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A Note on the Parameters of PBIB Association Schemes

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A NOTE ON THE PARAMETERS OF PBIB ASSOCIATION SCHEMES[1]

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1. Introduction. In an m-class partially balanced incomplete block (PBIB) design [3], any two distinct treatments are related as first, second, $\cdots$, or mth associates in accordance with certain rules, and the resulting classification of pairs of treatments is called an association scheme [1], [4]. Parameters, including v, n_i, p^i_{jk}, which depend only on the association relation between treatments and are common to all designs having a given association scheme, are called *association scheme parameters*. Other parameters, including b, r, k, λ_i, depend, in addition, on the arrangement of the treatments into blocks. Known results on two-class association scheme parameters, reviewed in this section with some changes in arrangement and notation, are used in Section 2 to prove some new relations. Dependent as they are on known necessary conditions, our theorems will not provide any new proofs of the nonexistence of particular designs. However, they are in a form which is convenient for application and are oriented toward the fundamental problem of the connection between number-theoretic properties of the parameters and combinatorial structure of the designs.

The association scheme parameters are non-negative integers which satisfy the familiar relations

$$n_1 + n_2 = v - 1, \tag{1.1}$$

$$p^1_{11} + p^1_{12} + 1 = p^2_{11} + p^2_{12} = n_1, \tag{1.2a}$$

$$p^1_{12} + p^1_{22} = p^2_{12} + p^2_{22} + 1 = n_2, \tag{1.2b}$$

$$p^1_{21} = p^1_{12}, \qquad p^2_{21} = p^2_{12}, \tag{1.3}$$

$$n_1 p^1_{12} = n_2 p^2_{11}, \qquad n_1 p^1_{22} = n_2 p^2_{12}. \tag{1.4}$$

The following are immediate consequences.

$$n_1 p^1_{12} + n_2 p^2_{12} = n_1 n_2, \tag{1.5}$$

$$0 \leqq p^1_{12} \leqq n_1 - 1, \qquad 0 \leqq p^2_{12} \leqq n_2 - 1. \tag{1.6}$$

We take the four integers n_i, p^i_{12} as fundamental parameters, subject to (1.5) and (1.6). If (1.1), (1.2a), (1.2b) are then taken as definitions of the remaining parameters it is easy to verify that they are all non-negative integers and that (1.4) is satisfied.

If N is the $v \times b$ incidence matrix of the design, then [5] the $v \times v$ symmetric

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matrix NN' has only three distinct characteristic roots θ_0, θ_1, θ_2, with multiplicities α_0, α_1, α_2 respectively, where $\sum \alpha_i = v$. Then θ_0 may be expressed

$$(1.7) \qquad \theta_0 = r + \lambda_1 n_1 + \lambda_2 n_2,$$

and $\alpha_0 = 1$ if NN' is irreducible (equivalently if the design is connected). Also,

$$(1.8) \qquad \theta_1 = r + \lambda_1 t + \lambda_2(-t - 1), \qquad \theta_2 = r + \lambda_1(-s - 1) + \lambda_2 s,$$

$$(1.9) \qquad \alpha_1 = [sn_1 + (s + 1)n_2]/\Delta^{\frac{1}{2}}, \qquad \alpha_2 = [(t + 1)n_1 + tn_2]/\Delta^{\frac{1}{2}}$$

where

$$(1.10) \qquad s = \tfrac{1}{2}(\Delta^{\frac{1}{2}} - \gamma - 1), \qquad t = \tfrac{1}{2}(\Delta^{\frac{1}{2}} + \gamma - 1),$$

$$(1.11) \qquad \gamma = p_{12}^2 - p_{12}^1,$$

$$(1.12) \qquad \Delta = \gamma^2 + 2p_{12}^1 + 2p_{12}^2 + 1.$$

Here s, t, α_1, α_2, γ, and Δ are association scheme parameters. The fact that

$$(1.13) \qquad \alpha_1 \text{ and } \alpha_2 \text{ are integral}$$

turns out to be an additional constraint on n_i, p_{12}^i.

We need the following remarks about two special families of association schemes.

