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Combinatorial and Commutative Manipulations in Feynman's Operational Calculi for Noncommuting Operators

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COMBINATORIAL AND COMMUTATIVE MANIPULATIONS IN FEYNMAN'S OPERATIONAL CALCULI FOR NONCOMMUTING OPERATORS

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In Feynman's Operational Calculi, a function of indeterminates in a commutative space is mapped to an operator expression in a space of (generally) noncommuting operators; the image of the map is determined by a choice of measures associated with the operators, by which the operators are 'disentangled'. Results in this area of research include formulas for disentangling in particular cases of operators and measures. We consider two ways in which this process might be facilitated. First, we develop a set of notations and operations for handling the combinatorial arguments that tend to arise. Second, we develop an intermediate space for the disentangling map, where commutativity might be exploited more extensively.

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Chapter 1

Overview of Feynman's Operational Calculi. Motivation for the present work

In a 1951 paper, "An Operator Calculus Having Applications in Quantum Electrodynamics" [10], physicist Richard Feynman introduced a new notation for multiplying noncommuting operators, with the intent that this notation would make operator manipulation easier and would, in appropriate cases, make more transparent the underlying physical theory that the operators represent. His approach was the following (this is a paraphrase of Gerald Johnson and Michel Lapidus' description of "Feynman's heuristic 'Rules,' " [24, p. 377]):

(1) Express the order of operation of a product of noncommuting operators not by means of conventional right-to-left order of operation, but instead by attaching time indices to the operators (an earlier time means earlier operation).

(2) Form functions of the operators, with the indices attached, and then manipulate

the operators as though they were commuting.

(3) Finally, 'disentangle' the resulting expressions; that is, restore the conventional ordering of the operators.

We will illustrate with a very simple example: Suppose A and B are two non-commuting operators, and say we want to look at their product. The product AB in conventional notation means that B operates first, then A, whereas BA means A operates before B. The idea of working with operators that occur in a certain order may be motivated by physics, in which earlier operation may correspond to an actual, physical time sequence of two events, or it may have mathematical motivation (see for example [20]).

In Feynman's notation, the order of operation is not denoted by the right-to-left order on the page. Instead, using his system we might represent the product AB as A_2B_1, where the fact that the subscript 1 is smaller than the subscript 2 is the means by which we indicate that B operates before A. Were we to exchange the order of the operators in Feynman's notation, keeping the indices attached, we would get B_1A_2, and then the indices would still tell us (because $1 < 2$) that B operates first, and A operates second. Hence $A_2B_1 = B_1A_2$; both expressions represent what in conventional notation is BA, so operators with Feynman time indices are commutative. After manipulating these expressions in ways that interest us, taking advantage of that commutativity, we eventually restore the conventional notation in the resulting expressions, so that, for example A_2B_1 becomes AB and A_1B_2 becomes BA (and AB is not the same as BA).

The result of the above rules is a functional calculus, in the sense that functions of the operators are manipulated as though they were functions of commuting indeterminates, and in the expressions that result after these manipulations they are again treated like noncommuting operators. Feynman did not formalize the mathematics of his ideas in the article; other individuals have since then developed formalizations of a number of his ideas.

Here we will work in the context of Feynman's Operational Calculi (FOCi) as initiated by B. Jefferies and G. W. Johnson [13], [15], [16], [14] (and also developed especially by L. Nielsen, B. S. Kim, and M. Lapidus). Instead of Feynman's heuristic approach of attaching indices to noncommuting operators and then acting as though the operators commute, the approach here is to begin in a commutative space of indeterminates, each of which is associated with one of the noncommuting operators, then to map from there to the space of operators. For example, noncommuting operators A and B will be related to indeterminates $\tilde{A}$ and $\tilde{B}$ that are elements of a space $\mathbb{D}$ where $\tilde{A}\tilde{B} = \tilde{B}\tilde{A}$. We will also associate 'time indices' s_1, s_2 which take values in $[0, 1]$ to the operators A, B, respectively (we will on occasion work in another interval, such as $[0, T]$). Thus $\tilde{A}(s_1)$ is thought of as referring to the indeterminate $\tilde{A}$ operating at the time s_1, and consequently, $\tilde{A}(s_1)\tilde{B}(s_2)$ is thought of as representing that the indeterminate $\tilde{A}$ operates before $\tilde{B}$ if $s_1 < s_2$, or that $\tilde{B}$ operates before $\tilde{A}$ if $s_2 < s_1$. (Truth be told, since indeterminates commute, the results are the same. However, we will map $\tilde{A}\tilde{B}$ to expressions involving the operators A and B that also have time indices s_1 and s_2, where if $s_1 < s_2$ we will have $B(s_2)A(s_1) = BA$, and if $s_2 < s_1$ we will have $A(s_1)B(s_2) = AB$, and those are not equal.)

A significant difference in this approach from Feynman's is that in his, a given expression involving operators with time indices results in a unique operator expression after disentangling. For the example above, if the beginning expression is $\int_0^1 A(s)ds \int_0^1 B(t)dt$, it can only result in the operator $\frac{1}{2}BA + \frac{1}{2}AB$. However, whereas Feynman only used, in effect, Lebesgue measure, in the Jefferies-Johnson approach each operator (together with its associated indeterminate and time index) is assigned a Borel measure. The measure tells (roughly speaking) which relation $s_1 < s_2$ or $s_2 < s_1$ occurs more often. This association produces the different weights for the terms AB and BA after disentangling. For example, if continuous probability measures μ and ν are associated with A and B, respectively, then the expression $\tilde{A}\tilde{B}$ will map to

$$\begin{aligned}
&\int_{\{(s_1,s_2):\, 0<s_1<s_2<1\}} BA\, d(\mu \times \nu) + \int_{\{(s_1,s_2):\, 0<s_2<s_1<1\}} AB\, d(\mu \times \nu) \\
&= (\mu \times \nu)(\{(s_1, s_2) :\ 0 < s_1 < s_2 < 1\})BA \\
&\qquad\qquad + (\mu \times \nu)(\{(s_1, s_2) :\ 0 < s_2 < s_1 < 1\})AB. \qquad (1.1)
\end{aligned}$$

In the case when μ and ν are both Lebesgue measure on $[0, 1]$, this will equal $\frac{1}{2}BA + \frac{1}{2}AB$, as before. However, we can choose probability measures μ and ν to produce any linear combination of the terms AB and BA with total probability equal to 1. We will not limit our attention to probability measures, but the intuition of relative weights is still helpful in other cases. (Recall, a probability measure is a measure for which the measure of the entire space is 1. A continuous Borel measure is a Borel measure for which the measure of any set consisting of a single point is zero.) If we choose a different set of measures, then the resulting operator expression will possibly be different, a different linear combination, which is why we speak of Feynman's Operational Calculi in the plural.

(For other work on and approaches to functional calculi for noncommuting operators, see for example [4], [5], [6], [21], [28], [29], [30], [31], [33], [35], [36], [41], [42], and [43]. Most of the other works in the bibliography regard FOCi and are cited elsewhere in this thesis; those not cited elsewhere are [25] and [23].)

In the disentangling process we often move from an expression in commuting indeterminates to a formula involving noncommuting operators (rather than indeterminates), and at that point we like to further simplify the operator formula if we can, and where possible to write out a completely disentangled expression in terms of operators.

Perhaps a good question at this point is what use it is to relate noncommuting objects to commuting objects, or why one would associate measures to operators and to time indices. To answer the first question, there appear to be at least two possible reasons to relate noncommuting objects to commuting objects. The first is that it might facilitate calculations; working with commuting operators is easier in some ways than working with noncommuting operators. The second is that there may be physical relevance for doing so. In Feynman's notation, the factors in the expression $\int_0^1 A(s)ds \int_0^1 B(t)dt = \int_0^1 B(t)dt \int_0^1 A(s)ds$ commute; one can represent this way that an event symbolized by an operator A occurs sometime within a time interval $[0, 1]$, while the other represents the occurrence of an event B in that interval. The product in this notation turns out to equal $\frac{1}{2}BA + \frac{1}{2}AB$, which combines the event of A occurring before B with the event of B occurring before A, a combination which may be physically meaningful. Associating time indices and measures to the operators may correspond to increasing or decreasing the likelihood of each event A or B within a given time interval, as though turning a physical apparatus to a higher or lower setting. (Questions also arise of more mathematical interest, such as whether an expression disentangled under one choice of measures can be approximated by expressions disentangled under a "nearby" choice of measures. There are stability theorems regarding when that is the case, see e.g. [26], [37], [38], [39], [40].)

This is the context of the current work. Part of the focus of research into FOCi is to establish formulas that yield disentangled operator expressions in certain special cases (such as physically and/or mathematically meaningful cases), and our objective here is to consider ways that the process of developing and proving disentangling operator formulas and expressions may be facilitated.

One part of the process we concentrate on is combinatorial aspects of disentangling. As with the example(s) we have discussed, the definition we will give of the disentangling map involves a sum of products of a finite set of operators in all possible orders the operators can occur. These orders will be represented by subscripts of time indices (as in '$s_1, \ldots, s_m$') attached to the operators; looking at all possible orders of operation means looking at all possible permutations, or what we will call 'orderings', of the subscripts. Since proofs in FOCi often rely on being able to express that set of orderings in different ways, we develop three operations that can be performed on sets of orderings to relate the sets to each other. The purpose of doing so is to create a vocabulary of these operations for use in FOCi proofs, so that for simple proofs one does not need to just appeal to the reader's intuition about the combinatorics involved, and for harder proofs one does not need to create entirely new terminology for each proof separately.

After that, our other major objective is to develop a context in which Feynman's suggestion of treating noncommuting operators as though they commute (if they are labeled with time indices) may be used more fully. In Feynman's paper [10, p. 216, abstract] he wrote, "An alteration in the notation used to indicate the order of operation of noncommuting quantities is suggested. Instead of the order being defined by the position on the page, an ordering subscript is introduced so that $A_s B_{s'}$ means AB or BA depending on whether s exceeds s' or *vice versa*. . . . An increase in ease of manipulating some operator expressions results." In Feynman's system, attaching time indices provided the freedom to treat operators as though they commute; once he reached the desired form in that context, he would return immediately to the noncommuting context (again we note Feynman did not make this process rigorous). That particular feature of his system is reflected to some extent in the Jefferies-Johnson approach, but we would like to extend it. We will consider how a space may be added to the Jefferies-Johnson approach, intermediate between the commuting and noncommuting spaces, so that calculations in the intermediate space may both make use of commutativity to reach a desired form (time-ordered form), and be able to map readily to the desired form in the noncommuting space.

In addition, prior to addressing those two objectives, we will develop two or so small results that will be useful in proofs and examples to follow (and may be useful more generally), namely a theorem about permuting factors of a product measure within an integral, and a disentangling theorem for use when a measure in a monomial is replaced by a sum of two measures.

Chapter 2

The definition of the disentangling map

Feynman's Operational Calculi involves mapping elements of a particular commutative function space ($\mathbb{D}$) into a noncommutative space of operators ($\mathcal{L}(X)$), where the actions of the maps depend on a choice of measures. We start, after a preliminary definition, by defining $\mathbb{D}$.

Definition 2.0.1. Let $r_1, \ldots, r_n$ be positive real numbers, and let $\mathbb{A} = \mathbb{A}(r_1, \ldots, r_n)$ be the space of complex-valued functions $(z_1, \ldots, z_n) \mapsto f(z_1, \ldots, z_n)$ of n complex variables which are analytic at $(0, \ldots, 0)$, and have power series

$$f(z_1, \ldots, z_n) = \sum_{m_1, \ldots, m_n = 0}^{\infty} c_{m_1, \ldots, m_n} z_1^{m_1} \cdots z_n^{m_n} \tag{2.1}$$

that are absolutely convergent at least on the closed polydisk $|z_j| \leq r_j$. A norm

$$\|f\| = \|f\|_{\mathbb{A}} := \sum_{m_1, \ldots, m_n = 0}^{\infty} |c_{m_1, \ldots, m_n}| r_1^{m_1} \cdots r_n^{m_n} \tag{2.2}$$

can be defined on $\mathbb{A}$, making $\mathbb{A}$ into a commutative Banach algebra with identity under pointwise multiplication. (We can say this because "$\mathbb{A}(r_1, \ldots, r_n)$ can be identified with the weighted l_1-space, where the weight on the index $(m_1, \ldots, m_n)$ is $r_1^{m_1} \cdots r_n^{m_n}$"; see [13, p. 5, Proposition 1.1].)

Definition 2.0.2 (The disentangling algebra $\mathbb{D}$)**.** Given a Banach space X and nonzero operators $A_1, \ldots, A_n \in \mathcal{L}(X)$ (where $\mathcal{L}(X)$ is the space of all bounded linear operators from X into X), we define the **disentangling algebra** $\mathbb{D} = \mathbb{D}(\tilde{A}_1, \ldots, \tilde{A}_n)$ to be the space $\mathbb{A}(r_1, \ldots, r_n)$, where we stipulate that $r_j = \|A_j\|$ for all j. We will commonly use the symbols $\tilde{A}_1, \ldots, \tilde{A}_n$ to represent the formal indeterminates $z_1, \ldots, z_n$, in order to make an association between the indeterminates $\tilde{A}_1, \ldots, \tilde{A}_n \in \mathbb{D}$ and the operators $A_1, \ldots, A_n \in \mathcal{L}(X)$, respectively.

(A remark on notation here: The space denoted '$\mathbb{D}(\tilde{A}_1, \ldots, \tilde{A}_n)$' has at times been denoted '$\mathbb{D}(A_1, \ldots, A_n)$'; the literature appears not to be settled about which is better. The parameters in the expression $\mathbb{D}(A_1, \ldots, A_n)$ seem to indicate both the convergence radii, $r_j = \|A_j\|$ for all j, and the operators $A_1, \ldots, A_n$ associated with the indeterminates $z_1, \ldots, z_n$. However, since the parameters are operators, expressions such as $\mathbb{D}(A_1 \|\mu_1\|, \ldots, A_n \|\mu_n\|)$ might be allowed, where multiplying an operator by the total value of a measure (or by some other scalar) is also an operator, in which case it is not clear whether, for example, an indeterminate $\tilde{A}_1$ should be associated with the operator A_1 or the operator $A_1 \|\mu_1\|$. For that reason, the present author has chosen to include the tildes in the expression $\mathbb{D}(\tilde{A}_1, \ldots, \tilde{A}_n)$, in order to be completely unambiguous how indeterminates and operators are associated; whatever operator the tilde is applied to, whether $\tilde{A}_1$ or $(A_1 \|\mu_1\|)^{\sim}$, is the operator associated with the corresponding indeterminate, and its norm is the convergence radius—technically, then, we should probably write $\widetilde{A_j}$ [with a wide tilde] to show that A_j is the operator and not just A, but let us just say that that will be understood and we will be more careful when we need to be. Additionally, keeping the tildes reminds us what the members of the space $\mathbb{D}(\tilde{A}_1, \ldots, \tilde{A}_n)$ look like. Should we want to associate an operator with an indeterminate, but also to incorporate both the operator and a measure in the radius of convergence of the indeterminate, we can refer to each parameter as $(A_j, \mu_j)^\sim$ to say that the indeterminate is associated with A_j and that the radius is $r_j = \|A_j\| \, \|\mu_j\|$.)

Therefore, $\mathbb{D}$ is a commutative Banach algebra with identity, where the norm is given by

$$\|f(\tilde{A}_1, \ldots, \tilde{A}_n)\| = \|f(\tilde{A}_1, \ldots, \tilde{A}_n)\|_{\mathbb{D}} := \sum_{m_1,\ldots,m_n=0}^{\infty} |c_{m_1,\ldots,m_n}| \|A_1\|^{m_1} \cdots \|A_n\|^{m_n} < \infty \tag{2.3}$$

(see [13, p. 5, Proposition 1.2]).

Functions in $\mathbb{D}$ are thus infinite sums of monomials, but in this thesis we will focus only on polynomials, though there may not be great difficulty in applying similar results to infinite series. Mostly we will work with monomials, and it is useful to implement a special notation for monomials. Specifically, given any nonnegative integers $m_1, \ldots, m_n$ we define

$$P^{m_1,\ldots,m_n}(z_1, \ldots, z_n) := z_1^{m_1} \cdots z_n^{m_n}, \tag{2.4}$$

or alternatively,

$$P^{m_1,\ldots,m_n}(\tilde{A}_1, \ldots, \tilde{A}_n) = \tilde{A}_1^{m_1} \cdots \tilde{A}_n^{m_n} \in \mathbb{D}. \tag{2.5}$$

The disentangling maps we will define momentarily will map such a monomial

$P^{m_1,\ldots,m_n}(\tilde{A}_1,\ldots,\tilde{A}_n) \in \mathbb{D}$ to an operator in $\mathcal{L}(X)$ that is a function of the operators $A_1,\ldots,A_n$. More specifically, it is mapped to a linear combination of products that consist of m_1 factors of the operator A_1, m_2 factors of the operator A_2, etc., in some order. For example, a disentangling map will take the monomial $\tilde{A}^2\tilde{B}^1 = \tilde{A}\tilde{B}\tilde{A} = \tilde{B}\tilde{A}^2$ to some linear combination of A^2B, ABA, and BA^2.

In fact, there will be more than one disentangling map (and therefore more than one operational calculus), yielding possibly different linear combinations from the same monomial. For instance, although one disentangling map might take the monomial $\tilde{A}\tilde{B}$ to the operator $\frac{1}{2}AB+\frac{1}{2}BA$, another will take it to a different linear combination, say, $\frac{2}{3}AB+\frac{1}{3}BA$, for example. The values of the coefficients assigned to the different terms will be determined by a selected set of measures, each of which is associated with an operator. In this example, we might choose to associate the continuous Borel probability measure μ with the operator A and the continuous Borel probability measure ν with the operator B. The effect is that the times in an interval where the measure is larger will produce a greater contribution of the associated operator at that time. For example, if μ has its entire support in the lower half of the interval $[0,1]$ and ν has its support in the upper half of the interval $[0,1]$, then that will force the operator A (associated with μ) to occur before the operator B (associated with ν), which means that the monomial $\tilde{A}\tilde{B}$ will map under the disentangling map $\mathcal{T}_{\mu,\nu}$ with those respective measures to the operator BA, without any contribution from an AB term. (As a possible model of a physical situation, this could represent that apparatus A is turned on for the first half of an interval, and apparatus B is turned on for the second half of the interval.) In all that follows here, we will consider only continuous, finite positive Borel measures, usually on the interval $[0,1]$.

Given a monomial of several factors, some possibly repeated, we will want to distinguish the different instances of each operator. Thus when dealing with the monomial $P^{m_1,\ldots,m_n}(\tilde{A}_1,\ldots,\tilde{A}_n) \in \mathbb{D}$, which has m_1 copies of $\tilde{A}_1$, m_2 copies of $\tilde{A}_2$, etc., we will rename the operators as follows: Let $m := m_1 + m_2 + \cdots + m_n$, and let $\mathrm{Bl}(1), \ldots, \mathrm{Bl}(n)$ denote blocks of integers

$$\begin{aligned}
\mathrm{Bl}(1) &:= \{1, \ldots, m_1\}, \qquad (2.6)\\
\mathrm{Bl}(2) &:= \{m_1 + 1, \ldots, m_1 + m_2\},\\
&\vdots\\
\mathrm{Bl}(n) &:= \{m_1 + \cdots + m_{n-1} + 1, \ldots, m\}.
\end{aligned}$$

Then define

$$C_k := \begin{cases} A_1, & k \in \mathrm{Bl}(1) \\ A_2, & k \in \mathrm{Bl}(2) \\ \vdots & \\ A_n, & k \in \mathrm{Bl}(n). \end{cases} \qquad (2.7)$$

The monomial $P^{m_1,\ldots,m_n}(\tilde{A}_1,\ldots,\tilde{A}_n)$ will be mapped to a sum of products of copies of the operators $A_1, \ldots, A_n$, or as we have now renamed the operators, a sum of products of the operators $C_1, \ldots, C_m$. To discuss the order in which $C_1, \ldots, C_m$ operate, we will think of them as operating at different times $s_1, \ldots, s_m$, respectively, with these times lying in the (time) interval $[0, 1]$, with earlier time corresponding to earlier operation. (Starting from the case of the interval $[0, 1]$ it is not difficult to generalize to an interval $[0, T]$ for an arbitrary time T. However, we will stay with the case $[0, 1]$.) That is, we assign to each operator C_j a time index s_j that takes values in $[0, 1]$, giving us a set of time-indexed operators $\{C_j(s_j)\}_{j=1}^m$.

For example, if an expression includes time-indexed operators $C_1(s_1), C_2(s_2)$, and

$C_3(s_3)$ with $s_2 < s_1 < s_3$, then $C_2(s_2)$ operates before $C_1(s_1)$, which operates before $C_3(s_3)$. These time-indexed operators are naturally taken to be time-dependent for certain kinds of problems; however, we will always limit ourselves here to the case of time-independent operators: $C_j(s_j) \equiv C_j,\ j = 1, \ldots, m$. Thus, each operator is still viewed as possibly operating at various times, but it remains the same operator at all times. Moreover, we will consider only operators that are bounded.

The last step before defining the disentangling maps is to introduce a notation to represent the different possible orders in which the time-indices in the m-tuple $(s_1, \ldots, s_m) \in [0,1]^m$ can occur. We will usually want to arrange operators so that their time indices $s_1, \ldots, s_m$ are in increasing time order from right to left. As the various indices range throughout $[0,1]^m$, their time order relative to each other will change. (For example, sometimes $s_1 < s_2$, and other times $s_2 < s_1$.) With that in mind, we let S_m be the set of permutations of the set of numbers $\{1, \ldots, m\}$ (that is, bijections from the set $\{1, \ldots, m\}$ to itself), and for each permutation $\pi \in S_m$ we define the set

$$\Delta_m(\pi) := \{(s_1, \ldots, s_m) \in [0,1]^m : 0 < s_{\pi(1)} < \cdots < s_{\pi(m)} < 1\}.$$

Each permutation, then, gives us one ordering of the time indices, and different permutations give us different orderings of the time indices. We will sometimes abbreviate the above set as

$$\{s_{\pi(1)} < s_{\pi(2)} < \cdots < s_{\pi(m)}\} := \Delta_m(\pi) \quad \text{or} \quad \{s_{\pi(m)} > \cdots > s_{\pi(2)} > s_{\pi(1)}\} := \Delta_m(\pi).$$

It may be noticed that under this definition of $\Delta_m(\pi)$, the union of all such sets will include nearly all, but not quite all of the points in the set $[0,1]^m$, because points in $[0,1]^m$ for which two time indices s_i, s_j are equal to each other or are equal to 0 or 1 are omitted from the union. For example, in the case $m = 2$, the the union of the sets $\{(s_1, s_2) : \ 0 < s_1 < s_2 < 1\}$ and $\{(s_1, s_2) : \ 0 < s_2 < s_1 < 1\}$ will not be all of $[0,1] \times [0,1]$, because the points (s_1, s_2) with $s_1 = s_2$, $s_1 = 0$, $s_1 = 1$, $s_2 = 0$, or $s_2 = 1$ are omitted. For our purposes those particular points are unimportant, so this definition will suffice. The reason they are unimportant is that the sets $\Delta_m(\pi)$ will serve as regions of integration with respect to some product measure $\mu_1^{m_1} \times \cdots \times \mu_n^{m_n}$, where $\mu_1, \ldots, \mu_n$ are continuous Borel measures and $\sum_{i=1}^n m_i = m$. As noted in [13, p. 7, Lemma 2.1], since the measures are continuous, this has the consequence that any subset of $[0,1]^m$ having two or more coordinates equal, or having a fixed value (such as 0 or 1) for some particular coordinate, has measure zero.

Since in what follows the measures we will deal with will exclusively be finite, continuous, positive Borel measures on a finite interval (usually $[0,1]$), we will define

$$\mathcal{M}_{cb}[a,b] := \{\text{all finite, continuous, positive Borel measures on the interval } [a,b]\}.$$

(It is possible in FOCi to consider more than just continuous measures; see e.g. [27], [3].) We will often deal with product measures that have as factors a set of these Borel measures. It may be worth noting here that given two topological spaces X and Y and their Borel classes $\mathcal{B}(X), \mathcal{B}(Y)$, it is not in general true that $\mathcal{B}(X) \otimes \mathcal{B}(Y) = \mathcal{B}(X \times Y)$ (see [8, p. 240]). Thus, given Borel measures μ on X and ν on Y, and their product measure $\mu \times \nu$ defined on $B(X) \otimes \mathcal{B}(Y)$, the product measure might not be defined on the entire Borel class $\mathcal{B}(X \times Y)$. However, for the cases we will be dealing with, the measures and their products will always be Borel measures. (That the component spaces are separable metric spaces is sufficent to establish that $B(X) \otimes \mathcal{B}(Y) = \mathcal{B}(X \times Y)$; see [11, p. 23], Proposition 1.5.)

Often it will be useful for the measures we use to be probability measures, and whether we are working with time indices in the interval $[0,1]$ or more generally $[0,T]$, we are able to scale the measures to yield probability measures. If we ever intend to use probability measures in this thesis, we will say so explicitly. (For a discussion of the relationship between disentangling using probability measures and disentangling using other measures, see [15, Section 3].)

The last thing we will do in preparation for defining the disentangling map is to do a calculation in the disentangling algebra $\mathbb{D}$ that motivates the definition of the map. Let us consider for a moment the case when the measures $\mu_1, \ldots, \mu_n$ are continuous Borel probability measures. We define indeterminates $\tilde{C}_1, \ldots, \tilde{C}_m$ in much the same way as the operators $C_1, \ldots, C_m$ were defined:

$$\tilde{C}_k := \begin{cases} \tilde{A}_1, & k \in \mathrm{Bl}(1) \\ \tilde{A}_2, & k \in \mathrm{Bl}(2) \\ \vdots & \\ \tilde{A}_n, & k \in \mathrm{Bl}(n). \end{cases} \tag{2.8}$$

We attach time indices, say $\tilde{C}_k(s_k) \equiv \tilde{C}_k$ for $k = 1, \ldots, m$. It can then be shown (using commutativity, [13, Proposition 2.2, p. 8]) that the monomial $P^{m_1,\ldots,m_n}(\tilde{A}_1, \ldots, \tilde{A}_n) \in \mathbb{D}$ can be rewritten as

$$\begin{aligned} P^{m_1,\ldots,m_n}(\tilde{A}_1, \ldots, \tilde{A}_n) &= \tilde{A}_1^{m_1} \cdots \tilde{A}_n^{m_n} = \tilde{C}_1 \tilde{C}_2 \cdots \tilde{C}_m \\ &= \sum_{\pi \in S_m} \int_{\Delta_m(\pi)} \tilde{C}_{\pi(m)}(s_{\pi(m)}) \cdots \tilde{C}_{\pi(1)}(s_{\pi(1)}) \, (\mu_1^{m_1} \times \cdots \times \mu_n^{m_n})(ds_1, \ldots, ds_m). \end{aligned} \tag{2.9}$$

Remark 1. Related to (2.9) we have, for example, $\tilde{A}_1 = \int_0^1 \tilde{A}_1(s_1)\mu_1(ds_1)$. It is also the case that $A_1 = \int_0^1 A_1(s_1)\mu_1(ds_1)$, given $A_1(s_1) \equiv A_1$ (since μ_1 is a probability measure). In these expressions it is clear that if μ_1 is zero over some region within the interval $[0,1]$, then the associated operator $A_1(s_1)$ has no effect at values of s_1 within that region; it can be viewed as not operating there; $A_1(s_1)$ operates only for values of s_1 within the support of the measure μ_1. A similar statement can be made for multiple time-indexed operators and their associated measures; the measures affect the times at which the operators operate, so we speak of "using measures to attach time indices to operators."

Imitating the form of the expression (2.9), we make the following definition for any set of finite, continuous Borel measures on $[0,1]$ (not only probability measures):

Definition 2.0.3 (The disentangling map)**.** Given $\mathbb{D} = \mathbb{D}(\tilde{A}_1, \ldots, \tilde{A}_n)$, and given $\mu_1, \mu_2, \ldots, \mu_n \in \mathcal{M}_{cb}[0,1]$ together with nonnegative integers $m_1, \ldots, m_n$, the **disentangling map**

$$\mathcal{T}_{\mu_1,\ldots,\mu_n} : \mathbb{D}(\tilde{A}_1, \ldots, \tilde{A}_n) \to \mathcal{L}(X)$$

is defined on monomials by

$$\begin{aligned} &\mathcal{T}_{\mu_1,\ldots,\mu_n}[P^{m_1,\ldots,m_n}(\tilde{A}_1, \ldots, \tilde{A}_n)] \\ &\qquad := \sum_{\pi \in S_m} \int_{\Delta_m(\pi)} C_{\pi(m)} \cdots C_{\pi(1)} \, (\mu_1^{m_1} \times \cdots \times \mu_n^{m_n})(ds_1, \ldots, ds_m), \end{aligned} \tag{2.10}$$

also denoted

$$P^{m_1,\ldots,m_n}_{\mu_1,\ldots,\mu_n}(A_1, \ldots, A_n) := \mathcal{T}_{\mu_1,\ldots,\mu_n}[P^{m_1,\ldots,m_n}(\tilde{A}_1, \ldots, \tilde{A}_n)]. \tag{2.11}$$

If $m = 0$ (i.e., $m_1 = \cdots = m_n = 0$), we interpret $P^{m_1,\ldots,m_n}_{\mu_1,\ldots,\mu_m}(A_1, \ldots, A_n)$ to be the identity operator $I \in \mathcal{L}(X)$. In (2.16) and throughout this thesis, the integral of a product of operators is defined by equating it to the same expression but with the operators factored out of the integral:

$$\int_{\Delta_m(\pi)} C_{\pi(m)} \cdots C_{\pi(1)} \, (\mu_1^{m_1} \times \cdots \times \mu_n^{m_n})(ds_1, \ldots, ds_m)$$
$$:= C_{\pi(m)} \cdots C_{\pi(1)} \int_{\Delta_m(\pi)} (\mu_1^{m_1} \times \cdots \times \mu_n^{m_n})(ds_1, \ldots, ds_m). \tag{2.12}$$

It is the fact that the operators are time-independent that allows us to use this definition, pulling the operators outside the integral. (It is, however, possible to make sense of the integral and the disentangling map in the time-dependent case as well. See for example [19], and especially the work of Byung Moo Ahn *et al.*, [1], [2], [3].)

Theorem 2.0.4. *Let* $A_1, \ldots, A_n \in \mathcal{L}(X)$ *be associated with measures* $\mu_1, \ldots, \mu_n \in \mathcal{M}_{cb}[0,1]$*, let* $m_1, \ldots, m_n$ *be nonnegative integers and* $m := m_1 + \cdots + m_n$*, and let blocks of integers* $Bl(1), \ldots, Bl(n)$ *be as in* (2.6) *and operators* $C_1, \ldots, C_m$ *be as in* (2.7)*. Moveover, define the measures* $\nu_1, \ldots, \nu_m$ *by*

$$\nu_k := \begin{cases} \mu_1, & k \in Bl(1) \\ \mu_2, & k \in Bl(2) \\ \vdots & \\ \mu_n, & k \in Bl(n). \end{cases} \tag{2.13}$$

Then

$$P^{m_1, \ldots, m_n}_{\mu_1, \ldots, \mu_n}(A_1, \ldots, A_n) = P^{1, \ldots, 1}_{\nu_1, \ldots, \nu_m}(C_1, \ldots, C_m), \tag{2.14}$$

where if $m = 0$ *we interpret* $P^{1, \ldots, 1}_{\nu_1, \ldots, \nu_m}(C_1, \ldots, C_m)$ *to be the identity operator* $I \in \mathcal{L}(X)$.

Proof. The case $m = 0$ is immediate. Otherwise, by Definition 2.0.3 we have

$$P^{m_1, \ldots, m_n}_{\mu_1, \ldots, \mu_n}(A_1, \ldots, A_n) = \sum_{\pi \in S_m} \int_{\Delta_m(\pi)} C_{\pi(m)} \cdots C_{\pi(1)} \, (\mu_1^{m_1} \times \cdots \times \mu_n^{m_n})(ds_1, \ldots, ds_m)$$

$$= \sum_{\pi \in S_m} \int_{\Delta_m(\pi)} C_{\pi(m)} \cdots C_{\pi(1)} \, (\nu_1 \times \cdots \times \nu_m)(ds_1, \ldots, ds_m)$$
$$= P^{1,\ldots,1}_{\nu_1,\ldots,\nu_m}(C_1, \ldots, C_m). \tag{2.15}$$

□

The disentangling map therefore maps the monomial $P^{m_1,\ldots,m_n}(\tilde{A}_1, \ldots, \tilde{A}_n)$ to a sum of integrals, each integral involving a product of all of the operators in a different order. Whatever product we began with, the integrals we have mapped to are in 'disentangled' form; that is, they are time-ordered. To see this, we write the definition in terms of the time-indexed operators we defined earlier, $C_j(s_j) \equiv C_j$, $j = 1, \ldots, m$:

$$\mathcal{T}_{\mu_1,\ldots,\mu_n}[P^{m_1,\ldots,m_n}(\tilde{A}_1, \ldots, \tilde{A}_n)]$$
$$= \sum_{\pi \in S_m} \int_{\Delta_m(\pi)} C_{\pi(m)}(s_{\pi(m)}) \cdots C_{\pi(1)}(s_{\pi(1)}) \, (\mu_1^{m_1} \times \cdots \times \mu_n^{m_n})(ds_1, \ldots, ds_m). \tag{2.16}$$

Here, when integrating over the set $\Delta_m(\pi)$, in which $s_{\pi(1)} < s_{\pi(2)} < \cdots < s_{\pi(m)}$, the operators appear in the corresponding order $C_{\pi(m)}(s_{\pi(m)}) \cdots C_{\pi(1)}(s_{\pi(1)})$, with the lowest time index on the far right and the highest time index on the far left.

