathbb{N}$, set $A_i := A \cap f^{-1}(Q_i)$. Since $A \in \mathcal{B}_{\mathbb{R}^n}$ and f is Borel measurable, each $A_i \in \mathcal{B}_{\mathbb{R}^n}$. Also, the A_i 's are mutually disjoint and $A = \cup_{i=1}^\infty A_i$. By Theorem 31a, for each $i \in \mathbb{N}$ there exists a compact set $K_i \subset \mathbb{R}^n$ such that $K_i \subseteq A_i$ and $m(A_i \setminus K_i) < \frac{\epsilon}{2^{i+k}}$. Now $\{K_i\}_{i=1}^\infty$ are disjoint and so $m(A \setminus \cup K_i) = m(\cup A_i \setminus \cup K_i) = m(\cup (A_i \setminus K_i)) = \sum m(A_i \setminus K_i) < \frac{\epsilon}{2^k}$. Thus there exists N_k such that $m(A \setminus \cup_{i=1}^{N_k} K_i) < \frac{\epsilon}{2^k}$. Set $D_k = \cup_{i=1}^{N_k} K_i$. Then D_k is compact and $D_k \subseteq A$. For each $i = 1, \dots, N_k$, select $b_i \in Q_i$ (note $Q_i \supseteq f(K_i)$) and define $g_k : D_k \rightarrow \mathbb{R}$ by $g_k(x) = \sum_{i=1}^{N_k} b_i \chi_{K_i}(x)$. Now $\{K_i\}_{i=1}^{N_k}$ are compact disjoint sets, so there exists a strictly positive distance between K_i and K_j whenever $i \neq j$. It follows that g_k is continuous on D_k . Moreover $|f(x) - g_k(x)| \leq 2^{-k}$ for all $x \in D_k$. Put $K = \cap_{k=1}^\infty D_k$, so K is compact and $m(A \setminus K) \leq \sum_{k=1}^\infty m(A \setminus D_k) < \epsilon$. Since $|f(x) - g_k(x)| \leq 2^{-k}$ for all $x \in D_k$, we see $g_k \xrightarrow{unif} f$ on K which implies f is continuous on K . \square

Remark. Sometimes Theorem 42 is called Lusin's Theorem.

Theorem 43. Suppose $K \subseteq \mathbb{R}^n$ is a compact set and $f : K \rightarrow \mathbb{R}$ is continuous. Then there exists a continuous function $\bar{f} : \mathbb{R}^n \rightarrow \mathbb{R}$ such that $\sup_{x \in \mathbb{R}^n} |\bar{f}(x)| \leq \sup_{x \in K} |f(x)|$ and $\bar{f}(x) = f(x)$ for all $x \in K$.

Proof. Note that this follows from Tietze's Extension Theorem, but we will prove it without. First, we construct a candidate for the extension of f to \mathbb{R}^n : Put $U = \mathbb{R}^n \setminus K$, so that U is open. For each $s \in K$, put $v_s(x) = \max\{2 - \frac{\|x-s\|}{\text{dist}(x,K)}, 0\}$. Notice that $0 \leq v_s(x) \leq 1$ for all $x \in U$. Note that $x \mapsto v_s(x)$ is continuous (as $2, \|x-s\|, \text{dist}(x,K)$ are continuous and $\text{dist}(x,K) > 0$). Let $\{s_j\}_{j=1}^\infty$ be a countably dense subset of K . Define $\sigma : U \rightarrow [0, 1]$ by $\sigma(x) = \sum_{j=1}^\infty v_{s_j}(x) \frac{1}{2^j}$. Since $\{s_j\}$ are dense, we see $\sigma(x) > 0$. Now, we define $w_k : U \rightarrow [0, 1]$ by $x \mapsto \frac{\frac{1}{2^k} v_{s_k}(x)}{\sigma(x)}$. Observe that $\{w_k\}_{k=1}^\infty$ forms a partition of unity in U :

- $x \mapsto w_k(x)$ is continuous on U
- $0 \leq w_k \leq 1$
- $\sum w_k(x) = 1$ for all $x \in U$.

Now, our candidate for \bar{f} is $\bar{f}(x) = \begin{cases} f(x) & \text{if } x \in K \\ \sum_{k=1}^\infty w_k(x) f(s_k) & \text{if } x \in U. \end{cases}$ Observe that $\sup_{x \in \mathbb{R}^n} |\bar{f}(x)| \leq \sup_{x \in K} |f(x)|$. To show \bar{f} is

continuous on U , let $E \subseteq U$ be compact. Then $\min_{x \in E} \sigma(x) > 0$ which implies $\max_{x \in E} |w_k(x) f(s_k)| \leq \frac{\sup_{x \in K} |f(x)|}{2^k \min_{x \in E} \sigma(x)} =: M_k$. Now $\sum_{k=1}^\infty M_k = \frac{\sup |f(x)|}{\min \sigma(x)} \sum \frac{1}{2^k} < \infty$. By the M-test and the fact that if $\sum u_k(X)$ is a uniform convergent series of continuous functions on E then the function $x \mapsto \sum u_k(x)$ is continuous on E , we conclude $\sum_{k=1}^\infty w_k(x) f(s_k)$ is continuous on E . Hence \bar{f} is continuous on U .

We now show \bar{f} is an extension of f . We need only show that for all $x \in K$ we have $\lim_{x \rightarrow a, x \in U} \bar{f}(x) = f(a)$. Fix $\alpha > 0$. Since f is continuous, there exists $\delta > 0$ such that $|f(s) - f(a)| < \alpha$ for all $s \in K$ satisfying $\|s - a\| < \delta$. Suppose $x \in U$ and $\|x - a\| < \frac{\delta}{4}$. Notice that whenever $\|a - s_k\| \geq \delta$, we have

$$\delta \leq \|a - s_k\| < \|a - x\| + \|x - s_k\| < \frac{\delta}{4} + \|x - s_k\|.$$

This says $\|x - s_k\| > \frac{3\delta}{4} > 3\|x - a\| > 3\text{dist}(x, K)$. Thus $2 - \frac{\|x - s_k\|}{\text{dist}(x, K)} < -1$ which implies $v_{s_k}(x) = 0$ and so $w_k(x) = 0$. Since $\sum w_k(x) = 1$ for all $x \in U$ and $w_k(x) = 0$ whenever $\|x - a\| < \frac{\delta}{4}, \|a - s_k\| \geq \delta$, and $|f(a) - f(s_k)| < \alpha$ when $\|s_k - a\| < \delta$, we see that

$$|\bar{f}(x) - f(a)| = \left| \sum_{k=1}^\infty w_k(x) [f(s_k) - f(a)] \right| \leq \sum_{k=1}^\infty w_k(x) |f(s_k) - f(a)| \leq \sum_{k \in \mathbb{N}, \|s_k - a\| < \delta} w_k(x) |f(s_k) - f(a)| < \sum_{k \in \mathbb{N}} w_k(x) \alpha = \alpha.$$

Since α was arbitrary, we see $\lim_{x \rightarrow a, x \in U} \bar{f}(x) = f(a)$. □

Lemma 7. Let $K \subseteq \mathbb{R}^n$ be compact and $U \subseteq \mathbb{R}^n$ be an open set such that $K \subseteq U$. Then there exists a continuous function $\Psi : \mathbb{R}^n \rightarrow \mathbb{R}$ such that $\Psi(x) = 1$ if $x \in K$, $\Psi(x) = 0$ if $x \notin U$, and $0 \leq \Psi(x) \leq 1$ for $x \in \mathbb{R}^n$.

Proof. Let $d := \min\{\text{dist}(x, \mathbb{R}^n \setminus U) : x \in K\}$. Since $K, \mathbb{R}^n \setminus U$ are closed, $d > 0$. Set $\tilde{K} = \cup_{x \in K} B(\frac{d}{2}, x)$ and note $\text{dist}(x, \mathbb{R}^n \setminus U) > \frac{d}{2}$ for all $x \in \tilde{K}$. Define $\Psi : \mathbb{R}^n \rightarrow \mathbb{R}$ by $\Psi(x) = \int_{B(\frac{d}{2}, x)} \chi_{\tilde{K}}(y) dy$. By HW3 #4, the function $\Psi(x)$ is continuous. Noting $B(\frac{d}{2}, x) \subseteq \tilde{K}$ whenever $x \in K$ and $B(\frac{d}{2}, x) \cap \tilde{K} = \emptyset$ if $x \in \mathbb{R}^n \setminus U$, we see all the properties for Ψ as stated in the lemma are verified. □

Proof. (Of Lusin's Theorem) Let $\epsilon > 0$. By Theorem 42, there exists a compact set $K \subseteq \mathbb{R}^n$ such that $m(A \setminus K) < \frac{\epsilon}{2}$ and $f|_K$ is continuous. By Theorem 43, there exists an $\bar{f} : \mathbb{R}^n \rightarrow \mathbb{R}$ such that \bar{f} is continuous and $\bar{f}(x) = f(x)$ for all $x \in K$ and $\sup_{x \in \mathbb{R}^n} |\bar{f}(x)| \leq \sup_{x \in K} |f(x)| \leq \sup_{x \in \mathbb{R}^n} |f(x)|$. By Theorem 31a and since A is compact, we may find an open set U and $R > 0$ such that $K \subseteq U \subseteq B(R, 0)$ and $m(U) \leq m(K) + \frac{\epsilon}{2}$. Then $m(U \setminus K) \leq \frac{\epsilon}{2}$. Let ψ be continuous such that $\psi = 1$ on K , $\psi = 0$ on $\mathbb{R}^n \setminus U$ and $0 \leq \psi \leq 1$ by Lemma 7. Put $g = \bar{f}\psi$. Then

- $g(x) = 0$ for all $x \in \mathbb{R}^n \setminus B(R, 0)$
- $\sup_{x \in \mathbb{R}^n} |g(x)| \leq \sup_{x \in \mathbb{R}^n} |\bar{f}(x)| \leq \sup_{x \in \mathbb{R}^n} |f(x)|$
- If $x \in K$, then $f(x) = g(x)$. If $x \in A^C \cap U^C$, then $f(x) = 0 = g(x)$. So

$$\begin{aligned}
\{x \in \mathbb{R}^n | f(x) \neq g(x)\} &\subseteq (\mathbb{R}^n \setminus K) \cap (U^C \cap A^C)^C \\
&= (\mathbb{R}^n \setminus K) \cap (U \cup A) \\
&= ((\mathbb{R}^n \setminus K) \cap U) \cup (\mathbb{R}^n \setminus K) \cap A \\
&= (U \setminus K) \cup (A \setminus K)
\end{aligned}$$

Hence $m(\{x \in \mathbb{R}^n | f(x) \neq g(x)\}) \leq m((U \setminus K) \cup (A \setminus K)) \leq m(U \setminus K) + m(A \setminus K) < \epsilon$. \square

2.6 Functions of Bounded Variation

Recall. There exists a correspondence between regular Borel measures and increasing right continuous functions. We just established a nice differentiation theorem for regular Borel measures in \mathbb{R}^n . We can use this to establish differentiation theorems for the distribution function F . Recall that if μ is a regular Borel measure, then $\lim_{r \rightarrow 0^+} \frac{\mu_F(E_r)}{m(E_r)} = F(x)$ for almost every $x \in \mathbb{R}^n$. Now, in \mathbb{R}^1 , this implies $\lim_{r \rightarrow 0^+} \frac{\mu_F((x, x+r))}{r} = \frac{F(x+r) - F(x)}{r}$, the derivative.

Theorem 44 (3.23). *Let $F : \mathbb{R} \rightarrow \mathbb{R}$ be an increasing function and define $G : \mathbb{R} \rightarrow \mathbb{R}$ by $G(x) := F(x+) = \lim_{a \rightarrow x^+} F(a)$. Then*

1. *The set of points of discontinuity for F is countable (and thus has measure 0)*
2. *F and G are differentiable and $F' = G'$ a.e.*

Proof. 1. Define intervals $I_x := (F(x-), F(x+)) \subseteq \mathbb{R}$ for all $x \in \mathbb{R}$. Then $I_x \neq \emptyset$ if and only if x is a point of discontinuity for F . Since F is increasing, $\{I_x\}_{x \in \mathbb{R}}$ is a disjoint family of intervals. For each $N \in \mathbb{N}$, we have

$$\begin{aligned}
\sum_{x \in (-N, N)} m(I_x) &= \sup\{\sum_{x \in E} m(I_x) : E \subseteq (-N, N) \text{ is finite}\} \\
&= \sup\{m(\cup_{x \in E} I_x) | E \subseteq (-N, N) \text{ is finite}\} \\
&\leq m(F((-N, N))) \\
&\leq m((F(-N-), F(N+))) = F(N+) - F(-N-) < \infty
\end{aligned}$$

Thus there exists countable many $x \in (-N, N)$ such that $m(I_x) > 0$. It follows that $F(x-) = F(x+)$ except for at most a countable number of $x \in (-N, N)$.

2. By definition, G is increasing and right continuous. Also, $G = F$ at all points of continuity for F . In particular, $G = F$

a.e. For all $h \neq 0$, we have $G(x+h) - G(x) = \begin{cases} \mu_G((x, x+h]) & \text{if } h > 0 \\ -\mu_G((x+h, x]) & \text{if } h < 0. \end{cases}$ Observe $\{(x-r, x]\}_{r>0}, \{(x, x+r]\}_{r>0}$

shrink nicely to x as $r \rightarrow 0^+$. Also, by Theorem 21a(1.18), μ_G is a regular Borel measure on \mathbb{R} . Thus by Theorem 41 (3.22), we have

$$\lim_{r \rightarrow 0^+} \frac{\mu_G((x-r, x])}{m((x-r, x])} = \lim_{r \rightarrow 0^+} \frac{G(x) - G(x-r)}{r} = G' \text{ and } \lim_{r \rightarrow 0^+} \frac{\mu_G((x, x+r])}{m((x, x+r])} = \lim_{r \rightarrow 0^+} \frac{G(x+r) - G(x)}{r} = G'$$

for almost every $x \in \mathbb{R}$. Put $H := G - F$. If we show $H' = 0$ a.e., then $F' = G'$ a.e. By part 1, we know $H = 0$ a.e. Let $\{x_j\}_{j=1}^\infty$ be an enumeration of those points for which $H \neq 0$. (We assume x_j are distinct and note that j may be over a finite index set). Since $G \geq F$, we see $H \geq 0$ and for all $j \in \mathbb{N}$, we have $H(x_j) = G(x_j) - F(x_j) = \lim_{x \rightarrow x_j^+} F(x) - F(x_j) \leq \lim_{x \rightarrow x_j^+} F(x) - \lim_{x \rightarrow x_j^-} F(x) = F(x_j+) - F(x_j-)$. As in part 1, we see $0 \leq \sum_{x_j \in (-N, N)} H(x_j) < \infty$ for all $N \in \mathbb{N}$. Put $\nu := \sum_{j=1}^\infty H(x_j) \delta_{x_j}$. We see that if K is compact, then $K \subseteq (-N, N)$ for some $N \in \mathbb{N}$. Then

$$\nu(K) = \sum_{x_j \in K} \nu(\{x_j\}) \leq \sum_{x \in (-N, N)} H(x) < \infty.$$

By Proposition 2.6 (1.16) and Theorem 21a, ν is regular. We can use the Lebesgue Differentiation Theorem. Moreover, $\nu \perp m$ and so by Theorem 41, $\lim_{r \rightarrow 0^+} \frac{\nu(E_r)}{m(E_r)} = 0$ for almost every $x \in \mathbb{R}$. (Here $\{E_r\}$ shrink nicely to x). Thus $\lim_{h \rightarrow 0} \left| \frac{H(x+h) - H(x)}{h} \right| \leq \lim_{h \rightarrow 0} \frac{H(x+h) - H(x)}{|h|} \leq \lim_{h \rightarrow 0} \frac{2r((x-2|h|, x+2|h|))}{|h|} = 0$ a.e. by above as $\{(x-2|h|, x+2|h|)\}$ shrinks nicely to x . Thus $H'(x) = 0$ a.e. \square

Definition. Let $F : \mathbb{R} \rightarrow \mathbb{R}$. Define $T_F : \mathbb{R} \rightarrow \mathbb{R}$ by $T_F(x) = \sup\{\sum_{i=1}^n |F(x_i) - F(x_{i-1})| : n \in \mathbb{N}, -\infty < x_0 < x_1 < \dots < x_n = x\}$. We call T_F the **total variation function** of F . If $a < b$, then we call $T_F(b) - T_F(a)$ the total variation of F over $[a, b]$.

