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Class Notes for Math 918: Homological Conjectures, Instructor Tom Marley

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This course was an overview of what are known as the "Homological Conjectures," in particular, the Zero Divisor Conjecture, the Rigidity Conjecture, the Intersection Conjectures, Bass' Conjecture, the Superheight Conjecture, the Direct Summand Conjecture, the Monomial Conjecture, the Syzygy Conjecture, and the big and small Cohen Macaulay Conjectures. Many of these are shown to imply others.

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Math 918: The Homological Conjectures

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This document contains notes for a course taught by Tom Marley during the 2009 spring semester at the University of Nebraska-Lincoln. The notes loosely follow the treatment given in Chapters 8 and 9 of *Cohen-Macaulay Rings*, by W. Bruns and J. Herzog, although many other sources, including articles and monographs by Peskine, Szpiro, Hochster, Huneke, Griffith, Evans, Lyubeznik, and Roberts (to name a few), were used. Special thanks to Laura Lynch for putting these notes into LaTeX. [1]

[1]Last edit: August 5, 2009

Introduction

Hochster has written "The existence of non-trivial modules of finite projective dimension is almost entirely due to the present of regular sequences in the ring."

As evidence of this, consider the following results:

- Koszul Complex: Let (R, m) be local and $x_1, ..., x_n \in m$. Then $K.(x_1, ..., x_n; R)$ is a finite free resolution of $R/(x_1, ..., x_n)$ if and only if $x_1, ..., x_n$ is an $R-$sequence.
- Auslander-Buchsbaum: Let (R, m) be local, M a finitely generated $R-$module with $\operatorname{pd}_R M < \infty$. Then $\operatorname{depth} M + \operatorname{pd}_R M = \operatorname{depth} R$. In particular, $\operatorname{pd}_R M \leq \operatorname{depth} R$.
- Buchsbaum-Eisenbud: Let R be Noetherian and suppose $F. : 0 \to F_s \xrightarrow{\phi_s} F_{s-1} \xrightarrow{\phi_{s-1}} \cdots \to F_0$ is a complex of finitely generated free $R-$modules. For $i = 1, ..., s$, set $r_i := \sum_{j=i}^{s} (-1)^{j-i} \operatorname{rank} F_i$. Then $F.$ is acyclic if and only if $\operatorname{grade} I_{r_i}(\phi_i) \geq i$ for $i = 1, ..., s$ where if $F \xrightarrow{\phi} G$ is a map of free modules then $I_r(\phi)$ is the ideal in R generated by all $r \times r$ minors of any matrix representation of ϕ.

Theorem 1 (**Zero Divisor Conjecture (ZDC)**, Auslander 1961)**.** *Let (R, m) be local and suppose M is a module of finite projective dimension. Any non-zero-divisor on M is a non-zero-divisor on R.*

Definition. *Let (R, m) be local, M a finitely generated $R-$module. Say M is **rigid** if whenever $\operatorname{Tor}_i^R(M, N) = 0$ for some finitely generated $R-$module N, then $\operatorname{Tor}_j^R(M, N) = 0$ for all $j \geq i$.*

Theorem 2 (**Rigidity Theorem**)**.** *Let (R, m) be a regular local ring. Then any finitely generated $R-$module M is rigid.*

The Rigidity Theorem was proved by Auslander in the unramified case (and in particular for regular local rings containing a field) in 1961. Lichtenbaum proved the theorem for arbitrary regular local rings in 1966.

Conjecture (**Rigidity Conjecture (RC)**, Auslander)**.** *Let (R, m) be local, M a finitely generated $R-$module with finite projective dimension. Then M is rigid.*

Auslander proved that RC implies ZDC. Unfortunately the RC was shown to be false by an example of R. Heitmann in 1993 of a non-rigid module of projective dimension 3. However, if one modifies the definition of rigid to force N to have finite projective dimension as well, the conjecture is still open.

Intersection Theorems

If U, V are subspaces of a finite dimensional vector space W, then $\dim U \cap V \geq \dim U + \dim V - \dim W$. Similarly, if X, Y are algebraic varieties in $\mathbb{A}_k^n$ (for $k = \overline{k}$), then $\dim X \cap Y \geq \dim X + \dim Y - n$. In fact this holds when $X \cap Y$ is replaced by any irreducible component of $X \cap Y$,: Let $X = Z(P)$ and $Y = Z(Q)$ where P, Q are primes in $R = k[x_1, ..., x_n]$. Then an irreducible component of $X \cap Y$ is of the form $W = Z(J)$ where J is a prime minimal over $P + Q$. Translating this, since $\dim W = \dim R/J = \dim R - \operatorname{ht} J$, $\dim X = \dim R - \operatorname{ht} P$ and $\dim Y = \dim R - \operatorname{ht} Q$, we have $\operatorname{ht} J \leq \operatorname{ht} P + \operatorname{ht} Q$. Thus in $k[x_1, ..., x_n]$, we see $\operatorname{ht}(P+Q) \leq \operatorname{ht} P + \operatorname{ht} Q$ for all primes P, Q. This formula does not hold for arbitrary rings, however. For example, take $R = k[x, y, u, v]/(xu - yv)$ with $p = (x, y)R$ and $q = (u, v)R$. Here $\operatorname{ht} p = \operatorname{ht} q = 1$ but $\operatorname{ht}(p + q) = 3$.

Theorem 3 (**Serre's Intersection Theorem**, 1961)**.** *Let (R, m) be a regular local ring, $p, q \in \operatorname{Spec} R$. Then* $\operatorname{ht}(p + q) \leq \operatorname{ht} p + \operatorname{ht} q$.

Corollary 4. *Let (R, m) be a regular local ring, M, N finitely generated $R-$modules such that $\lambda(M \otimes_R N) < \infty$. Then* $\dim M + \dim N \leq \dim R$.

Proof. Recall $\sqrt{\operatorname{Ann}_R M \otimes_R N} = \sqrt{\operatorname{Ann}_R M + \operatorname{Ann}_R N}$ and $\lambda(M \otimes_R N) < \infty$ if and only if $\operatorname{Ann}_R M \otimes_R N$ is $m-$primary. Thus $\lambda(M \otimes_R N) < \infty$ if and only if $\sqrt{\operatorname{Ann} M + \operatorname{Ann} N} = m$ which is if and only if $\lambda(R/\operatorname{Ann} M \otimes$

$R/\operatorname{Ann} N) < \infty$. By taking primes minimal over $\operatorname{Ann} M$ and $\operatorname{Ann} N$ we can assume $M = R/p$ and $N = R/q$ for some $p, q \in \operatorname{Spec} R$. As $p+q$ is $m-$primary, $\dim R = \operatorname{ht}(p+q) \leq \operatorname{ht} p + \operatorname{ht} q = 2\dim R - \dim M - \dim N$. $\square$

One might try to generalize the corollary by removing the regular local ring assumption. In this case, one could conjecture that for (R, m) local with $\operatorname{pd} M < \infty$ and $\lambda(M \otimes_R N) < \infty$ that $\dim M + \dim N \leq \dim R$. (This was conjectured by Peskine and Szpiro and is still open.) From here, we can slightly tweak the conjecture to just having $\dim N \leq \operatorname{depth} R - \operatorname{depth} M = \operatorname{pd}_R M$.

Theorem 5 (**Intersection Conjecture (IC)**, Peskine-Szpiro 1974, Roberts 1987)**.** *Let* (R, m) *be local,* $\operatorname{pd}_R M < \infty$ *and* $\lambda(M \otimes_R N) < \infty$*. Then* $\dim N \leq \operatorname{pd}_R M$.

By the above arguments, IC is true for regular local rings. Peskine and Szpiro proved it for local rings of characteristic p and for a large class of rings of equicharacteristic zero. IC was proved for arbitrary local rings by Paul Roberts in 1987.

Proposition 6. *IC implies ZDC.*

Proof. Suppose IC holds. We wish to show that if $\operatorname{pd}_R M < \infty$ and $p \in \operatorname{Ass}_R R$, then $p \subseteq q$ for some $q \in \operatorname{Ass}_R M$. (Then if x is a zerodivisor on R it is also on M). If $\dim M = 0$, done. So assume $\dim M > 0$ and induct on $\dim M$. Let $p \in \operatorname{Ass} R$.

Case 1. There exists $q \in \operatorname{Supp} M$ with $q \neq m$ such that $q \supseteq p$. Then $\dim M_q < \dim M$ and $\operatorname{pd}_{R_q} M_q < \infty$. By induction, there exists $q' \in \operatorname{Ass}_R M$ such that $q'R_q \supseteq pR_q$, which implies $q' \supseteq p$.

Case 2. $p + \operatorname{Ann} M$ is $m-$primary. Then $\lambda(R/p \otimes_R M) < \infty$ (since $\sqrt{\operatorname{Ann}_R(R/p \otimes M)} = \sqrt{(p + \operatorname{Ann}_R M)}$) and so $\dim R/p \leq \operatorname{pd}_R M = \operatorname{depth} R - \operatorname{depth} M$. Then $\operatorname{depth} M \leq \operatorname{depth} R - \dim R/p \leq 0$ since $\operatorname{depth} R \leq \dim R/p$ for all $p \in \operatorname{Ass}_R R$ (see [BH] Proposition 1.2.13). Thus $m \in \operatorname{Ass} M$ and clearly $p \subseteq m$. $\square$

Definition. *Let M be a finitely generated $R-$module. Define* $\operatorname{grade} M := \inf\{i \geq 0 | \operatorname{Ext}^i_R(M, R) \neq 0\}$.

Note that $\operatorname{grade} M = \operatorname{depth}_{\operatorname{Ann}_R M} R$ (see [Mats] Theorem 16.6) and $\operatorname{grade} M \leq \operatorname{pd}_R M$.

Conjecture (**Strong Intersection Conjecture (SIC)**)**.** *Let* (R, m) *be local,* $\lambda(M \otimes_R N) < \infty$, $\operatorname{pd}_R M < \infty$*. Then* $\dim N \leq \operatorname{grade} M$.

One consequence of SIC would be that if $\lambda(M \otimes_R N) < \infty$ and $\operatorname{pd}_R M < \infty$, then $\dim M + \dim N \leq \dim R$. This holds as $\dim N \leq \operatorname{grade} M = \operatorname{depth}_{\operatorname{Ann}_R M} R \leq \operatorname{ht} \operatorname{Ann}_R M \leq \dim R - \dim R/\operatorname{Ann}_R M = \dim R - \dim M$.

Theorem 7 (**Bass' Conjecture (BC)**, 1961)**.** *Let (R, m) be local and suppose there exists a finitely generated $R-$module of finite injective dimension. Then R is Cohen Macaulay.*

Proof. In 1972, Peskine and Szpiro showed IC implies BC. We will prove this later in the course. $\square$

Definition. *Let R be Noetherian, I an ideal of R. Let*

$$\operatorname{superht}(I) = \sup\{\operatorname{ht}(IS) | R \to S \text{ is a ring homomorphism}, S \text{ is Noetherian}, IS \neq S\}.$$

Example. Let $R = k[x, y]/(xy), I = (x)$. Then $\operatorname{ht} I = 0$ as I is minimal. For $S = R/(y) \cong k[x]$, we see $\operatorname{ht}(IS) = 1$. By Krull's PIT, $\operatorname{ht}(IS') \leq 1$ for all S Noetherian. Thus $\operatorname{superht}(I) = 1$. In general, $\operatorname{superht}(I) \leq \mu_R(I)$ by Krull's PIT.

Theorem 8 (**Superheight Conjecture (SC)**, Hochster 1970s)**.** *Let (R, m) be local, M a finitely generated $R-$module such that* $\operatorname{pd}_R M < \infty$*. Then* $\operatorname{superht}(\operatorname{Ann}_R M) \leq \operatorname{pd}_R M$.

Proof. We will see below that this is a consequence of the New Intersection Theorem. $\square$

Remark. SC implies KPIT

Proof. Let $I = (x_1, ..., x_n)$ in S and $R = \mathbb{Z}[T_1, ..., T_n]$ for T_i variables. For $J = (T_1, ..., T_n)$, we see $\operatorname{pd}_R R/J = n$. Then SC gives superht$(J) \leq n$. Map $\phi : R \to S$ by $T_i \mapsto x_i$. Then $JS = I$ and so $\operatorname{ht} I \leq \operatorname{superht} J \leq n$. $\square$

Proposition 9. *SC implies IC*

Proof. Let $\lambda(M \otimes_R N) < \infty$ and $\operatorname{pd}_R M < \infty$. Want to show $\dim N = \operatorname{pd}_R M$. Let $I = \operatorname{Ann}_R N$. So $\dim N = \dim R/I$. Then $\lambda(M \otimes_R N)M\infty$ if and only if $\sqrt{\operatorname{Ann} M + \operatorname{Ann} N} = m$ which is if and only if $\lambda(M \otimes_R R/I) < \infty$ as $I = \operatorname{Ann} N$. Without loss of generality, we may assume $N = R/I$. By SC, $\operatorname{superht}(\operatorname{Ann}_R M) \leq \operatorname{pd}_R M$. Consider the map $R \to R/I$. We have $\operatorname{ht}(\operatorname{Ann}_R M)R/I = \dim R/I \leq \operatorname{superht}(\operatorname{Ann}_R M) \leq \operatorname{pd}_R M$. $\square$

Theorem 10 (**New Intersection Conjecture (NIC)**, Roberts 1975)**.** *Let (R, m) be local and $F. : 0 \to F_s \to F_{s-1} \to \cdots \to F_0 \to 0$ a complex of finitely generated free $R-$modules. Suppose $F.$ is not exact and $\lambda(H_i(F.)) < \infty$ for all i (that is, $F.$ becomes exact when localizing at any prime $\neq m$). Then $s \geq \dim R$.*

Proposition 11. *NIC implies SC*

Proof. Let (R, m) be local, $\operatorname{pd}_R M < \infty$. Let $R \to S$ be a ring homomorphism, S Noetherian. Let Q be a minimal prime over $(\operatorname{Ann}_R M)S$ such that $\operatorname{ht} Q = \operatorname{ht}(\operatorname{Ann}_R M)S = \operatorname{ht}((\operatorname{Ann}_R M)S)_Q$. Let $q = \phi^{-1}(Q)$. Then we have a homomorphism $R_q \to S_Q$. Note $((\operatorname{Ann}_R M)S)_Q = (\operatorname{Ann}_{R_q} M_q)S_Q$. Thus $\operatorname{ht}(\operatorname{Ann}_{R_q} M_q)S_Q = \operatorname{ht}(\operatorname{Ann}_R M)S$. Also $\operatorname{pd}_{R_q} M_q \leq \operatorname{pd}_R M$. Hence we may assume $\phi : (R, m) \to (S, n)$ is a homomorphism of local rings and $\sqrt{(\operatorname{Ann}_R M)S} = n$. I n particular, $\operatorname{ht}(\operatorname{Ann}_R M)S = \dim S$.

Let $F.$ be a minimal free resolution of M as an $R-$module. Say $F. = 0 \to F_r \to F_{r-1} \to \cdots \to F_0 \to 0$ where $r = \operatorname{pd}_R M$. Let $Q \in \operatorname{Spec} S$ with $Q \neq n$ and set $q = \phi^{-1}(q)$. Since $Q \not\supset (\operatorname{Ann}_R M)S$, $q \not\supseteq \operatorname{Ann}_R M$ and thus $M_q = 0$. Hence $F. \otimes_R R_q$ is exact. Since $F.$ is free, $F. \otimes R_q$ is in fact split exact. Thus $(F. \otimes_R R_q) \otimes_{R_q} S_Q = F. \otimes_R S_Q$ is split exact. Consider $F. \otimes_R S$, a complex of free $S-$modules. Note $H_0(F. \otimes_R S) = M \otimes_R S \neq 0$ as $M \neq 0$ and the map $R \to S$ is local. So $F. \otimes_R S$ is not exact. Since $F. \otimes_R S_Q$ is exact for all $Q \neq n$, $\lambda(H_i(F. \otimes_R S)) < \infty$ for all i. By NIC, $r \geq \dim S = \operatorname{ht}(\operatorname{Ann}_R M)S$. $\square$

Conjecture (**Direct Summand Conjecture (DSC)**, Hochster 1971)**.** *Let (R, m) be a regular local ring and S a module finite ring extension of R. Then R is a direct summand of S as an $R-$module; that is, there exists an $R-$module map $\phi : S \to R$ such that $\phi(r) = r$ for all $r \in R$.*

Conjecture (**Monomial Conjecture (MC)**, Hochster 1970s)**.** *Let (R, m) be local and $x_1, ..., x_d$ a system of parameters for R. Then for all $t \geq 1$, $x_1^t \cdots x_d^t \notin (x_1^{t+1}, ..., x_d^{t+1})$.*

Exercise. Prove MC holds for all Cohen Macaulay rings.

In 1983, Hochster proved DSC was equivalent to MC and that DSC implies NIC. He also proved DSC and MC hold for Noetherian local rings containing a field. Hochster also proved DSC for arbitrary local rings of dimension at most two. In 2002, DSC was proved for arbitrary local rings of dimension three by R. Heitmann.

Definition. *Let R be a ring, Q the total quotient ring (that is, $Q = R_W$ for $W = \{$non-zerodivisors of $R\}$). An R-module M has a rank if $M \otimes_R Q$ is a free module. If so, we set the* $\operatorname{rank} M$ *to be* $\operatorname{rank}_Q M \otimes_R Q$.

The rank is not always defined, but if for example M has a finite free resolution then it is.

Definition. *Let (R, m) be local, M a finitely generated $R-$module. Let $\cdots \to F_i \xrightarrow{\phi_i} F_{i-1} \to \cdot \xrightarrow{\phi_1} F_0 \xrightarrow{\phi_0} M \to 0$ be a minimal free resolution of M. The i^{th}* ***syzygy of*** *M is defined to be $\ker \phi_{i-1} = \operatorname{im} \phi_i$. This is unique up to isomorphism and we denote the i^{th} syzygy of M by $\operatorname{syz}_i^R(M)$. If L is an $R-$module such that $L \cong \operatorname{syz}_i^R(M)$ for some M, then we say L is an i^{th} syzygy.*

Conjecture (**Syzygy Conjecture**, Evans-Griffiths 1981)**.** *Let (R, m) be local, L a non-free finitely generated i^{th} syzygy of finite projective dimension. Then* $\operatorname{rank} L \geq i$.

This was proved for rings containing fields by Evans and Griffiths in 1981. In 1983, Hochster showed DSC implies the Syzygy Conjecture.

Definition. *Let (R, m) be local, $\underline{x} = x_1, ..., x_d$ a system of parameters for R. An $R-$module M (not necessarily finitely generated) is called a **(Big) Cohen Macaulay Module** for $\underline{x}$ if $(x_1, .., x_d)M \neq M$ and $x_1, ..., x_d$ is $M-$regular.*

Conjecture (**Big CM Conjecture**)**.** *Every system of parameters in any local ring has a big CM module.*

Hochster proved this result for rings containing a field in 1974 and proved it implies DSC in 1983. In 1992, Hochster and Huneke proved if R is an excellent local ring of characteristic p, then R has a Big CM algebra. [For R a domain and R^+ the integral closure of R in an algebraic closure of the quotient field of R, R^+ is a big Cohen Macaulay algebra.] In 2003, Hochster showed R has a big Cohen Macaulay algebra for $\dim R \leq 3$ using Heitmann's proof of DSC in dimension 3.

Conjecture (**Small CM Conjecture**)**.** *If (R, m) is a complete local ring, then R has a finitely generated maximal Cohen Macaulay module.*

It is clear that the Small CM Conjecture implies the Big CM Conjecture: If M is a finitely generated maximal Cohen Macaulay module for $\widehat{R}$ then it is a Big Cohen Macaulay module for R.

In summary, we have the following conjectures/theorems and implications thus far.

$$\begin{tikzcd}
& \text{Small CM} \arrow[d, Rightarrow, "*"] & \\
& \text{Big CM} \arrow[d, Rightarrow] & \\
& \text{(DSC)} \Leftrightarrow \text{(MC)} \arrow[dl, Rightarrow] \arrow[d, Rightarrow] & \\
\text{Syzygy Conj} & \text{(NIC)} \arrow[d, Rightarrow, "*"] & \text{(SIC)} \arrow[dl, Rightarrow, "*"] \\
& \text{(IC)} \arrow[dl, Rightarrow, "*"] \arrow[d, Rightarrow] & \\
\text{(ZDC)} & \text{Bass' Conjecture} &
\end{tikzcd}$$

Furthermore, we have proved all of the implications with an asterisk. Our goal now is to prove the other implications, and to give a proof in characteristic p of the Big CM Conjecture. To do that, we first need to build up some necessary machinery.

Let R be Noetherian, Q the total quotient ring ($Q = R_W$ for $W = R \setminus \{P_1 \cup \cdots \cup P_n\}$ where P_i are the maximal associated primes of R). Note the maximal ideals of Q are P_iQ and so Q is semilocal.

Exercise. (From [BH]) If R is semilocal and M is a finitely generated $R-$module then M is free if and only if M_m is a free R_m-module for all maximal ideals m and $\operatorname{rank} M_{m_i} = \operatorname{rank} M_{m_j}$ for all maximal ideals m_i, m_j.

Definition. *If $\phi : M \to N$ is an $R-$linear map, define* $\operatorname{rank} \phi := \operatorname{rank} \operatorname{im} \phi$ *(if $\operatorname{im} \phi$ has a rank).*

Proposition 12. *Let R be Noetherian and $F_1 \xrightarrow{\phi} F_0 \to M \to 0$ exact where F_0, F_1 are finitely generated free $R-$modules. The following statements are equivalent:*

(1) $\operatorname{rank} M = r$.
(2) *There exists an exact sequence* $0 \to R^r \to M \to T \to 0$*. where T is torsion*
(3) *For all* $p \in \operatorname{Ass}_R R$, $M_p \cong R_p^r$.
(4) $\operatorname{rank} \phi = \operatorname{rank} F_0 - r$.

Proof. (1)⇒(2): Suppose $Q^r = M \otimes_R Q = M_W = (R_W)^r$. Choose $x_1, ..., x_r \in M$ such that $\frac{x_1}{1}, ..., \frac{x_r}{1}$ is an R_W-basis. Then $x_1, ..., x_r$ are $R-$linearly independent. So we have $0 \to R^r \to M \to M/R^r \to 0$ where the first map is defined by $r_i \mapsto x_i$. Localizing at W gives us $(M/R^r)_W = 0$. Thus M/R^r is torsion.

(2)⇒(1): Localize at W to get $0 \to R_W^r \to M_W \to 0$. Thus M_W is a free R_W-module of rank r.

(1)⇒(3): Let $M_W \cong R_W^r$. Now $M_p = (M_W)_{pR_W} \cong (R_W)^r_{pR_W} \cong R_p^r$ for all $p \in \operatorname{Ass}_R R$.

(3)⇒(1): The maximal ideals of R_W are pR_W for $p \in \operatorname{Ass}_R R$ maximal. So M_m is free of rank r for all maximal ideals of R_W. Now apply the exercise.

(3)⇒(4): We have $0 \to \operatorname{im} \phi \to F_0 \to M \to 0$ is exact. Localize to get $0 \to (\operatorname{im} \phi)_p \to (F_0)_p \to M_p \to 0$. Since M_p is free of rank r, the sequence splits. Thus $(\operatorname{im} \phi)_p$ is free of rank equal to $\operatorname{rank} F_0 - r$. By (3)⇒(1), this says $\operatorname{rank} \operatorname{im} \phi = \operatorname{rank} F_0 - r$.

(4)⇒(3) We first need to prove the following fact.

Fact. Let (R, m) be local. If $\operatorname{depth} R = 0$ and $\operatorname{pd}_R M < \infty$, then M is free.

Proof. Take a minimal free resolution of $M : 0 \to R^{t_r} \xrightarrow{(a_{ij})} R^{t_{r-1}} \to \cdots \to R^{t_0} \to M \to 0$. Then $a_{ij} \in m = (0 :_R x)$ for some $x \in m$ as $m \in \operatorname{Ass} R$. Let $\mathbf{x}$ denote a column vector with x in every row. Then $(a_{ij})\mathbf{X} = 0$ and thus a_{ij} is not injective. This is a contradiction unless $r = 0$ and $M \cong R^{t_0}$. □

Now $\operatorname{rank} \phi = \operatorname{rank} F_0 - r$. We have $0 \to (\operatorname{im} \phi)_p \to (F_0)_p \to M_0 \to 0$ where $(\operatorname{im} \phi)_p$ is a free R_p-module. Thus $\operatorname{pd}_{R_p} M_p < \infty$ but $p \in \operatorname{Ass}_R R$. So $\operatorname{depth} R_p = 0$. By the fact, M_p is free of rank $F_0 - \operatorname{rank} \phi$ for all $p \in \operatorname{Ass} R$.

□

Proposition 13. *Let R be Noetherian, $0 \to A \to B \to C \to 0$ an exact sequence of finitely generated modules. If any two of A, B, and C have a rank, then so does the third and* $\operatorname{rank} B = \operatorname{rank} A + \operatorname{rank} C$.

Proof. Without loss of generality, we may assume (R, m) is local and $\operatorname{depth} R = 0$. If C is free, the sequence splits and we are done. If A, B are free, then $\operatorname{pd}_R C < \infty$. Then C is free as $\operatorname{depth} R = 0$ and again the sequence splits. □

Corollary 14. *Suppose $0 \to F_r \xrightarrow{\phi_r} F_{r-1} \xrightarrow{\phi_{r-1}} \cdots \xrightarrow{\phi_1} F_0$ is an acyclic sequence of finitely generated free $R-$modules. Then* $\operatorname{rank} \phi_i = \sum_{j=i}^r (-1)^{j-i} \operatorname{rank} F_j$.

Proof. Recall $\operatorname{rank} \phi_i = \operatorname{rank} \operatorname{im} \phi_i$. Use the sequence $0 \to F_r \to \cdots \to F_i \to \operatorname{im} \phi_i \to 0$ along with the proposition and induction to show that $\operatorname{im} \phi_i$ has a rank and it is equal to $\sum_{j=i}^r (-1)^{j-i} \operatorname{rank} F_j$. □

<u>Question.</u> When does a finitely generated R-module M have a rank? If:

- R is a domain.
- M has a finite free resolution (that is, M has a free resolution of finite length consisting of finitely generated free R-modules).
- (R, m) is local and $\operatorname{pd}_R M < \infty$
- M is a finitely generated projective module and R has no nontrivial idempotents.
- M is projective and $\operatorname{rank} M_q = \operatorname{rank} M_p$ for all minimal primes p, q.

Definition. *Let A be an $m \times n$ matrix with coefficients in some ring R. For $1 \leq r \leq \min\{m,n\}$, let $I_r(A)$ be the ideal generated by the r-sized minors of A. [For $r \leq 0$ we let $I_r(A) = R$ and for $r > \min\{m,n\}$ we let $I_r(A) = 0$.]*

Facts.

- $I_r(A) \subseteq I_{r-1}(A)$ for all r (since an r-sized minor can be written as a linear combination of $r-1$-sized minors).
- $I_r(AB) \subseteq I_r(A) \cap I_r(B)$
- Suppose $\phi : F \to G$ is a homomorphism of finitely generated free $R-$modules and A, B are two matrices representing ϕ with respect to bases of F and G. Then $A = UBV$ where U, V are invertible. Thus $I_r(A) = I_r(B)$ for all r (by the preceding fact). Thus, we may define $I_r(\phi)$ to be $I_r(A)$ where A is any matrix representing ϕ.
- If R is a field and $\operatorname{rank} A = r$, then $I_r(A) = R$ and $I_{r+1}(A) = 0$.
- If S is an $R-$algebra, $\phi \otimes 1 : F \otimes_R S \to G \otimes_R S$, then $I_r(\phi \otimes 1) = I_r(\phi)S$ for all r.

Proposition 15. *Let R be a ring, $F_1 \xrightarrow{\phi} F_0 \to M \to 0$ a finite presentation (so F_1, F_0 are finitely generated free modules). Let $p \in \operatorname{Spec} R, t \in \mathbb{Z}$. The following are equivalent.*

- $I_t(\phi) \not\subseteq p$.
- $(\operatorname{im}\phi)_p$ *contains a free direct summand of* $(F_0)_p$ *of rank* t.
- $\mu(M_p) \leq \operatorname{rank} F_0 - t$.

Proof. Without loss of generality, we may assume R is local and $p = m$. Let the bar notation represent passage to R/m. Then

- $I_t(\phi) = R$ if and only if $I_t(\overline{\phi}) = R/m$.
- $\mu(M) = \mu(M/mM) = \dim_{R/m} M/mM$ and $\operatorname{rank}_R F_0 = \operatorname{rank}_{R/m} F_0/mF_0$.
- $\operatorname{im}\phi$ contains a free summand of F_0 of rank t if and only if $\operatorname{im}\overline{\phi}$ contains a free summand of $\overline{F_0}$ of rank t.

  Proof. One direction is clear. Suppose $\operatorname{im}\overline{\phi} = (\operatorname{im}\phi + mF_0)/mF_0$ contains a free summand of F_0/mF_0 of rank t. Then there exists $u_1, ..., u_t \in \operatorname{im}\phi$ such that $\overline{u_1}, ..., \overline{u_t} \in F_0/mF_0$ are part of a basis for F_0/mF_0. By Nakayama's Lemma, $u_1, ..., u_t$ form part of a basis for F_0. Thus $U = Ru_1 + ... + Ru_t \subseteq \operatorname{im}\phi$ is a direct summand of F_0 of rank t. □

Hence the proposition holds if and only if it does over a field, which is clear. □

Proposition 16. *Let R be a ring, $F_1 \xrightarrow{\phi} F_0 \to M \to 0$ a presentation. The following are equivalent*

(1) $I_t(\phi) \not\subseteq p, I_{t+1}(\phi)_p = 0$.
(2) $(\operatorname{im}\phi)_p$ *is a direct summand of* $(F_0)_p$ *of rank* t.
(3) M_p *is free of rank* $F_0 - t$.

Proof. Assume (R, m) is local, $p = m$. Note that (2) $\Leftrightarrow$ (3) follows from the sequence $0 \to \operatorname{im}\phi \to F \to M \to 0$. For (2) $\Rightarrow$ (1), choose a basis so ϕ has the identity matrix in the upper left corner and zeros everywhere else. For (1) $\Rightarrow$ (2), the previous proposition says $\operatorname{im}\phi$ contains a free direct summand of rank t. Thus $\phi = \left(\begin{smallmatrix} I_{t\times t} & 0 \\ 0 & B \end{smallmatrix}\right)$. If $B \neq 0$, there exists a nonzero $(t+1)-$sized minor, a contradiction. So $B = 0$ and $\operatorname{im}\phi$ is a direct summand of rank t. □

Corollary 17. *Let R be a ring, $F_1 \xrightarrow{\phi} F_0 \to M \to 0$ a presentation. The following are equivalent*

(1) M *is projective of rank equal to* $\operatorname{rank} F_0 - t$
(2) $I_t(\phi) = R, I_{t+1}(\phi) = 0$.

Proof. For (1) $\Rightarrow$ (2), since $r := \operatorname{rank} M = \operatorname{rank} F_0 - t = r$, we have $M_p \cong R_p^r$ for all $p \in \operatorname{Min}_R R$. As M is projective, $M_q \cong R_q^r$ for all $q \in \operatorname{Spec} R$ (as each q contains a minimal prime). By the proposition, $I_t(\phi) \not\subseteq q$ for all $q \in \operatorname{Spec} R$ and $I_{t+1}(\phi)_q = 0$ for all $q \in \operatorname{Spec} R$. Thus $I_t(\phi) = R$ and $I_{t+1}(\phi) = 0$.

For (2) $\Rightarrow$ (1), we have $I_t(\phi) \not\subseteq q$ and $I_{t+1}(\phi)_q = 0$ for all $q \in \operatorname{Spec} R$. Thus M_q is free of rank equal to $\operatorname{rank} F_0 - t$ by the proposition. Therefore, M is projective. Additionally, if R is Noetherian, then M has a rank. □

Corollary 18. *Let R be Noetherian, $\phi : F \to G$ a map of finitely generated free $R-$modules. The following are equivalent*

(1) $\operatorname{rank} \phi = r$.
(2) $\operatorname{grade} I_r(\phi) \geq 1$ *and* $I_{r+1}(\phi) = 0$.

Proof. For (1) $\Rightarrow$ (2), $(\operatorname{im} \phi)_p$ is a free R_p-module for all $p \in \operatorname{Ass}_R R$. Therefore $0 \to (\operatorname{im} \phi)_p \to G_p \to \operatorname{coker} \phi_p \to 0$. Now $\operatorname{pd}(\operatorname{coker} \phi_p) < \infty$ and $\operatorname{depth} R_p = 0$ imply $\operatorname{coker} \phi_p$ is free. Thus $(\operatorname{im} \phi)_p$ is a direct summand of G_p of rank r. Hence $I_r(\phi) \not\subset p$ for all $p \in \operatorname{Ass} R$ and $I_{r+1}(\phi)_p = 0$ for all $p \in \operatorname{Ass} R$. Therefore, $I_r(\phi)$ contains a non-zerodivisor and $I_{r+1}(\phi) = 0$.

Note (2) $\Rightarrow$ (1) follows directly from the second proposition. $\square$

Definition. *Let R be a ring and $G. : 0 \to G_s \xrightarrow{\phi_s} G_{s-1} \to \cdots \xrightarrow{\phi_1} G_0 \to 0$ be a complex of $R-$modules. Say $G.$ is* ***split acyclic*** *if it is acyclic and $\phi_i(G_i)$ is a direct summand of G_{i-1} for all $i \geq 1$. Equivalently, G is split acyclic if $0 \to \operatorname{im} \phi_i \to G_{i-1} \to \operatorname{im} \phi_{i-1} \to 0$ is split exact for all $i \geq 2$ and $0 \to \operatorname{im} \phi \to G_0 \to H_0(G.) \to 0$ is split exact.*

Remark. If $G.$ is split acyclic, so is $G. \otimes_R M$ for any $R-$module M.

Definition. *Let R be a ring, M an $R-$module, and $p \in \operatorname{Spec} R$. Say $p \in \operatorname{Ass}_R M$ if $p = (0 :_R x)$ for $x \in M$. Equivalently, if there exists a injective map $R/p \to M$.*

Note. If R is Noetherian and M arbitrary, then $\operatorname{Ass}_R M = \emptyset$ if and only if $M = 0$.

Lemma 19. *Let (R, m) be quasi-local, M and $R-$modules, and suppose $m \in \operatorname{Ass}_R M$. Let $\phi : F \to G$ be a map of finitely generated free $R-$modules. The following are equivalent:*

(1) *ϕ is a split injection.*
(2) *$\phi \otimes_R 1_M : F \otimes_R M \to G \otimes_R M$ is injective.*
(3) *$\overline{\phi} : F/mF \to G/mG$ is injective.*

Proof. Note that (1) $\Rightarrow$ (2) is clear and we leave (3) $\Rightarrow$ (1) as an exercise. For (2) $\Rightarrow$ (3), note that there exists a map $0 \to R/m \to M$ as $m \in \operatorname{Ass}_R M$. So we have the commutative diagram

$$\begin{tikzcd} & F \otimes_R R/m \arrow[r, "\overline{\phi}"] \arrow[d] & G \otimes_R R/m \arrow[d] \\ 0 \arrow[r] & F \otimes_R M \arrow[r, "\phi\otimes 1"] & G \otimes_R M \end{tikzcd}$$

where the down arrows are injective as F, G are flat. Thus the top horizontal arrow is injective by commutativity of the diagram. $\square$

Proposition 20. *Let R be a ring, M an $R-$module, $p \in \operatorname{Ass}_R M$. Let $F. = 0 \to F_s \xrightarrow{\phi_s} F_{s-1} \to \cdots \xrightarrow{\phi_1} F_0 \to 0$ be a complex of finitely generated free modules. The following are equivalent:*

(1) *$F. \otimes_R M_p$ is acyclic.*
(2) *$(F.)_p$ is split acyclic.*
(3) *For all $i = 1, ..., s$, $I_{r_i}(\phi_i) \not\subset p$ where $r_i = \sum_{j=i}^{s} (-1)^{j-i} \operatorname{rank} F_j$.*

Furthermore, if (1), (2), or (3) is satisfied, then for all i we have $I_t(\phi_i)_p = 0$ for $t > r_i$.

Proof. Without loss of generality, we may assume (R, m) is quasilocal and $p = m \in \operatorname{Ass}_R M$.