That is the usual pattern for introducing the disentangling map (a similar development is given in, for example, [13],[15], and [24]), and it is patterned after Feynman's description of an operator calculus given in [10]: Beginning with a product of commuting objects, each object is assigned a time index taking values in $[0, 1]$, and the expression is manipulated into time-ordered form (as in Equation (2.9)). Then the expression is, in a manner of speaking, converted to an operator expression that has operators in the corresponding order (as in Equation (2.16)); it is as though the tildes have been erased.

However, the reader should be cautioned about one element of this process which may be somewhat misleading (though our mathematics here is valid). Specifically, it relates to 'erasing' the tildes. By what we have said in Equations (2.9) and (2.16), it is entirely correct for us to say in the case of probability measures $\mu_1, \ldots, \mu_n \in \mathcal{M}_{cb}[0,1]$ that

$$\begin{aligned}\mathcal{T}_{\mu_1,\ldots,\mu_n}\left[\sum_{\pi\in S_m}\int_{\Delta_m(\pi)} \tilde{C}_{\pi(m)}(s_{\pi(m)})\cdots\tilde{C}_{\pi(1)}(s_{\pi(1)})\,(\mu_1^{m_1}\times\cdots\times\mu_n^{m_n})(ds_1,\ldots,ds_m)\right]\\ = \sum_{\pi\in S_m}\int_{\Delta_m(\pi)} C_{\pi(m)}\cdots C_{\pi(1)}\,(\mu_1^{m_1}\times\cdots\times\mu_n^{m_n})(ds_1,\ldots,ds_m). \quad (2.17)\end{aligned}$$

The difference in appearance between the inside of the left-hand expression and the entire right-hand expression is that in the latter, the time indices have been removed and the tildes have been erased. Since the map $\mathcal{T}_{\mu_1,\ldots,\mu_n}$ is linear, and since there is such a similarity between the two expressions—in fact, the map was defined to create this similarity—one might expect the map to hold not only for the entire sum, but also term-by-term. However, that is not the case. That is, generally speaking,

$$\begin{aligned}\mathcal{T}_{\mu_1,\ldots,\mu_n}\left[\int_{\Delta_m(\pi)} \tilde{C}_{\pi(m)}(s_{\pi(m)})\cdots\tilde{C}_{\pi(1)}(s_{\pi(1)})\,(\mu_1^{m_1}\times\cdots\times\mu_n^{m_n})(ds_1,\ldots,ds_m)\right]\\ \neq \int_{\Delta_m(\pi)} C_{\pi(m)}\cdots C_{\pi(1)}\,(\mu_1^{m_1}\times\cdots\times\mu_n^{m_n})(ds_1,\ldots,ds_m) \quad (2.18)\end{aligned}$$

for the various choices of $\pi \in S_m$.

Let us take a simple example. Given operators $A, B \in \mathcal{L}(X)$ associated with continuous Borel probability measures μ, ν, respectively, on $[0,1]$, and letting $\tilde{A}(s) \equiv \tilde{A}$ and $\tilde{B}(s) \equiv \tilde{B}$ on $[0,1]$, we have that

$$\mathcal{T}_{\mu,\nu}\left[\tilde{A}\tilde{B}\right] = \mathcal{T}_{\mu,\nu}\left[\int_{\{s<t\}} \tilde{B}(t)\tilde{A}(s)(\mu\times\nu)(ds,dt) + \int_{\{t<s\}} \tilde{A}(s)\tilde{B}(t)(\mu\times\nu)(ds,dt)\right]$$

$$= \int_{\{s<t\}} BA(\mu \times \nu)(ds, dt) + \int_{\{t<s\}} AB(\mu \times \nu)(ds, dt). \quad (2.19)$$

However, often

$$\mathcal{T}_{\mu,\nu} \left[\int_{\{s<t\}} \tilde{B}(t)\tilde{A}(s) \ (\mu \times \nu)(ds, dt) \right] \neq \int_{\{s<t\}} BA \ (\mu \times \nu)(ds, dt) \quad (2.20)$$

and

$$\mathcal{T}_{\mu,\nu} \left[\int_{\{t<s\}} \tilde{A}(s)\tilde{B}(t) \ (\mu \times \nu)(ds, dt) \right] \neq \int_{\{t<s\}} AB \ (\mu \times \nu)(ds, dt). \quad (2.21)$$

We will calculate a specific example in Section 6.1. (One exception, when equality does hold, is when the operators commute.)

On the other hand, in Feynman's notation (where $A(s)$ and $B(t)$ commute), the time-ordered expression

$$\int_0^1 \int_s^1 B(t)A(s) \ dt\, ds \qquad \text{does become} \qquad \int_0^1 \int_s^1 BA \ dt\, ds, \quad (2.22)$$

while

$$\int_0^1 \int_0^s A(s)B(t) \ dt\, ds \qquad \text{becomes} \qquad \int_0^1 \int_0^s AB \ dt\, ds. \quad (2.23)$$

Seeing this distinction between the definitions we are using and Feynman's notation, the present author believes it may be beneficial to have a process in which expressions in a commutative space (similar to $\mathbb{D}$) and in $\mathcal{L}(X)$ *that have the same form* can be mapped term-by-term as in Feynman's process, and that were we to have this, we could possibly take greater advantage of commutativity. This possibility will be discussed in Chapter 6, where we consider another way of performing the disentangling procedure. The definition of the disentangling map as we have stated it (Definition 2.0.3) remains the same, though, and we will continue to use the same definition throughout this thesis.

Having defined the disentangling maps—which may well be different maps if the measures are chosen differently—we have laid enough groundwork to start focusing on specific techniques in the theory of Feynman's Operational Calculi. We begin with simple examples.

Example 1. Let $A, B \in \mathcal{L}(X)$ be operators, let μ, ν be finite, continuous Borel measures on the interval $[0, 1]$ associated with the operators A and B, respectively, and let $m_1 := m_2 := 1$. We wish to disentangle the monomial $P^{1,1}(\tilde{A}, \tilde{B})$. The definition gives us

$$
\begin{aligned}
&\mathcal{T}_{\mu,\nu}[P^{1,1}(\tilde{A}, \tilde{B})] = P^{1,1}_{\mu,\nu}(\tilde{A}, \tilde{B}) \\
&= \int_{\{(s_1,s_2)\colon 0<s_1<s_2<1\}} A(s_1)B(s_2)(\mu \times \nu)(ds_1, ds_2) + \int_{\{(s_1,s_2)\colon 0<s_2<s_1<1\}} B(s_2)A(s_1)(\mu \times \nu)(ds_1, ds_2) \\
&= AB \int_{\{(s_1,s_2)\colon 0<s_2<s_1<1\}} (\mu \times \nu)(ds_1, ds_2) + BA \int_{\{(s_1,s_2)\colon 0<s_2<s_1<1\}} (\mu \times \nu)(ds_1, ds_2) \\
&= (\mu \times \nu)\{(s_1, s_2)\colon 0<s_2<s_1<1\}AB + (\mu \times \nu)\{(s_1, s_2)\colon 0<s_2<s_1<1\}BA.
\end{aligned}
\tag{2.24}
$$

The result is a linear combination of products of the operators A and B with coefficients that depend on the measures μ and ν associated with A and B, respectively. In particular, if μ and ν are both Lebesgue measure on $[0, 1]$, then the result of the disentangling is $\frac{1}{2}AB + \frac{1}{2}BA$.

Example 2. For a second example, let $A_1, A_2 \in \mathcal{L}(X)$ be operators, let μ_1, μ_2 be finite, continuous Borel measures on the interval $[0, 1]$ associated with the operators A_1, A_2, respectively, and let $m_1 := 2, m_2 := 1$. Let $\mathbb{D} := \mathbb{D}(\tilde{A}_1, \tilde{A}_2)$. Then assigning the names $C_1 := A_1,\ C_2 := A_1,\ C_3 := A_2$, we have that

$$\begin{aligned}
&\mathcal{T}_{\mu_1,\mu_2}[P^{2,1}(\tilde{A}_1, \tilde{A}_2)] \\
&:= \sum_{\pi \in S_3} \int_{\Delta_m(\pi)} C_{\pi(3)} C_{\pi(2)} C_{\pi(1)}\ (\mu_1^2 \times \mu_2^1)(ds_1, ds_2, ds_3) \\
&= \sum_{\pi \in S_3} \int_{\{(s_1,s_2,s_3)\in[0,1]^3:\ 0<\pi(s_1)<\pi(s_2)<\pi(s_3)<1\}} C_{\pi(3)} C_{\pi(2)} C_{\pi(1)}\ (\mu_1^2 \times \mu_2^1)(ds_1, ds_2, ds_3)
\end{aligned}$$

$$\begin{aligned}
= \quad & \int_{\{s_1<s_2<s_3\}} C_3C_2C_1\ (\mu_1^2 \times \mu_2^1)(ds_1, ds_2, ds_3) + \int_{\{s_1<s_3<s_2\}} C_2C_3C_1\ (\mu_1^2 \times \mu_2^1)(ds_1, ds_2, ds_3) \\
& + \int_{\{s_2<s_1<s_3\}} C_3C_1C_2\ (\mu_1^2 \times \mu_2^1)(ds_1, ds_2, ds_3) + \int_{\{s_2<s_3<s_1\}} C_1C_3C_2\ (\mu_1^2 \times \mu_2^1)(ds_1, ds_2, ds_3) \\
& + \int_{\{s_3<s_1<s_2\}} C_2C_1C_3\ (\mu_1^2 \times \mu_2^1)(ds_1, ds_2, ds_3) + \int_{\{s_3<s_2<s_1\}} C_1C_2C_3\ (\mu_1^2 \times \mu_2^1)(ds_1, ds_2, ds_3)
\end{aligned}$$

$$\begin{aligned}
= \quad & \mu_1^2 \times \mu_2(\{s_1 < s_2 < s_3\}) A_2 A_1^2 + \mu_1^2 \times \mu_2(\{s_1 < s_3 < s_2\}) A_1 A_2 A_1 \\
& + \mu_1^2 \times \mu_2(\{s_2 < s_1 < s_3\}) A_2 A_1^2 + \mu_1^2 \times \mu_2(\{s_2 < s_3 < s_1\}) A_1 A_2 A_1 \\
& + \mu_1^2 \times \mu_2(\{s_3 < s_1 < s_2\}) A_1^2 A_2 + \mu_1^2 \times \mu_2(\{s_3 < s_2 < s_1\}) A_1^2 A_2
\end{aligned}$$

$$\begin{aligned}
= \quad & (\mu_1^2 \times \mu_2)(\{s_3 < s_1 < s_2\} \cup \{s_3 < s_2 < s_1\})\ A_1^2 A_2 \\
& + (\mu_1^2 \times \mu_2)(\{s_1 < s_3 < s_2\} \cup \{s_2 < s_3 < s_1\})\ A_1 A_2 A_1 \\
& + (\mu_1^2 \times \mu_2)(\{s_1 < s_2 < s_3\} \cup \{s_2 < s_1 < s_3\})\ A_2 A_1^2. \qquad (2.25)
\end{aligned}$$

In the case where μ_1 and μ_2 are both Lebesgue measure, the result will equal $\frac{1}{3}A_1^2A_2 + \frac{1}{3}A_1A_2A_1 + \frac{1}{3}A_2A_1^2$.

Example 3. The examples so far have yielded mostly symmetric-looking results (in the sense that the form of the result does not clearly favor one order of the operators versus another order). Next we do an example with specific measures, where the choice of measures clearly affects the outcome and produces something less symmetric. Let $A, B \in \mathcal{L}(X)$ be operators, and let $\mu_1 := l|_{[0,\frac{1}{2}]}, \mu_2 := l|_{[\frac{1}{2},1]}$ be Borel measures on $[0,1]$, where l denotes Lebesgue measure. (That is, $\mu_1(E) = l(E \cap [0, \frac{1}{2}])$ and $\mu_2(E) = l(E \cap [\frac{1}{2}, 1])$ for any Lebesgue measurable set $E \subseteq [0,1]$. These are not probability measures, and the coefficients will not add to 1 when we are finished.) We will also choose exponents $m_1 := 1, m_2 := 1$. Then defining $C_1 := A, C_2 := B$ we have

$$\begin{aligned}
&\mathcal{T}_{\mu_1,\mu_2}[P^{1,1}(\tilde{A}, \tilde{B})] \\
&= \sum_{\pi \in S_2} \int_{\Delta_2(\pi)} C_{\pi(2)} C_{\pi(1)} (\mu_1 \times \mu_2)(ds_1, ds_2) \\
&= \int_{\{s_2 > s_1\}} C_2 C_1 (\mu_1 \times \mu_2)(ds_1, ds_2) + \int_{\{s_1 > s_2\}} C_1 C_2 (\mu_1 \times \mu_2)(ds_1, ds_2) \\
&= C_2 C_1 \int_0^1 \int_0^{s_2} \mu_1(ds_1) \mu_2(ds_2) + C_1 C_2 \int_0^1 \int_{s_2}^1 \mu_1(ds_1) \mu_2(ds_2) \\
&= C_2 C_1 \int_0^1 \int_0^{s_2} l|_{[0,\frac{1}{2}]}(ds_1) l|_{[\frac{1}{2},1]}(ds_2) + C_1 C_2 \int_0^1 \int_{s_2}^1 l|_{[0,\frac{1}{2}]}(ds_1) l|_{[\frac{1}{2},1]}(ds_2) \\
&= C_2 C_1 \int_0^1 \min\{s_2, \tfrac{1}{2}\} l|_{[\frac{1}{2},1]}(ds_2) + C_1 C_2 \int_0^1 \max\{\tfrac{1}{2} - s_2, 0\} l|_{[\frac{1}{2},1]}(ds_2) \\
&= C_2 C_1 \int_{\frac{1}{2}}^1 \min\{s_2, \tfrac{1}{2}\}\, ds_2 + C_1 C_2 \int_{\frac{1}{2}}^1 \max\{\tfrac{1}{2} - s_2, 0\}\, ds_2 \\
&= C_2 C_1 \int_{\frac{1}{2}}^1 \frac{1}{2}\, ds_2 + C_1 C_2 \int_{\frac{1}{2}}^1 0\, ds_2 \\
&= \frac{1}{4} C_2 C_1 = \frac{1}{4} BA. \qquad (2.26)
\end{aligned}$$

Chapter 3

Useful properties for disentangling monomials

3.1 Permuting factors of a product measure

Before we begin to develop ways of manipulating operators and disentangling maps, it would be helpful to establish a few properties of product measures. The first of these ensures what one might expect, that under quite general hypotheses, if we have a complex-valued $\mu\times\nu$-integrable function f defined on a space $X\times Y$, and a function g defined on $Y\times X$, where $g(y,x)=f(x,y)$, and if F is a $\mu\times\nu$-measurable subset of $X\times Y$, then $G:=\{(y,x):\ (x,y)\in F\}$ is a $\nu\times\mu$-measurable subset of $Y\times X$, and we may write

$$\int_F f(x,y)\,d(\mu\times\nu)(x,y)=\int_G g(y,x)\,d(\nu\times\mu)(y,x). \tag{3.1}$$

Moreover, this generalizes to product measures with any finite number of factors. Essentially all we are saying is that if we look at two product spaces that are the same except for the order of the factors, then we can in effect do all the same integrating and get the same results on the two different product spaces, provided we make the corresponding changes in the orders of the coordinates and of the measures. We will prove this using a change of variables theorem, but we first need to describe the relationship between measurable sets in the two product spaces of interest; specifically, we will show that there is a measurable bijection between them which preserves measures.

We consider a product measure space $(X = X_1 \times \cdots \times X_n,\ \mathcal{M} = \mathcal{M}_1 \otimes \cdots \otimes \mathcal{M}_n,\ \mu = \mu_1 \times \cdots \times \mu_n)$ formed from measure spaces $(X_j, \mathcal{M}_j, \mu_j)$ for $j = 1, \ldots, n$, where $\mu_1, \ldots, \mu_n$ are σ-finite positive Borel measures on the respective spaces $X_1, \ldots,$ X_n. (The proof of the main theorem of this section does not require that the measures be Borel measures, but generally we will be considering Borel measures throughout this thesis.) Let $\sigma \in S_n$ (that is, $\sigma : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$ is a bijection) be fixed. Given σ, we will use the letter h to represent various relations involving σ. (The effect of h depends on the choice of σ, but for our purposes we are dealing with only one σ, and hence only one h.) We want to define a map h from the space $(X, \mathcal{M}, \mu)$ to a space we will call $(h(X), \widehat{\mathcal{M}}, \hat{\mu})$, where the measure $\hat{\mu}$ is a product of the same measures as μ, but the factors are permuted. (Note, the 'hat' notation here ['$\widehat{\mathcal{M}}$' and '$\hat{\mu}$'] has nothing to do with Fourier transforms.) Specifically, we define the following:

- Let $h(X) := X_{\sigma(1)} \times \cdots \times X_{\sigma(n)}$.

- For any $x = (x_1, \ldots, x_n) \in X$, define $h(x) := (x_{\sigma(1)}, \ldots, x_{\sigma(n)}) \in h(X)$. It is not difficult to show that this map h is a bijection from X to $h(X)$.

- For any subset $E \subseteq X$, define (as usual) $h(E) := \{h(x) :\ x \in E\}$.

Because h is a bijection, this implies that h and h^{-1} preserve subsets and distribute set differences of sets and over arbitrary intersections and unions of subsets of X and $h(X)$, respectively. (See [34, p. 20, Problems 2, 3].)

- Define $\widehat{\mathcal{M}} := \mathcal{M}_{\sigma(1)} \otimes \cdots \otimes \mathcal{M}_{\sigma(n)}$. (Later we will establish that the collection $\widehat{\mathcal{M}}$ is the same as the collection $h(\mathcal{M}) := \{h(E) : E \in \mathcal{M}\}$.)

- Define the measure $\hat{\mu} := \mu_{\sigma(1)} \times \cdots \times \mu_{\sigma(n)}$. (Later we will argue that $\hat{\mu}(F) = \mu(h^{-1}(F))$ for all sets $F \in \widehat{\mathcal{M}}$, implying that $\hat{\mu} = \mu h^{-1}$, where μh^{-1} is the image measure of μ under h.)

Under those definitions, we will show that $(h(X), \widehat{\mathcal{M}}, \hat{\mu}) = (h(X), h(\mathcal{M}), \mu h^{-1})$ (as noted parenthetically above), in order to establish the following:

Theorem 3.1.1. *If $\mu = \mu_1 \times \cdots \times \mu_n$, where $\mu_1, \ldots, \mu_n$ are σ-finite Borel measures on $X_1, \ldots, X_n$, respectively, and if $f : X = X_1 \times \ldots \times X_n \to \mathbb{C}$ is a μ-integrable function, then $f \circ h^{-1}$ is μh^{-1}-integrable, and*

$$\int_X f \; d\mu = \int_{h(X)} f \circ h^{-1} \; d(\mu h^{-1}). \tag{3.2}$$

Remark 2. Note that since $\mu_1, \ldots, \mu_n$ are σ-finite, the product of the measures is associative, and is also σ-finite.

Proof. We offer only a sketch of the actual proof.

Our main objective in this proof is to establish that the bijection $h : X \to h(X)$ is a measurable map and preserves measures. We will do so using the definition of a product measure space, as applied to both $(X, \mathcal{M}, \mu)$ and $(h(X), \widehat{\mathcal{M}}, \hat{\mu})$.

Examining the definition of the measure space $(h(X), \widehat{\mathcal{M}}, \hat{\mu})$. To recall the definition of a product measure, we quote Folland [11, p. 65], where he has just defined the product of two measures μ and ν:

“The same construction works for any finite number of factors. That is, suppose $(X_j, \mathcal{M}_j, \mu_j)$ are measure spaces for $j = 1, \ldots, n$. If we define a rectangle to be a set of the form $A_1 \times \cdots \times A_n$ with $A_j \in \mathcal{M}_j$, then the collection $\mathcal{A}$ of finite disjoint unions of rectangles is an algebra, and the same procedure as above produces a measure $\mu_1 \times \cdots \times \mu_n$ on $\mathcal{M}_1 \otimes \cdots \otimes \mathcal{M}_n$ such that

$$\mu_1 \times \cdots \times \mu_n(A_1 \times \cdots \times A_n) = \prod_1^n \mu_j(A_j).$$

Moreover, if the μ_j's are σ-finite so that the extension from $\mathcal{A}$ to $\bigotimes_1^n \mathcal{M}_j$ is uniquely determined, the obvious associativity properties hold.”

We use Folland's definition of the algebra $\mathcal{A}$ (from which $\mathcal{M}$ is generated), and then we construct the measure space $(h(X), \widehat{\mathcal{M}}, \hat{\mu})$ similarly to $(X, \mathcal{M}, \mu)$. We start by defining $\widehat{\mathcal{A}}$ to be the collection of all finite disjoint unions of ‘measurable rectangles’ in $\mathcal{M}$ of the form $A_{\sigma(1)} \times \cdots \times A_{\sigma(n)}$, with sets $A_j \in \mathcal{M}_j$ for all j. The collection $\widehat{\mathcal{A}}$ is then an algebra.

Let $h(\mathcal{A}) := \{h(E) : E \in \mathcal{A}\}$. Since (one can show) each measurable rectangle $A_{\sigma(1)} \times \cdots \times A_{\sigma(n)}$ in $\widehat{\mathcal{A}}$ is the image under h of a measurable rectangle $A_1 \times \cdots \times A_n$ in $\mathcal{A}$, and h preserves disjoint unions, we have that $\widehat{\mathcal{A}} \subseteq h(\mathcal{A})$. Similar reasoning gives us the reverse inclusion, and thus $\widehat{\mathcal{A}} = h(\mathcal{A})$, so in what follows we will use the designation $h(\mathcal{A})$ for this collection of sets.

The collection $h(\mathcal{A})$ is thus an algebra, which by the process described above produces the measure $\hat{\mu} := \mu_{\sigma(1)} \times \cdots \times \mu_{\sigma(n)}$ on the space $\widehat{\mathcal{M}} = \mathcal{M}_{\sigma(1)} \otimes \cdots \otimes \mathcal{M}_{\sigma(n)}$, where

$$\mu_{\sigma(1)} \times \cdots \times \mu_{\sigma(n)}(A_{\sigma(1)} \times \cdots \times A_{\sigma(n)}) = \prod_{1}^{n} \mu_{\sigma(j)}(A_{\sigma(j)}) \tag{3.3}$$

for every measurable rectangle $A_{\sigma(1)} \times \cdots \times A_{\sigma(n)} \in \mathcal{M}_{\sigma(1)} \times \cdots \times \mathcal{M}_{\sigma(n)}$.

Comparing the measures μ and $\hat{\mu}$ on their respective algebras, and comparing their related outer measures. With this definition of the measure $\hat{\mu}$, given any measurable rectangle $B = B_1 \times \cdots \times B_n \in \mathcal{A}$, with $B_j \in \mathcal{M}_j$ for $j = 1, \ldots, n$, we have

$$\begin{aligned}
\mu(B) &= \mu_1 \times \cdots \times \mu_n(B_1 \times \cdots \times B_n) \\
&= \prod_{1}^{n} \mu_j(B_j) \\
&= \prod_{1}^{n} \mu_{\sigma(j)}(B_{\sigma(j)}) \\
&= \mu_{\sigma(1)} \times \cdots \times \mu_{\sigma(n)}(B_{\sigma(1)} \times \cdots \times B_{\sigma(n)}) \\
&= \hat{\mu}(h(B)). \qquad (3.4)
\end{aligned}$$

The measures μ and $\hat{\mu}$ therefore agree on corresponding measurable rectangles in their respective algebras $\mathcal{A}$ and $h(\mathcal{A})$, and therefore on each pair of corresponding sets in the two algebras (since h and h^{-1} preserve unions and by finite additivity of the measures). The extension of the algebra $h(\mathcal{A})$ to the σ-algebra $\widehat{\mathcal{M}}$ is uniquely determined, because the measures $\mu_1, \ldots, \mu_n$ here are taken to be σ-finite (and so we may refer to "the" σ-algebra $\widehat{\mathcal{M}}$). We now intend to show that the product measure $\hat{\mu}$ applied to sets in that σ-algebra $\widehat{\mathcal{M}}$ agrees with the product measure μ applied to corresponding sets in the σ-algebra $\mathcal{M}$. That is, we will show that for any μ-measurable set E (that is, for any $E \in \mathcal{M}$) we have that $h(E) \in \widehat{\mathcal{M}}$ and $\mu(E) = \hat{\mu}(h(E))$. We will use the method for extending a pre-measure to a measure that is described in Folland [11, pp.30-31].

Using the measures μ on X and $\hat{\mu}$ on $h(X)$ as pre-measures on the algebras $\mathcal{A}$ and $\widehat{\mathcal{A}}$, respectively, they induce outer measures μ^* on X and $\hat{\mu}^*$ on $h(X)$, given by

$$\mu^*(E) = \inf\left\{\sum_1^\infty \mu(C_i) :\ C_i \in \mathcal{A},\ E \subseteq \bigcup_1^\infty C_i\right\} \tag{3.5}$$

and

$$\hat{\mu}^*(F) = \inf\left\{\sum_1^\infty \hat{\mu}(D_i) :\ D_i \in h(\mathcal{A}),\ F \subseteq \bigcup_1^\infty D_i\right\} \tag{3.6}$$

for all subsets $E \subseteq X$ and $F \subseteq h(X)$. One can show that set relationships involving the map h together with the equalities we have established for corresponding sets in the algebras $\mathcal{A}$ and $\widehat{\mathcal{A}}$ imply that for any $E \subseteq X$ we have $\hat{\mu}^*(h(E)) = \mu^*(E)$.

Now, the restriction of the outer measure μ^* to the σ-algebra $\mathcal{M}$ generated by $\mathcal{A}$ and the restriction of the outer measure $\hat{\mu}^*$ to the σ-algebra $\widehat{\mathcal{M}}$ generated by $h(\mathcal{A})$ are measures ([11, Theorem 1.14, p. 31]), which we have called μ and $\hat{\mu}$, respectively. Therefore, the measures μ on $\mathcal{M}$ and $\hat{\mu}$ on $\widehat{\mathcal{M}}$ agree on corresponding sets E and $h(E)$ provided $E \in \mathcal{M}$ and $h(E) \in \widehat{\mathcal{M}}$. To finally establish the truth of Theorem 3.1.1, we would like to be able to say that a set in $\mathcal{M}$ always corresponds to a set in $\widehat{\mathcal{M}}$ under the map h—both that $h(\mathcal{M}) \subseteq \widehat{\mathcal{M}})$ and that $h^{-1}(\widehat{\mathcal{M}}) \subseteq \mathcal{M}$—equivalently, that $h(\mathcal{M}) = \widehat{\mathcal{M}}$.

Demonstrating the equality of σ-algebras $h(\mathcal{M}) = \widehat{\mathcal{M}}$. We claim that a set $E \subseteq X$ is in the σ-algebra $\mathcal{M} = \mathcal{M}_1 \otimes \cdots \otimes \mathcal{M}_n$ if and only if $h(E)$ is in the σ-algebra $\widehat{\mathcal{M}} = \mathcal{M}_{\sigma(1)} \otimes \cdots \otimes \mathcal{M}_{\sigma(n)}$. Since $\mathcal{M}$ is the σ-algebra generated by $\mathcal{A}$, $\mathcal{M}$ is the intersection of all σ-algebras containing $\mathcal{A}$;

$$\mathcal{M} := \bigcap\{\mathcal{S} :\ \mathcal{S} \text{ is a } \sigma\text{-algebra on } X \text{ with } \mathcal{A} \subseteq \mathcal{S}\}. \tag{3.7}$$

Similarly, $\widehat{\mathcal{M}}$ is the σ-algebra generated by $h(\mathcal{A})$;

$$\widehat{\mathcal{M}} := \bigcap \{\mathcal{S}' : \ \mathcal{S}' \text{ is a } \sigma\text{-algebra on } h(X) \text{ with } h(\mathcal{A}) \subseteq \mathcal{S}'\}. \tag{3.8}$$

We hope to show that $E \in \mathcal{M}$ if and only if $h(E) \in \widehat{\mathcal{M}}$, and for that we first note that any collection $\mathcal{S}$ is a σ-algebra on X if and only if $h(\mathcal{S})$ is a σ-algebra on $h(X)$. Similarly, $\mathcal{S}'$ is a σ-algebra on $h(X)$ if and only if $h^{-1}(\mathcal{S}')$ is a σ-algebra on $h^{-1}(h(X)) = X$. (We will not demonstrate these; they follow from the preservation of set relationships by h and h^{-1}.)

We may therefore rewrite the definition of $\widehat{\mathcal{M}}$, the σ-algebra generated by $h(\mathcal{A})$, letting $\mathcal{S} := h^{-1}(\mathcal{S}')$ (so $h(\mathcal{S}) = h(h^{-1}(\mathcal{S}')) = \mathcal{S}'$), as

$$\begin{aligned} \widehat{\mathcal{M}} &:= \bigcap \{h(\mathcal{S}) : \ h(\mathcal{S}) \text{ is a } \sigma\text{-algebra on } h(X) \text{ with } h(\mathcal{A}) \subseteq h(\mathcal{S})\} \\ &= \bigcap \{h(\mathcal{S}) : \ \mathcal{S} \text{ is a } \sigma\text{-algebra on } X \text{ with } \mathcal{A} \subseteq \mathcal{S}\}, \end{aligned} \tag{3.9}$$

because $h(\mathcal{A}) \subseteq h(\mathcal{S})$ is equivalent to $\mathcal{A} \subseteq \mathcal{S}$, and then, taking h outside the intersection,

$$\begin{aligned} \widehat{\mathcal{M}} &= h\left(\bigcap \{\mathcal{S} : \ \mathcal{S} \text{ is a } \sigma\text{-algebra on } X \text{ with } \mathcal{A} \subseteq \mathcal{S}\}\right) \\ &= h(\mathcal{M}). \end{aligned} \tag{3.10}$$

This yields $\widehat{\mathcal{M}} = h(\mathcal{M})$, as we claimed. Moreover, $h^{-1}(\widehat{\mathcal{M}}) = h^{-1}(h(\mathcal{M})) = \mathcal{M}$.

Measurability of h and h^{-1}, demonstration that $\hat{\mu}$ is an image measure, and integration. Because of this result, we may now refer to $(h(X), \widehat{\mathcal{M}}, \hat{\mu})$ as $(h(X), h(\mathcal{M}), \hat{\mu})$. Also as a consequence, given any set $E \in \mathcal{M}$ we have that $h(E) \in h(\mathcal{M})$, so h^{-1} is a measurable function. Similarly, h is a measurable function.

Therefore, we have the agreement of measures that we wanted, namely that given any set $E \in \mathcal{M}$, we have $h(E) \in \widehat{\mathcal{M}}$, and hence $\mu(E) = \mu^*(E) = \hat{\mu}^*(h(E)) = \hat{\mu}(h(E))$, so $\mu(E) = \hat{\mu}(h(E))$; moreover, given any set $F \in h(\mathcal{M})$, we have that $F = h(E)$ for some $E \in \mathcal{M}$, and therefore $\hat{\mu}(F) = \hat{\mu}(h(E)) = \mu(E) = \mu(h^{-1}(F))$. The latter fact tells us that $\hat{\mu}$ is the image measure of μ under h, as we claimed earlier. That is, $\hat{\mu} = \mu h^{-1}$. Thus we may write $h : (X, \mathcal{M}, \mu) \to (h(X), h(\mathcal{M}), \mu h^{-1})$. (We will therefore refer to μh^{-1} rather than $\hat{\mu}$ in what follows.)

Now we are ready to address integration. We will appeal to the following change-of-variable theorem [8, p. 82], where μh^{-1} refers to the image measure of μ under h:

Theorem 3.1.2 (Change of variables)**.** *Let $(X, \mathcal{M}, \mu)$ be a measure space, let $(Y, \mathcal{N})$ be a measurable space, and let $h : (X, \mathcal{M}) \to (Y, \mathcal{N})$ be measurable. Let g be an extended real-valued $\mathcal{N}$-measurable function on Y. Then g is μh^{-1}-integrable if and only if $g \circ h$ is μ-integrable. If these functions are integrable, then*

$$\int_X (g \circ h)\, d\mu = \int_Y g\, d(\mu h^{-1}). \tag{3.11}$$

Continuing the proof of Theorem 3.1.1, we consider a μ-integrable function $f : X \to \mathbb{R}$, and we look at the integral $\int_X f\, d\mu$. In order to apply Theorem 3.1.2, we let $Y := h(X)$, and $\mathcal{N} := h(\mathcal{M})$. We observe that $(X, \mathcal{M}, \mu)$ is a measure space. We have shown that $(h(X), h(\mathcal{M}))$ is a measurable space and that $h : (X, \mathcal{M}) \to (h(X), h(\mathcal{M}))$ is a measurable function. Now define the function $g : Y = h(X) \to \mathbb{R}$ by $g = f \circ h^{-1}$. Then given any Borel-measurable set $F \in \mathbb{R}$, we have $f^{-1}(F) \in \mathcal{M}$ (since f is assumed integrable on X, and hence measurable). Thus $h(f^{-1}(F)) \in h(\mathcal{M})$; that is, $(f \circ h^{-1})^{-1}(F) \in h(\mathcal{M})$, and therefore $g = f \circ h^{-1}$ is $h(\mathcal{M})$-measurable. Also, by hypothesis, $g \circ h = f \circ h^{-1} \circ h = f$ is μ-integrable.