Remarks.

- T_F is nondecreasing.
- It can be shown that $T_F(b) - T_F(a) = \sup\{\sum_{i=1}^n |F(x_i) - F(x_{i-1})| : n \in \mathbb{N}, a = x_0 < x_1 < \dots < x_n = b\}$.

Definition. Let $F : \mathbb{R} \rightarrow \mathbb{R}$ be given. If $\lim_{x \rightarrow \infty} T_F(x) < \infty$, we say F is of **bounded variation** on \mathbb{R} . We set $BV(\mathbb{R}) = \{F : \mathbb{R} \rightarrow \mathbb{R} \mid \lim_{x \rightarrow \infty} T_F(x) < \infty\}$. Let $F : [a, b] \rightarrow \mathbb{R}$. If $T_F(b) - T_F(a) < \infty$, then F is of bounded variation on $[a, b]$ and we set $BV([a, b]) = \{F : [a, b] \rightarrow \mathbb{R} \mid T_F(b) - T_F(a) < \infty\}$.

Remarks.

- $BV(\mathbb{R}), BV([a, b])$ are vector spaces.

- If $F \in BV([a, b])$, then $\bar{F} : \mathbb{R} \rightarrow \mathbb{R}$ defined by $\bar{F}(x) = \begin{cases} F(x) & x \in [a, b] \\ F(a) & x < a \\ F(b) & x > b \end{cases}$ is in $BV(\mathbb{R})$. Thus any $F \in BV([a, b])$ can be extended to $F \in BV(\mathbb{R})$.

- If $F \in BV(\mathbb{R})$, then $F \in BV([a, b])$.

Lemma 8. If $F \in BV(\mathbb{R})$, then $T_F + F, T_F - F$ are increasing.

Proof. Let $x, y \in \mathbb{R}$ with $x < y$. We want to show $T_F(y) \pm F(y) \geq T_F(x) \pm F(x)$. Let $\epsilon > 0$. We may find $\{x_j\}_{j=0}^n \subset (-\infty, x]$ such that $-\infty < x_0 < \dots < x_n = x$ and $T_F(x) - \epsilon \leq \sum_{j=1}^n |F(x_j) - F(x_{j-1})|$. Then

$$T_F(y) \geq \sum_{j=1}^n |F(x_j) - F(x_{j-1})| + |F(y) - F(x)| \geq T_F(x) - \epsilon + |F(y) - F(x)|.$$

Now, $T_F(y) \geq T_F(x) - \epsilon + F(x) - F(y)$ and $T_F(y) \geq T_F(x) - \epsilon + F(y) - F(x)$. Thus $T_F(y) \pm F(y) \geq T_F(x) \pm F(x) - \epsilon$. Since ϵ was arbitrary, done. \square

Theorem 45 (3.27b). Suppose $F : \mathbb{R} \rightarrow \mathbb{R}$. Then $F \in BV(\mathbb{R})$ if and only if F can be written as the difference of two bounded increasing functions.

Proof. Since $BV(\mathbb{R})$ is a vector space and any bounded increasing function is in $BV(\mathbb{R})$, the backward direction is done. Let $F \in BV(\mathbb{R})$. Then $\frac{1}{2}(T_F \pm F)$ are increasing and their difference is F . Just need to show they are bounded. Of course, since $F \in BV(\mathbb{R})$, we have T_F is bounded by definition. Also, $|F(x)| \leq |F(x) - F(0)| + |F(0)| \leq T_F(x) + |F(0)| < \infty$. So F is (uniformly) bounded. Thus $\frac{1}{2}(T_F \pm F)$ is bounded. \square

Theorem 46 (3.27). Let $F \in BV(\mathbb{R})$. Then

1. $F(x+), F(x-)$ exist for all $x \in \mathbb{R}$ and $\lim_{x \rightarrow \infty} F(x), \lim_{x \rightarrow -\infty} F(x)$ exists.
2. The set of points of discontinuity of F is countable.
3. If $F : \mathbb{R} \rightarrow \mathbb{R}$ is defined by $G(x) = F(x+)$, then F', G' exist a.e. and $F' = G'$ a.e.

Proof. Note that (1) follows from Theorem 45. To show $F(x+)$ exists, just note there exists increasing bounded functions f_1, f_2 such that $F = f_1 - f_2$. Then $\lim_{y \rightarrow x+} f_1(y), \lim_{y \rightarrow x+} f_2(y)$ exist and thus $\lim_{y \rightarrow x+} f(y)$ exists. For the limits, use the fact that F is bounded. Now parts (2) and (3) follow from Theorem 44 and 45. \square

Definition. If $F \in BV(\mathbb{R})$, then the representation $F = \frac{1}{2}(T_F + F) - \frac{1}{2}(T_F - F)$ is called the **Jordan representation** of F . We say $\frac{1}{2}(T_F + F)$ is the **positive variation** and $\frac{1}{2}(T_F - F)$ is the **negative variation** for F .

Remark. If $-\infty < x_0 < \dots < x_n = x$, then $F(x) = F(x_0) + \sum_{j=1}^n F(x_j) - F(x_{j-1})$. Thus

$$\begin{aligned} \frac{1}{2}(T_f + F) &= \frac{1}{2} \sup\{\sum |F(x_j) - F(x_{j-1})| + F(x)\} \\ &= \frac{1}{2} \sup\{\sum |F(x_j) - F(x_{j-1})| + F(x_0) + \sum_{j=0}^n F(x_j) - F(x_{j-1})\} \\ &= \sup\{\sum_{j=1}^n [F(x_j) - F(x_{j-1})]^+ + \frac{1}{2}F(x_0)\} \\ &= \sup\{\sum_{j=1}^n [F(x_j) - F(x_{j-1})]^+\} + \frac{1}{2} \lim_{x \rightarrow -\infty} F(x). \end{aligned}$$

Define $NBV(\mathbb{R}) = \{F \in BV(\mathbb{R}) : \lim_{x \rightarrow -\infty} F(x) = 0 \text{ and } F \text{ is right cont}\}$.

Lemma 9 (3.28). If $F \in BV(\mathbb{R})$, then $\lim_{x \rightarrow -\infty} T_F(x) = 0$. If F is right continuous, then T_F is also right continuous.

Proof. Let $\epsilon > 0$. With $x \in \mathbb{R}$, select $\{x_j\}_{j=1}^n \subset \mathbb{R}$ such that $-\infty < x_0 < \dots < x_n = x$ and $T_F(x) - \epsilon \leq \sum_{j=1}^n |F(x_j) - F(x_{j-1})|$. Thus $T_F(x) - T_F(x_0) = \sup\{\sum_{j=1}^m |F(y_j) - F(y_{j-1})| : m \in \mathbb{N}, x_0 = y_0 < \dots < y_m = x\} \geq \sum_{j=1}^m |F(x_j) - F(x_{j-1})| \geq T_F(x) - \epsilon$. Thus $T_f(x_0) \leq \epsilon$ which implies $T_F(y) \leq \epsilon$ for all $y \leq x_0$. Hence $\lim_{x \rightarrow -\infty} T_F(x) = 0$. Now, suppose F is right continuous. We want to show T_F is right continuous. Put $\alpha = \lim_{y \rightarrow x+} T_F(y) - T_F(x) \geq 0$. Let $\epsilon > 0$. Choose $\delta > 0$ such that $|F(y) - F(x)| < \epsilon$ whenever $x \leq y \leq x + \delta$ and (by definition) when $T_F(y) - \lim_{z \rightarrow x+} T_F(z) < \epsilon$ whenever $x \leq y \leq x + \delta$. (Note that δ may initially be different, but then choose the smaller one). Let $y \in (x, x + \delta)$ be given. We may choose $\{x_j\}_{j=1}^n \subset \mathbb{R}$ such that $x = x_0 < \dots < x_n = y$ and

$$\sum_{j=1}^n |F(x_j) - F(x_{j-1})| \geq [T_F(y) - T_F(x)] - \frac{1}{4}[T_F(y) - T_F(x)] \geq \frac{3}{4}[\lim_{z \rightarrow x+} T_F(z) - T_F(x)] = \frac{3}{4}\alpha.$$

Since $x_1 \in (x, x + \delta)$, we have $|F(x_1) - F(x_0)| < \epsilon$. So $\sum_{j=2}^n |F(x_j) - F(x_{j-1})| \geq \frac{3}{4}\alpha - \epsilon$. Now, we may find $\{t_j\}_{j=1}^m \subset \mathbb{R}$ such that $x = t_0 < \dots < t_m = x_1$ and $\sum |F(t_j) - F(t_{j-1})| \geq \frac{3}{4}\alpha$. Now, $x = t_0 < \dots < t_m = x_1 < \dots < x_n = y$ is a partition. So $T_F(y) - T_F(x) \geq \sum_{j=1}^m |F(t_j) - F(t_{j-1})| + \sum_{j=2}^n |F(x_j) - F(x_{j-1})| \geq \frac{3}{2}\alpha - \epsilon$. Now, $\alpha + \epsilon \geq T_F(y) - T_F(x) \geq \frac{3}{2}\alpha - \epsilon$, which implies $\alpha \leq \epsilon$. Since $\epsilon > 0$ was arbitrary, $\alpha = 0$. \square

Theorem 47 (3.29). If μ is a finite signed Borel measure on \mathbb{R} and $F : \mathbb{R} \rightarrow \mathbb{R}$ is defined by $F(x) = \mu((-\infty, x])$, then $F \in NBV(\mathbb{R})$. If $F \in NBV(\mathbb{R})$, then there exists a unique finite signed regular measure μ_F on \mathbb{R} such that $\mu_F((-\infty, x]) = F(x)$ for all $x \in \mathbb{R}$. Moreover, $|\mu_F| = \mu_{T_F}$, that is, $\mu_{T_F}((-\infty, x]) = T_F(x)$ for all $x \in \mathbb{R}$.

Proof. Suppose μ is a finite signed Borel measure. By the Jordan Decomposition Theorem, we may find positive Borel measures μ^+, μ^- such that $\mu = \mu^+ - \mu^-$. By Proposition 26, the functions $F^+, F^- : \mathbb{R} \rightarrow \mathbb{R}$ given by $F^\pm(x) = \mu^\pm((-\infty, x])$ are right continuous and increasing. Moreover, $\lim_{x \rightarrow -\infty} F^\pm(x) = 0$ by Theorem 5e (1.8d) and $\lim_{x \rightarrow \infty} F^\pm(x) = \mu^\pm(\mathbb{R})$. So $F^\pm \in NBV(\mathbb{R})$. Since $F = F^+ - F^-$ and $NBV(\mathbb{R})$ is a vector space, we see $F \in NBV(\mathbb{R})$. For the other direction, put $F^\pm = \frac{1}{2}(T_F \pm F)$. So F^\pm are bounded increasing and right continuous. Let μ_{F^\pm} be the associated Lebesgue Stieltjes measures (note they are regular) and put $\mu_F := \mu_{F^+} - \mu_{F^-}$. Thus μ_F is a signed regular Borel measure and $\mu_F((-\infty, x]) = \mu_{F^+}((-\infty, x]) - \mu_{F^-}((-\infty, x]) = F^+(x) - F^-(x) = F(x)$ as they are in $NBV(\mathbb{R})$. To show $|\mu_F| = \mu_{T_F}$, observe that for $[a, b] \in \mathbb{R}$, $|\mu_F|([a, b]) = \mu_{F^+}([a, b]) + \mu_{F^-}([a, b]) = F^+(b) - F^+(a) + F^-(b) - F^-(a) = T_F(b) - T_F(a) = \mu_{T_F}([a, b])$. Since they are equal on the semialgebra of left open-right closed intervals, they are equal on $\mathcal{B}_{\mathbb{R}}$. Uniqueness is an exercise. \square

Remarks. Folland proves everything for \mathbb{C} -valued functions. Also, compare Propositions 25 and 26 to Theorem 1.16 in Folland.

Proposition 40 (3.30). Let $F \in NBV(\mathbb{R})$. Then F' exists a.e. and there exists $f \in L^1$ such that $F' = f$ a.e. Moreover, $\mu_F \perp m$ if and only if $F' = 0$ a.e. and $\mu_F \ll m$ if and only if $F(x) = \int_{(-\infty, x]} f(t)dt$.

Proof. If $F \in NBV(\mathbb{R})$, then μ_F is a signed regular Borel measure by Theorem 47 (3.29) and F' exists a.e. by Theorem 46. Let f be the Radon-Nikodym derivative for μ_F so that $f \in L^1(m)$ by the LRN Theorem and $f = F'$ a.e. by Theorem 41 (3.22). The rest follows from the LRN Theorem. \square

Definition. We say $F : \mathbb{R} \rightarrow \mathbb{R}$ is **absolutely continuous** if for all $\epsilon > 0$ there exists $\delta > 0$ such that for any family $\{(a_j, b_j)\}_{j=1}^n$ of disjoint intervals, we have $\sum_{j=1}^n |F(b_j) - F(a_j)| < \epsilon$ whenever $\sum_{j=1}^n (b_j - a_j) < \delta$.

Proposition 41 (3.32). If $F \in NBV(\mathbb{R})$, then F is absolutely continuous if and only if $\mu_F \ll m$.

Proof. See Folland. \square

Corollary 12 (3.33). If $f \in L^1(m)$, then $F(x) := \int_{(-\infty, x]} f(t)dt$ is in $NBV(\mathbb{R})$, is absolutely continuous, and $F'(x) = f(x)$ a.e. If $F \in NBV(\mathbb{R})$ is absolutely continuous, then $F' \in L^1(m)$ (a.e.) and $F(x) = \int_{(-\infty, x]} F'(t)dt$.

Proof. The second part follows immediately from Proposition 40 and 41. For the first part, if $f \in L^1(m)$, then $f^+, f^- \in L^1(m)$ and $F(x) = \int_{(-\infty, x]} f^+(t)dt - \int_{(-\infty, x]} f^-(t)dt$, the difference of two increasing bounded functions. Thus $F \in BV(\mathbb{R})$. Of course, F is clearly continuous and $\lim_{x \rightarrow -\infty} F(x) = 0$. So $F \in NBV(\mathbb{R})$. We see F is absolutely continuous by Proposition 41 and $F'(x) = f(x)$ a.e. follows from Proposition 40. \square

Theorem (Fundamental Theorem of Calculus). Let $[a, b] \subset \mathbb{R}$ and $F : [a, b] \rightarrow \mathbb{R}$ be given. TFAE

1. F is absolutely continuous on $[a, b]$.
2. $F(x) = F(a) + \int_{(a, x]} f(t)dt$ for some $f \in L^1([a, b], m)$.
3. F' exists a.e. in $[a, b]$ and there exists $f \in L^1([a, b], m)$ such that $f = F'$ a.e. and $F(x) = F(a) + \int_{(a, x]} f(t)dt$.

Theorem (Integration By Parts). Suppose $F, G \in NBV(\mathbb{R})$ and either F or G is continuous. Then for all $[a, b] \subset \mathbb{R}$, we have $\int_{(a, b]} Fd\mu_G = F(b)G(b) - F(a)G(a) - \int_{(a, b]} Gd\mu_F$.