(1)$\Rightarrow$(2): For $s = 1$, we have $0 \to F_1 \xrightarrow{\phi_1} F_0 \to 0$. By assumption, $0 \to F_1 \otimes_R M \to F_0 \otimes_R M$ is injective. By the lemma, ϕ is a split injection. Suppose $s \geq 2$. Let $F'_. : 0 \to F_s \to \cdots \to F_1 \to 0$. Then $F'_. \otimes M$ is acyclic. By induction, $F'_.$ is split acyclic. Therefore $\phi_i(F_i)$ is a direct summand of F_{i-1} for all $i \geq 2$ and $0 \to \operatorname{im} \phi_2 \to F_1 \to \operatorname{coker} \phi_2 \to 0$ is split exact. Thus $\operatorname{coker} \phi_2$ is free. We have

$F_2 \otimes_R M \xrightarrow{\phi_2 \otimes 1} F_1 \otimes_R M \to F_0 \otimes_R M$ is exact. Now $(\operatorname{coker} \phi_2) \otimes_R M \cong F_1 \otimes_R M / \operatorname{im}(\phi_2 \otimes 1) = F_1 \otimes_R M / \ker(\phi_1 \otimes 1) \hookrightarrow F_0 \otimes_R M$. Consider the following commutative diagram:

$$\begin{tikzcd}
\operatorname{coker} \phi_2 \arrow[r, "\phi_1"] \arrow[d] & F_0 \\
F_1 / \ker \phi_1 \arrow[ur, "\phi_1"'] &
\end{tikzcd}$$

where again recall that $\operatorname{coker} \phi_2$ is free. By the argument above, tensoring the top arrow with M yields an injection. By the lemma, we have that $\operatorname{coker} \phi_2 \to F_0$ is a split injection. Thus, the natural surjection $\operatorname{coker} \phi_2 = F_1 / \operatorname{im} \phi_2 \to F_1 / \ker \phi_1$ is injective as well, which implies $\operatorname{im} \phi_2 = \ker \phi_1$ and $\phi_1(F_1)$ is a direct summand of F_0.

(2)⇒(1): Clear by the remark

(2)⇒(3): As $F.$ is split acyclic, $\overline{F.} := F. \otimes_R R/m$ is split acyclic. Now $0 \to \overline{F_s} \to \overline{F_{s-1}} \to \cdots \to \overline{F_i} \to \operatorname{im} \overline{\phi_i} \to 0$ is exact with rank $\overline{\phi_i} = \dim \overline{\phi_i} = r_i$. Thus $I_{r_i}(\overline{\phi_i}) \neq 0$, which implies $I_{r_i}(\phi_i) \not\subset m$ for all i.

(3)⇒(2): We use induction on s. For $s = 1$, we have $0 \to F_1 \xrightarrow{\phi_1} F_0 \to 0$. Now $r_1 = \operatorname{rank} F_1$. By assumption $I_{r_1}(\phi_1) \not\subset m$ and so $I_{r_1}(\phi_1) = R$. Of course, $I_{r_1+1}(\phi_1) = 0$ as $r_1 = \operatorname{rank} F_1$. Thus $\operatorname{im} \phi_1$ is a direct summand of F_0 of rank r_1, which implies ϕ_1 is injective (as F_1 and $\phi(F_1)$ have the same rank) and ϕ_1 splits.

Let $s > 1$ and $F'_. := 0 \to F_s \to \cdots \to F_1 \to 0$. By induction, $F'_.$ is split acyclic. Thus it is enough to show $\operatorname{im} \phi_2 = \ker \phi_1$ and $\phi_1(F_1)$ is a direct summand of F_0. Now $\operatorname{coker} \phi_2 = F_1 / \operatorname{im} \phi_2$ is free of rank r_1 (since $F'_.$ is split acyclic). By assumption, $I_{r_1}(\phi_1) \not\subset m$. By a previous proposition, $\operatorname{im} \phi_1$ contains a direct summand U of F_0 of rank r_1. Let ψ be the composition of the following maps:

$$\operatorname{coker} \phi_2 = F_1 / \operatorname{im} \phi_2 \twoheadrightarrow F_1 / \ker \phi_2 \twoheadrightarrow \operatorname{im} \phi_1 \twoheadrightarrow U.$$

Since ψ is a surjective homomorphism of free modules of the same rank, ψ is an isomorphism. Thus, $\operatorname{im} \phi_2 = \ker \phi_1$ and $\operatorname{im} \phi_1 = U$, a direct summand of F_0. Thus $F.$ is split acyclic.

For the last statement, note that since $0 \to F_s \to \cdots \to F_i \to \operatorname{im} \phi_i \to 0$ is split exact for all i, $\operatorname{im} \phi_i$ is free of rank r_i. By one of the previous propositions, $I_t(\phi_i) = 0$ for all $t > r_i$.

□

Remark. Let R be Noetherian and suppose $F. : 0 \to F_s \xrightarrow{\phi_s} F_{s-1} \to \cdots \xrightarrow{\phi_1} F_0 \xrightarrow{\phi_0} M \to 0$ is exact with F_i finitely generated free. Then $\operatorname{rank} \phi_i = \sum_{j=i}^{s} (-1)^{j-i} \operatorname{rank} F_j$.

Proof. By truncating, it is enough to show the $i = 0$ case. Let $p \in \operatorname{Ass}_R R$ and localize to get $(F.)_p$ is exact. Since depth $R_p = 0$, $(\operatorname{im} \phi_0)_p = M_p$ is a free R_p-module. Thus the sequence $(F.)_p$ splits. □

Corollary 21. *Let R be a ring, $F. : 0 \to F_s \xrightarrow{\phi_s} F_{s-1} \to \cdots \xrightarrow{\phi_1} F_0 \xrightarrow{\phi_0} M \to 0$ is exact. Then $r_i = \sum_{j=i}^{s} (-1)^{j-i} \operatorname{rank} F_j \geq 0$ for all i.*

Proof. Fix bases for all the F_i and let A_i be the matrix representation of ϕ_i. Let S be the subring of R generated by the prime subring of R together with all entries from the A_i's. Then S is Noetherian. Let G_i denote the free $S-$module of rank equal to rank F_i and let $\psi_i : G_i \to G_{i+1}$ be defined by multiplication by A_i. Certainly $A_{i-1}A_i = 0$ (in R and thus in S). So $G.$ is a complex of finitely generated free $S-$modules and $G. \otimes_S R = F.$ is acyclic. Let $p \in \operatorname{Ass}_S R$. Then $G. \otimes_S R_p$ is acyclic, which implies by the proposition that $(G.)_p$ is split acyclic. By the Noetherian case, we have $0 \leq \operatorname{rank} \psi_i = \sum_{j=i}^{s} (-1)^{j-i} \operatorname{rank} G_j = r_i$ for all i. □

Let R be a ring, M an R-module and $\underline{x} = x_1, \ldots, x_n$ elements of R. The Čech complex $C^{\cdot}(\underline{x}; R)$ of R with respect to $\underline{x}$ is defined to be the cochain complex

$$\bigotimes_{i=1}^{n}(0 \to R \xrightarrow{x_i} R \to 0).$$

The Čech complex $C^{\cdot}(\underline{x}; M)$ is defined to be $C^{\cdot}(\underline{x}; R) \otimes_R M$. It is easily seen (by induction) that

$$C^{\cdot}(\underline{x}; M)^i = \bigoplus_{1 \le j_1 < j_2 < \cdots < j_i \le n} M_{x_{j_1} \cdots x_{j_i}}.$$

The ith cohomology of $C^{\cdot}(\underline{x}; M)$ is called the ith Čech cohomology of M with respect to $\underline{x}$ and is denoted $H^i_{(\underline{x})}(M)$. If R is Noetherian then $H^i_{(\underline{x})}(M)$ is isomorphic to the ith local cohomology of M with support in the ideal $(\underline{x})$. This is not the case in general. However, one can still show that Čech cohomology with respect to $\underline{x}$ and $\underline{y}$ are isomorphic if $(\underline{x}) = (\underline{y})$, or in fact even if $\sqrt{(\underline{x})} = \sqrt{(\underline{y})}$.

Proposition 22. *Let $\underline{x} = x_1, ..., x_n \in R$ and M an $R-$module. Then for all i and for all $u \in H^i_{(\underline{x})}(M)$ there exists ℓ such that $(\underline{x})^{\ell} u = 0$ (that is, $H^i_{(\underline{x})}(M)$ is $(\underline{x})-$torsion).*

Proof. It is enough to show there exists ℓ_j such that $x_j^{\ell_j} u = 0$. Equivalently, $H^i_{(\underline{x})}(M)_{x_j} = 0$. Since localization is flat, $H^i_{(\underline{x})}(M)_{x_j} \cong H^i_{(\underline{x})R}(M_{x_j}) \cong H^i_{(\underline{x})R_{x_j}}(M_{x_j}) \cong H^i_{R_{x_j}}(M_{x_j}) = 0$ for all i. $\square$

Proposition 23. *Let $\underline{x} = x_1, ..., x_n \in R$ and M an $R-$module. Suppose $(\underline{x})M \neq M$. Then there exists i such that $H^i_{(\underline{x})}(M) \neq 0$.*

Proof. Suppose $H^i_{(\underline{x})}(M) = 0$ for all i. Then $0 \to M \xrightarrow{\phi_0} \oplus M_{x_i} \xrightarrow{\phi_1} \cdots \xrightarrow{\phi_{n-1}} M_{x_1 \cdots x_n} \xrightarrow{\phi_n} 0$ is exact. Let $K_i = \ker \phi_i$. We will show by induction that $\operatorname{Tor}^R_j(R/(\underline{x}), K_{n-i}) = 0$ for all j and $i \geq 0$. [Note that $M = K_0$. If the claim holds, then $M/\underline{x}M = \operatorname{Tor}^R_0(R/(\underline{x}), K_0) = 0$ implies $M = (\underline{x})M$.] For $i = 0$, we see $K_n = M_{x_1 \cdots x_n}$. So $\operatorname{Tor}^R_j(R/(\underline{x}), M_{x_1 \cdots x_n}) = \operatorname{Tor}^R_j(R/(\underline{x}), M)_{x_1 \cdots x_n} = \operatorname{Tor}^R_j(0, M_{x_1 \cdots x_n}) = 0$ (where the first equality holds as $R_{x_1 \cdots x_n}$ is flat). For $i > 0$, let $C^{i-1} = C^{i-1}(\underline{x}; R)$. Then we have $0 \to K_{i-1} \to C^{i-1} \otimes_R M \to K_i \to 0$. Now $\operatorname{Tor}^R_j(R/(\underline{x}), C^{i-1} \otimes_R M) \cong \operatorname{Tor}^R_j(R/(\underline{x}); M) \otimes_R C^{i-1} \cong 0$ for all j as the Tor is annihilated by $(\underline{x})$ and C^{i-1} is a direct sum of localizations at subproducts of $x_1 \cdots x_n$. By induction and the long exact sequence on Tor, we have $\operatorname{Tor}^R_j(R/(\underline{x}), K_{i-1}) = 0$ for all j. $\square$

Definition. *Let $I = (\underline{x})$ be a finitely generated ideal and M an $R-$module. Define* $\operatorname{grade}(I, M) = \sup\{k | H^i_I(M) = 0$ *for all* $i < k\}$.

Note that by the Proposition, if I is a finitely generated ideal and $IM \neq M$ then $\operatorname{grade}(I, M) < \infty$. Also $\operatorname{grade}(I, M) > 0$ if and only if $H^0_I(M) = 0$ if and only if $(0 :_M I) = 0$ which is if and only if $\operatorname{Hom}_R(R/I, M) = 0$. If R is Noetherian and M is finitely generated, we know (by primary decomposition) that $\operatorname{grade}(I, M) > 0$ if and only if I contains a non-zero-divisor on M. However, this does not hold if R is not Noetherian or M is not finitely generated, as the following examples show:

Example. Let $R = k[x, y]_{(x,y)}$, $m = (x, y)R$, and $M = \oplus\{R/p$ where the sum is over all height one primes p of R. Note every element of m is a zero-divisor on M (for $f \in m \setminus \{0\}$, we have $f \in p$ for some height 1 prime p and so $f(u_q) = 0$ where $u_q = 0$ if $q \neq p$ and $u_q = 1$ if $q = p$). However, $\operatorname{grade}(m, M) = 0$, that is $(0 :_M m) = 0$. Let $(u_q) \in (0 :_M m)$ so $m u_q = \overline{0}$ in R/q for all q. As $m \not\subset q$, we have $u_q = \overline{0}$.

Example. Let R and M be as above and set $S = R \times M = \{(r, m) | r \in R, m \in M\}$ with $(r_1, m_1) \cdot (r_2, m_2) = (r_1 r_2, r_1 m_2 + r_2 m_1)$, the idealization of M. Then S is a commutative quasi-local ring with maximal ideal $m \times M$. Then $n = mS$ consists of zerodivisors on S, yet $\operatorname{grade}(n, S) > 0$.

Lemma 24. *Suppose $I \subseteq J$ are finitely generated ideals, M any $R-$module. Then* $\operatorname{grade}(I, M) \leq \operatorname{grade}(J, M)$.

Proof. By induction, it is enough to show the case $J = (I, x)$. Then we have the long exact sequence

$$\cdots \to H_I^{i-1}(M)_x \to H_J^i(M) \to H_I^i(M) \to H_I^i(M)_x \to \cdots .$$

If $i < \operatorname{grade}(I, M)$ then $H_J^i(M) = 0$ which implies $i < \operatorname{grade}(J, M)$. $\square$

By virtue of this lemma, we can make the following definition:

Definition. *Let I be an ideal of a ring R and M an R-module. We set*

$$\operatorname{grade}(I, M) := \sup\{\operatorname{grade}(J, M) | J \subseteq I, J \text{ f.g.}\}.$$

If (R, m) is quasilocal, we define $\operatorname{depth} M := \operatorname{grade}(m, M)$.

Proposition 25. *Let R be a ring, I an ideal and M an $R-$module.*

(1) $\operatorname{grade}(I, M) = \operatorname{grade}(\sqrt{I}, M)$
(2) *If $R \to S$ is flat,* $\operatorname{grade}(I, M) \leq \operatorname{grade}(IS, M \otimes_R S)$
(3) *If $R \to S$ is faithfully flat,* $\operatorname{grade}(I, M) = \operatorname{grade}(IS, M \otimes_R S)$
(4) *For any ring homomorphism $R \to S$ and $S-$module M,* $\operatorname{grade}_R(I, M) = \operatorname{grade}_S(IS, M)$.
(5) *Suppose $0 \to A \to B \to C \to 0$ is exact. Then*

$$\begin{aligned} \operatorname{grade}(I, B) &\leq \min\{\operatorname{grade}(I, A), \operatorname{grade}(I, C)\}, \\ \operatorname{grade}(I, A) &\leq \min\{\operatorname{grade}(I, B), \operatorname{grade}(I, C) + 1\}, \\ \operatorname{grade}(I, C) &\leq \min\{\operatorname{grade}(I, C), \operatorname{grade}(I, A) - 1\}. \end{aligned}$$

(6) *If $x \in I$ is a non-zerodivisor on M, then* $\operatorname{grade}_{R/(x)}(I/(x), M/xM) = \operatorname{grade}_R(I, M/xM) = \operatorname{grade}_R(I, M) - 1$.
(7) *If I is finitely generated, then there exists $p \in \operatorname{Spec} R$ with $p \supseteq I$ such that* $\operatorname{grade}(I, M) = \operatorname{grade}(pR_p, M_p) = \operatorname{depth} M_p$.

Definition. *Let M be an $R-$module and $x_1, ..., x_n \in R$. We say $x_1, ..., x_n$ is a* ***weak*** *$M-$**sequence** (or* ***weakly*** *$M-$**regular**) if x_i is a non-zerodivisor on $M/(x_1, ..., x_n)M$ for all i.*

Note that any $M-$sequence is a weak $M-$sequence. Furthermore, if $M = 0$ then any sequence is a weak M-sequence. Now let

$$\operatorname{Grade}(I, M) = \sup\{n | \text{ there exists a weak } M-\text{sequence of length } n \text{ in } I\}.$$

Note that $\operatorname{Grade}(I, 0) = \infty = \operatorname{Grade}(R, M)$ for any ideal I and R-module M. Furthermore, by part (6) of the Proposition on grade, $\operatorname{Grade}(I, M) \leq \operatorname{grade}(I, M)$. If $R \to S$ is faithfully flat, $\operatorname{Grade}(I, M) \leq \operatorname{Grade}(IS, M \otimes_R S)$ and $\operatorname{grade}(I, M) = \operatorname{grade}(IS, M \otimes_R S)$.

Notice that if $x \in I$ is a non-zero-divisor on M and $xM = M$, then $\operatorname{Grade}(I, M) = \infty = \operatorname{grade}(I, M)$. Perhaps because of this, Bruns and Herzog adopt the convention that $\operatorname{grade}(I, M) = \operatorname{Grade}(I, M) = \infty$ if $IM = M$. However, this differs from our conventions, as shown by the following example:

Example. Let $R = \mathbb{Z}_{(2)}$, $m = (2)R$, and $M = \mathbb{Q}/\mathbb{Z}_{(2)}$. Then every element of m is a zero-divisor on M, $mM = M$ and $\operatorname{grade}(m, M) = 0 = \operatorname{Grade}(m, M)$.

Lemma 26. *Let R be a ring, $I \subset R$, M an $R-$module. Let T be an indeterminate over R. If* $\operatorname{grade}(I, M) > 0$ *then* $\operatorname{Grade}(IR[T], M[T]) > 0$ *where $M[T] = M \otimes_R R[T]$.*

Proof. Note that $\operatorname{grade}(I, M) > 0$ implies $\operatorname{grade}(J, M) > 0$ for some finitely generated ideal J contained in I. Thus, $(0 :_M J) = 0$. Let $J = (a_1, ..., a_t)$.

Claim. $a_1 t + ... + a_n t^n$ is a non-zero-divisor on $M[T]$.

Proof. If $a_1 t + ... + a_n t^n$ is a zero-divisor on $M[T]$ then Nick's exercise (Homework set 1) says there exists $m \in M \setminus \{0\}$ such that $(a_1 t + ... + a_n t^n)m = 0$. Then $Jm = 0$, a contradiction.

Thus $\operatorname{Grade}(IR[T], M[T]) > 0$ as $a_1 t + ... + a_n t^n \in IR[T]$. $\square$

Proposition 27. *If* $\operatorname{grade}(I, M) \geq s$, *then* $\operatorname{Grade}(IR[T_1, ..., T_s], M[T_1, ..., T_s]) \geq s$.

Proof. The $s = 1$ case was proved in the lemma. So suppose $s > 1$. By the $s - 1$ case, there exists $f_1, ..., f_{s-1} \in \tilde{I} := IR[T_1, ..., T_{s-1}]$ which is a weak $\tilde{M} = M[T_1, ..., T_{s-1}]-$sequence. Then $\operatorname{grade}(\tilde{I}, \tilde{M}/(f_1, ..., f_{s-1})\tilde{M}) = \operatorname{grade}(\tilde{I}, \tilde{M}) - s - 1 = \operatorname{grade}(I, M) - s - 1 \geq 1$ since $R \to R[T_1, \ldots, T_{s-1}]$ is a faithfully flat extension. By the lemma, $\operatorname{Grade}(\tilde{I}R[T_s], \tilde{M}/(f_1, ..., f_{s-1})[T_s]) \geq 1$. As $\tilde{M}/(f_1, ..., f_{s-1})[T_s] = M[T_1, ..., T_s]/(f_1, ..., f_{s-1})M[T_1, ..., T_s]$, we see that $\operatorname{Grade}(IR[T_1, ..., T_s], M[T_1, ..., T_s]) \geq s$. $\square$

Corollary 28. *With I, M as above, we have*

$$\begin{aligned}\operatorname{grade}(I, M) &= \lim_{n\to\infty} \operatorname{Grade}(IR[T_1, ..., T_n], M[T_1, ..., T_n]) \\ &= \sup\{\operatorname{Grade}(IS, M \otimes_R S) \mid R \to S \textit{ faithfully flat}\}.\end{aligned}$$

Remark.

(1) Let R be a ring, I finitely generated ideal, M an $R-$module. Let S be the subring of R generated over the prime subring by a generating set for $x_1, \ldots, x_n$ for I. Let $J = (x_1, \ldots, x_n)S$. Then S is Noetherian and $\operatorname{grade}_R(I, M) = \operatorname{grade}_S(J, N)$.

(2) Suppose R is Noetherian of dimension d. Then for every ideal I of R and R-module M such that $IM \neq M$, we have $\operatorname{grade}(I, M) \leq d$. In particular, if R is local and $mM \neq M$, then $\operatorname{depth} M \leq \dim R$.

 Proof. Without loss of generality, we may assume R is local. Then $H^i_I(M) = 0$ for all $i > d$ and for all R-modules M. Hence, $\operatorname{grade}(I, M) \leq d$. $\square$

(3) Suppose R is Noetherian, $I \subset R$ and M an $R-$module. Then $\operatorname{grade}(I, M) > 0$ if and only if $I \not\subseteq p$ for all $p \in \operatorname{Ass}_R M$.

 Proof. $\operatorname{grade}(I, M) > 0$ if and only if $(0 :_M I) = 0$. If $(0 :_M I) = 0$ then $I \not\subset p$ for all $p \in \operatorname{Ass}_R M$. Conversely, suppose $(0 :_M I) \neq 0$. Then $Iz = 0$ for some $z \in M \setminus \{0\}$. Consider $N = Rz \subseteq M$. As N is finitely generated, $I \subseteq p$ for some $p \in \operatorname{Ass}_R N \subseteq \operatorname{Ass}_R M$. $\square$

Definition. *Let R be a ring and M an $R-$module and $\phi : F \to G$ where F, G are finitely generated free. Then* $\operatorname{rank}(\phi, M) = r$ *if and only if* $\operatorname{grade}(I_r(\phi), M) \geq 1$ *and* $I_{r+1}(\phi)M = 0$. *If $M = 0$, set* $\operatorname{rank}(\phi, M) = 0$.

Note by a previous result we have $\operatorname{rank}\phi = \operatorname{rank}(\phi, R)$ if R is Noetherian.

Lemma 29. *Let R be a ring, M an $R-$module, $\phi : F \to G$ a map of finitely generated free $R-$modules and $r = \operatorname{rank} F$. Then $F_. \otimes_R M \to G_. \otimes_R M$ is injective if and only if* $\operatorname{grade}(I_r(\phi), M) \geq 1$.

Proof. Let S be the Noetherian subring generated by the entries of a matrix representing ϕ. Let F', G' be free $S-$modules of the same rank as F, G respectively and let $\psi : F' \to G'$ be given by the same matrix as the one representing ϕ. Clearly, the following diagram commutes:

$$\begin{tikzcd} F'_. \otimes_S R \arrow[r, "\psi\otimes 1"] \arrow[d, "\cong"] & G'_. \otimes_S R \arrow[d, "\cong"] \\ F \arrow[r, "\phi"] & G \end{tikzcd}$$

Thus $I_r(\psi)R = I_r(\phi)$ which implies $\text{grade}(I_r(\psi), M) = \text{grade}(I_r(\phi), M)$. Now, consider the commutative squares

$$\begin{array}{ccc} F' \otimes_S M & \longrightarrow & G' \otimes_S M \\ \downarrow \cong & & \downarrow \cong \\ (F' \otimes_S R) \otimes_R M & \longrightarrow & (G' \otimes_S R) \otimes_R M \\ \downarrow \cong & & \downarrow \cong \\ F \otimes_R M & \longrightarrow & G \otimes_R M. \end{array}$$

Hence $F \otimes_R M \to G \otimes_R M$ is injective if and only if $F' \otimes_S M \to G' \otimes_S M$ is injective. Thus we may assume R is Noetherian.

Let K be the kernel of the map $F \otimes_R M \xrightarrow{\phi \otimes 1} G \otimes_R M$. Then $\text{Ass}_R K \subseteq \text{Ass}_R M$. So

$$\begin{aligned} \phi \otimes_R 1 \text{ is injective} \quad &\Leftrightarrow \quad K = 0 \\ &\Leftrightarrow \quad K_p = 0 \text{ for all } p \in \text{Ass}_R M \\ &\Leftrightarrow \quad (\phi \otimes 1)_p : F \otimes_R M_p \to G \otimes_R M_p \text{ is injective for all } p \in \text{Ass}_R M \\ &\Leftrightarrow \quad I_r(\phi) \not\subset p \text{ for all } p \in \text{Ass}_R M \qquad \text{(by Prop 20)} \\ &\Leftrightarrow \quad \text{grade}(I_r(\phi), M) \geq 1. \end{aligned}$$

□

Proposition 30. *Let R be a ring, $M \neq 0$ an $R-$module, $F.$ the complex $0 \to F_s \xrightarrow{\phi_s} F_{s-1} \to \cdots \xrightarrow{\phi_1} F_0 \to 0$. Suppose $F. \otimes_R M$ is acyclic. Then $\text{rank}(\phi_i, M) = r_i$ for $i = 1, ..., s$.*

Proof. As above, we reduce to the case where R is Noetherian. Let $p \in \text{Ass}_R M$. Then $F. \otimes_R M_p$ is acyclic, which implies $I_{r_i}(\phi_i) \not\subset p$ and $I_{r_i+1}(\phi_i)_p = 0$ for all i by Proposition 20. Thus $\text{grade}(I_{r_i}(\phi_i), M) \geq 1$. Fix i and let $I = I_{r_i+1}(\phi_i)$. We want to show $IM = 0$. If $IM \neq 0$, choose $z \in M$ such that $Iz \neq 0$. Let $p \in \text{Ass}_R Iz \subseteq \text{Ass}_R M$. Then $(Iz)_p \neq 0$ implies $I_p \neq 0$, a contradiction. Thus $\text{rank}(\phi_i, M) = r_i$. □

Exercise. Let R be a ring, I, J ideals, and M an $R-$module. Then $\text{grade}(I \cap J, M) = \min\{\text{grade}(I, M), \text{grade}(J, M)\}$.

Exercise. Suppose $N. : \cdots \to N_i \to N_{i-1} \to \cdots \to N_1 \to N_0 \to 0$ is an exact sequence of $R-$modules. Let $x \in R$ be weakly N_i-regular for all i. Then $N. \otimes R/(x)$ is exact.

Lemma 31. *Let (R, m) be a quasi-local ring, $\phi : F \to G$ a map of finitely generated free R-modules, and M an $R-$module. Let $C = \text{coker}(F \otimes_R M \to G \otimes_R M)$. Suppose that $I_r(\phi) = R$ and $I_{r+1}(\phi)M = 0$ for some r. Then C is isomorphic to a direct sum of finitely many copies of M.*

Proof. As $I_r(\phi) = R$, $\text{im}\, \phi$ contains a direct summand of G of rank r. By choosing an appropriate basis, ϕ has the form $\begin{pmatrix} 1_r & 0 \\ 0 & B \end{pmatrix}$, where 1_r denotes the $r \times r$ identity matrix. With respect to this basis, let $\psi : F \to G$ be the map given by $\begin{pmatrix} 1_r & 0 \\ 0 & 0 \end{pmatrix}$. The result follows if we show $\text{im}(\phi \otimes 1_M) = \text{im}(\psi \otimes 1_M)$. Let $B = (b_{ij})$. Its enough to show $b_{ij}M = 0$ for all i, j. But note that, with respect to this basis, each b_{ij} is an $r+1$-sized minor of ϕ. Hence, $b_{ij}M = 0$ by hypothesis. □

Theorem 32 (Buchsbaum-Eisenbud, Northcott)**.** *Let R be a ring, M an $R-$module. Let $F.$ denote the complex $0 \to F_s \xrightarrow{\phi_s} \cdots \to F_1 \xrightarrow{\phi_1} F_0 \to 0$. Then $F. \otimes_R M$ is acyclic if and only if $\text{grade}(I_{r_i}(\phi_i), M) \geq i$ for $i = 1, ..., s$ where r_i are the expected ranks.*

Proof. As usual, we may assume R is Noetherian (by adjoining the entries of the matrices to the prime subring of R). First, we assume that $F. \otimes_R M$ is acyclic and use induction on s. The case when $s = 1$ is done by Lemma 29. So suppose $s > 1$. We want to show $\text{grade}(I_{r_i}(\phi_i), M) \geq i$ for $i = 1, ..., s$. By Proposition 30, $\text{rank}(\phi_i, M) = r_i$. Hence $\text{grade}(I_{r_i}(\phi_i), M) \geq 1$ for all i. By an exercise, $\text{grade}(\cap_{i=1}^s I_{r_i}(\phi_i), M) \geq 1$. By passing to a faithfully flat extension S

of R we can assume that there exists $x \in \cap_{i=1}^{s} I_{r_i}(\phi_i)$ which is weakly $M-$regular. (Note that the hypotheses and the conclusion are stable under passage to faithfully flat extensions.) Consider $0 \to F_s \otimes_R M \to \cdots \xrightarrow{\phi_2 \otimes 1} F_1 \otimes_R M \to \operatorname{coker} \phi_2 \otimes 1 \to 0$, which is exact. Also $0 \to \operatorname{coker}(\phi_2 \otimes I) \to F_0 \otimes_R M$ is exact. As x is weakly $M-$regular, x is weakly regular on $F_i \otimes_R M$ for all i and is weakly regular on $\operatorname{coker}(\phi_2 \otimes 1)$. By the second exercise above, $0 \to F_s \otimes_R M/xM \xrightarrow{\phi_s \otimes 1} \cdots \xrightarrow{\phi_2 \otimes 1} F_1 \otimes M/xM \to 0$ is acyclic. By induction on s, $\operatorname{grade}(I_{r_i}(\phi_i), M/xM) \geq i-1$ for $i = 2, ..., s$. Thus $\operatorname{grade}(I_{r_i}(\phi_i), M) \geq i$ for $i = 2, .., s$. Since we already have $\operatorname{grade}(I_{r_1}(\phi_1), M) \geq 1$, we are done.

Conversely, assume that $\operatorname{grade}(I_{r_i}(\phi_i), M) \geq i$ for all $i = 1, ..., s$. We will use induction on the length s of the complex. The case when $s = 1$ is again done by Lemma 29, so we assume $s > 1$. Let $F'_\cdot$ denote the complex $0 \to F_s \xrightarrow{\phi_s} F_{s-1} \to \cdots \xrightarrow{\phi_2} F_1 \to 0$. By induction, $F'_\cdot \otimes_R M$ is acyclic. For each $i = 1, .., s$, let $M_i = \operatorname{coker}(\phi_{i+1} \otimes 1_M)$. We need to show $F_2 \otimes_R M \to F_1 \otimes_R M \to F_0 \otimes_R M$ is exact at $F_1 \otimes_R M$. Its enough to show the induced map $M_1 \to F_0 \otimes_R M$ is injective. Note by exactness of $F'_\cdot$, that $0 \to M_{i+1} \to F_i \otimes_R M \to M_i \to 0$ is exact for all $i \geq 1$.

Claim. For all $p \in \operatorname{Spec} R$ and for all $i \geq 1$, $\operatorname{depth}(M_i)_p \geq \min\{\operatorname{depth} M_p, i\}$.

Proof. We use induction on $s - i$. Note that $M_s = F_s \otimes_R M$, and hence $\operatorname{depth}(M_s)_p = \operatorname{depth} M_p$ for all primes p. Now suppose $i < s$ and assume that the claim holds for M_{i+1}. By localizing, we may assume (R, m) is local and $p = m$. (Note that if $M = 0$ we are done.) By the short exact sequence above,

$$\operatorname{depth} M_i \geq \min\{\operatorname{depth}(F_i \otimes_R M), \operatorname{depth} M_{i+1} - 1\} = \min\{\operatorname{depth} M, \operatorname{depth} M_{i+1} - 1\}.$$

Suppose first that $\operatorname{depth} M \geq i+1$. Then, as $\operatorname{depth} M_{i+1} \geq \min\{\operatorname{depth} M, i+1\}$, we have $\operatorname{depth} M_i \geq i$. Suppose now that $\operatorname{depth} M \leq i$. Since $\operatorname{grade}(I_{r_{i+1}}(\phi_{i+1}), M) \geq i+1$ (by assumption), $I_{r_{i+1}}(\phi_{i+1}) = R$. Since $i+1 \geq 2$ and $F'_\cdot \otimes_R M$ is exact, $\operatorname{rank}(\phi_{i+1}, M) = r_{i+1}$ by Proposition 30 and so $I_t(\phi_{i+1})M = 0$ for all $t > r_{i+1}$. By Lemma 31, M_i is isomorphic to a direct sum of finitely many copies of M, and hence $\operatorname{depth} M_i = \operatorname{depth} M$.

Let $N = \ker(M_1 \to F_0 \otimes_R M) = H_1(F_\cdot \otimes_R M)$. We want to show $N = 0$. Its enough to show $N_p = 0$ for all $p \in \operatorname{Ass}_R M_1$. Let $p \in \operatorname{Ass}_R M_1$. Then $0 = \operatorname{depth}(M_1)_p \geq \min\{\operatorname{depth} M_p, 1\}$ which implies $\operatorname{depth} M_p = 0$. Then $p \in \operatorname{Ass}_R M$. Since $\operatorname{grade}(I_{r_i}(\phi_i), M) \geq i$ for all $i \geq 1$, $I_{r_i}(\phi_i) \not\subset p$ for all i. By Proposition 20, $(F_\cdot \otimes_R M)_p$ is (split) acyclic which implies $N_p = H_1(F_\cdot \otimes_R M)_p = 0$ for all $p \in \operatorname{Ass}_R M_1$. Thus $N = 0$. ☐

Corollary 33. *Let $0 \to F_s \xrightarrow{\phi_s} F_{s-1} \to \cdots \xrightarrow{\phi_1} F_0 \xrightarrow{\phi_0} N \to 0$ be exact where F_i are finitely generated free R-modules. Let $K_i = \ker \phi_{i-1}$. Then for all $p \in \operatorname{Spec} R$ and all $i \geq 1$, $\operatorname{depth}(K_i)_p \geq \min\{\operatorname{depth} R_p, i\}$.*

Proof. In the Claim in the proof of Buchsbaum-Eisenbud, $M_i = K_i$ for all i. ☐

Theorem 34 (Hilbert-Burch). *Let (R, m) be a Noetherian, local ring and $I \subset R$ an ideal such that $\operatorname{pd}_R R/I = 2$. Then the minimal resolution of R/I has the form $0 \to R^n \xrightarrow{\phi} R^{n+1} \to R \to R/I \to 0$ where $n+1 = \mu_R(I)$. Moreover, $I = xI_n(\phi)$ for some non-zero-divisor x. Conversely, let A be an $(n+1) \times n$ matrix with entries in m and suppose $\operatorname{grade} I_n(A) \geq 2$. Let $\psi : R^{n+1} \to R$ be the map which sends the ith standard basis element e_i to $(-1)^i \Delta_i$, where Δ_i is the $n \times n$ minors of A obtained by deleting the i^{th} row. Then the sequence $0 \to R^n \xrightarrow{A} R^{n+1} \xrightarrow{\psi} R \to R/I_n(A) \to 0$ is exact.*

Proof. We prove the 'converse' first, so suppose A is as above with $\operatorname{grade} I_n(A) \geq 2$. Using cofactor expansion, one can show that $[(-1)\Delta_1, ..., (-1)^n \Delta_n]A = 0$. (This is left as an exercise.) Thus $0 \to R^n \xrightarrow{A} R^{n+1} \xrightarrow{\psi} R$ is a complex. But $I_1(\psi) = I_n(A)$ and so $\operatorname{grade} I_1(\psi) \geq 1$. By Buchsbaum Eisenbud, the complex is acyclic.

Now suppose I is an ideal and $\operatorname{pd} R/I = 2$. A minimal resolution of R/I has the form $(\#)0 \to R^m \xrightarrow{\phi} R^{n+1} \xrightarrow{\delta} R \xrightarrow{\pi} R/I \to 0$. Since this sequence is exact, we must have $1 - (n+1) + m \geq 0$, or $m \geq n$. Also $n + 1 - m \geq 0$, so $n \leq m \leq n+1$. If $m = n+1$ then $\operatorname{rank} \pi = 0$, which implies $I_p = 0$ for all $p \in \operatorname{Ass}_R R$. This means $I = 0$, a contradiction. Hence we must have $m = n$. By Buchsbaum-Eisenbud, we have $\operatorname{grade} I_n(\phi) \geq 2$. Now, fix bases for

R^n and R^{n+1} and let A be the matrix which represents ϕ. Let $\psi : R^{n+1} \to R$ be the map defined above. Consider the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & R^n \arrow[r, "\phi"] \arrow[d, "="] & R^{n+1} \arrow[r, "\delta"] \arrow[d, "="] & I \arrow[r] & 0 . \\
0 \arrow[r] & R^n \arrow[r, "A"] & R^{n+1} \arrow[r, "\psi"] & I_n(A) \arrow[r] & 0
\end{tikzcd}$$

The top row is exact (by hypothesis) and the bottom row is exact by the 'converse' part (note $I_n(\phi) = I_n(A)$). By the five-lemma, there exists an isomorphism $\tau : I_n(\phi) \to I \hookrightarrow R$. We claim that every map $I_n(\phi) \to R$ is multiplication by some element of R. If so, then $I = xI_n(\phi)$ for some non-zero-divisor x, since τ is an isomorphism. Since $\operatorname{grade} I_n(\phi) \geq 2$, we have that $\operatorname{Ext}^i_R(R/I_n(\phi), R) = 0$ for $i = 0, 1$. Applying $\operatorname{Hom}_R(-, R)$ to $0 \to I_n(\phi) \to R \to R/I_n(\phi) \to 0$, we have

$$\cdots \to \underbrace{\operatorname{Hom}_R(R/I_n(\phi), R)}_{=0} \to \operatorname{Hom}_R(R, R) \xrightarrow{\alpha} \operatorname{Hom}_R(I_n(\phi), R) \to \underbrace{\operatorname{Ext}^1_R(R/I_n(\phi), R)}_{=0}.$$

This says α is an isomorphism, but of course $\alpha : \mu_r \mapsto \mu_r|_{I_n(\phi)}$. $\square$

Note. By Buchsbaum-Eisenbud, $\operatorname{grade} I_n(\phi) \geq 2$. But we always have $\operatorname{grade} I_n(\phi) \leq \operatorname{pd}_R R/I_n(\phi)$. Thus $\operatorname{grade} I_n(\phi) = \operatorname{pd} R/I_n(\phi) = 2$; that is, $I_n(\phi)$ is a perfect ideal.