We may therefore apply Theorem 3.1.2, and the result is that

$$\int_X f\ d\mu = \int_{h(X)} f \circ h^{-1}\ d(\mu h^{-1}). \tag{3.12}$$

We therefore have the desired result in the case f is real-valued. The case of complex-valued f follows from that without much difficulty, establishing Theorem 3.1.1. □

Corollary 3.1.3. *Using the same notation as in the previous theorem for a given $\sigma \in S_n$, if $\mu = \mu_1 \times \cdots \times \mu_n$, where $\mu_1, \ldots, \mu_n$ are σ-finite measures on $X_1, \ldots, X_n$, respectively, if $E \subseteq X$ is a μ-measurable subset of $X = X_1 \times \cdots \times X_n$, and if $f : X \to \mathbb{C}$ is a μ-integrable function, then $f \circ h^{-1}$ is μh^{-1}-integrable, and*

$$\int_E f\ d\mu = \int_{h(E)} f \circ h^{-1}\ d(\mu h^{-1}). \tag{3.13}$$

(Again, because $\mu_1, \ldots, \mu_n$ are σ-finite, the product of the measures is associative.)

Proof. We note first that since E is a μ-measurable set and h^{-1} is measurable (as shown in the proof of Theorem 3.1.1), $h(E)$ is a μh^{-1}-measurable set, so we may integrate over it. Second, we can say of the product function $f\chi_E$ (recall χ_E is the characteristic function of the set E) that for all $x \in X$,

$$\begin{aligned}[(f\chi_E) \circ h^{-1}](h(x)) &= (f\chi_E)(x) = f(x)\chi_E(x) \\ &= (f \circ h^{-1})(h(x))\chi_{h(E)}(h(x)) = [(f \circ h^{-1})\chi_{h(E)}](h(x)),\end{aligned} \tag{3.14}$$

and hence $(f\chi_E) \circ h^{-1} = (f \circ h^{-1})\chi_{h(E)}$. Therefore, applying Theorem 3.1.1 to $f\chi_E$,

$$\int_E f\ d\mu = \int_X f\chi_E\ d\mu$$

$$= \int_{h(X)} (f\chi_E) \circ h^{-1} \; d(\mu h^{-1})$$
$$= \int_{h(X)} (f \circ h^{-1})\chi_{h(E)} \; d(\mu h^{-1})$$
$$= \int_{h(E)} f \circ h^{-1} \; d(\mu h^{-1}), \tag{3.15}$$

establishing the corollary. □

We will now define a convenient notation for using the above theorems in connection with Feynman's Operational Calculi.

Definition 3.1.4. Given the product measure space ($X = X_1 \times \cdots \times X_n$, $\mathcal{M} = \mathcal{M}_1 \otimes \cdots \otimes \mathcal{M}_n$, $\mu = \mu_1 \times \cdots \times \mu_n$) formed from measure spaces $(X_j, \mathcal{M}_j, \mu_j)$ for $j = 1, \ldots, n$, where $\mu_1, \ldots, \mu_n$ are σ-finite (positive) Borel measures on $X_1, \ldots, X_n$, respectively, and given a fixed permutation $\sigma \in S_m$, we make the following definitions:

- We define the product space $X^\sigma := X_{\sigma(1)} \times \cdots \times X_{\sigma(n)}$.
- Given $x = (x_1, \ldots, x_n) \in X$, we define a map from the space X to the space X^σ by $x^\sigma := (x_{\sigma(1)}, \ldots, x_{\sigma(n)})$.
- Given any subset $E \subseteq X$ we define $E^\sigma := \{x^\sigma : \; x \in E\} \subseteq X^\sigma$.
- We define $\mu^\sigma := \mu_{\sigma(1)} \times \cdots \times \mu_{\sigma(n)}$. (Thus μ^σ is identical to $\hat{\mu}$ and μh^{-1} defined earlier.)
- Given a function $f : X \to \mathbb{C}$, we define the function $f^\sigma : X^\sigma \to \mathbb{C}$ by $f^\sigma(y) := f(y^{(\sigma^{-1})})$. (Thus we have $f^\sigma(x^\sigma) = f((x^\sigma)^{(\sigma^{-1})}) = f(x)$ for all $x \in X$. Note that this f^σ is identical to $f \circ h^{-1}$ that was defined earlier.)

Using this notation, we may write Corollary 3.1.3 in the following form:

Corollary 3.1.5 (Integration in a permuted product space)**.** *Given any product measure space* $(X = X_1 \times \cdots \times X_n, \mathcal{M} = \mathcal{M}_1 \otimes \cdots \otimes \mathcal{M}_n, \mu = \mu_1 \times \cdots \times \mu_n)$*, where* $\mu_1, \ldots, \mu_n$ *are* σ*-finite Borel measures on* $X_1, \ldots, X_n$*, respectively, and given any permutation* $\sigma \in S_n$*, if* $E \subseteq X$ *is a* μ*-measurable set, and if* $f : X \to \mathbb{C}$ *is a* μ*-integrable function, then* f^σ *is* μ^σ*-integrable, and*

$$\int_E f \; d\mu = \int_{E^\sigma} f^\sigma \; d\mu^\sigma, \tag{3.16}$$

or equivalently,

$$\int_E f(x) \; d\mu(x) = \int_{E^\sigma} f(x) \; d\mu^\sigma(x^\sigma). \tag{3.17}$$

(Again, because $\mu_1, \ldots, \mu_n$ *are* σ*-finite, the product of the measures is associative.)*

Proof. To prove this, we simply define the function $h : X \to X^\sigma$ as before. This gives $h(x) = x^\sigma$ for all $x \in X$, $h(E) = E^\sigma$, $f \circ h^{-1} = f^\sigma$, and $\mu h^{-1} = \mu^\sigma$. Making these substitutions establishes Equation (3.16) directly. From there we have

$$\int_E f(x) \; d\mu(x) = \int_{E^\sigma} f^\sigma \; d\mu^\sigma = \int_{E^\sigma} f^\sigma(x^\sigma) \; d\mu^\sigma(x^\sigma) = \int_{E^\sigma} f(x) \; d\mu^\sigma(x^\sigma), \tag{3.18}$$

and the corollary is proved. □

Remark 3. For the applications we will have for Corollary 3.1.5, we will be integrating operators. Since, however, the operators are time-independent, they may be factored out of the integral, and then back in, for a similar result. That is, given any operator $A \in \mathcal{L}(Y)$ for a Banach space Y and the conditions of Corollary 3.1.5, we have that

$$\begin{aligned} \int_E f(x)A \; d\mu(x) &= \int_E f(x) \; d\mu(x)A \\ &= \int_{E^\sigma} f(x) \; d\mu^\sigma(x^\sigma)A \end{aligned}$$

$$= \int_{E^\sigma} f(x)A \; d\mu^\sigma(x^\sigma). \tag{3.19}$$

Example 4. Consider the set $E := \{(s_1, s_2) : 1 > s_1 > s_2 > 0\}$. Let the permutation $\sigma \in S_2$ be given by $\sigma(1) = 2, \sigma(2) = 1$. We then have that $(s_1, s_2)^\sigma = (s_2, s_1)$ for all $s_1, s_2 \in [0, 1]$, and hence

$$\begin{aligned} E^\sigma &= \{(s_1, s_2) : 1 > s_1 > s_2 > 0\}^\sigma \\ &= \{(s_1, s_2)^\sigma \in [0,1]^2 : \; (s_1, s_2) \in E\} \\ &= \{(s_2, s_1) \in [0,1]^2 : \; 1 > s_1 > s_2 > 0\}. \end{aligned} \tag{3.20}$$

Given an integrable function $f : [0, 1] \times [0, 1] \to \mathbb{C}$ and σ-finite Borel measures μ, ν on $[0, 1]$, we therefore have by Corollary 3.1.5 that

$$\begin{aligned} \int_{\{s_1 > s_2\}} f(s_1, s_2) \; d(\mu \times \nu)(s_1, s_2) &= \int_{\{(s_1,s_2):\, 1>s_1>s_2>0\}} f(s_1, s_2) \; d(\mu \times \nu)(s_1, s_2) \\ &= \int_{\{(s_1,s_2)\in[0,1]^2:\, 1>s_1>s_2>0\}^\sigma} f(s_1, s_2) \; d(\mu \times \nu)^\sigma(s_1, s_2)^\sigma \\ &= \int_{\{(s_2,s_1)\in[0,1]^2:\, 1>s_1>s_2>0\}} f(s_1, s_2) \; d(\nu \times \mu)(s_2, s_1) \\ &= \int_{\{s_1 > s_2\}} f(s_1, s_2) \; d(\nu \times \mu)(s_2, s_1). \end{aligned} \tag{3.21}$$

Note that in the last expression, integration is with respect to the measure $\nu\times\mu$, which means the integrand should be a $\nu\times\mu$-measurable function, whereas the function f is defined to be $\mu \times \nu$-measurable, not $\nu \times \mu$-measurable. However, we can view the set of function values $f(s_1, s_2)$ as being images of a $\nu\times\mu$-measurable function, namely the function f^σ, since $f(s_1, s_2) = f^\sigma(s_2, s_1)$ for all values of s_1, s_2; thus the last integral makes sense. Also note that we have introduced a small change in the meaning of '$\{s_1 > s_2\}$'; in the first integral it refers to $\{(s_1, s_2) \in [0, 1]^2 : \; 1 > s_1 > s_2 > 0\}$, whereas in the last integral it refers to $\{(s_2, s_1) \in [0,1]^2 : \ 1 > s_1 > s_2 > 0\}$. However, the context makes this clear, since the product measure $\mu \times \nu$ in the first integral expression (defined on $[0,1] \times [0,1]$) is applied to ordered pairs (s_1, s_2), while the product measure $\nu \times \mu$ in the last integral expression (defined on $[0,1] \times [0,1]$) is applied to ordered pairs (s_2, s_1).

Theorem 3.1.6. *Given any permutations* $\pi, \sigma \in S_m$, $m \geq 1$, *we have* $[\Delta_m(\pi)]^\sigma = \Delta_m(\sigma^{-1}\pi)$. *In particular, if* $e_m \in S_m$ *is the identity permutation, then* $[\Delta_m(\pi)]^\pi = \Delta_m(e_m)$.

Proof. Let $\pi, \sigma \in S_m$, $m \geq 1$. Then

$$
\begin{aligned}
[\Delta_m(\pi)]^\sigma &= \{(s_1, \ldots, s_m) \in [0,1]^m : \ 0 < s_{\pi(1)} < \cdots < s_{\pi(m)} < 1\}^\sigma \\
&= \{(s_1, \ldots, s_m)^\sigma \in [0,1]^m : \ 0 < s_{\pi(1)} < \cdots < s_{\pi(m)} < 1\} \\
&= \{(s_{\sigma(1)}, \ldots, s_{\sigma(m)}) \in [0,1]^m : \ 0 < s_{\pi(1)} < \cdots < s_{\pi(m)} < 1\}. \qquad (3.22)
\end{aligned}
$$

We rename the variables, replacing $s_{\sigma(1)}, \ldots, s_{\sigma(m)}$ with $s_1, \ldots, s_m$, respectively, which is to say that we apply the permutation σ^{-1} to every subscript in the last expression above. We obtain

$$
\begin{aligned}
[\Delta_m(\pi)]^\sigma &= \{(s_1, \ldots, s_m) : \ 0 < s_{\sigma^{-1}(\pi(1))} < \cdots < s_{\sigma^{-1}(\pi(m))} < 1\} \\
&= \Delta_m(\sigma^{-1}(\pi)). \qquad (3.23)
\end{aligned}
$$

This gives us the first statement of the theorem, and the second statement follows immediately. $\square$

Theorem 3.1.7 (Composing permutations of a product measure)**.** *Given any permutations* $\pi, \sigma \in S_m, m \geq 1$, *and given any measures* $\nu_1, \ldots, \nu_m \in \mathcal{M}_{cb}[0,1]$, *we* *have*

$$(\nu_{\sigma(1)} \times \cdots \times \nu_{\sigma(m)})^{\pi} = \nu_{\sigma\pi(1)} \times \cdots \times \nu_{\sigma\pi(m)}; \tag{3.24}$$

that is,

$$[(\nu_1 \times \cdots \times \nu_m)^{\sigma}]^{\pi} = (\nu_1 \times \cdots \times \nu_m)^{\sigma\pi}. \tag{3.25}$$

Proof. Letting $\mu_j := \nu_{\sigma(j)}$ for $j = 1, \ldots, m$, we have

$$\begin{aligned}(\nu_{\sigma(1)} \times \cdots \times \nu_{\sigma(m)})^{\pi} &= (\mu_1 \times \cdots \times \mu_m)^{\pi} \\ &= \mu_{\pi(1)} \times \cdots \times \mu_{\pi(m)} \\ &= \nu_{\sigma\pi(1)} \times \cdots \times \nu_{\sigma\pi(m)}.\end{aligned} \tag{3.26}$$

□

By similar reasoning we can establish for points $x \in X_1 \times \cdots \times X_m$ and sets $E \subseteq X_1 \times \cdots \times X_m$ that $(x^{\sigma})^{\pi} = x^{\sigma\pi}$ and $(E^{\sigma})^{\pi} = E^{\sigma\pi}$.

One thing Corollary 3.1.3 allows us to do is to rewrite the definition of the disentangling of a monomial (Definition 2.0.3) in another useful way:

Theorem 3.1.8 (Alternate definition of disentangling a monomial)**.** *Given* $\mathbb{D} = \mathbb{D}(A_1, \ldots, A_n)$*, and given* $\mu_1, \mu_2, \ldots, \mu_n \in \mathcal{M}_{cb}[0,1]$ *together with nonnegative integers* $m_1, \ldots, m_n$*, let blocks of integers* $Bl(1), \ldots, Bl(n)$ *and operators* $C_1, \ldots, C_m$ *be defined in the usual way, as well as measures*

$$\nu_k := \begin{cases} \mu_1, & k \in Bl(1) \\ \mu_2, & k \in Bl(2) \\ \vdots & \\ \mu_n, & k \in Bl(n), \end{cases} \tag{3.27}$$

so that the measures $\nu_1, \dots, \nu_m$ *are associated with the operators* $C_1, \dots, C_m$*, respectively. We then have*

$$\begin{aligned}
&\mathcal{T}_{\mu_1,\dots,\mu_n}[P^{m_1,\dots,m_n}(\tilde{A}_1,\dots,\tilde{A}_n)] = \\
&= \sum_{\pi \in S_m} \int_{\Delta_m(e_m)} C_{\pi(m)} \cdots C_{\pi(1)} \, (\nu_{\pi(1)} \times \cdots \times \nu_{\pi(m)})(ds_{\pi(1)}, \dots, ds_{\pi(m)}), \qquad (3.28)
\end{aligned}$$

where $e_m \in S_m$ *is the identity permutation, so that the set* $\Delta_m(e_m)$ *is given by* $\Delta_m(e_m) = \{(s_1, \dots, s_m) : \; 0 < s_1 < \cdots < s_m < 1\} = \{(s_{\pi(1)}, \dots, s_{\pi(m)}) : \; 0 < s_{\pi(1)} < \cdots < s_{\pi(m)} < 1\}$.

Proof. Under the given hypotheses, we have by Corollary 3.1.5 that

$$\begin{aligned}
&\mathcal{T}_{\mu_1,\dots,\mu_n}[P^{m_1,\dots,m_n}(\tilde{A}_1,\dots,\tilde{A}_n)] \\
&= P^{m_1,\dots,m_n}_{\mu_1,\dots,\mu_n}(A_1,\dots,A_n) = P^{1,\dots,1}_{\nu_1,\dots,\nu_m}(C_1,\dots,C_m) \\
&= \sum_{\pi \in S_m} \int_{\Delta_m(\pi)} C_{\pi(m)} \cdots C_{\pi(1)} \, (\nu_1 \times \cdots \times \nu_m)(ds_1, \dots, ds_m) \\
&= \sum_{\pi \in S_m} \int_{[\Delta_m(\pi)]^\pi} C_{\pi(m)} \cdots C_{\pi(1)} \, (\nu_1 \times \cdots \times \nu_m)^\pi(ds_1, \dots, ds_m)^\pi \\
&= \sum_{\pi \in S_m} \int_{\Delta_m(e_m)} C_{\pi(m)} \cdots C_{\pi(1)} \, (\nu_{\pi(1)} \times \cdots \times \nu_{\pi(m)})(ds_{\pi(1)}, \dots, ds_{\pi(m)}). \qquad (3.29)
\end{aligned}$$

□

A result of this theorem is the following corollary, which we will have use for later:

Corollary 3.1.9. *Given* $\mathbb{D} = \mathbb{D}(A_1, \dots, A_n)$*, where* $A_1, \dots, A_n \in \mathcal{L}(X)$*, and given finite, continuous Borel measures* $\mu_1, \dots, \mu_n$ *on the interval* $[0, 1]$*, together with non-negative integers* $m_1, \dots, m_n$*, we have for any permutation* $\sigma \in S_n$ *that*

$$P^{m_1,\dots,m_n}_{\mu_1,\dots,\mu_n}(A_1,\dots,A_n) = P^{m_{\sigma(1)},\dots,m_{\sigma(n)}}_{\mu_{\sigma(1)},\dots,\mu_{\sigma(n)}}(A_{\sigma(1)},\dots,A_{\sigma(n)}). \qquad (3.30)$$

Remark 4. The result in Corollary 3.1.9 is given in [13, Proposition 2.11, p. 14] for the case of probability measures. The proposition is stated there without proof, introduced by the statement that it follows directly from the definition of the disentangling map. Were the article's definition the one we have just discussed here in Theorem 3.1.8, it would indeed follow directly, but it does not seem to follow so directly from the original definition of the disentangling map, which our Definition 2.0.3 matches. Perhaps the authors simply recognized Corollary 3.1.5 intuitively as an application of a change of variables theorem. Nevertheless, it seems good in the current section of this thesis to attempt to outline a justification of this proposition.

Proof. Assigning the same names as before for $C_1, \ldots, C_m$ and $\nu_1, \ldots, \nu_m$, we may take any fixed permutation $\tau \in S_m$ and, letting $e \in S_m$ be the identity permutation, say using Theorems 3.1.7 and 3.1.8 (and then just changing the index of summation) that

$$\begin{aligned}
&P^{1,\ldots,1}_{\nu_{\tau(1)},\ldots,\nu_{\tau(m)}}(C_{\tau(1)}, \ldots, C_{\tau(m)}) \\
&= \sum_{\pi \in S_m} \int_{\Delta_m(e_m)} C_{\tau\pi(m)} \cdots C_{\tau\pi(1)} \, (\nu_{\tau\pi(1)} \times \cdots \times \nu_{\tau\pi(m)})(ds_{\tau\pi(1)}, \ldots, ds_{\tau\pi(m)}) \\
&= \sum_{\tau\pi \in S_m} \int_{\Delta_m(e_m)} C_{\tau\pi(m)} \cdots C_{\tau\pi(1)} \, (\nu_{\tau\pi(1)} \times \cdots \times \nu_{\tau\pi(m)})(ds_{\tau\pi(1)}, \ldots, ds_{\tau\pi(m)}). \quad (3.31)
\end{aligned}$$

Letting $\rho := \tau\pi$ each time we select $\tau\pi \in S_m$ gives

$$\begin{aligned}
&P^{1,\ldots,1}_{\nu_{\tau(1)},\ldots,\nu_{\tau(m)}}(C_{\tau(1)}, \ldots, C_{\tau(m)}) \\
&= \sum_{\rho \in S_m} \int_{\Delta_m(e_m)} C_{\rho(m)} \cdots C_{\rho(1)} \, (\nu_{\rho(1)} \times \cdots \times \nu_{\rho(m)})(ds_{\rho(1)}, \ldots, ds_{\rho(m)}) \\
&= P^{1,\ldots,1}_{\nu_1,\ldots,\nu_m}(C_1, \ldots, C_m). \quad (3.32)
\end{aligned}$$

In particular, any rearrangement of the A_j's corresponds to a rearrangement of the

C_k's, so for any permutation $\sigma \in S_n$ there is some permutation $\tau \in S_m$ for which the list $C_{\tau(1)}, \ldots, C_{\tau(m)}$ has every $A_{\sigma(1)}$ preceding every $A_{\sigma(2)}$, preceding every $A_{\sigma(3)}$, etc., and similarly for the measures. We can then say

$$\begin{aligned} P^{m_1,\ldots,m_n}_{\mu_1,\ldots,\mu_n}(A_1, \ldots, A_n) &= P^{1,\ldots,1}_{\nu_1,\ldots,\nu_m}(C_1, \ldots, C_m) \\ &= P^{1,\ldots,1}_{\nu_{\tau(1)},\ldots,\nu_{\tau(m)}}(C_{\tau(1)}, \ldots, C_{\tau(m)}) \\ &= P^{m_{\sigma(1)},\ldots,m_{\sigma(n)}}_{\mu_{\sigma(1)},\ldots,\mu_{\sigma(n)}}(A_{\sigma(1)}, \ldots, A_{\sigma(n)}). \end{aligned} \tag{3.33}$$

(Note: The way permutations are handled in this proof is a technique that will be used again, such as in Remark 24, Section 6.2.) □

3.2 Disentangling a monomial that involves a sum of two measures

Besides the sort of "commutativity" properties we have just shown for product measures, we would like to establish what is effectively a distributive law for measures in the context of disentangling a monomial. For example, we would like to calculate disentanglings of the form $P^{1,m_2,\ldots,m_n}_{\nu+\eta,\mu_2,\ldots,\mu_n}(A_1, \ldots, A_n)$. (Later, in Theorem 4.3.7, we will consider a distributive law for disentangling a monomial involving a sum of operators, $P^{1,m_2,\ldots,m_n}_{\mu_1,\mu_2,\ldots,\mu_n}(B + C, A_2, \ldots, A_n)$.)

To do so, we begin by noting two straightforward results. The first is that the sum of two measures is a measure: If μ and ν are (positive) measures on the measurable space $(X, \mathcal{M})$, and the function $\mu + \nu : \mathcal{M} \to [0, \infty]$ is defined by $(\mu + \nu)(E) = \mu(E) + \nu(E)$ for all $E \in \mathcal{M}$, then $\mu + \nu$ is a measure on $(X, \mathcal{M})$.

The second is a distributive law that holds for product measures: If μ is a σ-finite (positive) measure on $(X_1, \mathcal{M}_1)$ and ν, η are σ-finite (positive) measures on $(X_2, \mathcal{M}_2)$, then

$$\mu \times (\nu + \eta) = \mu \times \nu \ + \ \mu \times \eta. \tag{3.34}$$

For the rest of this thesis we will assume X is a Banach space unless otherwise stated. Using the above facts, we are able to prove the following disentangling theorem:

Theorem 3.2.1 (Distributive law for disentangling a monomial that involves a sum of two measures)**.** *Let $A_1, A_2, \ldots, A_n \in \mathcal{L}(X)$, and let $m_2, \ldots, m_n$ be non-negative integers. Then given finite, continuous Borel measures $\nu, \eta, \mu_2, \mu_3, \ldots, \mu_n$ on the interval $[0,1]$ associated with the operators $A_1, A_1, A_2, A_3, \ldots, A_n$, respectively, we have that*

$$P^{1,m_2,\ldots,m_n}_{\nu+\eta,\mu_2,\ldots,\mu_n}(A_1,\ldots,A_n) = P^{1,m_2,\ldots,m_n}_{\nu,\mu_2,\ldots,\mu_n}(A_1,\ldots,A_n) + P^{1,m_2,\ldots,m_n}_{\eta,\mu_2,\ldots,\mu_n}(A_1,\ldots,A_n). \tag{3.35}$$

Remark 5. Since, when disentangling a monomial, we may by Corollary 3.1.9 permute the operators and correspondingly permute the measures and exponents without changing the value of the expression, the theorem will hold as well if the sum of measures appears later in the list of measures, with an exponent 1 corresponding to them.

Proof. We start by assigning blocks of integers almost the same as before: Let $m := 1 + m_1 + m_2 + \cdots + m_n$, and let

$$\begin{aligned} \mathrm{Bl}(1) &:= \{1\}, \\ \mathrm{Bl}(2) &:= \{2, 3, \ldots, 1 + m_2\}, \\ &\vdots \\ \mathrm{Bl}(n) &:= \{1 + m_2 + \cdots + m_{n-1} + 1, \ldots, m\}. \end{aligned} \tag{3.36}$$

We again assign names of operators by $C_k := A_j$ for $k \in \mathrm{Bl}(j)$, where $j = 1, \dots, n$ $k = 1, \dots, m$. Then

$$
\begin{aligned}
&P^{1,m_2,\dots,m_n}_{\nu+\eta,\mu_2,\dots,\mu_n}(A_1, \dots, A_n) \\
&= \sum_{\pi \in S_m} \int_{\Delta_m(\pi)} C_{\pi(m)} \cdots C_{\pi(1)}\, ((\nu + \eta) \times \mu_2^{m_2} \times \cdots \times \mu_n^{m_n})(ds_1, ds_2, \dots, ds_m) \\
&= \sum_{\pi \in S_m} \int_{\Delta_m(\pi)} C_{\pi(m)} \cdots C_{\pi(1)}(\nu \times \mu_2^{m_2} \times \cdots \times \mu_n^{m_n})(ds_1, ds_2, \dots, ds_m) \\
&\qquad + \sum_{\pi \in S_m} \int_{\Delta_m(\pi)} C_{\pi(m)} \cdots C_{\pi(1)}(\eta \times \mu_2^{m_2} \times \cdots \times \mu_n^{m_n})(ds_1, ds_2, \dots, ds_m) \\
&= P^{1,m_2,\dots,m_n}_{\nu,\mu_2,\dots,\mu_n}(A_1, \dots, A_n) \; + \; P^{1,m_2,\dots,m_n}_{\eta,\mu_2,\dots,\mu_n}(A_1, \dots, A_n).
\end{aligned}
\tag{3.37}
$$

Although the finiteness of the measures may not have been needed in these steps, σ-finiteness was used both in applying the distributive law established in the previous theorem, and in guaranteeing that the product of the measures is associative. □

We may extend this result using Corollary 3.1.9:

Theorem 3.2.2 (Disentangling a monomial that involves a sum of two measures)**.** *Given a Banach space X, together with operators $A_1, \dots, A_n \in \mathcal{L}(X)$, non-negative integers $m_1, \dots, m_n$, and finite, continuous Borel measures $\nu, \eta, \mu_2, \mu_3, \dots, \mu_n$ on the interval $[0, 1]$ associated with $A_1, A_1, A_2, A_3, \dots, A_n$, respectively, we have that*

$$
\begin{aligned}
&P^{m_1,m_2,\dots,m_n}_{\nu+\eta,\mu_2,\dots,\mu_n}(A_1, A_2, A_3, \dots, A_n) \\
&\qquad = \sum_{k=0}^{m_1} \binom{m_1}{k} P^{k,m_1-k,m_2,m_3,\dots,m_n}_{\nu,\eta,\mu_2,\mu_3\dots,\mu_n}(A_1, A_1, A_2, A_3, \dots, A_n).
\end{aligned}
\tag{3.38}
$$

Remark 6. Although the theorem here introduces a summation while making changes in the first argument of the monomial being disentangled, Corollary 3.1.9 allows us to apply the theorem to other arguments, and of course we may also apply the theorem repeatedly if there is more than one sum of measures appearing among the subscripts of P.

Proof. (A later result, Theorem 4.3.8, will be proved in much the same way.) We observe first that if $m_1 = 0$, then both sides of Equation (3.38) reduce to

$$P^{m_2,\ldots,m_n}_{\mu_2,\ldots,\mu_n}(A_2, \ldots, A_n).$$

Let us therefore assume that $m_1 > 0$.

We observe next that

$$P^{m_1,m_2,\ldots,m_n}_{\nu+\eta,\mu_2,\ldots,\mu_n}(A_1, A_2, \ldots, A_n) = P^{j,m_1-j,m_2,\ldots,m_n}_{\nu+\eta,\nu+\eta,\mu_2,\ldots,\mu_n}(A_1, A_1, A_2, \ldots, A_n) \tag{3.39}$$

for any $j = 0, \ldots, m_1$. (The reason we can say this is that when we rename the copies of the operators $A_1, \ldots, A_n$ with the names $C_1, \ldots, C_m$, the names $C_1, \ldots, C_{m_1}$ all refer to the operator A_1, whether we are calculating the disentangling on the left-hand side of Equation (3.39) or the disentangling on the right-hand side. Moreover, the operator $\mu + \nu$ is associated with each of those operators. Thus the definition of the disentangling map will yield the same expression for both. In effect this argument is Proposition 3.6 of [13], applied more generally than to probability measures.) We will show by mathematical induction that, for any $j = 0, \ldots, m_1$,

$$\begin{aligned} &P^{m_1,m_2,\ldots,m_n}_{\nu+\eta,\mu_2,\ldots,\mu_n}(A_1, A_2, \ldots, A_n) \\ &\qquad = \sum_{k=0}^{j} \binom{j}{k} P^{k,j-k,m_1-j,m_2,m_3\ldots,m_n}_{\nu,\eta,\nu+\eta,\mu_2,\mu_3\ldots,\mu_n}(A_1, A_1, A_1, A_2, A_3, \ldots, A_n). \end{aligned} \tag{3.40}$$

For the case $j = 0$, we have that the right-hand side of Equation (3.40) is just the one term

$$P^{0,0,m_1-0,m_2,m_3\dots,m_n}_{\nu,\eta,\nu+\eta,\mu_2,\mu_3\dots,\mu_n}(A_1, A_1, A_1, A_2, A_3, \dots, A_n)$$
$$= P^{m_1,m_2,m_3\dots,m_n}_{\nu+\eta,\mu_2,\mu_3\dots,\mu_n}(A_1, A_2, A_3, \dots, A_n), \quad (3.41)$$

which is identical to the left-hand side.

Let us now suppose that Equation (3.40) holds for each $j = 0, 1, \dots, h$, where $h < m_1$. Then applying Corollary 3.1.9 and Theorem 3.2.1 in various ways (as well as splitting and combining exponents), we have

$$\begin{aligned}
&P^{m_1,m_2,\dots,m_n}_{\nu+\eta,\mu_2,\dots,\mu_n}(A_1, A_2, \dots, A_n)\\
=\;& \sum_{k=0}^{h} \binom{h}{k} P^{k,h-k,m_1-h,m_2,m_3\dots,m_n}_{\nu,\eta,\nu+\eta,\mu_2,\mu_3\dots,\mu_n}(A_1, A_1, A_1, A_2, A_3, \dots, A_n)\\
=\;& \sum_{k=0}^{h} \binom{h}{k} P^{1,k,h-k,m_1-h-1,m_2,m_3\dots,m_n}_{\nu+\eta,\nu,\eta,\nu+\eta,\mu_2,\mu_3\dots,\mu_n}(A_1, A_1, A_1, A_1, A_2, A_3, \dots, A_n)\\
=\;& \sum_{k=0}^{h} \binom{h}{k} P^{1,k,h-k,m_1-h-1,m_2,m_3\dots,m_n}_{\nu,\nu,\eta,\nu+\eta,\mu_2,\mu_3\dots,\mu_n}(A_1, A_1, A_1, A_1, A_2, A_3, \dots, A_n)\\
&+ \sum_{k=0}^{h} \binom{h}{k} P^{1,k,h-k,m_1-h-1,m_2,m_3\dots,m_n}_{\eta,\nu,\eta,\nu+\eta,\mu_2,\mu_3\dots,\mu_n}(A_1, A_1, A_1, A_1, A_2, A_3, \dots, A_n)\\
=\;& \sum_{k=0}^{h} \binom{h}{k} P^{k+1,h-k,m_1-h-1,m_2,m_3\dots,m_n}_{\nu,\eta,\nu+\eta,\mu_2,\mu_3\dots,\mu_n}(A_1, A_1, A_1, A_2, A_3, \dots, A_n)\\
&+ \sum_{k=0}^{h} \binom{h}{k} P^{k,h+1-k,m_1-h-1,m_2,m_3\dots,m_n}_{\nu,\eta,\nu+\eta,\mu_2,\mu_3\dots,\mu_n}(A_1, A_1, A_1, A_2, A_3, \dots, A_n)
\end{aligned}$$

$$= \sum_{k=1}^{h+1} \begin{pmatrix} h \\ k-1 \end{pmatrix} P^{k,h+1-k,m_1-h-1,m_2,m_3\ldots,m_n}_{\nu,\eta,\nu+\eta,\mu_2,\mu_3\ldots,\mu_n}(A_1, A_1, A_1, A_2, A_3, \ldots, A_n)$$

$$+ \sum_{k=0}^{h} \begin{pmatrix} h \\ k \end{pmatrix} P^{k,h+1-k,m_1-h-1,m_2,m_3\ldots,m_n}_{\nu,\eta,\nu+\eta,\mu_2,\mu_3\ldots,\mu_n}(A_1, A_1, A_1, A_2, A_3, \ldots, A_n). \tag{3.42}$$

Applying the properties of binomial coefficients that

$$\begin{pmatrix} h \\ 0 \end{pmatrix} = \begin{pmatrix} h+1 \\ 0 \end{pmatrix} = 1, \qquad \begin{pmatrix} h \\ h \end{pmatrix} = \begin{pmatrix} h+1 \\ h+1 \end{pmatrix} = 1,$$

and for $0 < k \leq h$,

$$\begin{pmatrix} h \\ k-1 \end{pmatrix} + \begin{pmatrix} h \\ k \end{pmatrix} = \begin{pmatrix} h+1 \\ k \end{pmatrix},$$

we get

$$P^{m_1,m_2,\ldots,m_n}_{\nu+\eta,\mu_2,\ldots,\mu_n}(A_1, A_2, \ldots, A_n)$$
$$= \sum_{k=0}^{h+1} \begin{pmatrix} h+1 \\ k \end{pmatrix} P^{k,h+1-k,m_1-h-1,m_2,m_3\ldots,m_n}_{\nu,\eta,\nu+\eta,\mu_2,\mu_3\ldots,\mu_n}(A_1, A_1, A_1, A_2, A_3, \ldots, A_n). \tag{3.43}$$

By mathematical induction, this proves Equation (3.40) for $j = 0, 1, \ldots, m_1$. The case $j = m_1$ yields

$$P^{m_1,m_2,\ldots,m_n}_{\nu+\eta,\mu_2,\mu_3,\ldots,\mu_n}(A_1, A_2, \ldots, A_n)$$
$$= \sum_{k=0}^{m_1} \begin{pmatrix} m_1 \\ k \end{pmatrix} P^{k,m_1-k,0,m_2,m_3\ldots,m_n}_{\nu,\eta,\nu+\eta,\mu_2,\mu_3\ldots,\mu_n}(A_1, A_1, A_1, A_2, A_3, \ldots, A_n)$$

$$= \sum_{k=0}^{m_1} \begin{pmatrix} m_1 \\ k \end{pmatrix} P_{\nu,\eta,\mu_2,\mu_3\ldots,\mu_n}^{k,m_1-k,m_2,m_3\ldots,m_n}(A_1, A_1, A_2, A_3, \ldots, A_n), \tag{3.44}$$

which establishes Equation (3.38). □

An example of the use of Theorem 3.2.2 will be given below, in Section 4.3, Example 15.