Proof. WLOG, assume G is continuous. By considering $H(x) := G(-x)$, we see T_G is continuous. Thus $G^\pm := \frac{1}{2}(T_G \pm G)$ are continuous. Also $F^\pm := \frac{1}{2}(T_F \pm F)$ are right continuous as $F \in NBV(\mathbb{R})$. Set $\Omega = \{(x, y) : a < x \leq y \leq b\} \subseteq \mathbb{R}^2$. Now, μ_{F^\pm}, μ_{G^\pm} are all positive finite Borel measures. Thus by Fubini's Theorem, we have

$$\begin{aligned} (\mu_{F^+} \times \mu_{G^+})(\Omega) &= \int_{\Omega} d(\mu_{F^+} \times \mu_{G^+}) \\ &= \int_{(a, b]} \int_{[x, b]} d\mu_{G^+}(y) d\mu_{F^+}(x) \\ &= \int_{(a, b]} \mu_{G^+}(b) - \mu_{G^+}(x) d\mu_{F^+}(x) \\ &= \int_{(a, b]} G^+(b) - G^+(x) d\mu_{F^+}(x) \\ &= G^+(b)[F^+(b) - F^+(a)] - \int_{(a, b]} G^+(x) d\mu_{F^+}(x). \end{aligned}$$

and similarly

$$\begin{aligned} (\mu_{F^+} \times \mu_{G^+})(\Omega) &= \int_{(a, b]} \int_{(a, y]} d\mu_{F^+}(x) d\mu_{G^+}(y) \\ &= \int_{(a, b]} F^+(y) - F^+(a) d\mu_{G^+}(y) \\ &= \int_{(a, b]} F^+(y) d\mu_{G^+}(y) - F^+(a)(G^+(b) - G^+(a)). \end{aligned}$$

Combining these two equations, we get

$$\int_{(a, b]} F^+ d\mu_{G^+} = F^+(b)G^+(b) - F^+(a)G^+(a) - \int_{(a, b]} G^+ d\mu_{F^+}. (*)$$

Of course, we could easily show $(*)$ holds for F^-, G^+ . Then, subtracting these we get that $(*)$ holds for F, G^+ . Repeat with G to get that $(*)$ holds for F, G . \square

2.7 Measurable Transformations

Recall. Change of Variable Formula: If $g : (a, b] \rightarrow (c, d]$ is continuously differentiable and monotone and f is continuous on $(c, d]$, then $\int_a^b f(g(x))|g'(x)|dx = \int_c^d f(y)dy$. We want to generalize this idea to Lebesgue integrals.

Definition. Let $(X, \mathcal{M}), (Y, \mathcal{N})$ be measurable spaces. A mapping $T : X \rightarrow Y$ is called a $(\mathcal{M}, \mathcal{N})$ -measurable transformation if $T^{-1}(F) \in \mathcal{M}$ whenever $F \in \mathcal{N}$.

Remarks.

- This is the same definition as a measurable function. The point is to note we are not restricting ourselves to \mathbb{R} .
- If $(\mathbb{R}, \mathcal{O})$ is a measurable space and $f : Y \rightarrow \mathbb{R}$ is a $(\mathcal{N}, \mathcal{O})$ -measurable function, then $f \circ T : X \rightarrow \mathbb{R}$ is a $(\mathcal{M}, \mathcal{O})$ -measurable function.

Proposition 42. Let T be a measurable transformation from $(X, \mathcal{M}) \rightarrow (Y, \mathcal{N})$. Let μ be a positive measure on \mathcal{M} and define $\mu \circ T^{-1} : \mathcal{N} \rightarrow [0, \infty]$ by $\mu \circ T^{-1}(F) = \mu(T^{-1}(F))$. Then $\mu \circ T^{-1}$ is a measure on \mathcal{N} .

Definition. The measure $\mu \circ T^{-1}$ above is called the **measure induced** by μ and T , or the **pushforward** of μ through T .

Example. Let $T : [0, 2\pi) \rightarrow \mathbb{R}^2$ be given by $T(t) := (\cos t, \sin t)$. So T is a bijection between $[0, 2\pi)$ and the unit circle S' . Let $m = m|_{\mathcal{L}([0, 2\pi))}$. Let \mathcal{N} be the σ -algebra on S' generated by T . Then T is a measurable function. By definition, T is an $(\mathcal{M}, \mathcal{N})$ -measurable transformation. The Lebesgue measure on S' is the pushforward of m through T , that is, $m_{S'} = m \circ T^{-1}$.

General Change of Variable Formula. Let (X, \mathcal{M}, μ) be a measure space. Let (Y, \mathcal{N}) be a measurable space. Suppose T is an $(\mathcal{M}, \mathcal{N})$ -measurable transformation. Then for all \mathcal{N} -measurable functions $f : Y \rightarrow \mathbb{R}$, we have

$$\int_X f(T(x))d\mu = \int_Y f(y)d(\mu \circ T^{-1})$$

in the sense that if one exists, they both do and are equal.

Proof. Exercise using simple function technique. □

Corollary 13. Under the same hypotheses, for all $F \in \mathcal{N}$, we have

$$\int_{T^{-1}(F)} f(T(x))d\mu = \int_F f(y)d(\mu \circ T^{-1})$$

provided one of the integrals exist.

Corollary 14. Let (X, \mathcal{M}, μ) be a measure space. Let (Y, \mathcal{N}, ν) be a σ -finite measure space. Suppose T is a $(\mathcal{M}, \mathcal{N})$ -measurable transformation such that $\mu \circ T^{-1} \ll \nu$. Then for all \mathcal{N} -measurable functions $f : Y \rightarrow \mathbb{R}$, we have

$$\int_X f(T(x))d\mu = \int_Y f(y) \frac{d(\mu \circ T^{-1})}{d\nu} d\nu.$$

Theorem 48. Suppose $T : [a, b] \rightarrow [c, d]$ is an increasing bijection that is absolutely continuous on $[a, b]$. Let $f \in L^1([a, b], m)$ be given. Then $\int_{[a, b]} f(T(x))T'(x)dx = \int_{[c, d]} f(y)dy$.

Proof. Want to use Corollary 14. Define $\mu : \mathcal{B}_{[a, b]} \rightarrow [0, \infty]$ by $\mu(E) = m(T(E))$. Observe that for $(x, y] \subseteq [a, b]$, we have $\mu((x, y]) = m(T((x, y])) = m((T(x), T(y))) = T(y) - T(x)$ as T is monotone and continuous. Upon appropriately extending T , we get a Lebesgue-Stieltjes measure $\mu_{\tilde{T}}$ where $\tilde{T} : \mathbb{R} \rightarrow [0, d - c]$ is defined by

$$\tilde{T}(x) := \begin{cases} T(x) - c & \text{if } x \in [a, b] \\ 0 & \text{if } x < a \\ d - c & \text{if } x > b \end{cases}.$$

Then \tilde{T} is absolutely continuous and is in $NBV(\mathbb{R})$. So $\mu_{\tilde{T}} \ll m$. Also $\mu_{\tilde{T}}|_{\mathcal{B}_{[a,b]}} = \mu$. Moreover, $\mu_{\tilde{T}} \circ T^{-1}|_{\mathcal{B}_{[c,d]}} = m|_{\mathcal{B}_{[c,d]}}$ since for $(x, y) \subseteq [c, d]$ we have

$$\mu_{\tilde{T}} \circ T^{-1}((x, y)) = \mu_{\tilde{T}}(T^{-1}((x, y))) = \mu_{\tilde{T}}((T^{-1}(x), T^{-1}(y))) = T(T^{-1}(y)) - T(T^{-1}(x)) = y - x = m((x, y)).$$

Now, by Corollary 14, $\int_{[a,b]} f(T(x))d\mu_{\tilde{T}} = \int_{[c,d]} f(y) \frac{d((\mu_{\tilde{T}}) \circ T^{-1})}{dm} dy$ which implies, by the LRN Theorem that $\int_{[a,b]} f(T(x)) \frac{d\mu_{\tilde{T}}}{dm} dx = \int_{[c,d]} f(y) dy$. By the Fundamental Theorem of Calculus and the LRN Theorem, we find that $\tilde{T}'(x)$ exists for almost every $x \in \mathbb{R}$ and $\frac{d\mu_{\tilde{T}}}{dm} = \tilde{T}'$ almost everywhere. Since $\tilde{T}'(x) = T'(x)$ for almost every $x \in [a, b]$, $\int_{[a,b]} f(T(x))T'(x)dx = \int_{[c,d]} f(y)dy$. \square

Note. This holds if T is decreasing, but then we want $-T'$.

We use a similar idea in higher dimensions for the case where $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a linear transformation. We needed transformations $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$ such that $\mu = m \circ T$ is a measure and $\mu \ll m$. So, in particular, we need $m(T(E)) = 0$ whenever $m(E) = 0$. Recall Theorem 35: If $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a linear transformation, then there is a number $\delta < \infty$ such that $m(T(E)) = \delta m(E)$ for all $E \in \mathcal{L}^n$. If $E = Q_0$, the unit cube, then $m(T(Q_0)) = \delta m(Q_0) = \delta$. Thus $\delta = |\det T|$.

Theorem 49 (Linear Change of Variables Formula - 2.44). *Let $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a bijective linear transformation. If $f \in L^1(m)$, then $f \circ T \in L^1(m)$ and $\int_{\mathbb{R}^n} f(y)dy = |\det T| \int_{\mathbb{R}^n} f(T(x))dx$.*

Proof. Define $\mu : \mathcal{B}_{\mathbb{R}^n} \rightarrow [0, \infty]$ by $\mu(E) = m(T(E))$ for all $E \in \mathcal{B}_{\mathbb{R}^n}$. Since T is a bijection and T^{-1} is continuous, we find that μ is a measure on $\mathcal{B}_{\mathbb{R}^n}$. (check!) Also, if $m(E) = 0$, then $\mu(E) = m(T(E)) = |\det T|m(E) = 0$. Thus $\mu \ll m$. Also, $\mu \circ T^{-1} = m$. Since $\frac{\mu(E)}{m(E)} = |\det T|$ for all $E \in \mathcal{B}_{\mathbb{R}^n}$ with $m(E) \neq 0$, it follows that $\frac{d\mu}{dm} = |\det T|$ (by the theorem on nicely shrinking sets). Now T is a measurable transformation, so by Corollary 14, $\int_{\mathbb{R}^n} f(T(x))d\mu = \int_{\mathbb{R}^n} f(y) \frac{d(\mu \circ T^{-1})}{dm} dy$, which implies $\int_{\mathbb{R}^n} f(T(x)) \frac{d\mu}{dm} dx = \int_{\mathbb{R}^n} f(y)dy$ and thus $|\det T| \int_{\mathbb{R}^n} f(T(x))dx = \int_{\mathbb{R}^n} f(y)dy$. \square

Corollary 15. *If $R : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a rotation or a reflection across the $(n - 1)$ -dimensional plane, then $m(R(E)) = m(E)$ for all $E \in \mathcal{L}^n$.*

Notation. If $G : \Omega \rightarrow \mathbb{R}^n$, for $\Omega \subseteq \mathbb{R}^n$ open, has continuously differentiable components, that is $G = (G_1, \dots, G_n)$ with $G_j \in \mathcal{C}'$, then $D_x G : \Omega \rightarrow \mathbb{R}^{n \times m}$ is given by $[D_x G(x)]_{ij} = \frac{\partial G_j}{\partial x_i}(x)$.

Definition. We say $G : \Omega \rightarrow G(\Omega) \subseteq \mathbb{R}^n$ is a \mathcal{C}' -**diffeomorphism** if G is bijective and both G and G^{-1} are continuously differentiable.

Theorem (Change of Variables Formula - 2.47). *Suppose $\Omega \subseteq \mathbb{R}^n$ is open and $G : \Omega \rightarrow G(\Omega)$ is a \mathcal{C}' -diffeomorphism. Then*

1. *If $f \in L^1(G(\Omega), m)$, then $\int_{G(\Omega)} f(x)dx = \int_{\Omega} f(G(x))|\det D_x G(x)|dx$.*
2. *If $E \subseteq \Omega$ and $E \in \mathcal{L}^n$, then $G(E) \in \mathcal{L}^n$ and $m(G(E)) = \int_E |\det D_x G(x)|dx$.*

2.7.1 Integration in Polar Coordinates

2-dimensions: Set $\Omega := \{(r, \theta) \in \mathbb{R}^2 : r > 0, \theta \in (0, 2\pi)\}$ and $W := \mathbb{R}^2 \setminus \{(r, 0) : r > 0\}$. So Ω and W are open. Now, define $G : \Omega \rightarrow W$ by $(r, \theta) \mapsto (r \cos \theta, r \sin \theta)$. Then G is a \mathcal{C}' -diffeomorphism. Indeed, $G^{-1} : (x, y) \mapsto (\sqrt{x^2 + y^2}, \theta(x, y))$ where

$$\theta(x, y) = \begin{cases} \tan^{-1}(y/x) & x > 0, y > 0 \\ \pi + \tan^{-1}(y/x) & x \leq 0 \\ 2\pi + \tan^{-1}(y/x) & x > 0, y < 0. \end{cases} \quad \text{Also, } |\det D_{(r,\theta)} G(r, \theta)| = r \text{ for all } (r, \theta) \in \Omega. \text{ Suppose } f \in L^1(\mathbb{R}^2, m). \text{ Then the}$$

change of variables formula yields

$$\int_W f(x, y)d(x, y) = \int_{\Omega} f(G(r, \theta))|\det D_{(r,\theta)} G(r, \theta)|d(r, \theta) = \int_{\Omega} f(r \cos \theta, r \sin \theta)rd(r, \theta).$$

Noting that $W = \mathbb{R}^2 \setminus \{0\}$ (a null set) and using Fubini's Theorem, we see

$$\int_{\mathbb{R}^2} f(x, y) d(x, y) = \int_{(0, \infty)} \int_{(0, 2\pi)} f(r \cos \theta, r \sin \theta) r d\theta dr.$$

Higher dimensions: The same formula can be derived in higher dimensions. Set $S^{n-1} := \{x \in \mathbb{R}^n : \|x\| = 1\}$. For $x \in \mathbb{R}^n \setminus \{0\}$, define $r(x) := \|x\|$ and $\theta(x) = \frac{x}{\|x\|} \in S^{n-1}$. It can be verified that the map $G : (0, \infty) \times S^{n-1} \rightarrow \mathbb{R}^n \setminus \{0\}$ defined by $(r, \theta) \mapsto r\theta$ is a \mathcal{C}' -diffeomorphism. (Note that $G^{-1}(x) = (r(x), \theta(x))$ and both components are differentiable). Since G is a bijection and has a continuous inverse, we may define the measure m_* on $\mathcal{B}_{(0, \infty) \times S^{n-1}}$ by $m_*(E) = m(G(E))$. Now, define the measure ρ_n on $\mathcal{B}_{(0, \infty)}$ by $\rho_n(E) = \int_E r^{n-1} dr$. We want to define a measure σ_{n-1} on S^{n-1} such that $m_* = \rho_n \times \sigma_{n-1}$.

Theorem 50 (2.49). *There exists a unique Borel measure σ_{n-1} on $\mathcal{B}_{S^{n-1}}$ such that $m_* = \rho_n \times \sigma_{n-1}$.*

Proof. Let $E \in \mathcal{B}_{S^{n-1}}$. For each $\alpha > 0$, set $E_\alpha := G((0, \alpha] \times E)$. So $E_\alpha = \{r\theta : 0 < r \leq \alpha, \theta \in E\}$. Define $\sigma_{n-1}(E) := nm(E_1)$. Consider the map $\lambda : \mathcal{P}(S^{n-1}) \rightarrow \mathcal{P}(B(1, 0)) \subset \mathcal{P}(\mathbb{R}^n)$. Since λ commutes with unions, intersections, and complements, λ maps Borel sets to Borel sets and thus σ_{n-1} is a Borel measure on S^{n-1} . Also, given $\alpha \in (0, \infty)$, $E_\alpha = T(E_1)$ where $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a linear transformation such that $T(x) = \alpha x$. Thus $m(E_\alpha) = |\det T| m(E_1) = \alpha^n m(E_1)$. For any $(a, b] \subset (0, \infty)$ and $E \in \mathcal{B}_{S^{n-1}}$, we see

$$\begin{aligned} m_*((a, b] \times E) = m(E_b \setminus E_a) &= m(E_b) - m(E_a) \\ &= (b^n - a^n) m(E_1) \\ &= \frac{b^n - a^n}{n} \sigma_{n-1}(E) \\ &= \left(\int_{(a, b]} r^{n-1} dr \right) \sigma_{n-1}(E) \\ &= (\rho_n \times \sigma_{n-1})((a, b] \times E). \end{aligned}$$

Also, $m_*((a, \infty) \times E)$ is 0 when $\sigma_{n-1}(E) = 0$ and ∞ otherwise. This agrees with $\rho_n \times \sigma_{n-1}$ which implies the same formula works. Fix $E \in \mathcal{B}_{S^{n-1}}$. Set $\mathcal{C}_E := \{(a, b] \times E : 0 \leq a \leq b\} \cup \{(a, \infty) : 0 \leq a\}$. Then \mathcal{C}_E is a semialgebra on $(0, \infty) \times E$. Since $m_* = \rho_n \times \sigma_{n-1}$ on \mathcal{C}_E , Caratheodory's Extension process and uniqueness (as m_* is σ -finite on $(0, \infty) \times E$) imply that $m_* = \rho_n \times \sigma_{n-1}$ on the σ -algebra $\mathcal{M}_E = \{A \times E : A \in \mathcal{B}_{(0, \infty)}\}$. By Proposition 35 (1.5), $\mathcal{B}_{(0, \infty) \times S^{n-1}} = \mathcal{B}_{(0, \infty)} \otimes \mathcal{B}_{S^{n-1}}$ and by Proposition 34 (1.7), $\mathcal{B}_{(0, \infty)} \otimes \mathcal{B}_{S^{n-1}}$ is generated by $\{\mathcal{M}_E\}_{E \in \mathcal{B}_{S^{n-1}}}$. Thus Caratheodory's Extension process and uniqueness imply that $m_* = \rho_n \times \sigma_{n-1}$ on $\mathcal{B}_{(0, \infty) \times S^{n-1}}$. \square

Thus by Theorem 50 and the simple function technique, if $f \in L^1(\mathbb{R}^n, m)$, then

$$\int_{\mathbb{R}^n} f(x) dx = \int_{(0, \infty)} \int_{S^{n-1}} f(r\theta) r^{n-1} d\sigma_{n-1}(\theta) dr$$

(as $d\rho = r^{n-1} dr$).