Definition. *Let (R, m) be Noetherian, local. Let $\underline{x} = x_1, ..., x_d$ be a system of parameters for R. An $R-$module M is called a **big Cohen Macaulay module** (for $\underline{x}$) if $\underline{x}$ is $M-$regular. An $R-$module M is called a **balanced** big Cohen Macaulay module if $\underline{x}$ is $M-$regular for all system of parameters $\underline{x}$ of R.*

Note. If M is a big Cohen-Macaulay R-module then $\operatorname{depth} M = \dim R$. For clearly, $\operatorname{depth} M \geq \dim R$. Let $\underline{x}$ be an M-regular sequence. Then $(\underline{x})M \neq M$. Since $(\underline{x})$ is m-primary, we have $mM \neq M$. Hence, $\operatorname{depth} M \leq \dim R$.

<u>A brief review of completions</u>

Let R be a ring, M an $R-$module. Let $\{M_i\}_{i=1}^{\infty}$ be a filtration of M by submodules: $M_1 \supseteq M_2 \supseteq \cdots$. Any such filtration defines a linear topology on M by letting the cosets $\{x + M_i\}_{i=1}$ be a fundamental system of open neighborhoods for all $x \in M$. The topology on M is *separated* (or *Hausdorff*) if $\cap M_i = (0)$. Given a submodule N of M there exists an induced linear topology on N given by the filtration $\{N \cap M_i\}$ and an induced topology on M/N by $\{M_i + N/N\}$. Given an ideal I of R, the **I-adic topology** on M is the one given by the filtration $\{I^n M\}$. The module M is said to be complete if every Cauchy sequence in M has a limit in M.

Definition. *Let M be an $R-$module with a linear topology. The **completion of** M is a linearly topologized $R-$module $\widehat{M}$ which is separated and complete, together with a continuous homomorphism $\phi : M \to \widehat{M}$ with the following universal property: If $f : M \to M'$ is a continuous map and M' is complete and separated, then there exists a unique continuous map $g : \widehat{M} \to M'$ such that the following diagram commutes*

$$\begin{tikzcd}
M \arrow[r, "\phi"] \arrow[d, "f"] & \widehat{M} \arrow[dl, "\exists! g"] \\
M' &
\end{tikzcd}$$

Fact. Completions exist and are unique.

Note that the map $\phi : M \to \widehat{M}$ is injective if and only if M is separated. Clearly, if M has the discrete topology then M is separated and complete (and hence, isomorphic to its completion).

If $\{M_i\}$ and $\{M_i'\}$ are two filtrations on M and are cofinal, the resulting induced topologies on M are the same. In particular, if I and J are finitely generated ideals and $\sqrt{I} = \sqrt{J}$, the $I-$adic and $J-$adic topologies on any module are the same. Let M be a module with a topology defined by $\{M_i\}_{i\geq 1}$. We have an inverse system $M/M_i \twoheadrightarrow M/M_j$ for all $i \geq j$. Then $\widehat{M} = \varprojlim M/M_i$.

Alternatively, let $T = \{\{x_i\} \mid \{x_i\}$ is a Cauchy sequence in $M\}$. Then T has a natural $R-$module structure with a naturally induced linear topology from M. Let $T_0 = \{\{x_i\} \in T \mid \lim x_i = 0\}$ and $\widehat{M} = T/T_0$. Let $\phi : M \to \widehat{M}$ be given by $x \mapsto \overline{\{x\}}$, where $\{x\}$ is the constant sequence and $^{-}$ denotes modulo T_0. Then $\ker\phi = \cap_i M_i$. For each i, let $\widehat{M_i} = \{\overline{\{x_j\}} \in \widehat{M} \mid x_j \in M_j$ for all $j\}$. We get a filtration $\widehat{M_1} \supseteq \widehat{M_2} \supseteq \cdots$. One can show $\widehat{M}$ is complete and separated with respect to the topology induced by this filtration and that $\phi : M \to \widehat{M}$ is continuous and has the required universal property.

Proposition 35. *Let $A \subseteq B$ be modules where B has a linear topology and A has the induced topology from B. Then $0 \to \widehat{A} \to \widehat{B} \to \widehat{B/A} \to 0$ is exact.*

Exercise. Suppose M has the $I-$adic topology. Then $\widehat{I^nM} = I^n\widehat{M}$ for all n. Furthermore, $\widehat{M}$ also has the $I-$adic topology.

Remark. Suppose M has the $I-$adic topology. Then $I^n\widehat{M}/I^{n+1}\widehat{M} = \widehat{I^nM}/\widehat{I^{n+1}M} \cong I^n\widehat{M/I^{n+1}}M \cong I^nM/I^{n+1}M$. This is because the topology induced by the I-adic topology on the last module is discrete. Thus $\operatorname{gr}_I(M) \cong \operatorname{gr}_I(\widehat{M})$.

Definition. *Let R be a ring, M an $R-$module, $x_1, ..., x_n \in R$. Let $I = (x_1, ..., x_n)$. There is a natural graded homomorphism $\psi : M/IM[T_1, ..., T_n] \to \operatorname{gr}_I(M)$ defined by $T_i \mapsto x_i + I^2M \in IM/I^2M$. Since $\{x_1 + I^2M, ..., x_n + I^2M\}$ generates $\operatorname{gr}_I(M)$ (as a $\operatorname{gr}_I(R)-$module), ψ is surjective. We say $x_1, ..., x_n$ is M-**quasiregular** if ψ is an isomorphism and $IM \neq M$.*

Facts.

(1) If $x_1, ..., x_n$ is $M-$quasiregular, so is any permutation.
(2) If $\underline{x}$ is $M-$regular then $\underline{x}$ is $M-$quasiregular.
(3) If (R, m) is quasilocal, $\underline{x} \in m$, and M is finite, then the converse of (2) is true.

Theorem 36. *Let R be a ring, $\underline{x} = x_1, ..., x_n \in R$, $I = (x_1, ..., x_n)$, M an $R-$module. Let $\widehat{M}$ denote the $I-$adic completion of M. The following are equivalent:*

(1) *$\underline{x}$ is $M-$quasiregular.*
(2) *$\underline{x}$ is $\widehat{M}-$quasiregular.*
(3) *$\underline{x}$ is $\widehat{M}-$regular.*

Proof. Note that (1) $\Leftrightarrow$ (2) follows from the fact that $\operatorname{gr}_I(M) \cong \operatorname{gr}_I(\widehat{M})$ and (3) $\Rightarrow$ (2) is true by one of the facts above. So we need only prove (2) $\Rightarrow$ (3). We assume M is I-adically complete and proceed by induction on n. For $n = 1$, suppose x_1 is $M-$quasiregular and $x_1y = 0$. For a given j, $x_1y \in I^jM$. Note $\widehat{x_1} := x_1 + I^2M$ is a non-zero-divisor on $\operatorname{gr}_I(M)$. Suppose $y \in I^kM$ for some k and let $\widehat{y} = y + I^{k+1}M$. Then $\widehat{x}\widehat{y} \in I^{k+1}M/I^{k+2}M$. Of course $\widehat{x_1}\widehat{y} = 0$ by assumption, so $\widehat{y} = 0$. Thus, $y \in I^{k+1}M$. Hence $y \in \cap I^jM = 0$. The same argument shows that $(I^{k+1}M :_M x_1) = I^kM$ for all k.

Now suppose $n > 1$. We know x_1 is $M-$regular and $x_2, ..., x_n$ is M/x_1M-quasiregular. We are done by induction provided M/x_1M is $I-$adically complete. Consider $0 \to x_1M \to M \to M/x_1M \to 0$ and complete: $0 \to \widehat{x_1M} \to M \to \widehat{M/x_1M} \to 0$. Here $\widehat{x_1M}$ is the completion of x_1M with respect to the filtration $\{I^nM \cap x_1M\}_{n\geq 1}$. By the above, $I^nM \cap x_1M = x_1(I^nM :_M x_1) = x_1I^{n-1}M$. Therefore the topologies on x_1M given by $\{I^nM \cap x_1M\}$ and $\{I^nx_1M\}$ are the same, the latter being the I-adic topology on x_1M. Now, as x_1 is M-regular, $M \cong x_1M$ as R-modules. Since M is I-adically complete, so is x_1M. Thus we must have $M/x_1M = \widehat{M/x_1M}$. By induction, $x_2, ..., x_n$ is M/x_1M-regular, which implies $x_1, ..., x_n$ is $M-$regular. ☐

Theorem 37. *Let (R, m) be local Noetherian and M a big Cohen-Macaulay module for $\underline{x} = x_1, ..., x_d$. Let $\widehat{M}$ denote the $m-$adic completion of M. Then $\widehat{M}$ is a balanced big Cohen-Macaulay module.*

Proof. Let $y_1, ..., y_d$ be any system of parameters for R. We need to show $y_1, ..., y_d$ is $\widehat{M}-$regular. Since $m = \sqrt{(y_1, ..., y_d)}$, the $m-$adic and $(\underline{y})-$adic topologies on M are the same. So $\widehat{M}$ is also the $(\underline{y})-$adic completion of M. Now $x_1, ..., x_d$ is $M-$regular, which implies $x_1, ..., x_d$ is $\widehat{M}-$regular. We use induction on d to show $y_1, ..., y_d$ is $\widehat{M}-$regular. If $d = 1$, then $\sqrt{(x_1)} = \sqrt{(y_1)}$. Since x_1 is a non-zero-divisor on $\widehat{M}$, so is y_1 (as $x_1^n \in (y_1)$ for some n). So suppose $d > 1$. By prime avoidance, choose w not in any minimal prime over $(x_1, ..., x_{d-1})$ or $(y_1, ..., y_{d-1})$. Then $(x_1, ..., x_{d-1}, w)$ and $(y_1, ..., y_{d-1}, w)$ are systems of parameters for R. In $R/(x_1, ..., x_{d-1})$, $\overline{x_d}$ and $\overline{w}$ are both systems of parameters. Since $\overline{x_d}$ is $\widehat{M}/(x_1, ..., x_d)\widehat{M}-$regular, so is $\overline{w}$ (by the same argument as $d = 1$ case). Thus $x_1, ..., x_{d-1}, w$ is $\widehat{M}-$regular which implies $w, x_1, ..., x_{d-1}$ is $\widehat{M}-$quasiregular (and thus $\widehat{M}-$regular by lemma). Thus $\overline{x_2}, ..., \overline{x_d}$ is $\widehat{M}/w\widehat{M}-$regular. Both $\overline{x_1}, ..., \overline{x_{d-1}}$ and $\overline{y_1}, ..., \overline{y_{d-1}}$ are systems of parameters for $R/(w)$. By induction, $\overline{y_1}, ..., \overline{y_{d-1}}$ is $\widehat{M}/w\widehat{M}-$regular. Lift to get $w, y_1, ..., y_{d-1}$ is $\widehat{M}-$regular, which implies $y_1, ..., y_{d-1}, w$ is $\widehat{M}-$quasiregular. In $R/(y_1, ..., y_{d-1})$, $\sqrt{\overline{w}} = \sqrt{\overline{y_d}}$ which implies $\overline{y_d}$ is $\widehat{M}/(y_1, ..., y_{d-1})\widehat{M}-$regular. $\square$

Example. Let $R = k[[x, y]]$ where k is a field. Let $M = R \oplus Q$ where Q is the quotient field of $R/(y)$. Then x, y is $M-$regular but y, x is not. So M is a big Cohen-Macaulay module, but not a balanced one.

Definition. *Let R be a ring, I an ideal. Set* $\operatorname{codim} I := \dim R - \dim R/I$.

Remarks. Suppose R is Noetherian.

(1) $\operatorname{ht} I \leq \operatorname{codim} I$ (since $\operatorname{ht} I + \dim R/I \leq \dim R$ for all ideals I) with equality if R is equidimensional, catenary, and all maximal ideals have the same height (e.g., $R = k[x_1, ..., x_d]$).

(2) If R is a Cohen-Macaulay local ring then $\operatorname{ht} I = \operatorname{grade} I = \operatorname{codim} I$.

Exercise. Suppose R is Noetherian local and I is an ideal. Then $\operatorname{codim} I \geq i$ if and only if I contains $x_1, ..., x_i$ which form part of a system of parameters for R.

Definition. *Let $F_{.} : 0 \to F_s \xrightarrow{\phi_s} F_{s-1} \to \cdots \xrightarrow{\phi_1} F_0$ be a complex of finitely generated free $R-$modules. Define* $\operatorname{codim} F_{.} := \inf\{\operatorname{codim} I_{r_i}(\phi_i) - i \mid i = 1, ..., s\}$, *where the r_i are the expected ranks.*

Remarks.

(1) If $F_{.}$ is acyclic, then $\operatorname{codim} F_{.} \geq 0$ (by Buchsbaum-Eisenbud, $\operatorname{grade} I_{r_i}(\phi_i) - i \geq 0$ for all i).

(2) If R is Cohen-Macaulay and local and $\operatorname{codim} F_{.} \geq 0$ then $F_{.}$ is acyclic (again by Buchsbaum-Eisenbud and the remarks above).

(3) Cohen-Macaulay is crucial in (2). For example, let $R = k[[x, y]]/(x^2, xy)$ and $F_{.} : 0 \to R \xrightarrow{y} R$. Then $I_1(\phi_1) = y$, $\operatorname{codim} y = 1$, but $F_{.}$ is not acyclic.

Proposition 38. *Let (R, m) be local, $F_{.}$ a complex as above. Suppose* $\operatorname{codim} F_{.} \geq 0$. *Then $F_{.} \otimes_R M$ is acyclic for every balanced big Cohen-Macaulay module M.*

Proof. For each i, we have $\operatorname{codim} I_{r_i}(\phi_i) \geq i$. By the exercise, $I_{r_i}(\phi_i)$ contains part of a system of parameters $x_1, ..., x_i$. Then $x_1, ..., x_i$ is $M-$regular (as M is balanced) and so $\operatorname{grade}(I_{r_i}(\phi_i), M) \geq i$. By Buchsbaum Eisenbud, we have $F_{.} \otimes_R M$ is acyclic. $\square$

Theorem 39. *Let (R, m) be a Noetherian local ring possessing a big Cohen-Macaulay module. Let $F_{.} : 0 \to F_s \xrightarrow{\phi_s} F_{s-1} \to \cdots \xrightarrow{\phi_1} F_0 \to 0$ with* $\operatorname{codim} F_{.} \geq 0$. *Let $C = \operatorname{coker} \phi_1$ and assume $C \neq 0$. Then for every $e \in C \setminus mC$, we have* $\operatorname{codim}(\operatorname{Ann}_R e) \leq s$.

Proof. By Theorem 37, we may assume R has a balanced big Cohen-Macaulay module M. We use induction on $\dim R/\operatorname{Ann}_R e$. Suppose $\dim R/\operatorname{Ann}_R e = 0$ and let M be a balanced Cohen-Macaulay module. Then $F_{.} \otimes_R M$ is acyclic. As before, let $M_i = \operatorname{coker}(\phi_{i+1} \otimes 1_M)$ for $i = 0, ..., s$. (Note $M_s = F_s \otimes_R M$ and $M_0 = C \otimes_R M$). We have $0 \to M_i \to F_{i-1} \otimes_R M \to M_{i-1} \to 0$ is exact for $i = 1, ..., s$ as $F_{.} \otimes_R M$ is acyclic. Note $\operatorname{depth} F_{i-1} \otimes_R M = \operatorname{depth} M$ and so $\operatorname{depth} M_{i-1} \geq \min\{\operatorname{depth} M, \operatorname{depth} M_i - 1\}$. Thus for all $i = 0, ..., s$, $\operatorname{depth} M_{s-i} \geq \operatorname{depth} M - i$. So depth $M \otimes_R C = \operatorname{depth} M_0 \geq \operatorname{depth} M - s = \dim R - s$ as M is a balanced big Cohen-Macaulay module. It suffices to show depth $M \otimes_R C = 0$ as then $s \geq \dim R = \operatorname{codim}(\operatorname{Ann}_R e)$. Let $M \otimes e$ denote the submodule of $M \otimes_R C$ consisting of those elements of the form $u \otimes e$ for some $u \in M$. Let $w \in M$, $w \notin mM$. Then the image of $w \otimes e$ in $M/mM \otimes_R C/mC$ is nonzero. Hence, $M \otimes e \neq 0$. As $\dim R/\operatorname{Ann} e = 0$, we have $m^\ell e = 0$ for some ℓ, and so $m^\ell(M \otimes e) = 0$. Thus $m \in \operatorname{Ass}_R M \otimes e \subseteq \operatorname{Ass}_R M \otimes C$ and hence depth $M \otimes C = 0$.

Now suppose $\dim R/\operatorname{Ann}_R e > 0$. Since $\operatorname{codim} \operatorname{Ann}_R e \leq \dim R$, we can assume $s < \dim R$. Let $\Lambda_0 = \{p \in \operatorname{Spec} R \mid \dim R/p = \dim R\}$ and $\Lambda_1 = \{p \in \operatorname{Spec} R \mid \operatorname{Ann} e \subseteq p, \dim R/p = \dim R/\operatorname{Ann}_R e\}$. As all the primes in Λ_1 are minimal over $\operatorname{Ann}_R e$, Λ_1 is finite. Further, since $\dim R/\operatorname{Ann}_R e > 0$ we see $m \notin \Lambda_1$. Let $\Lambda_2 = \{p \in \operatorname{Spec} R \mid p \supseteq I_{r_i}(\phi_i), \operatorname{codim} p = i \text{ for some } i\}$. By assumption on codim $F.$, Λ_2 is a finite set. Also as $s < \dim R$, $m \notin \Lambda_2$. By prime avoidance, choose an element $x \notin p$ for all $p \in \Lambda_1 \cup \Lambda_2 \cup \Lambda_0$. Let $\overline{(\ )}$ denote modulo (x), so $\overline{F.} = F. \otimes R/(x)$.

Claim. $\operatorname{codim}_{\overline{R}} \overline{F.} \geq 0$

Proof. There are two cases. First suppose $\operatorname{codim} I_{r_i}(\phi_i) \geq i+1$. Then $\dim R/I_{r_i}(\phi_i) \leq \dim R - i - 1$ and so $\dim R/(I_{r_i}(\phi_i), x) \leq \dim R - i - 1 \leq \dim R/(x) - i$. Thus $\operatorname{codim} I_{r_i}(\overline{\phi_i}) \geq i$. Next suppose $\operatorname{codim} I_{r_i}(\phi_i) = i$. Then $\dim R/I_{r_i}(\phi_i) = \dim R - i$. As $x \notin p$ for all $p \in \Lambda_2$, we see $\dim R/(I_{r_i}(\phi_i), x) = \dim R - i - 1 \leq \dim R/(x) - i$.

Since $e \notin mC$, we have $\overline{e} \notin \overline{mC}$. Now $\operatorname{Ann}_{\overline{R}} \overline{e} \supseteq (\operatorname{Ann}_R e + (x))/(x)$. Therefore, $\dim \overline{R}/\operatorname{Ann}_{\overline{R}} \overline{e} \leq \dim R/(\operatorname{Ann}_R e + (x)) = \dim(R/\operatorname{Ann}_R e) - 1$ since $x \notin p$ for any $p \in \Lambda_1$. As $x \notin p$ for any $p \in \Lambda_0$, we have that x is part of a system of parameters for R. Hence, $R/(x)$ has a big Cohen-Macaulay module (namely M/xM). By induction, $s \geq \operatorname{codim}_{\overline{R}}(\operatorname{Ann}_{\overline{R}} \overline{e}) = \dim \overline{R} - \dim R/\operatorname{Ann}_{\overline{R}} \overline{e} \geq \dim R - 1 - (\dim R/\operatorname{Ann}_R e - 1) = \operatorname{codim} \operatorname{Ann}_R e$. □

Corollary 40 (**Improved New Intersection Theorem**, Evans-Griffith '81)**.** *Let (R, m) is a local ring possessing a big Cohen-Macaulay module. Let $F.$ be as in Theorem 39 and $C = \operatorname{coker} \phi_1 \neq 0$. Choose $e \in C \setminus mC$. Suppose $(F.)_p$ is acyclic for all $p \neq m$ and $\lambda(Re) < \infty$. Then $s \geq \dim R$.*

Proof. Suppose $s < \dim R$. We claim $\operatorname{ht} I_{r_i}(\phi_i) \geq i$ for all i. If not, then there exists $j \in \{1, ..., s\}$ and a prime $p \supseteq I_{r_j}(\phi_j)$ such that $\operatorname{ht}(p) < j \leq s < \dim R$. Clearly $p \neq m$, so $(F.)_p$ is acyclic and thus grade $I_{r_j}(\phi_j)_p \geq j$. But this is a contradiction, since grade $I_{r_j}(\phi_j)_p \leq \operatorname{ht} pR_p < j$. Thus codim $F. \geq 0$. By Theorem 39, $\operatorname{codim}(\operatorname{Ann} e) \leq s$. Since $\lambda(Re) < \infty$, we have $\operatorname{codim}(\operatorname{Ann}_R e) = \dim R$, a contradiction. □

Corollary 41 (**New Intersection Theorem**)**.** *Let (R, m) be a local ring possessing a big Cohen-Macaulay module. Let $F.$ be as in Theorem 39 and suppose $H_i(F.)$ has finite length for all i. If $s < \dim R$, then $F.$ is exact.*

Proof. If $s = 0$, we have $\lambda(F_0) = \lambda(H_0(F.)) < \infty$. If $F_0 \neq$ then $\lambda(R) < \infty$ and thus $\dim R = 0$, a contradiction since $s < \dim R$. Thus $F_0 = 0$ and $F.$ is exact.

Now assume $s > 0$. Suppose first that $H_0(F.) = 0$, that is, $F_1 \xrightarrow{\phi_1} F_0 \to 0$ is exact. Then ϕ_1 splits and $\ker \phi_1$ is a (free) direct summand of F_1. Let $F'_. : 0 \to F_s \to \cdots \to F_2 \to \ker \phi_1 \to 0$. Then $H_0(F'_.) = H_1(F.)$, which has finite length. By induction on s, $F'_.$ is exact and thus $F.$ is exact.

Now suppose $H_0(F.) \neq 0$. Let $e \in H_0(F.) \setminus mH_0(F.)$. Certainly $\lambda(Re) < \infty$ and $(F.)_p$ is exact for all $p \neq m$. By the Improved New Intersection Theorem, $s \geq \dim R$, a contradiction. Hence, $F.$ is exact. □

Exercise. (cf. Matsumura, p. 129) Let R be a ring, M an $R-$module, $x_1, ..., x_n \in R$. Let $I = (x_1, ..., x_n)$ and assume $IM \neq M$. Then $x_1, ..., x_n$ is $M-$quasiregular if and only if for every homogenous polynomial $F(T_1, ..., T_n) \in M[T_1, ..., T_n]$ of degree v such that $F(x_1, ..., x_n) \in I^{v+t}M$ for some t, all the coefficients of F has lie in I^tM.

Theorem 42 (**Monomial Conjecture**)**.** *Let (R, m) be a local ring possessing a big Cohen Macaulay module and $x_1, ..., x_d$ a system of parameters for R. Then for all $n \geq 1$ we have $x_1^n \cdots x_d^n \notin (x_1^{n+1}, ..., x_d^{n+1})$.*

Proof. Let M be a balanced big Cohen Macaulay module. Then $x_1, ..., x_d$ is $M-$quasiregular. Suppose $x_1^n \cdots x_d^n \in (x_1^{n+1}, ..., x_d^{n+1})$ for some n. Then $x_1^n \cdots x_d^n M \subseteq (x_1^{n+1}, ..., x_d^{n+1})M$.

Claim. For all $t \geq 0$, $(x_1^n \cdots x_d^n)I^t M \subseteq (x_1^{n+1}, ..., x_d^{n+1})I^{nd-n-1+t}M$.

Proof. By multiplication by I^t it is enough to show for $t = 0$. Let $u \in M$. We know $u(x_1^n \cdots x_d^n) = m_1 x_1^{n+1} + ... + m_d x_d^{n+1}$ for some $m_1, ..., m_d \in M$. Let $F(T_1, ..., T_d) = m_1 T_1^{n+1} + ... + m_d T_d^{n+1}$. Then $F(I)$ is homogenous of degree $n+1$. Now $F(x_1, ..., x_d) = (x_1 \cdots x_d)^n u \in I^{nd}M$. By the exercise, $m_i \in I^{nd-n-1}M$ for all i.

Give $\operatorname{gr}_I(M)$ the natural $R/I[T_1, ..., T_d]-$module structure where $T_i f = x_i^* f$ for all $f \in \operatorname{gr}_I(M)$ where $x_i^* = x_i + I^2 \in I/I^2 \subseteq \operatorname{gr}_I(R)$. By claim, $(*)(T_1^n \cdots T_d^n)\operatorname{gr}_I(M) \subseteq (T_1^{n+1}, ..., T_d^{n+1})\operatorname{gr}_I(M)$ (the degree t piece of the right hand side is $[(x_1^{n+1}, ..., x_d^{n+1})I^{n-t-1}M + I^{n-t}M]/I^{n-t}M$ and of the left hand side is $[(x_1^n \cdots x_d^n)I^{t-nd}M + I^{t-nd+1}M]/I^{t-nd+1}M$.) As $x_1, ..., x_d$ is $M-$quasiregular, $\operatorname{gr}_I(M) \cong M/IM[T_1, ..., T_d]$ as an $R/I[T_1, ..., T_d]-$module. Thus $(*)$ implies $(T_1 \cdots T_d)^n M/IM[T_1, ..., T_d] \subseteq (T_1^{n+1}, ..., T_d^{n+1})M/IM[T_1, ..., T_d]$, a contradiction to polynomial division. □

Theorem 43 (**Acyclicity Lemma**, Peskine-Szpiro '74)**.** *Let R be a ring of characteristic $p > 0$ and $G. : 0 \to G_s \xrightarrow{\phi_s} \cdots \to G_0$ be a complex of finitely generated free $R-$modules. Then $G.$ is acyclic if and only if $F(G.)$ is acyclic.*

Proof. For any $\phi : R^m \to R^n$ and any $r \geq 0$ we see $I_r(\phi^{[p]}) = I_r(\phi)^{[p]}$ (where $\phi^{[p]} = F(\phi)$). In particular, $\sqrt{I_r(\phi)} = \sqrt{I_r(\phi^{[p]})}$ and so $\operatorname{grade} I_r(\phi) = \operatorname{grade} I_r(\phi^{[p]})$. By Buchsbaum Eisenbud, $G.$ is acyclic if and only if $\operatorname{grade} I_{r_i}(\phi_i) \geq i$, which is if and only if $\operatorname{grade} I_{r_i}(\phi_i^{[p]}) \geq i$ which is if and only if $F(G.)$ is acyclic. □

Corollary 44. *(R, m) local, M a finitely generated $R-$module and $\operatorname{pd}_R M < \infty$. Then $\operatorname{Tor}_i^R(R^F, M) = 0$ for all $i \geq 1$.*

Proof. Let $G.$ be a finite free resolution of M. By the acyclicity lemma, $F(G.) = R^F \otimes G.$ is a free resolution of $F(M)$. In particular $\operatorname{Tor}_i^R(R^F, M) = H_i(R^F \otimes G.) = 0$ for all $i \geq 1$. □

Corollary 45 (Kunz, '68)**.** *Let (R, m) be a regular local ring. Then R^F is a flat $R-$module (that is, F is an exact functor).*

Note that the converse is also true (but harder).

Suppose we have a complex $F. : 0 \to F_s \to \cdots \to F_0$ with codimension $F. \geq 0$. If R is Cohen Macaulay, then $F.$ is acyclic by Buchsbaum-Eisenbud. If R is the homomorphic image of a Cohen Macaulay ring, then there exists $c \in R$ (not contained in any minimal prime) such that R_c is Cohen Macaulay. Then $(F.)_c$ is acyclic and there exists n such that $C^n H_i(F.) = 0$ for all $i > 0$.

Q: Does there exist a non nilpotent element c such that $cH_i(F.) = 0$ for all $i > 0$ for any complex $F.$ such that $\operatorname{codim} F. \geq 0$?

The answer is yes in the case that R is the homomorphic image of a Gorenstein ring. The proof of this fact uses Spectral Sequences, which are discussed in the appendix.

Theorem 46. *Let (R, m) be a local ring and $F. : 0 \to F_s \to F_{s-1} \to \cdots \to F_0 \to 0$ a complex of finitely generated free $R-$modules such that $\lambda(H_i(F.)) < \infty$ for all i. Let $I_i = \operatorname{Ann}_R H_m^i(R)$ for $i \geq 0$. Then for $0 \leq i \leq s$, $I_0 I_1 \cdots I_{s-i} H_i(F.) = 0$.*

Proof. Let $\underline{x} = (x_1, ..., x_d)$ be a system of parameters and $K^{\cdot}$ the Čech complex of R with respect to $\underline{x}$. Then $H^i(K^{\cdot}) = H^i_{(\underline{x})}(R) = H^i_m(R)$. Reindex $F.$ as $F^{\cdot} : 0 \to F^0 \to F^1 \to \cdots \to F^s \to 0$ (so $F^i = F_{s-i}$). Then $H^i(F^{\cdot}) = H_{s-i}(F.)$. We want to prove $I_0 \cdots I_j H^j(F^{\cdot}) = 0$ for all $j \geq 0$.

Let C be the first quadrant double complex $K^{\cdot} \otimes F^{\cdot}$. First filter by the columns:

$$\begin{aligned}
{}^{I}E_1^{pq} &= H_v^q(K^p \otimes F^{\cdot}) \\
&= K^p \otimes H_v^q(F^{\cdot}) \text{ as } K^p \text{ is flat for all } p \\
&= \begin{cases} H^q(F^{\cdot}) & \text{if } p = 0 \\ 0 & \text{if } p > 0 \end{cases} \text{as } R_{x_i} \otimes H^i(F^{\cdot}) = 0 \text{ for all } i
\end{aligned}$$

Thus the sequence ${}^{I}E_1^{pq}$ collapses and we get $H^{p+q}(F^{\cdot}) = {}^{I}E_\infty^{pq} = H^{p+q}(\mathrm{Tot}(C))$. Now filter by the rows:

$$\begin{aligned}
{}^{II}E_1^{pq} &= H_h^q(K^{\cdot} \otimes F^p) \\
&= H^q(K^{\cdot}) \otimes F^p \text{ as } F^p \text{ is free, hence flat} \\
&= H_m^q(R)^{n_p}, \text{ for } n_p = \operatorname{rank} F_p
\end{aligned}$$

By definition of I_q, we see $I_q\ {}^{II}E_1^{pq} = 0$ and so $I_q\ {}^{II}E_\infty^{pq} = 0$ for all p, q as ${}^{II}E_\infty^{pq}$ is a subquotient of ${}^{II}E_1^{pq}$.

By the main convergence theorem of spectral sequences, ${}^{II}E_1^{pq} \Rightarrow H^{p+q}(\mathrm{Tot}(C)) \cong H^{p+q}(F^{\cdot})$. Thus for any $n \in \mathbb{Z}$, there exists a filtration $\{F^pH^n\}_{p\in\mathbb{Z}}$ where $H^n = H^n(F^{\cdot})$ such that $F^pH^n/F^{p+1}H^n \cong {}^{II}E_\infty^{p,n-p}$ for all p. As ${}^{II}E_1^{pq}$ is a first quadrant spectral sequence, ${}^{II}E_1^{p,n-p} = 0$ if $p < 0$ or $p > m$. Hence the filtration of H^n has the form $0 = F^{n+1}H^n \subseteq F^nH^n \subseteq \cdots \subseteq F^1H^n \subseteq F^0H^n = H^n$. Since $I_{n-p}\ {}^{II}E_\infty^{p,n-p} = 0$, we have $I_{n-p}F^pH^n \subseteq F^{p+1}H^n$ and hence $I_nI_{n-1}\cdots I_0H^n = 0$. □

Recall for a ring R and $\underline{x} = x_1, ..., x_n \in R$ that the Koszul Complex is defined by $K.(\underline{x}) = \otimes_{i=1}^n(0 \to R \xrightarrow{x_i} R \to 0)$. The i^{th}Koszul homology is written $H_i(\underline{x}) = H_i(K.(\underline{x}))$ for all i, where $H_i(\underline{x}) = 0$ for $i < 0$ and $i > n$. Also recall the following basic facts of Koszul Homology:

(1) $H_0(\underline{x}) = R/(\underline{x})$
(2) $H_n(\underline{x}) = (0 :_R (\underline{x}))$
(3) $(\underline{x})H_i(\underline{x}) = 0$ for all i. In particular, if (R, m) is local and $\sqrt{(\underline{x})} = m$, then $\lambda(H_i(\underline{x})) < \infty$ for all i.
(4) Let $\underline{x}' = x_1, ..., x_{n-1}$. Then there is a long exact sequence

$$\cdots H_i(\underline{x}' \xrightarrow{\pm x_n}) H_i(\underline{x}') \to H_i(\underline{x}) \to H_{i-1}(\underline{x}') \xrightarrow{\mp x_n} \cdots .$$

(5) If (R, m) is local and $\underline{x} \in m$, then $\underline{x}$ is a regular sequence if and only if $H_i(\underline{x}) = 0$ for all $i \geq 1$.

Corollary 47. *Let (R, m) be a local Noetherian ring of dimension d and $x_1, ..., x_n$ part of a system of parameters for R. Let $I_i = \operatorname{Ann}_R H_m^i(R)$. Then*

(1) $I_0 \cdots I_{d-i}H_i(\underline{x}) = 0$
(2) $I_0 \cdots I_{d-1} \cdot [((x_1 \cdots x_{n-1}) : x)/(x_1, ..., x_{n-1})] = 0.$

Proof. (1) Extend $x_1, ..., x_n$ to a full system of parameters $x_1, ..., x_d$ and induct on $d-n$. For $n = d$, $\lambda(H_i(\underline{x})) < \infty$ for all i (by fact 3 above). Let $K^{\cdot} = F^{\cdot}$ in the previous theorem to get the result.

Suppose $n < d$. For a given $t \geq 1$, let $\underline{x}(t) = x_1, ..., x_n, x_{n+1}^t$. This is part of a system of parameters. By induction, $I_0 \cdots I_{d-i}H_i(\underline{x}(t)) = 0$. From the long exact sequence in fact 4 above, we have

$$\begin{tikzcd}
H_i(\underline{x}) \arrow[r, "\pm x_n^t"] & H_i(\underline{x}) \arrow[rr] \arrow[dr] & & H_i(\underline{x}(t)) \\
& & K \arrow[ur] \arrow[dr] & \\
& 0 \arrow[ur] & & 0
\end{tikzcd}$$

where $K \cong H_i(\underline{x})/x_{n+1}^tH_i(\underline{x}) \subseteq H_i(\underline{x}(t))$. Since $H_i(\underline{x}(t))$ is annihilated by $I_0 \cdots I_{d-i}$, we have $I_0 \cdots I_{d-i}H_i(\underline{x}) \subseteq x_{n+1}^tH_i(\underline{x})$ for all t. Thus by Krull's Intersection Theorem, $I_0 \cdots I_{d-i}H_i(\underline{x}) = 0$.

(2) Induct on n. For $n = 1$, we have $(0 : x_1) \cong H_1(x_1)$. By part 1, $I_0 \cdots I_{d-1} H_1(x_1) = 0$. So suppose $n > 1$. From the long exact sequence

$$\begin{tikzcd}
H_1(\underline{x}) \arrow[rr] \arrow[dr] & & H_0(\underline{x}') \arrow[r, "\pm x_n"] & H_0(\underline{x}') \\
& K \arrow[ur] \arrow[dr] & & \\
0 \arrow[ur] & & 0 &
\end{tikzcd}$$

we know $K = (\underline{x}' : x_n)/(\underline{x}')$. Of course by part 1, we have $I_0 \cdots I_{d-1} H_n(\underline{x}) = 0$ and so $I_0 \cdots I_{d-1} K = 0$. $\square$

Lemma 48. *Let (S, n) be Gorenstein of dimension d and M a finitely generated $S-$module. Then $\dim \operatorname{Ext}^i_s(M, S) \leq d - i$.*

Proof. Let $p \in \operatorname{Supp}_R \operatorname{Ext}^i_S(M, S)$. So $\operatorname{Ext}^i_S(M, S)_p \cong \operatorname{Ext}^i_{S_p}(M_p, S_p) \neq 0$. Then $\operatorname{Ext}^i_{S_p}(M_p, S_p)^\vee \neq 0$, which implies $H^{\dim S_p - i}_{pR_p}(M_p) \neq 0$ by Local Duality. Then $\dim S_p - i \geq 0$, which implies $\dim S - \dim S/p \geq i$. Thus $\dim S/p \leq d - i$. $\square$

Theorem 49. *Let (R, m) be the homomorphism image of a Gorenstein ring, $d = \dim R$. Let $I_i = \operatorname{Ann}_R H^i_m(R)$. Then $\dim R/I_i \leq i$ for all i. In particular, $\dim R/I_0 \cdots I_{d-1} < R$ (so $I_0 \cdots I_{d-1}$ contain no nilpotent elements).*

Proof. Let $R = S/J$ where (S, n) is a local Gorenstein ring of dimension t. By local duality, $I_i = \operatorname{Ann}_R H^i_m(R) = \operatorname{Ann}_R H^i_m(R)^\vee = \operatorname{Ann}_R \operatorname{Ext}^{t-i}_S(R, S)$. By the lemma, $\dim S/\operatorname{Ann}_R \operatorname{Ext}^{t-i}_S(R, S) \leq t - (t - i) = i$. $\square$

Exercise. Let (R, m) be the homomorphic image of a Gorenstein ring with $\dim R > 0$. Then there exists $c \in m$ such that $\dim R/(c) < \dim R$ and $c \cdot [(x_1, ..., x_{n-1}) : x_n/(x_1, ..., x_{n-1})] = 0$ for all partial system of parameters $x_1, ..., x_n$.