Chapter 4

Orderings and operations on orderings

4.1 Orderings

As we have discussed, in Feynman's Operational Calculi one finds products of time-indexed operators, and thus expressions such as:

$$C_{\pi(m)}(s_{\pi(m)})\cdots C_{\pi(2)}(s_{\pi(2)})C_{\pi(1)}(s_{\pi(1)}),$$

where $\{C_j\}_{j=1}^m$ are operators, $\{s_j\}_{j=1}^m$ are time indices that we are taking to have values in the interval $[0,1]$, and $\pi \in S_m = \mathrm{Perm}\{1,2,\ldots,m\}$ (that is, S_m is the collection of all bijections from $\{1,\ldots,m\}$ to $\{1,\ldots,m\}$). For our purposes here we will handle only the time-independent operator case, $C_j(s_j) \equiv C_j$, allowing us to drop the time indices:

$$C_{\pi(m)}\cdots C_{\pi(2)}C_{\pi(1)}.$$

We find this, for example, in our definition of the disentangling map of a monomial, Definition 2.0.3:

$$\mathcal{T}_{\mu_1,\cdots,\mu_n}[P^{m_1,\cdots,m_n}(\tilde{A}_1,\ldots,\tilde{A}_n)]$$
$$:= \sum_{\pi\in S_m}\int_{\Delta_m(\pi)} C_{\pi(m)}\cdots C_{\pi(1)}\,(\mu_1^{m_1}\times\cdots\times\mu_n^{m_n})(ds_1,\ldots,ds_m). \quad (4.1)$$

Since, in expressions like this, we deal with all the permutations of the subscripts on the operators $C_1,\ldots,C_m$, the proofs of various FOCi results often involve combinatorial arguments about the order in which the subscripts $1,2,\ldots,m$ appear. For example, it is useful at times to regard the permutation π that specifies that order as the joining of two shorter permutations σ and τ in some way. In order to handle such concepts more easily, we would like to develop a way to express relationships among permutations of different lengths, which we will do in terms of operations on 'orderings'.

Definition 4.1.1 (Orderings)**.** Given any finite set P, the set of **orderings** of P is the set $\mathcal{O}_P := \{\text{all bijections } \sigma : \{1,2,\ldots,\text{card}(P)\}\to P\}$. We represent an individual ordering $\sigma\in\mathcal{O}_P$ as $\sigma = [\sigma(1),\sigma(2),\ldots,\sigma(\text{card}(P))]$, and we say that σ **orders** the set P. The **length** of an ordering $\sigma\in\mathcal{O}_P$ is defined to be $\text{length}(\sigma) := \text{card}(P)$. Two orderings of two sets are said to be **disjoint** if the sets they order are disjoint.

(By $\text{card}(P)$ we mean the cardinality of the set P.)

Note that this definition includes the case of an empty set $P=\emptyset$, in which case there is only one ordering, namely the empty map from $P=\emptyset$ to $P=\emptyset$. Thinking of maps as sets of ordered pairs, the empty map is the set consisting of no ordered pairs; that is, it is the empty set. So we will represent the empty map by the empty set symbol, $\emptyset:\emptyset\to\emptyset$. The empty map $\emptyset:\emptyset\to\emptyset$ is trivially a bijection, so we are indeed able to say that $\emptyset\in\mathcal{O}_\emptyset$, and we will call it the **null ordering** (or sometimes the **empty ordering**); in fact, $\mathcal{O}_\emptyset = \{\emptyset\}$.

Example 5. Let $P = \{2, 3, 5\}$. Then the map σ given by $\sigma(1) = 3, \sigma(2) = 2, \sigma(3) = 5$ is an ordering of P; that is, $\sigma = [\sigma(1), \sigma(2), \sigma(3)] = [3, 2, 5] \in \mathcal{O}_P$. In fact, $\mathcal{O}_P = \{[2, 3, 5], [2, 5, 3], [3, 2, 5], [3, 5, 2], [5, 2, 3], [5, 3, 2]\}$. Since $\mathcal{O}_P$ is a set, we may of course list its elements in any order.

Remark 7. The notation $[s_1, s_2, \dots, s_m]$ gives the full description of an ordering map. It implies the associated domain $\{1, 2, \dots, m\}$ and range $\{s_1, s_2, \dots, s_m\}$.

Remark 8. Orderings are the same as permutations in the case when the set to be ordered or permuted is the set $\{1, 2, \dots, m\}$ for some nonnegative integer m. That is, $\mathcal{O}_{\{1,2,\dots,m\}} = S_m$. (We will therefore at times apply ordering notation to permutations.)

Remark 9. Since $\mathcal{O}_P := \{\text{all bijections } \sigma : \{1, 2, \dots, \operatorname{card}(P)\} \to P\}$, we have that the number of those bijections is

$$\operatorname{card}(\mathcal{O}_P) = \operatorname{card}(P)!.$$

This holds even in the case $P = \emptyset$, where we have $\operatorname{card}(\mathcal{O}_\emptyset) = \operatorname{card}(\{\emptyset\}) = 1 = 0! = \operatorname{card}(\emptyset)!$.

Remark 10. One caution about the notation for orderings: In other, contexts, permutations are often represented in 'cycle notation', involving a string of elements in which each maps to the next. For example, in cycle notation the expression '$\pi = [1\ 3\ 2]$' would refer to the map $1 \mapsto 3 \mapsto 2 \mapsto 1$. This is *not* what we mean here by the expression $\pi = [1, 3, 2]$. Instead, for us '$[1, 3, 2]$' is simply an ordered list of the images of $1, 2, 3$, respectively, under the map; that is, '$\pi = [1, 3, 2]$' means $\pi(1) = 1, \pi(2) = 3, \pi(3) = 2$. In general, '$\pi = [p_1, p_2, \dots, p_m]$' will mean $\pi(1) = p_1,\ \pi(2) = p_2,\ \dots,\ \pi(m) = p_m$.

Before we discuss the 'merge' operation on orderings, it may be helpful to note the following: Suppose that we have an ordering $\pi \in \mathcal{O}_P$, where P is a finite set, and suppose that $x, y \in P$. Then x and y appear in the representation $\pi = [\pi(1), \pi(2), \ldots, \pi(\mathrm{card}(P))]$. The element x is to the left of y if and only if there exist $j, k \in \{1, 2, \ldots, \mathrm{card}(P)\}$, $j < k$, with $\pi(j) = x$ and $\pi(k) = y$. Equivalently, x is to the left of y if and only if $\pi^{-1}(x) < \pi^{-1}(y)$; that expression will appear in the definition of the merge operation. (Also, x and y are the same if and only if $\pi^{-1}(x) = \pi^{-1}(y)$.)

4.2 The merge operation

Our next objective is to express relationships among orderings, such as expressing a set of orderings of several objects in terms of sets of orderings of fewer objects. For example we might want to think of the orderings of five objects as a kind of combination of the orderings of three of those objects and the orderings of the other two objects. Our reason for wanting to think this way is that if we are working with five linear operators $A_1, \ldots, A_5$ and their associated measures $\mu_1, \ldots, \mu_5$, it may be that two of the operators are distinguished from the other three in some way. For example, maybe A_1 and A_2 commute with all the others, but A_3, A_4, A_5 do not. Or maybe μ_1 and μ_2 have their support in a proper subinterval of $[0, 1]$, while μ_3, μ_4 and μ_5 have their support in the rest of the interval. (The latter situation allows disentangling to occur in two steps, first in the subinterval, and then over the whole interval using the operator that results from the first step, as described in Theorem 2.1 of [22], which is related to the 'autonomous bracket' concept of V. Maslov, described in [35, p. 15].) In order to handle a variety of relationships, we will use operations on orderings that are defined below.

We will start by defining the 'merge' operation $\odot$ on sets of orderings. Two other operations will be defined later, but the merge operation is the main operation we will deal with. We will also provide examples of how these operations may be applied to FOCi.

Definition 4.2.1 (The merge operation)**.** Given disjoint, finite sets P and Q and orderings $\sigma \in \mathcal{O}_P, \tau \in \mathcal{O}_Q$, we define $\{\sigma\} \odot \{\tau\}$ to be the set of all orderings $\pi \in \mathcal{O}_{P \mathbin{\dot\cup} Q}$ with the two properties that

(i) $\pi^{-1}(x) < \pi^{-1}(y)$ if and only if $\sigma^{-1}(x) < \sigma^{-1}(y)$ for all $x, y \in P$, and

(ii) $\pi^{-1}(x) < \pi^{-1}(y)$ if and only if $\tau^{-1}(x) < \tau^{-1}(y)$ for all $x, y \in Q$.

Given sets of orderings $\mathcal{U} \subseteq \mathcal{O}_P, \mathcal{V} \subseteq \mathcal{O}_Q$, we define $\mathcal{U} \odot \mathcal{V}$ ('$\mathcal{U}$ **merge** $\mathcal{V}$') by

$$\mathcal{U} \odot \mathcal{V} := \bigcup_{\sigma \in \mathcal{U}, \tau \in \mathcal{V}} \left(\{\sigma\} \odot \{\tau\}\right). \tag{4.2}$$

(We will show in Theorem 4.2.5 that the union in Equation 4.2 is disjoint. The symbol '$\dot\cup$', as in "$\pi \in \mathcal{O}_{P \mathbin{\dot\cup} Q}$" above, represents a disjoint union of sets.)

Often P and Q will be sets of positive integers.

Remark 11. The definition is interpreted so that if either $\mathcal{U} = \emptyset$ or $\mathcal{V} = \emptyset$ or both, then the union $\bigcup_{\sigma \in \mathcal{U}, \tau \in \mathcal{V}} \{\sigma\} \odot \{\tau\}$ is empty, hence $\mathcal{U} \odot \mathcal{V} = \emptyset$.

Remark 12. The merge operation can be regarded as a special case of the 'shuffle' of two languages in the theory of formal languages—see [12, pp. 292-293]—as was pointed out to the author by a colleague, Scott Dyer. The statement of the definition of the shuffle is much like part (iii) of Theorem 4.3.5 below. This special case in effect applies the shuffle to disjoint languages, each of which has equal-length strings of exactly the same distinct symbols. The 'concatenation' and 'excerption' operations defined below similarly have counterparts in that field; the excerption operation selects a subword. Our focus for all three operations is on developing propositions that may be applied to the subscripts of operators in FOCi.

To clarify Definition 4.2.1, we repeat a comment from the end of Section 4.1. Given an ordering $\sigma = [\sigma(1), \sigma(2), \ldots, \sigma(\mathrm{card}(P))] \in \mathcal{O}_P$ and elements $x, y \in P$, the statement that $\sigma^{-1}(x) < \sigma^{-1}(y)$ means that x is to the left of y in the explicit representation $\sigma = [\sigma(1), \sigma(2), \ldots, \sigma(\mathrm{card}(P))]$. When we say then in property (i) of the definition that $\pi^{-1}(x) < \pi^{-1}(y)$ if and only if $\sigma^{-1}(x) < \sigma^{-1}(y)$ for all $x, y \in P$, we are saying that the order of the two elements x and y is the same in π as in σ. We may therefore think of the merge operation as an order-preserving operation; when applied to the singleton sets $\{\sigma\}$ and $\{\tau\}$, $\{\sigma\} \odot \{\tau\}$, it produces all possible orderings of the objects that σ and τ together order that preserve the relative order of the objects σ orders and that preserve the relative order of the objects that τ orders.

Properties (i) and (ii) in the definition of the merge operation both use the phrase "if and only if," which makes them useful properties to apply when we have a set of orderings that satisfies the definition. However, it happens that when we want to prove that a set of orderings satisfies the definition of a merging of two sets, proving weaker statements is enough:

Theorem 4.2.2 (Equivalent statements for the merge definition)**.** *Property (i) in Definition 4.2.1 may be replaced by either of these equivalent statements:*

(i′) If $\pi^{-1}(x) < \pi^{-1}(y)$ then $\sigma^{-1}(x) < \sigma^{-1}(y)$ for all $x, y \in P$.

(i″) If $\sigma^{-1}(x) < \sigma^{-1}(y)$ then $\pi^{-1}(x) < \pi^{-1}(y)$ for all $x, y \in P$.

Property (ii) in the definition may be replaced by either of these equivalent statements:

(ii′) If $\pi^{-1}(x) < \pi^{-1}(y)$ then $\tau^{-1}(x) < \tau^{-1}(y)$ for all $x, y \in Q$.

(ii″) If $\tau^{-1}(x) < \tau^{-1}(y)$ then $\pi^{-1}(x) < \pi^{-1}(y)$ for all $x, y \in Q$.

Proof. All of these equivalences follow from the fact that σ, π, and τ are bijections. We will show that (i) is equivalent to (i′).

Comparing (i) and (i′), we note first that (i) clearly implies (i′). We can see that (i′) implies (i) by contraposition: If (i′) is true, that is, if $\pi^{-1}(x) < \pi^{-1}(y)$ implies that $\sigma^{-1}(x) < \sigma^{-1}(y)$ for all $x, y \in P$, then the supposition that for some $x, y \in P$ we have $\pi^{-1}(x) \geq \pi^{-1}(y)$ gives us either that $\pi^{-1}(x) = \pi^{-1}(y)$, in which case $x = y$ and therefore $\sigma^{-1}(x) = \sigma^{-1}(y)$, or else that $\pi^{-1}(x) > \pi^{-1}(y)$, in which case (by (i′) itself) $\sigma^{-1}(x) > \sigma^{-1}(y)$. Hence, $\pi^{-1}(x) \geq \pi^{-1}(y)$ implies that $\sigma^{-1}(x) \geq \sigma^{-1}(y)$, and by contraposition, $\sigma^{-1}(x) < \sigma^{-1}(y)$ implies $\pi^{-1}(x) < \pi^{-1}(y)$. Therefore, (i) is equivalent to (i′). We omit proofs of the other equivalences, as they are similar. □

Example 6. Let $P := \{1, 2\}$, $Q := \{3, 4\}$. Then we have $\mathcal{O}_P := \{[1, 2], [2, 1]\}, \mathcal{O}_Q := \{[3, 4], [4, 3]\}$. Letting $\sigma := [1, 2], \tau := [3, 4]$, we have

$$\{\sigma\} \odot \{\tau\} = \{[1, 2, 3, 4], [1, 3, 2, 4], [3, 1, 2, 4], [1, 3, 4, 2], [3, 1, 4, 2], [3, 4, 1, 2]\}. \quad (4.3)$$

Note that the elements of $\{\sigma\} \odot \{\tau\}$ all preserve the relative order of the entries in σ and the relative order of the entries in τ, while allowing any other variation in the order.

To apply this example of the merge operation, consider a situation in which we would like to find the sum of products of four operators A_1, A_2, A_3, A_4 in all possible orders, except that A_1 always operates before A_2, and A_3 always operates before A_4. (This is similar to what we might do in FOCi, but greatly simplified.) The sum of the operators in all orders without restriction would be

$$\sum_{\pi \in \mathcal{O}_{\{1,2,3,4\}}} A_{\pi(4)} A_{\pi(3)} A_{\pi(2)} A_{\pi(1)}. \tag{4.4}$$

The sum of the operators with the restriction listed above would be

$$\sum_{\pi \in \{[1,2]\} \odot \{[3,4]\}} A_{\pi(4)} A_{\pi(3)} A_{\pi(2)} A_{\pi(1)}. \tag{4.5}$$

Another possible situation in which to apply the merging of two (singleton) sets of orderings would be if we want to add up the possible products of two copies of the operator A and two copies of the operator B. More specifically, let us say that $C_1 := C_2 := A$ and $C_3 := C_4 := B$, and that we are interested in the sum

$$\sum_{\pi \in \mathcal{O}_{\{1,2,3,4\}}} C_{\pi(4)} C_{\pi(3)} C_{\pi(2)} C_{\pi(1)}. \tag{4.6}$$

As the reader may verify, the set $\mathcal{O}_{\{1,2,3,4\}}$ can be rewritten as

$$\mathcal{O}_{\{1,2,3,4\}} = \{[1,2],[2,1]\} \odot \{[3,4],[4,3]\} = \biguplus_{\substack{\sigma \in \{[1,2],[2,1]\} \\ \tau \in \{[3,4],[4,3]\}}} \{\sigma\} \odot \{\tau\}. \tag{4.7}$$

We can therefore write the sum as

$$\sum_{\pi \in \mathcal{O}_{\{1,2,3,4\}}} C_{\pi(4)} C_{\pi(3)} C_{\pi(2)} C_{\pi(1)} = \sum_{\substack{\sigma \in \{[1,2],[2,1]\} \\ \tau \in \{[3,4],[4,3]\}}} \sum_{\pi \in \{\sigma\} \odot \{\tau\}} C_{\pi(4)} C_{\pi(3)} C_{\pi(2)} C_{\pi(1)}. \tag{4.8}$$

At this point we note that the sum over σ and τ will be the same whether we choose $\sigma = [1,2]$ or $\sigma = [2,1]$, because $C_1 = C_2 = A$. Also the sum over σ and τ will be the same whether we choose $\tau = [3,4]$ or $\tau = [4,3]$, because $C_3 = C_4 = B$. We will therefore rewrite the sum by choosing one of each ($\sigma = [1,2]$ and $\tau = [3,4]$) and multiplying by the number of duplicate choices:

$$\begin{aligned}\sum_{\pi\in\mathcal{O}_{\{1,2,3,4\}}} C_{\pi(4)}C_{\pi(3)}C_{\pi(2)}C_{\pi(1)} &= 2\times 2\times \sum_{\substack{\sigma=[1,2]\\ \tau=[3,4]}} \sum_{\pi\in\{\sigma\}\odot\{\tau\}} C_{\pi(4)}C_{\pi(3)}C_{\pi(2)}C_{\pi(1)} \\ &= 4\times \sum_{\pi\in\{[1,2]\}\odot\{[3,4]\}} C_{\pi(4)}C_{\pi(3)}C_{\pi(2)}C_{\pi(1)}. \qquad (4.9)\end{aligned}$$

This reduces the number of terms in the sum. In the example here, the number of terms is reduced from $4! = 24$ to $2!2! = 4$. (The set $\{[1,2]\}\odot\{[3,4]\}$ is given the designation $\mathcal{P}_{2,2}$ in [24, p. 575], where it is used as an index set in an application similar to this. We will give the general definition of $\mathcal{P}_{m_1,\ldots,m_n}$ below. This notation simplifies expressions in FOCi in, for example, the context of evolution equations—see [9, pp.24ff]—where the number of terms in a sum can be significantly reduced.)

Example 7. Let

$$P := \{1,3,5\}, Q := \{6,9\}.$$

Then $\mathcal{O}_P = \{[1,3,5],[1,5,3],[3,1,5],[3,5,1],[5,1,3],[5,3,1]\}, \mathcal{O}_Q = \{[6,9],[9,6]\}$. Let

$$\sigma := [3,5,1] \in \mathcal{O}_P, \quad \tau := [6,9] \in \mathcal{O}_Q.$$

Then

$$\begin{aligned}\{\sigma\}\odot\{\tau\} = \{&[3,5,1,6,9],[3,5,6,1,9],[3,6,5,1,9],[6,3,5,1,9],[3,5,6,9,1],\\ &[3,6,5,9,1],[6,3,5,9,1],[3,6,9,5,1],[6,3,9,5,1],[6,9,3,5,1]\}. \qquad (4.10)\end{aligned}$$

(Again, the elements of a set may of course be listed in any order, but we will use the same order in each of our examples, according to the order they are generated by Theorem 4.3.6, below.) At this point we will not demonstrate that $\{\sigma\} \odot \{\tau\}$ is exactly the set shown in Equation (4.10). We will however, by choosing $\pi :=$ $[6, 3, 5, 9, 1] \in \mathcal{O}_{\{1,3,5,6,9\}}$, indicate the validity of (4.10) using Theorem 4.2.2. We claim that $\pi \in \{\sigma\} \odot \{\tau\}$. By the definition of $\{\sigma\} \odot \{\tau\}$, π needs to satisfy both

(i′) If $\sigma^{-1}(x) < \sigma^{-1}(y)$ then $\pi^{-1}(x) < \pi^{-1}(y)$ for all $x, y \in P$, and

(ii″) If $\tau^{-1}(x) < \tau^{-1}(y)$ then $\pi^{-1}(x) < \pi^{-1}(y)$ for all $x, y \in Q$.

Theorem 4.2.2 says in effect, "An ordering π will be in the set $\{\sigma\} \odot \{\tau\}$ if and only if when we look at π, the elements it has in common with σ (the elements of P) appear in the same relative (left-to-right) order in π as in σ, and the elements it has in common with τ (the elements of Q) appear in the same relative order in π as in τ." We will do this below.

We first note that by the notation we are using,

$$\sigma = [3, 5, 1] = [\sigma(1), \sigma(2), \sigma(3)], \quad \tau = [6, 9] = [\tau(1), \tau(2)],$$
$$\pi = [6, 3, 5, 9, 1] = [\pi(1), \pi(2), \pi(3), \pi(4), \pi(5)].$$

Consider first the elements of $P = \{1, 3, 5\}$, which are ordered by σ. Taking them in the order specified by σ, we see that $\sigma^{-1}(3) = 1, \sigma^{-1}(5) = 2$ and $\sigma^{-1}(1) = 3$. The pairs $x, y \in P$ for which $\sigma^{-1}(x) < \sigma^{-1}(y)$ are then $x = 3, y = 5$; $x = 3, y = 1$; and $x = 5, y = 1$. For these pairs we have:

$$\begin{aligned}
x = 3, y = 5: &\quad \sigma^{-1}(3) = 1 < 2 = \sigma^{-1}(5) \text{ and } \pi^{-1}(3) = 2 < 3 = \pi^{-1}(5),\\
x = 3, y = 1: &\quad \sigma^{-1}(3) = 1 < 3 = \sigma^{-1}(1) \text{ and } \pi^{-1}(3) = 2 < 5 = \pi^{-1}(1), \text{ and}\\
x = 5, y = 1: &\quad \sigma^{-1}(5) = 2 < 3 = \sigma^{-1}(1) \text{ and } \pi^{-1}(5) = 3 < 5 = \pi^{-1}(1).
\end{aligned}$$

For τ, which orders $Q = \{6, 9\}$, there is only the pair $x = 6, y = 9$, for which:

$$x = 6, y = 9: \quad \tau^{-1}(6) = 1 < 2 = \tau^{-1}(9) \text{ and } \pi^{-1}(6) = 1 < 4 = \pi^{-1}(9).$$

This confirms that $\pi \in \{\sigma\} \odot \{\tau\}$.

In some cases an easier method for checking whether an ordering π is an element of the merging of two singleton sets of orderings $\{\sigma\} \odot \{\tau\}$ is contained in the following theorem:

Theorem 4.2.3 (Alternate definition of merge). *Let P and Q be disjoint, finite sets and $\sigma \in \mathcal{O}_P$, $\tau \in \mathcal{O}_Q$ be orderings. Let $p_1, p_2, \ldots, p_{\mathrm{card}(P)}, q_1, q_2, \ldots, q_{\mathrm{card}(Q)}$ be defined by*

$$\begin{aligned} [p_1, p_2, \ldots, p_{\mathrm{card}(P)}] &:= [\sigma(1), \sigma(2), \ldots, \sigma(\mathrm{card}(P))] = \sigma, \qquad \textit{and} \\ [q_1, q_2, \ldots, q_{\mathrm{card}(Q)}] &:= [\tau(1), \tau(2), \ldots, \tau(\mathrm{card}(Q))] = \tau. \end{aligned}$$

(This will imply that $P = \{p_1, p_2, \ldots, p_{\mathrm{card}(P)}\}$ and $Q = \{q_1, q_2, \ldots, q_{\mathrm{card}(Q)}\}$.) Then $\{\sigma\} \odot \{\tau\}$ is the set of all $\pi \in \mathcal{O}_{P \cup Q}$ for which both

(i) $\pi^{-1}(p_1) < \pi^{-1}(p_2) < \cdots < \pi^{-1}(p_{\mathrm{card}(P)})$ *and*

(ii) $\pi^{-1}(q_1) < \pi^{-1}(q_2) < \cdots < \pi^{-1}(q_{\mathrm{card}(Q)})$.

Note that property (i) will be vacuously satisfied if $\mathrm{card}(P) = 0$ or 1, and property (ii) will be vacuously satisfied if $\mathrm{card}(Q) = 0$ or 1.

Proof. With the hypotheses as stated, let us suppose that $\pi \in \{\sigma\} \odot \{\tau\}$. Since $p_i = \sigma(i)$ for all $i \in \{1, 2, \ldots, \mathrm{card}(P)\}$, we have that $\sigma^{-1}(p_i) = i$ for all i. Therefore,

$$\sigma^{-1}(p_1) = 1 < \sigma^{-1}(p_2) = 2 < \ldots < \sigma^{-1}(p_{\mathrm{card}(P)}) = \mathrm{card}(P).$$

Consequently, by definition of the merge operation,

$$\pi^{-1}(p_1) < \pi^{-1}(p_2) < \ldots < \pi^{-1}(p_{\mathrm{card}(P)}).$$

Similarly we may establish that $\pi^{-1}(q_1) < \pi^{-1}(q_2) < \ldots < \pi^{-1}(q_{\mathrm{card}(Q)})$, and therefore (i) and (ii) of the theorem hold.

On the other hand, with the given hypotheses, let us suppose that (i) and (ii) hold. Then taking any $x, y \in P$ with $\sigma^{-1}(x) < \sigma^{-1}(y)$ we let $j := \sigma^{-1}(x)$ and $k := \sigma^{-1}(y)$. But then $x = \sigma(j) = p_j$ and $y = \sigma(k) = p_k$ with $j < k$, so by property (i) we have that $\pi^{-1}(x) = \pi^{-1}(p_j) < \pi^{-1}(p_k) = \pi^{-1}(y)$. This establishes property (i) of the definition of the merge operation applied to $\pi \in \{\sigma\} \odot \{\tau\}$. Property (ii) of the definition is established similarly. Therefore, $\pi \in \{\sigma\} \odot \{\tau\}$. $\square$

Example 8. Continuing Example 7, but now using Theorem 4.2.3, we are claiming for $\sigma = [3, 5, 1]$ and $\tau = [6, 9]$ that

$$\begin{aligned}\{\sigma\} \odot \{\tau\} = \{&[3,5,1,6,9], [3,5,6,1,9], [3,6,5,1,9], [6,3,5,1,9], [3,5,6,9,1],\\ &[3,6,5,9,1], [6,3,5,9,1], [3,6,9,5,1], [6,3,9,5,1], [6,9,3,5,1]\}.\end{aligned} \tag{4.11}$$

Although the sets are in fact equal, for now we will only show the inclusion

$$\begin{aligned}\{\sigma\} \odot \{\tau\} \supseteq \{&[3,5,1,6,9], [3,5,6,1,9], [3,6,5,1,9], [6,3,5,1,9], [3,5,6,9,1],\\ &[3,6,5,9,1], [6,3,5,9,1], [3,6,9,5,1], [6,3,9,5,1], [6,9,3,5,1]\},\end{aligned} \tag{4.12}$$

and equality will then follow from a cardinality argument once we have proved Theorem 4.2.9.

Defining p_1, p_2, p_3, q_1, q_2 as in Theorem 4.2.3, we have $[p_1, p_2, p_3] := [3, 5, 1] = \sigma$

and $[q_1, q_2] = [6, 9] := \tau$. We first look at each of the orderings π on the right-hand side of Equation (4.12) to see whether $\pi^{-1}(p_1) < \pi^{-1}(p_2) < \pi^{-1}(p_3)$; that is, whether $\pi^{-1}(3) < \pi^{-1}(5) < \pi^{-1}(1)$. In the first ordering listed, $[3, 5, 1, 6, 9]$, we have that $\pi^{-1}(3) = 1 < \pi^{-1}(5) = 2 < \pi^{-1}(1) = 3$, so that satisfies the desired property (property (i) of Theorem 4.2.3). It is easy to see in the remaining orderings that 3 precedes 5, which precedes 1. Similarly, checking whether $\pi^{-1}(q_1) < \pi^{-1}(q_2)$—that is, whether $\pi^{-1}(6) < \pi^{-1}(9)$—we see that in all of the orderings π, 6 is in fact to the left of 9. Therefore, all of these orderings are in $\{\sigma\} \odot \{\tau\}$ as claimed.

Example 9. We have given an example of the merge operation applied to singleton sets. For an example using more general sets, let

$$\mathcal{U} := \{[1, 5, 3], [3, 5, 1]\} \subseteq \mathcal{O}_P, \ \mathcal{V} := \{[6, 9]\} \subseteq \mathcal{O}_Q.$$

The set $\mathcal{U} \odot \mathcal{V}$ is defined to be the union of all sets of the form $\{\sigma\} \odot \{\tau\}$ where $\sigma \in \mathcal{U}$ and $\tau \in \mathcal{V}$. Hence,

$$\begin{aligned}
\mathcal{U} \odot \mathcal{V} = &\Big(\{[1, 5, 3]\} \odot \{[6, 9]\}\Big) \cup \Big(\{[3, 5, 1]\} \odot \{[6, 9]\}\Big) \\
= &\{[1, 5, 3, 6, 9], [1, 5, 6, 3, 9], [1, 6, 5, 3, 9], [6, 1, 5, 3, 9], [1, 5, 6, 9, 3], \\
&[1, 6, 5, 9, 3], [6, 1, 5, 9, 3], [1, 6, 9, 5, 3], [6, 1, 9, 5, 3], [6, 9, 1, 5, 3], \\
&[3, 5, 1, 6, 9], [3, 5, 6, 1, 9], [3, 6, 5, 1, 9], [6, 3, 5, 1, 9], [3, 5, 6, 9, 1], \\
&[3, 6, 5, 9, 1], [6, 3, 5, 9, 1], [3, 6, 9, 5, 1], [6, 3, 9, 5, 1], [6, 9, 3, 5, 1]\}.
\end{aligned}$$

We now go on to state theorems involving the merge operation. We begin by showing the effect of the merge operation when one of the sets consists of the empty ordering.

Theorem 4.2.4 (Merging with the empty ordering)**.** *If P is a finite set and $\mathcal{U} \subseteq \mathcal{O}_P$, then $\mathcal{U} \odot \{\emptyset\} = \{\emptyset\} \odot \mathcal{U} = \mathcal{U}$.*

Proof. Since we have already established that the merge of the empty set and any set is empty, we will assume throughout this proof that $\mathcal{U}$ is nonempty. In order to apply the definition of merging (Definition 4.2.1), we will name the empty set $Q = \emptyset$, which yields that $\mathcal{O}_Q = \mathcal{O}_\emptyset = \{\emptyset\}$.

First we will consider the case $P = \emptyset$. In this case, $\mathcal{U} \subseteq \mathcal{O}_P$ implies that $\mathcal{U} = \{\emptyset\}$ (since we are assuming $\mathcal{U}$ is nonempty), so we want to look at $\mathcal{U} \odot \{\emptyset\} = \{\emptyset\} \odot \{\emptyset\}$, and we claim $\{\emptyset\} \odot \{\emptyset\} = \{\emptyset\}$.