Remarks.

- $\sigma(S^1) = 2\pi$, the circumference of the unit circle.
- For $E \in S^{n-1}$, one can show $\sigma_{n-1}(E) = \mathcal{H}_{n-1}(E)$, the $n - 1$ dimensional Hausdorff measure on \mathbb{R}^n . Thus

$$\int_{\mathbb{R}^n} f(x) dx = \int_{(0, \infty)} \int_{\partial B(1, 0)} f(r\theta) r^{n-1} d\mathcal{H}_{n-1}(\theta) dr.$$

- One can also show $r^{n-1} d\mathcal{H}_{n-1}$ is the $n - 1$ dimensional measure on $\partial B(r, 0)$. So

$$\int_{\mathbb{R}^n} f(x) dx = \int_{(0, \infty)} \int_{\partial B(r, 0)} f(y) d\mathcal{H}_{n-1}(y) dr.$$

- Notice the function $F(s) := \int_{(0, s)} \int_{\partial B(r, 0)} f(y) d\mathcal{H}_{n-1}(y) dr$ is absolutely continuous on $[0, \infty)$. So $F'(s)$ exists almost

everywhere and

$$F'(s) = \frac{d}{ds} \left[\int_{(0,s)} \int_{\partial B(r,0)} f(y) d\mathcal{H}_{n-1} dr \right] = \int_{\partial B(s,0)} f(y) d\mathcal{H}_{n-1}(y).$$

3 More about L^p Spaces

Let (X, \mathcal{M}, μ) be a measure space. For $f : X \rightarrow \mathbb{R}$ such that $|f| \in L^+$, we define for $p \in [1, \infty)$ $\|f\|_p := (\int_X |f|^p d\mu)^{1/p}$ and $\|f\|_\infty := \inf\{a \in \mathbb{R} : \mu(\{x \in X : |f(x)| \geq a\}) = 0\}$. Also, for $p \in [1, \infty]$, we see $L^p(\mu) := \{f : X \rightarrow \mathbb{R} : \|f\|_p < \infty\}$.

Properties of L^p Spaces:

- Banach Space
- L^2 is a Hilbert Space
- Simple functions are dense
- If $\mu = m$, then continuous functions are dense.
- Hölder's Inequality: Suppose $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. Then for measurable functions $f, g : X \rightarrow \mathbb{R}$, we have $\|fg\|_1 \leq \|f\|_p \|g\|_q$. (In particular, if $f \in L^p, g \in L^q$, then $fg \in L^1$.)

3.1 Dual Spaces

Definition. Let $(\mathfrak{X}, \|\cdot\|_{\mathfrak{X}})$ and $(\mathfrak{Y}, \|\cdot\|_{\mathfrak{Y}})$ be normed vector spaces. A linear map is **bounded** if there exists $C \in [0, \infty)$ such that $\|Tx\|_{\mathfrak{Y}} \leq C\|x\|_{\mathfrak{X}}$ for all $x \in \mathfrak{X}$.

Proposition 43 (5.2). If $(\mathfrak{X}, \|\cdot\|_{\mathfrak{X}})$ and $(\mathfrak{Y}, \|\cdot\|_{\mathfrak{Y}})$ are normed vector spaces and $T : \mathfrak{X} \rightarrow \mathfrak{Y}$ is a linear map, then TFAE

- T is continuous on \mathfrak{X} .
- T is continuous at a single point (generally use 0).
- T is bounded.

Proof. (1) \Rightarrow (2) : By definition.

(2) \Rightarrow (3) : There is a $\delta > 0$ such that $\|Tx\|_{\mathfrak{Y}} < 1$ whenever $\|x\|_{\mathfrak{X}} \leq \delta$. If $x \in \mathfrak{X} \setminus \{0\}$, then $\left\| \frac{\delta x}{\|x\|_{\mathfrak{X}}} \right\|_{\mathfrak{X}} = \delta$. Thus $\|Tx\|_{\mathfrak{Y}} = \frac{\|x\|_{\mathfrak{X}}}{\delta} \left\| T \frac{\delta x}{\|x\|_{\mathfrak{X}}} \right\|_{\mathfrak{Y}} \leq \frac{1}{\delta} \|x\|_{\mathfrak{X}}$.

(3) \Rightarrow (1) : There exists c such that $\|Tx\|_{\mathfrak{Y}} \leq c\|x\|_{\mathfrak{X}}$ for all $x \in \mathfrak{X}$. Let $\epsilon > 0$. If $\|x_1 - x_2\|_{\mathfrak{X}} < \frac{\epsilon}{c}$, then $\|Tx_1 - Tx_2\|_{\mathfrak{Y}} = \|T(x_1 - x_2)\|_{\mathfrak{Y}} \leq c\|x_1 - x_2\|_{\mathfrak{X}} < \epsilon$. \square

Definition. If $(\mathfrak{X}, \|\cdot\|_{\mathfrak{X}})$ and $(\mathfrak{Y}, \|\cdot\|_{\mathfrak{Y}})$ are normed vector spaces, then the space of all bounded linear maps from \mathfrak{X} to \mathfrak{Y} is denoted by $L(\mathfrak{X}, \mathfrak{Y})$. The function $\| \cdot \| : L(\mathfrak{X}, \mathfrak{Y}) \rightarrow [0, \infty)$ defined by $\|T\| = \sup\{\|Tx\|_{\mathfrak{Y}} : \|x\|_{\mathfrak{X}} = 1\}$ is called the **operator norm**.

Remarks.

- $(L(\mathfrak{X}, \mathfrak{Y}), \| \cdot \|)$ is a normed vector space.
- $T(\|x\|) = \sup\left\{ \frac{\|Tx\|_{\mathfrak{Y}}}{\|x\|_{\mathfrak{X}}} : x \neq 0 \right\} = \inf\{c \in \mathbb{R} : \|Tx\|_{\mathfrak{Y}} \leq c\|x\|_{\mathfrak{X}} \text{ for all } x \in \mathfrak{X}\}$.

Example. There do exist discontinuous linear maps (other than the obvious $T : \mathfrak{X} \rightarrow +\infty$). Let $\mathfrak{X} = \{\{x_k\}_{k=1}^\infty : x_k \in \mathbb{R}, \sum_{k=1}^\infty kx_k < \infty\}$. Define $\|\cdot\|_1 : \mathfrak{X} \rightarrow [0, \infty)$ by $\|x\|_1 = \sum_{k=1}^\infty |x_k|$ and $\|\cdot\|_2 : \mathfrak{X} \rightarrow [0, \infty)$ by $\|x\|_2 = \sum_{k=1}^\infty k|x_k|$. These are both norms. Define $T : (\mathfrak{X}, \|\cdot\|_1) \rightarrow (\mathfrak{X}, \|\cdot\|_2)$ by $Tx = x$. However, $\|T\| = \sup\{\|Tx\|_2 : \|x\|_1 = 1\}$. Define $x^{(n)} \in \mathfrak{X}$ by $x^{(n)} = (0, \dots, 0, 1, 0, \dots)$ where 1 appears in the n^{th} spot. Then $\|x^{(n)}\|_1 = 1$ but $\|x^{(n)}\|_2 = n$. Thus $\|T\| \geq \|x^{(n)}\|_2 = n$ for all n which implies $\|T\| = \infty$. Thus T is unbounded and discontinuous.

Proposition 44 (5.4). *If $(\mathfrak{Y}, \|\cdot\|_{\mathfrak{Y}})$ is complete, then so is $(L(\mathfrak{X}, \mathfrak{Y}), \|\cdot\|)$.*

Definition. *If $(\mathfrak{X}, \|\cdot\|_{\mathfrak{X}})$ is a normed vector space, $(L(\mathfrak{X}, \mathbb{R}), \|\cdot\|)$ is called the (**continuous**) **dual space** of \mathfrak{X} . It is denoted by \mathfrak{X}^* or $(\mathfrak{X}^*, \|\cdot\|_{\mathfrak{X}^*})$. The members of $L(\mathfrak{X}, \mathbb{R})$ are called **linear functionals**.*

Remark. By Proposition 44, \mathfrak{X}^* is complete.

Definition. *If \mathfrak{X} is a vector space, we see that $p : \mathfrak{X} \rightarrow \mathbb{R}$ is a **sublinear functional** if $p(x+y) \leq p(x) + p(y)$ and $p(\lambda x) = \lambda p(x)$ for $\lambda \geq 0$ and for all $x, y \in \mathfrak{X}$.*

Theorem (Hahn Banach Theorem). *Let \mathfrak{X} be a vector space over \mathbb{R} , p a sublinear functional on \mathfrak{X} and \mathcal{M} a linear subspace of \mathfrak{X} . If $f : \mathcal{M} \rightarrow \mathbb{R}$ is a linear functional such that $f(x) \leq p(x)$ for all $x \in \mathcal{M}$. Then there exists a linear functional $\bar{f} : \mathfrak{X} \rightarrow \mathbb{R}$ such that $\bar{f}(x) \leq p(x)$ for all $x \in \mathfrak{X}$ and $\bar{f}|_{\mathcal{M}} = f$.*

Proof. WLOG, assume $\mathcal{M} \subseteq \mathfrak{X}$. Let $x \in \mathcal{M}^C$. Want to extend f to $\mathcal{M}_x = \{y + \alpha x : y \in \mathcal{M}, \alpha \in \mathbb{R}\}$. To do this, we want to define $f_x : \mathcal{M}_x \rightarrow \mathbb{R}$ such that f_x is linear on \mathcal{M}_x , $f_x(y) \leq p(y)$ for all $y \in \mathcal{M}_x$ and $f_x|_{\mathcal{M}} = f$. Suppose there exists $\beta \in \mathbb{R}$ such that $\alpha\beta \leq p(\alpha x + y) - f(y)$ for all $\alpha \in \mathbb{R}, y \in \mathcal{M}$. (*) Then we could define $f_x(\alpha x + y) = \alpha\beta + f(y)$ for all $\alpha \in \mathbb{R}, y \in \mathcal{M}$ and f_x would be the desired extension to \mathcal{M}_x . For each $y_1, y_2 \in \mathcal{M}$, we have

$$\begin{aligned} f(y_1) + f(y_2) &= f(y_1 + y_2) \leq p(y_1 + y_2 - x + x) \leq p(y_1 - x) + p(y_2 + x) \\ &\Rightarrow f(y_1) - p(y_1 - x) \leq p(y_2 + x) - f(y_2) \end{aligned}$$

Since $y_1, y_2 \in \mathcal{M}$ were arbitrary, we have

$$\sup\{f(y) - p(y - x) : y \in \mathcal{M}\} \leq \inf\{p(y + x) - f(y) : y \in \mathcal{M}\}.$$

So select β such that $\sup\{f(y) - p(y - x) : y \in \mathcal{M}\} \leq \beta \leq \inf\{p(y + x) - f(y) : y \in \mathcal{M}\}$. If $\alpha = 0$, then β satisfies (*). If $\alpha > 0$, then

$$\begin{aligned} \alpha\beta &\leq \alpha \inf\{p(y + x) - f(y) : y \in \mathcal{M}\} \\ &= \inf\{\alpha p(y + x) - \alpha f(y) : y \in \mathcal{M}\} \\ &= \inf\{p(\alpha y + \alpha x) - f(\alpha y) : y \in \mathcal{M}\} \\ &= \inf\{p(\alpha y + \alpha x) - f(\alpha y) : \alpha y \in \mathcal{M}\} \\ &= \inf\{p(y + \alpha x) - f(y) : y \in \mathcal{M}\} \end{aligned}$$

and lastly, if $\alpha < 0$, then

$$\begin{aligned} \alpha\beta &\leq \alpha \sup\{f(y) - p(y - x) : y \in \mathcal{M}\} \\ &= (-\alpha) \inf\{p(y - x) - f(y) : y \in \mathcal{M}\} \\ &= \inf\{p(-\alpha y + \alpha x) - f(-\alpha y) : y \in \mathcal{M}\} \\ &= \inf\{p(y + \alpha x) - f(y) : y \in \mathcal{M}\}. \end{aligned}$$

Thus β satisfies (*) and we have thus extended f to \mathcal{M}_x . Now, consider \mathcal{F} , the collection of all pairs $(f|_V, V)$ where V is a linear subspace of \mathfrak{X} containing \mathcal{M} and f_V is a linear functional on V such that $f_V|_{\mathcal{M}} = f$ and $f_V(y) \leq p(y)$ for all $y \in V$. Define a partial order \leq on \mathcal{F} by $(f_{V_1}, V_1) \leq (f_{V_2}, V_2)$ if $V_1 \subset V_2$ and $f_{V_2}|_{V_1} = f_{V_1}$. Note that $(f_x, \mathcal{M}_x) \in \mathcal{F}$ and thus it is non empty. Let $\mathcal{G} \subseteq \mathcal{F}$ be a totally ordered subset of \mathcal{F} . Since $W = \cup_{(f_V, V) \in \mathcal{G}} V$ is a linear subspace of \mathfrak{X} , we see $(f_W, W) \in \mathcal{F}$ and is an upper bound for the chain. Thus, by Zorn's Lemma, there exists a maximal element, call it (\bar{f}, V) . If $V \neq \mathfrak{X}$, then there exists $x \in \mathfrak{X} \setminus V$ and \bar{f}_x extends \bar{f} , a contradiction to maximality. \square

Example. (Generalized/Banach limits). Set $\ell^\infty = \{\{x_k\}_1^\infty \subset \mathbb{R} : \sup |x_k| < \infty\}$ and $\mathcal{M} = \{x \in \ell^\infty : \lim_{k \rightarrow \infty} x_k \text{ exists}\}$. So \mathcal{M} is a linear subspace. Consider the linear function $L_0 : \mathcal{M} \rightarrow \mathbb{R}$ defined by $x \mapsto \lim_{k \rightarrow \infty} x_k$. Define $p : \ell^\infty \rightarrow \mathbb{R}$ by $x \mapsto \limsup_{k \rightarrow \infty} \frac{1}{k} \sum_{j=1}^k x_j$. So p is a sublinear function on ℓ^∞ . We can verify $L_0(x) = p(x)$ for $x \in \mathcal{M}$. So by the Hahn Banach Theorem, there exists a linear function $\bar{L} : \ell^\infty \rightarrow \mathbb{R}$ such that $\bar{L}|_{\mathcal{M}} = L_0$ and $\bar{L}(x) \leq p(x)$ for $x \in \ell^\infty$. Since $p(x) \leq \limsup_{k \rightarrow \infty} x_k$, we have

$$\liminf x_k = -\limsup(-x_k) \leq -p(-x) \leq -\bar{L}(-x) = \bar{L}(x) \leq p(x) \leq \limsup x_k.$$

Also, if we define, for each $x \in \ell^\infty$, the sequence $x^{(n)} \in \ell^\infty$ by $x_k^{(n)} = x_{k+n}$, then you can verify $\overline{L}(x^{(n)}) = \overline{L}(x)$.