Given v treatments and an integer n, $1 < n < v$, which is a divisor of v, an association scheme of group divisible (GD) type may be constructed by arranging the treatments into disjoint groups of size n and taking two treatments to be first associates if and only if they are in the same group. We remark that $n_1 < n_2$ and $p_{12}^1 = 0$. A scheme in which the same construction is used to define second associates will also be defined to be a GD scheme; in this case $n_1 > n_2$ and $p_{12}^2 = 0$. It is shown in [2] that the parameters of a GD scheme are sufficient to determine its structure uniquely (apart from permutation of treatments). In particular, a necessary and sufficient condition for a two-class scheme to have GD structure is

$$(1.14) \qquad p_{12}^i = 0, \qquad i = 1 \text{ or } 2.$$

It is clear that for a given v, the number of GD schemes with $n_1 < n_2$ is equal to the number of proper divisors of v.

Association schemes of cyclic type are defined in [4] in terms of their combinatorial structure. A somewhat restricted class which includes all known cyclic schemes has parameters which can be expressed as follows in terms of an integer q. $p_{12}^1 = p_{12}^2 = q$, $n_1 = n_2 = \alpha_1 = \alpha_2 = 2q$, $v = \Delta = 4q + 1$. Knowledge of their existence is incomplete, though they are known to exist whenever v is a prime. Following a usage suggested by R. H. Bruck, we use the name *pseudo-cyclic* for all two-class designs having these parameters; the existence of pseudo-cyclic designs which do not have the structure of cyclic designs has not been investigated. It follows from Theorems 5.3 and 5.5 of [5] that

(1.15) In a two-class association scheme not of pseudo-cyclic type, Δ must be a perfect square.

2. Relations among parameters. Using (1.10), (1.11) and (1.12), we observe that s and t are non-negative. We calculate

$$\Delta^{\frac{1}{2}} = s + t + 1, \qquad \gamma = t - s, \tag{2.1}$$

$$p_{12}^1 = s(t + 1), \qquad p_{12}^2 = (s + 1)t. \tag{2.2}$$

If $\Delta^{\frac{1}{2}}$ is an integer, (1.10) or (2.1) shows that $2s$ and $2t$ are integers which must be even in view of (2.2). Hence, using (1.15), s and t are non-negative integers for all two-class association schemes not of pseudo-cyclic type, and the parameters of such schemes may be expressed in terms of s, t, n_1, n_2, subject to (1.5), (1.6) and (1.13). From (1.14), the scheme is of GD type if and only if $st = 0$.

Among various consequences of (2.2), we note that the product of p_{12}^1 and p_{12}^2 is divisible by 4 and that if neither is zero, their ratio is between $\frac{1}{2}$ and 2, since

$$\tfrac{1}{2} \leqq s/(s + 1) < s(t + 1)/(s + 1)t < (t + 1)/t \leqq 2.$$

While (2.2) clarifies the nature of p_{12}^i, it does not make the other restrictions unnecessary. In (a), (b), (c) below, (1.5), (1.6), (1.13) respectively are stated for the p_{12}^i values corresponding to $s = 1$, $t = 3$, with examples to show that each of the three conditions may be violated by n_1, n_2 values which satisfy the other two.

(a) $4n_1 + 6n_2 = n_1n_2$, violated by $n_1 = 9$, $n_2 = 8$;
(b) $n_1 \geqq 5$, $n_2 \geqq 7$, violated by $n_1 = 18$, $n_2 = 6$;
(c) $(n_1 + 2n_2)/5$ is an integer, violated by $n_1 = 12$, $n_2 = 8$.

All three conditions are satisfied by the values $n_1 = 8$, $n_2 = 16$ and $n_1 = n_2 = 10$, which correspond to known association schemes.

The following makes use of (1.9), (2.2) and (1.1)–(1.4).

$$\begin{aligned}
\Delta\alpha_1\alpha_2 &= [n_1s + n_2(s + 1)][n_1(t + 1) + n_2t] \\
&= (n_1)^2p_{12}^1 + n_1n_2(2st + s + t + 1) + (n_2)^2p_{12}^2 \\
&= n_1n_2p_{11}^2 + n_1n_2(p_{12}^1 + p_{12}^2 + 1) + n_1n_2p_{22}^1 \\
&= n_1n_2(n_1 + n_2 + 1),
\end{aligned}$$

giving

$$vn_1n_2 = \Delta\alpha_1\alpha_2, \tag{2.3}$$

an interesting relation which seems to have received little notice.