Theorem 50 (New Intersection Theorem). *Suppose (R, m) is a local ring of characteristic $p > 0$. Let $F. : 0 \to F_s \xrightarrow{\phi_s} \cdots F_1 \xrightarrow{\phi_1} F_0 \to 0$ be a complex of finitely generated free $R-$modules such that $\lambda(H_i(F.)) < \infty$. If $s < \dim R$, then $F.$ is exact.*

Proof. Use induction on s. If $s = 0$, we have $0 \to F_0 \to 0$. Since $\lambda(H_0(F.)) \leq \infty$, we have $\lambda(F_0) < \infty$. Since F_0 is a free module, $F_0 = 0$ or $\lambda(R) < \infty$. If $\lambda(R) < \infty$, then $\dim R = 0$, a contradiction as $s < \dim R$. Thus $F_0 = 0$ and $F.$ is exact. So suppose $s > 0$. Note that we can complete R as $F.$ is exact if and only if $\hat{F}.$ is exact. Thus we may assume R is the homomorphic image of a Gorenstein ring.

Case 1. $H_0(F.) = 0$. Then ϕ_1 splits and we can form the complex $F'_. : 0 \to F_s \to \cdots \to F_2 \to \ker \phi_1 \to 0$ where $\ker \phi_1$ is free. So $\lambda(H_i(F'_.)) < \infty$ for all i. By induction, $F'_.$ is exact and thus $F.$ is exact.

Case 2. $H_0(F.) \neq 0$. Suppose $\phi_1(F_1) \not\subset mF_0$. Then $I_1(\phi_1) \not\subset m$, which implies $\operatorname{im} \phi_1$ contains a free direct summand of rank 1. Then we can define ϕ'_1 as $F_1 = F'_1 \oplus R \xrightarrow{A} F_0 = F'_0 \oplus R$ with $A = \begin{pmatrix} \phi'_1 & 0 \\ 0 & 1 \end{pmatrix}$. Now replace $F_1 \xrightarrow{\phi_1} F_0$ with $F'_1 \xrightarrow{\phi'_1} F'_0$ and repeat this process until $\phi_1(F_1) \subseteq mF_0$. Then apply the Frobenius functor to $F.$ Note that $H_i(F(F.))$ has finite length for all i as

$$\begin{aligned}
\lambda(H_i(F(F.))) < \infty \text{ for all } i \quad &\Leftrightarrow \quad (F_R(F.))_p \text{ is exact for all } p \neq m \\
&\Leftrightarrow \quad F_{R_p}((F.)_p) \text{ is exact for all } p \neq m \\
&\Leftrightarrow \quad (F.)_p \text{ is exact for all } p \neq m \\
&\Leftrightarrow \quad \lambda(H_i(F.)) < \infty \text{ for all } i.
\end{aligned}$$

Let F^e denote the Frobenius functor applied e times. Then $\lambda(H_i(F^e(F.))) < \infty$ for all i. Also $F^e(\phi_1)(F_1) \subseteq m^{[p^e]} F_0$ for all e as $\phi_1(F_1) \subseteq mF_0$. Now $H_0(F.) \cong F_0/\operatorname{im} \phi_1 \twoheadrightarrow F_0/mF_0$ and

$H_0(F^e(F.)) \cong F_0/\operatorname{im} F^e(\phi_1) \twoheadrightarrow F_0/m^{[p^e]}F_0$. Thus $\operatorname{Ann}_R H_0(F^e(F.)) \subseteq \operatorname{Ann}_R F_0/m^{[p^e]}F_0 = m^{[p^e]}$. By the theorem, $I_0 \cdots I_s H_0(F^e(F.)) = 0$ for all e and thus by Krull's Intersection Theorem, $I_0 \cdots I_s = (0)$. □

Lemma 51. *For a local ring (R, m) of dimension d and a system of parameters $\underline{x} = x_1, ..., x_d$, there exists k such that for all $t > k$ and all n $(x_1 \cdots x_d)^n \notin (x_1^{n+t}, ..., x_d^{n+t})$.*

Proof. Recall $H_m^d(R) = R_{x_1 \cdots x_d}/\sum_{i=1}^d R_{x_1 \cdots \hat{x_i} \cdots x_d}$ is generated by $\left\{\overline{\frac{1}{(x_1 \cdots x_d)^t}} | t \geq 1\right\}$. Now

$$\begin{array}{rcl} \overline{\frac{1}{(x_1 \cdots x_d)^t}} = \overline{0} & \Leftrightarrow & \frac{1}{(x_1 \cdots x_d)^t} = \sum \frac{r_i}{(x_1 \cdots \hat{x_i} \cdots x_d)^s} \text{ for } r_i \in R, s \in \mathbb{Z} \text{ in } R_{x_1 \cdots x_d} \\ & \Rightarrow & \text{there exists } q \text{ such that } (x_1 \cdots x_d)^s + q = \sum r_i x_i^s (x_1 \cdots x_d)^{t+q} \in (x_1^{s+q+t}, ..., x_d^{s+q+t}) \text{ in } R \\ & \Leftrightarrow & \text{there exists } n \text{ such that } (x_1 \cdots x_d)^n \in (x_1^{n+t}, ..., x_d^{n+t}) \end{array}$$

Now suppose $(x_1 \cdots x_d)^n \in (x_1^{n+t}, ..., x_d^{n+t})$. Then $(x_1 \cdots x_d)^n = r_1 x_1^{n+1} + \cdots + r_d x_d^{n+t}$. In $R_{x_1 \cdots x_d}$, we have $\frac{1}{(x_1 \cdots x_d)^t} = \sum \frac{r_i}{(x_1 \cdots \hat{x_i} \cdots x_d)^{n+t}} \in \sum R_{x_1 \cdots \hat{x_i} \cdots x_d}$. Thus in $H_m^d(R)$, $\overline{\frac{1}{(x_1 \cdots x_d)^t}} = 0$. Thus

$$\overline{\frac{1}{(x_1 \cdots x_d)^t}} = 0 \text{ in } H_m^d(R) \Leftrightarrow \text{ there exists } n \text{ such that } (x_1 \cdots x_d)^n \in (x_1^{n+t}, ..., x_d^{n+t}).$$

Since $H_m^d(R) \neq 0$, $\overline{\frac{1}{(x_1 \cdots x_d)^t}} \neq \overline{0}$ for some t (and thus for all $t > k$ for some k by multiplication). Thus there exists k such that for all $t > k$ and for all n $(x_1 \cdots x_d)^n \notin (x_1^{n+t}, ..., x_d^{n+t})$. □

Theorem 52 (Monomial Conjecture). *Let (R, m) be local of characteristic p and $\underline{x} = x_1, ..., x_d$ a system of parameters for r. Then for all n we have $(x_1 \cdots x_d)^n \notin (x_1^{n+1}, ..., x_d^{n+1})$.*

Proof. Suppose for some n that $(x_1 \cdots x_d)^n = r_1 x_1^{n+1} + \cdots + r_d x_d^{n+1}$. Take p^e-th powers to get $(x_1 \cdots x_d)^{np^e} = r_1^{p^e} x_1^{p^e(n+1)} + \cdots + r_d^{p^e} x_d^{p^e(n+1)} \in (x_1^{p^e n + p^e}, ..., x_d^{p^e n + p^e})$. This contradicts the lemma as we can choose e so p^e is as large as necessary. □

Definition. *Let R be a ring, $x_1, ..., x_n \in R$ and M an $R-$module. Suppose there exists $y \in M \setminus (x_1, ..., x_s)M$ for some $s \leq n$ such that $x_{s+1} y \in (x_1, ..., x_s)M$. Let $M' = M \oplus R^s/Rw$ where $w = y - (x_1 e_1 + ... + x_s e_s)$ (here $e_1, ..., e_s$ is a basis for R^s and identify M and R^s with their images in $M \oplus R^s$). There is an obvious map $M \to M'$ defined by $m \mapsto \overline{(m, 0)}$. Given $f \in M$, let f' be the image of f under $M \to M'$. We say $(M, f) \to (M', f')$ is an* ***$\underline{x}-$modification of type*** *s.*

More generally,, a sequence of $\underline{x}-$modifications $(M, f) = (M_0, f_0) \to (M_1, f_1) \to \cdots \to (M_r, f_r) = (N, g)$ where (M_{i+1}, f_{i+1}) is an $\underline{x}-$modification of (M, f_i) of type s_{i+1} is called an ***$\underline{x}-$modification of*** *(M, f)* ***of type*** *$(s_1, ..., s_r)$. We say this modification is* ***non-degenerate*** *if $g \notin (\underline{x})N$.*

Lemma 53. *Let $(N, f) \to (N', f')$ be an $\underline{x}-$modification. Suppose there exists an $\underline{x}-$regular modification m. Then for any $R-$module homomorphism $\phi : N \to M$ there exists a map $\phi' : N' \to M$ such that the diagram below commutes.*

$$\begin{tikzcd} N \arrow[r] \arrow[d, "\phi"'] & N' \arrow[dl, dashed, "\phi'"] \\ M & \end{tikzcd}$$

Proof. By definition, there exists $y \in N \setminus (x_1, ..., x_s)N$ such that $x_{s+1} y \in (x_1, ..., x_s)N$ and $N' = N \oplus R^s/Rw$ where $w = y - (x_1 e_1 + ... + x_s e_s)$. Then $x_{s+1} \phi(y) \in (x_1, ..., x_s)M$. As M is $\underline{x}-$regular, $\phi(y) = x_1 m_1 + ... + x_s m_s$ for $m_i \in M$. Define $\psi : N \oplus R^s \to M$ by $(n, \sum r_i e_i) \mapsto \phi(n) + \sum r_i m_i$. Then $\psi(w) = \psi(y - \sum x_i e_i) = \phi(y) - \sum r_i m_i = 0$. Then $\overline{\psi} : N' \to M$ clearly extends to ϕ. Define $\overline{\psi} = \phi'$. □

Proposition 54. *Suppose there exists an $\underline{x}-$regular $R-$module M. Then every $\underline{x}-$modification of type $(s_1, ..., s_r)$ of $(R, 1)$ is non-degenerate.*

Proof. Consider the following diagram obtained by the lemma

$$\begin{tikzcd}
(R,1) = (M_0, f_0) \arrow[r] \arrow[d, "\phi_0"'] & (M_1, f_1) \arrow[r] \arrow[dl, "\phi_1"] & \cdots \arrow[r] & (M_r, f_r) \arrow[dlll, "\phi_r"] \\
M & & &
\end{tikzcd}$$

where ϕ_0 is defined by by $1 \mapsto g$ for some $g \in M \setminus \underline{x}M$. As the diagram commutes, $\phi_r(f_r) = \phi_0(1) = g \notin (\underline{x})M$. So $f_r \notin (\underline{x})M_r$. So the modification is non-degenerate. ◻

Theorem 55. *Suppose R is Noetherian, $\underline{x} = x_1, ..., x_n \in R$, and every $\underline{x}-$modification of $(R, 1)$ is non-degenerate. Then there exists an $R-$module M such that M is $\underline{x}-$regular.*

Proof. First define a direct system $\{M_j, \phi_{ij}\}_{i,j\in\mathbb{N}}$ where $\phi_{ij} : M_i \to M_j$ for $i \leq j$ is defined with $M_0 = R$ and $\phi_{00} = id$. Suppose $M_1, ..., M_j$ and ϕ_{jk} for $i \leq k \leq j$ have been defined.

> *Case 1.* $\underline{x}$ is weakly M_j-regular. Then stop.
>
> *Case 2.* $\underline{x}$ is not weakly M_j-regular. Choose i least and then s least such that there exists $y \in M_i$ with $\phi_{ij}(y) \notin (x_1, ..., x_s)M_j$ but $x_{s+1}\phi_{ij}(y) \in (x_1, ..., x_s)M_j$. Let $M_{j+1} = M_j \oplus R^s/Rw$ where $w = \phi_{ij}y - (\sum_1^r x_i e_i)$. Then M_{j+1} is an $\underline{x}-$modification of M_j of type s. We say step $j+1$ ($M_j \to M_{j+1}$) has index (i, s).

Note that, by construction, every $(M_j, \phi_{0j}(1))$ is an $\underline{x}-$modification of $(R, 1)$. Now if case 1 occurs, we have $\underline{x}$ is weakly M_j-regular and (by hypothesis) $\phi_{0j}(1) \notin (\underline{x})M_j$. So $M_j \neq (\underline{x})M_j$ which implies $\underline{x}$ is M_j-regular and we are done. Thus we are in the case that the process iterated indefinitely, gives us a direct system. Note for all $i \leq j$ that $(M_i, f) \to (M_j, \phi_{ij}(f))$ is a (multistep) $\underline{x}-$modification. In particular, $(M_j, \phi_{0j}(1))$ is an $\underline{x}-$modification of $(R, 1)$ and is thus non-degenerate. Also, each M_j is finitely generated and therefore Noetherian. Let $M = \varinjlim M_i$ and $\psi : M_i \to M$ the direct limit maps. So for all $i \leq j$ we have $\psi_i = \psi_j\phi_{ij}$. Recall that every element in M has the form $\psi_i(m_i)$ for some $m_i \in M_i$ and $\psi_i(m_i) = 0$ if and only if there exists $j \geq i$ such that $\phi_{ij}(M_i) = 0$.

> *Claim 1.* $M \neq (\underline{x})M$
>
> *Proof.* We will show $\psi_0(1) \notin (\underline{x})M$. Suppose $\psi_0(1) = x_1m_1 + ... + x_nm_n$. Now there exists j and $u_1, ..., u_n \in M_j$ such that $\psi_j(\phi_{0j}(1)) = \psi_0(1) = x_1\psi_j(u_1) + ... + x_n\psi_j(u_n) = \psi_j(x_1u_1 + ... + x_nu_n)$. Thus $\psi_j(\phi_{0j}(1) - (x_1u_1 + ... + x_nu_n)) = 0$. Therefore there exists $k \geq j$ such that $\phi_{jk}(\phi_{0j}(1) - \sum x_iu_i) = 0$ and so $\phi_{0k}(1) = \phi_{jk}(\sum x_iu_i) \in (\underline{x})M_k$. This contradicts the fact that $(M_k, \phi_{0k}(1))$ is a non-degenerate $x-$modification of $(R, 1)$.
>
> *Claim 2.* For each (i, s) there are only finitely many steps of index (i, s).
>
> *Proof.* Suppose steps $j_1 < j_2 < \cdots$ have index (i, s). Consider the maps $M_i \xrightarrow{\phi_{ij_1}} M_{j_1} \to M_{j_2} \to \cdots$. For all $k \geq 1$ there exist elements $y_k \in M_i$ with $\phi_{ij_k-1}(y_k) \notin (x_1, ..., x_s)M_{j_k-1}$ but $\phi_{ij_k}(y_k) \in (x_1, ..., x_s)M_{jk}$. Consider the chain of submodules in M_i :
>
> $$(x_1, ..., x_s)M_i \subsetneq \phi_{ij_1}((x_1, ..., x_s)M_{j_1}) \subsetneq \phi_{ij_2}((x_1, ..., x_s)M_{j_2}) \subsetneq \cdots$$
>
> The containments are proper as $y_j \in \phi_{ij_k}((x_1, ..., x_s)M_{jk}) \setminus \phi_{ij_{k-1}}((x_1, ..., x_s)M_{j_{k-1}})$.
>
> *Claim 3.* Fix i, s. Suppose there exists $b \in M_i$ such that $x_{s+1}b$ $in(x_1, ..., x_s)M_i$. Then $\phi_{ij}(b) \in (x_1, ..., x_s)M_j$ for $j >> 0$.
>
> *Proof.* Its easy to see that if it is true for some j, then it is true for all $j' \geq j$. So suppose $\phi_{ij}(b) \notin (x_1, ..., x_s)M_j$ for all $j \geq i$. This would mean infinitely many steps of index (i, s), contrary to claim 2.

We will show M is $\underline{x}-$regular. Suppose $x_{s+1}m = x_1m_1 + ... + x_sm_s$ for $m, m_1, ..., m_s \in M$. As before, we get $b, b_1, ..., b_s \in M_j$ such that $x_{s+1}\underbrace{\psi_j(b)}_{=m} = x_1\underbrace{\psi_j(b_1)}_{=m_1} + ... + x_s\underbrace{\psi_j(b_s)}_{=m_s}$. Then $x_{s+1}\phi_{jk}(b) = x_1\phi_{jk}(b_1) + ... + x_s\phi_{jk}(b_s)$ for $k \geq j$. By claim 3, $\phi_{j\ell} \in (x_1, ..., x_s)M_\ell$ for $\ell >> 0$ and so applying ψ_ℓ gives $m \in (x_1, ..., x_s)M$. ☐

Theorem 56. *Suppose R is Noetherian and $\underline{x} = x_1, ..., x_n \in R$. TFAE*

(1) *There exists an $R-$module M such that $\underline{x}$ is $M-$regular (that is, M is $\underline{x}-$regular)*
(2) *Every $(\underline{x})-$modification of $(R, 1)$ is non-degenerate.*

Definition. *Let $x_1, ..., x_n \in R$ and M and $R-$module. Suppose $x_{s+1}y \in (x_1, ..., x_s)M$ for some $y \in M$. Let $M' = (M + R^s)/Rw$ where $w = y - (x_1e_1 + ... + x_se_s)$. Then M' is called a* ***quasi$-\underline{x}-$modification*** *of M.*

Note. This is a weaker condition than for an $\underline{x}-$modification as we do not require $y \notin (x_1, ..., x_s)M$. As we will see, this weaker definition is necessary when using the Frobenius map.

Proposition 57. *Let (R, m) be a homomorphic image of a Gorenstein ring. Then there exists $c \in m$ such that $\dim R/(c) < \dim R$ and for all system of parameters $\underline{x}$ of R and every sequence $(R, 1) = (M_0, f_0) \to \cdots \to (M_r, f_r)$ of quasi-$\underline{x}-$modifications, one has a commutative diagram*

$$\begin{tikzcd}
(M_0, f_0) \arrow[r] \arrow[d, "\phi_0"] & (M_1, f_1) \arrow[r] \arrow[d, "\phi_1"] & \cdots \arrow[r] & (M_r, f_r) \arrow[d, "\phi_r"] \\
(R, 1) \arrow[r, "c"] & (R, c) \arrow[r, "c"] & \cdots \arrow[r, "c"] & (R, c^r)
\end{tikzcd}$$

By commutativity, $\phi_i(f_i) = c^i$ for all i.

Proof. Let $d = \dim R$ (assume $d > 0$). Let $I_i = \mathrm{Ann}_R H^i_m(R)$. By a previous result, $\dim R/I_0 \cdots I_{d-1} < \dim R$. Choose $c \in (I_0 \cdots I_{d-1} \cap m) \setminus \cup_{\dim R/p = d} p$. By another result, for all system of parameters $\underline{x} = x_1, ..., x_d$ of R, we have $c((x_1, ..., x_s) : x_{s+1}) \subset (x_1, ..., x_s)$ for all $1 \leq s \leq d-1$. Construct ϕ_i inductively. Let $\phi_0 = 1_R$. Suppose $\phi_0, ..., \phi_i$ have been chosen. We have $M_{i+1} = M_i \oplus R^s/Rw$ for some s where $x_{s+1}y \in (x_1, ..., x_s)M$ and $w = y - (x_1e_1 + ... + x_se_s)$. Then $phi_i : M_i \to R_i$ has $x_{s+1}\phi_i(y) \in (x_1, ..., x_s)\phi_i(M_i) \subseteq (x_1, ..., x_s)$. This implies $\phi_i(y) \in (x_1, ..., x_s :_R x_{s+1})$ and so $c\phi_i(y) \in (x_1, ..., x_s)$. Thus $c\phi_i(y) = x_1u_1 + ... + x_su_s$ for some $u_i \in R$. Define $\widetilde{\phi}_{i+1} : M_i \oplus R^s$ by $m + \sum r_ie_i \mapsto c\phi_i(m) + \sum r_iu_i$. Note $\widetilde{\phi}_{i+1}(w) = \widetilde{\phi}_{i+1}(y - \sum x_ie_i) = c\phi_iy) - \sum x_iu_i = 0$. Therefore, we get an induced map $\phi_{i+1} : M_i \oplus R^2/Rw \to R$. Note for $m \in M_i$ that $\phi_{i+1}(m) = c\phi_i(m)$. This makes the square commute. ☐

Notation. Let R be a ring of characteristic $p > 0$. Given an $R-$module M, let $F(M) := R^F \otimes_R M$, viewed as a left $R-$module. Given $f \in M$, let $F(f)$ denote $1 \otimes f \in F(M)$. If $M = R^n$, say $f = \sum r_ie_i$. Then $F(f) = 1 \otimes f = 1 \otimes (\sum r_ie_i) = \sum r_i^p(1 \otimes e_i) = \sum r_i^pe_i \in R^n = F(R^n)$. For this reason, denote $F(f)$ by f^p and similarly $F^e(f)$ by f^{p^e}.

Note that if $f = r_1u_1 + ... + r_nu_n$ for $f, u_i \in M$ and $r_i \in R$, then $f^p = 1 \otimes f = 1 \otimes (\sum r_iu_i) = \sum r_i^p(1 \otimes u_i) = \sum r_i^pu_i^p$.

Lemma 58. *Let $\mathrm{char}\, R = p > 0$. Suppose $(M, f) \to (M', f')$ is a quasi-$\underline{x}$-modification for $\underline{x} = x_1, ..., x_n \in R$. Then $(F(M), f^p) \to (F(M'), (f')^p)$ is a quasi$-\underline{x}^p-$modification.*

Proof. Let $M' = M \oplus R^s/Rw$ where $x_{s+1}y = x_1z_1 + ... + x_sz_s$ for $y, z_i \in M$ and $q = y - (x_1e_1 + ... + x_se_s)$. We have a short exact sequence $0 \to Rw \to M \oplus R^s \to M' \to 0$. Apply F we have

$$\underbrace{F(Rw)}_{=RF(w)} \xrightarrow{\psi} \underbrace{F(M) \oplus F(R^s)}_{=F(M) \oplus R^s} \to F(M') \to 0$$

Now $\mathrm{im}\, \psi = RF(w) = Rw^p$. Thus $F(M') \cong F(M) \oplus R^s/Rw^p$ where $w^p = y^p - \sum x_i^pe_i^p = y^p - \sum x_i^pe_i$ (since we identified $F(R^s)$ with R^s, we must identify the basis elements e_i^p with e_i) and $x_{s+1}^py^p = x_1^pz_1^p + ... + x_s^pz_s^p$. ☐

Theorem 59 (Hochster, '70s)**.** *Let (R, m) be a local Noetherian ring of characteristic $p > 0$. Then R has a balanced big Cohen Macaulay module.*

Proof. It is enough to show R has a big Cohen Macaulay module. Since any system of paramters for $\hat{R}$ is a system of parameters for R, we may assume R is complete and therefore the homomorphic image of a Gorenstein ring. Fix a system of parameters $\underline{x} = x_1, ..., x_d \in R$. It is enough to show every $\underline{x}-$modification for $(R, 1)$ is non-degenerate. Suppose not. Then there exists a sequence of $\underline{x}-$modifications $(R, 1) = (M_0, f_0) \to \cdots \to (M_r, f_r)$ where $f_r \in (x_1, ..., x_d)M_r$. For any $e \geq 1$, $(R, 1) = (F^e(M_0), f_0^{p^e}) \to \cdots \to (F^e(M_r), f_r^{p^e})$ is a quasi-$\underline{x}^{p^e}-$modification of $(R, 1)$ and $f_r^{p^e} \in (x_1^{p^e}, ..., x_d^{p^e})F^p(M_r)$. By the proposition, there exists $c \in R$ such that $\dim R/(c) < \dim R$ and for all $e \geq 1$ there exists a diagram

$$\begin{array}{ccccc} (F^e(M), f_0^{p^e}) & \longrightarrow & \cdots & \longrightarrow & (F^e(M_r), f_r^{p^e}) \\ \downarrow & & & & \downarrow{\scriptstyle \phi_{re}} \\ (R, 1) & \longrightarrow & \cdots & \longrightarrow & (R, c^r) \end{array}$$

By commutativity of the diagram, $\phi_{re}(f_r^{p^e}) = c^r$. On the other hand, $c^r = \phi_{re}(f_r^{p^e}) \in (x_1^{p^e}, ..., x_d^{p^e})\phi_{re}(F^e(M_r)) \subseteq R$, which implies $c^r \in \cap_e(x_1^{p^e}, ..., x_d^{p^e}) = (0)$, a contradiction as c is not nilpotent. □

Definition. *Let R be a domain. The* ***absolute integral closure*** *of R, denoted R^+, is the integral closure of R in a fixed algebraic closure of $Q(R)$, the quotient field of R.*

Except in trivial cases, R^+ is non-Noetherian.

Theorem 60 (Hochster-Huneke, '92)**.** *If (R, m) is a local excellent Noetherian domain of characteristic $p > 0$, then R^+ is a balanced big Cohen Macaulay module (in fact, algebra) for R.*

Examples of excellent rings include finitely generated algebras over a field and complete rings. Most rings that we encounter are excellent.

Theorem 61 (Huneke-Lyubeznik, '06)**.** *Let (R, m) be a local domain which is the homomorphic image of a Gorenstein ring with characteristic $p > 0$. Then R^+ is a balanced big Cohen Macaulay algebra.*

We will prove this latter result, but first we must prove some preliminary results.

Remarks.

(1) The two theorems above give the existence of balanced big Cohen Macaulay algebras for arbitrary local rings of characteristic $p > 0$.

Proof. Let $p \in \operatorname{Spec} \hat{R}$. Then $\dim \hat{R}/p = \dim \hat{R} = \dim R$. Now $\hat{R}/p$ meets the requirements of one of the above theorems and thus $(\hat{R}/p)^+$ is a balanced big Cohen Macaulay algebra for $\hat{R}/p$ and therefore R (as any system of parameters for R is one for $\hat{R}/p$). □

(2) For a domain R, we have $(R^+)_p = (R_p)^+$ for all $p \in \operatorname{Spec} R$ (as $Q(R) = Q(R_p)$ and localization commutes with integral closures)

(3) For (R, m) a domain, $I \subseteq m$, we have $IR^+ \neq R^+$.

Proof. Suppose $IR^+ = R^+$. Since R^+ is a ring, this says $1 = i_1s_1 + ... + i_ks_k$ for $s_i \in R^+$. Let $S = R[s_1, ..., s_k]$. Then S is a finitely generated $R-$module and $IS = S$, a contradiction to NAK. □

Proposition 62. *Let Λ be a class of catenary Noetherian local domains which is closed under localization. (e.g. $\Lambda = \{$excellent local rings of characteristic $p\}$ or $\Lambda = \{$local rings which are homomorphic images of Gorenstein rings$\}$. TFAE*

(1) *For all local rings $(R,m) \in \Lambda, H^i_m(R^+) = 0$ for all $i < \dim R$.*
(2) *For all $(R,m) \in \Lambda$, R^+ is a balanced big Cohen Macaulay algebra.*

Proof. For (2) $\Rightarrow$ (1), let $(R,m) \in \Lambda$ and $\underline{x}$ a system of parameters for R. So $H^i_m(R^+) = \check{H}^i_{(\underline{x})}(R^+)$. By (2), $\underline{x}$ is regular on R^+ and so $\operatorname{grade}(\underline{x}, R^+) \geq \operatorname{Grade}(\underline{x}, R^+) \geq d$. By definition of grade, $H^i_{\underline{x}}(R^+) = 0$ for all $i < d$.

For (1) $\Rightarrow$ (2), let $(R,m) \in \Lambda$.

Claim 1. Let $x_1, ..., x_j \in m$ be R^+-regular. Then $H^i_m(R^+/(x_1,...,x_j)R^+) = 0$ for all $i < \dim R - j$.

Proof. Induct on j. For $j = 0$, done by (1). For $j \geq 1$, use the short exact sequence

$$0 \to R^+/(x_1,...,x_{j-1})R^+ \xrightarrow{x_j} R^+/(x_1,...,x_{j-1})R^+ \to R^+/(x_1,...,x_j)R^+ \to 0.$$

Using (1), the long exact sequence on homology and the induction hypothesis

$$H^i_m(R^+/(x_1,...,x_{j-1})R^+) \xrightarrow{x_j} \underbrace{H^i_m(R^+/(x_1,...,x_{j-1})R^+)}_{=0 \text{ for } i<\dim R-j+1} \to H^i_m(R^+/(x_1,...,x_j)R^+) \to \underbrace{H^{i+1}_m(R^+/(x_1,...,x_{j-1})R^+)}_{=0 \text{ for } i<\dim R-j} \xrightarrow{x_j} \cdots$$

Thus $H^i_m(R^+/(x_1,...,x_j)R^+) = 0$ for $i < \dim R - j$.

Consequently, we have the following claim.

Claim 2. If $x_1,...,x_j$ is R^+-regular and $j < \dim R$, then $H^0_m(R^+/(x_1,...,x_j)R^+) = 0$.

Let $x_1,...,x_d \in m$ be a system of parameters for R. Induct on j to show $x_1,...,x_j$ is R^+-regular. As R^+ is a domain, the $j = 1$ case is done. So suppose $j \geq 1$. Assume $x_1,...,x_j$ is R^+-regular and suppose x_{j+1} is a zerodivisor on $R^+/(x_1,...,x_j)R^+$. Then there exists $p \in \operatorname{Ass}_R R^+/(x_1,...,x_j)R^+$ with $x_{j+1} \in p$. Then $\frac{x_{j+1}}{1} \in pR_p \in \operatorname{Ass}_{R_p}(R^+)_p/(x_1,...,x_j)(R^+)_p$. So $(*) H^0_{pR_p}((R_p)^+/(x_1,...,x_j)(R_p)^+) \neq 0$.

Claim 3. $j < \dim R_p$.

Proof. Since $x_1,...,x_d$ is a system of parameters for R, x_{j+1} is not in any minimal prime of $(x_1,...,x_j)$ of dimension $\dim R - j$. Suppose $\dim R_p \leq j$. Since R is a catenary local domain, $\dim R_p + \dim R/p = \dim R = \dim R/p \geq \dim R - j$. So $x_{j+1} \in p$ and $\dim R/p - \dim R - j$, a contradiction.

Now $(*)$ contradicts claim 2 applied to $R_p \in \Lambda$. $\square$

We will show for $R \in \Lambda = \{\text{local domains of char } p \text{ which are homomorphic images of Gorenstein rings}\}$ that $H^i_m(R^+) = 0$ for all $i < \dim R$.

Notation. Let R be a ring, $\underline{x} = x_1,...,x_n$ and $C^\cdot(\underline{x}; R)$ the Čech complex. So $C^i(\underline{x}; R) = \bigoplus_{1 \leq j_1 \leq \cdots \leq j_i \leq n} R_{x_{j_1} \cdots x_{j_i}}$. Fix k with $1 \leq k \leq n$ and let $\Lambda_k = \{(j_1,...,j_k) | 1 \leq j_1 \leq \cdots \leq j_k \leq n\}$. For $J = (j_1,...,j_k) \in \Lambda_k$, set $x_J := x_{j_1} \cdots x_{j_k}, x^e_J = x^e_{j_1} \cdots x^e_{j_k}$, and $R_{x_J} := R_{x_{j_1} \cdots x_{j_k}}$. So $C^i(\underline{x}; R) = \oplus_{J \in \Lambda_k} R_{x_J}$ and $\alpha \in C^i(\underline{x}; R)$ has the form $\left(\frac{r_J}{x_J} p\right)_{J \in \Lambda_i}$. Suppose $\phi : R \to S$ is a ring homomorphism. Then ϕ induces a chain map

$$\begin{tikzcd}
0 \arrow[r] & R \arrow[r] \arrow[d, "\phi"] & R_{x_i} \arrow[r] \arrow[d, "\phi"] & 0 \\
0 \arrow[r] & S \arrow[r] & S_{\phi(x_i)} \arrow[r] & 0
\end{tikzcd}$$

Tensoring gives a natural chain map $\hat{\phi}(\alpha) = \left(\frac{\phi(r_J)}{\phi(x_J)^e}\right)$. As $\hat{\phi}$ is a chain map, it induces a map on cohomology $\overline{\phi} : H^i_{(\underline{x})}(R) \to H^i_{\phi(\underline{x})}(S)$ defined by $\overline{\alpha} \mapsto \overline{\phi(\alpha)}$.

Let $\phi : R \to R$ defined by $r \mapsto r^p$ be the Frobenius map. This gives a natural map on local cohomology: $\overline{\phi} : H^i_I(R) \to H^i_I(R)$ defined by $\alpha = \left(\frac{r_J}{x^e_J}\right) \mapsto \overline{\alpha^p} = \left(\frac{r^p_j}{(x^e_J)^p}\right)$.

If $R \hookrightarrow S$ (that is, R is a subring of S), then we can consider $C^\cdot(\underline{x}; R)$ as a subcomplex of $C^\cdot(\underline{x}; S)$. This gives rise to natural maps $H^i_{(\underline{x})}(R) \to H^i_{(\underline{x})}(S)$ for all i.

Remark. Let R be a domain, $\underline{x} = x_1, ..., x_n \in R$ and $\underline{y} = y_1, ..., y_n \in R$. Suppose $y_i | x_i$ for all i. Then there are natural chain maps

$$\begin{tikzcd} 0 \arrow[r] & R \arrow[r] \arrow[d] & R_{y_i} \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & R \arrow[r] & R_{x_i} \arrow[r] & 0 \end{tikzcd}$$

where the diagram commutes. Tensoring gives a natural chain map $C^{\cdot}(\underline{y}; R) \to C^{\cdot}(\underline{x}; R)$. Since R is a domain, $R_{y_J} \to R_{x_J}$ is injective for all J. Thus $C^{\cdot}(\underline{y}; R) \to C^{\cdot}(\underline{x}; R)$ is injective, that is $C^{\cdot}(\underline{y}; R)$ is a subcomplex of $C^{\cdot}(\underline{x}; R)$.

- Special Case: Let $y_i = 1$ for all i. Then $C^{\cdot}(\underline{1}; R)$ is a subcomplex of $C^{\cdot}(\underline{x}; R)$ for all $\underline{x}$ But the i^{th} cohomology of $C^{\cdot}(\underline{1}; R)$ is $H^i_{(1)R}(R) = 0$ for all i. Thus $C^{\cdot}(\underline{1}; R)$ is an exact complex.

Hence if $\alpha \in C^i(\underline{x}; R)$ has the form $\left(\frac{r_J}{1}\right)_{J \in \Lambda_i}$ and is a cycle, then α is a boundary.

Proposition 63. *Let R be a Noetherian domain of characteristic p. Let $K = Q(R)$ and $\overline{K}$ a fixed algebraic closure. Let $I = (x_1, .., x_n)$ be an ideal of R. Let $w = H^i_I(R)$ and suppose the submodule $\sum_{i=0}^{\infty} Rw^{p^i}$ is finitely generated. Then there exists $R \subseteq S \subseteq \overline{K}$ where S is a finite $R-$module such that w goes to zero under the natural map $H^i_I(R) \to H^i_I(S)$.*

Proof. Since $\sum Rw^{p^i}$ is finitely generated (and hence Noetherian), there exists an equation of the form $w^{p^s} = r_{s-1}w^{p^{s-1}} + ... + r_1 w$ for $r_i \in R$, that is, $w^{p^s} - (r_{s-1}w^{p^{s-1}} + ... + r_1 w) = 0$. Let α be a cycle in $C^i(\underline{x}, R)$ which lifts w. Then $\alpha^{p^s} - (r_{s-1}\alpha^{p^{s-1}} + ... + r_1\alpha) = \partial(\beta)$ for some $\beta = \left(\frac{r_J}{x_J^e}\right) \in C^{i-1}(\underline{x}; R)$.

We need to find a finite extension S or R such that the image of α in $C^{\cdot}(\underline{x}; S)$ is a boundary. Let $g(T) = T^{p^s} - (r_{s-1}T^{p^{s-1}} + ... + r_1 T) \in R[T]$. So $g(\alpha) - \partial\left(\frac{r_J}{x_J^e}\right) = 0$. For each $J \in \Lambda_{i-1}$, let z_J be an indeterminate over R. Consider the equation $(x_J^e)^{p^s}\left(g\left(\frac{z_J}{x_J^e}\right) - \frac{r_J}{x_J^e}\right) = 0$, a monic polynomial in $R[z_J]$. Let $u_J \in \overline{K}$ be a root of this polynomial. Then u_J is integral over R. Thus $g\left(\frac{u_J}{x_J^e}\right) = \frac{r_J}{x_J^e}$ for all J. Let $\beta' = \left(\frac{u_J}{x_J^e}\right) \in C^{i-1}(\underline{x}; R')$ for $R' = R[u_J | J \in \lambda_{i-1}]$. Therefore $g(\beta') = \left(\frac{r_j}{x_J^e}\right) = \beta$.