Theorem 51 (5.8). *Let $(\mathfrak{X}, \|\cdot\|_{\mathfrak{X}})$ be a normed vector space.*

1. *If \mathcal{M} is a closed subspace of \mathfrak{X} and $x \in \mathfrak{X} \setminus \mathcal{M}$, then there exists $f \in \mathfrak{X}^*$ such that $f(x) \neq 0$ and $f|_{\mathcal{M}} = 0$. Moreover, $\|f\|_{\mathfrak{X}^*} = 1$ and $f(x) = \inf_{y \in \mathcal{M}} \|x - y\|_{\mathfrak{X}}$.*
2. *If $x \in \mathfrak{X} \setminus \{0\}$, then there exists $f \in \mathfrak{X}^*$ such that $\|f\|_{\mathfrak{X}^*} = 1$ and $f(x) = \|x\|_{\mathfrak{X}}$.*
3. *Bounded linear functionals in \mathfrak{X}^* separate points in \mathfrak{X} , that is, if $x_1, x_2 \in \mathfrak{X}$ and $x_1 \neq x_2$, then there exists $f \in \mathfrak{X}^*$ such that $f(x_1) \neq f(x_2)$.*
4. *For each $x \in \mathfrak{X}$, define $\hat{x} : \mathfrak{X}^* \rightarrow \mathbb{R}$ by $\hat{x}(f) = f(x)$. The map $x \mapsto \hat{x}$ is a linear isometry from \mathfrak{X} into \mathfrak{X}^{**} , that is, $\|x\|_{\mathfrak{X}} = \|\hat{x}\|_{\mathfrak{X}^{**}} = \sup\{|\hat{x}(f)| : \|f\|_{\mathfrak{X}^*} = 1\} = \sup\{|f(x)| : \|f\|_{\mathfrak{X}^*} = 1\}$.*

Proof. 1. Let $\mathcal{M}_x = \{y \in \mathfrak{X} : y = z + \lambda x \text{ for some } z \in \mathcal{M}, \lambda \in \mathbb{R}\}$ and define $f : \mathcal{M}_x \rightarrow \mathbb{R}$ by $f(z + \lambda x) = \lambda \inf_{y \in \mathcal{M}} \|x - y\|_{\mathfrak{X}} - f(z)$. So $f(x) = \inf_{y \in \mathcal{M}} \|x - y\|_{\mathfrak{X}}$, $f|_{\mathcal{M}} = 0$ and f is linear on \mathcal{M}_x :

$$\text{Let } a, b \in \mathbb{R}, y_1, y_2 \in \mathcal{M}_x. \text{ Say } y_i = z_i + \lambda_i x. \text{ Then } f(ay_1 + by_2) = f(az_1 + bz_2 + a\lambda_1 x + b\lambda_2 x) = (a\lambda_1 + b\lambda_2) \inf_{y \in \mathcal{M}} \|x - y\|_{\mathfrak{X}} + af(y_1) + bf(y_2).$$

For all $\lambda \neq 0$, we see $|f(z + \lambda x)| = |\lambda| \inf_{y \in \mathcal{M}} \|x - y\|_{\mathfrak{X}} \leq |\lambda| \|x + \lambda^{-1}z\|_{\mathfrak{X}} = \|\lambda x + z\|_{\mathfrak{X}}$ (take $y = -\lambda^{-1}z$). So f is linear on \mathcal{M} and $f(y) \leq \|y\|_{\mathfrak{X}}$ for all $y \in \mathcal{M}_x$. Thus by the Hahn Banach Theorem, there exists an extension $\bar{f} \in \mathfrak{X}^*$ that is linear and satisfies

$$\bar{f}|_{\mathcal{M}_x} = f, \bar{f}(x) = \inf_{y \in \mathcal{M}} \|x - y\|_{\mathfrak{X}} \text{ and } \|f\|_{\mathfrak{X}^*} \leq 1.$$

since \mathcal{M} is closed, $f(x) > 0$. Let $\epsilon > 0$. Then there exists $y^* \in \mathcal{M}$ such that $\|x - y^*\|_{\mathfrak{X}} \leq \inf_{y \in \mathcal{M}} \|x - y\|_{\mathfrak{X}} + \epsilon$. Since $x - y^* \in \mathcal{M}_x$, we see

$$\bar{f}(x - y^*) = f(x - y^*) = \inf_{y \in \mathcal{M}} \|x - y\|_{\mathfrak{X}} \geq \|x - y^*\|_{\mathfrak{X}} - \epsilon.$$

Thus

$$\bar{f} \left(\frac{x - y^*}{\|x - y^*\|_{\mathfrak{X}}} \right) \geq \frac{\|x - y^*\|_{\mathfrak{X}} - \epsilon}{\|x - y^*\|_{\mathfrak{X}}} \geq 1 - \frac{\epsilon}{\inf_{y \in \mathcal{M}} \|x - y\|_{\mathfrak{X}}}.$$

Thus $\|f\|_{\mathfrak{X}^*} = \sup\{|f(x)| : \|x\|_{\mathfrak{X}} = 1\} \geq 1$. Hence $\|f\|_{\mathfrak{X}^*} = 1$.

2. Take $\mathcal{M} = \{0\}$ and apply 1.
3. Since $x_1 \neq x_2$, we have $x_1 - x_2 \in \mathfrak{X} \setminus \{0\}$. Now apply 2.
4. If $f, g \in \mathfrak{X}^*$ and $a, b \in \mathbb{R}$, then $\hat{x}(af + bg) = (af + bg)(x) = af(x) + bg(x) = a\hat{x}(f) + b\hat{x}(g)$. So \hat{x} is a linear functional on \mathfrak{X}^* . Moreover, if $x_1, x_2 \in \mathfrak{X}$ and $a, b \in \mathbb{R}$, then $\widehat{ax_1 + bx_2}(f) = f(ax_1 + bx_2) = af(x_1) + bf(x_2) = a\hat{x}_1(f) + b\hat{x}_2(f)$. Thus $x \mapsto \hat{x}$ is a linear map from $\mathfrak{X} \rightarrow \mathfrak{X}^{**}$. Now for all $f \in \mathfrak{X}^*$, we have $|\hat{x}(f)| = |f(x)| \leq \|f\|_{\mathfrak{X}^*} \|x\|_{\mathfrak{X}} = \|x\|_{\mathfrak{X}}$ if $\|f\|_{\mathfrak{X}^*} = 1$. Thus $\|\hat{x}\|_{\mathfrak{X}^{**}} \leq \|x\|_{\mathfrak{X}}$. By 2, there exists $f \in \mathfrak{X}^*$ such that $\|f\|_{\mathfrak{X}^*} = 1$ and $f(x) = \|x\|_{\mathfrak{X}}$. So, for this particular f , we see $|\hat{x}(f)| = |f(x)| = \|x\|_{\mathfrak{X}}$. Then $\|\hat{x}\|_{\mathfrak{X}^{**}} \geq \|x\|_{\mathfrak{X}}$ and so $\|\hat{x}\|_{\mathfrak{X}^{**}} = \|x\|_{\mathfrak{X}}$. \square

Remarks.

1. \mathfrak{X}^{**} is complete.
2. Define $\hat{\mathfrak{X}} := \{\hat{x} \in \mathfrak{X}^{**} | x \in \mathfrak{X}\}$. Then $\hat{\mathfrak{X}}$ is a subspace of \mathfrak{X}^{**} . Since we've shown $x \mapsto \hat{x}$ is a linear isometry, we can identify $\hat{\mathfrak{X}}$ with \mathfrak{X} so that $\mathfrak{X} \hookrightarrow \mathfrak{X}^{**}$. By definition, \mathfrak{X} is a dense subspace of $\overline{\hat{\mathfrak{X}}}$ (the closure of $\hat{\mathfrak{X}}$). Thus $\overline{\hat{\mathfrak{X}}}$ has to be a subset of \mathfrak{X}^{**} as \mathfrak{X}^{**} is complete. Call $\overline{\hat{\mathfrak{X}}}$ the completion of \mathfrak{X} .
3. \mathfrak{X} is called **reflexive** if $\mathfrak{X} \rightarrow \mathfrak{X}^{**}$ defined by $x \mapsto \hat{x}$ is surjective as well. It is standard to identify \hat{x} with x itself.

Definition. (p 125) A **directed set** is a nonempty set A with a relation \lesssim such that

- $\alpha \lesssim \alpha$ for all $\alpha \in A$.
- $\alpha \lesssim \beta$ and $\beta \lesssim \gamma$ implies $\alpha \lesssim \gamma$.
- If $\alpha, \beta \in A$, then there exists $\gamma \in A$ such that $\alpha \lesssim \gamma$ and $\beta \lesssim \gamma$.

An element of A is called an **index**.

Examples.

- Any nonempty subset of \mathbb{R} with the usual order relation is a directed set. In particular, \mathbb{N} is a directed set.
- Let \mathcal{B} be a neighborhood basis for a topology \mathcal{T} on X , that is $\mathcal{B} \subseteq \mathcal{T}$ and for all $x \in X$ there exists $\mathcal{N} \in \mathcal{B}$ such that $x \in V$ for all $V \in \mathcal{N}$ and if $U \in \mathcal{T}$ and $x \in U$, then there exists $V \in \mathcal{N}$ such that $V \subseteq U$. Set $\mathcal{N}_x := \{U \in \mathcal{B} : x \in U\}$ with $x \in X$ fixed. If we say for $U, V \in \mathcal{N}_x$ that $U \lesssim V$ when $U \supseteq V$, then \mathcal{N}_x is a directed set. (Note: If $U, V \in \mathcal{N}_x$, then $x \in U \cap V \in \mathcal{T}$ which implies there exists $W \in \mathcal{B}$ with $x \in W$ such that $W \subseteq U \cap V$. So $W \gtrsim U$ and $W \gtrsim V$.)

Definition. Let V be a set. A **net** in X is a function from a directed set A into X . We denote the mapping $\alpha \mapsto x_\alpha$ by $\langle x_\alpha \rangle_{\alpha \in A}$. The set A is called the **index set**.

Definition. Let (X, \mathcal{T}) be a topological space and $E \subseteq X$. Let $\langle x_\alpha \rangle_{\alpha \in A}$ be a net. Then

- $\langle x_\alpha \rangle_{\alpha \in A}$ is **eventually in** E if there exists α_0 such that $x_\alpha \in E$ for all $\alpha \gtrsim \alpha_0$.
- $\langle x_\alpha \rangle_{\alpha \in A}$ is **frequently in** E if for every $\alpha \in A$, there exists $\beta \in A$ such that $\beta \gtrsim \alpha$ and $x_\beta \in E$.
- $\langle x_\alpha \rangle_{\alpha \in A}$ converges to a point $x \in X$ (that is, $x_\alpha \rightarrow x$) if for all neighborhoods U of x , $\langle x_\alpha \rangle_{\alpha \in A}$ is eventually in U .
- A point $x \in X$ is a **cluster point** of $\langle x_\alpha \rangle_{\alpha \in A}$ if for all neighborhoods U of x $\langle x_\alpha \rangle_{\alpha \in A}$ is frequently in U .

Examples.

- Let \mathcal{B} be a neighborhood basis for \mathcal{T} and set $\mathcal{N}_x = \{U \in \mathcal{B} : x \in U\}$. Suppose $\langle x_U \rangle_{U \in \mathcal{N}_x} \subseteq X$ satisfies $x_U \in U$ for all $U \in \mathcal{N}_x$. Then $x_U \rightarrow x$.
- Let $f \in L^1([a, b]) \cap L^+$. Let S be the set of all nonnegative simple functions on $[a, b]$ that are dominated by f . Order S with the usual ordering of functions (that is, $\phi_1 \lesssim \phi_2$ if $\phi_1(x) \leq \phi_2(x)$ for all $x \in [a, b]$). So S is a directed set. For $\phi \in S$, put $y_\phi = \int_{[a, b]} \phi dx$. Then $\langle y_\phi \rangle_{\phi \in S}$ is a net and $y_\phi \mapsto \int_{[a, b]} f dx$.

Definition. A **subnet** of a net $\langle x_\alpha \rangle_{\alpha \in A}$ is a net $\langle y_\beta \rangle_{\beta \in B}$ together with a map $\beta \mapsto \alpha_\beta$ from B into A such that

- For all $\alpha_0 \in A$ there exists $\beta_0 \in B$ such that $\alpha_\beta \gtrsim \alpha_0$ whenever $\beta \gtrsim \beta_0$.
- $y_\beta = x_{\alpha_\beta}$.

Note. The map $\beta \mapsto \alpha_\beta$ need not be injective.

Proposition 45 (4.18). If (X, \mathcal{T}) is a topological space and $E \subseteq X$, then $x \in X$ is an **accumulation point** of E if and only if there exists a net $\langle x_\alpha \rangle_{\alpha \in A} \subseteq E \setminus \{x\}$ that converges to x and $x \in \overline{E}$ if and only if there exists $\langle x_\alpha \rangle_{\alpha \in A} \subseteq E$ that converges to x .

Proposition 46 (4.19). If (X, \mathcal{T}) and (Y, \mathcal{S}) are topological spaces and $f : X \rightarrow Y$, then f is continuous at $x \in X$ if and only if for all nets $\langle x_\alpha \rangle_{\alpha \in A}$ converging to x , we have $\langle f(x_\alpha) \rangle_{\alpha \in A} \rightarrow f(x)$.

Proof. (Sketch)

Claim: Let $\mathcal{N} := \{\bigcap_{f \in \mathcal{D}} f^{-1}(W_f) : W_f \text{ is open in } Y_f, \mathcal{D} \text{ is a finite subset of } \mathcal{F}\}$. Then the topology induced by \mathcal{N} is the weak topology and \mathcal{N} is a neighborhood basis.

Proof: Suppose $\langle f(x_\alpha) \rangle$ converges to $f(x)$. Let $\mathcal{O} \in \mathcal{T}$ be an open set such that $x \in \mathcal{O}$. There exists a finite collection $f_n \in \mathcal{F}$ and $W_{f_j} \in \mathcal{T}_f$ and $f(x) \in W_f$ such that $\bigcap_{j=1}^n f_j^{-1}(W_{f_j}) \subseteq \mathcal{O}$. Thus $x \in \bigcap_{j=1}^n f_j^{-1}(W_{f_j})$. Since $f_j(x_\alpha) \rightarrow f(x)$ for all $j = 1, \dots, n$ there exists α_j such that $\alpha \succ \alpha_j$ and $f_j(x_\alpha) \in W_{f_j}$. Note the α 's are in a directed set. So there exists $\alpha_0 \succ \alpha_j$ for all $j = 1, \dots, n$. Thus for all $\alpha \succ \alpha_0$, we have $f_j(x_\alpha) \in W_{f_j}$ which implies $x_\alpha \in f_j^{-1}(W_{f_j})$. Thus $x_\alpha \in \bigcap_{j=1}^n f_j^{-1}(W_{f_j}) \subseteq \mathcal{O}$. Since \mathcal{O} was arbitrary, done. \square

Definition. Let X be a set and $\{(Y_\alpha, \mathcal{T}_\alpha)\}_{\alpha \in A}$ a family of topological spaces. Given a family $\{f_\alpha : X \rightarrow Y_\alpha\}_{\alpha \in A}$, there exists a unique smallest topology that makes each f_α continuous. Call this the **weak topology** generated by $\{f_\alpha\}_{\alpha \in A}$.

Proposition 47. Let (X, \mathcal{T}) be a topological space and $\langle x_\alpha \rangle_{\alpha \in A} \subseteq X$ a net.