THEOREM 1. *A two-class association scheme with v equal to a prime must be of pseudo-cyclic type.*

PROOF. Let $v = p$, a prime. Then n_1, n_2, α_1, α_2 are positive integers less than p. p but not p^2 is a divisor of vn_1n_2; from (2.3) the same is true of $\Delta\alpha_1\alpha_2$ and hence of Δ. Therefore Δ is not a square and by (1.15) the scheme, if it exists, is of pseudo-cyclic type.

COROLLARY. *There are no two-class association schemes with v equal to a prime of the form $4m + 3$.*

THEOREM 2. *In a two-class association scheme the products vn_i, n_1n_2, $n_ip_{jk}^i$ are even integers, $i, j, k = 1, 2$.*

PROOF. n_1 and n_2 are both even for pseudo-cyclic type schemes. For a scheme not of pseudo-cyclic type, first suppose that n_1 and n_2 are both odd. Then $sn_1 + (s + 1)n_2$ is odd and from (1.9), $\Delta^{\frac{1}{2}}$ is odd. (2.1) shows that s and t are of the same parity, from (2.2) p_{12}^i are both even and from (1.5) n_1n_2 is even, a contradiction. Therefore n_1 and n_2 are not both odd, and two and hence all three terms of (1.5) are even. If either of n_i, say n_1, is odd, the remaining products involving it may be expressed $vn_1 = n_1(n_1 + 1) + n_1n_2$, $n_1p_{22}^1 = n_2p_{12}^2$, $n_1p_{11}^1 = n_1(n_1 - 1) - n_1p_{12}^1$ with the aid of (1.1), (1.4), and (1.2a), where the right hand side in each equation is even.

We remark that parity conditions related to those of Theorem 2 can be proved for association schemes with any number of classes by suitable enumeration of elements in the symmetric matrix NN'.

THEOREM 3. *In a two-class association scheme not of* GD *type, n_1 and n_2 are not relatively prime.*

PROOF. Let $n_1 = m_1 d$, $n_2 = m_2 d$, where d is the greatest common divisor of n_1, n_2. Then (1.5) leads to

$$m_1p_{12}^1 + m_2p_{12}^2 = m_1m_2 d. \tag{2.4}$$

Each term in this equation must be divisible by each of the relatively prime integers m_1, m_2. Hence there exist non-negative integers u, w such that

$$p_{12}^1 = um_2, \qquad p_{12}^2 = wm_1, \qquad u + w = d. \tag{2.5}$$

In a non-GD scheme p_{12}^1 and p_{12}^2 are positive; then u and w are positive and $d \geqq 2$.

COROLLARY. *In a two-class association scheme not of* GD *type, v cannot be of the form $p + 1$, p a prime.*

THEOREM 4. *If p is an odd prime, there are exactly two* GD *association schemes with $v = 2p$, $n_1 < n_2$, but no other two-class schemes with $v = 2p$ unless p is of the form*

$$p = 2s^2 + 2s + 1, \tag{2.6}$$

in which case the only possible parameters with $n_1 \leqq n_2$ are given by

$$p_{12}^1 = p_{12}^2 = s(s + 1), \tag{2.7}$$

$$n_1 = s(2s + 1), \qquad n_2 = (s + 1)(2s + 1). \tag{2.8}$$

PROOF. The assertion about GD schemes follows from the discussion in Section 1, since 2 and p are the only proper divisors of v. Now assume the scheme not of GD type. From Theorem 3, n_1 and n_2 are not relatively prime and must be distinct from p and $p - 1$. Then p but not p^2 is a factor of $vn_1n_2 = \Delta\alpha_1\alpha_2$. Since $v \not\equiv 1 \pmod 4$, the scheme cannot be of pseudo-cyclic type. It follows from (1.15) that the integer Δ is a perfect square; not being divisible by p^2, it is not divisible by p. Therefore the product of α_1 and α_2 is divisible by p. Neither of α_1, α_2 is as large as $2p$, since their sum is $v - 1 = 2p - 1$; hence one of them is equal to p. We choose notation so that

$$\alpha_1 = p, \qquad \alpha_2 = p - 1. \tag{2.9}$$

Then (2.3) reduces to

$$2n_1n_2 = \Delta(p - 1). \tag{2.10}$$

The parameters s and t are integers. From (1.9),

$$p\Delta^{\frac{1}{2}} = s(n_1 + n_2) + n_2, \qquad (p - 1)\Delta^{\frac{1}{2}} = t(n_1 + n_2) + n_1. \tag{2.11}$$