Let $\alpha' = \alpha - \partial(\beta') \in C^i(\underline{x}; R')$. It remains to find a finite extension S of R' such that α' is a boundary in $C^{\cdot}(\underline{x}; S)$ as then α is a boundary. Since taking p^{th} powers induces a chain map on $C^{\cdot}(\underline{x}; R) \to C^{\cdot}(\underline{x}; R)$, we see that $g\partial(y) = \partial g(y)$ for all $y \in C^{\cdot}(\underline{x}; R)$. Then

$$g(\alpha') = g(\alpha) - g\partial(\beta') = g(\alpha) - \partial g(\beta') = g(\alpha) - \partial(\beta) = 0.$$

Let $\alpha' = (c_J)_{J \in \Lambda_i}$ for $c_J \in R'_{x_J}$. Now $g(c_J) = 0$ for all J and thus c_J are integral over R'. Let $S = R'[c_J | J \in \Lambda_i]$. This is a finite extension over R' and hence over R. Now $\alpha' = (c_J) \in C^i(\underline{x}'S)$ is a cycle. As all components of α' are in S, we see α' is also a boundary in $C^{\cdot}(\underline{x}; S)$. Thus the image of w in $H^i_J(S)$ is zero. □

Let $\phi : R \to S$ be a ring homomorphism and $\underline{x} = x_1, ..., x_n \in R$. Then one has a natural map of chain complexes $\tilde{\phi} : C^{\cdot}(\underline{x}; R) \to C^{\cdot}(\phi(\underline{x}); S)$. Let $f_R : R \to R$ and $f_S : S \to S$ be the Frobenius maps. Then we have a commutative diagram

$$\begin{tikzcd} R \arrow[r, "\phi"] \arrow[d, "f_R"] & S \arrow[d, "f_S"] \\ R \arrow[r, "\phi"] & S \end{tikzcd}$$

This yields a commutative square of cochain complexes and taking homology, we have for all i

$$\begin{tikzcd} H^i_{(\underline{x})}(R) \arrow[r, "\tilde{\phi}^*"] \arrow[d, "\tilde{f_R}^*"] & H^i_{(\phi(\underline{x}))}(S) \arrow[d, "\tilde{f_S}^*"] \\ H^i_{(\underline{x})}(R) \arrow[r, "\tilde{\phi}^*"] & H^i_{(\phi(\underline{x}))}(S) \end{tikzcd}$$

Let R be a ring. An $R-$algebra S which is finitely generated as an $R-$module will be called **finite $R-$algebra**. Let R be a domain, $K = Q(R)$, and $\overline{K}$ a fixed algebraic closure of K. Let

$$\Lambda(R) = \{S \text{ a finite } R\text{−algebra}, \ R \subset S \subset \overline{K}\}.$$

If $S \in \Lambda(R)$, then S is integral over R. Thus $Q(S)$ is algebraic over K which implies $\overline{Q(S)} = \overline{K}$. Therefore, $\Lambda(S) \subseteq \Lambda(R)$. Recall R^+ is the integral closure of R in $\overline{K}$, that is, $R^+ = \cup_{S\in\Lambda(R)} S = \varinjlim_{S\in\Lambda(R)} S$. So

$$C^{\cdot}(\underline{x}; R^+) = C^{\cdot}(\underline{x}; R) \otimes R^+ = C^{\cdot}(\underline{x}; R) \otimes_R (\varinjlim_{S\in\Lambda(R)} S) = \varinjlim (C^{\cdot}(\underline{x}; R) \otimes_R S) = \varinjlim_{S\in\Lambda(R)} H^i_{(\underline{x})}(S).$$

Thus $H^i_{(\underline{x})}(R^+) = H^i(\varinjlim C^{\cdot}(\underline{x}; S)) = \varinjlim_{S\in\Lambda(R)} H^i_{(\underline{x})}(S)$. In particular, $H^i_{(\underline{x})}(R^+) = 0$ if and only if for all $\alpha \in H^i_{(\underline{x})}(S)$ for $S \in \Lambda(R)$ there exists $T \in \Lambda(S) \subset \Lambda(R)$ such that α maps to zero in the map $H^i_{(\underline{x})}(S) \to H^i_{(\underline{x})}(T)$.

Theorem 64 (Huneke, Lyubeznik)**.** *Let (R, m) be a Noetherian local domain of characteristic $p > 0$, which is the homomorphic image of a Gorenstein local ring (A, n). Let $d = \dim R$. For each $i < d$ and $S \in \Lambda(R)$, there exists $T \in \Lambda(S)$ such that the natural map $H^i_m(S) \to H^i_n(T)$ is zero.*

Proof. Without loss of generality, we may assume $\dim A = d$. Induct on d. Since R is a domain, the $d = 0$ and $d = 1$ cases are trivial. So assume $d > 1$ and that the theorem holds for all R with $\dim R < d$ and the above hypotheses. Fix $i < d$ and $S \in \Lambda(R)$.

> *Claim.* For all $p \in \operatorname{Spec} A \setminus \{n\}$, there exists $S(p) \in \Lambda(S)$ such that for all $T \in \Lambda(S(p))$, the natural map $\operatorname{Ext}^{d-i}_A(T, A)_p \to \operatorname{Ext}^{d-i}_A(S, A)_r$ is zero, where the map is induced by the inclusion $S \hookrightarrow T$.
>
> *Proof.* Fix $p \in \operatorname{Spec} A \setminus \{n\}$ and let $t = \dim A/p > 0$. Then $\dim R_p = \dim A_p = \dim A - \dim A/p = d - t < d$. Note $S_p \in \Lambda(R_p)$ and $i - t < d - t = \dim R_p$. By the induction hypothesis, there exists $\tilde{S}_p \in \Lambda(S_p)$ such that $H^{i-t}_{pR_p}(S_p) \to H^{i-t}_{pR_p}(\tilde{S})$ is zero $(*)$. Write $\tilde{S}_p = S_p[z_1, ..., z_\ell]$ where z_i are integral over S_p and thus over R_p. We can multiply each z_i be any element in $R \setminus p$ and thus assume each z_i is integral over R. Let $S(p) = S[z_1, ..., z_\ell] \in \Lambda(S)$. Note $S(p)_p = \tilde{S}_p$. We'll show $S(p)$ works. Let $T \in \Lambda(S(p))$. The inclusions $S \to S(p) \to T$ induce natural maps
>
> $$\operatorname{Ext}^{d-i}_A(T, A) \to \operatorname{Ext}^{d-i}_A(S(p), A) \xrightarrow{\psi} \operatorname{Ext}^{d-i}_A(S, A).$$
>
> Now localize and note it is enough to show $\psi_p = 0$, that is, show the map $\psi_p : \operatorname{Ext}^{d-i}_{A_p}(\tilde{S}_p, A_p) \to \operatorname{Ext}^{d-i}_{A_p}(S_p, A_p)$ is zero. Let $(-)^\vee = \operatorname{Hom}_{A_p}(-, E_{A_p}(A_p, pA_p))$. Then it is enough to show $\psi_p^\vee = 0$, that is, $H^{(d-t)-(d-i)}_{pA_p}(S_p) \to H^{(d-t)-(d-i)}_{pA_p}(\tilde{S}_p)$ is zero. This is true by $(*)$ and thus the claim holds.

Now $\operatorname{Ext}^{d-i}(S, A)$ is a finitely generated $A-$module. Let $\Gamma = \{P_1, ..., P_\ell\} = \operatorname{Ass}_A \operatorname{Ext}^{d-i}_A(S, A) \setminus \{n\}$. If $\Gamma = \emptyset$, then $\operatorname{Ext}^{d-i}_A(S, A)$ has finite length. Otherwise, let $B = S[S(P_1), ..., S(P_\ell)] \subseteq \overline{K}$. As each $S(P_i)$ is a finite integral extension, B is and thus $B \in \Lambda(S)$. In fact, $B \in \Lambda(S(P_j))$ for all j. Thus the natural maps $\operatorname{Ext}^{d-i}_A(B, A)_{P_j} \to \operatorname{Ext}^{d-i}_A(S, A)_{P_j}$ are zero for all j by the claim.

Let $\phi : \operatorname{Ext}^{d-i}_A(B, A) \to \operatorname{Ext}^{d-i}_A(S, A)$ be the natural map induced by $S \hookrightarrow B$ and let $U := \operatorname{im} \phi$. Since $\operatorname{Ass}_A U \setminus \{n\} \subseteq \Gamma$ and $U_p = 0$ for all $P \in \Lambda$, we have $\operatorname{Ass}_A U \subseteq \{n\}$. Therefore $\lambda_A(U) < \infty$. Let $(-)^\vee = \operatorname{Hom}_A(-, E_A(A/n))$ and note $\lambda_A(U^\vee) < \infty$. We have

$$\begin{tikzcd}
& \operatorname{Ext}^{d-i}_A(B, A) \arrow[rr, "\phi"] \arrow[dr] & & \operatorname{Ext}^{d-i}_A(S, A) \\
& & U \arrow[dr] & \\
& 0 \arrow[ur] & & 0
\end{tikzcd}$$

and applying $(-)^\vee$ we get

$$
\begin{tikzcd}
H^i_m(S) \arrow[rr, "\psi=\phi^\vee"] \arrow[dr] & & H^i_m(B) \\
& U^\vee \arrow[dr] & \\
0 \arrow[ur] & & 0
\end{tikzcd}
$$

which implies $\operatorname{im}\psi \cong U^\vee$ and thus $\lambda(\operatorname{im}\psi) < \infty$. Recall ψ commutes with the Frobenius maps $f_S : H^i_m(S) \to H^i_m(S)$ defined by $\alpha \mapsto \alpha^p$. Therefore, for all $\alpha \in H^i_m(S)$, we see $\psi(\alpha)^p = f_T(\psi(\alpha)) = \psi(f_T(\alpha)) \in \operatorname{im}\psi$. Thus for all $\beta \in \operatorname{im}\psi$, we have $\beta^{p^e} \in \operatorname{im}\psi$ for all e. As $\lambda(\operatorname{im}\psi)M\infty$, we know $\operatorname{im}\psi$ is Noetherian and thus $\sum_{e\geq 0} R\beta^{p^e}$ is finitely generated for all β. By the proposition, for all $\beta \in \operatorname{im}\psi$, there exists $T_\beta \in \Lambda(\beta)$ such that $\beta \mapsto 0$ under the map $H^i_m(B) \to H^i_m(T_\beta)$. Let $\operatorname{im}\psi = R\beta_1 + ... + R\beta_t$ and $T = B[T_{\beta_1}, ..., T_{\beta_t}] \subset \Lambda(B) \subseteq \Lambda(S)$. Thus $\operatorname{im}\psi$ goes to zero under the map $H^i_m(B) \to H^i_m(T)$. Therefore, $H^i_m(B) \to H^i_m(T)$ is zero and thus $H^i_m(S) \xrightarrow{\psi} H^i_m(B) \to H^i_m(T)$ is zero. □

Corollary 65. *With R as above, $H^i_m(R^+) = 0$ for all $i < d$ and thus R^+ is a big Cohen Macaulay algebra.*

Let $\underline{x} = x_1, ..., x_n$ and $\underline{y} = y_1, ..., y_m$ be indeterminants over $\mathbb{Z}$. Let $S \subseteq \mathbb{Z}[\underline{x}, \underline{y}]$. We say S **has a solution of height** n in a Noetherian ring R if there exists $\underline{a} = a_1, ..., a_n \in R$ and $\underline{b} = b_1, ..., b_m \in R$ such that

(1) $f(\underline{a}, \underline{b}) = 0$ for all $f \in S$
(2) $\operatorname{ht}(\underline{a}R = n$.

Theorem 66 (**Hochster's Finiteness Theorem**). *Suppose a set $S \subseteq \mathbb{Z}[\underline{x}, \underline{y}]$ has a solution of height n in some Noetherian ring containing a field. Then S has a solution of height n in an affine domain R over a finite field (so $R = k[T_1, ..., T_\ell]/p$ for some finite k). In particular, S has a solution $(\overline{a}, \overline{b})$ in a Noetherian local domain R of characteristic $p > 0$ where $\underline{a}$ is a system of parameters for R.*

Proof. Uses Artin approximation and Henselization. □

For the following, we will make use of Proposition 54 and Theorem 56 where we replace $\underline{x}-$modification with quasi$-\underline{x}-$modification.

Proposition 67. *Fix $r, n \geq 1$ and integers $s_1, ..., s_r$ such that $1 \leq s_i \leq n-1$ for all i. Then there exists a set $S \subseteq \mathbb{Z}[x_1, ..., x_n, y_1, ..., y_m]$ (where S and m depend on $s_1, ..., s_r$) such that given any ring R and elements $\underline{a} = a_1, ..., a_n \in R$, TFAE*

(1) *There exists a degenerate quasi$-\underline{a}-$modification of $(R, 1)$ of type $(s_1, ..., s_r)$*
(2) *There exists $\underline{b} = b_1, ..., b_m \in R$ such that $f(\underline{a}, \underline{b}) = 0$ for all $f \in S$.*

Sketch of proof. Suppose $(R, 1) = (M_0, f_0) \to \cdots \to (M_r, f_r)$ is a degenerate quasi$-\underline{a}-$modification (that is, $f_r \in (\underline{a})M$) of type $(s_1, ..., s_r)$. Then $M_{i-1} \to M_i$ is an $\underline{a}-$modification of type s_i, that is, $M_i = M_{i-1} \oplus F_i/Rw_i$ where F_i is free with basis $\{e^i_1, .., e^i_{s_i}\}$ and $w_i = y_i \setminus \sum_{j=1}^{s_i} a_j e^i_j$ for $y_i \in M_{i-1}$ with $a_{s_i+1}y_i \in (a_1, ..., a_{s_i})M_{i-1}$. Now $M_i = \oplus^i_1 F_j / \sum^i_1 Rw_j$ where $F_0 = R$. Each y_i can be written in terms of the basis elements of $\oplus^r_0 F_j$. Then each w_i can be expressed similarly. The condition $a_{s_i+1}y_i \in (a_1, ..., a_{s_i})M_{i-1}$ can be expressed in terms of the basis elements. Degeneracy means $1 = e^0_1 \in (a_1, ..., a_n)M_r$ which gives another equation in terms of the basis elements. Each equation among the basis elements gives one equation in the ring R for each basis element. Replace all coefficients by variables (replace a_i's with x_i's and all other coefficients with y_i's). This gives a set of equations in $\mathbb{Z}[\underline{x}, \underline{y}]$. □

Corollary 68. *If (R, m) is a Noetherian local ring contain a field, then R has a balanced big Cohen Macaulay module.*

Proof. Let $\underline{x} = x_1, ..., x_n$ be a system of parameters for R. It is enough to show every quasi$-\underline{x}-$modification of $(R, 1)$ is non-degenerate. Suppose not. Then there exists a degenerate $\underline{x}-$modification of $(R, 1)$ of type $s_1, ..., s_r$. Then the set S described in the proposition has a solution in R of height n. By Hochster's Finiteness Theorem, S has a solution of height n in some local ring of characteristic p of dim n. This contradicts Proposition 54. $\square$

Theorem 69 (**Bass' Conjecture**). *Let (R, m) be a Noetherian local ring. Suppose R has a nonzero finitely generated module of finite injective dimension. Then R is Cohen Macaulay.*

To prove Bass' Conjecture, we need first need several lemmas and a proposition that will allow us to use the New Intersection Theorem. First, recall the following facts.

Facts.

(1) (R, m) Noetherian. If M is finitely generated and $\operatorname{id}_R M < \infty$, then $\operatorname{id}_R M = \operatorname{depth} R$.

(2) R Noetherian, M finitely generated, $q \subseteq p$ primes with $\operatorname{ht}(q/p) = n$. If $\mu_i(q, M) \neq 0$, then $\mu_{i+n}(p, M) \neq 0$ where $\mu_i(p, M) := \dim_{k(p)} \operatorname{Ext}^i_{R_p}(k(p), M_p)$ for $k(p) = R_p/pR_p$.

Lemma 70. *Let (R, m) be Noetherian and M a finitely generated $R-$modules such that $\operatorname{id}_R M < \infty$. Then for all $p \in \operatorname{Supp} M$, $\dim R/p + \operatorname{depth} R_p = \operatorname{depth} R$.*

Proof. Next time. $\square$

Lemma 71. *Let R be Noetherian and M a finitely generated $R-$module. Then $\operatorname{Supp}_R M = \cup_i \operatorname{Supp}_R \operatorname{Ext}^i_R(M, R)$.*

Proof. Recall $\operatorname{Grade} M := \inf\{i | \operatorname{Ext}^i_R(M, R) \neq 0\}$ and $\operatorname{Grade} M = \operatorname{Grade} R/\operatorname{Ann}_R M = \operatorname{depth}_{\operatorname{Ann}_R M} R < \infty$ if $\operatorname{Ann} M \neq R$, that is, if $M \neq 0$. Thus $M \neq 0$ if and only if $\operatorname{Ext}^i_R(M, R) \neq 0$ for some i. Therefore, $M_p \neq 0$ if and only if $\operatorname{Ext}^i_{R_p}(M_p, R_p) \neq 0$. $\square$

Lemma 72. *Let (R, m) be a complete Noetherian ring and M, N $R-$modules. Suppose N is finitely generated or Artinian. Then for all i there exist natural isomorphisms $\operatorname{Ext}^i_R(M, N) \cong \operatorname{Tor}^R_i(M, N^\vee)^\vee$ where $(-)^\vee = \operatorname{Hom}_R(-, E_R(R/m))$.*

Proof. Let $F.$ be a projective resolution of M. Then as $N^{\vee\vee} \cong N$, we have

$$\begin{aligned} \operatorname{Tor}^R_i(M, N^\vee)^\vee &\cong H_i(F. \otimes_R N^\vee)^\vee \cong H^i((F. \otimes_R N^\vee)^\vee) \cong H^i(\operatorname{Hom}_R(F. \otimes N^\vee, E)) \\ &\cong H^i(\operatorname{Hom}_R(F., \operatorname{Hom}_R(N^\vee, E))) \cong H^i(\operatorname{Hom}_R(F., N)) \cong \operatorname{Ext}^i_R(M, N). \end{aligned}$$

$\square$

Lemma 73. *Let (R, m) be a complete Noetherian local ring. Then for all finitely generated or Artinian $R-$modules C, there exists a natural isomorphism $\operatorname{Hom}_R(E, C) = \operatorname{Hom}_R(\operatorname{Hom}_R(C, E), R) \cong (C^\vee)^*$.*

Proof.

$$\begin{aligned} \operatorname{Hom}_R(E, C) &\cong \operatorname{Hom}_R(E, C^{\vee\vee}) \cong \operatorname{Hom}_R(E, \operatorname{Hom}_R(C^\vee, E)) \cong \operatorname{Hom}_R(E \otimes C^\vee, E) \cong \operatorname{Hom}_R(C^\vee \otimes E, E) \\ &= \operatorname{Hom}_R(C^\vee, \operatorname{Hom}_R(E, E)) \cong \operatorname{Hom}_R(C^\vee, R) \end{aligned}$$

$\square$

Proposition 74. *Let (R, m) be a complete Noetherian local ring and T a finitely generated $R-$module of finite injective dimension. Let $M = \operatorname{Ext}^r_R(E, T)$ where $r = \operatorname{depth} R$. Then M is finitely generated, $\operatorname{pd} M = r - \operatorname{depth} T$, and $\operatorname{Supp} M = \operatorname{Supp} T$.*

Proof. Let $I^\cdot : 0 \to I^0 \to \cdots \to I^r \to 0$ be a minimal injective resolution of T (since $r = \operatorname{depth} R = \operatorname{id}_R T$). Recall each $I^i = \oplus_{p \in \operatorname{Spec} R} E_R(R/p)^{\mu_i(p,T)}$. Let $E = E_R(R/m)$ and note $\operatorname{Hom}_R(E, E(R/p)) = \begin{cases} R, & \text{if } p = m \\ 0, & \text{if } p \neq m \end{cases}$. Apply $\operatorname{Hom}_R(E, -)$ to $I^\cdot$:

$$0 \to \underbrace{\operatorname{Hom}_R(E, I^0)}_{=R^{\mu_0(m,T)}} \to \cdots \to \underbrace{\operatorname{Hom}_R(E, I^r)}_{=R^{\mu_r(m,T)}} \to 0$$

This gives a free resolution $(1) F^{\cdot} = \operatorname{Hom}_R(E, I^{\cdot})$ where $F^i = R^{\mu_i(m,T)}$. Now $F^{\cdot}$ is a complex of finitely generated $R-$modules. Recall $H_m^0(E(R/p)) = \begin{cases} E, & \text{if } p = m \\ 0, & \text{if } p \neq m \end{cases}$. Thus $H_m^{\cdot}(I^i) = E^{\mu_i(m,T)}$ and so $(2) F^{\cdot} = \operatorname{Hom}_R(E, H_m^0(I^{\cdot}))$. Note that from (1), $\operatorname{Ext}_R^i(E,T) = H^i(F^{\cdot})$.

Claim 1. $\operatorname{Ext}_R^i(E,T) = 0$ for all $i < r$.

Proof. By Lemma 72, $\operatorname{Ext}_R^i(E,T) \cong \operatorname{Tor}(E, T^\vee)^\vee$. Now $T^\vee$ is Artinian and thus

$$T^\vee = \cup_{n \geq 1}(0 :_{T^\vee} m^n) = \varinjlim \operatorname{Hom}_R(R/m^n, T^\vee) = \varinjlim \underbrace{\operatorname{Hom}_R(R/m^n, \operatorname{Hom}_R(T,E))}_{=T_n}.$$

Note $\lambda(T_n) < \infty$ as $\lambda(T/m^nT)\infty$. Thus we have $\operatorname{Tor}_i^R(E, T^\vee)^\vee = \operatorname{Tor}_i(E, \varinjlim T_n)^\vee = (\varinjlim \operatorname{Tor}_i^R(E, T_n))^\vee = \varprojlim \operatorname{Tor}_i^R(E, T_n)^\vee = \varprojlim \operatorname{Tor}_i^R(T_n, E)^\vee \cong \varprojlim \operatorname{Ext}_R^i(T_n, R)$ by Lemma 72. As $\lambda(T_n) < \infty$, we see $\operatorname{Ext}_R^i(T_n, R) = 0$ for all $i < r$.

Thus $F^{\cdot}$ is a finite free resolution of $M = \operatorname{Ext}_R^r(E,T)$ and also M is finitely generated.

Claim 2. $\operatorname{Ext}_R^i(M,R) \cong H_m^{r-i}(T)^\vee$

Proof. From (2) and Lemma 73, we see $F^{\cdot} = \operatorname{Hom}_R(E, H_m^0(I^{\cdot})) = \operatorname{Hom}_R(H^0(I^{\cdot})^\vee, R)$ as $H_m^0(I^{\cdot})$ is a complex of Artinian modules. Thus

$$\operatorname{Hom}_R(F^{\cdot}, R) \cong \operatorname{Hom}_R(\operatorname{Hom}_R(H_m^0(I^{\cdot})^\vee, R), R) \cong [H_m^0(I^{\cdot})^\vee]^{**} \cong H_m^0(I^{\cdot})^\vee$$

as $H_m^0(I^{\cdot})^\vee$ is a bounded complex of finitely generated free $R-$modules. Since $F^{\cdot}$ is a free resolution of M,

$$\operatorname{Ext}_R^i(M,R) \cong H_{r-i}(\operatorname{Hom}_R(F^{\cdot}, R)) \cong H_{r-i}(H_m^0(I^{\cdot})^\vee) \cong H^{r-i}(H_m^0(I^{\cdot}))^\vee \cong H_m^{r-i}(T)^\vee.$$

Thus if $i > r - \operatorname{depth} T$, then $r - i < \operatorname{depth} T$ and so $\operatorname{Ext}_R^i(M,R) \cong H_m^{r-i}(T)^\vee = 0$. Since $H_m^{\operatorname{depth} T}(T) \neq 0$, we see $\operatorname{Ext}_0^{r-\operatorname{depth} T}(M,R) \neq 0$. Thus $\operatorname{pd}_R M = r - \operatorname{depth} T$ since $\operatorname{pd}_R M < \infty$.

By Lemma 71 and Claim 2, we see $\operatorname{Supp} M = \cup_i \operatorname{Supp} \operatorname{Ext}_R^i(M,R) = \cup_i \operatorname{Supp} H_m^{r-i}(T)^\vee$.

Claim 3. $\operatorname{Supp}_R T = \cup_i \operatorname{Supp}_R H_m^i(T)^\vee$.

Proof. Let $R = S/J$ where (S,n) is Gorenstein of dimension d. By local duality, $\underline{H_m^i(T)^\vee} \cong H_n^i(T)^\vee \cong \operatorname{Ext}_S^{d-i}(T,S)$. Let $\overline{(-)}$ denote mod J. So $\operatorname{Supp}_R T = \overline{\operatorname{Supp}_S T} = \cup_i \overline{\operatorname{Supp}_S \operatorname{Ext}_S^i(T,S)} = \cup_i \overline{\operatorname{Supp} H_n^i(T)^\vee} = \cup_i \operatorname{Supp}_R H_m^i(T)^\vee$.

Thus $\operatorname{Supp} M = \operatorname{Supp} T$. ∎

Recall for (R,m) local $T \neq 0$ a finite generated $R-$module with $\operatorname{id}_R T < \infty$ that $\operatorname{id}_R T = \operatorname{depth} R = \sup\{i | \operatorname{Ext}_R^i(R/m, T) \neq 0\}$.

Lemma 75. *Let (R,m) be Noetherian and T a finitely generated $R-$modules such that $\operatorname{id}_R T < \infty$. Then for all $p \in \operatorname{Supp}_R T$ with $\dim R/p = 1$, $\operatorname{depth} R \geq \operatorname{depth} R_p + 1$.*

Proof. Choose $x \in m \backslash p$. Consider the short exact sequence $(*) 0 \to R/p \xrightarrow{x} R/p \to R/(p,x) \to 0$. Note $\lambda(R/(p,x)) < \infty$ and $\dim R/p = 1$. Since $p \in \operatorname{Supp}_R T$ we see $T_p \neq 0$ and $s := \operatorname{id}_{R_p} T_p = \operatorname{depth} R_p = \sup\{i | \operatorname{Ext}_R^i(R/p, T)_p \neq 0\}$ by above. In particular, $\operatorname{Ext}_R^s(R/p,T) \neq 0$. Applying $\operatorname{Hom}_R(-,T)$ to $(*)$ gives

$$\underbrace{\operatorname{Ext}_R^s(R/p,T)}_{=0} \xrightarrow{x} \operatorname{Ext}_R^s(R/p,T) \to \operatorname{Ext}_R^{s+1}(R/(p,x),T) \to \cdots$$

By Nakayama's Lemma, multiplication by x is not surjective. Thus $\operatorname{Ext}_R^{s+1}(R/(p,x),T) \neq 0$. By induction on the length, we have $\operatorname{Ext}_R^{s+1}(R/m,T) \neq 0$. Thus $\operatorname{depth} R = \operatorname{id}_R T \geq s+1 = \operatorname{depth} R_p + 1$. ∎

Theorem 76. *Let (R,m) be Noetherian and $T \neq 0$ a finitely generated $R-$module such $\operatorname{id}_R T < \infty$. Then R is Cohen Macaulay.*

Proof. WLOG, assume R is complete. We will induct on the dimension of T. If $\dim T = 0$, then there exists a finitely generated $R-$module M such that $\operatorname{pd}_R M < \infty$ and $\dim M = 0$ by the Proposition. Let $F.$ be a minimal free resolution of M. Say $F. : 0 \to F_s \to \cdots \to F_0 \to 0$. Then $\lambda(H_i(F.)) = \lambda(M)$ if $i = 0$ and is zero for $i > 0$. As $M \neq 0$, $F.$ is not exact. So $\operatorname{pd}_R M = s \geq \dim R$. By the Auslander Buchsbaum formula, $\operatorname{pd}_R M \leq \operatorname{depth} R$ and thus $\dim R = \operatorname{depth} R$.

Assume $\dim T > 0$. Let M be a finitely generated $R-$module such that $\operatorname{pd} M < \infty$ and $\operatorname{Supp} M = \operatorname{Supp} T$ and let $q \in \operatorname{Spec} R$ such that $\dim R/q = \dim R$.

Case 1. $\dim M/qM > 0$. Choose $p \in \operatorname{Supp}_R M/qM$ such that $\dim R/p = 1$. Clearly $p \supseteq q$. As R is catenary, $\dim R/p + \operatorname{ht}(p/q) = \dim R/q = \dim R$. Then $\dim R_p \geq \operatorname{ht}(p/q) = \dim R - 1$. Since $\dim R_p \neq \dim R$, we have $\dim R_p = \dim R - 1$. Since $p \in \operatorname{Supp} M = \operatorname{Supp} T$, the lemma implies $\operatorname{depth} R \geq \operatorname{depth} R_p + 1$. Since $\dim T_p < \dim T$, induction gives that R_p is Cohen Macaulay. So $\dim R = \dim R_p + 1 = \operatorname{depth} R_p + 1 \leq \operatorname{depth} R$. Thus R is Cohen Macaulay.

Case 2. $\dim M/qM = 0$. Let $F.$ be a minimal free resolution of M. Say $F. : 0 \to F_s \to \cdots \to F_0 \to 0$. Apply $- \otimes R/q : 0 \to F_s/qF_s \to \cdots \to F_0/qF_0 \to 0$. Then $H_i(F. \otimes R/q) = \operatorname{Tor}_i^R(M, R/q)$, which implies $\operatorname{Supp} H_i(F. \otimes R/q) = m$ and thus $\lambda(H_i(F. \otimes R/q)) < \infty$. Also, $H_0(F. \otimes R/q) = M/qM \neq 0$. So $F. \otimes R/q$ is not exact. By New Intersection Theorem, $s = \operatorname{pd}_R M \geq \dim R/q = \dim R$. Then $\operatorname{depth} R \geq \operatorname{pd}_R M \geq \dim R$. Thus R is Cohen Macaulay. $\square$

Conjecture (Direct Summand Conjecture). *Let R be a regular local ring and suppose $R \subseteq S$ where S is a finite $R-$algebra. Then R is a direct summand of S as an $R-$module (that is, the inclusion map $i : R \hookrightarrow S$ splits).*

We say "DSC holds for R" when the direct summand conjecture is true for all $S \supseteq R$.

Proposition 77. *Let R be a regular local ring containing a field of characteristic 0. Then DSC holds for R.*

Proof. Let $R \subseteq S$ where S is a finite $R-$algebra.

Claim. It suffices to prove DSC holds in the case S is a domain.

Proof. Since S is integral over R, $\dim R = \dim S = d$. Let $p \in \operatorname{Spec} S$ such that $\dim R/p = d$. Then S/p is integral and finite over $R/p \cap R$. So $d = \dim S/p = \dim R/p \cap R = \dim R$. As R is a regular local ring, R is a domain. So $p \cap R = (0)$. We have $i' : R \xrightarrow{i} S \to S/p$ and so we may consider $R \subseteq S/p$ for S/p finite $R-$algebra. If DSC held for domains, then there would exist $\ell' : S/p \to R$ such that $\ell' i' = 1_R$. Let $\ell : S \xrightarrow{\pi} S/p \xrightarrow{\ell'} R$. Then $\ell i = 1_R$.

Now assume S is a domain. Let $K = Q(R), L = Q(S)$. Then $\ell := [L : K] < \infty$ and L/K is separable ($\operatorname{char} K = 0$). Let $\sigma_1, ..., \sigma_\ell$ be the distinct $k-$embeddings (that is, field maps that fix K) of L into $\overline{K}$. Then $\operatorname{Tr}_K^L : L \to K$ is given by $\operatorname{Tr}_K^L(\alpha) = \sum_{i=1}^{\ell} \sigma_i(\alpha)$.

Claim. For all $s \in S$, $\operatorname{Tr}_K^L(S) \subseteq R$.

Proof. As S/R is integral, each $s \in S$ satisfies an equation $s^n + r_{n-1}\sigma_i(s)^{n-1} + ... + r_0 = 0$. So each $\sigma_i(s)$ is integral over R and thus $\operatorname{Tr}_K^L(s)$ is integral over R. But regular local rings are integrally closed in their quotient fields. Since $\operatorname{Tr}_K^L(S) \subset K$, we get $\operatorname{Tr}_K^L(S) \subseteq R$.

For $r \in R$, $\operatorname{Tr}_K^L(r) = \ell r$. Let $\rho = \frac{1}{\ell} \operatorname{Tr}_K^L : S \to R$. Then ρ is $R-$linear and $\rho(r) = r$ for all $r \in R$, that is, $\rho i = 1_R$ for $i : R \hookrightarrow S$. $\square$

Remark. Let $R \hookrightarrow S$ be rings and suppose this inclusion splits (as $R-$modules). Then for all ideals $I \subset R$, we have $IS \cap R = I$.

Proof. Let $\rho : S \to R$ be the splitting map. Let $a \in IS \cap R$. Then $a = i_1 s_1 + ... + i_k s_k$ for $i_j \in I, s_j \in S$. Then $a = \rho(a) = i_1\rho(s_1) + ... + i_k\rho(s_k) \in I$ as $\rho(s_i) \in R$. Thus $IS \cap R \subset I$. As the other containment is clear, done. $\square$

Corollary 78. *The monomial conjecture holds for all local rings containing a field of characteristic zero.*

Proof. Let (S, m) be a local ring of dimension d and $x_1, ..., x_d$ a system of parameters. We want to show $(x_1, ..., x_d^n) \notin (x_1^{n+1}, ..., x_d^{n+1})$ for all $n \geq 1$. It suffices to show $(x_1, ..., x_d)^n \notin (x_1^{n+1}, ..., x_d^{n+1})\hat{S}$ where $\hat{S}$ is the $m-$adic completion. Since $x_1, ..., x_d$ form a system of parameters in $hatS$, we may assume S is complete. Let K be a coefficient field for S ($\operatorname{char} K = 0$). By a corollary to the Cohen Structure theorem, S over $K[\![x_1, ..., x_d]\!] =: R$ is a finite extension as $x_1, ..., x_d$ is a system of parameters. Since $\dim R = \dim S = d$, we see R is a regular local ring. Suppose $(x_1, ..., x_d)^n \in (x_1^{n+1}, ..., x_d^{n+1})S$ for some n. As DSC holds for R, the remark implies $(x_1, ..., x_d)^n \in (x_1^{n+1}, ..., x_d^{n+1})R$, a contradiction to #1 on Homework Set 1 as R is Cohen Macaulay. $\square$

Theorem 79. *Let (R, m) be a regular local ring of dimension d and let $x_1, ..., x_d$ form a regular system of parameters (so $m = (x_1, ..., x_d)$). Suppose $R \subseteq S$ for a finite $R-$algebra S. Then R is a direct summand of S if and only if $(x_1, ..., x_d)^n \notin (x_1^{n+1}, ..., x_d^{n+1})S$ for all n.*

Corollary 80. *Suppose the monomial conjecture holds for all local rings. Then DSC holds for all regular local rings.*

Proof. Let (R, m) be a regular local ring and $m = (x_1, ..., x_d)$ where $d = \dim R$. Let $R \subseteq S$ for a finite $R-$algebra S. By the theorem, $R \hookrightarrow S$ splits if and only if $(x_1, ..., x_d)^n \notin (x_1^{n+1}, ..., x_d^{n+1})S$ for all n. As before, we may assume S is a domain. Let $p \in \operatorname{Spec} S$ such that p is minimal over $(x_1, ..., x_d)S$. Since R is integrally closed in $Q(R)$, the going down theorem holds for S/R (see Mastumura, Theorem 9.4). Thus $\operatorname{ht} p = \operatorname{ht} p \cap R = \operatorname{ht} m = d$. So $x_1, ..., x_d$ is a system of parameters for S_p. By the monomial conjecture, $(x_1, ..., x_d)^n \notin (x_1^{n+1}, ..., x_d^{n+1})S_p$ for all n and so $(x_1, ..., x_d)^n \notin (x_1^{n+1}, ..., x_d^{n+1})S$. Thus DSC holds for all regular local rings. $\square$

Remark. The proof also shows that DSC holds for regular local rings of characteristic $p > 0$.

Corollary 81. *The existence of big Cohen Macaulay modules implies the Direct Summand Conjecture (via the Monomial Conjecture).*

Exercise. Let R be a ring and M an $R-$module. Let $a_1 + M \supseteq a_2 + M_2 \supseteq \cdots$ be a descending chain of cosets in M (so $a_i \in M$ and M_i are submodules of M). Then (1) $M_1 \supseteq M_2 \supseteq \cdots$ and (2) the chain of cosets stabilizes if and only if the chain of submodules stabilizes. In particular, if M is Artinian, every descending chain of cosets stabilizes. Thus $\cap a_i + M_i$ is a coset and thus is nonempty.