1. If \mathcal{T} is induced by a metric ρ on X , then $\langle x_\alpha \rangle \rightarrow x \in X$ if and only if $\langle \rho(x_\alpha, x) \rangle \rightarrow 0 \in \mathbb{R}$.
2. If the topological space is the weak topology generated by a family of functions $\mathcal{F} \subseteq \{f : X \rightarrow (Y, \rho)\}$, then $\langle x_\alpha \rangle \rightarrow x \in X$ if and only if $\langle f(x_\alpha) \rangle \rightarrow f(x) \in Y$ for all $f \in \mathcal{F}$.

Definition. Let $(\mathfrak{X}, \|\cdot\|)$ be a normed vector space. The **weak topology** on \mathfrak{X} is the weak topology generated by \mathfrak{X}^* . Convergence in this topology is called **weak convergence**.

Remark. If $\langle x_\alpha \rangle_{\alpha \in A} \subseteq \mathfrak{X}$ is a net, then we say $x_\alpha \rightarrow x$ strongly if and only if $\|x_\alpha - x\| \rightarrow 0$ and we say $x_\alpha \rightarrow x$ weakly (denoted $x_\alpha \rightarrow x$) if and only if $f(x_\alpha) \rightarrow f(x)$ for all $f \in \mathfrak{X}^*$.

Recall. $\mathfrak{X} \subseteq \mathfrak{X}^{**}$, so each $x \in \mathfrak{X}$ is a linear functional on \mathfrak{X}^* .

Definition. Let $(\mathfrak{X}, \|\cdot\|)$ be a normed vector space. The **weak* topology** on \mathfrak{X}^* is the weak topology generated by \mathfrak{X} . Convergence in this topology is called **weak* convergence**.

Remark. If $\langle f_\alpha \rangle_{\alpha \in A} \subseteq \mathfrak{X}^*$ is a net, then we say $f_\alpha \rightarrow f$ strongly if and only if $\|f_\alpha - f\|_{\mathfrak{X}^*} \rightarrow 0$ and we say $f_\alpha \rightarrow f$ weakly (denoted $f_\alpha \rightarrow^* f$) if and only if $f_\alpha(x) \rightarrow f(x)$ for all $x \in \mathfrak{X}$ (note that this is just like pointwise convergence).

Theorem (Alaoglu's Theorem). If $(\mathfrak{X}, \|\cdot\|)$ is a normed vector space, then the closed unit ball $\overline{B}^* := \{f \in \mathfrak{X}^* : \|f\| \leq 1\} \subseteq \mathfrak{X}^*$ is compact in the weak* topology.

Definition. Let $\{(X_\alpha, \mathcal{T}_\alpha)\}_{\alpha \in A}$ be a family of topological spaces. The **product topology** on $X = \prod_{\alpha \in A} X_\alpha$ is the weak topology generated by the coordinate maps $\{\pi_\alpha : X \rightarrow X_\alpha\}$.

Definition. A topological space (X, \mathcal{T}) is **compact** if whenever $\{U_\alpha\}_{\alpha \in A}$ is an open cover of X there exists a finite subset $B \subseteq A$ such that $X = \bigcup_{\alpha \in B} U_\alpha$. A subset $K \subseteq X$ is called **compact** if it is compact with respect to the relative topology on K .

Theorem (Tychonoff's Theorem. 4.42). If $\{(X_\alpha, \mathcal{T}_\alpha)\}_{\alpha \in A}$ is a family of compact topological spaces, then $X = \prod_{\alpha \in A} X_\alpha$ is compact with respect to the product topology.

Corollary 16. Suppose X has the weak topology generated by a family of functions \mathcal{F} and that the following hold:

1. $\overline{f(X)}$ is compact for all $f \in \mathcal{F}$
2. If $x \neq y$, then there exists $f \in \mathcal{F}$ such that $f(x) \neq f(y)$.
3. If $\langle f(x_\alpha) \rangle_{\alpha \in A}$ is a convergent net for all $f \in \mathcal{F}$, then there exists $x \in X$ such that $f(x_\alpha) \rightarrow f(x)$ for all $f \in \mathcal{F}$.

Then X is compact.

Proof. Each $f \in \mathcal{F}$ is a map into some topological space (Y_f, \mathcal{T}_f) . Condition 1 states that $\overline{f(X)}$ is a compact subset of Y_f for each $f \in \mathcal{F}$. Tychonoff's Theorem shows that $Z = \prod_{f \in \mathcal{F}} \overline{f(X)}$ is compact with respect to the product topology. Suppose there exists a map $h : X \rightarrow Z$ such that

- $h(X)$ is closed

- h is a continuous bijection from X to $h(X)$.
- h^{-1} is continuous on $h(X)$.

(So h is a homeomorphism from X to $h(X)$). Then if $\{U_\alpha\}_{\alpha \in A}$ is an open cover of X , then we have $\{h(U_\alpha)\}_{\alpha \in A}$ is an open cover of $h(X)$. Since $h(X)$ is a closed subset of the compact set Z , it is compact. So there exists a finite subcover $\{h(U_\alpha)\}_{\alpha \in B}$ for $h(X)$. Thus $\{U_\alpha\}_{\alpha \in B}$ is a finite subcover of X as $X = h^{-1}(h(X)) \subseteq h^{-1}(\cup_{\alpha \in B} h(U_\alpha)) = \cup_{\alpha \in B} U_\alpha$. Thus X is compact.

Define $h : X \rightarrow Z$ by $h_f(x) = f(x)$ (the f component of h) for all $f \in \mathcal{F}, x \in X$. To show $h(X)$ is closed, let $\langle y_\alpha \rangle_{\alpha \in A} \subseteq h(X)$ be a convergent net. Then there exists a net $\langle x_\alpha \rangle_{\alpha \in A} \subseteq X$ such that $h(x_\alpha) = y_\alpha$ for all $\alpha \in A$. Now $\langle f(x_\alpha) \rangle = \langle h_f(x_\alpha) \rangle$ is a convergent net (otherwise $\langle y_\alpha \rangle$ does not converge). By condition 3, there exists $x \in X$ such that $f(x_\alpha) \rightarrow f(x)$ for all $f \in \mathcal{F}$. But this implies $h(x_\alpha) \rightarrow h(x)$ and thus $y_\alpha \rightarrow h(x) \in h(X)$. By Proposition 45 (4.18), $h(X)$ is closed. We see that h is continuous, since each f is continuous and each coordinate map $\pi_f : Z \rightarrow \overline{f(X)}$ is continuous. Now $h : X \rightarrow h(X)$ is surjective. So we need only show it is 1-1. By the contrapositive of condition 2, this is clear. Thus h^{-1} exists. To show it is continuous on $h(X)$, we will show for a convergent net $\langle y_\alpha \rangle_{\alpha \in A}$, that $h^{-1}(y_\alpha) \rightarrow h^{-1}(y)$, where $y_\alpha \rightarrow y$. Let $x_\alpha = h^{-1}(y_\alpha)$. So $\langle h(x_\alpha) \rangle = \langle y_\alpha \rangle$ is convergent. By condition 3, there exists $x \in X$ such that $h(x_\alpha) \rightarrow h(x)$. Then $h(x) = y$ since $h(x_\alpha) = y_\alpha \rightarrow y$. Thus $h^{-1}(y) = x$. By Proposition 47b, $\langle x_\alpha \rangle_{\alpha \in A}$ is convergent in X if and only if $\langle h(x_\alpha) \rangle_{\alpha \in A}$ is convergent in $h(X)$. Since $h(x_\alpha) \rightarrow h(x)$, we see $x_\alpha \rightarrow x$ by the bijectivity of h . Thus $h^{-1}(y_\alpha) = x_\alpha \rightarrow x = h^{-1}(y)$. Thus h^{-1} is continuous. \square

Proof. (Of Alaoglu's Theorem) We want to use Corollary 16 with $X = \overline{B}^*$ and the relative weak* topology. Also, \mathfrak{X} is the family \mathcal{F} in the corollary (so we replace the x 's with $f \in \mathfrak{X}^*$ and the f 's with $x \in \mathfrak{X}$). Now, we just need to verify the three conditions of the corollary hold.

Condition 1: Observe for $f \in \overline{B}^*$ and $x \in \mathfrak{X}$ that $|x(f)| = |f(x)| \leq \|f\|_{\mathfrak{X}^*} \|x\|_{\mathfrak{X}} \leq \|x\|_{\mathfrak{X}}$. Thus $x(\overline{B}^*) \subseteq [-\|x\|_{\mathfrak{X}}, \|x\|_{\mathfrak{X}}]$ which implies $x(\overline{B}^*)$ is compact.

Condition 2: Clearly, if $x(f) = x(g)$ for all $x \in \mathfrak{X}$, then $f = g$.

Condition 3: Let $\langle f_\alpha \rangle_{\alpha \in A} \subseteq \overline{B}^*$ be given and suppose $\langle x(f_\alpha) \rangle$ is convergent for all $x \in \mathfrak{X}$. Then for all $x \in \mathfrak{X}$, there exists $\ell(x)$ such that $f_\alpha(x) \rightarrow \ell(x)$. We need to show $\ell \in \overline{B}^*$. Let $\beta, \gamma \in \mathbb{R}, x, y \in \mathfrak{X}$. Then $f_\alpha(\beta x + \gamma y) = \beta f_\alpha(x) + \gamma f_\alpha(y) \rightarrow \beta \ell(x) + \gamma \ell(y)$ and $f_\alpha(\beta x + \gamma y) \rightarrow \ell(\beta x + \gamma y)$. Thus $\beta \ell(x) + \gamma \ell(y) = \ell(\beta x + \gamma y)$. Thus ℓ is linear. Now, we show $\|\ell\|_{\mathfrak{X}^*} \leq 1$. We see $|f_\alpha(x)| \leq \|f_\alpha\|_{\mathfrak{X}^*} \|x\|_{\mathfrak{X}} \leq \|x\|_{\mathfrak{X}}$. Thus $|f_\alpha(x)| \rightarrow |\ell(x)| \leq \|x\|_{\mathfrak{X}}$ for all $x \in \mathfrak{X}$ and so $\|\ell\|_{\mathfrak{X}^*} \leq 1$. Therefore $\ell \in \overline{B}^*$. Hence $x(f_\alpha) = f_\alpha(x) \rightarrow \ell(x) = x(\ell)$ for all $x \in \mathfrak{X}$.

Thus, by Corollary 16, \overline{B}^* is compact. \square

Corollary 17. Suppose $\langle f_\alpha \rangle \subseteq \mathfrak{X}^*$ is a net such that $\sup \|f\|_{\mathfrak{X}^*} \leq M$ for some $M < \infty$. Then there exists a subnet $\langle g_\beta \rangle$ of $\langle f_\alpha \rangle$ that is weak* convergent.

Proof. The net $\langle \frac{1}{M} f_\alpha \rangle \subseteq \overline{B}^*$, which is compact by Alaoglu's Theorem. It follows that there exists a subnet $\langle \frac{1}{M} g_\beta \rangle$ of $\langle \frac{1}{M} f_\alpha \rangle$ converging weak* in \overline{B}^* . \square

Application (Sect 6.2) Since $L^p(\mu)$ is a Banach Space, Alaoglu's Theorem implies the closed unit ball in $[L^p]^*$ is weak* compact. In particular, bounded nets in $[L^p]^*$ have weak* convergent subnets. What is $[L^p]^*$?

- Suppose $p \geq 1$ and $\frac{1}{p} + \frac{1}{q} = 1$ (if $p = 1$, take $q = \infty$). For each $g \in L^q$, define $\phi_g : L^p(\mu) \rightarrow \mathbb{R}$ by $\phi_g(f) = \int_X gf \, d\mu$. This is a linear functional. To show $\phi_g \in [L^p]^*$, use Hölder's Inequality:

$$\begin{aligned} \|\phi_g\|_{[L^p]^*} &= \sup\{|\int_X fg \, d\mu| : f \in L^p, \|f\|_p = 1\} \\ &\leq \sup\{\int_X |fg| \, d\mu : f \in L^p, \|f\|_p = 1\} \\ &\leq \sup\{\|g\|_q \|f\|_p : f \in L^p, \|f\|_p = 1\} \\ &= \|g\|_q < \infty. \end{aligned}$$

Proposition 48. Suppose $\frac{1}{p} + \frac{1}{q} = 1$ and $1 < p \leq \infty$. If $g \in L^q$, then $\|g\|_{L^q} = \|\phi_g\|_{[L^p]^*} = \sup\{|\int_X fg d\mu| : \|f\|_p = 1\}$. If μ is semifinite, then we can allow for $p = 1$.

Recall. μ is semifinite if for all E with $\mu(E) = \infty$, there exists $F \subseteq E$ such that $0 < \mu(F) < \infty$.

Proof. If $g = 0$, done. By Hölder's Inequality, $\|\phi_g\|_{[L^p]^*} \leq \|g\|_{L^q}$. Define $\text{sgn}(g(x)) = \begin{cases} \frac{g(x)}{|g(x)|} & \text{if } g(x) \neq 0. \\ 0 & \text{if } g(x) = 0. \end{cases}$ If $p > 1$,

define $f : X \rightarrow \mathbb{R}$ by $f(x) = \frac{|g(x)|^{q-1} \text{sgn}(g(x))}{\|g\|_q^{q-1}}$. So $\|f\|_p^p = \frac{1}{\|g\|_q^{p(q-1)}} \int_X |g(x)|^{p(q-1)} d\mu = \frac{1}{\|g\|_q^q} \int_X |g(x)|^q d\mu = 1$. Thus $\|\phi_g\|_{[L^p]^*} \geq |\int_X fg d\mu| = \left| \int_X f(x) |g(x)| \text{sgn}(g(x)) d\mu \right| = \left| \int_X \frac{|g(x)|^q}{\|g\|_q^{q-1}} d\mu \right| = \|g\|_{L^q}$. Thus $\|\phi_g\|_{[L^p]^*} = \|g\|_{L^q}$. If $p = 1$, we assume μ is semifinite. Let $\epsilon > 0$ and set $E := \{x \in X : |g(x)| \geq \|g\|_\infty - \epsilon\}$. Since μ is semifinite, there exists $F \subseteq E$ such that $0 < \mu(F) < \infty$ (note $\mu(E) > 0$). Define $f : X \rightarrow \mathbb{R}$ by $f(x) = \frac{1}{\mu(F)} \chi_F(x) \text{sgn}(g(x))$. Thus $\|f\|_1 = \frac{1}{\mu(F)} \int_X \chi_F d\mu = 1$ and $\|\phi_g\|_{[L^1]^*} \geq |\int_X fg d\mu| = \frac{1}{\mu(F)} \int_F |g| d\mu \geq \frac{1}{\mu(F)} (\|g\|_\infty - \epsilon) \mu(F) = \|g\|_\infty - \epsilon$. Since $\epsilon > 0$ was arbitrary, $\|\phi_g\|_{[L^1]^*} \geq \|g\|_\infty$. So $\|\phi_g\|_{[L^p]^*} = \|g\|_{L^q}$. \square

Theorem 52 (6.14). Suppose $\frac{1}{p} + \frac{1}{q} = 1$ and $g : X \rightarrow \mathbb{R}$ is measurable. If

1. $fg \in L^1(\mu)$ for all $f \in \Sigma := \{\text{simple functions in } L^1(\mu) \text{ which are 0 outside a set of finite measure}\}$.
2. $M_q(g) = \sup\{|\int_X fg d\mu| : f \in \Sigma, \|f\|_p \leq 1\} < \infty$.
3. Either $S_g = \{x \in X : g(x) \neq 0\}$ is σ -finite or μ is semifinite.

Then $g \in L^q$ and $\|g\|_{L^q} = M_q(g)$.

Proof. Set $\bar{\Sigma} := \{f : X \rightarrow \mathbb{R} : f \text{ is bounded, measurable, } f = 0 \text{ outside a set of finite measure}\}$.

Claim: $\sup\{|\int_X fg d\mu| : f \in \bar{\Sigma}, \|f\|_p \leq 1\} \leq M_q(g)$.