To show the inclusion $\{\emptyset\} \odot \{\emptyset\} \subseteq \{\emptyset\}$ is straightforward, since given any $\pi \in \{\emptyset\} \odot \{\emptyset\}$, the definition of merging tells us that $\pi \in \mathcal{O}_{\emptyset \cup \emptyset} = \mathcal{O}_\emptyset = \{\emptyset\}$. To show the reverse inclusion, $\{\emptyset\} \odot \{\emptyset\} \supseteq \{\emptyset\}$, we consider any $\pi \in \{\emptyset\}$, which is to say that $\pi = \emptyset$, the null ordering. Let us also say $\sigma := \tau := \emptyset$. Using the definition of merging, we will be able to say that $\pi \in \{\emptyset\} \odot \{\emptyset\} = \{\sigma\} \odot \{\tau\}$ if we can show that $\pi \in \mathcal{O}_{\emptyset \cup \emptyset} = \mathcal{O}_\emptyset = \{\emptyset\}$, which we already know, and that conditions (i) and (ii) in the definition of merging hold, namely (i) $\pi^{-1}(x) < \pi^{-1}(y)$ if and only if $\sigma^{-1}(x) < \sigma^{-1}(y)$ for all $x, y \in P$, and (ii) $\pi^{-1}(x) < \pi^{-1}(y)$ if and only if $\tau^{-1}(x) < \tau^{-1}(y)$ for all $x, y \in Q$. In fact, both of these conditions are vacuously satisfied since P and Q are both empty sets. Thus $\pi \in \{\sigma\} \odot \{\tau\} = \{\emptyset\} \odot \{\emptyset\}$, and $\{\emptyset\} \odot \{\emptyset\} \supseteq \{\emptyset\}$. Therefore, $\{\emptyset\} \odot \{\emptyset\} = \{\emptyset\}$, so for the case $P = \emptyset$ we have $\mathcal{U} \odot \{\emptyset\} = \mathcal{U}$. It follows also that $\{\emptyset\} \odot \mathcal{U} = \{\emptyset\} \odot \{\emptyset\} = \{\emptyset\}$.

Second, we consider the case $\operatorname{card}(P) = 1$, say $P = \{a\}$. Then $\operatorname{card}(\mathcal{O}_P) = 1! = 1$ and $\mathcal{O}_P = \{[a]\}$. Consequently, since $\mathcal{U} \subseteq \mathcal{O}_P$ we have that $\mathcal{U} = \{[a]\}$ (because we are assuming $\mathcal{U}$ is nonempty). We are therefore interested in the set $\mathcal{U} \odot \{\emptyset\} = \{[a]\} \odot \{\emptyset\}$, and we claim $\{[a]\} \odot \{\emptyset\} = \{[a]\}$. Let $\sigma := [a] \in \mathcal{U}$, $\tau := \emptyset \in \mathcal{O}_Q = \mathcal{O}_\emptyset$.

To show the includion $\{[a]\} \odot \{\emptyset\} \subseteq \{[a]\}$ is straightforward, since given any $\pi \in \{[a]\} \odot \{\emptyset\}$, the definition of merging tells us that $\pi \in \mathcal{O}_{\{a\}\cup\emptyset} = \mathcal{O}_{\{a\}} = \{[a]\}$. To show the reverse inclusion, $\{[a]\} \odot \{\emptyset\} \supseteq \{[a]\}$, we consider an arbitrary $\pi \in \{[a]\}$, which can only be $\pi = [a]$. We wish to show that $\pi \in \{[a]\} \odot \{\emptyset\} = \{\sigma\} \odot \{\tau\}$ using the definition of merging. We already have that $\pi \in \mathcal{O}_{\{a\}} = \mathcal{O}_{\{a\}\cup\emptyset}$, so we have only to demonstrate that π satisfies properties (i) and (ii) of the definition of merging, namely (i) $\pi^{-1}(x) < \pi^{-1}(y)$ if and only if $\sigma^{-1}(x) < \sigma^{-1}(y)$ for all $x, y \in P$, and (ii) $\pi^{-1}(x) < \pi^{-1}(y)$ if and only if $\tau^{-1}(x) < \tau^{-1}(y)$ for all $x, y \in Q$. This time, P has only one element, and Q is empty, so again both conditions are vacuously satisfied, telling us that $\pi \in \{\sigma\} \odot \{\tau\} = \{[a]\} \odot \{\emptyset\}$. Thus $\{[a]\} \odot \{\emptyset\} \supseteq \{[a]\}$, and hence $\{[a]\} \odot \{\emptyset\} = \{[a]\}$. Therefore, we again have $\mathcal{U} \odot \{\emptyset\} = \mathcal{U}$. Establishing $\{\emptyset\} \odot \mathcal{U} = \mathcal{U}$ is similar.

Finally, consider the case $\operatorname{card}(P) \geq 2$. We are trying to prove $\mathcal{U} \odot \{\emptyset\} = \mathcal{U}$, which we will again break up into two subset relationships, first $\mathcal{U} \odot \{\emptyset\} \subseteq \mathcal{U}$. Choose any $\pi \in \mathcal{U} \odot \{\emptyset\}$. This implies that $\pi \in \{\sigma\} \odot \{\emptyset\}$ for some $\sigma \in \mathcal{U} \subseteq \mathcal{O}_P$. Again let $\tau := \emptyset \in \mathcal{O}_\emptyset = \mathcal{O}_Q$. Then by the definition of merging and Theorem 4.2.2, $\pi \in \mathcal{O}_{P\cup Q} = \mathcal{O}_P$, and

(i) if $\pi^{-1}(x) < \pi^{-1}(y)$ then $\sigma^{-1}(x) < \sigma^{-1}(y)$ for all $x, y \in P$, and

(ii) if $\pi^{-1}(x) < \pi^{-1}(y)$ then $\tau^{-1}(x) < \tau^{-1}(y)$ for all $x, y \in Q$.

Since Q is empty, the second condition provides us with nothing. But define $[p_1, p_2, \ldots, p_{\operatorname{card}(P)}] := \pi = [\pi(1), \pi(2), \ldots, \pi(\operatorname{card}(P))]$, so that

$$\pi^{-1}(p_1) = 1 \ < \ \pi^{-1}(p_2) = 2 \ < \ \cdots \ < \ \pi^{-1}(p_{\operatorname{card}(P)}) = \operatorname{card}(P),$$

and the first condition will then give us that

$$\sigma^{-1}(p_1) < \sigma^{-1}(p_2) < \cdots < \sigma^{-1}(p_{\mathrm{card}(P)}).$$

Since the domain of σ is the set $\{1, 2, \ldots, \mathrm{card}(P)\}$, this implies that

$$\sigma^{-1}(p_1) = 1,\ \sigma^{-1}(p_2) = 2,\ \cdots,\ \sigma^{-1}(p_{\mathrm{card}(P)}) = \mathrm{card}(P).$$

Consequently $\sigma(i) = p_i = \pi(i)$ for all $i \in \{1, 2, \ldots, \mathrm{card}(P)\}$, and thus $\pi = \sigma \in \mathcal{U}$. Therefore, $\mathcal{U} \odot \{\emptyset\} \subseteq \mathcal{U}$.

For the reverse inclusion, $\mathcal{U} \odot \{\emptyset\} \supseteq \mathcal{U}$, choose an arbitrary $\pi \in \mathcal{U}$. Say $\sigma := \pi$ and let $\tau := \emptyset$. We claim that $\pi \in \{\sigma\} \odot \{\tau\}$, which we will prove using the definition of the merge operation. Certainly we know that $\pi \in \mathcal{O}_P = \mathcal{O}_{P \cup Q}$. The two remaining conditions we need to satisfy are that (i) $\pi^{-1}(x) < \pi^{-1}(y)$ if and only if $\sigma^{-1}(x) < \sigma^{-1}(y)$ for all $x, y \in P$, and (ii) $\pi^{-1}(x) < \pi^{-1}(y)$ if and only if $\tau^{-1}(x) < \tau^{-1}(y)$ for all $x, y \in Q$. But (i) holds since $\pi = \sigma$, and (ii) holds vacuously because Q is empty. Therefore, $\pi \in \{\sigma\} \odot \{\tau\} = \{\sigma\} \odot \{\emptyset\}$, and consequently $\pi \in \bigcup_{\sigma \in \mathcal{U}, \tau \in \{\emptyset\}} \{\sigma\} \odot \{\tau\} = \mathcal{U} \odot \{\emptyset\}$. Therefore, $\mathcal{U} \odot \{\emptyset\} \supseteq \mathcal{U}$. Thus we have both inclusions, and hence $\mathcal{U} \odot \{\emptyset\} = \mathcal{U}$. Similarly, $\{\emptyset\} \odot \mathcal{U} = \mathcal{U}$. $\square$

Theorem 4.2.5. *For finite, disjoint sets P and Q, let $\sigma_1, \sigma_2 \in \mathcal{O}_P$ and $\tau_1, \tau_2 \in \mathcal{O}_Q$. If either $\sigma_1 \neq \sigma_2$ or $\tau_1 \neq \tau_2$ or both, then $\{\sigma_1\} \odot \{\tau_1\}$ and $\{\sigma_2\} \odot \{\tau_2\}$ are disjoint. Consequently, for any sets of orderings $\mathcal{U} \subseteq \mathcal{O}_P$ and $\mathcal{V} \subseteq \mathcal{O}_Q$, the union $\bigcup_{\sigma \in \mathcal{U}, \tau \in \mathcal{V}} \{\sigma\} \odot \{\tau\}$ is a disjoint union, $\dot{\bigcup}_{\sigma \in \mathcal{U}, \tau \in \mathcal{V}} \{\sigma\} \odot \{\tau\}$.*

Proof. We will prove this by the contrapositive. Let P, Q be finite, disjoint sets, let $\sigma_1, \sigma_2 \in \mathcal{O}_P$ and $\tau_1, \tau_2 \in \mathcal{O}_Q$, and suppose that $\pi \in \big(\{\sigma_1\} \odot \{\tau_1\}\big) \cap \big(\{\sigma_2\} \odot \{\tau_2\}\big)$. We will show that $\sigma_1 = \sigma_2$ and $\tau_1 = \tau_2$.

Using the definition of the merge operation (Definition 4.2.1) twice, for any $x, y \in P$, $\sigma_1^{-1}(x) < \sigma_1^{-1}(y)$ is equivalent to $\pi^{-1}(x) < \pi^{-1}(y)$, which in turn is equivalent to $\sigma_2^{-1}(x) < \sigma_2^{-1}(y)$. Now, for any $j, k \in \{1, 2, \ldots, \mathrm{card}(P)\}$ with $j < k$ we have $\sigma_1(j), \sigma_1(k) \in P$, so clearly $\sigma_1^{-1}(\sigma_1(j)) = j < k = \sigma_1^{-1}(\sigma_1(k))$. This will imply that $\sigma_2^{-1}(\sigma_1(j)) < \sigma_2^{-1}(\sigma_1(k))$. Consequently,

$$\sigma_2^{-1}(\sigma_1(1)) \ < \ \sigma_2^{-1}(\sigma_1(2)) \ < \ \sigma_2^{-1}(\sigma_1(3)) \ < \ \cdots \ < \ \sigma_2^{-1}(\sigma_1(\mathrm{card}(P))).$$

Since $\sigma_2^{-1}(\sigma_1(j))$ are elements of the domain of σ_2, which is the set $\{1, 2, \ldots, \mathrm{card}(P)\}$, we obtain

$$\sigma_2^{-1}(\sigma_1(1)) = 1,\ \sigma_2^{-1}(\sigma_1(2)) = 2,\ \sigma_2^{-1}(\sigma_1(3)) = 3,\ \ldots\ ,\ \sigma_2^{-1}(\sigma_1(\mathrm{card}(P))) = \mathrm{card}\, P,$$

hence $\sigma_1(i) = \sigma_2(i)$ for all $x \in \{1, 2, \ldots, \mathrm{card}(P)\}$, i.e., $\sigma_1 = \sigma_2$. Similarly $\tau_1 = \tau_2$. Hence $\sigma_1 = \sigma_2$ and $\tau_1 = \tau_2$, as required. □

Theorem 4.2.6 (Merging subsets of two sets)**.** *If P and Q are disjoint, finite sets, and if $\mathcal{U} \subseteq \mathcal{W} \subseteq \mathcal{O}_P$ and $\mathcal{V} \subseteq \mathcal{Z} \subseteq \mathcal{O}_Q$, then $\mathcal{U} \odot \mathcal{V} \subseteq \mathcal{W} \odot \mathcal{Z}$.*

Proof. Let $\pi \in \mathcal{U} \odot \mathcal{V}$. Then, by Definition 4.2.1, there exist $\sigma \in \mathcal{U}$, $\tau \in \mathcal{V}$ with $\pi \in \{\sigma\} \odot \{\tau\}$. But then $\sigma \in \mathcal{W}$, $\tau \in \mathcal{Z}$, so $\pi \in \{\sigma\} \odot \{\tau\} \subseteq \mathcal{W} \odot \mathcal{Z}$. Therefore, $\mathcal{U} \odot \mathcal{V} \subseteq \mathcal{W} \odot \mathcal{Z}$. □

Theorem 4.2.7 (Set relations and merge)**.** *Let P and Q be disjoint, finite sets, and let $\mathcal{U} \subseteq \mathcal{O}_P$ and $\mathcal{V}, \mathcal{W} \subseteq \mathcal{O}_Q$. Then*

(i) $\mathcal{U} \odot (\mathcal{V} \cup \mathcal{W}) = (\mathcal{U} \odot \mathcal{V}) \cup (\mathcal{U} \odot \mathcal{W})$,

(ii) $\mathcal{U} \odot (\mathcal{V} \cap \mathcal{W}) = (\mathcal{U} \odot \mathcal{V}) \cap (\mathcal{U} \odot \mathcal{W})$,

(iii) $\mathcal{U} \odot (\mathcal{V} \setminus \mathcal{W}) = (\mathcal{U} \odot \mathcal{V}) \setminus (\mathcal{U} \odot \mathcal{W})$, *and*

(iv) if $\mathcal{V} \cap \mathcal{W} = \emptyset$, then $\mathcal{U} \odot (\mathcal{V} \uplus \mathcal{W}) = (\mathcal{U} \odot \mathcal{V}) \uplus (\mathcal{U} \odot \mathcal{W})$.

Proof. (i) Claim: $\mathcal{U} \odot (\mathcal{V} \cup \mathcal{W}) = (\mathcal{U} \odot \mathcal{V}) \cup (\mathcal{U} \odot \mathcal{W})$.

(Proof of $\subseteq$) Let $\pi \in \mathcal{U} \odot (\mathcal{V} \cup \mathcal{W})$. Then there exist $\sigma \in \mathcal{U}, \tau \in \mathcal{V} \cup \mathcal{W}$ with $\pi \in \{\sigma\} \odot \{\tau\}$. But then $\tau \in \mathcal{V}$ or $\tau \in \mathcal{W}$, so $\pi \in \mathcal{U} \odot \mathcal{V}$ or $\pi \in \mathcal{U} \odot W$. Thus $\pi \in (\mathcal{U} \odot \mathcal{V}) \cup (\mathcal{U} \odot \mathcal{W})$.

(Proof of $\supseteq$) Since $\mathcal{V}, \mathcal{W} \subseteq \mathcal{V} \cup \mathcal{W}$, we have $\mathcal{U} \odot \mathcal{V} \subseteq \mathcal{U} \odot (\mathcal{V} \cup \mathcal{W})$ and $\mathcal{U} \odot \mathcal{W} \subseteq \mathcal{U} \odot (\mathcal{V} \cup \mathcal{W})$. Therefore, $(\mathcal{U} \odot \mathcal{V}) \cup (\mathcal{U} \odot \mathcal{W}) \subseteq \mathcal{U} \odot (\mathcal{V} \cup \mathcal{W})$.

(ii) Claim: $\mathcal{U} \odot (\mathcal{V} \cap \mathcal{W}) = (\mathcal{U} \odot \mathcal{V}) \cap (\mathcal{U} \odot \mathcal{W})$.

(Proof of $\subseteq$) Since $\mathcal{V} \cap \mathcal{W} \subseteq \mathcal{V}, \mathcal{W}$ we have $\mathcal{U} \odot (\mathcal{V} \cap \mathcal{W}) \subseteq \mathcal{U} \odot \mathcal{V}$ and $\mathcal{U} \odot (\mathcal{V} \cap \mathcal{W}) \subseteq \mathcal{U} \odot \mathcal{W}$. Therefore, $\mathcal{U} \odot (\mathcal{V} \cap \mathcal{W}) \subseteq (\mathcal{U} \odot \mathcal{V}) \cap (\mathcal{U} \odot \mathcal{W})$.

(Proof of $\supseteq$) Let $\pi \in (\mathcal{U} \odot \mathcal{V}) \cap (\mathcal{U} \odot \mathcal{W})$. Then $\pi \in \mathcal{U} \odot \mathcal{V}$ and $\pi \in \mathcal{U} \odot \mathcal{W}$, so there exist $\sigma \in \mathcal{U}, \tau \in \mathcal{V}$ with $\pi \in \{\sigma\} \odot \{\tau\}$, and there exist $\sigma' \in \mathcal{U}, \tau' \in \mathcal{W}$ with $\pi \in \{\sigma'\} \odot \{\tau'\}$. By Theorem 4.2.5, $\sigma = \sigma'$ and $\tau = \tau'$, so $\tau \in \mathcal{V} \cap \mathcal{W}$. Therefore, $\pi \in \{\sigma\} \odot \{\tau\} \subseteq \mathcal{U} \odot (\mathcal{V} \cap \mathcal{W})$, and hence $(\mathcal{U} \odot \mathcal{V}) \cap (\mathcal{U} \odot \mathcal{W}) \subseteq \mathcal{U} \odot (\mathcal{V} \cap \mathcal{W})$.

(iii) Claim: $\mathcal{U} \odot (\mathcal{V} \setminus \mathcal{W}) = (\mathcal{U} \odot \mathcal{V}) \setminus (\mathcal{U} \odot \mathcal{W})$.

Let $\pi \in \mathcal{U} \odot (\mathcal{V} \setminus \mathcal{W})$. There exist $\sigma \in \mathcal{U}$, $\tau \in \mathcal{V} \setminus \mathcal{W}$ such that $\pi \in \{\sigma\} \odot \{\tau\}$. Then $\tau \in \mathcal{V}$ and $\tau \notin \mathcal{W}$. It follows that $\pi \in \{\sigma\} \odot \{\tau\} \subseteq \mathcal{U} \odot \mathcal{V}$, and we claim that $\pi \notin \mathcal{U} \odot \mathcal{W}$. Assume to the contrary, i.e. that $\pi \in \mathcal{U} \odot \mathcal{W}$. It then follows that there are $\sigma' \in \mathcal{U}$, $\tau' \in \mathcal{W}$ with $\pi \in \{\sigma'\} \odot \{\tau'\}$. But then $(\{\sigma\} \odot \{\tau\}) \cap (\{\sigma'\} \odot \{\tau'\}) \neq \emptyset$, so $\sigma = \sigma'$, $\tau = \tau'$ by Theorem 4.2.5, and so $\tau = \tau' \in \mathcal{W}$, which is a contradiction. Therefore, $\pi \notin \mathcal{U} \odot \mathcal{W}$, and therefore $\pi \in (\mathcal{U} \odot \mathcal{V}) \setminus (\mathcal{U} \odot \mathcal{W})$. Thus $\mathcal{U} \odot (\mathcal{V} \setminus \mathcal{W}) \subseteq (\mathcal{U} \odot \mathcal{V}) \setminus (\mathcal{U} \odot \mathcal{W})$.

On the other hand, suppose that $\pi \in (\mathcal{U} \odot \mathcal{V}) \setminus (\mathcal{U} \odot \mathcal{W})$. Then $\pi \in \mathcal{U} \odot \mathcal{V}$, but $\pi \notin \mathcal{U} \odot \mathcal{W}$. Since $\pi \in \mathcal{U} \odot \mathcal{V}$, let us say $\pi \in \{\sigma\} \odot \{\tau\}$ for some $\sigma \in \mathcal{U}$, $\tau \in \mathcal{V}$, and we claim that $\tau \notin \mathcal{W}$. If we suppose that $\tau \in \mathcal{W}$, then $\pi \in \{\sigma\} \odot \{\tau\} \subseteq \mathcal{U} \odot \mathcal{W}$, a contradiction. Therefore $\tau \notin \mathcal{W}$, and we see that $\pi \in \{\sigma\} \odot \{\tau\} \subseteq \mathcal{U} \odot (\mathcal{V} \setminus \mathcal{W})$. Hence $(\mathcal{U} \odot \mathcal{V}) \setminus (\mathcal{U} \odot \mathcal{W}) \subseteq \mathcal{U} \odot (\mathcal{V} \setminus \mathcal{W})$.

(iv) Claim: If $\mathcal{V} \cap \mathcal{W} = \emptyset$, then $\mathcal{U} \odot (\mathcal{V} \dot\cup \mathcal{W}) = (\mathcal{U} \odot \mathcal{V}) \dot\cup (\mathcal{U} \odot \mathcal{W})$.

Let $\mathcal{V} \cap \mathcal{W} = \emptyset$. Then by (i) we have $\mathcal{U} \odot (\mathcal{V} \dot\cup \mathcal{W}) = (\mathcal{U} \odot \mathcal{V}) \cup (\mathcal{U} \odot \mathcal{W})$, so all we need to show is that the union on the right-hand side is disjoint. But by (ii) we have $(\mathcal{U} \odot \mathcal{V}) \cap (\mathcal{U} \odot \mathcal{W}) = \mathcal{U} \odot (\mathcal{V} \cap \mathcal{W}) = \mathcal{U} \odot \emptyset = \emptyset$, so the union on the right-hand side is a disjoint union.

□

Theorem 4.2.8. *If P and Q are disjoint, finite sets, and if $\mathcal{U}, \mathcal{W} \subseteq \mathcal{O}_P$ and $\mathcal{V}, \mathcal{Z} \subseteq \mathcal{O}_Q$ are nonempty sets of orderings with $\mathcal{U} \odot \mathcal{V} = \mathcal{W} \odot \mathcal{Z}$, then $\mathcal{U} = \mathcal{W}$ and $\mathcal{V} = \mathcal{Z}$.*

Proof. Choose an arbitrary $\sigma \in \mathcal{U}$. Because $\mathcal{V}$ is nonempty, there exists an ordering $\tau \in \mathcal{V}$. Choose any $\pi \in \{\sigma\} \odot \{\tau\}$. Then $\pi \in \{\sigma\} \odot \{\tau\} \subseteq \mathcal{U} \odot \mathcal{V} = \mathcal{W} \odot \mathcal{Z}$, so there exist $\sigma' \in \mathcal{W}, \tau' \in \mathcal{Z}$ with $\pi \in \{\sigma'\} \odot \{\tau'\}$. Then $\pi \in \big(\{\sigma\} \odot \{\tau\}\big) \cap \big(\{\sigma'\} \odot \{\tau'\}\big)$. It follows from Theorem 4.2.5 that $\sigma = \sigma'$ and $\tau = \tau'$. Therefore, $\sigma \in \mathcal{W}$. Hence, $\mathcal{U} \subseteq \mathcal{W}$. A similar argument gives that $\mathcal{V} \subseteq \mathcal{Z}$. The reverse inclusions follow in the same way, and therefore, $\mathcal{U} = \mathcal{W}$ and $\mathcal{V} = \mathcal{Z}$. □

Theorem 4.2.9 (Cardinality of merged sets)**.** *For finite, disjoint sets P, Q and sets of orderings $\mathcal{U} \subseteq \mathcal{O}_P, \mathcal{V} \subseteq \mathcal{O}_Q$, the cardinality of $\mathcal{U} \odot \mathcal{V}$ is given by*

$$\operatorname{card}(\mathcal{U} \odot \mathcal{V}) = \frac{[\operatorname{card}(P) + \operatorname{card}(Q)]!}{\operatorname{card}(P)! \operatorname{card}(Q)!} \operatorname{card}(\mathcal{U}) \operatorname{card}(\mathcal{V}). \tag{4.13}$$

Proof. For arbitrary $\sigma \in \mathcal{O}_P$, $\tau \in \mathcal{O}_Q$, we know that the length of σ is $\operatorname{card}(P)$, and that of τ is $\operatorname{card}(Q)$. The orderings in $\{\sigma\} \odot \{\tau\}$ will therefore have length $\operatorname{card}(P) + \operatorname{card}(Q)$. Since the elements of σ remain in a fixed relative order for all

$\pi \in \{\sigma\} \odot \{\tau\}$, as do the elements of τ, each such π can be specified by simply stating which positions in π are occupied by elements from σ. That is, we can specify π by choosing $\mathrm{card}(P)$ of the $\mathrm{card}(P) + \mathrm{card}(Q)$ positions to be occupied by elements of σ. Consequently, the cardinality of $\{\sigma\} \odot \{\tau\}$ is given by

$$\mathrm{card}(\{\sigma\} \odot \{\tau\}) = \frac{[\mathrm{card}(P) + \mathrm{card}(Q)]!}{\mathrm{card}(P)!\,\mathrm{card}(Q)!}.$$

We know that $\mathcal{U} \odot \mathcal{V} = \bigcup_{\sigma \in \mathcal{U}, \tau \in \mathcal{V}} \{\sigma\} \odot \{\tau\}$, and each $\{\sigma\} \odot \{\tau\}$ has the same cardinality. The union $\bigcup_{\sigma \in \mathcal{U}, \tau \in \mathcal{V}} \{\sigma\} \odot \{\tau\}$ is disjoint by Theorem 4.2.5, so $\mathrm{card}(\bigcup_{\sigma \in \mathcal{U}, \tau \in \mathcal{V}} \{\sigma\} \odot \{\tau\}) = \mathrm{card}(\{\sigma\} \odot \{\tau\}) \cdot \mathrm{card}(\mathcal{U}) \cdot \mathrm{card}(\mathcal{V})$, i.e.,

$$\mathrm{card}\,(\mathcal{U} \odot \mathcal{V}) = \frac{[\mathrm{card}(P) + \mathrm{card}(Q)]!}{\mathrm{card}(P)!\,\mathrm{card}(Q)!}\;\mathrm{card}(\mathcal{U})\,\mathrm{card}(\mathcal{V}).$$

□

Remark 13. The cardinality formula in the preceding theorem holds when either P or Q is empty, or when $\mathcal{U}$ or $\mathcal{V}$ is empty. For example, if $\mathcal{U}$ or $\mathcal{V}$ is empty, then both sides of the formula are zero. If both $\mathcal{U}$ and $\mathcal{V}$ are nonempty but $Q = \emptyset$, then $\mathcal{U} \odot \mathcal{V} = \mathcal{U}$, so the left-hand side is $\mathrm{card}(\mathcal{U})$, and the right-hand side is $\frac{\mathrm{card}(P)!}{\mathrm{card}(P)!}(\mathrm{card}(\mathcal{U}))(1) = \mathrm{card}(\mathcal{U})$.

Example 10. Continuing with the sets in Example 9 above, we have

$$\mathcal{U} = \{[1,5,3],[3,5,1]\} \subseteq \mathcal{O}_P,\ \mathcal{V} = \{[6,9]\} \subseteq \mathcal{O}_Q,$$

where $P = \{1,3,5\}, Q = \{6,9\}$. The cardinality of $\mathcal{U} \odot \mathcal{V}$ is therefore

$$\mathrm{card}(\mathcal{U} \odot \mathcal{V}) = \frac{(3+2)!}{3!\,2!}(2)(1) = 20,$$

which agrees with the set $\mathcal{U} \odot \mathcal{V}$ found in Example 9.

Remark 14. Johnson and Lapidus [24] have used the notation $\mathcal{P}_{m_1,m_2}$ to refer to the set of permutations of the integers $\{1, 2, \ldots, m_1 + m_2\}$ for which the first m_1 integers $\{1, 2, \ldots, m_1\}$ retain their canonical order relative to each other, and the last m_2 integers $\{m_1 + 1, m_1 + 2, \ldots, m_1 + m_2\}$ retain their canonical order relative to each other also. Using the merge notation, we are now able to represent that set as $\mathcal{P}_{m_1,m_2} = \{[1, \ldots, m_1]\} \odot \{[m_1+1, \ldots, m_1+m_2]\}$. The cardinality of the set $\mathcal{P}_{m_1,m_2}$ is then $\operatorname{card}(\mathcal{P}_{m_1,m_2}) = \operatorname{card}(\{[1, \ldots, m_1]\} \odot \{[m_1+1, \ldots, m_1+m_2]\}) = \frac{(m_1+m_2)!}{m_1!\, m_2!}(1)(1) = \frac{(m_1+m_2)!}{m_1!\, m_2!}$. The definition can be extended: $\mathcal{P}_{m_1,\ldots,m_n} := \{1, \ldots, m_1\} \odot \{m_1+1, \ldots, m_1+m_2\} \odot \cdots \odot \{m_1 + \cdots + m_{n-1} + 1, \ldots, m_1 + \cdots + m_n\}$.

Theorem 4.2.10. *The merge operation $\odot$ is commutative and associative.*

Proof. Commutativity is immediate from the fact that the definition of $\mathcal{U} \odot \mathcal{V}$ is symmetric with respect to the sets $\mathcal{U}$ and $\mathcal{V}$.

We will prove associativity first for merged singleton sets, and then for merged sets in general. To begin, we claim that given pairwise disjoint, finite sets P, Q, R and orderings $\mu \in \mathcal{O}_P$, $\nu \in \mathcal{O}_Q$, $\omega \in \mathcal{O}_R$, we have

$$(\{\mu\} \odot \{\nu\}) \odot \{\omega\} = \{\mu\} \odot (\{\nu\} \odot \{\omega\}). \tag{4.14}$$

Let $p = \operatorname{card}(P), q = \operatorname{card}(Q), r = \operatorname{card}(R)$.

It will be helpful to define an auxiliary set. Let $\Sigma(\mu, P;\, \nu, Q;\, \omega, R)$ be defined by

$$\begin{aligned} &\Sigma(\mu, P;\, \nu, Q;\, \omega, R) \\ &:= \Big\{\text{all } \lambda \in \mathcal{O}_{P \cup Q \cup R} \text{ such that} \\ &\qquad \text{(I) } \lambda^{-1}(x) < \lambda^{-1}(y) \text{ if and only if } \mu^{-1}(x) < \mu^{-1}(y) \text{ for all } x, y \in P, \end{aligned}$$

(II) $\lambda^{-1}(x) < \lambda^{-1}(y)$ if and only if $\nu^{-1}(x) < \nu^{-1}(y)$ for all $x, y \in Q$, and

(III) $\lambda^{-1}(x) < \lambda^{-1}(y)$ if and only if $\omega^{-1}(x) < \omega^{-1}(y)$ for all $x, y \in R\Big\}$.

We will prove associativity of singleton sets by showing that both sides of Equation (4.14) equal this auxiliary set. We start by proving that $(\{\mu\} \odot \{\nu\}) \odot \{\omega\} = \Sigma(\mu, P;\, \nu, Q;\, \omega, R)$.

First we show that $(\{\mu\} \odot \{\nu\}) \odot \{\omega\} \subseteq \Sigma(\mu, P;\, \nu, Q;\, \omega, R)$. Let

$$\lambda \in (\{\mu\} \odot \{\nu\}) \odot \{\omega\} = \bigcup_{\pi \in \{\mu\} \odot \{\nu\}} \{\pi\} \odot \{\omega\}.$$

Then $\lambda \in \{\pi\} \odot \{\omega\}$ for some $\pi \in \{\mu\} \odot \{\nu\}$. Note by the definition of the merge operation (Definition 4.2.1) we have $\pi \in \mathcal{O}_{P \cup Q}, \lambda \in \mathcal{O}_{P \cup Q \cup R}$.

The definition of the merge operation for $\lambda \in \{\pi\} \odot \{\omega\}$ also implies that

(i) $\lambda^{-1}(x) < \lambda^{-1}(y)$ if and only if $\pi^{-1}(x) < \pi^{-1}(y)$ for all $x, y \in P \cup Q$, and

(ii) $\lambda^{-1}(x) < \lambda^{-1}(y)$ if and only if $\omega^{-1}(x) < \omega^{-1}(y)$ for all $x, y \in R$,

and the definition of the merge operation for $\pi \in \{\mu\} \odot \{\nu\}$ implies that

(iii) $\pi^{-1}(x) < \pi^{-1}(y)$ if and only if $\mu^{-1}(x) < \mu^{-1}(y)$ for all $x, y \in P$, and

(iv) $\pi^{-1}(x) < \pi^{-1}(y)$ if and only if $\nu^{-1}(x) < \nu^{-1}(y)$ for all $x, y \in Q$.

Parts (i) and (iii) give us (I); parts (i) and (iv) give us (II), and part (ii) gives us (III). Therefore, $\lambda \in \Sigma(\mu, P;\, \nu, Q;\, \omega, R)$, and consequently, $(\{\mu\} \odot \{\nu\}) \odot \{\omega\} \subseteq \Sigma(\mu, P;\, \nu, Q;\, \omega, R)$.

To show the reverse inclusion $(\{\mu\} \odot \{\nu\}) \odot \{\omega\} \supseteq \Sigma(\mu, P;\, \nu, Q;\, \omega, R)$, let

$$\lambda \in \Sigma(\mu, P;\, \nu, Q;\, \omega, R).$$

Then λ is a bijection; $\lambda : \{1, 2, p + q + r\} \to P \mathbin{\dot\cup} Q \mathbin{\dot\cup} R$.