Proof of Theorem 79. The forward direction has already been shown as if $R \hookrightarrow S$ splits, then $IS \cap R = I$ for all ideals I of R. In particular, take $I = (x_1^{n+1}, ..., x_d^{n+1})$.

So we just need to show the backward implication. Recall that R a direct summand of S means there exists $\rho : S \to R$ such that $\rho \circ i = 1_R$ where $i : R \hookrightarrow S$ is the inclusio map. This happens if and only if the natural map $\operatorname{Hom}_R(S, R) \to \operatorname{Hom}_R(R, R) \cong R$ defined by $\rho \mapsto \rho \circ i = \rho|_R$ is surjective. That is, $\operatorname{Hom}_R(S, R) \otimes \hat{R} \to \operatorname{Hom}_R(R, R) \otimes \hat{R}$ is surjective. Since $\hat{S}$ is finitely presented and $R \to \hat{R}$ is faithfully flat, this is if and only if $\operatorname{Hom}_{\hat{R}}(\hat{S}, \hat{R}) \to \operatorname{Hom}_{\hat{R}}(\hat{R}, \hat{R})$ is surjective, that is, $\hat{i} : \hat{R} \to \hat{S}$ splits. Also, $m = (x_1, ..., x_d)R$ implies $\hat{m} = (x_1, ..., x_d)\hat{R}$. If $(x_1, ..., x_d)^n \in (x_1^{n+1}, ..., x_d^{n+1})\hat{S}$ for some n, then $(x_1, ..., x_d)^n \in (x_1^{n+1}, ..., x_d^{n+1})\hat{S} \cap S = (x_1^{n+1}, ..., x_d^{n+1})S$ (since $S \to \hat{S}$ is faithfully flat and $I\hat{S} \cap S = I$), a contradiction. Thus we may assume R is complete.

For $t \geq 1$, define $(\underline{x}^t) = (x_1^t, ..., x_d^t)$, $R_t = R/(\underline{x}^t)$, $S_t = S/(\underline{x}^t)S$, and $i_t : R_t \to S_t$.

Claim. For every t, i_t is a split injection.

Proof. As $(\underline{x}^t)$ is regular, R_t is a zero-dimensional Gorenstein ring. Thus $\operatorname{soc} R_t = (\overline{0} :_{R_t} m)$ is a one-dimensional $R/m-$vector space. Note $\overline{(x_1 \cdots x_d)^{t-1}} \neq 0$ in R_t as the monomial conjecture holds for regular local rings. However $m\overline{(x_1 \cdots x_d)^{t-1}} = \overline{0}$ in R_t. Thus $\operatorname{soc} R_t = R/m \cdot \overline{(x_1 \cdots x_d)^{t-1}}$. Suppose $\ker i_t \neq 0$. Then $\ker i_t$ contains something in the socle. Since $\dim_k \operatorname{soc} R_t = 1$, $\ker i_t \supseteq \operatorname{soc} R_t$. Thus $i_t((x_1 \cdots x_d)^{t-1}) = 0$ in $S/(\underline{x}^t)S$ and so $(x_1, ..., x_d)^{t-1} \in (\underline{x}^t)S$, a contradiction. Thus i_t is injective. As R_t is zero-dimensional Gorenstein, we see R is injective. Now $R_t \xrightarrow{i_t} S_t$ is exact with R_t injective and so it splits.

Note $\{(\underline{x}^t)\}_{t\geq 1}$ is cofinal with $\{m^n\}_{n\geq 1}$. Thus $R = \hat{R} = \varprojlim R_t$. Let $\delta_{t-1} : R_t \to R_{t-1}$ defined by $r + (\underline{x}^t) \mapsto r + (\underline{x}^{t-1})$ be the natural surjection. Then $\varprojlim R_t = \{(r_t) \in \prod R_t | \delta_{t-1}(r_t) = r_{t-1} \text{ for all } t\}$. Now $R \xrightarrow{\cong} R_t$ where $r \mapsto (r_t)$ for $r_t = r + (\underline{x}^t)$. As $\operatorname{Hom}_R(S, -)$ commutes with inverse limits, we have

$$\operatorname{Hom}_R(S,R) \cong \operatorname{Hom}_R(S, \varprojlim R_t) = \varprojlim \operatorname{Hom}_R(S, R_t) = \varprojlim \operatorname{Hom}_{R_t}(S_t, R_t).$$

As any map $\rho : S_t \to R_t$ induces a map $\overline{\rho} : S_{t-1} \to R_{t-1}$, let $\pi_{t-1} = \operatorname{Hom}_{R_t}(S_t, R_t) \to \operatorname{Hom}_{R_{t-1}}(S_{t-1}, R_{t-1})$. Now the inverse limit

$$\varprojlim \operatorname{Hom}_{R_t}(S_t, R_t) = \{(\psi_t) | \psi_t : S_t \to R_t, \pi_{t-1}\psi_t = \psi_{t-1} \text{ for all } t\}.$$

Finally we have an isomorphism $\operatorname{Hom}_R(S,R) \to \varprojlim \operatorname{Hom}_{R_t}(S_t, R_t)$ defined by $\psi \mapsto (\psi_t)$ where the inverse map is defined for $(\psi_t) \in \varprojlim(S_t, R_t)$ by $(\psi_t)(r) = (\psi_t(r_t)) \in \varprojlim R_t = R$ for $r = (r_t) \in \varprojlim R_t$. To show $i : R \to S$ splits, it suffices to find R_t-homomorphisms $\psi_t : S_t \to R_T$ such that $\pi_{t-1}\psi_t = \psi_{t-1}$ and $\psi_t i_t = 1_{R_t}$ for all t. If so, by (1) we have $(psi_t) \in \varprojlim \operatorname{Hom}_{R_t}(S_t, R_t) = \operatorname{Hom}_R(S,R)$. Let $\psi = (\psi_t) : S \to R$. Then $\psi i = 1_R$ as for $r = (r_t) \in R$, we see $\psi_t(r) = \psi(r_t) = (\psi_t(r_t)) = (\psi_t i_t(r_t)) = (r_t) = 1$ by (2) of the exercise. Thus we just need to find ψ_t.

We know $i_t : R_t \to S_t$ splits for all t. Let ρ_t be a splitting. We know $i_t^* : \operatorname{Hom}_{R_t}(S_t, R_t) \to \operatorname{Hom}_{R_t}(R_t, R_t)$ defined by $\psi_t \mapsto \psi_t i_t$ is surjective. Now $\psi_t : S_t \to R_t$ is a splitting map for i_t if and only if $\psi_t \in (i_t^*)^{-1}(1_{R_t}) = \rho_t + \ker i_t^* =: D_t$, a coset in $\operatorname{Hom}_{R_t}(S_t, R_t)$. Certainly, $\pi_{t-1}(D_t) \subseteq D_{t-1}$ for all t. For each t, let $E_t := \cap_{i\geq 0}\pi_t\pi_{t+1}\cdots\pi_{t+i}(D_{t+i+1})$. Note that we have a descending chain of cosets $D_t \supseteq \pi_t(D_{t+1}) \supseteq \pi_t\pi_{t+1}(D_{t+2}) \supseteq \cdots$ in $\operatorname{Hom}_{R_t}(S_t, R_t)$ which is Artinian (as $\dim R_t = 0$ and S_t is a finitely generated R_t-module). Therefore, by the exercise, $E_t \neq \emptyset$. Say $(*) E_t = \pi_t\pi_{t+1}\cdots\pi_{t+i}(D_{t+i+1})$ for some i (which depends on t. Note that $\pi_{t-1}(E_t) = E_{t-1}$ for all t (this follows from the definition and from $(*)$). Choose $\psi_1 \in E_1$. So $\psi_1 : S_1 \to R_1$ splits i_1. By the chain, there exists $\psi_2 \in E_2$ such that $\pi_2(\psi_2) = \psi_1$. Continue to get the desired maps. □

Conjecture (**Small Cohen Macaulay Module Conjecture** (SCM))**.** *If (R,m) is a complete local ring, then R has a maximal Cohen Macaulay module.*

Note that SCM implies the Big Cohen Macaulay module conjecture. The conjecture is easy to see in several cases:

- $\dim R = 0$ (then R is Cohen Macaulay)
- $\dim R = 1$ (then R/p is a maximal Cohen Macaulay module for $p \in \operatorname{Min} R$)
- $\dim R = 2$ Nagata gave an example of a two-dimensional local domain R which is not universally catenary (and thus does not have a maximal Cohen Macaulay module by [BH] 2.1.14).

Proposition 82. *Let (R,m) be complete of dimension two. Then R has a maximal Cohen Macaulay module.*

Proof. By passing to R/p for $p \in \operatorname{Spec} R$ with $\dim R/p = 2$, we may assume R is a complete domain. Let R' be the integral closure of R in $Q(R)$. Then R' is a finitely generated $R-$module and hence a Noetherian local domain. So R' is normal, which implies it is S_2 and R_1. As $\dim R' = 2$, it is Cohen Macaulay and thus $\operatorname{depth}_R R' = 2$. Thus R' is a maximal Cohen Macaulay algebra. □

Proposition 83. *Let R be a ring, I a finitely generated ideal. Suppose $n \in \mathbb{N}$ and $H_I^i(R) = 0$ for all $i > n$. Then*

(1) $H_I^i(M) = 0$ *for all* $i > n$ *and for all* $R-$*modules* M
(2) $H_I^n(M) \cong H_I^n(R) \otimes M$ *for all* $R-$*modules* M

Proof. (1) Let $c = \inf\{\ell | H_I^i(M) = 0 \text{ for all } i > \ell, R\text{-modules } M\}$. Since I is finitely generated, $H_I^i(M)$ for all $i > \mu(I)$. So $c \leq \mu(I)$. It suffices to show $c \leq n$. If not, $c > m$ and there exists an $R-$module M such that $H_I^c(M) \neq 0$. Consider the short exact sequence $0 \to L \to F \to M \to 0$ where F is free. Since $F = \oplus R$, we see $H_I^i(F) = 0$ for all $i > n$. Then the long exact sequence on homology gives $H_I^{c+1}(L) \neq 0$, a contradiction to the definition of C. Thus $c \leq n$.

(2) For any i, $H_I^i(-)$ is covariant, additive and multiplicative. So if $F \xrightarrow{(a_{ij})} G$ is a map of free $R-$modules, then

$$\begin{tikzcd}
H_I^i(F) \arrow[r, "H_I^i((a_{ij}))"] \arrow[d, "\cong"] & H_I^i(G) \arrow[d, "\cong"] \\
H_I^i(R) \otimes_R \arrow[r, "1\otimes(a_{ij})"] & H_I^i(R) \otimes RG
\end{tikzcd}$$

As $H_I^i(M) = 0$ for all $i > n$ for all $R-$modules M, $H_I^n(-)$ is right exact. Let M be an $R-$modules and $F \to G \to M \to 0$ exact with F, G free. Then we have

$$\begin{tikzcd}
H_I^n(F) \arrow[r] \arrow[d, "\cong"] & H_I^n(G) \arrow[r] \arrow[d, "\cong"] & H_I^n(M) \arrow[r] \arrow[d, dashed] & 0 \\
H_I^n(R) \otimes F & H_I^n(R) \otimes G \arrow[r] & H_I^n(R) \otimes M \to 0 &
\end{tikzcd}$$

$\square$

Corollary 84. *Let R be a ring, I an ideal, and $n \in \mathbb{Z}$. TFAE*

(1) $H_I^i(R) = 0$ *for all* $i > n$
(2) $H_I^i(R/p) = 0$ *for all* $p \in \operatorname{Spec} R$ *and* $i > n$
(3) $H_I^i(R/p) = 0$ *for all* $p \in \operatorname{Min} R$ *and* $i > n$

Proof. Note that (1) $\Rightarrow$ (2) and (3) $\Rightarrow$ (2) follow from the proposition. For (2) $\Rightarrow$ (1), take a prime filtration of R with factors isomorphic to R/p and take local cohomology. $\square$

Definition. *Let R be a ring and I an ideal containing a non-zerodivisor in R (that is, I is a* ***regular ideal****). Let Q be the total quotient ring of R. We define the* ***ideal transform of*** *I to be $D(I) := \cup_{n\geq 1}(R :_Q I^n) = \{q \in Q | qI^n \subseteq R \text{ for some } n\}$.*

Note.

(1) $D(I)$ is a subring of Q containing R.
(2) $D(I)$ is almost never Noetherian, even if R is.
(3) If $I = (a_1, ..., a_k)$, then $D(I) = \cup_n (R :_Q (a_1^n, ..., a_k^n))$

Proposition 85. *Let R be a Noetherian ring and I a finitely generated regular ideal. Let $S = D(I)$. Then*

(1) $H_I^0(S) = H_I^1(S) = 0$
(2) $H_I^i(S) \cong H_I^i(R)$ *for all* $i \geq 2$.

Proof. (1) Let $x \in I$ be a non-zerodivisor on R. Then x is a non-zerodivisor on Q and hence on S. So $H_I^0(S) = 0$. Consider the short exact sequence $0 \to S \xrightarrow{x} S \to S/xS \to 0$ and apply local cohomology: $0 \to H_I^0(S/xS) \to H_I^1(S) \xrightarrow{x} H_I^1(S)$. If $H_I^0(S/xS) = 0$, then multiplication by x is injective. But $x \in I$ and every element in $H_I^1(S)$ is annihilated by a power of I and thus by a power of x. Then $H_I^1(S) = 0$. So it is enough to show $H_I^0(S/xS) = 0$. Let $y \in S$ such that $I^n y \subseteq xS$ for some n (so $\overline{y} \in H_I^0(S/xS)$). If we show $y \in xS$ then $H_I^0(S/xS) = 0$. So write $I = (a_1, ..., a_k)$ where a_i is a non-zerodivisor for all i. Then for all i there exists s_i such that $a_i^n y = xs_i$. In Q, we see $y = \frac{xs_i}{a_i^n} = x\left(\frac{s_i}{a_i^n}\right)$ for all i. It is enough to show $\frac{s_i}{a_i^n} \in S$. Now $\frac{xs_i}{a_i^n} = \frac{xs_j}{a_j^n}$ for all i, j. As x is a non-zerodivisor, this says $u := \frac{s_i}{a_i^n} = \frac{s_j}{a_j^n}$ for all i, j. As $s_i \in S$, there exists ℓ such that $I^\ell s_i \subseteq R$ for all i. Then $a_i^n I^\ell u = I^\ell s_i \subseteq R$ for all i, which implies $(a_1^n, ..., a_k^n) I^\ell u \subseteq R$ and thus $(a_1^{n+\ell}, ..., a_k^{n+\ell}) u \subseteq R$. Thus $u \in S$ and $y = xu \in xS$. Therefore $H_I^0(S/xS) = 0$.

(2) Consider the short exact sequence $0 \to R \to S \to S/R \to 0$. For all $\overline{s} \in S/R$ there exists n such that $I^n \overline{s} = \overline{0}$. Thus $H_I^0(S/R) = S/R$ and $H_I^i(S/R) = 0$ for all $i \geq 1$. Applying $H_I^i(-)$ to our short exact sequence gives

$$0 \to S/R \to H_I^1(R) \to H_I^1(S) \to 0 \to H_I^2(R) \to H_I^2(S) \to 0 \to \cdots$$

$\square$

Theorem 86 (Hochster '83; Katz, Huneke, Marley '06)**.** *Let R be Noetherian and $I = (x, y)$. Then TFAE*

(1) $H^2_I(R) = 0$
(2) $(xy)^n \in (x^{n+1}, y^{n+1})$ *for some n.*

Proof. First suppose $H^2_I(R) = 0$. As $H^2_I(R) \cong R_{xy}/R_x + R_y$, we see $\frac{1}{xy} = 0$ in $R_{xy}/R_x + R_y$. Thus $\frac{1}{xy} = \frac{r}{x^n} + \frac{s}{y^n}$ for some n and for $r, s \in R$. Then there exists ℓ such that $(xy)^{n+\ell-1} = rx^\ell y^{n+\ell} + sx^{n+\ell}y^\ell \in (x^{n+\ell}, y^{n+\ell})$.

Now suppose (2) holds. Since $H^i_I(R) = 0$ for all $i > 2$ (as I is two-generated) and the corollary implies $H^2_I(R) = 0$ if and only if $H^2_I(R/p) = 0$ for all $p \in \operatorname{Spec} R$, we may assume R is a domain and I is a regular ideal. Let $S = D(I)$. Then its enough to show $H^2_I(S) = 0$ as $H^2_I(S) \cong H^2_I(R)$. To do so, it is enough to show $IS = S = (1)S$. We have $(xy)^n = rx^{n+1} + xy^{n+1}$ for some $r, s \in R$. So $1 = \frac{r}{y^n}x + \frac{s}{x^n}y$ in Q. To get $1 \in IS$, we need only show $\frac{r}{y^n}, \frac{s}{x^n} \in S$. Now $\frac{r}{y^n}x^{n+1} = x^n - sy \in R$. So $\frac{r}{y^n} \in (R :_Q (x^{n+1}, y^{n+1})) \subseteq S$. Similarly for $\frac{s}{x^n}$. $\square$

Corollary 87. *The monomial conjecture and the direct summand conjecture hold in dimension 2.*

Proof. For (R, m) local and $I = (x, y)$ a system of parameters, we see $H^2_I(R) \neq 0$. $\square$

Definition. *Let (C, d) and (D, d') be chain complexes. A **homotopy** s from C to D is a set of maps $s_n : C_n \to D_{n+1}$ for each n. Two chain maps $f, g : C \to D$ are called **homotopic** if there exists a homotopy s from C to D such that for all n $f_n - g_n = s_{n-1}d_n + d'_{n+1}s_n$.*

Theorem 88 (**Comparison Theorem**)**.** *Let C, D be chain complexes such that $C_i = D_i = 0$ for all $i < 0$. Let $\epsilon : C_0 \to X$ and $\delta : D_0 \to Y$ be augmentation maps. Suppose*

(1) *C_i is projective for all i*
(2) *$C_0 \xrightarrow{\epsilon} X \to 0$ is a complex*
(3) *$D_0 \xrightarrow{\delta} Y \to 0$ is exact*

Then given any map $f_{-1} : X \to Y$ there exists a chain map $f : C \to D$ lifting f_{-1}. Furthermore, any two liftings are homotopic.

Definition (Hochster '83)**.** *A local ring $R(, m)$ of dimension d **satisfies CE** if for every projective resolution $P.$ of $k = R/m$ and for every system of paramters $x_1, ..., x_n$ and every chain map $f : K.(\underline{x}) \to P.$ lifting the canonical surjection $f_{-1} : R/(\underline{x}) \to R/m$, one has $f_d \neq 0$.*

Conjecture. *Every local ring satisfies CE.*

Theorem 89. *If a local ring (R, m) has a big Cohen Macaulay module, then R satisfies CE (e.g., every local ring containing a field satisfies CE).*

Proof. Fix a system of parameters $\underline{x} = x_1, ..., x_d$ for R and let $P.$ be a projective resolution of k. Let $f : K.(\underline{x}) \to P.$ be a lifting of $f_{-1} : R/(\underline{x}) \to R/m$ and suppose $f_d = 0$. Let M be an $R-$module which is a big Cohen Macaulay module for $\underline{x}$. Note $K.(\underline{x}, M) = K.(\underline{x}) \otimes M$ is acyclic and $M \neq (\underline{x})M$. As $(\underline{x})$ is $m-$primatry, there exists $y \in M \setminus (\underline{x})M$ such that $my \subseteq (\underline{x})M$ (to find y, take $z \in M \setminus (\underline{x})M$ so $\lambda(R/z) < \infty$ and choose $y \in \operatorname{soc} z$). Let $g_{-1} : R/m \to M/(\underline{x})M$ be defined by $\overline{1} \mapsto \overline{y}$. By the comparison theorem, there exists $g_0 : P_0 \to K_0(\underline{x}, M)$ which lifts g_{-1}. Then $\alpha = g \circ f : K.(\underline{x}) \to K.(\underline{x}, M)$ lifts $\alpha_{-1} = g_{-1} \circ f_{-1} : R/(\underline{x}) \to M/(\underline{x})M$. Since $f_d = 0$, we see $\alpha_d = 0$. Let $\rho : R \to M$ be defined by $1 \mapsto y$. Consider the composition of chain maps $\alpha' : K.(\underline{x}) \cong K.(\underline{x}) \otimes_R R \xrightarrow{1 \otimes \rho} K.(\underline{x}) \otimes_R M \xrightarrow{\cong} K.(\underline{x}, M)$. Note $(\alpha')_0 : R \to M$ is defined by $1 \mapsto y$. So α' also lifts $g_{-1}f_{-1}$ and so α and α' are homotopic, say with homotopy s. Then since $\alpha_d = 0$ we see $\alpha'_d = \alpha'_d - \alpha d = \partial s_d + s_{d-1}\partial = s_{d-1}\partial$. Thus $y = \alpha'_d(1) = s_{d-1}\partial(1) = s_{d-1}(\sum_1^d x_i e_i) = \sum x_i s_{d-1}(e_i) \in (\underline{x})M$, a contradiction as y was chosen to be not in $(\underline{x})M$. Thus $f_d \neq 0$ and CE holds. $\square$

Proposition 90. *A local ring (R,m,k) of dimension d satisfies CE if and only if for every system of parameters $\underline{x} = x_1, ..., x_d$ and every complex $F. : \cdots \to F_{i+1} \to F_i \to \cdots \to F_0 \to 0$ where F_i is finitely generated free and for every chain map $f : K.(\underline{x}) \to F.$ such that the induced map $\overline{f_0^*} : H_0(K.(\underline{x})) \otimes R/m \to H_0(F.) \otimes R/m$ is not zero, we have $f_d \neq 0$.*

Proof. For the backward direction, it suffices to show that if CE holds where $P.$ is a minimal resolution of k, then CE holds for every resolution of k. Suppose CE holds for every chain map $f : K.(\underline{x}) \to F.$ which lifts $R/(\underline{x}) \to R/m$ and where $F.$ is a minimal resolution of k. Let $g : K.(\underline{x}) \to P.$ be a lifting where $P.$ is an arbitrary projective resolution of k. By the comparision theorem, there exists a chain map $h : P. \to F.$ which lifts the identity map on R/m. Then $hg : K.(\underline{x}) \to F.$ lifts $R/(\underline{x}) \to R/m$. As CE holds for $F.$, $h_d g_d \neq 0$. Therefore $g_d \neq 0$ and CE holds for $P.$.

For the forward direction, let $f : K.(\underline{x}) \to F.$ be as in the hypothesis. Let $y = f_0(1) \in F_0$. Then the image $\overline{y}$ of y in $H_0(F.) \otimes R/m$ is non-zero. Choose a projective $\pi : H_0(F.) \otimes R/m \to R/m$ such that $\pi(\overline{y}) = \overline{1} \neq 0$. Let $\epsilon : F_0 \to F_0/\operatorname{im}\phi_1 \to F_0/\operatorname{im}\phi_1 \otimes R/m \xrightarrow{\pi} R/m$. Then $\epsilon(y) = \overline{1}$. Let $P.$ be a projective resolution of k. By the comparison theorem, there exists a chain map $g : F. \to P.$ lifting 1_k. Then $gf : K.(\underline{x}) \to P.$ lifts the canonical surjection $R/(\underline{x}) \to R/m$. Since CE holds, $g_d f_d \neq 0$ and so $f_d \neq 0$. ☐

Corollary 91. *Let $\phi : (R,m) \to (S,n)$ be a local homomorphism such that $\dim R = \dim S$ and $\sqrt{mS} = n$. If CE holds for S, it holds for R.*

Proof. If there is a counter example to CE for R, then apply $- \otimes S$ to find a counterexample for S using the propostion. ☐

Corollary 92. *To show CE holds for R, it suffices to show CE holds for $\hat{R}/p$ for $p \in \operatorname{Min}\hat{R}$ with $\dim \hat{R}/p = \dim R$.*

Conjecture (**Improved New Interesection Conjecture** (INIC))**.** *Let (R,m) be local of dimension d. Suppose $F. : 0 \to F_s \xrightarrow{\phi_s} \cdots \to F_1 \to F_0 \to 0$ is a complex of finitely generated free $R-$modules such that $\lambda(H_i(F.)) < \infty$ for all $i > 0$, $H_0(F.) \neq 0$, and $H_0(F.)$ has a minimal generator z such that $\lambda(Rz) < \infty$. Then $s \geq d$.*

Theorem 93. *Suppose (R,m) satisfies CE. Then INIC holds for R.*

Proof. Let $F.$ be as in INIC. Let $M = \operatorname{coker}\phi_1 = H_0(F.)$ and $d = \dim R$. Let $z \in M \backslash mM$ such that $\lambda(Rz)M\infty$. Then there exists t_0 such that $(x_1, ..., x_d)^{t_0} \subseteq \operatorname{Ann}_R Rz$ $(*)$. Let $Z_i = \ker\phi_i$ and $B_i = \operatorname{im}\phi_{i+1}$ for $i \geq 1$. As $\lambda(Z_i/B_i)M\infty$, there exists c such that $(x_1, ..., x_d)^c Z_i \subseteq B_i$. By the Artin Rees Lemma, for all $i \geq 1$ there exists t_i such that $(x_1, ..., x_d)^{t_i} F_i \cap Z_i \subseteq (x_1, ..., x_d)^c Z_i \subseteq B_i$. Let $t = \max\{t_0, ..., t_s\}$. We will construct a chain map $f. : K.(\underline{x}^t) \to F.$. Let $y \in F_0$ such that $\overline{y} = z$ in $H_0(F.)$. Define $f_0 : R = K.(\underline{x}^t)_0 \to F_0$ by $1 \mapsto y$.

$$\begin{tikzcd}
F_1 \arrow[r, "\phi_1"] & F_0 \arrow[r] & 0 \\
K.(\underline{x}^t)_1 = R^d \arrow[u] \arrow[r, "\partial_1"] & K.(\underline{x}^t)_0 = R \to 0 &
\end{tikzcd}$$

Let $\{e_1, ..., e_d\}$ be a basis for $K.(\underline{x}^t)_1$. Then $f_0\partial_1(e_i) = f_0(x_1^t) = x_1^t y$. Note $(*)$ implies $(x_1, ..., x_d)^t y \subseteq B_0 = \operatorname{im}\phi_0$. Thus $f_0\partial(e_i) \in B_0$. So there exists $u_i \in F_1$ such that $\phi_1(u_i) = x_i^t y$. Define $f_1 : K.(\underline{x}^t) \to F_1$ by $f_1(e_i) = u_i$. Then the diagram commutes. Now suppose we have defined $f_0, ..., f_i$.

$$\begin{tikzcd}
\cdot \arrow[r] & F_{i+1} \arrow[r, "\phi_{i+1}"] & f_i \arrow[r, "\phi_i"] & F_{i-1} \arrow[r] & \cdots \\
\cdots \arrow[r] & K.(\underline{x}^t)_{i+1} \arrow[r, "\partial_{i+1}"] & K.(\underline{x}^t)_i \arrow[u, "f_i"] \arrow[r, "\partial_i"] & K.(\underline{x}^t)_{i-1} \arrow[u, "f_{i+1}"] \arrow[r] & \cdot
\end{tikzcd}$$

Let $\{w_1, ..., w_\ell\}$ be a basis for $K(\underline{x}^t)_{i+1}$. Then $\partial_{i+1}(w_j) \subseteq (x_1^t, ..., x_d^t)K.(\underline{x}^t)_i$, which together with diagram chasing implies that $f_i\partial_{i+1}(w_j) \subseteq (x_1^t, ..., x_d^t)F_i \cap Z_i \subseteq B_i$. Thus there exists $v_j \in F_{i+1}$ such that $\phi_{i+1}(v_j) = f_i\partial_{i+1}(w_j)$ for all j. Define f_{i+1} in the obvious way. This gives the desired chain map.

Note $f_0^* : H_0(K.(\underline{x}^t)) = R/(\underline{x}^t) \to H_0(F.) = M$ sends $\overline{1} \mapsto \overline{y} = z$. So $\overline{f_0^*} : H_0(K.(\underline{x}^t)) \otimes k \to M \otimes k$ sends $\overline{1} \otimes 1 \mapsto z \otimes 1 \neq 0$ as $z \in M \setminus mM$. By the above proposition since CE holds for R, we have $f_d \neq 0$ and thus $s \geq d$. $\square$

Lemma 94. *Let (R, m) be local. Then R satisfies CE if and only if for all system of parameters $\underline{x}$ and for all projective resolutions $P.$ of k and for all chain maps $f. : K.(\underline{x}) \to P.$ lifting $R/(\underline{x}) \to k$, we have $f_d(1) \notin (x_1, ..., x_d)P_d$.*

Proof. The backward direction is clear. For the forward direction, suppose $f_d(1) = x_1u_1 + ... + x_du_d, u_i \in P_d$. Consider

$$\begin{tikzcd} & P_d \arrow[r, "\phi_d"] & P_{d-1} \arrow[r, "\phi_{d-1}"] & \cdots \\ 0 \arrow[r] & K.(\underline{x})_d \arrow[u, "f_d"] \arrow[r, "\partial_d"] & K.(\underline{x})_{d-1} \arrow[u, "f_{d-1}"] \arrow[r, "\partial_{d-1}"] & \cdots \end{tikzcd}$$

where $\partial_d(1) = (x_1, ..., x_d)$. Define $s : K.(\underline{x})_{d-1} \to P_d$ by $e_i \mapsto u_i$. Define a map $\tilde{f} : K.(\underline{x}) \to F.$ by $\tilde{f}_d = f_d - s\partial_d = 0$ and $\tilde{f}_{d-1} = f_{d-1} - \phi_d s$ and $\tilde{f}_i = f_i$ for all $i < d-1$. Note

$$\tilde{f}\partial_d = f_{d-1}\partial_d - \phi_d s \partial_d = f_{d-1}\partial_d - \phi_d f_d = 0$$

as the square commutes. Since $\tilde{f}_d = 0$, the last square commutes. Now $\phi_{d-1}\tilde{f}_{d-1} = \phi_{d-1}(f_{d-1} - \phi_d s) = \phi_{d-1}f_{d-1} = f_{d-2}\partial_{d-1}$. Thus the second to last square commutes. So $\tilde{f} : K.(\underline{x}) \to F.$ is a chain map and lifts $R/(\underline{x}) \to k$. Thus $\tilde{f}_0^*$ is the canonical surjection. But $\tilde{f}_d = 0$, a contradiction as R satifies CE. $\square$

Recall if $\underline{x} = x_1, ..., x_n \in R$ and $t, s \geq 1$ then there exists a chain map $\mu(t, s) : K.(\underline{x}^{t+s}) \to K.(\underline{x}^t)$ such that $\mu(t, s)_0 = 1_R$ and $\mu(t, s)_n$ is given by multiplication by $(x_1 ... x_n)^s$.

Theorem 95. *Suppose CE holds for (R, m). Then the monomial conjecture holds for (R, m).*

Proof. Let $\underline{x} = x_1, ..., x_d$ be a system of parameters for R and $t \geq 1$. We need to show $(x_1 \cdots x_d)^t \notin (x_1^{t+1}, ..., x_d^{t+1})$. Let P be a projective resolution of k and $f : K.(\underline{x}) \to P.$ which lifts $R/(\underline{X}) \to R/m$. Let $\mu := \mu(1, t) : K.(\underline{x}^{t+1}) \to K.(\underline{x})$. Since $\mu_0 = 1_R$ we see $f\mu : K.(\underline{x}^{t+1}) \to P.$ lifts $R/(\underline{x}) \to R/m$. On the other hand

$$(f\mu)_d(1) = f_d\mu_d(1) = f_d((x_1 \cdots x_d)^t) = (x_1 \cdots x_d)^t f_d(1) \notin (x_1^{t+1}, ..., x_d^{t+1})P_d$$

by the Lemma. Thus $(x_1 \cdots x_d)^t \notin (x_1^{t+1}, ..., x_d^{t+1})$. $\square$

Remark. Hochster ('83) proves that if the Direct Summand Conjecture holds for all regular local rings (A, n) of characteristic $p > 0$ then CE holds for all local rings (R, m) of characteristic p. Since they are both true in the characteristic zero case, we have

$$\text{DSC} \Rightarrow \text{CE} \Rightarrow \text{MC} \Rightarrow \text{DSC}.$$

Another Construction of the Koszul Complex. Let R be a ring, $F = R^n$. Let $f : F \to R$ defined by $e_i \mapsto x_i$ be $R-$linear. For $i \geq 1$ define $\tilde{\partial}(f)_i : F^t \to \bigwedge^{i-1} F$ by $(u_1, ..., u_i) \mapsto \sum_{j=1}^{i} (-1)^{j+1} f(u_j) u_1 \wedge \cdots \wedge \hat{u_j} \wedge \cdots \wedge u_i$. One can check this map is multilinear and alternating. Thus we get an induced map $\partial(f)_i : \bigwedge^i F \to \bigwedge^{i-1} F$. The sequence $0 \to \bigwedge^n F \to \cdots \to \bigwedge^0 F \to 0$ is $K.(\underline{x}; R)$. Now suppose $\phi : G \to F$ is an $R-$linear map where $G = R^m$. By the Functorial property of $\bigwedge^i$, we get induced maps $\phi_i = \bigwedge^i(\phi) : \bigwedge^i(G) \to \bigwedge^i(F)$ defined by $u_1 \wedge \cdots \wedge u_i \mapsto \phi(u_1) \wedge \cdots \wedge \phi(u_i)$. Let $g = f\phi : G \to R$ and $y_i = g(e_i)$ where $\{e_1, ..., e_m\}$ is a basis for S.

Claim. $\phi : K.(\underline{y}) \to K.(\underline{x})$ is a chain map.

Proof. We need only show $\partial(f)_i \phi_i = \phi_{i-1}\partial(g)_i$ to show that the following diagram commutes:

$$\begin{tikzcd} \bigwedge^i G \arrow[r, "\partial(g)_i"] \arrow[d, "\phi_i"] & \bigwedge^i G \arrow[d, "\phi_{i-1}"] \\ \bigwedge^i F \arrow[r, "\partial(f)_i"] & \bigwedge^{i-1} F \end{tikzcd}$$

Chasing elements, we see

$$\begin{aligned}
\phi_{i-1}\partial(g)_i(u_1\wedge\cdots\wedge u_i) &= \phi_{i-1}\left(\sum(-1)^{j+1}g(u_j)u_1\wedge\cdots\wedge\hat{u_j}\wedge\cdots\wedge u_i\right)\\
&= \sum(-1)^{j+1}f\phi(u_j)\phi(u_1)\wedge\cdots\wedge\widehat{\phi(u_j)}\wedge\cdots\wedge\phi(u_i)\\
&= \partial(f)_i(\phi(u_1)\wedge\cdots\wedge\phi(u_i))\\
&= \partial(f)_i\phi_i(u_1\wedge\cdots\wedge u_i).
\end{aligned}$$

Note that for a domain R, if we have $h : L \xrightarrow{f} M \xrightarrow{g} N$ then $\operatorname{rank} h = \operatorname{rank}\operatorname{im} h \le \min\{\operatorname{rank} f, \operatorname{rank} g\} \le \min\{\operatorname{rank} M, \operatorname{rank} L\}(*)$. Let $f : F \to G$ be a map of finitely generated free $R-$modules. Then $\operatorname{rank} f = \min\{r \ge 0 | I_{r+1}(f) = 0\}$. Now f induces maps $\bigwedge^i f : \bigwedge^i F \to \bigwedge^i G$ defined by $u_1\wedge\cdots\wedge u_i \to f(u_1)\wedge\cdots\wedge f(u_i)$. If $\{e_1, ..., e_m\}$ is a basis for F then $\{e_{j_1}\wedge\cdots\wedge e_{j_i} | j_1 < \cdots < j_i\}$ is a basis for $\bigwedge^i F$. So $\bigwedge^i F$ is free of rank $\binom{m}{i}$. Fix a basis $\{u_1, ..., u_n\}$ for G and let $A = (a_{ij})$ be the matrix representation for f with respect to the chosen basis.

Exercise. The matrix representing $\bigwedge^i f$ with respect to the bases above is given by the $i\times i$ minors of A. Specifically, the coefficient of $u_{k_1}\wedge\cdots\wedge u_{k_i}$ in the expression of $(\bigwedge^i f)(e_{j_1}\wedge\cdots\wedge e_{j_i})$ is the $i\times i$ minor determined by rows $k_1, ..., k_i$ and columns $j_1, ..., j_i$. Thus $I_1\left(\bigwedge^i f\right) = I_i(f)$. Thus $I_i(F) = 0$ if and only if $\bigwedge^i f = 0$ and so $\operatorname{rank} f = \min\{r \ge 0 | \bigwedge^{r+1} f = 0\}$.

Definition. *Let R be a ring and M and $R-$module with $x \in M$. The order ideal of x is* $\mathcal{O}_R(x) = \{\phi(x) | \phi \in M^* = \operatorname{Hom}_R(M, R)\}$.

Remarks.

(1) $\mathcal{O}_R(x)$ is an ideal.

(2) If M is finitely presented, then $\operatorname{Hom}_R(M, R)_S \cong \operatorname{Hom}_{R_S}(M_S, R_S)$ for all multiplicatively closed sets S. Thus $\mathcal{O}_R(x)_S \cong \mathcal{O}_{R_S}\left(\frac{x}{1}\right)$.