Proof: Let $f \in \bar{\Sigma}$ such that $\|f\|_{L^p} \leq 1$. Select $\{\phi_n\}_{n=1}^\infty$ to be simple functions such that $0 \leq |\phi_1| \leq |\phi_2| \leq \dots \leq |f|$ and $\phi_n \rightarrow f$ a.e. Observe each $\phi_n \in \Sigma$ and $\|\phi_n\|_p \leq \|f\|_p \leq 1$. Suppose $f = 0$ for $x \in X \setminus E$ with $\mu(E) < \infty$. Then $|\phi_n g| \leq |fg| = \|f\|_\infty |g \chi_E| \in L^1$ by (1) since χ_E is a simple function. Also $\phi_n g \rightarrow fg$ a.e. So by the LDC Theorem,

$$\left| \int_X fg d\mu \right| = \left| \int \lim \phi_n g d\mu \right| = \lim \left| \int \phi_n g d\mu \right| \leq M_q(g).$$

Now suppose $1 \leq q \leq \infty$. Note that S_g is σ -finite (exercise). Let $\{E_n\}_{n=1}^\infty \subseteq \mathcal{M}$ satisfy $E_1 \subset E_2 \subset \dots$ with $S_g = \cup_{n=1}^\infty E_n$ and $\mu(E_n) < \infty$ for all $n \in \mathbb{N}$. Let $\{\phi_n\}_{n=1}^\infty$ be a sequence of simple functions such that $0 \leq |\phi_1| \leq |\phi_2| \leq \dots \leq |g|$ and $\phi_n \rightarrow g$ a.e. Put $g_n = \phi_n \chi_{E_n}$ so $0 \leq |g_1| \leq |g_2| \leq \dots \leq |g|$ and $g_n \rightarrow g$ with $g_n = 0$ for all $x \in X \setminus E_n$. Define $f_n(x) := \frac{|g_n|^{q-1} \text{sgn}(g(x))}{\|g_n\|_q^{q-1}}$. So $\|f_n\|_p = 1$ as in the previous proof and $f_n \in \bar{\Sigma}$. By Fatou's Lemma and the first claim,

$$\begin{aligned} \|g\|_q &\leq \liminf \|g_n\|_q &= \liminf \frac{1}{\|g_n\|_q^{q-1}} \int_X |g_n|^q d\mu \\ &= \liminf \int_X |f_n(x)| |g_n(x)| d\mu \\ &\leq \liminf \int_X |f_n| |g| d\mu \\ &= \liminf \int_X f_n g d\mu \leq M_q(g) < \infty. \end{aligned}$$

By Hölder's Inequality, $\|g\|_q \geq M_q(g)$. Thus $\|g\|_q = M_q(g)$.

Now, suppose $q = \infty$ and let $\epsilon > 0$. Set $A = \{x \in X : |g(x)| \geq M_\infty(g) + \epsilon\}$. If $\mu(A) > 0$, select $B \subseteq A$ such that $0 < \mu(B) < \infty$ (by 3) and put $f = \frac{1}{\mu(B)} \chi_B \text{sgn}(g(x))$. So $f \in \Sigma$ and $\|f\|_{L^1} = \int_X |f| d\mu = 1$. Thus $M_\infty(g) \geq |\int fg d\mu| = \frac{1}{\mu(B)} \int_B |g(x)| d\mu \geq \frac{1}{\mu(B)} (M_\infty(g) + \epsilon) \mu(B) = M_\infty(g) + \epsilon$, a contradiction. So $\mu(A) = 0$. Hence $\|g\|_\infty \leq M_\infty(g) < \infty$. So $f \in L^\infty$. Also $M_\infty(g) = \sup\{|\int fg d\mu|\} \leq \|g\|_\infty \sup\{|\int f d\mu|\} \leq \|g\|_\infty$. Thus $\|g\|_\infty = M_\infty(g)$. \square

Theorem (Riesz Representation Theorem for L^p Spaces (6.15)). Suppose $\frac{1}{p} + \frac{1}{q} = 1$ with $p \in (1, \infty)$. Then for all $\phi \in [L^p]^*$, there exists $g \in L^q$ such that $\phi(f) = \int_X fg d\mu$ for all $f \in L^p$. If μ is σ -finite, then we allow for $p = 1$.

Proof. First assume $\mu(X) < \infty$. So all simple functions belong to L^1 . Let $\phi \in [L^p]^*$. Define $\nu : \mathcal{M} \rightarrow (-\infty, \infty)$ by $\nu(E) = \phi(\chi_E)$. Want to show ν is a finite signed measure. Note that $|\nu(E)| \leq |\phi(\chi_E)| \leq \|\phi\|_{[L^p]^*} \|\chi_E\|_p \leq \|\phi\|_{[L^p]^*} \mu(E)^{1/p} \leq \|\phi\|_{[L^p]^*} \mu(X)^{1/p} < \infty$. Thus ν is uniformly bounded. To show it is countably additive, let $\{E_j\}_{j=1}^{\infty} \subseteq \mathcal{M}$ be mutually disjoint. So if $E = \bigcup_{j=1}^{\infty} E_j$, then $\chi_E = \sum \chi_{E_j}$. Also, $\sum_{j=1}^{\infty} \mu(E_j) \leq \mu(X) < \infty$ which implies $\sum_{j=1}^{\infty} \mu(E_j)$ is absolutely convergent. Since $p < \infty$ $\|\chi_E - \sum_{j=1}^N \chi_{E_j}\|_p = \|\sum_{j=N+1}^{\infty} \chi_{E_j}\|_p = (\mu(\bigcup_{j=N+1}^{\infty} E_j))^{1/p} = (\sum_{j=N+1}^{\infty} \mu(E_j))^{1/p} \rightarrow 0$ as $N \rightarrow \infty$ as $\mu(E_j)$ is absolutely convergent. Since ϕ is continuous and linear in L^p and $\sum_{j=1}^N \chi_{E_j} \rightarrow \chi_E$ in L^p , we have $\nu(E) = \phi(\chi_E) = \phi(\sum_{j=1}^{\infty} \chi_{E_j}) = \lim_{N \rightarrow \infty} \phi(\sum_{j=1}^N \chi_{E_j}) = \lim_{N \rightarrow \infty} \sum_{j=1}^N \phi(\chi_{E_j}) = \sum_{j=1}^{\infty} \nu(E_j)$. So ν is countably additive. Moreover, if $\mu(E) = 0$, then for $F \in \mathcal{M}$ such that $F \subseteq E$, we have $\|\chi_F\|_p = 0$. So $\phi(\chi_F) = 0$ as ϕ is continuous which implies $\phi(0) = 0$. Thus $\nu(F) = 0$ and thus $\nu \ll \mu$. By the Radon Nikodym Theorem, there exists $g \in L^1(\mu)$ such that $\phi(\chi_E) = \nu(E) = \int_E g d\mu$. If f is a simple function, by linearity of the integral and ϕ we have $\phi(f) = \int f g d\mu$. Since $|\int_X f g d\mu| = |\phi(f)| \leq \|\phi\|_{[L^p]^*} \|f\|_p$. By Theorem 51, we see $g \in L^q$. Need to show $\phi(f) = \int f g d\mu$ for all $f \in L^p$. By Proposition 19, there exists a sequence of simple functions $\{f_n\}_{n=1}^{\infty} \subseteq L^p$ such that $f_n \rightarrow f$ in L^p , that is $\|f_n - f\|_p \rightarrow 0$. By continuity of ϕ , we see

$$\phi(f) = \lim_{n \rightarrow \infty} \phi(f_n) = \lim_{n \rightarrow \infty} \int f_n g d\mu = \lim_{n \rightarrow \infty} \int (f_n - f) g d\mu + \int f g d\mu \leq \lim_{n \rightarrow \infty} \|f_n - f\|_p \|g\|_q + \int f g d\mu = \int f g d\mu.$$

(Note here that g is in fact unique a.e. by the Radon Nikodym Theorem).

Now we assume μ is a σ -finite measure. Select $\{E_n\}_{n=1}^{\infty} \subseteq \mathcal{M}$ such that $E_1 \subseteq E_2 \subseteq \dots, \bigcup E_n = X, \mu(E_n) < \infty$ for all $n \in \mathbb{N}$. For each $n \in \mathbb{N}$, find $g_n \in L^q(\mu, E_n)$ such that for all $f \in L^p(\mu, E_n)$ we have $\phi(f) = \int_{E_n} f g d\mu$ and $\|g_n\|_q = \|\phi\|_{[L^p(E_n)]^*} \leq \|\phi\|_{[L^p(X)]^*}$. We may assume for $m \leq n$ that $g_n|_{E_m} = g_m$ a.e. in E_m (by the uniqueness of the finite case.) Define g a.e. in X by $g(x) = g_n(x)$ if $x \in E_n$. Thus $g|_{E_n} = g_n$ a.e. in E_n . By the Monotone Convergence Theorem, $\|g\|_q^q = \int_X |g|^q d\mu = \int_X \lim_{n \rightarrow \infty} |g \chi_{E_n}|^q d\mu = \lim_{n \rightarrow \infty} \int_{E_n} |g_n|^q d\mu \leq \|\phi\|_{[L^p(X)]^*}^q < \infty$. Thus $g \in L^q$. For a given $f \in L^p(X)$, we have

- $g f \chi_{E_n} \rightarrow g f$ a.e. in X as $n \rightarrow \infty$.
- $|g f \chi_{E_n}(x)| \leq \underbrace{|g(x)| |f(x)|}_{L^1(x)}$ for a.e. $x \in X$.
- $f \chi_{E_n} - f \rightarrow 0$ for a.e. $x \in X$.
- $|f \chi_{E_n} - f|^p \leq 2|f(X)|^p$ for a.e. $x \in X$ (as $|f \chi_{E_n}(x)|^p \leq |f(x)|^p$ for a.e. $x \in X$).

Thus by the Lebesgue Dominated Convergence Theorem and the last two observations, $\|f \chi_{E_n} - f\|_{L^p} \rightarrow 0$ as $n \rightarrow \infty$, that is, $f \chi_{E_n} \rightarrow f$ in L^p . By the Lebesgue Dominated Convergence Theorem and the continuity of ϕ , we have $\phi(f) = \phi(\lim f \chi_{E_n}) = \lim \phi(f \chi_{E_n}) = \lim \int g f \chi_{E_n} d\mu = \int \lim g f \chi_{E_n} d\mu = \int g f d\mu$. Thus the Representation Theorem holds when μ is σ -finite and $p \in [1, \infty)$.

Finally, assume $p \in (1, \infty)$ and μ is an arbitrary measure. For any σ -finite set $E \subseteq X$, there exists an a.e. unique $g_E \in L^q(E)$ such that $\phi(f) = \int_E f g d\mu$ for all $f \in L^p(E)$. If $E \subseteq F$ and F is σ -finite, then $g_F|_E = g_E$ for a.e. $x \in E$ and $\|g_E\|_q \leq \|g_F\|_q \leq \|\phi\|_{[L^p(x)]^*}$. Put $M = \sup\{\|g_E\|_q : E \text{ is } \sigma\text{-finite}, \phi(f) = \int_E f g_E d\mu \text{ for all } f \in L^p(E)\}$. So $M \leq \|\phi\|_{[L^p(x)]^*} < \infty$. Let $\{E_n\}_{n=1}^{\infty} \subseteq M$ be σ -finite such that $\|g_{E_n}\|_q \rightarrow M$. Then $F = \bigcup_{n=1}^{\infty} E_n$ is σ -finite (as it is a countable union of σ -finite sets) and $\|g_{E_n}\|_q \leq \|g_F\|_q$ for all $n \in \mathbb{N}$. It follows that $M = \|g_F\|_q$. For any σ -finite set A such that $F \subseteq A$, we have $\int_F |g_F|^q + \int_{A \setminus F} |g_{A \setminus F}|^q = \int_F |g_A|^q + \int_{A \setminus F} |g_A|^q = \int_A |g_A|^q d\mu \leq M_q = \int_F |g_F|^q d\mu$. Thus $g_{A \setminus F} = 0$ a.e. Notice if $f \in L^p(X)$, then $\mu(\{x \in X : |f(x)| > \frac{1}{j}\}) = j^p (\frac{1}{j})^p \int_{\{|f(x)| > \frac{1}{j}\}} d\mu \leq j^p \int_X |f(x)|^p d\mu < \infty$. Thus $\{x \in X : f(x) \neq 0\} = \bigcup_{j=1}^{\infty} \{x \in X : |f(x)| > \frac{1}{j}\}$ is σ -finite and so is $A = F \cup \{x \in X : f(x) \neq 0\}$. Thus $g_F|_{A \setminus F} = 0$ a.e. and $f|_{X \setminus A} = 0$. We have $\phi(f) = \phi(f \chi_A + f \chi_{X \setminus A}) = \phi(f \chi_A) + \phi(f \chi_{X \setminus A})$. Now $f \chi_{X \setminus A} = 0$ implies $f \chi_{X \setminus A} = 0$ in L^p and thus $\phi(f \chi_{X \setminus A}) = 0$. Thus $\phi(f) = \phi(f \chi_A) = \int_A f g_A d\mu = \int_F f g_F d\mu$ as $g_A = 0$ on $A \setminus F$. Take $g(x) = \begin{cases} g_F(x) & \text{if } x \in F \\ 0 & \text{otherwise.} \end{cases} \quad \square$

Remark. By Proposition 47 (6.13), we see that L^q is isometrically isomorphic to $[L^p]^*$. Functionals in $[L^p]^*$ are usually just identified with functions in L^q (for $1 \leq p < \infty$).

Corollary 18. If $p \in (1, \infty)$, then L^p is reflexive.

Corollary 19. If μ is σ -finite and $\langle f_\alpha \rangle_{\alpha \in A}$ is a bounded net in L^∞ , then there exists a subnet $\langle g_\beta \rangle_{\beta \in B}$ and a function $g \in L^\infty$ such that $g_\beta \rightarrow^* g$ in L^∞ .

Corollary 20. If $p \in (1, \infty)$ and $\langle f_\alpha \rangle_{\alpha \in A}$ is a bounded net in L^p , then there exists a subnet $\langle g_\beta \rangle_{\beta \in B}$ and a function $g \in L^p$ such that $g_\beta \rightarrow g$ in L^p .

Example. (Fourier Series) Let $1 < p < \infty$. Consider the space $L^p([0, 2\pi])$. For each $k \in \mathbb{Z}$, put $e_k(x) = \frac{1}{\sqrt{2\pi}}e^{-ikx}$. Then $e_k \in L^\infty$ for all $k \in \mathbb{Q}$ which implies $e_k \in L^q$ for all $q \in [1, \infty]$. For each $f \in L^p$ and $k \in \mathbb{Z}$, put $\hat{f}(k) := \frac{1}{\sqrt{2\pi}} \int_{[0, 2\pi]} f(x)e_k(x)dx = \frac{1}{2\pi} \int_{[0, 2\pi]} f(x)e^{-ikx}dx$. Here, $\hat{f}(k)$ is the k^{th} fourier coefficient for f . The series $\sum_{k=-n}^n \hat{f}(k)e^{ikx}$ is the n^{th} partial sum of the Fourier Series for f . For each $x \in [0, 2\pi]$, define $\phi_x : L^p \rightarrow \mathbb{C}$ by $\phi_x(f) = \sum_{k=-n}^n \hat{f}(k)e^{ikx}$. From the definition of $\hat{f}(x)$, we see ϕ_x is linear and

$$\begin{aligned} |\phi_x(f)| &\leq \sum_{k=-n}^n |\hat{f}(k)| |e^{ikx}| \text{ note } |e^{ikx}| = 1 \\ &= \sum_{k=-n}^n |\hat{f}(k)| = \sum_{k=-n}^n \frac{1}{2\pi} \left| \int_{[0, 2\pi]} f(x)e^{-ikx} dx \right| \\ &\leq \sum_{k=-n}^n \frac{1}{2\pi} \|f\|_{L^p} \left(\int_{[0, 2\pi]} |e^{-ikx}|^{p/(p-1)} dx \right)^{p-1/p} \text{ By Holder} \\ &= \sum_{k=-n}^n \frac{\|f\|_{L^p}}{(2\pi)^{1/p}} = \frac{2n+1}{(2\pi)^{1/p}} \|f\|_p. \end{aligned}$$

Hence $\|\phi_x\|_{[L^p]^*} < \infty$ and so $\phi_x \in [L^p]^*$. By the Riesz Representation Theorem, there exists $g_x \in L^{p/(p-1)}([0, 2\pi])$ such that $\phi_x(f) = \int_{[0, 2\pi]} f(y)g_x(y)dy$. What is g_x ? Note

$$\phi_x(f) = \sum_{k=-n}^n \frac{1}{2\pi} \int f(y)e^{-iky}e^{ikx}dy = \int_{[0, 2\pi]} f(y) \underbrace{\sum_{k=-n}^n \frac{1}{2\pi} e^{ik(x-y)} dy}_{:=g_x(y)}$$

Using trig,

$$g_x(y) = k_n(x-y) = \begin{cases} \frac{\sin(\frac{n+1}{2}(x-y))}{2\pi \sin(\frac{x-y}{2})} & \text{if } x \neq y \\ \frac{2n+1}{2\pi} & \text{if } x = y. \end{cases}$$

Thus $\phi_x(f) = \int_{[0, 2\pi]} f(y)k_n(x-y)dy$.