Choose distinct $i_1, i_2, \ldots, i_{p+q} \in \{1, 2, \ldots, p + q + r\}$ (the domain of λ) so that $\lambda(i_\alpha) \in P \mathbin{\dot\cup} Q$ for each $\alpha \in \{1, 2, \ldots, p + q\}$. Without loss of generality, assume that $i_1 < i_2 < \ldots < i_{p+q}$. Define $\pi : \{1, 2, \ldots, p + q\} \to P \mathbin{\dot\cup} Q$ by $\pi(\alpha) = \lambda(i_\alpha)$ for all α. Then π is a bijection (because λ is bijective and each i_α is distinct for distinct α), so $\pi \in \mathcal{O}_{P \dot\cup Q}$. We would like to show that $\pi \in \{\mu\} \odot \{\nu\}$.

To show this, we look first at property (i) in the definition of the merge operation as it pertains to $\pi \in \{\mu\} \odot \{\nu\}$. We select any $x, y \in P$. Let $j := \pi^{-1}(x)$, $k := \pi^{-1}(y)$. Then

$$\begin{aligned}
& & \pi^{-1}(x) < \pi^{-1}(y) & \\
&\text{if and only if} & j < k & \\
&\text{if and only if} & i_j < i_k & \\
&\text{if and only if} & \lambda^{-1}(\pi(j)) < \lambda^{-1}(\pi(k)) & \\
&\text{if and only if} & \lambda^{-1}(x) < \lambda^{-1}(y) & \\
&\text{if and only if} & \mu^{-1}(x) < \mu^{-1}(y) & \qquad (4.15)
\end{aligned}$$

by (I). Thus part (i) of the merge definition holds. A similar argument replacing P with Q and μ with ν gives us part (ii). Therefore, $\pi \in \{\mu\} \odot \{\nu\}$.

We also claim that $\lambda \in \{\pi\} \odot \{\omega\}$. In order to check property (i) of the merge definition applied to $\lambda \in \{\pi\} \odot \{\omega\}$, we take any $x, y \in P \mathbin{\dot\cup} Q$ and let $j := \pi^{-1}(x)$, $k :=$

$\pi^{-1}(y)$. Then

$$\begin{aligned} & \pi^{-1}(x) < \pi^{-1}(y) \\ \text{if and only if} \quad & j < k \\ \text{if and only if} \quad & i_j < i_k \\ \text{if and only if} \quad & \lambda^{-1}(\pi(j)) < \lambda^{-1}(\pi(k)) \\ \text{if and only if} \quad & \lambda^{-1}(x) < \lambda^{-1}(y), \end{aligned} \tag{4.16}$$

establishing property (i) of the definition of the merge operation (the definition of $\lambda \in \{\pi\} \odot \{\omega\}$). As for property (ii) of the same definition, that is identical to (III) and therefore also holds. Consequently,

$$\lambda \in \{\pi\} \odot \{\omega\} \subseteq (\{\mu\} \odot \{\nu\}) \odot \{\omega\},$$

and hence, $\Sigma(\mu, P;\, \nu, Q;\, \omega, R) \subseteq (\{\mu\} \odot \{\nu\}) \odot \{\omega\}$.

Therefore,

$$(\{\mu\} \odot \{\nu\}) \odot \{\omega\} = \Sigma(\mu, P;\, \nu, Q;\, \omega, R). \tag{4.17}$$

Now we will use this fact to prove associativity of the merge operation for singleton sets. To do this, note that if we relabel the variables, Equation (4.17) can be written

$$(\{\nu\} \odot \{\omega\}) \odot \{\mu\} = \Sigma(\nu, Q;\, \omega, R;\, \mu, P). \tag{4.18}$$

Second, note that the set $\Sigma(\mu, P;\, \nu, Q;\, \omega, R)$ is defined symmetrically with respect to the three pairs $(\mu, P), (\nu, Q), (\omega, R)$, meaning that any rearrangement of those pairs will produce the same set. In particular,

$$\Sigma(\mu, P;\ \nu, Q;\ \omega, R) = \Sigma(\nu, Q;\ \omega, R;\ \mu, P). \tag{4.19}$$

Combining Equations (4.17)–(4.19), we have that

$$\begin{aligned}
(\{\mu\} \odot \{\nu\}) \odot \{\omega\} &\overset{(4.17)}{=} \Sigma(\mu, P;\ \nu, Q;\ \omega, R) \\
&\overset{(4.19)}{=} \Sigma(\nu, Q;\ \omega, R;\ \mu, P) \\
&\overset{(4.18)}{=} (\{\nu\} \odot \{\omega\}) \odot \{\mu\} \\
&= \{\mu\} \odot (\{\nu\} \odot \{\omega\}),
\end{aligned} \tag{4.20}$$

where an appeal to commutativity of merging gives that last step, and this establishes associativity of the merge operation for singleton sets.

To prove associativity for general finite sets, we start with the observation that given any disjoint, finite sets P, Q, and any set of orderings $\mathcal{U} \subseteq \mathcal{O}_P$, and any ordering $\tau \in \mathcal{O}_Q$, we have

$$U \odot \{\tau\} = \bigcup_{\sigma \in \mathcal{U}, \tau' \in \{\tau\}} \{\sigma\} \odot \{\tau'\} = \bigcup_{\sigma \in \mathcal{U}} \{\sigma\} \odot \{\tau\}. \tag{4.21}$$

Now, let P, Q, R be pairwise disjoint, finite sets, and let $\mathcal{U} \subseteq \mathcal{O}_P$, $\mathcal{V} \subseteq \mathcal{O}_Q$, $\mathcal{W} \subseteq \mathcal{O}_R$. We claim that $(\mathcal{U} \odot \mathcal{V}) \odot \mathcal{W} = \mathcal{U} \odot (\mathcal{V} \odot \mathcal{W})$.

Beginning from the left-hand side, we have:

$$\begin{aligned}
(\mathcal{U} \odot \mathcal{V}) \odot \mathcal{W} &= \bigcup_{\omega \in \mathcal{W}} \bigcup_{\pi \in \mathcal{U} \odot \mathcal{V}} \{\pi\} \odot \{\omega\} \\
&= \bigcup_{\omega \in \mathcal{W}} \bigcup_{\mu \in \mathcal{U}, \nu \in \mathcal{V}} \bigcup_{\pi \in \{\mu\} \odot \{\nu\}} \{\pi\} \odot \{\omega\}
\end{aligned}$$

$$= \bigcup_{\omega\in\mathcal{W}} \bigcup_{\mu\in\mathcal{U},\nu\in\mathcal{V}} (\{\mu\}\odot\{\nu\})\odot\{\omega\}, \tag{4.22}$$

where the index change in the next-to-last step is valid because $\mathcal{U}\odot\mathcal{V} = \bigcup_{\mu\in\mathcal{U},\nu\in\mathcal{V}} \{\mu\}\odot\{\nu\}$, and the last step is due to the observation we just made (Equation (4.21)). Continuing from Equation (4.22), adjusting the indices and applying associativity for singleton sets, we have

$$\begin{aligned}
(\mathcal{U}\odot\mathcal{V})\odot\mathcal{W} &= \bigcup_{\mu\in\mathcal{U}} \bigcup_{\nu\in\mathcal{V},\omega\in\mathcal{W}} (\{\mu\}\odot\{\nu\})\odot\{\omega\} \\
&= \bigcup_{\mu\in\mathcal{U}} \bigcup_{\nu\in\mathcal{V},\omega\in\mathcal{W}} \{\mu\}\odot(\{\nu\}\odot\{\omega\}) \\
&= \bigcup_{\mu\in\mathcal{U}} \bigcup_{\nu\in\mathcal{V},\omega\in\mathcal{W}} \bigcup_{\rho\in\{\nu\}\odot\{\omega\}} \{\mu\}\odot\{\rho\} \\
&= \bigcup_{\mu\in\mathcal{U}} \bigcup_{\rho\in\mathcal{V}\odot\mathcal{W}} \{\mu\}\odot\{\rho\} \\
&= \mathcal{U}\odot(\mathcal{V}\odot\mathcal{W}).
\end{aligned} \tag{4.23}$$

This establishes associativity of the merge operation. □

Theorem 4.2.11. *If P and Q are disjoint, finite sets, then $\mathcal{O}_P\odot\mathcal{O}_Q = \mathcal{O}_{P\cup Q}$.*

Proof. By the definition of merging (Definition 4.2.1), if $\pi\in\mathcal{O}_P\odot\mathcal{O}_Q$, then $\pi\in\{\sigma\}\odot\{\tau\}$ for some $\sigma\in\mathcal{O}_P$, $\tau\in\mathcal{O}_Q$. But then by the same definition, $\pi\in\mathcal{O}_{P\cup Q}$. Therefore, $\mathcal{O}_P\odot\mathcal{O}_Q\subseteq\mathcal{O}_{P\cup Q}$.

For the reverse inclusion, let $\pi\in\mathcal{O}_{P\cup Q}$. Say $\pi=[\pi(1),\pi(2),\ldots,\pi(m+n)]$, where $\mathrm{card}(P)=:m$, $\mathrm{card}(Q)=:n$. The range of π is then $P\cup Q=\{\pi(1),\pi(2),\ldots,\pi(m+n)\}$.

We now split up $P\cup Q$ as $P=\{\pi(j_1),\ldots,\pi(j_m)\}$, $Q=\{\pi(k_1),\ldots,\pi(k_n)\}$ with $\{j_1,j_2,\ldots,j_m,k_1,k_2,\ldots,k_n\}=\{1,\ldots,m+n\}$ (the elements of the set on the left and the set on the right are not necessarily in the same order). Without loss of generality, $j_1 < j_2 < \cdots < j_m$ and $k_1 < k_2 < \cdots < k_n$.

Define $\sigma : \{1, \ldots, m\} \to P$ and $\tau : \{1, \ldots, n\} \to Q$ by

$$\sigma(1) = \pi(j_1),\ \sigma(2) = \pi(j_2),\ \ldots,\ \sigma(m) = \pi(j_m),$$
$$\tau(1) = \pi(k_1),\ \tau(2) = \pi(k_2),\ \ldots,\ \tau(n) = \pi(k_n).$$

Then $\sigma = [\sigma(1), \ldots, \sigma(m)] \in \mathcal{O}_P$, and $\tau = [\tau(1), \ldots, \tau(n)] \in \mathcal{O}_Q$. We show that $\pi \in \{\sigma\} \odot \{\tau\}$ using the definition of the merge operation and Theorem 4.2.2. To do so, suppose that $x, y \in P$ with $\sigma^{-1}(x) < \sigma^{-1}(y)$. The way we have defined σ gives us that $\pi^{-1}(\sigma(i)) = j_i$ for all $i \in \{1, 2, \ldots, m\}$. Thus

$$\begin{aligned}\pi^{-1}(x) = \pi^{-1}(\sigma(\sigma^{-1}(x))) &= j_{\sigma^{-1}(x)} \\ &< j_{\sigma^{-1}(y)} = \pi^{-1}(\sigma(\sigma^{-1}(y))) \\ &= \pi^{-1}(y).\end{aligned}$$

We therefore have that property (i$''$) of Theorem 4.2.2 holds, and thus that property (i) of Definition 4.2.1 is satisfied. Similarly, if $x, y \in P$ with $\tau^{-1}(x) < \tau^{-1}(y)$, then $\pi^{-1}(x) < \pi^{-1}(y)$, satisfying property (ii) of Definition 4.2.1. Therefore, by definition, $\pi \in \{\sigma\} \odot \{\tau\}$, and therefore $\pi \in \mathcal{O}_P \odot \mathcal{O}_Q$.

$\square$

Corollary 4.2.12. *For any positive integer m we have that $S_{m+1} = S_m \odot \{[m+1]\}$, and therefore for any positive integer m, $S_m = \{[1]\} \odot \{[2]\} \odot \cdots \odot \{[m]\}$.*

Proof. Using the relationship between permutations and orderings (Remark 8) and applying Theorem 4.2.11, we have $S_m \odot \{[m+1]\} = \mathcal{O}_{\{1,\ldots,m\}} \odot \mathcal{O}_{\{m+1\}} = \mathcal{O}_{\{1,\ldots,m,m+1\}}$

$= S_{m+1}$, establishing the first statement.

We prove the second part by induction: Certainly $S_1 = \mathcal{O}_{\{1\}} = \{[1]\}$ ([1] is just the map $1 \mapsto 1$). Suppose that for some positive integer k we have $S_k = \{[1]\} \odot \{[2]\} \odot \cdots \odot \{[k]\}$. Then $\{[1]\} \odot \{[2]\} \odot \cdots \odot \{[k]\} \odot \{[k+1]\} = S_k \odot \{[k+1]\} = S_{k+1}$ by what was just shown. By induction this establishes the desired result. $\square$

Theorem 4.2.13. *Let $P_1, P_2, \ldots, P_n$ be pairwise disjoint, finite sets, and let $\mathcal{U}_1 \subseteq \mathcal{O}_{P_1}, \mathcal{U}_2 \subseteq \mathcal{O}_{P_2}, \ldots, \mathcal{U}_n \subseteq \mathcal{O}_{P_n}$ be sets of orderings. Then*

$$\mathcal{U}_1 \odot \mathcal{U}_2 \odot \cdots \odot \mathcal{U}_n = \bigcupdot_{\substack{\sigma_1 \in \mathcal{U}_1 \\ \sigma_2 \in \mathcal{U}_2 \\ \vdots \\ \sigma_n \in \mathcal{U}_n}} \{\sigma_1\} \odot \{\sigma_2\} \odot \cdots \odot \{\sigma_n\}. \tag{4.24}$$

Proof. That the union on the right-hand side of Equation (4.24) is disjoint is a direct consequence of Theorem 4.2.5 (every different choice of σ_1 will change all of the elements of the merge, and similarly for different choices of $\sigma_2, \ldots, \sigma_n$). It will therefore suffice to prove the statement rewritten as a union (without specifying that it is disjoint):

$$\mathcal{U}_1 \odot \mathcal{U}_2 \odot \cdots \odot \mathcal{U}_n = \bigcup_{\substack{\sigma_1 \in \mathcal{U}_1 \\ \sigma_2 \in \mathcal{U}_2 \\ \vdots \\ \sigma_n \in \mathcal{U}_n}} \{\sigma_1\} \odot \{\sigma_2\} \odot \cdots \odot \{\sigma_n\}. \tag{4.25}$$

To use induction on n, we begin by noting that for the case $n = 1$ we have the claim that $\mathcal{U}_1 = \bigcup_{\sigma \in \mathcal{U}_1} \{\sigma\}$, which is true by definition of union. For the case $n = 2$ we have, by definition of the merge operation,

$$\mathcal{U}_1 \odot \mathcal{U}_2 = \bigcup_{\substack{\sigma_1 \in \mathcal{U}_1 \\ \sigma_2 \in \mathcal{U}_2}} \{\sigma_1\} \odot \{\sigma_2\}. \tag{4.26}$$

Now we suppose that Equation (4.25) holds for $n = k$, where $k \geq 2$. Looking at the case when $n = k + 1$ we have

$$
\begin{aligned}
\mathcal{U}_1 \odot \mathcal{U}_2 \odot \cdots \odot \mathcal{U}_k \odot \mathcal{U}_{k+1} &= \left(\bigcup_{\substack{\sigma_1 \in \mathcal{U}_1 \\ \sigma_2 \in \mathcal{U}_2 \\ \vdots \\ \sigma_k \in \mathcal{U}_k}} \{\sigma_1\} \odot \{\sigma_2\} \odot \cdots \odot \{\sigma_k\} \right) \odot \mathcal{U}_{k+1} \\
&= \bigcup_{\tau \in \bigcup_{\substack{\sigma_1 \in \mathcal{U}_1 \\ \sigma_2 \in \mathcal{U}_2 \\ \vdots \\ \sigma_k \in \mathcal{U}_k}} \{\sigma_1\} \odot \{\sigma_2\} \odot \cdots \odot \{\sigma_k\},\ \sigma_{k+1} \in \mathcal{U}_{k+1}} \{\tau\} \odot \{\sigma_{k+1}\} \\
&= \bigcup_{\substack{\sigma_1 \in \mathcal{U}_1 \\ \sigma_2 \in \mathcal{U}_2 \\ \vdots \\ \sigma_k \in \mathcal{U}_k}} \bigcup_{\substack{\tau \in \{\sigma_1\} \odot \{\sigma_2\} \odot \cdots \odot \{\sigma_k\} \\ \sigma_{k+1} \in \mathcal{U}_{k+1}}} \{\tau\} \odot \{\sigma_{k+1}\} \\
&= \bigcup_{\substack{\sigma_1 \in \mathcal{U}_1 \\ \sigma_2 \in \mathcal{U}_2 \\ \vdots \\ \sigma_k \in \mathcal{U}_k}} \bigcup_{\sigma_{k+1} \in \mathcal{U}_{k+1}} (\{\sigma_1\} \odot \{\sigma_2\} \odot \cdots \odot \{\sigma_k\}) \odot \{\sigma_{k+1}\} \\
&= \bigcup_{\substack{\sigma_1 \in \mathcal{U}_1 \\ \sigma_2 \in \mathcal{U}_2 \\ \vdots \\ \sigma_k \in \mathcal{U}_k \\ \sigma_{k+1} \in \mathcal{U}_{k+1}}} \{\sigma_1\} \odot \{\sigma_2\} \odot \cdots \odot \{\sigma_k\} \odot \{\sigma_{k+1}\}. \qquad (4.27)
\end{aligned}
$$

By induction, Equation 4.25 holds, and therefore so does Equation 4.24. □

We now want to give a few examples using the merge operation. These examples have appeared in the work of other authors (without the merge operation), and in presenting them here in terms of merging we will not necessarily use all the properties of merging (such as commutativity and associativity), but these examples give some idea of the possible benefits of writing disentangling expressions in terms of the merge operation. The benefits may sometimes be in terms of clarity more than efficiency, but conceivably there could be applications where efficiency would be improved by use of the merge operation.

In the following example and thereafter, the notation '$\mathcal{M}_{cb}[0,1]$' will be used to represent the set of all finite, continuous Borel measures on the interval $[0,1]$.

Example 11. (A result from [13], extended beyond probability measures.) Consider operators $A_1, \ldots, A_n \in \mathcal{L}(X)$ associated with measures $\mu_1, \ldots, \mu_n \in \mathcal{M}_{cb}[0,1]$, respectively. Let $m_1, \ldots, m_n$, be nonnegative integers and $m := \sum_{i=1}^n m_i$.

Define blocks of integers $\mathrm{Bl}(1), \ldots, \mathrm{Bl}(n)$ by

$$\begin{aligned} \mathrm{Bl}(1) &:= \{1, 2, \ldots, m_1\} \\ \mathrm{Bl}(2) &:= \{m_1+1, m_1+2, \ldots, m_1+m_2\} \\ &\vdots \\ \mathrm{Bl}(n) &:= \{m_1 + \cdots + m_{n-1} + 1, \ldots, m\}, \end{aligned} \tag{4.28}$$

and define operators $C_1, \ldots, C_m$ and measures $\nu_1, \ldots, \nu_m$ by

$$C_j := \begin{cases} A_1, & j \in \mathrm{Bl}(1) \\ A_2, & j \in \mathrm{Bl}(2) \\ \vdots & \\ A_n, & j \in \mathrm{Bl}(n), \end{cases} \qquad \text{and} \qquad \nu_j := \begin{cases} \mu_1, & j \in \mathrm{Bl}(1) \\ \mu_2, & j \in \mathrm{Bl}(2) \\ \vdots & \\ \mu_n, & j \in \mathrm{Bl}(n), \end{cases} \tag{4.29}$$

or briefly, $C_j = A_i$ and $\nu_j = \mu_i$ whenever $j \in \mathrm{Bl}(i)$, for $i = 1, \ldots, n$, $j = 1, \ldots, m$.

We would like to prove the following formula for disentangling a monomial:

$$\begin{aligned} &P^{m_1,\ldots,m_n}_{\mu_1,\ldots,\mu_n}(A_1, \ldots, A_n) \\ &= m_1! m_2! \ldots m_n! \sum_{\pi \in \mathcal{P}_{m_1,\ldots,m_n}} \int_{\{s_{\pi(m)} > \cdots > s_{\pi(1)}\}} C_{\pi(m)} \ldots C_{\pi(1)} (\mu_1^{m_1} \times \cdots \times \mu_n^{m_n})(ds_1, \ldots, ds_m). \end{aligned}$$

The proof, which relies on Corollary 3.1.5, is as follows: By definition,

$$P^{m_1,\ldots,m_n}_{\mu_1,\ldots,\mu_n}(A_1,\ldots,A_n) := \sum_{\pi\in S_m}\int_{\{s_{\pi(m)}>\cdots>s_{\pi(1)}\}} C_{\pi(m)}\ldots C_{\pi(1)}(\mu_1^{m_1}\times\cdots\times\mu_n^{m_n})(ds_1,\ldots,ds_m). \tag{4.30}$$

The index of summation is

$$\begin{aligned}\pi\in S_m = \mathcal{O}_{\{1,2,\ldots,m\}} &= \mathcal{O}_{\mathrm{Bl}(1)}\odot\mathcal{O}_{\mathrm{Bl}(2)}\odot\cdots\odot\mathcal{O}_{\mathrm{Bl}(n)}\\ &= \dot{\bigcup_{\substack{\tau_1\in\mathcal{O}_{\mathrm{Bl}(1)}\\ \tau_2\in\mathcal{O}_{\mathrm{Bl}(2)}\\ \cdots\\ \tau_n\in\mathcal{O}_{\mathrm{Bl}(n)}}}}\{\tau_1\}\odot\{\tau_2\}\odot\cdots\odot\{\tau_n\}. \end{aligned}\tag{4.31}$$

Hence,

$$\begin{aligned}&P^{m_1,\ldots,m_n}_{\mu_1,\ldots,\mu_n}(A_1,\ldots,A_n)\\ &=\sum_{\substack{\tau_1\in\mathcal{O}_{\mathrm{Bl}(1)}\\ \tau_2\in\mathcal{O}_{\mathrm{Bl}(2)}\\ \cdots\\ \tau_n\in\mathcal{O}_{\mathrm{Bl}(n)}}}\ \sum_{\pi\in\{\tau_1\}\odot\{\tau_2\}\odot\cdots\odot\{\tau_n\}}\int_{\{(s_1,\ldots,s_m):\, s_{\pi(m)}>\cdots>s_{\pi(1)}\}} C_{\pi(m)}\ldots C_{\pi(1)}(\mu_1^{m_1}\times\cdots\times\mu_n^{m_n})(ds_1,\ldots,ds_m)\\ &=\sum_{\substack{\tau_1\in\mathcal{O}_{\mathrm{Bl}(1)}\\ \tau_2\in\mathcal{O}_{\mathrm{Bl}(2)}\\ \cdots\\ \tau_n\in\mathcal{O}_{\mathrm{Bl}(n)}}}\ \sum_{\pi\in\{\tau_1\}\odot\{\tau_2\}\odot\cdots\odot\{\tau_n\}}\int_{\{(s_1,\ldots,s_m):\, s_{\pi(m)}>\cdots>s_{\pi(1)}\}} C_{\pi(m)}\ldots C_{\pi(1)}(\nu_1\times\cdots\times\nu_n)(ds_1,\ldots,ds_m).\end{aligned}\tag{4.32}$$

In the latter expression, consider what the integral is when an ordering τ_1 is chosen from $\mathcal{O}_{\mathrm{Bl}(1)}$, compared with what it will be if a different ordering τ_1' is chosen from $\mathcal{O}_{\mathrm{Bl}(1)}$. Let us say (regarding the index of the inner sum) that π' is identical to π, except that π corresponds to the choice τ_1, and π' corresponds to the choice τ_1'. We can express π' as a composition of π and another permutation, say $\pi' = \sigma\pi$ for some

$\sigma \in S_m$. Then

$$\int_{\{(s_1,\ldots,s_m)\colon s_{\pi'(m)}>\cdots>s_{\pi'(1)}\}} C_{\pi'(m)}\ldots C_{\pi'(1)}(\nu_1\times\cdots\times\nu_n)(ds_1,\ldots,ds_m)$$
$$= \int_{\{(s_1,\ldots,s_m)\colon s_{\sigma\pi(m)}>\cdots>s_{\sigma\pi(1)}\}} C_{\pi'(m)}\ldots C_{\pi'(1)}(\nu_1\times\cdots\times\nu_m)(ds_1,\ldots,ds_m). \quad (4.33)$$

Since both τ_1 and τ_1' order only the elements in the block Bl(1), the only operators they affect are those C_j with subscripts $j \in$ Bl(1), all of which are equal to A_1. Rearranging the copies of the operator A_1 has no effect on the product of operators—that is, $C_{\pi'(m)}\ldots C_{\pi'(1)} = C_{\pi(m)}\ldots C_{\pi(1)}$—so the right-hand expression becomes

$$\int_{\{(s_1,\ldots,s_m)\colon s_{\sigma\pi(m)}>\cdots>s_{\sigma\pi(1)}\}} C_{\pi(m)}\ldots C_{\pi(1)}(\nu_1\times\cdots\times\nu_m)(ds_1,\ldots,ds_m)$$
$$= \int_{\{(s_{\sigma(1)},\ldots,s_{\sigma(m)})\colon s_{\sigma\pi(m)}>\cdots>s_{\sigma\pi(1)}\}} C_{\pi(m)}\ldots C_{\pi(1)}(\nu_1\times\cdots\times\nu_m)^{\sigma}(ds_1,\ldots,ds_m)^{\sigma}$$
$$= \int_{\{(s_{\sigma(1)},\ldots,s_{\sigma(m)})\colon s_{\sigma\pi(m)}>\cdots>s_{\sigma\pi(1)}\}} C_{\pi(m)}\ldots C_{\pi(1)}(\nu_{\sigma(1)}\times\cdots\times\nu_{\sigma(m)})(ds_{\sigma(1)},\ldots,ds_{\sigma(m)}) \quad (4.34)$$

by Corollary 3.1.5. Next we rename variables, $s_{\sigma(j)} \mapsto s_j$ for $j = 1,\ldots,m$, and get

$$\int_{\{(s_1,\ldots,s_m)\colon s_{\pi(m)}>\cdots>s_{\pi(1)}\}} C_{\pi(m)}\ldots C_{\pi(1)}(\nu_{\sigma(1)}\times\cdots\times\nu_{\sigma(m)})(ds_1,\ldots,ds_m). \quad (4.35)$$

Finally, we note that the permutation σ, given by $\pi' = \sigma\pi$, affects only the indices in Bl(1), and so changes only the order of the m_1 copies of the measure μ_1, thus having no effect. We may therefore write $\nu_{\sigma(1)}\times\cdots\times\nu_{\sigma(m)} = \nu_1\times\cdots\times\nu_m$, yielding the expression

$$\int_{\{(s_1,\ldots,s_m)\colon s_{\pi(m)}>\cdots>s_{\pi(1)}\}} C_{\pi(m)}\ldots C_{\pi(1)}(\nu_1\times\cdots\times\nu_m)(ds_1,\ldots,ds_m). \quad (4.36)$$

In other words, the change from τ_1 to τ_1', and the resulting change from π to π', have no effect on the integral.

Consequently, the inner sum on the right-hand side of Equation (4.32) is the same for all $m_1!$ orderings τ_1 that we can select from $\mathcal{O}_{\mathrm{Bl}(1)}$. We can therefore just choose one such ordering and multiply the result by $m_1!$. We will choose the ordering $\tau_1 = [1, 2, \ldots, m_1]$ and then rewrite the sum as

$$P^{m_1,\ldots,m_n}_{\mu_1,\ldots,\mu_n}(A_1, \ldots, A_n)$$
$$= \sum_{\substack{\tau_1=[1,2,\ldots,m_1] \\ \tau_2\in\mathcal{O}_{\mathrm{Bl}(2)} \\ \cdots \\ \tau_n\in\mathcal{O}_{\mathrm{Bl}(n)}}} m_1! \sum_{\pi\in\{\tau_1\}\odot\{\tau_2\}\odot\cdots\odot\{\tau_n\}} \int_{\{s_{\pi(m)}>\cdots>s_{\pi(1)}\}} C_{\pi(m)} \ldots C_{\pi(1)}(\mu_1^{m_1} \times \cdots \times \mu_n^{m_n})(ds_1, \ldots, ds_m). \tag{4.37}$$

The same argument works for the other blocks, enabling us to rewrite the sum as

$$P^{m_1,\ldots,m_n}_{\mu_1,\ldots,\mu_n}(A_1, \ldots, A_n)$$
$$= \sum_{\substack{\tau_1=[1,2,\ldots,m_1] \\ \tau_2=[m_1+1,\ldots,m_1+m_2] \\ \cdots \\ \tau_n=[m_1+\cdots+m_{n-1}+1,\ldots,m]}} m_1!m_2!\cdots m_n! \sum_{\pi\in\{\tau_1\}\odot\{\tau_2\}\odot\cdots\odot\{\tau_n\}} \int_{\{s_{\pi(m)}>\cdots>s_{\pi(1)}\}} C_{\pi(m)} \ldots C_{\pi(1)}(\mu_1^{m_1}\times\cdots\times\mu_n^{m_n})(ds_1, \ldots, ds_m)$$

$$= m_1!m_2!\cdots m_n! \sum_{\substack{\pi\in\{[1,2,\ldots,m_1]\} \\ \odot\{[m_1+1,\ldots,m_1+m_2]\} \\ \cdots \\ \odot\{[m_1+\cdots+m_{n-1}+1,\ldots,m]\}}} \int_{\{s_{\pi(m)}>\cdots>s_{\pi(1)}\}} C_{\pi(m)} \ldots C_{\pi(1)}(\mu_1^{m_1} \times \cdots \times \mu_n^{m_n})(ds_1, \ldots, ds_m)$$

$$= m_1!m_2!\ldots m_n! \sum_{\pi\in\mathcal{P}_{m_1,\ldots,m_n}} \int_{\{s_{\pi(m)}>\cdots>s_{\pi(1)}\}} C_{\pi(m)} \ldots C_{\pi(1)}(\mu_1^{m_1} \times \cdots \times \mu_n^{m_n})(ds_1, \ldots, ds_m). \tag{4.38}$$

4.3 The concatenation operation

A second operation we can define relative to orderings is concatenation. The effect is to take two orderings and place them next to each other to make a longer ordering, or in the case of two sets, the effect is to form the set of all orderings that can be formed by taking one ordering from the first set and one ordering from the second set and placing them next to each other the same way.

Concatenation is therefore a less complicated operation than the merge operation. However, the concept is useful; we will show that the merging of orderings can be expressed in terms of concatenations, and several proofs will rely on arguments involving concatenation.

Definition 4.3.1 (The concatenation operation)**.** Given disjoint, finite sets P, Q with $\mathrm{card}(P) = m$ and $\mathrm{card}(Q) = n$ and orderings $\sigma = [\sigma(1), \sigma(2), \ldots, \sigma(m)] \in \mathcal{O}_P, \tau = [\tau(1), \tau(2), \ldots, \tau(n)] \in \mathcal{O}_Q$, we define the **ordering concatenation** $\sigma.\tau \in \mathcal{O}_{P \cup Q}$ by

$$\sigma.\tau := [\sigma(1), \sigma(2), \ldots, \sigma(m), \tau(1), \tau(2), \ldots, \tau(n)]. \tag{4.39}$$

That is,

$$(\sigma.\tau)(j) := \begin{cases} \sigma(j) & \text{if } \ 1 \leq j \leq m \\ \tau(j-m) & \text{if } \ m+1 \leq j \leq m+n. \end{cases} \tag{4.40}$$

Given sets of orderings $\mathcal{U} \subseteq \mathcal{O}_P, \mathcal{V} \subseteq \mathcal{O}_Q$, we define the **set concatenation** $\mathcal{U}.\mathcal{V} \subseteq \mathcal{O}_{P \cup Q}$ by

$$\mathcal{U}.\mathcal{V} := \bigcup_{\sigma \in \mathcal{U}, \tau \in \mathcal{V}} \{\sigma.\tau\}. \tag{4.41}$$

(In fact the union is disjoint, which is straightforward to establish.)

Remark 15. Concatenation also makes sense in the case when one or both of the orderings are the null ordering. In either case, the definition is interpreted to say that if $\sigma \in \mathcal{O}_P$ is an ordering of a finite set P, then $\sigma.\emptyset = \emptyset.\sigma = \sigma \in \mathcal{O}_P$.

If P is a finite set and $\mathcal{U} \subseteq \mathcal{O}_P$, then

$$\mathcal{U}.\{\emptyset\} = \bigcup_{\sigma\in\mathcal{U},\tau\in\{\emptyset\}} \{\sigma.\tau\} = \bigcup_{\sigma\in\mathcal{U}} \{\sigma.\emptyset\} = \bigcup_{\sigma\in\mathcal{U}} \{\sigma\} = \mathcal{U}. \tag{4.42}$$

Similarly, $\{\emptyset\}.\mathcal{U} = \mathcal{U}$.

It ought to be remarked that, in constrast to (4.42), for P a finite set and $\mathcal{U} \subseteq \mathcal{O}_P$ we have $\mathcal{U}.\emptyset = \bigcup_{\sigma\in\mathcal{U},\tau\in\emptyset}\{\sigma\}.\{\tau\} = \emptyset$, and similarly for $\emptyset.\mathcal{U}$, i.e., $\mathcal{U}.\emptyset = \emptyset.\mathcal{U} = \emptyset$.

(The reader might find reason for concern here, since the statement '$\emptyset.\emptyset = \emptyset$' could be a statement about null orderings or a statement about empty sets. However, happily, the statement is true in both interpretations. The context should stipulate which is intended.)