(3) More generally, let $f : R \to S$ be a ring homomorphism. Then there exists a natural map $\operatorname{Hom}_R(M, R)\otimes_R S \to \operatorname{Hom}_S(M\otimes_R S, S)$. Thus for $x \in M$, $\mathcal{O}_R(x)S \subseteq \mathcal{O}_S(x\otimes 1)$ (note that when S is flat, this become an equality). In particular, if $I \subset R$, then $\mathcal{O}_R(x)\cdot R/I \subseteq \mathcal{O}_{R/I}(\overline{x})$.

(4) Suppose $M = A\oplus B$ and $x = (a, b)$. Then $M^* = A^*\oplus B^*$ and so $\mathcal{O}_R(x) = \mathcal{O}_R(a) + \mathcal{O}_R(b)$.

(5) If $x \in IM$ for an ideal I, then $\mathcal{O}_R(x) \subseteq I$. In particular, if $x \in mM$ for some maximal ideal of R then $\mathcal{O}_R(x)$ is a proper ideal.

(6) Let $M = R^n$ and $x = (x_1, ..., x_n$. Then $\mathcal{O}_R(x) = (x_1, ..., x_n)$.

(7) If R is Noetherian, $M = R^n$ and $x \in mM$ for some maximal ideal then $\operatorname{ht}\mathcal{O}_R(x) \le n = \operatorname{rank} M$ (by Krulls Principal Ideal Theorem and Remark 6).

Definition. *For a Noetherian ring R and finitely generated $R-$module M, define* $\operatorname{bigrank} M = \max\{\mu_{R_p}(M_p) | p \in \operatorname{Min} R\}$.

If R is a domain, then $\operatorname{bigrank} M = \operatorname{rank} M$.

Theorem 96 (Eisenbud-Evans, '76)**.** *Let (R, m) be a local ring, M a finitely generated $R-$module and $x \in mM$. Suppose R satisfies CE, then* $\operatorname{ht}\mathcal{O}_R(X) = \operatorname{bigrank} M$.

Proof. Let $p \in \operatorname{Min} R$ such that $\operatorname{ht}\mathcal{O}_R(x) = \operatorname{ht}\mathcal{O}_R(x)R/p$. By Remark 3, $\mathcal{O}_R(x)R/p \subseteq \mathcal{O}_{R/p}(\overline{x})$. Thus $\operatorname{ht}\mathcal{O}_R(x) \le \operatorname{ht}\mathcal{O}_{R/p}(\overline{x})$. Note also

$$\operatorname{bigrank} M \ge \mu_{R_p}(M_p) = \mu_{R_p}(M_p/pM_p) = \mu_{R_p/pR_p}(M_p/pM_p) = \operatorname{rank}_{R/p}(M/p) = \operatorname{bigrank}_{R/p}(M/pM).$$

Thus we may assume R is a domain. Let $h = \operatorname{ht}\mathcal{O}_R(X)$. Then $\operatorname{codim}\mathcal{O}_R(x) \ge h$. So there exists a system of parameters $x_1, ..., x_d$ for R such that $x_1, ..., x_n \in \mathcal{O}_R(x)$. Let $M' = M\oplus R^{d-h}$ and $x' = x + (x_{h+1}, ..., x_d)$. Then $\mathcal{O}_R(x') = \mathcal{O}_R(x) + \mathcal{O}_R(x_{h+1}, ..., x_d) = \mathcal{O}_R(x) + (x_{h+1}, ..., x_d)$, which is $m-$primary. Clearly $\operatorname{rank} M' = \operatorname{rank} M + d - h$.

So if we prove $\operatorname{ht} \mathcal{O}_R(x') \leq \operatorname{rank} M'$ then $d \leq \operatorname{rank} M + d - h$, that is, $h \leq \operatorname{rank} M$. So without loss of generality, suppose $\operatorname{ht} \mathcal{O}_R(x) = d$. We need to show $\operatorname{rank} M \geq d$. Let $x_1, ..., x_d$ be a system of parameters such that $x_1, ..., x_d \subseteq \mathcal{O}_R(x)$. Then there exists $\alpha_i \in M^*$ such that $\alpha_i(x) = x_i$ for all i. Define $\alpha : M \to R^d =: F$ by $u \mapsto (\alpha_1(u), ..., \alpha_d(u))$. Let $m = (y_1, ..., y_n)$. Since $x \in mM$ there exists $u_1, ..., u_n \in M$ such that $x = \sum y_i u_i$. Define $\pi : R^n =: G \to M$ by $e_i \mapsto u_i$. Note $\pi(y_1, ..., y_n) = x$. Let $f = \alpha\pi : G \to F$ and note that the following squares commute.

$$\begin{tikzcd} R \arrow[r, "(x_1,\ldots,x_d)"] & F & & F^* \arrow[r, "(x_1,\ldots,x_d)"] \arrow[d, "f^*"] & R \arrow[d, "1_R"] \\ R \arrow[u, "1_R"] \arrow[r, "(y_1,\ldots,y_n)"'] & G \arrow[u, "f"'] & \xrightarrow{\text{Apply } (-)^*} & g^* \arrow[r, "(y_1,\ldots,y_n)"'] & R \end{tikzcd}$$

Note $\operatorname{rank} f^* = \operatorname{rank} f \leq \operatorname{rank} M$ by $(*)$. By the remarks on the Koszul complex, f induces a chain map $\tilde{f} : K.(\underline{x}) \to K.(\underline{y})$ given by $\bigwedge^i(f) : \bigwedge^i F^* \to \bigwedge^i G^*$. Let $P.$ be a projective resolution of $k = R/m$ and $\phi : K.(\underline{y}) \to P.$ lift the identity map $R/(\underline{y}) \to k$. Then $\phi\tilde{f} : K.(\underline{x}) \to P.$ is a chain map lifting $R/(\underline{x} \to k$. By CE, $\phi_d \bigwedge^d f = (\phi\tilde{f})_d \neq 0$. Thus $\bigwedge^d f \neq 0$ and so $\operatorname{rank} f \geq d$. Thus $\operatorname{rank} M \geq \operatorname{rank} f \geq d$. $\square$

Appendix A. Homework Problems

A.1. **Homework Set 1.**

(1) (Justin) Prove the monomial conjecture for Cohen-Macaulay local rings.

Proof. Since R is Cohen Macaulay, $x_1, ..., x_d$ is $R-$regular. Let $I = (x_1, ..., x_d)$. Recall $\text{gr}_I(R) = \oplus_{i=0}^{\infty} I^i/I^{i+1}$ is $\mathbb{Z}-$graded and in degree zero is R/I.

Claim. The map $\phi : (R/I)[X_1, ..., X_d] \to \text{gr}_I(R)$ defined by $X_i \mapsto \overline{x_i} \in I/I^2$ is an isomorphism.

Proof. This map is homogenous, thus we only need to check the isomorphism for homogenous elements. For surjectivity, note that a homogenous element of $\text{gr}_I(R)$ lives in I^s/I^{s+1} for some s. The element is an $(R/I)-$linear combination of $s-$fold products of $x_1, ..., x_d$. The same combination in $(R/I)[X_1, ..., X_d]$ works.

For injectivity, supposed $F \in (R/I)[X_1, ..., X_d]$ is homogenous of degree s. Say $F = \sum_{n \in \mathbb{Z}^d} a_{n_i} X^{n_i}$. Under ϕ, F maps to $I^s/^{s+1}$. If $\phi(F) = 0$, then $\phi(F) \in I^{s+1}$ when we life to R. Thus $\sum a_{n_i} x^{n_i} \in I^{s+1}$. By the following theorem, $a_i \in I$ when we lift to R. Thus $a_i = 0 \in R/I$.

Theorem (Rees). If $I = (x_1, ..., x_d)$ is an $R-$regular sequence and $F \in R[X_1, ..., X_d]$ is homogenous of degree s with $F(x_1, ..., x_d) \in I^{s+1}$ then F has coefficients in I.

Suppose $x_1^t \cdots x_d^t \in (x_1^{t+1}, ..., x_d^{t+1})$. Look at $\text{gr}_I(R)/(x_1^{t+1}, ..., x_d^{t+1})\,\text{gr}_I(R) \cong (R/I)[X_1, ..., X_d]/(X_1^{t+1}, ..., X_d^{t+1})$. By the isomorphism, we know $X_1^t \cdots X_d^t \notin (X_1^{t+1}, ..., X_d^{t+1})$ as they are variables. Thus $X_1^t \cdots X_d^t \in 0$ on the right hand side, yet $x_1^t \cdots x_d^t = 0$ on the left hand side, a contradiction. □

(2) (Hamid) Let (R, m) be a quasi-localr ing. Let M be an $R-$module and suppose $F.$ and $G.$ are two free resolutions of M consisting of finitely generated free $R-$modules. Suppose $F.$ is minimal. Prove that there exists an exact complex $H.$ of finitely generated free $R-$modules such that $G. \cong F. \oplus H.$ as complexes.

Proof. Consider the following diagram

$$\begin{tikzcd} F. \arrow[r] \arrow[d, "\alpha"] & M \arrow[d, "="] \\ G. \arrow[r] \arrow[d, "\beta"] & M \arrow[d, "="] \\ F. \arrow[r] & M \end{tikzcd}$$

Composing gives us $\beta \circ \alpha$, which is null homotopic to 1 by the comparison theorem. Thus $\beta \circ \alpha$ is an isomorphism which implies α splits. □

(3) (Laura) Let M be a finitely presented $R-$module and $F_1 \xrightarrow{\phi} F_0 \to M \to 0$ and $G_1 \xrightarrow{\psi} G_0 \to M \to 0$ two presentations of M. Let $r = \text{rank}\, F_0$ and $s = \text{rank}\, G_0$. Prove that $I_{r-i}(\phi) = I_{s-i}(\psi)$. [Note: This result allows us to call $I_{r-i}(\phi)$ the i^{th} Fitting Ideal of M.]

Proof. We may assume R is local as $I_{r-i}(\phi) = I_{s-i}(\psi)$ if and only if they are locally equal. Furthermore, since we can compare both of these presentations to a fixed minimal one, we may assume $F_1 \to F_0 \to M \to 0$ is minimal. Extend these presentations to free resolutions of M of finitely generated free $R-$modules $F.$ and $G.$. By exercise 2, there exists an exact complex $H.$ of finitely generated free $R-$modules such that $G. \cong F. \oplus H.$. Say $H. : \cdots \to H_1 \xrightarrow{\tau} H_0 \to 0$. Note $G_0 \cong F_0 \oplus H_0$ which implies $p : \text{rank}\, H_0 = s - r$. As τ is surjective, choose bases for H_1 and H_0 such that τ is represented by the matrix $(I_p 0)$. Note that

$F_. \oplus H_. : \cdots \to F_1 \oplus H_1 \xrightarrow{A} F_0 \oplus H_0 \to 0$ where $A = \begin{pmatrix} \phi & 0 \\ 0 & \tau \end{pmatrix}$ is a free resolution of M as $H_.$ is exact. Since $G_. \cong F_. \oplus H_.$, we see $\psi = \begin{pmatrix} \phi & 0 & 0 \\ 0 & I_p & 0 \end{pmatrix}$. $\square$

(4) (Xuan) Let R be a ring and M a finitely presented $R-$module. Let $F_1 \xrightarrow{\phi} F_0 \to M \to 0$ be a presentation for M. Prove that M is projective if and only if $I_j(\phi)$ is generated by an idempotent for each j.

Proof. For the forward direction, suppose M is projective. Then M_m is a free R_m-module, hence projective with rank. By Corollary 17, there exists r such that $I_r(\phi)_m = R_m$ and $I_{r+1}(\phi)_m = 0$. Note $\cdots \subseteq \underbrace{I_{r+2}(\phi)_m}_{=0} \subseteq \underbrace{I_{r+1}(\phi)_m}_{=0} = \underbrace{I_r(\phi)_m}_{=0} \subseteq \cdots$ and so $I_j(\phi)_m = R_m$ or 0 for all j and all m.

Claim. J a finitely generated ideal, $J_m = 0$ or R_m. Then J is generated by an idempotent.

Proof. If $J_m = 0$ then $(0 : J) \not\subseteq m$. IF $J_m = R_m$ then $J \not\subseteq m$. So $(0 : J) + J \not\subseteq m$ which implies $(0 : J) = R$. Choose $i \in (0 : J)$ and $j \in J$ so that $i + j = 1$ and $ij = 0$. Then $j(1 - j) = 0$ which implies $j^2 = j$. For all $x \in J$ we see $(1 - j)x = 0$ which implies $x = jx$ and so $J = (j)$.

For the backward direction, let $p \in \operatorname{Spec} R$. We claim idempotents in R_p are either 0 or 1. Note $I_r(\phi)_p = I_r(\phi)R_p$ where the left side is generated by idempotents by assumption. Now M_p is projective R_p-module and is thus locally free. So M is finitely generated. Take r equal to the maximum such that $I_r(\phi)_p = R_p$ and $I_{r+1}(\phi)_p = 0$. $\square$

(5) (Brian) Let A be an $n \times m$ matrix with entries from a commutative ring R. Prove that the system $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution if and only if there exists a nonzero element $z \in R$ such that $zI_m(A) = 0$. (This is a theorem due to McCoy.)

Proof. Recall from last time for F, G finitely generated free and $\phi : F \to G$ with $r = \operatorname{rank} F$ that $F \otimes_R M \to G \otimes_R M$ is injective if and only if $\operatorname{grade}(I_r(\phi), M) \geq 1$. Now its enough to show $R^m \xrightarrow{A} R^n$ is injective if and only if there does not exists $z \in R \setminus \{0\}$ such that $zI_m(A) = 0$. Replace M with R in our recall statement and note $\operatorname{grade}(I_m(A), R) \geq 1$ if and only if $\operatorname{Ann}_R(I_m(A)) = 0$. $\square$

(6) (Katie) Let R be a ring, $F_1 \xrightarrow{\phi} F_0 \to M \to 0$ a presentation, and $r = \operatorname{rank} F_0$. Prove that $I_r(\phi) \subseteq \operatorname{Ann}_R M$.

Proof. Let $m = \operatorname{rank} F_.$. If $r > m$, then $I_r(\psi) = 0$. So assume $r \leq m$. Let ϕ' be an $r \times r$ submatrix of ϕ and $M' = \operatorname{coker} \phi'$. Then $\operatorname{Ann}(M') \subseteq \operatorname{Ann}(M)$. So it is enough to show $I_r(\phi) \subseteq \operatorname{Ann}(M)$ and thus we may assume $m = r$. Note $I_r(\psi) = \det \psi$ and so its enough to show $\det(\psi) \subseteq \operatorname{Ann}(M)$. Now $\det \psi(I_r) = \psi \cdot adj(\psi)$.

$$\begin{tikzcd}
R^r \arrow[r, "\phi"] \arrow[d] & R^r \arrow[r] \arrow[d, "(\det\phi)I_r"] \arrow[dl, "adj(\phi)"] & M \arrow[r, "*"] \arrow[d, "\det|_M"] & 0 \\
R^r \arrow[r] & R^r \arrow[r] & M \arrow[r, "**"] & 0
\end{tikzcd}$$

By diagram chasing, we see $*$ is zero and thus $**$ is zero. Thus $\det \phi \in \operatorname{Ann} M$. $\square$

(7) (Lori) Let R be a ring, $F_1 \xrightarrow{\phi} F_0 \to M \to 0$ a presentation, and $r = \operatorname{rank} F_0$. Prove that $I_r(\phi) \subseteq \operatorname{Ann}_R M$.

Proof. Let A be a matrix representation for ϕ. Now $F_0 \cong R^r$ and say $F_1 \cong R^m$. Let A_j be a $j \times j$ submatrix of A, $d = \det A_j$, and $x \in \operatorname{Ann} M$. We want to show $dx \in I_{j+1}(\phi)$. Let B be the $r \times (m+r)$ matrix (AxI_r). This gives $R^{m+r} \xrightarrow{B} R^r \to \operatorname{coker} B \to 0$ where $\operatorname{coker} B = R^r / \operatorname{im} B = R^r / \operatorname{im} A + xR^r \cong \operatorname{coker} A$ and $xR^r \in \operatorname{im} A$. By exercise 3, $I_{r-i}(\phi) = I_{r-i}(B)$ for all i. Consider $I_{j+1}(B)$ and take the $(j+1) \times (j+1)$ submatrix $\begin{pmatrix} A_j & 0 \\ * & x \end{pmatrix}$

which has determinant equal to $(\det A_j)x = dx$. Thus $dx \in I_{j+1}(\phi)$. Thus $I_r(\phi) \supseteq \operatorname{Ann} M I_{r-1}(\phi) \supseteq \cdots \supseteq (\operatorname{Ann} M)^r I_0(\phi) = (\operatorname{Ann} M)^r$. ☐

(8) (Silvia) Let R be a semi-local ring and P a finitely generated projective $R-$module. Prove that P is free if and only if for all maximal ideals m and n of R, $\operatorname{rank}_{R_m} P_m = \operatorname{rank}_{R_n} P_n$.

Proof. Let R be a semi-local ring and let $\mathfrak{m}_1, \ldots, \mathfrak{m}_t$ be the maximal ideals of R. Let P be a finitely generated projective R-module.

($\Rightarrow$) Suppose P is free, i.e. $P \cong R^n$ for some $n > 0$. Since localization commutes with direct sums, we have: $P_{\mathfrak{m}_i} \cong (R^n)_{\mathfrak{m}_i} \cong R^n_{\mathfrak{m}_i}$ which implies $\operatorname{rank}_{R_{\mathfrak{m}_i}} P_{\mathfrak{m}_i} = n$ for all $i = 1, \ldots, t$.

($\Leftarrow$) Conversely, suppose $\operatorname{rank}_{R_{\mathfrak{m}_i}} P_{\mathfrak{m}_i} = n$ for all $i = 1, \ldots, t$. As P is a finitely generated projective R-module, P is locally free, i.e. for each i we have $P_{\mathfrak{m}_i} \cong R^n_{\mathfrak{m}_i}$ for all i. Use Lemma 12.2 in [BH] with $N = P$ to find $u \in P$ such that $\frac{u}{1} \notin \mathfrak{m}_i P_{\mathfrak{m}_i}$ for all i (note that the condition $P_{\mathfrak{m}_i} \nsubseteq \mathfrak{m}_i P_{\mathfrak{m}_i}$ is satisfied). Thus $\frac{u}{1}$ is in a minimal generating set for $P_{\mathfrak{m}_i}$ (by NAK) for all i. Use induction on n.

(i) Assume $n = 1$.
Then $P_{\mathfrak{m}_i}$ is free of rank 1 for all i, and $\left\{\frac{u}{1}\right\}$ is a basis for $P_{\mathfrak{m}_i}$ and we can write $P_{\mathfrak{m}_i} = R_{\mathfrak{m}_i}\frac{u}{1}$ for all i. Let $\phi : R \to P$ be the R-module homomorphism given by $\phi(1) = u$, and consider the following exact sequence $0 \to K \to R \xrightarrow{\phi} P \to C \to 0$. Localize at a maximal ideal $\mathfrak{m}_i$ to get:

$$0 \longrightarrow K_{\mathfrak{m}_i} \longrightarrow R_{\mathfrak{m}_i} \xrightarrow{\phi_{\mathfrak{m}_i}} P_{\mathfrak{m}_i} \longrightarrow C_{\mathfrak{m}_i} \longrightarrow 0,$$

where $\phi_{\mathfrak{m}_i}$ is an isomorphism. Thus $K_{\mathfrak{m}_i} = 0 = C_{\mathfrak{m}_i}$ for all maximal ideals $\mathfrak{m}_i$ of R and hence $K = 0 = C$. Thus $P \cong R$, i.e. P is free of rank 1.

(ii) Assume the claim holds for $n-1$, i.e. if M is a finitely generated projective R-module such that $\operatorname{rank}_{R_\mathfrak{m}} P_\mathfrak{m} = n-1$ for all maximal ideals $\mathfrak{m}$ of R, then M is free.
Since $P_{\mathfrak{m}_i} \cong R^n_{\mathfrak{m}_i}$ and $\left\{\frac{u}{1}\right\}$ is part of a basis, $(P/Ru)_{\mathfrak{m}_i}$ is free of rank $n-1$ for all i. As P is finitely generated, so is P/Ru. Since P/Ru is finitely generated and locally free (**and** R Noetherian), P/Ru is projective. By induction, P/Ru is free of rank $n-1$. Moreover $Ru \cong R$ is free of rank 1. Now consider the exact sequence $0 \to Ru \to P \to P/Ru \to 0$. As P/Ru is projective, the sequence splits. Thus $P \cong Ru \oplus P/Ru$ is free of rank n. ☐

(9) (Nick) Let R be a ring, M an $R-$module, and x an indeterminate over R. Suppose $f(x) \in R[x]$ is a zero-divisor on $M[x] = M \otimes_R R[x]$. Prove there exists a nonzero element $u \in M$ such that $f(x)u = 0$ (that is, all the coefficients of f annihilate u).

Proof. There exists $g(x) \in M(x)$ such that $f(x)g(x) = 0$. Then $\sum_{i=0}^k f_i x^i \cdot \sum_{j=0}^\ell g_j x^j = 0$. We will induct on $k+\ell$. If $k+\ell = 0$, then take $u = g_0$. Suppose $k+\ell > 0$. Then $f_k g_k = 0$. Set $\overline{g}(x) = f_k g(x)$. If $\overline{g}(x) \neq 0$, then $\deg \overline{g}(x) < \deg g(x)$. Now $f(x)\overline{g}(x) = f(x) \cdot g(x) f_k$. Thus there exists $0 \neq \overline{u} \in Rf_k g_0 + \cdots + Rf_k g_{\ell-1} \subseteq Rg_0 + \cdots + Rg_\ell$ with $\overline{u} f(x) = 0$. If $\overline{g}(x) = 0$, then $f_k \cdot g_i = 0$ for all i. Let $\overline{f}(x) = f(x) - f_k x^k$. If $\overline{f}(x) = 0$ then $f(x) = f_k x^k$ and $u = g_\ell$. If not, $\deg(\overline{f}(x)) < \deg f(x)$ and $\overline{f}(x)g(x) = f(x) - f_k x^k g(x) = f(x)g(x) - f_k g(x) x^k = 0$. So there exists $\overline{u} \in Rg_0 + \ldots + Rg_\ell$ such that $\overline{u}\overline{f}(x) = 0$. Then $f_k \overline{u} = 0$ since $f_k g_i = 0$ for all i. Set $u = \overline{u}$. ☐

A.2. **Homework Set 2.**

(1) (Katie) Let R be a Noetherian local ring of dimension d and I an ideal of R. Prove that $\operatorname{codim} I \geq i$ if and only if I contains $x_1, \ldots, x_i$ which form part of a system of parameters for R.

(2) (Justin) Let (R, m) be a Noetherian local ring and $F.$ a complex $0 \to F_s \to F_{s-1} \to \cdots \to F_0 \to 0$ consisting of finitely generated free modules in each degree and such that all the homology has finite length. Let M be an $R-$module and $J_i := \operatorname{Ann}_R H^i_m(M)$ for $i \geq 0$. Prove that for each $i \geq 0$, $J_0 J_1 \cdots J_{s-i}$ annihilates $H_i(F. \otimes_R M)$.

Proof. Take $K^\cdot$ to be the Čech complex on a system of parameters $\underline{x}$ and reindex $F_\cdot$ by $F^\cdot$ where $F^i = F_{s-i}$. Define a double complex by $C^{\cdot\cdot} := K^\cdot \otimes F^\cdot \otimes M$. We will now examine spectral sequences.

First filter by the columns:

$$\begin{aligned} {}^I E_1^{pq} &= H^q(K^p \otimes F^\cdot \otimes M) \\ &= K^p \otimes H^q(F^\cdot \otimes M) \text{ as } K^p \text{ is flat for all } p \\ &= \begin{cases} H^q(F^\cdot \otimes M) & \text{if } p = 0 \\ K^p \otimes H^q(F^\cdot \otimes M) & \text{if } p > 0 \end{cases} \end{aligned}$$

We want to show $K^p \otimes H^q(F^\cdot \otimes M) = 0$ for $p > 0$. Then the sequence ${}^I E_1^{pq}$ will collapse and we will get $H^{p+q}(F^\cdot \otimes M) = {}^I E_\infty^{pq} = H^{p+q}(\mathrm{Tot}(C))$.

Claim. $H^q(F^\cdot \otimes M)$ is $m-$torsion (and so $K^p \otimes H^q(F^\cdot \otimes M) = 0$ for $p > 0$).

Proof. Let $G^\cdot \xrightarrow{\sim} M$ be a projective resolution of M, indexed cohomologically. Consider the double complex $F^\cdot \otimes G^\cdot$. Filtering by columns gives us

$${}^I E_1^{p,q} = H^q(F^p \otimes G^\cdot) = F^p \otimes H^q(G^\cdot) = \begin{cases} F^p \otimes M, & \text{if } q = 0 \\ 0, & \text{if } q > 0 \end{cases}$$

as $G^\cdot$ is a projective resolution of M. Thus ${}^{II} E_2^{p,q} = H^{p+q}(F^\cdot \otimes M)$. Filtering by rows gives us

$${}^{II} E_1^{p,q} = H^q(F^\cdot \otimes G^p) \cong H^q(F^\cdot) \otimes G^p \cong (H^q(F^\cdot))^{m_p}$$

where $m_p = \operatorname{rank} G^p$. So ${}^{II} E_1^{p,q}$ is $m-$torsion as each $H^q(F^\cdot)$ has finite length. Thus ${}^{II} E_\infty^{p,q}$ are $m-$torsion.

Now, consider the filtration of $H^n = H^n(F^\cdot \otimes M)$:

$$0 = F^{n+1} H^n \subset F^n H^n \subset \cdots F^0 H^n = H^n$$

with $F^i H^n / F^{i+1} H^n = {}^{II} E_\infty^{i,n-i}$. For each i, there exists ℓ_i such that $m^{\ell_i}\ {}^{II} E_\infty^{i,n-i} = 0$. So $m^{\ell_0} \cdots m^{\ell_n} H^n = 0$, that is, H^n is $m-$torsion. □

Now filter by rows:

$$\begin{aligned} {}^{II} E_1^{p,q} &= H^q(K^\cdot \otimes (F^p \otimes M)) \\ &= H^q(K^\cdot \otimes M^{n_p}), \text{ where } n_p = \operatorname{rank} F^p \\ &= H^q_m(M^{n_p}) \end{aligned}$$

By hypothesis, $J_q \cdot {}^{II} E_1^{p,q} = 0$. Since ${}^{II} E_\infty^{p,q}$ is a subquotient of ${}^{II} E_1^{p,q}$, we also have $J_q \cdot {}^{II} E_\infty^{p,q} = 0$.

By the main convergence theorem of spectral sequences, ${}^{II} E_1^{pq} \Rightarrow H^{p+q}(\mathrm{Tot}(C)) \cong H^{p+q}(F^\cdot \otimes M)$. Thus for any $n \in \mathbb{Z}$, there exists a filtration $\{F^p H^n\}_{p \in \mathbb{Z}}$ where $H^n = H^n(F^\cdot)$ such that $F^p H^n / F^{p+1} H^n \cong {}^{II} E_\infty^{p,n-p}$ for all p. As ${}^{II} E_1^{pq}$ is a first quadrant spectral sequence, ${}^{II} E_1^{p,n-p} = 0$ if $p < 0$ or $p > m$. Hence the filtration of H^n has the form $0 = F^{n+1} H^n \subseteq F^n H^n \subseteq \cdots \subseteq F^1 H^n \subseteq F^0 H^n = H^n$. Since $J_{n-p}\ {}^{II} E_\infty^{p,n-p} = 0$, we have $J_{n-p} F^p H^n \subseteq F^{p+1} H^n$ and hence $J_n J_{n-1} \cdots J_0 H^n = 0$. □

(3) (Nick) Let (R, m) be a Cohen Macaulay ring and $x_1, ..., x_d$ a system of parameters for R. Prove that for any positive integers $n_1, ..., n_d$,

$$\lambda(R/(x_1^{n_1}, ..., x_d^{n_d}) = \left(\prod_{i=1}^d n_i\right) \lambda(R/(x_1, ..., x_d)).$$

Proof. Induct on $N := \sum n_i \geq d$. For $N = d$, we see $n_i = 1$ for all i and we are done. So suppose $N > d$. Then there exists i with $n_i \geq 2$. Without loss of generality, reindex so $n_d \geq 2$. Let

$$I = (x_1^{n_1}, ..., x_d^{n_d}), I' = (x_1^{n_1}, ..., x_{d-1}^{n_{d-1}}, x_d^{n_d - 1}), I'' = (x_1^{n_1}, ..., x_{d-1}^{n_{d-1}}, x_d).$$

We have a short exact sequence $0 \to I/I' \to R/I \to R/I' \to 0$.

Claim. $I/I' \cong R/I''$

Proof. Define $\phi : R \to I/I'$ by $r \mapsto rx^{n_d - 1} + I$. Note ϕ is surjective. Thus it is enough to show $\ker \phi = I''$. Clearly, $\ker \phi \supseteq I''$. So let $y \in \ker \phi$. Then $yx^{n_d-1} \in I$. Say $yx^{n_d-1} = \sum_{i=1}^{d} a_i x_i^{n_i}$. Then $(y - a_d x_d)x_d^{n_d-1} = \sum_{i=1}^{d-1} a_i x_i^{n_i} \in (x_1^{n_1}, ..., x_{d-1}^{n_{d-1}})$, which is regular. Furthermore, $(x_1^{n_1}, ..., x_{d-1}^{n_{d-1}}, x_d^{n_d-1})$ is regular. Thus $y - a_D x_d \in (x_1^{n_1}, ..., x_{d-1}^{n_{d-1}})$ and so $y \in I''$. □

Now $\lambda(R/I) = \lambda(R/I') + \lambda(R/I'') = n_1 \cdots n_{d-1}(n_d - 1)\lambda(R/(\underline{x})) + n_1 \cdots n_{d-1}\lambda(R/(\underline{x})) = n_1 \cdots n_d \lambda(R/(\underline{x}))$. □

(4) (Laura) Let (R, m) be a regular local ring of characteristic $p > 0$ and M an $R-$module of finite length. Prove that $\lambda(F(M)) = p^d \lambda(M)$, where $d = \dim R$.

Proof. Induct on $\lambda(M)$. If $\lambda(M) = 1$, then $M \cong R/m$. As R is a regular local ring, $m = (x_1, ..., x_d)$ where $\underline{x}$ form a system of parameters. Then

$$\lambda(F(M)) = \lambda(F(R/(x_1, ..., x_d))) = \lambda(R/(x_1^p, ..., x_d^p)) = p^d \lambda(R/(x_1, ..., x_d)) = p^d \lambda(M)$$

by Nick's exercise. Now suppose $\lambda(M) > 0$ and choose $N \subset M$ with $\lambda(N) < \lambda(M)$. We have a short exact sequence $0 \to N \to M \to M/N \to 0$. As R is regular, F is exact and thus $0 \to F(N) \to F(M) \to F(M/N) \to 0$ is exact. By assitivity of length, we thus have

$$\lambda(F(M)) = \lambda(F(N)) + \lambda(F(M/N)) = p^d \lambda(N) + p^d \lambda(M/N) = p^d \lambda(M).$$

□

(5) (Lori) Let R be a regular local ring and I an ideal of R. Prove that $F(H_I^i(R)) \cong H_I^i(R)$ for all i.

Proof. Let $I = (x_1, ..., x_n)$ and recall $H_I^i(R) = H^i(C^{\cdot}(\underline{x}))$ where $C^{\cdot}$ is the Čech complex. As R is regular, F is exact. Note that $0 \to \ker \phi_i \to C_i \xrightarrow{\phi_i} C_{i+1} \to \operatorname{coker} \phi_i \to 0$ is exact. This yields the following commutative diagram with exact rows.

$$\begin{tikzcd}
0 \arrow[r] \arrow[d, "="] & F(\ker \phi_i) \arrow[r] & F(C^i) \arrow[r] \arrow[d, "="] & F(C^{i+1}) \arrow[r] \arrow[d, "="] & F(\operatorname{coker} \phi_i) \arrow[r] & 0 \arrow[d, "="] \\
0 \arrow[r] & \ker F(\phi_i) \arrow[r] & F(C^i) \arrow[r] & F(C^{i+1}) \arrow[r] & \operatorname{coker} F(\phi_i) \arrow[r] & 0
\end{tikzcd}$$

By the Five Lemma, we have $F(\ker \phi_i) = \ker F(\phi_i)$ and $F(\operatorname{coker} \phi_i) = \operatorname{coker} F(\phi_i)$. So $F(C^{i+1})/F(\operatorname{im} \phi_i) = F(C^{i+1}/\operatorname{im} \phi_i) = F(\operatorname{coker} \phi_i) = \operatorname{coker} F(\phi_i) = F(C^{i+1})/\operatorname{im} F(\phi_i)$, which implies $F(\operatorname{im} \phi_i) = \operatorname{im} F(\phi_i)$. This yields another commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & F(\operatorname{im} \phi_i) \arrow[r] \arrow[d, "\cong"] & F(\ker \phi_i) \arrow[r] \arrow[d, "\cong"] & F(H^i(C^{\cdot})) \arrow[r] & 0 \\
0 \arrow[r] & \operatorname{im}(F(\phi_i)) \arrow[r] & \ker F(\phi_i) \arrow[r] & H^i(F(C^{\cdot})) \arrow[r] & 0
\end{tikzcd}$$

By the Five Lemma, we have $F(H^i(C^{\cdot})) \cong H^i(F(C^{\cdot}))(*)$.

Furthermore, we have the following commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & \ker \phi_i \arrow[r] & C^i \arrow[r, "\phi_i"] \arrow[d, "="] & C^{i+1} \arrow[r] \arrow[d, "="] & \operatorname{coker} \phi_i \arrow[r] & 0 \\
0 \arrow[r] & \ker F(\phi_i) \arrow[r] & F(C^i) \arrow[r] & F(C^{i+1}) \arrow[r] & \operatorname{coker} F(\phi_i) \arrow[r] & 0
\end{tikzcd}$$

By the Five Lemma, we have $\ker \phi_i = \ker F(\phi_i)$ and $\operatorname{coker} \phi_i = \operatorname{coker} F(\phi_i)$. Thus we have $H^i(C^{\cdot}) = H^i(F(C^{\cdot})) = F(H^i(C^{\cdot}))$ by $(*)$. □

(6) (Brian) Let $\phi : R \to S$ be a homomorphism of commutative rings. For an $R-$module M, let $S \otimes_\phi M$ denote the left $S-$module $S \otimes_R M$ where S is viewed as a right $R-$module via ϕ (i.e., $s \otimes rm = s\phi(r) \otimes m$). In this context, if $\phi : R \to R$ is a Frobenius map, then $R \otimes_\phi M$ is $F(M)$, the Frobenius functor applied to M. By the associative property of tensor products, if $\phi : R \to S$ and $\psi : S \to T$ are ring homomorphisms, then $T \otimes_\psi (S \otimes_\phi M) \cong T \otimes_{\psi\phi} M$. Use this approach to show that Frobenius commutes with localization and completion.

Proof. We will prove the result for localization. The proof for completions is similar. Let $\phi : R \to R$ be the Frobenius map, $\psi : R \to R_S$ the natural map, and $\tilde{\phi} : R_S \to R_S$ the Frobenius map of R_S. Note that $\psi\phi = \tilde{\phi}\psi$ as $\psi\phi(r) = \psi(r^p) = \frac{r^p}{1} = \tilde{\phi}\left(\frac{r}{1}\right) = \tilde{\phi}\psi(r)$. Now,

$$F_{R_s}(M_S) = R_S \otimes_{\tilde{\phi}} (R_S \otimes_\psi M) = R_S \otimes_{\tilde{\phi}\psi} M = R_S \otimes_{\psi\phi} M = R_S \otimes_\psi (R \otimes_\phi M) = (F_R(M))_S. \qquad \square$$

(7) (Xuan) Let (R, m) be a local ring of characteristic $p > 0$ and M a finitely generated $R-$module such that $M \cong F(M)$. Prove that M is free.

Proof. As M is finitely presented, we have free modules F and G so that $F \to G \to M \to 0$ is a minimal presentation. Applying Frobenius, we get $F \to G \to F(M) \to 0$. Now $I_j = I_j^{[p]} \subseteq I_j^p \subseteq I_j^2 \subseteq I_j$. Thus $I_j = I_j^2$ which implies $I_j = 0$ by NAK. Thus M is projective and hence free. $\square$

(8) (Silvia) Let R be a ring of characteristic $p > 0$ and S a multiplicatively closed set of R. Prove that $F(R_S) \cong R_S$. More generally, let M be a flat $R-$module. Prove that $F(M)$ is flat.