3.2 Dual Spaces for Spaces of Continuous Functions (Ch 7)

Definition. If (X, \mathcal{T}) is a topological space and $f : X \rightarrow \mathbb{R}$, then the **support** of f is $\text{supp}(f) := \overline{\{x \in X : f(x) \neq 0\}}$.

Notation. Let $\mathcal{C}(X)$ denote the vector space of all continuous functions from $(X, \mathcal{T}) \rightarrow \mathbb{R}$. Set $\mathcal{C}_c(X) := \{f \in \mathcal{C}(X) : \text{supp}(f) \text{ is compact}\}$. Define $\|\cdot\|_{\mathcal{U}} : \mathcal{C}_c(X) \rightarrow (0, \infty)$ by $\|f\|_{\mathcal{U}} := \sup\{|f(x)| : x \in X\}$ (the **uniform norm**). We will assume $\mathcal{C}_c(X)$ is endowed with $\|\cdot\|_{\mathcal{U}}$, making it a normed vector space. Note that, in general, it is not complete.

Definition. If $I : \mathcal{C}_c(X) \rightarrow \mathbb{R}$, then I is a **positive linear function** if I is linear and $I(f) \geq 0$ whenever $f \geq 0$.

Definition. If μ is a Borel measure on X and $E \in \mathcal{B}_X$, then

- μ is **outerregular** on E if $\mu(E) = \inf\{\mu(U) : U \text{ is open, } E \subseteq U\}$.
- μ is **innerregular** on E if $\mu(E) = \sup\{\mu(K) : K \text{ is compact, } K \subseteq E\}$.
- μ is **regular** if it is both outer and inner regular on all Borel sets.

Definition. A Borel measure μ is called a **Radon measure** if

- $\mu(K) < \infty$ for all compact $K \subseteq X$.

- μ is outer regular on all Borel sets.
- μ is inner regular on all open sets.

Remark. If (X, \mathcal{T}) is σ -compact and σ -finite, then μ is a Radon measure if and only if μ is regular.

Notation. If U is open and $f \in \mathcal{C}_c(X)$, then we write $f \prec U$ if $0 \leq f \leq \chi_U$ and $\text{supp}(f) \subseteq U$ and say f is **subordinate** to U .

Theorem (Riesz Representation Theorem for $\mathcal{C}_c(X)$). If I is a positive linear function on $\mathcal{C}_c(X)$, then there exists a unique Radon measure μ on X such that $I(f) = \int_X f d\mu$ for all $f \in \mathcal{C}_c(X)$. Also

- For all open sets U , $\mu(U) = \sup\{I(f) : f \in \mathcal{C}_c(X), f \prec U\}$.
- For all compact K , $\mu(K) = \inf\{I(f) : f \in \mathcal{C}_c(X), f \geq \chi_K\}$.

Facts. Suppose μ is a Radon measure on X .

- If $1 \leq p < \infty$, then $\mathcal{C}_c(X)$ is dense in $L^p(\mu)$.
- (Lusin's Theorem for Radon measures) If $f : X \rightarrow \mathbb{R}$ is measurable and 0 outside a set of finite measure, then for all $\epsilon > 0$, there exists $g \in \mathcal{C}_c(X)$ such that $\sup_{x \in X} |g(x)| \leq \sup_{x \in X} |f(x)|$ and $\mu(\{x \in X : f(x) \neq g(x)\}) < \epsilon$.

Definition. A topological space (X, \mathcal{T}) is called a **locally compact Hausdorff space** if

1. it is **locally compact**, that is, for each $x \in X$, there exists $A \subseteq X$ such that x is in the interior of A and A is compact.
2. it is **Hausdorff**, that is, whenever $x, y \in X$ and $x \neq y$, we may find open $U, V \in \mathcal{T}$ such that $U \cap V = \emptyset$ and $x \in U, y \in V$.

Let (X, \mathcal{T}) be a locally compact Hausdorff space. Let $\mathcal{C}_0(X)$ be the closure of $\mathcal{C}_c(X)$ with respect to the uniform metric. It can be shown that $f \in \mathcal{C}_0(X)$ if and only if $\{x \in X : |f(x)| > \epsilon\}$ is compact for all $\epsilon > 0$ (so f is "vanishing at ∞ ").

Definition. We say that a signed Borel measure μ on \mathcal{B}_X is a **signed Radon measure** if $|\mu|$ is a Radon measure.

Notation. Let $M(X)$ be the vector space of all finite signed Radon measures. Define $\|\cdot\|_M : M(X) \rightarrow [0, \infty)$ by $\|\mu\|_M = |\mu|(X)$.

Riesz Representation Theorem for $\mathcal{C}_0(X)$. Let (X, \mathcal{T}) be a locally compact Hausdorff space. For each $\mu \in M(X)$ and $f \in \mathcal{C}_0(X)$, define $I_\mu(f) = \int_X f d\mu$. Then the map $\mu \mapsto I_\mu$ is an isometric isometry from $M(X) \rightarrow \mathcal{C}_0(X)^*$.

Corollary 21. If $\langle \mu_\alpha \rangle_{\alpha \in A} \subseteq M(X)$ is a bounded net, then there exists a subnet $\langle \nu_\beta \rangle_{\beta \in B}$ of $\langle \mu_\alpha \rangle$ and a Radon measure $\nu \in M(X)$ such that $\nu_\beta \rightharpoonup^* \nu$ in $M(X)$, that is, for all $f \in \mathcal{C}_0(X)$, we have $\int_X f d\nu_\beta \rightarrow \int_X f d\nu$.

Corollary 22. Let μ be a positive Radon measure and $\langle g_\alpha \rangle \subseteq L^1(\mu)$ satisfies $\sup_{\alpha \in A} \|g_\alpha\|_{L^1} < \infty$. Then there exists a subnet $\langle h_\beta \rangle_{\beta \in B}$ of $\langle g_\alpha \rangle$ and a Radon measure $\nu \in M(X)$ such that $h_\beta d\mu \rightharpoonup^* d\nu$ in $M(X)$.

Proof. First, define $\mu_\alpha \in M(X)$ by $\mu_\alpha(E) = \int_E g_\alpha d\mu$. By the previous corollary, there exists a subnet $\langle \nu_\beta \rangle$ such that $\nu_\beta \rightharpoonup^* \nu$. By definition of μ_α , we have $h_\beta d\mu \rightharpoonup^* d\nu$. □

Note. Although each $g_\alpha d\mu$ is absolutely continuous with respect to μ , this may not be true for $d\nu$.

Example. $X = [-1, 1]$ with the usual topology. Let μ be the Lebesgue measure on $[-1, 1]$ restricted to the Borel sets. Then μ is a Radon measure. Consider the sequence $g_n(x) = \begin{cases} 0 & \text{if } x \notin [-\frac{1}{n}, \frac{1}{n}] \\ \frac{n}{2} & \text{if } x \in (-\frac{1}{n}, \frac{1}{n}). \end{cases}$ Notice that $\|g_n\|_{L^1} = 1 < \infty$.

Claim: $g_n dx \rightharpoonup^* d\delta_0$ in $M(X)$.

Proof: Consider $\int_{[-1,1]} g_n(x)f(x)dx$ for $f \in \mathcal{C}_0([-1,1])$. We have

$$\frac{n}{2} \int_{(-\frac{1}{n}, \frac{1}{n})} f(x)dx = \frac{n}{2} \int_{(-\frac{1}{n}, \frac{1}{n})} [f(x) - f(0)]dx + f(0) \leq \|f(x) - f(0)\|_{L^\infty(-\frac{1}{n}, \frac{1}{n})} + f(0) \rightarrow f(0)$$

by continuity of f .

This implies $g_n dx \rightharpoonup^* d\delta_0$, which is not absolutely continuous with respect to m .

How do we fix this? If $\sup \|f_\alpha\| < \infty$ and for all $\epsilon > 0$ there exists $\delta > 0$ for all α such that $\int |f_\alpha| d\mu < \epsilon$ whenever $\mu(E) < \delta$, then there exists a subnet $\langle g_\beta \rangle \rightharpoonup g$ in L^1 . So if $h_\beta d\mu$ is “**uniformly absolutely continuous**”, then we are good.

Note. The above example also works to show that weak convergence does not imply strong convergence (consider the sequence $g_n^{1/p}$).

3.3 Baire Category Theorem

Definition. Let (X, \mathcal{T}) be a topological space. A set $E \subseteq X$ is of **first category** (meager) if E is the countable union of nowhere dense sets (in particular, it does not contain any open sets) and E is of **second category** if it is not meager.

Example. \mathbb{Q} is meager as it is a countable union of points.

Baire Category Theorem [p 161]. Let X be a complete metric space.

1. If $\{U_n\}_{n=1}^\infty$ is a sequence of open dense subsets of X , then $\bigcap_{n=1}^\infty U_n$ is dense in X .
2. X is not the countable union of nowhere dense subsets of X .

Proof. Since X is a metric space, if a set E is not dense in X , then there exists $x \in X$ such that $x \notin \overline{E}$, which implies there exists an open set $A \subseteq X$ such that $x \in A$ and $A \cap E = \emptyset$. To prove (1), it suffices to show that for any open set $W \subseteq X$ we have $W \cap (\bigcap_{j=1}^\infty U_j) \neq \emptyset$. Since each U_j is dense, we must have $A \cap U_j \neq \emptyset$ for all open sets A . In particular, $W \cap U_1 \neq \emptyset$. Thus there exists a ball $B(r_1, x_1) \subseteq W \cap U_1$. Now, $B(r_1, x_1)$ is open, so we may find $B(r_2, x_2) \subseteq B(\frac{1}{2}r_1, x_1) \cap U_2$ and note $\overline{B(r_2, x_2)} \subseteq B(r_1, x_1) \cap U_2$. Continuing inductively, we obtain $\{B(r_j, x_j)\}_{j=1}^\infty$ such that $B(r_j, x_j) \subseteq B(\frac{1}{2}r_{j-1}, x_{j-1}) \cap U_j$ and $\overline{B(r_j, x_j)} \subseteq B(r_{j-1}, x_{j-1}) \cap U_j$. Note $r_1 \geq 2r_2 \geq 2^2r_3 \geq \dots \geq 2^{j-1}r_j \geq \dots$. Hence $r_j \rightarrow 0$. Also, if $m, n \geq N \in \mathbb{N}$, then $B(r_n, x_m) \subseteq 2^{1-N}r_1$. Thus $\{x_n\}_{n=1}^\infty$ is a Cauchy sequence. As X is complete, there exists $x \in X$ such that $\rho(x_n, x) \rightarrow 0$. In particular, $x \in \overline{B(r_N, x_N)} \subseteq U_N \cap B(r_1, x_1) \subseteq U_N \cap W$ for all $N \in \mathbb{N}$. Thus $x \in (\bigcap_{j=1}^\infty U_j) \cap W$. Hence, $\bigcap_{j=1}^\infty U_j$ is dense in X . For (2), if $\{E_j\}_{j=1}^\infty$ are each nowhere dense sets, then $\{X \setminus \overline{E_j}\}_{j=1}^\infty$ would be a sequence of open dense sets. By (1), $\bigcap_{j=1}^\infty (X \setminus \overline{E_j})$ is dense and thus $\neq \emptyset$. Of course

$$\emptyset \neq \bigcap_{j=1}^\infty (X \setminus \overline{E_j}) = X \setminus (\bigcup_{j=1}^\infty \overline{E_j}) \subseteq X \setminus \bigcup_{j=1}^\infty E_j.$$

So $\bigcup_{j=1}^\infty E_j \neq X$. □

Consequences of the Baire Category Theorem

Open Mapping Theorem. Let $\mathfrak{X}, \mathfrak{Y}$ be Banach Spaces. If $T \in L(\mathfrak{X}, \mathfrak{Y})$ is surjective, then T is an open mapping, that is $T(U)$ is open whenever U is open in \mathfrak{X} .

Corollary 23. If T is a bijection between \mathfrak{X} and \mathfrak{Y} , then T and T^{-1} are continuous and so T is an isomorphism.

Closed Graph Theorem. If $T : \mathfrak{X} \rightarrow \mathfrak{Y}$ is a closed linear map of Banach Spaces, then T is bounded (and so continuous).

Definition. A linear map T is **closed** if the **graph of T** , $\{(x, y) \in \mathfrak{X} \times \mathfrak{Y} : Tx = y\}$, is a closed subspace in $\mathfrak{X} \times \mathfrak{Y}$.

Remark. To show $Tx_n \rightarrow Tx$ whenever $x_n \rightarrow x$, it is sufficient to show Tx_n converges to some y in the range of T .

Example. There exists a nowhere differentiable function on $[0, 1]$.

Proof: Endow $\mathcal{C}([0, 1])$ with the uniform norm. Let $D = \{f \in \mathcal{C}([0, 1]) : f'(x_0) \text{ exists for some } x_0 \in [0, 1]\}$. We will show that D is a countable union of nowhere dense sets in $\mathcal{C}([0, 1])$ and thus $D \neq \mathcal{C}([0, 1])$ by the Baire Category Theorem.

If $f'(x_0)$ exists then $\lim_{x \rightarrow x_0} \frac{f(x) - f(x_0)}{x - x_0}$ exists. Furthermore, if $f'(x_0)$ exists, then for some $m, n \in \mathbb{N}$, we must have $\left| \frac{f(x) - f(x_0)}{x - x_0} \right| \leq n$ for all x with $0 < |x - x_0| \leq \frac{1}{m}$. Define $E_{n,m} = \{f \in \mathcal{C}([0, 1]) : \text{for some } x_0 \in [0, 1], |f(x) - f(x_0)| \leq n|x - x_0| \text{ for all } x \in [0, 1] \text{ with } 0 < |x - x_0| \leq \frac{1}{m}\}$. So $D \subseteq \cup_{n=1}^{\infty} \cup_{m=1}^{\infty} E_{n,m}$.

Claim: $E_{n,m}$ are closed.

Proof: Let $\{f_j\}_{j=1}^{\infty} \subseteq E_{n,m}$ be such that $\|f - f_j\|_{\mathcal{U}} \rightarrow 0$ for some $f \in \mathcal{C}([0, 1])$. For each j , we have $|f_j(x) - f_j(x_j)| \leq n|x - x_j|$ for all x such that $0 < |x - x_k| \leq \frac{1}{m}$. Extract a subsequence (unrelabeled, for simplicity) so that $x_n \rightarrow x_0 \in [0, 1]$. Then

$$\begin{aligned} |f(x) - f(x_0)| &\leq \underbrace{|f(x) - f_j(x)|}_{\rightarrow 0} + |f_j(x) - f_j(x_j)| + \underbrace{|f_j(x_j) - f_j(x_0)|}_{\rightarrow 0} + \underbrace{|f_j(x_0) - f(x_0)|}_{\rightarrow 0} \\ &\leq \frac{1}{m}|x - x_j| \\ &\rightarrow \frac{1}{m}|x - x_0|. \end{aligned}$$

To show that $E_{n,m}$ are nowhere dense, note that it is enough to show that for all $\epsilon > 0$ and $f \in E_{n,m}$ that there exists $g \in \mathcal{C}([0, 1])$ such that $\|g - f\|_{\mathcal{U}} < \epsilon$, but $g \notin E_{n,m}$. Of course, we can do this- just take g to be a piecewise linear continuous function with slope $> n$ everywhere.