Example 12. Let $\sigma = [1,2,3], \tau = [6,7,8,9]$ be orderings. Then their concatenation is $\sigma.\tau = [1,2,3].[6,7,8,9] = [1,2,3,6,7,8,9]$.

Let $\mathcal{U} = \{[1,2,3],[3,2,1]\}, \mathcal{V} = \{[4,5,6],[5,6,4],[6,4,5]\}$ be sets of orderings. Then

$$\begin{aligned}\mathcal{U}.\mathcal{V} = \{&[1,2,3,4,5,6],[1,2,3,5,6,4],[1,2,3,6,4,5],\\ &[3,2,1,4,5,6],[3,2,1,5,6,4],[3,2,1,6,4,5]\}.\end{aligned}$$

It is relatively straightforward to see that the length of $\sigma.\tau$ is the length of σ plus the length of τ. The cardinality of the set concatenation $\mathcal{U}.\mathcal{V}$ for sets of orderings $\mathcal{U} \subseteq \mathcal{O}_P, \mathcal{V} \subseteq \mathcal{O}_Q$ is given by $\operatorname{card}(\mathcal{U}.\mathcal{V}) = \operatorname{card}(\mathcal{U})\operatorname{card}(\mathcal{V})$.

Remark 16. Both concatenation operations (concatenation of orderings and concatenation of sets of orderings) are associative but not commutative. Non-commutativity is simple: $[1].[2] = [1,2] \neq [2,1] = [2].[1]$, and similarly for sets. For associativity of ordering concatenation, giving orderings $\sigma \in \mathcal{O}_P, \tau \in \mathcal{O}_Q, \rho \in \mathcal{O}_R$ of pairwise disjoint, finite sets P, Q, R, we have

$$\begin{aligned}
(\sigma.\tau).\rho &= ([\sigma(1), \ldots, \sigma(\mathrm{card}(P))].[\tau(1), \ldots, \tau(\mathrm{card}(Q))]).[\rho(1), \ldots, \rho(\mathrm{card}(R))] \\
&= [\sigma(1), \ldots, \sigma(\mathrm{card}(P)), \tau(1), \ldots, \tau(\mathrm{card}(Q))].[\rho(1), \ldots, \rho(\mathrm{card}(R))] \\
&= [\sigma(1), \ldots, \sigma(\mathrm{card}(P)), \tau(1), \ldots, \tau(\mathrm{card}(Q)), \rho(1), \ldots, \rho(\mathrm{card}(R))] \\
&= [\sigma(1), \ldots, \sigma(\mathrm{card}(P))].[\tau(1), \ldots, \tau(\mathrm{card}(Q)), \rho(1), \ldots, \rho(\mathrm{card}(R))] \\
&= [\sigma(1), \ldots, \sigma(\mathrm{card}(P))].([\tau(1), \ldots, \tau(\mathrm{card}(Q))].[\rho(1), \ldots, \rho(\mathrm{card}(R))]) \\
&= \sigma.(\tau.\rho).
\end{aligned} \tag{4.43}$$

For set concatenation, say $\mathcal{U} \subseteq \mathcal{O}_P, \mathcal{V} \subseteq \mathcal{O}_Q, \mathcal{W} \subseteq \mathcal{O}_R$, we have

$$\begin{aligned}
(\mathcal{U}.\mathcal{V}).\mathcal{W} &= \bigcup_{\pi \in \mathcal{U}.\mathcal{V}, \rho \in \mathcal{W}} \{\pi.\rho\} \\
&= \bigcup_{\pi \in \bigcup_{\sigma \in \mathcal{U}, \tau \in \mathcal{V}} \{\sigma.\tau\},\, \rho \in \mathcal{W}} \{\pi.\rho\} \\
&= \bigcup_{\sigma \in \mathcal{U}, \tau \in \mathcal{V}} \bigcup_{\pi \in \{\sigma.\tau\},\, \rho \in \mathcal{W}} \{\pi.\rho\} \\
&= \bigcup_{\sigma \in \mathcal{U}, \tau \in \mathcal{V}, \rho \in \mathcal{W}} \{(\sigma.\tau).\rho\} \\
&= \bigcup_{\sigma \in \mathcal{U}, \tau \in \mathcal{V}, \rho \in \mathcal{W}} \{\sigma.(\tau.\rho)\} \\
&= \bigcup_{\tau \in \mathcal{V}, \rho \in \mathcal{W}} \bigcup_{\sigma \in \mathcal{U}, \eta \in \{\tau.\rho\}} \{\sigma.\eta\} \\
&= \bigcup_{\sigma \in \mathcal{U},\, \eta \in \bigcup_{\tau \in \mathcal{V}, \rho \in \mathcal{W}} \{\tau.\rho\}} \{\sigma.\eta\}
\end{aligned}$$

$$= \bigcup_{\sigma\in\mathcal{U},\eta\in\mathcal{V}.\mathcal{W}}$$

$$= \mathcal{U}.(\mathcal{V}.\mathcal{W}). \tag{4.44}$$

Theorem 4.3.2. *Given pairwise disjoint, finite sets* $P_1, \ldots, P_n$ *and sets of orderings* $\mathcal{U}_1 \subseteq \mathcal{O}_{P_1}, \ldots, \mathcal{U}_n \subseteq \mathcal{O}_{Q_1}$*, we have*

$$\mathcal{U}_1.\mathcal{U}_2.\cdots.\mathcal{U}_n = \dot{\bigcup_{\substack{\sigma_1\in\mathcal{U}_1\\ \sigma_2\in\mathcal{U}_2\\ \vdots \\ \sigma_n\in\mathcal{U}_n}}} \{\sigma_1.\sigma_2.\cdots.\sigma_n\}. \tag{4.45}$$

Proof. That the union on the right-hand side of Equation (4.45) is disjoint is seen directly, since the concatenation $\sigma_1.\sigma_2.\cdots.\sigma_n$ can be written as one long string of elements, and any different choice of some σ_i will change the order of the elements within that part of the larger ordering $\sigma_1.\sigma_2.\cdots.\sigma_n$, which means it is a different ordering. It will therefore suffice to prove the statement as a union (without specifying that it is disjoint):

$$\mathcal{U}_1.\mathcal{U}_2.\cdots.\mathcal{U}_n = \bigcup_{\substack{\sigma_1\in\mathcal{U}_1\\ \sigma_2\in\mathcal{U}_2\\ \vdots \\ \sigma_n\in\mathcal{U}_n}} \{\sigma_1.\sigma_2.\cdots.\sigma_n\}. \tag{4.46}$$

To use mathematical induction on n, we begin by noting that for the case $n = 1$ we have the claim that $\mathcal{U}_1 = \bigcup_{\sigma\in\mathcal{U}_1} \{\sigma\}$, which is true by the definition of a union. For the case $n = 2$ we claim

$$\mathcal{U}_1.\mathcal{U}_2 = \bigcup_{\substack{\sigma_1\in\mathcal{U}_1\\ \sigma_2\in\mathcal{U}_2}} \{\sigma_1.\sigma_2\}, \tag{4.47}$$

and in fact that is true by the definition of set concatenation.

Now we suppose that Equation (4.46) holds for $n = k$, where $k \geq 2$. Then looking at the case when $n = k + 1$ we have

$$\begin{aligned}
\mathcal{U}_1.\mathcal{U}_2.\cdots.\mathcal{U}_k.\mathcal{U}_{k+1} &= \left(\bigcup_{\substack{\sigma_1 \in \mathcal{U}_1 \\ \sigma_2 \in \mathcal{U}_2 \\ \vdots \\ \sigma_k \in \mathcal{U}_k}} \{\sigma_1.\sigma_2.\cdots.\sigma_k\} \right) .\mathcal{U}_{k+1} \\
&= \bigcup_{\tau \in \bigcup_{\substack{\sigma_1 \in \mathcal{U}_1 \\ \sigma_2 \in \mathcal{U}_2 \\ \vdots \\ \sigma_k \in \mathcal{U}_k}} \{\sigma_1.\sigma_2.\cdots.\sigma_k\},\ \sigma_{k+1} \in \mathcal{U}_{k+1}} \{\tau.\sigma_{k+1}\} \\
&= \bigcup_{\substack{\sigma_1 \in \mathcal{U}_1 \\ \sigma_2 \in \mathcal{U}_2 \\ \vdots \\ \sigma_k \in \mathcal{U}_k}} \bigcup_{\substack{\tau \in \{\sigma_1.\sigma_2.\cdots.\sigma_k\} \\ \sigma_{k+1} \in \mathcal{U}_{k+1}}} \{\tau.\sigma_{k+1}\} \\
&= \bigcup_{\substack{\sigma_1 \in \mathcal{U}_1 \\ \sigma_2 \in \mathcal{U}_2 \\ \vdots \\ \sigma_k \in \mathcal{U}_k}} \bigcup_{\sigma_{k+1} \in \mathcal{U}_{k+1}} \{(\sigma_1.\sigma_2.\cdots.\sigma_k).\sigma_{k+1}\} \\
&= \bigcup_{\substack{\sigma_1 \in \mathcal{U}_1 \\ \sigma_2 \in \mathcal{U}_2 \\ \vdots \\ \sigma_k \in \mathcal{U}_k \\ \sigma_{k+1} \in \mathcal{U}_{k+1}}} \{\sigma_1.\sigma_2.\cdots.\sigma_k.\sigma_{k+1}\}. \qquad (4.48)
\end{aligned}$$

By mathematical induction, therefore, Equation 4.46 holds, and therefore so does Equation 4.45. □

Theorem 4.3.3 (Concatenating subsets of two sets)**.** *If P and Q are disjoint, finite sets, and if $\mathcal{U} \subseteq \mathcal{W} \subseteq \mathcal{O}_P$ and $\mathcal{V} \subseteq \mathcal{Z} \subseteq \mathcal{O}_Q$, then $\mathcal{U}.\mathcal{V} \subseteq \mathcal{W}.\mathcal{Z}$.*

Proof. Let $\pi \in \mathcal{U}.\mathcal{V}$. Then by definition of set concatenation, $\pi = \sigma.\tau$ for some $\sigma \in \mathcal{U}, \tau \in \mathcal{V}$. But then $\sigma \in \mathcal{W}, \tau \in \mathcal{Z}$, so $\pi \in \mathcal{W}.\mathcal{Z}$. Therefore, $\mathcal{U}.\mathcal{V} \subseteq \mathcal{W}.\mathcal{Z}$. □

Theorem 4.3.4. *If P and Q are disjoint, finite sets, and if $\mathcal{U}, \mathcal{W} \subseteq \mathcal{O}_P$ and $\mathcal{V}, \mathcal{Z} \subseteq \mathcal{O}_Q$ are nonempty sets of orderings with $\mathcal{U}.\mathcal{V} = \mathcal{W}.\mathcal{Z}$, then $\mathcal{U} = \mathcal{W}$ and $\mathcal{V} = \mathcal{Z}$.*

Proof. Let $m := \mathrm{card}(P), n := \mathrm{card}(Q)$. Choose an arbitrary ordering $\sigma \in \mathcal{U}$. Because $\mathcal{V}$ is nonempty, there exists an ordering $\tau \in \mathcal{V}$. Let $\pi := \sigma.\tau$. Then $\pi \in \mathcal{U}.\mathcal{V} = \mathcal{W}.\mathcal{Z}$. So $\pi = \sigma'.\tau'$ for some $\sigma' \in \mathcal{W}, \tau' \in \mathcal{Z}$. But then $\pi = [\sigma(1), \ldots, \sigma(m), \tau(1), \ldots, \tau(n)] = [\sigma'(1), \ldots, \sigma'(m), \tau'(1), \ldots, \tau'(n)]$, and equating these term-by-term, $\sigma = \sigma'$ and $\tau = \tau'$. Therefore, $\sigma \in \mathcal{W}$. Hence, $\mathcal{U} \subseteq \mathcal{W}$. A similar argument gives $\mathcal{V} \subseteq \mathcal{Z}$. Similarly, the reverse inclusions follow, and therefore, $\mathcal{U} = \mathcal{W}$ and $\mathcal{V} = \mathcal{Z}$. □

The following theorem will be useful for proving several subsequent results in this section (Theorems 4.3.6, 4.3.9, 4.3.11, and 4.3.13 appeal to it directly, and other results in the section follow from those):

Theorem 4.3.5 (Merging in terms of concatenation)**.** *Let P and Q be disjoint, finite sets, let $m := \mathrm{card}(P)$, and let $\sigma \in \mathcal{O}_P, \tau \in \mathcal{O}_Q$. Then the following are equivalent:*

(i) $\pi \in \{\sigma\} \odot \{\tau\}$,

(ii) $\pi = \tau_1.[\sigma(1)].\tau_2.[\sigma(2)].\cdots.\tau_m.[\sigma(m)].\tau_{m+1}$ *for some pairwise disjoint orderings* $\tau_1, \tau_2, \ldots, \tau_{m+1}$ *of subsets of Q with* $\tau = \tau_1.\tau_2.\cdots.\tau_{m+1}$, *and*

(iii) $\pi = \sigma_1.\tau_1.\sigma_2.\tau_2.\cdots.\sigma_k.\tau_k$ *for some $k \geq 1$, where $\sigma_1, \sigma_2, \ldots, \sigma_k$ are pairwise disjoint orderings of subsets of P and $\tau_1, \tau_2, \ldots, \tau_k$ are pairwise disjoint orderings of subsets of Q with $\sigma = \sigma_1.\sigma_2.\cdots.\sigma_k$ and $\tau = \tau_1.\tau_2.\cdots.\tau_k$.*

[Note that the expression $\pi = \sigma_1.\tau_1.\sigma_2.\tau_2.\cdots.\sigma_k.\tau_k$ in (iii) is not unique for a given π, and some of the orderings above, especially in (ii) and (iii), can be the null ordering.]

Proof. Define $n := \mathrm{card}(Q)$.

(i)⇒(ii): Let $\pi \in \{\sigma\} \odot \{\tau\}$. Then $\pi = [\pi(1), \pi(2), \ldots,$
$\pi(m+n)] \in \mathcal{O}_{P \cup Q}$. Let $[p_1, p_2, \ldots, p_m] := \sigma = [\sigma(1), \sigma(2), \ldots, \sigma(m)]$, so that $P =$

$\{p_1, p_2, \dots, p_m\}$. Since the elements of P are images of π, we may define

$$j_1 := \pi^{-1}(p_1),\ j_2 := \pi^{-1}(p_2),\ \dots,\ j_m := \pi^{-1}(p_m),$$

and so

$$\sigma = [p_1, p_2, \dots, p_m] = [\pi(j_1), \pi(j_2), \dots, \pi(j_m)].$$

Since $\sigma^{-1}(p_1) < \sigma^{-1}(p_2) < \cdots < \sigma^{-1}(p_m)$, it follows from Definition 4.2.1 that $\pi^{-1}(p_1) < \pi^{-1}(p_2) < \cdots < \pi^{-1}(p_m)$, which is to say, $j_1 < j_2 < \cdots < j_m$.

The remaining entries in $\pi = [\pi(1), \pi(2), \dots, \pi(m+n)]$ make up the set

$$\begin{aligned} Q = \{&\pi(1), \pi(2), \dots, \pi(j_1 - 1), \pi(j_1 + 1), \dots, \pi(j_2 - 1), \pi(j_2 + 1), \\ &\dots, \quad \pi(j_m - 1), \pi(j_m + 1), \dots, \pi(m + n - 1), \pi(m + n)\}; \end{aligned} \tag{4.49}$$

the set of images of τ. Working with τ as we just did with σ, we let $[q_1, q_2, \dots, q_n] := \tau = [\tau(1), \tau(2), \dots, \tau(n)]$. Then $\tau^{-1}(q_1) < \tau^{-1}(q_2) \cdots < \tau^{-1}(q_n)$, so by the definition of merging, $\pi^{-1}(q_1) < \pi^{-1}(q_2) < \cdots < \pi^{-1}(q_n)$. That is,

$$\pi^{-1}(\tau(1)) < \pi^{-1}(\tau(2)) < \cdots < \pi^{-1}(\tau(n)). \tag{4.50}$$

Since the pre-images of Q under the map π are the set

$$\{1, 2, \dots, j_1 - 1, j_1 + 1, \dots, j_2 - 1, j_2 + 1, \quad \dots, \quad j_m - 1, j_m + 1, \dots, m + n\}, \tag{4.51}$$

we see that the elements in the set shown in (4.51) are identical in the given order to the order given in (4.50); hence,

$$\tau(1) = \pi(1),\ \tau(2) = \pi(2),\ \dots,\ \tau(j_1 - 1) = \pi(j_1 - 1),$$
$$\tau(j_1) = \pi(j_1 + 1),\ \dots,\ \tau(j_2 - 2) = \pi(j_2 - 1),$$
$$\tau(j_2 - 1) = \pi(j_2 + 1), \quad \dots, \quad \tau(j_m - m) = \pi(j_m - 1),$$
$$\tau(j_m - m + 1) = \pi(j_m + 1),\ \dots,\ \tau(n) = \pi(m + n). \quad (4.52)$$

This enables us to say that

$$\begin{aligned}\tau = {} & [\pi(1), \pi(2), \dots, \pi(j_1 - 1), \pi(j_1 + 1), \dots, \pi(j_2 - 1),\\ & \pi(j_2 + 1), \dots, \pi(j_m - 1), \pi(j_m + 1), \dots, \pi(m + n)]\\ = {} & [\pi(1), \pi(2), \dots, \pi(j_1 - 1)].[\pi(j_1 + 1), \dots, \pi(j_2 - 1)].[\pi(j_2 + 1),\ \dots\ ].\cdots\\ & .[\ \dots\ , \pi(j_m - 1)].[\pi(j_m + 1), \dots, \pi(m + n)]. \end{aligned} \quad (4.53)$$

Choosing $\tau_1 := [\pi(1), \dots, \pi(j_1 - 1)]$, $\tau_2 := [\pi(j_1 + 1),\ \dots,\ \pi(j_2 - 1)]$, $\dots$, $\tau_{m+1} := [\pi(j_m + 1), \dots, \pi(m + n)]$ (which are pairwise disjoint) gives

$$\tau = \tau_1.\tau_2.\cdots\tau_{m+1}. \quad (4.54)$$

We then have

$$\begin{aligned}\pi = {} & [\pi(1), \pi(2), \dots, \pi(j_1 - 1), \pi(j_1), \pi(j_1 + 1), \dots, \pi(j_2 - 1), \pi(j_2), \pi(j_2 + 1),\\ & \dots, \pi(j_m - 1)\pi(j_m), \pi(j_m + 1), \dots, \pi(m + n)]\\ = {} & [\pi(1), \pi(2), \dots, \pi(j_1 - 1)].[\sigma(1)].[\pi(j_1 + 1) \dots, \pi(j_2 - 1)].[\sigma(2)].[\pi(j_2 + 1),\ \dots\ ].\cdots\\ & .[\ \dots\ , \pi(j_m - 1)].[\sigma(m)].[\pi(j_m + 1), \dots, \pi(m + n)]\\ = {} & \tau_1.[\sigma(1)].\tau_2.[\sigma(2)].\cdots.[\sigma(m)].\tau_{m+1}. \end{aligned} \quad (4.55)$$

(ii)⇒(iii): Given that $\pi = \tau_1.[\sigma(1)].\tau_2.[\sigma(2)].\cdots.\tau_m.[\sigma(m)].\tau_{m+1}$ for some pairwise disjoint orderings $\tau_1, \tau_2, \ldots, \tau_{m+1}$, with $\tau = \tau_1.\tau_2.\cdots.\tau_{m+1}$, we let $k := m + 1 \geq 1$ (we know $m \geq 0$ because $m = \text{card}(P)$), and we let $\sigma_1 := \emptyset, \sigma_2 := [\sigma(1)], \sigma_3 := [\sigma(2)], \ldots, \sigma_{m+1} := [\sigma(m)]$ (these singleton orderings are pairwise disjoint). Then $\pi = \emptyset.\sigma_1.\tau_1.\sigma_2.\tau_2.\cdots.\sigma_{m+1}.\tau_{m+1} = \sigma_1.\tau_1.\sigma_2.\tau_2.\cdots.\sigma_k.\tau_k$, with $\sigma = \emptyset.[\sigma(1)].[\sigma(2)].\cdots.[\sigma(m)] = \sigma_1.\sigma_2.\cdots.\sigma_k$ and $\tau = \tau_1.\tau_2.\cdots.\tau_k$.

(iii)⇒(i): Let $\pi = \sigma_1.\tau_1.\sigma_2.\tau_2.\cdots.\sigma_k.\tau_k$ for some $k \geq 1$, where $\sigma_1, \sigma_2, \ldots, \sigma_k$ are pairwise disjoint orderings of subsets of P with $\sigma = \sigma_1.\sigma_2.\cdots.\sigma_k$, and $\tau_1, \tau_2, \ldots, \tau_k$ are pairwise disjoint orderings of subsets of Q with $\tau = \tau_1.\tau_2.\cdots.\tau_k$. Define $m_1, \ldots, m_k$, $n_1, \ldots, n_k$ to be the lengths of the orderings $\sigma_1, \sigma_2, \ldots, \sigma_k, \tau_1, \tau_2, \ldots, \tau_k$, respectively.

Then $m_1 + m_2 + \cdots + m_k = m,\ n_1 + n_2 + \cdots + n_k = n$, and

$$\begin{aligned}
\pi &= \sigma_1.\tau_1.\sigma_2.\tau_2.\cdots.\sigma_k.\tau_k \\
&= [\sigma_1(1), \sigma_1(2), \ldots, \sigma_1(m_1),\ \tau_1(1), \tau_1(2), \ldots, \tau_1(n_1), \\
&\quad\ \sigma_2(1), \sigma_2(2), \ldots, \sigma_2(m_2),\ \tau_2(1), \tau_2(2), \ldots, \tau_2(n_2), \\
&\qquad\qquad \cdots \\
&\quad\ \sigma_k(1), \sigma_k(2), \ldots, \sigma_k(m_k),\ \tau_k(1), \tau_k(2), \ldots, \tau_k(n_k)].
\end{aligned} \tag{4.56}$$

We will show that $\pi \in \{\sigma\} \odot \{\tau\}$ using the definition of the merge operation and Theorem 4.2.2. We observe first that the entries in the representation of π above are exactly the elements of the set $P \cup Q$, so $\pi \in \mathcal{O}_{P \cup Q}$.

Second, we suppose that $x, y \in P$ with $\pi^{-1}(x) < \pi^{-1}(y)$. Then $x = \sigma_\alpha(j)$ and $y = \sigma_\beta(l)$ for some $\alpha, \beta \in \{1, 2, \ldots, k\}$, $j \in \{1, 2, \ldots, m_\alpha\}$, and $l \in \{1, 2, \ldots, m_\beta\}$. Looking at (4.56), the fact that $\pi^{-1}(x) < \pi^{-1}(y)$ (so x is to the left of y) tells us that either $\alpha < \beta$ or else $\alpha = \beta$ and $j < l$. Since

$$\sigma = [\sigma(1), \ldots, \sigma(m)] = [\sigma_1(1), \ldots, \sigma_1(m_1), \sigma_2(1), \ldots, \sigma_2(m_2), \sigma_3(1), \quad \ldots \quad, \sigma_k(m_k)],$$

this will imply that $\sigma^{-1}(\sigma_\alpha(j)) < \sigma^{-1}(\sigma_\beta(l))$, in other words, that $\sigma^{-1}(x) < \sigma^{-1}(y)$.

A similar demonstration tells us that if $x, y \in Q$ and $\pi^{-1}(x) < \pi^{-1}(y)$, then $\tau^{-1}(x) < \tau^{-1}(y)$. Together these facts and the definition of the merge operation show that $\pi \in \{\sigma\} \odot \{\tau\}$. □

In order to list all the elements of the merging of two orderings, the following theorem is useful (and it is what we have been using up to this point for listing orderings, though we have not said so):

Theorem 4.3.6 (Recursive expression for merging singletons)**.** *If P and Q are disjoint, finite sets with* $\operatorname{card}(P) = m \geq 1$ *and* $\operatorname{card}(Q) = n \geq 0$*, and if $\sigma \in \mathcal{O}_P$ and $\tau \in \mathcal{O}_Q$, then*

$$\begin{aligned} &\{\sigma\} \odot \{\tau\} \\ &= \dot{\bigcup_{k=0}^{n}} \Big(\Big(\{[\sigma(1), \ldots, \sigma(m-1)]\} \odot \{[\tau(1), \ldots, \tau(k)]\}\Big).\{[\sigma(m)].[\tau(k+1), \ldots, \tau(n)]\}\Big). \end{aligned} \tag{4.57}$$

Before we prove this theorem, a few words are appropriate about how to interpret the theorem. As indicated by the left-hand side of Equation (4.57), we are merging $\sigma = [\sigma(1), \ldots, \sigma(m)]$ and $\tau = [\tau(1), \ldots, \tau(n)]$. We want to list all orderings in that merging, and to do so we will use the last entry in the ordering σ, namely $\sigma(m)$, as a "pivot," meaning simply that we will interpret the terms on the right-hand side of (4.57) by considering separately the parts to the left of $\sigma(m)$ and the parts to the right of $\sigma(m)$. Since $\sigma(m)$ is the last entry in σ, the entries to the right of $\sigma(m)$ are the end of the ordering τ. We work through all possible orderings by starting with the entire ordering τ to the right of $\sigma(m)$, then all but the first entry of τ, then all but the first two entries of τ, etc. To the left of $\sigma(m)$ will then be all orderings we can construct with the rest of the entries of σ and the rest of the entries of τ, preserving the relative orders of σ entries among themselves and τ entries among themselves. Moreover, we can view the expression (4.57) as being recursive; it expresses a merging of two orderings as a union of terms that involve merging smaller orderings.

It is worth noting that we could perform the recursion in the opposite direction, using $\sigma(1)$ as the pivot. This would give the formula

$$\begin{aligned}&\{\sigma\} \odot \{\tau\}\\ &= \dot{\bigcup_{k=0}^{n}} \left(\{[\tau(1), \ldots, \tau(k)].[\sigma(1)]\}. \left(\{[\sigma(2), \ldots, \sigma(m)]\} \odot \{[\tau(k+1), \ldots, \tau(n)]\} \right)\right).\end{aligned} \tag{4.58}$$

Proof of Theorem 4.3.6. First we will show that Equation (4.57) holds if we replace the disjoint union symbol with simply a union symbol; that is, we will first show

$$\{\sigma\} \odot \{\tau\}$$
$$= \bigcup_{k=0}^{n} \Big(\Big(\{[\sigma(1), \ldots, \sigma(m-1)]\} \odot \{[\tau(1), \ldots, \tau(k)]\} \Big) . \{[\sigma(m)].[\tau(k+1), \ldots, \tau(n)]\} \Big). \tag{4.59}$$

To show the left-hand side of (4.59) is a subset of the right-hand side, we let $\pi \in \{\sigma\} \odot \{\tau\}$. By Theorem 4.3.5, $\pi = \tau_1.[\sigma(1)].\tau_2.[\sigma(2)].\cdots.\tau_m.[\sigma(m)].\tau_{m+1}$ for some pairwise disjoint orderings $\tau_1, \tau_2, \ldots, \tau_{m+1}$ of subsets of Q with $\tau = \tau_1.\tau_2.\cdots.\tau_{m+1}$. But then $\tau_{m+1} = [\tau(k+1), \ldots, \tau(n)]$ for some $k \in \{0, 1, \ldots, n\}$. The fact that part (ii) of that theorem implies part (i) of the theorem gives us

$$\tau_1.[\sigma(1)].\tau_2[\sigma(2)].\cdots.[\sigma(m-1)].\tau_m \in \{[\sigma(1), \ldots, \sigma(m-1)]\} \odot \{[\tau(1), \ldots, \tau(k)]\}.$$

Therefore,

$$\pi = (\tau_1.[\sigma(1)].\tau_2.[\sigma(2)].\cdots.[\sigma(m-1)].\tau_m).([\sigma(m)].\tau_{m+1})$$
$$\in \Big(\{[\sigma(1), \ldots, \sigma(m-1)]\} \odot \{[\tau(1), \ldots, \tau(k)]\} \Big) . \{[\sigma(m)].[\tau(k+1), \ldots, \tau(n)]\}$$
$$\subseteq \bigcup_{k=1}^{n} \Big(\Big(\{[\sigma(1), \ldots, \sigma(m-1)]\} \odot \{[\tau(1), \ldots, \tau(k)]\} \Big) . \{[\sigma(m)].[\tau(k+1), \ldots, \tau(n)]\} \Big),$$

giving the desired inclusion.

For the reverse inclusion, consider any element π in the right-hand side of Equation (4.59). For this π, there exist $k \in \{0, 1, \ldots, n\}$ and $\rho \in \{[\sigma(1), \ldots, \sigma(m-1)]\} \odot \{[\tau(1), \ldots, \tau(k)]\}$ for which $\pi = \rho.[\sigma(m)].[\tau(k+1), \ldots, \tau(n)]$. But then the implication (i)⇒(ii) of Theorem 4.3.5 gives us, when applied to ρ, that

$$\rho = \tau_1.[\sigma(1)].\tau_2[\sigma(2)].\cdots.[\sigma(m-1)].\tau_m$$

for some pairwise disjoint orderings $\tau_1, \tau_2, \ldots, \tau_m$ of subsets of $\{\tau(1), \tau(2), \ldots, \tau(k)\}$ with $[\tau(1), \ldots, \tau(k)] = \tau_1.\tau_2.\cdots.\tau_m$. Then

$$\pi = \tau_1.[\sigma(1)].\tau_2.[\sigma(2)].\cdots.\tau_m.[\sigma(m)].\tau_{m+1},$$

where $\tau_{m+1} := [\tau(k+1), \ldots, \tau(n)]$ (which is disjoint with each of $\tau_1, \ldots, \tau_m$), and so $\tau = \tau_1.\ldots,\tau_{m+1}$. Thus (by Theorem 4.3.5) $\pi \in \{\sigma\} \odot \{\tau\}$. Therefore, the union of the elements on the right-hand side of Equation (4.59) is contained in the left-hand side. Therefore, the two sides of (4.59) are equal.