Proof. Define $\phi : R^F \times S \to R_S$ by $\left(r, \frac{a}{b}\right) \mapsto \frac{ra^p}{b^p}$. Then ϕ is $R-$balanced (that is, ϕ is additive in each component and for all $u \in R$ we have $\phi\left(ru, \frac{a}{b}\right) = \phi\left(r, \frac{ua}{b}\right)$).

$$\begin{tikzcd} R^F \times R_S \arrow[r] \arrow[d, "\phi"] & R^F \otimes R_S \arrow[dl, dashed] \\ R_S & \end{tikzcd}$$

By definition of tensor product, there exists a unique group homomorphism $\alpha : R^F \otimes R_S \to R_S$ such that the diagram above commutes. Also $\alpha\left(ur \otimes \frac{a}{b}\right) = u\alpha\left(r \otimes \frac{a}{b}\right)$ and so α is an $R-$module homomorphism. Define $\beta : R_S \to R^F \otimes_R R_S$ by $\frac{a}{b} \mapsto ab^{p-1} \otimes \frac{1}{b}$. Then β is an $R-$module homomorphism and one can check $\alpha\beta = 1$ and $\beta\alpha = 1$. Thus $R_S \cong R^F \otimes R_S = F(R_S)$.

Now assume M is flat. By Lazard's Theorem, $M = \varinjlim(M_i, \phi_j^i)$ where M_i are finitely generated free modules. Thus

$$F(M) = R^F \otimes \varinjlim(M_i, \phi_j^i) = \varinjlim R^F \otimes (M_i, \phi_j^i) = \varinjlim(F(M_i), F(\phi_j^i)) = \varinjlim(M_i, (\phi_j^i)^{[p]}).$$

As M_i are finitely generated free, M_i is flat. As the direct limit of flat modules is flat, we are done. $\square$

(9) (Hamid) Let R be a Noetherian ring of characteristic $p > 0$. Prove that the Frobenius functor is faithful; i.e., $F(M) = 0$ if and only if $M = 0$.

Proof. Clearly, if $M = 0$ then $F(M) = 0$. So suppose $F(M) = 0$. Recall $M = 0$ if and only if $M_p = 0$ and Frobenius commutes with localization. Thus we may assume (R, m) is local. Similarly, $M = 0$ if and only if $\hat{R} \otimes R = 0$ and Frobenius commutes with completion. Thus we may assume R is a complete local ring and hence the homomorphic image of a regular local ring Q of characteristic p. Say $R = Q/I$ and consider M as a $Q-$module. We have the following commutative diagram where f_Q and f_R are the Frobenius maps and π

is the natural surjection

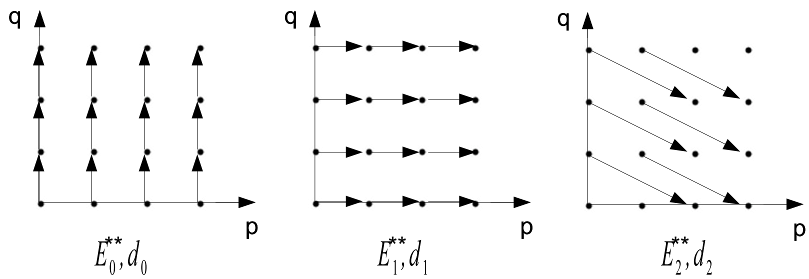
$$\begin{array}{ccc} Q & \xrightarrow{\pi} & Q/I = R \\ \Big\downarrow{\scriptstyle f_Q} & & \Big\downarrow{\scriptstyle f_R} \\ Q & \xrightarrow{\pi} & Q/I = R \end{array}$$

Now $0 = F(M) = (M \otimes_R Q/I) \otimes R^F$. Also $0 = (M \otimes_Q Q^F) \otimes_Q Q/I$ as $M \otimes Q^F$ is $I-$torsion ($\oplus Q/I \to M \to 0$ exact implies $\oplus M/I^{[p]} \to M \otimes Q^F \to 0$ is exact). Thus $M \otimes Q^F = 0$. If $M \neq 0$, then there exists $0 \neq x \in M$. Then $0 \to (x) \otimes Q^F \to M \otimes Q^F$ is exact, which implies $x = 0$, a contradiction. $\square$

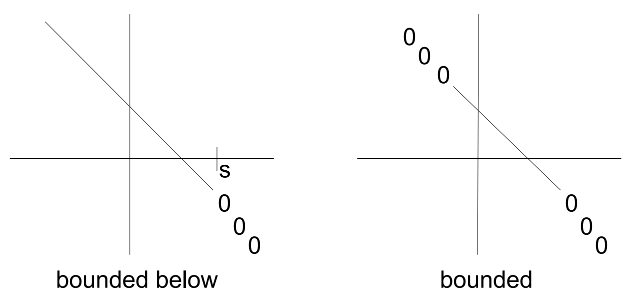
Appendix B. Cohomological Spectral Sequences

The following is based from notes taken from Weibel's An Introduction to Homological Algebra.

Definition. *A **cohomological spectral sequence** starting with $\{E_a\}$ is a family $\{E_r^{pq}\}_{r\geq a}$ of objects, together with maps $d_r^{pq}: E_r^{pq} \to E_r^{p+r,q-r+1}$ such that $d_r d_r = 0$ and $E_{r+1}^{pq} \cong H(E_r) = \ker(d_r^{pq})/\operatorname{im}(d_r^{p-r,q+r-1})$.*



Definition. *A cohomological spectral sequence $\{E_r^{pq}\}_{r\geq a}$ is said to be **bounded below** if for each n there exists $s = s(n)$ such that $E_a^{pq} = 0$ for all $p < s$. The spectral sequence is said to be **bounded** if for each n there are only finitely many non-zero terms E_a^{pq} with $p + q = n$.*



Note that 1^{st} and 3^{rd} quadrant spectral sequences are bounded, and 2^{nd} quadrant spectral sequences are bounded below.

B.1. **Convergence.** Note that E_{r+1}^{pq} is a subquotient of the previous term E_r^{pq}. Define $Z_{r+1}^{pq} = \ker(d_r^{pq})$ and $B_{r+1}^{pq} = \operatorname{im}(d_r^{p-r,q+r-1})$ for $r \geq a$. Further set $Z_a^{pq} = E_a^{pq}$ and $B_a^{pq} = 0$. Then $E_r^{pq} \cong Z_r^{pq}/B_r^{pq}$.

Claim. The following is a nested family of subobjects of E_a^{pq} :

$$0 = B_a^{pq} \subseteq \cdots \subseteq B_r^{pq} \subseteq B_{r+1}^{pq} \subseteq \cdots \subseteq Z_{r+1}^{pq} \subseteq Z_r^{pq} \subseteq \cdots \subseteq Z_a^{pq} = E_a^{pq}$$

Proof. Induct on r. For $r = q$, we have $0 = B_a^{pq} \subseteq Z_a^{pq} = E_a^{pq}$. For $r > a$, we know $Z_{r+1}^{pq} \subseteq E_{r+1}^{pq} \subseteq Z_r^{pq}$ and $B_r^{pq} \subseteq B_{r+1}^{pq}$. Thus we have

$$\begin{tikzcd}
0 \arrow[r] & B_r^{pq} \arrow[r] & Z_r^{pq} \arrow[r] & E_{r+1}^{pq} \arrow[r] & 0 \\
0 \arrow[r] \arrow[u] & B_{r+1}^{pq} \arrow[r] \arrow[u, dashed] & Z_{r+1}^{pq} \arrow[r] \arrow[u, hook] & E_{r+2}^{pq} \arrow[r] \arrow[u, hook] & 0
\end{tikzcd}$$

By a generalization of the Five Lemma, done.

Define $B_\infty^{pq} = \cup_{r=a}^{\infty} B_r^{pq}$ and $Z_\infty^{pq} = \cap_{r=a}^{pq} Z_r^{pq}$. Then set $E_\infty^{pq} = Z_\infty^{pq}/B_\infty^{pq}$.

Note that if $\{E_r^{pq}\}$ is bounded below, then $Z_\infty^{pq} = Z_r^{pq}$ for all large r. If $\{E_r^{pq}\}$ is bounded, then $E_\infty^{pq} = E_r^{pq}$ for all large r.

Definition. *Let $\{E_r^{pq}\}_{r\geq a}$ be a bounded below spectral sequence. We say E_r^{pq} **converges** to $H^* = \{H^n\}$ if for each n we have a filtration (i.e., a chain of submodules of H^n)*

$$0 = F^t H^n \subseteq F^{t-1}H^n \subseteq \cdots \subseteq F^{p+1}H^n \subseteq F^p H^n \subseteq \cdots \subseteq H^n$$

such that $E_\infty^{pq} \cong F^p H^{p+1}/F^{p+1}H^{p+q}$ and $\cup_p F^p H^n = H^n$. In this case, we write $E_a^{pq} \Rightarrow H^{p+q}$.

Remark. H^* need not be unique, even if the spectral sequence is bounded. For example, let $E_0^{pq} = \begin{cases} \mathbb{Z}/2\mathbb{Z}, & \text{if } p,q \geq 0 \\ 0, & \text{otherwise} \end{cases}$ and $d_r^{pq} = 0$ for all p,q,r. Then $\{E_r^{pq}\}$ is a first quadrant spectral sequence with $E_\infty^{pq} = \mathbb{Z}/2\mathbb{Z}$ for all $p,q \geq 0$. So $E_0^{pq} \Rightarrow (\mathbb{Z}/2\mathbb{Z})^{p+q+1}$ and $E_0^{pq} \Rightarrow \left(\mathbb{Z}/2^{p+q+1}\mathbb{Z}\right)$.

Definition. *The spectral sequence $\{E^p q_r\}$ **collapses** at E_r $(r \geq 2)$ if there is exactly one non-zero row or column in the lattice E_r^{pq}.*

Notes.

(1) If the spectral sequence collapses at E_r, then $E_r = E_\infty$.

(2) Suppose $E_a^{pq} \Rightarrow H^n$ and the spectral sequence collapses at E_r. Then H^* is unique. In fact, H^n is the unique non-zero E_r^{pq} with $p+q=n$.

Proof. As $E_r^{pq} = 0$ for all $p \neq c$, we see $F^p H^n = 0$ for all $p \neq c$. So $H^n = F \subseteq H^n$. Since $E_r^{cq} = F^c H^{p+1}/F^{c+1}H^{p+q} = F^c H^n$, we see $E_r^{cq} = H^n$. □

Remarks.

(1) Suppose a spectral sequence converting to H^* has $E_2^{pq} = 0$ unless $p = h$ or $h+1$ (i.e., we have two non-zero columns). So $E_2^{pq} \Rightarrow H^{p+q}$. Note $d_2^{pq} = 0$ for all p,q and so $E_2^{pq} = E_\infty^{pq}$ for all p,q. Also, $E_2^{h+1,n-h-1} \subseteq H^n$ and $H^n/E_2^{h+1,n-h-1} \cong E_2^{h,n-h}$.

Proof. By definition, $E_2^{t,h-t} \cong F^t H^h/F^{t+1}H^h$. Since $E^{h+2,h-(h+2)} = 0$, we see $F^{h+2}H^h = 0$. Thus $E^{h+1}, h-(h+1) \cong F^{h+1}H^n \subseteq H^n$. Similarly, $H^n/E_2^{h+1,n-h-1} \cong E_2^{h,n-h}$. □

Thus we have the exact sequence $0 \to E_2^{h+1,n-h-1} \to H^n \to E_2^{h,n-h} \to 0$ for all n.

(2) Suppose a spectral sequence converging to H^* has $E_2^{pq} = 0$ unless $q = s$ or $s+1$ (i.e., we have two non-zero rows). So $E_2^{pq} \Rightarrow Hp+q$. Note $d_2^{pq} = 0$ for all $q \neq s+1$. Then $E_\infty^{pq} = E_3^{pq} = \ker(d_2^{pq})/\operatorname{im}(d_2^{p-2,q+1})$. As above, $E_3^{n-s,s} \subseteq H^n$ and $H^n/E_3^{n-s,s} \cong E_3^{n-s-1,s+1}$. In addition, $E^{n-2-1,s+1} = \ker(E_2^{n-s-1,s+1} \xrightarrow{d_2} E_s^{n-s+1,s})$ and $E_3^{n-s,s} = E_2^{n-s,s}/\operatorname{im}(E_2^{n-s-2,s+1} \xrightarrow{d_2} E_2^{n-s,s})$. Putting this together, we get a long exact for all n : sequence

$$\cdots \to H^{n-1} \to E_2^{n-s-2,s+1} \xrightarrow{d_2} E_2^{n-s,s} \to H^n \to E_2^{n-s-1,s+1} \xrightarrow{d_2} E_2^{n-s+1,s} \to \cdots$$

(3) Suppose $\{E_r\}^{pq}$ is a first quadrant spectral sequence converging to H^*. Then $H^0 = E_2^{0,0}$ and there is an exact sequence $0 \to E_2^{1,0} \to H^1 \to E_2^{0,1} \xrightarrow{d_2} E_2^{2,0} \to H^2$. Similarly, suppose $\{E_r^{p,q}\}$ is a third quadrant spectral sequence converging to H^*. Then $H^0 = E_2^{0,0}$ and there is an exact sequence $H^{-2} \to E_2^{-2,0} \xrightarrow{d_2}\to E_2^{0,-1} \to H^{-1} \to E^{-1,0} \to 0$. These are called the **exact sequences of low degree**.

Definition. *A **filtration** F on a chain complex $C^\cdot$ is an ordered family of chain subcomplexes $\cdots \subseteq F^{p+1}C \subseteq F^p C \subseteq \cdots$ of C. The filtration is **exhaustive** if $\cup_p F^p C = C$. The filtration is **bounded below** if for each n there exists $s = s(n)$ such that $F^p C^n = 0$ for $p > s$.*

Theorem 97. *A filtration F of a chain complex $C^\cdot$ naturally determines a spectral sequence starting with $E_0^{pq} = F^p C^{p+q}/F^{p+1}C^{p+q}$ and $E_1^{pq} = H^{p+q}(E_0^{p*})$. The maps d_r^{pq} are induced by the differential of C.*

Proof. Weibel, page 133. □

Theorem 98. *Suppose $C^{\cdot}$ is a chain complex and F is a filtration of $C^{\cdot}$. Suppose F is bounded below and exhaustive. Then the spectral sequence E_1^{pq} associated to F is bounded below and converges to $H^*(C)$:*

$$E_1^{pq} = H^{p+q}(F^pC/F^{p+1}C) \Rightarrow H^{p+q}(C).$$

Proof. Weibel, page 136. $\square$

B.2. **Spectral sequences of Double Complexes.** Let $C = C^{**}$ be a double complex:

$$\begin{tikzcd}
 & \vdots & \vdots & \vdots & \\
\cdots \arrow[r] & C^{p-1,q+1} \arrow[u] \arrow[r, "d_h"] & C^{p,q+1} \arrow[u] \arrow[r, "d_h"] & C^{p+1,q+1} \arrow[u] \arrow[r] & \cdots \\
\cdots \arrow[r] & C^{p-1,q} \arrow[u, "d_v"] \arrow[r, "d_h"] & C^{p,q} \arrow[u, "d_v"] \arrow[r, "d_h"] & C^{p+1,q} \arrow[u, "d_v"] \arrow[r] & \cdots \\
\cdots \arrow[r] & C^{p-1,q-1} \arrow[u, "d_v"] \arrow[r, "d_h"] & C^{p,q-1} \arrow[u, "d_v"] \arrow[r, "d_h"] & C^{p+1,q-1} \arrow[u, "d_v"] \arrow[r] & \cdots \\
 & \vdots \arrow[u] & \vdots \arrow[u] & \vdots \arrow[u] &
\end{tikzcd}$$

such that $d_vd_v = d_hd_h = d_hd_v + d_vd_h = 0$. The **total complex** $\mathrm{Tot}(C)$ of C is defined by $\mathrm{Tot}(C)^n = \oplus_{p+q=n} C^{p,q}$ and $d : \mathrm{Tot}(C)^n \to \mathrm{Tot}(C)^n$ is given by $d = d_h + d_v$. There are two natural filtrations of $\mathrm{Tot}(C)$ which give rise to two spectral sequences.

First, we may filter the total complex by columns: For each n, let X_n^{**} be the double subcomplex of C^{**} defined by $X_n^{pq} = \begin{cases} C^{p,q}, & \text{if } p \geq n \\ 0, & \text{otherwise.} \end{cases}$ Let ${}^IF^n\,\mathrm{Tot}(C)$ be the total complex of X_n^{**}. Clearly, ${}^IF^n\,\mathrm{Tot}(C)$ is a subcomplex of $\mathrm{Tot}(C)$ and ${}^IF^{n+1}\,\mathrm{Tot}(C) \subseteq^I F^n\,\mathrm{Tot}(C)$ for all n. As $\mathrm{Tot}(C)$ is a direct sum of C^{pq}'s, this filtration is always exhaustive. Also ${}^IF^n\,\mathrm{Tot}(C)$ is bounded below provided C^{**} is. This filtration gives rise to a spectral sequence $\{{}^IE_r^{pq}\}$ starting with

$${}^IE_0^{pq} =^I F^p\,\mathrm{Tot}(C)^{p+q}/{}^IF^{p+1}\,\mathrm{Tot}(C)^{p+q} = \bigoplus_{i+j=p+q, i\geq p} C^{ij}/\bigoplus_{i+j=p+q, i\geq p+1} C^{ij} = C^{pq}.$$

The maps d_0 are just the vertical differentials d_v of C^{**} and so ${}^IE_1^{pq} = H_v^q(C^{p*})$. The maps $d_1 : H_v^q(C^{p*}) \to H_v^q(C^{p+1,*})$ are induced by the horizontal differentials and so ${}^IE_2^{pq} = H_h^pH_v^q(C)$. By the theorem above, if C is a bounded below double complex, then this spectral sequence converged to $H^*(\mathrm{Tot}(C))$:

$${}^IE_2^{pq} = H_h^pH_v^q(C) \Rightarrow H^{p+q}(\mathrm{Tot}(C)).$$

Similarly, we can filter $\mathrm{Tot}(C)$ by the rows of C : for each n let Y_n^{**} be the double subcomplex of C^{**} defined by $Y_n^{pq} = \begin{cases} C^{pq}, & \text{if } q \geq n \\ 0, & \text{otherwise.} \end{cases}$ Let ${}^{II}F^n\,\mathrm{Tot}(C)$ be the total complex of Y_n^{**}. Then ${}^{II}F^n\,\mathrm{Tot}(C)$ is a subcomplex of $\mathrm{Tot}(C)$ and ${}^{II}F^{n+1}\,\mathrm{Tot}(C) \subseteq^{II} F^n\,\mathrm{Tot}(C)$. As before, this is an exhaustive filtration of $\mathrm{Tot}(C)$ and is bounded below if C^{**} is bounded above. This filtration gives rise to another spectral sequence $\{{}^{II}E_r^{pq}\}$ beginning with

$${}^{II}E_0^{pq} =^{II} F^p\,\mathrm{Tot}(C)^{p+q}/{}^{II}F^{p+1}\,\mathrm{Tot}(C)^{p+q} = \bigoplus_{i+j=p+q, j\geq p} C^{ij}/\bigoplus_{i+j=p+q, j\geq p+1} C^{ij} = C^{qp}.$$

The differentials d_0 are the horizontal differentials and so ${}^{II}E_1^{pq} = H_h^q(C^{*p})$. The maps d_1 are the vertical differentials of C and so ${}^{II}E_2^{pq} = H_v^pH_h^q(C)$. Again by the theorem above, if C is a bounded above double complex, then this spectral sequence converges to $H^*(\mathrm{Tot}(C))$:

$$^{II}E_2^{pq} = H_v^p H_h^q(C) \Rightarrow H^{p+q}(\mathrm{Tot}(C)).$$

B.3. **Applications.**

Theorem 99 (**Universal Coefficient Theorem for Cohomology**). *Let $P.$ be a bounded below chain complex of projective $R-$modules such that each $d(P_n)$ is also projective. Then for every n and every $R-$module M there exists an exact sequence* $0 \to \mathrm{Ext}_R^1(H_{n-1}(P), M) \to H^n(\mathrm{Hom}_R(P, M)) \to \mathrm{Hom}_R(H_n(P), M) \to 0$.

Proof. Let $P. = \cdots \to P_{n+1} \to P_n \emptyset P_{n-1} \to \cdots \to P_t \to 0$. Let $I^\cdot$ be an injective resolution of M and C^{**} the double complex defined by $C^{pq} = \mathrm{Hom}_R(P_p, I^q)$. Note that C is a bounded double complex. Now $C^{p*} = \mathrm{Hom}_R(P_p, I^\cdot)$ and so

$$^{I}E_1^{pq} = H^q(C^{p,*}) = \mathrm{Ext}_R^q(P_p, M) = \begin{cases} \mathrm{Hom}_R(P_p, M), & \text{if } q = 0 \\ 0, & \text{otherwise.} \end{cases}$$

since P_p is projective. Thus $^{I}E_1^{pq}$ collapses and H^* is unique. Now

$$^{I}E_2^{pq} = H_h^p(H_v^q(C)) = \begin{cases} H^p(\mathrm{Hom}_R(P, M)), & \text{if } q = 0 \\ 0, & \text{otherwise.} \end{cases}$$

Since $^{I}E_2^{pq} \Rightarrow H^{p+q}(\mathrm{Tot}(C))$, we see $H^n(\mathrm{Tot}(C)) \cong H^n(\mathrm{Hom}_R(P, M))$ as H^* is unique.

Now $^{II} = H_h^q(C^{*,p}) = H^q(\mathrm{Hom}_R(P., I^p)) = \mathrm{Hom}_R(H_q(P), I^p)$ since $\mathrm{Hom}_R(-, I^p)$ is exact. Then $^{II}E_2^{pq} = H_v^p(\mathrm{Hom}_R(H_q(P), I^\cdot)) = \mathrm{Ext}_R^p(H_q(P), M)$. Thus we have $\mathrm{Ext}_R^p(H_q(P), M) \Rightarrow H^{p+q}(\mathrm{Hom}_R(P, M))$.

Recall that each $d(P_n)$ is projective and thus $\ker(d_n)$ is projective for all n and $\mathrm{pd}_R\, H_n(P) \leq 1$ for all n (consider the short exact sequence $0 \to d(P_{n+1}) \to \ker(d_n) \to H_n(P) \to 0$). Hence $\mathrm{Ext}_R^p(H_q(P), M) = 0$ for all $p \geq 2$. So $^{II}E_2^{pq} = 0$ for all $p \neq 0, 1$. Therefore we have a two column spectral sequence. By Remark 1, there exists exact sequences

$$0 \to \mathrm{Ext}_R^1(H_{n-1}(P), M) \to H^n(\mathrm{Hom}_R(P, M)) \to \mathrm{Hom}_R(H_n(P), M) \to 0$$

for all n. □

Theorem 100 (**Base-change for Ext**). *Let $f : R \to S$ be a ring map. Then there is a first quadrant spectral sequence $E_2^{pq} = \mathrm{Ext}_S^p(A, \mathrm{Ext}_R^q(S, B)) \Rightarrow \mathrm{Ext}_R^{p+q}(A, B)$ for all $S-$modules A and $R-$modules B.*

Proof. Let $P. \to A \to 0$ be a projective $S-$resolution of A and $0 \to B \to I^\cdot$ an injective $R-$resolution of B. Let $C^{pq} = \mathrm{Hom}_R(P_p, I^q)$. Then C^{**} is a first quadrant double complex.

$$\begin{aligned} ^{II}E_1^{pq} = H_h^q(C^{*p}) &= H_h^q(\mathrm{Hom}_R(P., I^p)) \\ &= H_h^q(\mathrm{Hom}_R(P. \otimes_R S, I^p)) \text{ as } P. \text{ is projective} \\ &= H_h^q(\mathrm{Hom}_S(P., \mathrm{Hom}_R(S, I^p))) \\ &= \mathrm{Ext}_S^q(A, \mathrm{Hom}_R(S, I^p)) \end{aligned}$$

Since I^p is an injective $R-$module, $\mathrm{Hom}_R(S, I^p)$ is an injective $S-$module. Thus

$$^{II}E_1^{pq} = \begin{cases} \mathrm{Hom}_S(A, \mathrm{Hom}_R(S, I^p)), & \text{if } q = 0 \\ 0, & \text{otherwise.} \end{cases}$$

Thus the spectral sequence collapses at E_1. Now

$$\begin{aligned} ^{II}E_2^{pq} &= H_v^p(H_h^q(C)) \\ &= H_v^p(\mathrm{Hom}_S(A, \mathrm{Hom}_R(S, I^\cdot))) \text{ if } q = 0 \\ &= H^p(\mathrm{Hom}_R(A, I^\cdot)) \\ &= \mathrm{Ext}_R^p(A, B) \end{aligned}$$

So ${}^{II}E_2^{pq} = \begin{cases} \mathrm{Ext}^p_R(A,B), & \text{if } q=0 \\ 0, & \text{otherwise.} \end{cases}$ and therefore $H^n(\mathrm{Tot}(C)) = \mathrm{Ext}^n_R(A,B)$.

Similarly, we have ${}^{I}E_1^{pq} = H^q_v(C^{p*}) = H^q_v(\mathrm{Hom}_R(P_p, I^{\cdot})) = H^q_v(\mathrm{Hom}_R(P_p \otimes_R S, I^{\cdot})) = H^q_v(\mathrm{Hom}_S(P_p, \mathrm{Hom}_R(S, I^{\cdot})))$. As P_p is a projective $S-$module, $\mathrm{Hom}_S(P_p, -)$ is an exact functor. Thus

$$ {}^{I}E_1^{pq} = \mathrm{Hom}_S(P_p, H^q(\mathrm{Hom}_R(S, I^{\cdot}))) = \mathrm{Hom}_S(P_p, \mathrm{Ext}^q_R(S,B)). $$

Now ${}^{I}E_2^{pq} = H^p_h H^q_v(C) = H^p_h(\mathrm{Hom}_S(P_{\cdot}, \mathrm{Ext}^q_R(S,B))) = \mathrm{Ext}^p_S(A, \mathrm{Ext}^q_R(S,B))$. Therefore,

$$ \mathrm{Ext}^p_S(A, \mathrm{Ext}^q_R(S,B)) \Rightarrow \mathrm{Ext}^{p+q}_R(A,B). $$

□

B.4. **Grothendieck Spectral Sequences.**

Definition. *Let $C^{\cdot}$ be a complex. A* ***right Cartan-Eilenberg resolution*** *of $C^{\cdot}$ is an upper half-plane double complex I^{**} together with an augmentation chain map $C^* \to I^{*0}$ such that*

1. *Each I^{pq} is an injective module*
2. *If $C^p = 0$ then the column $I^{p*} = 0$*
3. *The induced maps on the boundaries and cohomology $0 \to B^p(C) \to B^{p0}(I) \to B^{p1}(I) \to \cdots$ and $0 \to H^p(C) \to H^{p0}(C) \to H^{p1}(C) \to \cdots$ are injective resolutions, where $B^{rq}(I) = \mathrm{im}(I^{p-1,q} \xrightarrow{d_h} I^{pq}, Z^{pq}(I) = \ker(I^{pq} \xrightarrow{d_h} I^{p+1,q})$, and $H^{pq}(I) = Z^{pq}(I)/B^{pq}(I)$.*

Remark. If I is a right Cartan-Eilenberg resolution of C then $0 \to Z^p(C) \to Z^{p0}(I) \to Z^{p1}(I) \to \cdots$ and $0 \to C^p \to I^{p0} \to I^{p1} \to \cdots$ are injective resolutions.

Lemma 101. *Every complex has a right Cartan Eilenberg resolution.*

Proof. The analogous statement for left Cartan Eilenberg resolutions is proved in Wiebel. □

Definition. *An object B of a category $\mathcal{B}$ is $F-$acyclic if the right derived functor of F vanishes on B, that is $R^iF(B) = 0$ for all $i \neq 0$.*

Theorem 102 (**Grothendieck Spectral Sequence #1**)**.** *Let $\mathcal{A}, \mathcal{B}, \mathcal{C}$ be abelian categories such that $\mathcal{A}, \mathcal{B}$ have enough injectives. Suppose $G : \mathcal{A} \to \mathcal{B}$ and $F : \mathcal{B} \to \mathcal{C}$ are left exact covariant functors. Suppose G sends injective objects of $\mathcal{A}$ to $F-$acyclic objects of $\mathcal{B}$. Then there is a convergent first quadrant spectral sequence ${}^{II}E_2^{pq}(R^pF)(R^qG)(A) \Rightarrow R^{p+q}(FG)(A)$ for every object A in $\mathcal{A}$. The exact sequence of low degree terms is*

$$ 0 \to (R^1F)(GA) \to R^1(FG)(A) \to F(R^1G(A)) \to (R^2F)(GA) \to R^2(FG)(A). $$

Proof. The exact sequence of low degree terms follows from Remark 3 above. Let $0 \to A \to J^{\cdot}$ be an injective resolution of A (in the category $\mathcal{A}$). Apply the functor G to $J^{\cdot}$ and let I^{**} be a Cartan-Eilenberg resolution of $G(J^{\cdot})$ in the category $\mathcal{B}$. Let X^{**} be the double complex $F(I^{**})$. Now I^{p*} is an injective resolution of $G(J^p)$ and so

$$ {}^{I}E_1^{pq} = H^q_v(X^{p*}) = H^q_v(F(I^{p*})) = R^qF(G(J^p)). $$

As J^p is an injective object of $\mathcal{A}$, $G(J^p)$ is $F-$acyclic, that is $R^iF(G(J^p)) = 0$ for $i > 0$. Thus the spectral sequence collapses at E_1 and we have ${}^{I}E_1^{pq} = \begin{cases} (FG)(J^p), & \text{if } q = 0, \\ 0, & \text{otherwise.} \end{cases}$ So

$$ {}^{I}E_2^{pq} = H^p_h(H^q_v(X)) = \begin{cases} R^p(FG)(A), & \text{if } q = 0, \\ 0, & \text{otherwise.} \end{cases} $$

Therefore $H^{p+q}(\mathrm{Tot}(X)) \cong R^{p+q}(FG)(A)$. Now ${}^{II}E_1^{pq} = H_h^q(X^{*p}) = H_h^q(F(I^{*p}))$. As I^{**} is a right Cartan-Eilenberg resolution of $G(J^{\cdot})$, the kernels, boundaries, and homologies of the complex I^{*p} are all injective objects of $\mathcal{B}$. Thus $H_h^q(F(I^{*p})) \cong F(H_h^q(I^{*p}))$. Now $H_h^q(I)$ is an injective resolution of $H^q(G(J^{\cdot})) = R^q(G)(A)$. Therefore

$$ {}^{II}E_2^{pq} = H_v^p H_h^q(X) = H_v^p(F(H_h^q(I))) = (R^pF)(R^qG)(A). $$

Hence $(R^pF)(R^qG)(A) \Rightarrow R^{p+q}(FG)(A)$. □

Examples.

(1) Let $\mathcal{A} = \mathcal{B} = \mathcal{C}$ be the category of $R-$modules and $J \subset I$ ideals of R. Let $F = \mathrm{Hom}_R(R/I, -)$ and $G = H_J^0(-)$. Then G sends injectives to injectives. Since $FG = \mathrm{Hom}_R(R/I, -)$, we get

$$ E_2^{pq} = \mathrm{Ext}_R^p(R/I, H_J^q(M)) \Rightarrow \mathrm{Ext}_R^{p+q}(R/I, M) $$

for all $R-$modules M.

(2) (Base-change of Ext) Let $\mathcal{A}$ be the category of $R-$modules and $\mathcal{B} = \mathcal{C}$ the category of $R/J-$modules for some ideal J of R. Suppose $I \supset J$. Let $F = \mathrm{Hom}_{R/J}(R/I, -)$ and $G = \mathrm{Hom}_R(R/J, -)$. Then $FG = \mathrm{Hom}_R(R/I, -)$ and thus

$$ E_2^{pq} = \mathrm{Ext}_{R/J}^p(R/I, \mathrm{Ext}_R^q(R/J, M)) \Rightarrow \mathrm{Ext}_R^{p+q}(R/I, M). $$

Definition. *Let $C.$ be a complex. A* ***left Cartan-Eilenberg resolution*** *of $C.$ is an upper-half plane double complex P_{**} together with an augmentation map $P_{*0} \to C_*$ such that*

(1) *Each P_{pq} is projective*
(2) *If $C_p = 0$ then the column $P_{p*} = 0$*
(3) *The induced maps on the boundaries and homology $\cdots \to B_{pq}(P) \to B_{p0}(P) \to B_p(C) \to 0$ and $\cdots H_{p1}(P) \to H_{p0}(P) \to H_p(C) \to 0$ are projective resolutions (and thus the induced maps $\cdots \to Z_{p1}(P) \to Z_{p0}(P) \to Z_p(C) \to 0$ and $\cdots \to P_{p1} \to P_{p0} \to C_p \to 0$ are projective resolutions).*

Lemma 103. *Every complex has a left Cartan-Eilenberg resolution.*

Theorem 104 (Grothendieck Spectral Sequence #2). *Let $\mathcal{A}, \mathcal{B}$, and $\mathcal{C}$ be abelian categories such that $\mathcal{A}$ and $\mathcal{B}$ have enough projectives. Suppose $G : \mathcal{A} \to \mathcal{B}$ is a right covariant functor and $F : \mathcal{B} \to \mathcal{C}$ a contravariant left exact functor. Suppose G sends projective objects of $\mathcal{A}$ to $F-$acyclic objects of $\mathcal{B}$. Then there is a first quadrant spectral sequence $E_2^{pq} = (R^p)(L_qG)(M) \Rightarrow R^{p+q}(FG)(M)$ for every object M in $\mathcal{A}$. The exact sequence of low degree terms is*

$$ 0 \to (R^1F)(GM) \to R^1(FG)(M) \to F(L_1G(M)) \to (R^2F)(GM) \to R^2(FG)(M). $$

Proof. The exact sequence of low degree terms follows from Remark 3 above. Let $P. \to m$ be a projective resolution of M in the category $\mathcal{A}$. Let Q_{**} be a left Cartan-Eilenberg resolution of $G(P.)$ in the category $\mathcal{B}$. Let X^{**} be the double complex $F(Q_{**})$. Then X^{**} is a first quadrant double complex and ${}^IE_1^{pq} = H_v^q(X^{p*})$. since Q_{p*} is a projective resolution of $G(P_p)$, $H_v^q(X^{p*}) = H_v^q(F(Q_{p*})) = (R^qF)(G(P_p))$. Since P_p is projective, $G(P_p)$ is $F-$acyclic and so ${}^IE_1^{pq} = \begin{cases} FG(P_p), & \text{if } q = 0 \\ 0, & \text{otherwise.} \end{cases}$ Thus the spectral sequence collapses and

$$ {}^IE_2^{pq} = H_h^p H_v^q(X) = \begin{cases} R^p(FG)(M), & \text{if } q = 0 \\ 0, & \text{otherwise.} \end{cases} $$

Therefore $H^n(\mathrm{Tot}(X)) \cong R^n(FG)(M)$. Now ${}^{II}E_1^{pq} = H_h^q(X^{*p}) = H_h^q(F(Q_{*p}))$. As Q_{**} is a left Cartan Eilenberg resolution of $G(P)$, the horizontal kernels, boundaries, and homology of Q_{*p} are all projective objects of $\mathcal{B}$. Thus $H_h^q(F(Q_{*p})) \cong F(H_q^h(Q_{*p}))$. Now $H_q^h(Q)$ is a projective resolution of $H_q(G(P)) = L_qG(M)$. Therefore

${}^{II}E_2^{pq} H_v^p H_h^q(X) = H_v^p(F(H_q^h(Q))) = (R^pF)(L_qG)(M)$ and

$$(R^pF)(L_qG)(M) \Rightarrow R^{p+q}(FG)(M).$$

□

Example. Let $\phi : R \to S$ be a ring map. Let $\mathcal{A}$ be the category of $R-$modules and $\mathcal{B} = \mathcal{C}$ the category of $S-$modules. Let $G : \mathcal{A} \to \mathcal{B}$ be $- \otimes_R S$ and $F : \mathcal{B} \to \mathcal{C}$ be $\mathrm{Hom}_S(-, N)$ for some $S-$module N. For any $R-$module M, $(FG)(M) = \mathrm{Hom}_S(M \otimes_R S, N) = \mathrm{Hom}_R(M, \mathrm{Hom}_S(S, N)) = \mathrm{Hom}_R(M, N)$. Also G takes projective $R-$modules to projective $S-$modules. Thus there exists a first quadrant spectral sequence

$$E_2^{pq} = \mathrm{Ext}_S^p(\mathrm{Tor}_q^R(S, M), N) \Rightarrow \mathrm{Ext}_R^{p+q}(M, N)$$

for all $r-$modules M and $S-$modules N.

Theorem 105 (**Grothendieck Spectral Sequence #3**)**.** *Let $\mathcal{A}, \mathcal{B}, \mathcal{C}$ be abelian categories such that $\mathcal{A}$ has enough projectives and $\mathcal{B}$ has enough injectives. Suppose $G : \mathcal{A} \to \mathcal{B}$ is a contravariant left exact functor and $F : \mathcal{B} \to \mathcal{C}$ is a covariant left exact functor. Suppose G sends projective objects of $\mathcal{A}$ to $F-$acyclic objects of $\mathcal{B}$. Then there is a first quadrant spectral sequence $E_2^{pQ} = (R^pF)(R^qG)(M) \Rightarrow R^{p+q}(FG)(M)$ for all objects M in $\mathcal{A}$. The exact sequence of low degree terms is*

$$0 \to (R^1F)(GM) \to R^1(FG)(M) \to F(R^1G(M)) \to (R^2F)(GM) \to R^2(FG)(M).$$

Proof. Similar to that of Grothendieck Spectral Sequence #1, except start with a projective resolution of M instead of an injective resolution. □