Subtracting,

$$\Delta^{\frac{1}{2}} = (s - t)(n_1 + n_2) + n_2 - n_1. \tag{2.12}$$

The integer $s - t$ is non-negative since $\Delta^{\frac{1}{2}}$ is positive. If $s - t$ is positive, then (2.12) shows $\Delta^{\frac{1}{2}} \geqq 2n_2$ and $\Delta \geqq 4(n_2)^2$. Then

$$4(n_2)^2(p - 1) \leqq \Delta(p - 1) = 2n_1n_2 = 2(2p - 1 - n_2)n_2,$$

reducing to $n_2 \leqq 1$, which is impossible for a non-GD association scheme. Therefore $s - t$ is non-positive, we have $s = t$, and (2.2) gives (2.7). Simplifying (2.12) and using (1.1) we have

$$n_2 - n_1 = \Delta^{\frac{1}{2}} = s + t + 1 = 2s + 1,$$

$$n_2 + n_1 = v - 1 = 2p - 1,$$

which can be solved to give $n_1 = p - s - 1$, $n_2 = p + s$. Using this in (2.10), we obtain this quadratic equation in p:

$$2(p - s - 1)(p + s) = (2s + 1)^2(p - 1).$$

The solutions are $p = \frac{1}{2}$, extraneous to this problem, and $p = 2s^2 + 2s + 1$, proving (2.6) and leading to (2.8) to complete the proof.

Theorem 4 excludes two-class schemes not of GD type and with $v = 2p$ for many primes, including those of the form $4m + 3$. The only primes less than 300 of the form (2.6) are 5, 13, 41, 61, 113, 181. Association schemes of the family specified by (2.7) and (2.8) have the special property that $v = \Delta + 1$ and are known for many values of s [7], including some in which $2s^2 + 2s + 1$ is composite; in the latter case, however, other non-GD schemes for the same v may be possible. For example, $s = 3$ gives $2s^2 + 2s + 1 = 25$, and two non-GD schemes are known with $v = 50$, one [7] in the present family with parameters $n_1 = 21$, $n_2 = 28$, $p_{12}^1 = p_{12}^2 = 12$, and another [6] with parameters $n_1 = 7$, $n_2 = 42$, $p_{12}^1 = p_{12}^2 = 6$.

THEOREM 5. *In a two-class association scheme not of* GD *type*, $n_i \geqq (v-1)^{\frac{1}{2}}$, $i = 1, 2$.

PROOF. The following proof for n_1 uses (1.1)–(1.4); it follows from (1.14) and (1.4) that we may assume $p_{11}^2 \geqq 1$. Interchanging the indices 1 and 2 gives a proof for n_2.

$$\begin{aligned} n_1 &= p_{11}^1 + p_{12}^1 + 1, \\ (n_1)^2 &= n_1 p_{11}^1 + n_1 p_{12}^1 + n_1 \\ &= n_1 + n_2 + n_1 p_{11}^1 + n_2(p_{11}^2 - 1) \geqq n_1 + n_2 = v - 1. \end{aligned}$$

The inequality of Theorem 5 need not hold for GD schemes and is the best possible for other schemes, as shown for example by the triangular scheme with $v = 10$, $n_1 = 3$ and by the above parameters with $v = 50$, $n_1 = 7$. On the other hand, this equality is possible only in isolated cases. To see this, assume $(n_1)^2 = v - 1$. It follows from the above proof that $p_{11}^1 = 0$ and $p_{11}^2 = 1$, which is enough to determine the parameters with $p_{12}^1 = p_{12}^2 = n_1 - 1 = s^2 + s$. Then, using (1.9),

$$\begin{aligned} \alpha_1 &= [s(s^2 + s + 1) + (s + 1)(s^2 + s + 1)(s^2 + s)]/(2s + 1) \\ &= 2s + 5s^3 - 7s^4 + 15s^5/(2s + 1), \end{aligned}$$

which is integral only if $2s + 1$ is a divisor of 15. The three possibilities $s = 1, 2, 7$ lead to the examples just mentioned and to one more with $v = 3250$.

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