All that remains is to note the the union on the right-hand side of (4.59) is in fact disjoint: Suppose that two different terms of the union—that is, terms with different values of k, say $k = k_1$ and $k = k_2$—contain a common ordering, call it π. The first of these ends with $[\sigma(m)].[\tau(k_1+1), \ldots, \tau(n)]$, and the other ends with $[\sigma(m)].[\tau(k_2+1), \ldots, \tau(n)]$. For that same ordering to be in both of these terms of the sum, then, it is necessary that endings match entry-by-entry starting at some entry (since these endings appear in every ordering in that term). Because $\sigma(m)$ is the furthest-right element of P in each of the orderings for these two terms, the entries must match from there to the end. Thus $[\tau(k_1+1), \ldots, \tau(n)] = [\tau(k_2+1), \ldots, \tau(n)]$. But then $k_1 = k_2$, which implies that these orderings are in the same term of the union. Consequently, different choices of k give disjoint terms of the union. Thus (4.59) becomes (4.57). $\square$

Example 13. We apply the preceding theorem to two examples. The first is merging a singleton with another ordering, say $\{[2]\} \odot \{[4,5,6,7]\}$. Here $m = 1$ and $n = 4$, so k runs from 0 to 4. For the various values of k, the right-hand side of Equation (4.57) will be as follows:

$$\begin{aligned}
k = 0: \quad & (\{\emptyset\} \odot \{\emptyset\}).\{[2].[4,5,6,7]\} && = \{[2,4,5,6,7]\},\\
k = 1: \quad & (\{\emptyset\} \odot \{[4]\}).\{[2].[5,6,7]\} && = \{[4,2,5,6,7]\},\\
k = 2: \quad & (\{\emptyset\} \odot \{[4,5]\}).\{[2].[6,7]\} && = \{[4,5,2,6,7]\},\\
k = 3: \quad & (\{\emptyset\} \odot \{[4,5,6]\}).\{[2].[7]\} && = \{[4,5,6,2,7]\},\\
k = 4: \quad & (\{\emptyset\} \odot \{[4,5,6,7]\}).\{[2].\emptyset\} && = \{[4,5,6,7,2]\}.
\end{aligned} \tag{4.60}$$

Second is an example we have already discussed: We claim

$$\begin{aligned}
\{[3,5,1]\} \odot \{[6,9]\} = \{&[3,5,1,6,9],[3,5,6,1,9],[3,6,5,1,9],[6,3,5,1,9],[3,5,6,9,1],\\
&[3,6,5,9,1],[6,3,5,9,1],[3,6,9,5,1],[6,3,9,5,1],[6,9,3,5,1]\}.
\end{aligned}$$

Applying the theorem, the second ordering has length 2, so k runs from 0 to 2. The different values of k give us the following:

$$\begin{aligned}
k = 0: \quad & (\{[3,5]\} \odot \{\emptyset\}).\{[1].[6,9]\} = \{[3,5,1,6,9]\},\\
k = 1: \quad & (\{[3,5]\} \odot \{[6]\}).\{[1].[9]\} = \{[3,5,6,1,9],[3,6,5,1,9],[6,3,5,1,9]\},\\
k = 2: \quad & (\{[3,5]\} \odot \{[6,9]\}).\{[1].\emptyset\} = \{[3,5,6,9,1],\\
& \qquad [3,6,5,9,1],[6,3,5,9,1],\\
& \qquad [3,6,9,5,1],[6,3,9,5,1],[6,9,3,5,1]\}.
\end{aligned}$$

We can use the preceding results to prove the following:

Theorem 4.3.7 (Distributive law for disentangling a monomial that involves a sum of two operators)**.** *Given operators $B, C, A_2, A_3, \ldots, A_n \in \mathcal{L}(X)$, non-negative integers $m_2, \ldots, m_n$, and measures $\mu_1, \ldots, \mu_n \in \mathcal{M}_{cb}[0,1]$ (finite, continuous Borel measures on $[0,1]$), where μ_1 is associated with B and C, and $\mu_2, \ldots, \mu_n$ are associated with $A_2, \ldots, A_n$, respectively, we have*

$$\begin{aligned} &P^{1,m_2,\ldots,m_n}_{\mu_1,\mu_2,\ldots,\mu_n}(B+C, A_2, \ldots, A_n) \\ &\qquad = P^{1,m_2,\ldots,m_n}_{\mu_1,\mu_2,\ldots,\mu_n}(B, A_2, \ldots, A_n) + P^{1,m_2,\ldots,m_n}_{\mu_1,\mu_2,\ldots,\mu_n}(C, A_2, \ldots, A_n). \end{aligned} \tag{4.61}$$

Proof. First we note that given any operators $A, B, C, D \in \mathcal{L}(X)$ we have

$$A(B+C)D = A(BD+CD) = ABD + ACD.$$

Second, we let $m := 1+m_2+\cdots+m_n$ and define blocks of integers $\mathrm{Bl}(2), \ldots, \mathrm{Bl}(n)$ by

$$\begin{aligned} &\mathrm{Bl}(1) := \{1\} \\ &\mathrm{Bl}(2) := \{2, \ldots, 1+m_2\}, \\ &\mathrm{Bl}(3) := \{1+m_2+1, \ldots, 1+m_2+m_3\}, \\ &\quad \cdots \\ &\mathrm{Bl}(n) := \{1+m_2+\cdots+m_{n-1}+1, \ldots, m\}, \end{aligned} \tag{4.62}$$

and we define

$$C_k := \begin{cases} B+C, & k \in \mathrm{Bl}(1) \text{ (that is, } k=1) \\ A_2, & k \in \mathrm{Bl}(2) \\ A_3, & k \in \mathrm{Bl}(3) \\ \quad\vdots & \\ A_n, & k \in \mathrm{Bl}(n). \end{cases} \tag{4.63}$$

Then

$$\begin{aligned} &P^{1,m_2,\ldots,m_n}_{\mu_1,\mu_2,\ldots,\mu_n}(B+C, A_2, \ldots, A_n) \\ &= \sum_{\pi\in S_m} \int_{\Delta_m(\pi)} C_{\pi(m)} \ldots C_{\pi(1)} \left(\mu_1^{m_1} \times \ldots \times \mu_n^{m_n}\right)(ds_1, \ldots, ds_m). \end{aligned} \tag{4.64}$$

Working with S_m, we apply the definition of the merging of sets and then Theorem 4.3.6 to get

$$\begin{aligned} S_m = \{[1]\} \odot \mathcal{O}_{\{2,\ldots,m\}} &= \dot{\bigcup_{\tau\in\mathcal{O}_{\{2,\ldots,m\}}}} \{[1]\} \odot \{\tau\} \\ &= \dot{\bigcup_{\tau\in\mathcal{O}_{\{2,\ldots,m\}}}} \dot{\bigcup_{k=0}^{m-1}} \{[\tau(1), \ldots, \tau(k), 1, \tau(k+1), \ldots, \tau(m-1)]\} \\ &= \dot{\bigcup_{\tau\in\mathcal{O}_{\{2,\ldots,m\}}}} \dot{\bigcup_{j=1}^{m}} \{[\tau(1), \ldots, \tau(j-1), 1, \tau(j), \ldots, \tau(m-1)]\}. \end{aligned} \tag{4.65}$$

We therefore have

$$P^{1,m_2,\ldots,m_n}_{\mu_1,\mu_2,\ldots,\mu_n}(B+C, A_2, \ldots, A_n)$$

$$
\begin{aligned}
&= \sum_{\tau\in\mathcal{O}_{\{2,\ldots,m\}}} \sum_{j=1}^{m} \int_{\{(s_1,\ldots,s_m):\, 1>s_{\tau(m-1)}>\cdots>s_{\tau(j)}>s_1>s_{\tau(j-1)}>\cdots>s_{\tau(1)}>0\}} C_{\tau(m-1)}\cdots C_{\tau(j)}C_1C_{\tau(j-1)}\cdots C_{\tau(1)}\,(\mu_1^{m_1}\times\ldots\times\mu_n^{m_n})(ds_1,\ldots,ds_m)\\
&= \sum_{\tau\in\mathcal{O}_{\{2,\ldots,m\}}} \sum_{j=1}^{m} \int_{\{(s_1,\ldots,s_m):\, 1>s_{\tau(m-1)}>\cdots>s_{\tau(j)}>s_1>s_{\tau(j-1)}>\cdots>s_{\tau(1)}>0\}} C_{\tau(m-1)}\cdots C_{\tau(j)}BC_{\tau(j-1)}\cdots C_{\tau(1)}\,(\mu_1^{m_1}\times\ldots\times\mu_n^{m_n})(ds_1,\ldots,ds_m)\\
&\quad + \sum_{\tau\in\mathcal{O}_{\{2,\ldots,m\}}} \sum_{j=1}^{m} \int_{\{(s_1,\ldots,s_m):\, 1>s_{\tau(m-1)}>\cdots>s_{\tau(j)}>s_1>s_{\tau(j-1)}>\cdots>s_{\tau(1)}>0\}} C_{\tau(m-1)}\cdots C_{\tau(j)}CC_{\tau(j-1)}\cdots C_{\tau(1)}\,(\mu_1^{m_1}\times\ldots\times\mu_n^{m_n})(ds_1,\ldots,ds_m)\\
&= P^{1,m_2,\ldots,m_n}_{\mu_1,\mu_2,\ldots,\mu_n}(B,A_2,\ldots,A_n) + P^{1,m_2,\ldots,m_n}_{\mu_1,\mu_2,\ldots,\mu_n}(C,A_2,\ldots,A_n). \qquad (4.66)
\end{aligned}
$$

$\square$

Extending this result, we have the following theorem:

Theorem 4.3.8 (Disentangling a monomial that involves a sum of two operators)**.** *Given operators $B, C, A_2, A_3, \ldots, A_n \in \mathcal{L}(X)$, non-negative integers $m_1, \ldots, m_n$, and measures $\mu_1, \ldots, \mu_n \in \mathcal{M}_{cb}[0,1]$, where μ_1 is associated with B and C, and $\mu_2, \ldots, \mu_n$ are associated with $A_2, \ldots, A_n$, respectively, we have*

$$
\begin{aligned}
&P^{m_1,m_2,\ldots,m_n}_{\mu_1,\mu_2,\ldots,\mu_n}(B+C, A_2, A_3, \ldots, A_n)\\
&\qquad = \sum_{k=0}^{m_1} \binom{m_1}{k} P^{k,m_1-k,m_2,m_3\ldots,m_n}_{\mu_1,\mu_1,\mu_2,\mu_3\ldots,\mu_n}(B, C, A_2, A_3, \ldots, A_n), \qquad (4.67)
\end{aligned}
$$

where $\binom{m_1}{k}$ *is the binomial coefficient,* $\binom{m_1}{k} = \dfrac{m_1!}{k!(m_1-k)!}$.

Proof. We observe first that if $m_1 = 0$, then both sides of Equation (4.67) reduce to

$$P^{m_2,\ldots,m_n}_{\mu_2,\ldots,\mu_n}(A_2, \ldots, A_n).$$

Let us therefore assume that $m_1 > 0$.

We observe next that

$$P^{m_1,m_2,\ldots,m_n}_{\mu_1,\mu_2,\ldots,\mu_n}(B+C,A_2,\ldots,A_n) = P^{j,m_1-j,m_2,\ldots,m_n}_{\mu_1,\mu_1,\mu_2,\ldots,\mu_n}(B+C,B+C,A_2,\ldots,A_n) \quad (4.68)$$

for any $j = 0,\ldots,m_1$, and we will show by induction that for any $j = 0,\ldots,m_1$,

$$\begin{aligned} &P^{m_1,m_2,\ldots,m_n}_{\mu_1,\mu_2,\ldots,\mu_n}(B+C,A_2,\ldots,A_n) \\ &\qquad = \sum_{k=0}^{j} \binom{j}{k} P^{k,j-k,m_1-j,m_2,m_3\ldots,m_n}_{\mu_1,\mu_1,\mu_1,\mu_2,\mu_3\ldots,\mu_n}(B,C,B+C,A_2,A_3,\ldots,A_n). \quad (4.69)\end{aligned}$$

For the case $j = 0$, we see that the right-hand side of Equation (4.69) is just the one term

$$\begin{aligned} &P^{0,0,m_1-0,m_2,m_3\ldots,m_n}_{\mu_1,\mu_1,\mu_1,\mu_2,\mu_3\ldots,\mu_n}(B,C,B+C,A_2,A_3,\ldots,A_n) \\ &\qquad = P^{m_1,m_2,m_3\ldots,m_n}_{\mu_1,\mu_2,\mu_3\ldots,\mu_n}(B+C,A_2,A_3,\ldots,A_n), \quad (4.70)\end{aligned}$$

which is identical to the left-hand side.

Let us now suppose that Equation (4.69) holds for each $j = 0,1,\ldots,h$, where $h < m_1$:

$$\begin{aligned} &P^{m_1,m_2,\ldots,m_n}_{\mu_1,\mu_2,\ldots,\mu_n}(B+C,A_2,\ldots,A_n) \\ = &\sum_{k=0}^{h} \binom{h}{k} P^{k,h-k,m_1-h,m_2,m_3\ldots,m_n}_{\mu_1,\mu_1,\mu_1,\mu_2,\mu_3\ldots,\mu_n}(B,C,B+C,A_2,A_3,\ldots,A_n).\end{aligned}$$

We split the third exponent and apply Corollary 3.1.9:

$$\sum_{k=0}^{h} \binom{h}{k} P^{1,k,h-k,m_1-h-1,m_2,m_3\ldots,m_n}_{\mu_1,\mu_1,\mu_1,\mu_1,\mu_2,\mu_3\ldots,\mu_n}(B+C,B,C,B+C,A_2,A_3,\ldots,A_n).$$

Apply Theorem 4.3.7:

$$\begin{gathered}\sum_{k=0}^{h} \binom{h}{k} P^{1,k,h-k,m_1-h-1,m_2,m_3\ldots,m_n}_{\mu_1,\mu_1,\mu_1,\mu_1,\mu_2,\mu_3\ldots,\mu_n}(B,B,C,B+C,A_2,A_3,\ldots,A_n)\\ +\sum_{k=0}^{h} \binom{h}{k} P^{1,k,h-k,m_1-h-1,m_2,m_3\ldots,m_n}_{\mu_1,\mu_1,\mu_1,\mu_1,\mu_2,\mu_3\ldots,\mu_n}(C,B,C,B+C,A_2,A_3,\ldots,A_n).\end{gathered}$$

Apply Corollary 3.1.9 and combine B terms, C terms:

$$\begin{gathered}\sum_{k=0}^{h} \binom{h}{k} P^{k+1,h-k,m_1-h-1,m_2,m_3\ldots,m_n}_{\mu_1,\mu_1,\mu_1,\mu_2,\mu_3\ldots,\mu_n}(B,C,B+C,A_2,A_3,\ldots,A_n)\\ +\sum_{k=0}^{h} \binom{h}{k} P^{k,h+1-k,m_1-h-1,m_2,m_3\ldots,m_n}_{\mu_1,\mu_1,\mu_1,\mu_2,\mu_3\ldots,\mu_n}(B,C,B+C,A_2,A_3,\ldots,A_n).\end{gathered}$$

Finally, we adjust the first summation index to get

$$\begin{aligned}&P^{m_1,m_2,\ldots,m_n}_{\mu_1,\mu_2,\ldots,\mu_n}(B+C,A_2,\ldots,A_n)\\ =\ &\sum_{k=1}^{h+1} \binom{h}{k-1} P^{k,h+1-k,m_1-h-1,m_2,m_3\ldots,m_n}_{\mu_1,\mu_1,\mu_1,\mu_2,\mu_3\ldots,\mu_n}(B,C,B+C,A_2,A_3,\ldots,A_n)\\ &+\sum_{k=0}^{h} \binom{h}{k} P^{k,h+1-k,m_1-h-1,m_2,m_3\ldots,m_n}_{\mu_1,\mu_1,\mu_1,\mu_2,\mu_3\ldots,\mu_n}(B,C,B+C,A_2,A_3,\ldots,A_n). \qquad (4.71)\end{aligned}$$

Applying the properties of binomial coefficients that

$$\binom{h}{0} = \binom{h+1}{0} = 1, \qquad \binom{h}{h} = \binom{h+1}{h+1} = 1,$$

and for $0 < k \leq h$,

$$\binom{h}{k-1} + \binom{h}{k} = \binom{h+1}{k},$$

we get

$$\begin{aligned} &P^{m_1,m_2,\ldots,m_n}_{\mu_1,\mu_2,\ldots,\mu_n}(B+C, A_2, \ldots, A_n) \\ &\quad = \sum_{k=0}^{h+1} \binom{h+1}{k} P^{k,h+1-k,m_1-h-1,m_2,m_3\ldots,m_n}_{\mu_1,\mu_1,\mu_1,\mu_2,\mu_3\ldots,\mu_n}(B, C, B+C, A_2, A_3, \ldots, A_n). \end{aligned} \tag{4.72}$$

By induction, this proves Equation (4.69) for $j = 0, 1, \ldots, m_1$. The case $j = m_1$ yields

$$\begin{aligned} P^{m_1,m_2,\ldots,m_n}_{\mu_1,\mu_2,\ldots,\mu_n}&(B+C, A_2, \ldots, A_n) \\ &= \sum_{k=0}^{m_1} \binom{m_1}{k} P^{k,m_1-k,0,m_2,m_3\ldots,m_n}_{\mu_1,\mu_1,\mu_1,\mu_2,\mu_3\ldots,\mu_n}(B, C, B+C, A_2, A_3, \ldots, A_n) \\ &= \sum_{k=0}^{m_1} \binom{m_1}{k} P^{k,m_1-k,m_2,m_3\ldots,m_n}_{\mu_1,\mu_1,\mu_2,\mu_3\ldots,\mu_n}(B, C, A_2, A_3, \ldots, A_n), \end{aligned} \tag{4.73}$$

which establishes Equation (4.67). $\square$

An alternate characterization of the concatenation of two orderings is possible, as follows:

Theorem 4.3.9 (A concatenation as a particular element of a merging)**.** *If P and Q are disjoint, finite sets, then*

(i) If $\sigma \in \mathcal{O}_P$ and $\tau \in \mathcal{O}_Q$, then $\sigma.\tau \in \{\sigma\} \odot \{\tau\}$, and $(\sigma.\tau)^{-1}(a) < (\sigma.\tau)^{-1}(b)$ for all $a \in P$, $b \in Q$. Conversely, if $\sigma \in \mathcal{O}_P$ and $\tau \in \mathcal{O}_Q$, and if $\pi \in \{\sigma\} \odot \{\tau\}$ and $\pi^{-1}(a) < \pi^{-1}(b)$ for all $a \in P$, $b \in Q$, then $\pi = \sigma.\tau$.

(ii) If $\mathcal{U} \subseteq \mathcal{O}_P$, $\mathcal{V} \subseteq \mathcal{O}_Q$, then $\pi \in \mathcal{U}.\mathcal{V}$ if and only if both $\pi \in \mathcal{U} \odot \mathcal{V}$ and $\pi^{-1}(a) < \pi^{-1}(b)$ for all $a \in P$, $b \in Q$.

Proof. (Proof of (i)) Let $\operatorname{card}(P) = m, \operatorname{card}(Q) = n$.

Let $\pi = \sigma.\tau$, and let $a \in P, b \in Q$. Then by Theorem 4.3.5 parts (iii) and (i), $\pi \in \{\sigma\} \odot \{\tau\}$. Secondly, there exist $j \in \{1, 2, \ldots, m\}, k \in \{1, 2, \ldots, n\}$ with $\sigma(j) = a, \tau(k) = b$. But then by the definition of concatenation (Definition 4.3.1), $\pi(j) = \sigma(j) = a$ and $\pi(m + k) = \tau(k) = b$, so

$$\pi^{-1}(a) = j < m + k = \pi^{-1}(b).$$

On the other hand, suppose that $\rho \in \{\sigma\} \odot \{\tau\}$ and that for every $a \in P, b \in Q$ we have that $\rho^{-1}(a) < \rho^{-1}(b)$. Then the same theorem tells us (part (i) implies (ii)) that $\rho = \tau_1.[\sigma(1)].\tau_2.[\sigma(2)].\cdots.\tau_m.[\sigma(m)].\tau_{m+1}$ for some pairwise disjoint orderings $\tau_1, \tau_2, \ldots, \tau_{m+1}$ of subsets of Q with $\tau = \tau_1.\tau_2.\cdots.\tau_{m+1}$. As a result, if $\tau_m \neq \emptyset$, then $\tau_m(1) \in Q$ and $\sigma(m) \in P$ with $\tau_m(1)$ to the left of $\sigma(m)$ in the ordering ρ, that is, $\rho^{-1}(\tau_m(1)) < \rho^{-1}(\sigma(m))$; this contradicts our assumption that for every $a \in P, b \in Q$ we have $\rho^{-1}(a) < \rho^{-1}(b)$. Therefore, $\tau_m = \emptyset$, and similarly, $\tau_1 = \cdots = \tau_{m-1} = \emptyset$, leaving us with $\tau_{m+1} = \tau$. Consequently, $\rho = [\sigma(1)].[\sigma(2)].\cdots.[\sigma(m)].\tau_{m+1} = \sigma.\tau$.

(Proof of (ii)) Suppose $\pi \in \mathcal{U}.\mathcal{V}$. Then $\pi = \sigma.\tau$ for some orderings $\sigma \in \mathcal{U}, \tau \in \mathcal{V}$, so $\pi = \sigma.\tau \in \{\sigma\} \odot \{\tau\} \subseteq \mathcal{U} \odot \mathcal{V}$, and for all $a \in P$, $b \in Q$ we have $\pi^{-1}(a) < \pi^{-1}(b)$, both by part (i).

On the other hand, suppose that $\pi \in \mathcal{U} \odot \mathcal{V}$ and that $\pi^{-1}(a) < \pi^{-1}(b)$ for all $a \in P$, $b \in Q$. Then $\pi \in \{\sigma\} \odot \{\tau\}$ for some $\sigma \in \mathcal{U}, \tau \in \mathcal{V}$, and therefore $\pi = \sigma.\tau \subseteq \mathcal{U}.\mathcal{V}$ by part (i). $\square$

Example 14 (Disentangling with ordered supports)**.** We can apply the above theorem to prove a result for ordered supports [22, Corollary 2.7]; the proof here is substantially different from that in [22], where it involves probability measures, and where it is a corollary of a theorem on disentangling by means of the 'extraction of a linear factor' (which can be applied if a number of measures have their support in a subinterval $[a, b]$ of $[0, T]$ and the rest have their supports in $[0, a] \cup [b, T]$.). Also, to simplify the exposition, the result below is proved first in the case of disentangling when every exponent equals 1, after which we prove a theorem involving general exponents. The technique used in the early steps of the proof relies on expressing the supports of the measures in terms of characteristic functions.

Let $C_1, \ldots, C_m \in \mathcal{L}(X)$, and associate measures $\nu_1, \ldots, \nu_m \in \mathcal{M}_{cb}[0, 1]$ to $C_1, \ldots, C_m$, respectively. Suppose further that there is an $a \in (0, 1)$ for which $\operatorname{supp}[\nu_1], \ldots, \operatorname{supp}[\nu_k] \subseteq [0, a]$ and $\operatorname{supp}[\nu_{k+1}], \ldots, \operatorname{supp}[\nu_m] \subseteq [a, 1]$ for some $k \in \{0, 1, \ldots, m\}$. Then

$$P^{1,\ldots,1}_{\nu_1,\ldots,\nu_m}(C_1, \ldots, C_m) = P^{1,\ldots,1}_{\nu_{k+1},\ldots,\nu_m}(C_{k+1}, \ldots, C_m) P^{1,\ldots,1}_{\nu_1,\ldots,\nu_k}(C_1, \ldots, C_k), \tag{4.74}$$

where if $k = 0$ we understand $P^{1,\ldots,1}_{\nu_1,\ldots,\nu_k}(C_1, \ldots, C_m)$ to refer to the identity operator $I \in \mathcal{L}(X)$, and if $k = m$ we understand $P^{1,\ldots,1}_{\nu_{k+1},\ldots,\nu_m}(C_{k+1}, \ldots, C_m)$ to refer to the identity operator $I \in \mathcal{L}(X)$.

The reasoning behind Equation (4.74) is as follows: The cases $k = 0$ and $k = m$ are immediate. For the others, we note first that for $j = 1, \ldots, k$ we have $\nu_j = \nu_j|_{[0,a]} =$

$\nu_j|_{(0,a)}$ by hypothesis and because the measures are continuous, and similarly for $j = k+1, \ldots, m$, we have $\nu_j = \nu_j|_{[a,1]} = \nu_j|_{(a,1)}$. By the definition of the disentangling map we have

$$\begin{aligned}
P^{1,\ldots,1}_{\nu_1,\ldots,\nu_m}(C_1, \ldots, C_m) &= \sum_{\pi \in S_m} \int_{\Delta_m(\pi)} C_{\pi(m)} \cdots C_{\pi(1)} (\nu_1 \times \cdots \times \nu_m)(ds_1, \ldots, ds_m) \\
&= \sum_{\pi \in S_m} \int_{\Delta_m(\pi)} C_{\pi(m)} \cdots C_{\pi(1)} (\nu_1|_{(0,a)} \times \cdots \times \nu_k|_{(0,a)} \times \nu_{k+1}|_{(a,1)} \times \cdots \times \nu_m|_{(a,1)})(ds_1, \ldots, ds_m) \\
&= \sum_{\pi \in S_m} \int_{\Delta_m(\pi)} \chi_{(0,a)^k \times (a,1)^{m-k}}(s_1, \ldots, s_m) C_{\pi(m)} \cdots C_{\pi(1)} (\nu_1 \times \cdots \times \nu_m)(ds_1, \ldots, ds_m) \\
&= \sum_{\pi \in S_m} \int_{\Delta_m(\pi) \cap [(0,a)^k \times (a,1)^{m-k}]} C_{\pi(m)} \cdots C_{\pi(1)} (\nu_1 \times \cdots \times \nu_m)(ds_1, \ldots, ds_m). \qquad (4.75)
\end{aligned}$$

The terms of the sum will survive only if the region of integration $\Delta_m(\pi) \cap [(0,a)^k \times (a,1)^{m-k}]$ is nonempty. That region is the set of all points $(s_1, \ldots, s_m)$ for which $s_{\pi(1)} < s_{\pi(2)} < \cdots < s_{\pi(m)}$ and for which $s_1, \ldots, s_k \in (0,a)$ and $s_{k+1}, \ldots, s_m \in (a,1)$. The latter two facts tell us that for all $i \in \{1, \ldots, k\}$ and for all $j \in \{k+1, \ldots, m\}$ we have $s_i \leq a < s_j$. Combining this with the string of inequalities $s_{\pi(1)} < s_{\pi(2)} < \cdots < s_{\pi(m)}$, we can say of the subscripts that $\{\pi(1), \ldots, \pi(k)\} = \{1, \ldots, k\}$ and $\{\pi(k+1), \ldots, \pi(m)\} = \{k+1, \ldots, m\}$. Therefore, $\pi^{-1}(x) < \pi^{-1}(y)$ for all $x \in \{1, \ldots, k\}, y \in \{k+1, \ldots, m\}$. Hence since $S_m = \mathcal{O}_{\{1,\ldots,m\}} = \mathcal{O}_{\{1,\ldots,k\}} \odot \mathcal{O}_{\{k+1,\ldots,m\}}$, by Theorem 4.3.9(ii) we have for the nonzero terms in (4.75) that

$$\pi \in \mathcal{O}_{\{1,\ldots,k\}}.\mathcal{O}_{\{k+1,\ldots,m\}} = \dot{\bigcup_{\substack{\sigma \in \mathcal{O}_{\{1,\ldots,k\}} \\ \tau \in \mathcal{O}_{\{k+1,\ldots,m\}}}}} \{\sigma.\tau\}, \qquad (4.76)$$

and thus

$$P^{1,\ldots,1}_{\nu_1,\ldots,\nu_m}(C_1, \ldots, C_m)$$

$$= \sum_{\substack{\sigma \in \mathcal{O}_{\{1,\dots,k\}} \\ \tau \in \mathcal{O}_{\{k+1,\dots,m\}}}} \int_{\Delta_m(\sigma.\tau)\cap[(0,a)^k \times (a,1)^{m-k}]} C_{\tau(m-k)} \cdots C_{\tau(1)} C_{\sigma(k)} \cdots C_{\sigma(1)} (\nu_1 \times \cdots \times \nu_m)(ds_1, \dots, ds_m). \tag{4.77}$$

(Note since $\sigma.\tau$ is a permutation, we have $\Delta_m(\sigma.\tau) = \{(s_1, \dots, s_m) : 0 < s_{\sigma(1)} < \cdots < s_{\sigma(k)} < s_{\tau(1)} < \cdots < s_{\tau(m-k)} < 1\}$.)

The region of integration in (4.77) is

$$\begin{aligned} &\Delta_m(\sigma.\tau) \cap [(0,a)^k \times (a,1)^{m-k}] \\ &= \{(s_1, \dots, s_m) :\ 0 < s_{\sigma(1)} < \cdots < s_{\sigma(k)} < a < s_{\tau(1)} < \cdots < s_{\tau(m-k)} < 1\} \\ &= \{(s_1, \dots, s_k) :\ 0 < s_{\sigma(1)} < \cdots < s_{\sigma(k)} < a\} \\ &\qquad\qquad \times \{(s_{k+1}, \dots, s_m) :\ a < s_{\tau(1)} < \cdots < s_{\tau(m-k)} < 1\}. \end{aligned} \tag{4.78}$$

For $\tau \in \mathcal{O}_{\{k+1,\dots,m\}}$ we define $\Delta_{k+1,m}(\tau) := \{(s_{k+1}, \dots, s_m) : 0 < s_{\tau(1)} < \cdots < s_{\tau(m-k)} < 1\}$, and then up to a set of $\nu_1 \times \cdots \times \nu_m$-measure zero, we have

$$\Delta_m(\sigma.\tau) \cap [(0,a)^k \times (a,1)^{m-k}] = \Delta_k(\sigma) \times \Delta_{k+1,m}(\tau).$$

Therefore, using the equality just above and applying the Fubini-Tonelli Theorem, we have

$$\begin{aligned} &P^{1,\dots,1}_{\nu_1,\dots,\nu_m}(C_1, \dots, C_m) \\ &= \sum_{\substack{\sigma \in \mathcal{O}_{\{1,\dots,k\}} \\ \tau \in \mathcal{O}_{\{k+1,\dots,m\}}}} \int_{\Delta_k(\sigma)\times\Delta_{k+1,m}(\tau)} C_{\tau(m-k)} \cdots C_{\tau(1)} C_{\sigma(k)} \cdots C_{\sigma(1)} (\nu_1 \times \cdots \times \nu_m)(ds_1, \dots, ds_m) \end{aligned}$$

$$= \sum_{\substack{\sigma\in\mathcal{O}_{\{1,\ldots,k\}}\\ \tau\in\mathcal{O}_{\{k+1,\ldots,m\}}}} \int_{\Delta_{k+1,m}(\tau)} \left[\int_{\Delta_k(\sigma)} C_{\tau(m-k)}\cdots C_{\tau(1)}C_{\sigma(k)}\cdots C_{\sigma(1)}\right.$$

$$\left. \times (\nu_1\times\cdots\times\nu_k)(ds_1,\ldots,ds_k)\right](\nu_{k+1}\times\cdots\times\nu_m)(ds_{k+1},\ldots,ds_m)$$

$$= \left[\sum_{\tau\in\mathcal{O}_{\{k+1,\ldots,m\}}} \int_{\Delta_{k+1,m}(\tau)} C_{\tau(m-k)}\cdots C_{\tau(1)}(\nu_{k+1}\times\cdots\times\nu_m)(ds_{k+1},\ldots,ds_m)\right]$$

$$\times \left[\sum_{\sigma\in\mathcal{O}_{\{1,\ldots,k\}}} \int_{\Delta_k(\sigma)} C_{\sigma(k)}\cdots C_{\sigma(1)}(\nu_1\times\cdots\times\nu_k)(ds_1,\ldots,ds_k)\right]$$

$$= P^{1,\ldots,1}_{\nu_{k+1},\ldots,\nu_m}(C_{k+1},\ldots,C_m)P^{1,\ldots,1}_{\nu_1,\ldots,\nu_k}(C_1,\ldots,C_k). \tag{4.79}$$

This establishes Equation (4.74).

From the result in the preceding example, we may immediately state the following theorem, which is found in [22, Corollary 2.7] (there, it is a corollary of a theorem on the 'extraction of a linear factor', a concept pursued further in [18]):

Theorem 4.3.10 (Disentangling with ordered supports). *Let $A_1,\ldots,A_n \in \mathcal{L}(X)$ be operators to which we associate finite, continuous Borel measures $\mu_1,\ldots,\mu_n$ on* $[0,1]$*, and suppose that there is an $a\in(0,1)$ and an $l\in\{1,\ldots,n\}$ with* $\operatorname{supp}[\mu_1],\ldots,$ $\operatorname{supp}[\mu_l]\subseteq[0,a]$ *and* $\operatorname{supp}[\mu_{l+1}],\ldots,\operatorname{supp}[\mu_n]\subseteq[a,1]$. *Let $m_1,\ldots,m_n$ be nonnegative integers. Then*

$$P^{m_1,\ldots,m_n}_{\mu_1,\ldots,\mu_n}(A_1,\ldots,A_n) = P^{m_{l+1},\ldots,m_n}_{\mu_{l+1},\ldots,\mu_n}(A_{l+1},\ldots,A_n)P^{m_1,\ldots,m_l}_{\mu_1,\ldots,\mu_l}(A_1,\ldots,A_l). \tag{4.80}$$

Proof. The theorem is proved by making the same assignment of names of operators as in Equation 4.29, as well as assignments of the names $\nu_1,\ldots,\nu_m$ to the measures $\mu_1,\ldots,\mu_n$ corresponding to the way $C_1,\ldots,C_m$ are assigned to $A_1,\ldots,A_n$, letting $k := m_1+\cdots+m_l$, and then applying Equation (4.74). □

Example 15. With Theorem 4.3.10 in hand, we are now able to illustrate Theorem 3.2.2, as promised above, employing it to reproduce a 'decomposing disentanglings' result from [17, p. 4]. Let $A_1, A_2 \in \mathcal{L}(X)$ be associated with measures $\mu_1, \mu_2 \in \mathcal{M}_{cb}[0,1]$, respectively, let m_1, m_2 be nonnegative integers, and let $a \in (0,1)$. Define $\mu_{1,1} := \mu_1|_{[0,a]}, \mu_{1,2} := \mu_1|_{[a,1]}, \mu_{2,1} := \mu_2|_{[0,a]}$, and $\mu_{2,2} := \mu_2|_{[a,1]}$. Then we claim that

$$P^{m_1,m_2}_{\mu_1,\mu_2}(A_1,A_2) = \sum_{\substack{i_1+j_1=m_1\\ i_2+j_2=m_2}} \frac{m_1!}{i_1!j_1!}\frac{m_2!}{i_2!j_2!} P^{j_1,j_2}_{\mu_{1,2},\mu_{2,2}}(A_1,A_2) P^{i_1,i_2}_{\mu_{1,1},\mu_{2,1}}(A_1,A_2). \tag{4.81}$$

To see this, first note that $\mu_1 = \mu_{1,1} + \mu_{1,2}$ and $\mu_2 = \mu_{2,1} + \mu_{2,2}$. Then using Theorem 3.2.2 twice, we have

$$\begin{aligned}
P^{m_1,m_2}_{\mu_1,\mu_2}(A_1,A_2) &= P^{m_1,m_2}_{\mu_{1,1}+\mu_{1,2},\mu_{2,1}+\mu_{2,2}}(A_1,A_2)\\
&= \sum_{i_1=0}^{m_1} \binom{m_1}{i_1} P^{i_1,m_1-i_1,m_2}_{\mu_{1,1},\mu_{1,2},\mu_{2,1}+\mu_{2,2}}(A_1,A_1,A_2)\\
&= \sum_{i_1=0}^{m_1} \binom{m_1}{i_1} \sum_{i_2=0}^{m_2} \binom{m_2}{i_2} P^{i_1,m_1-i_1,i_2,m_2-i_2}_{\mu_{1,1},\mu_{1,2},\mu_{2,1},\mu_{2,2}}(A_1,A_1,A_2,A_2)\\
&= \sum_{i_1=0}^{m_1}\sum_{i_2=0}^{m_2} \frac{m_1!}{i_1!(m_1-i_1)!}\frac{m_2!}{i_2!(m_2-i_2)!} P^{i_1,i_2,m_1-i_1,m_2-i_2}_{\mu_{1,1},\mu_{2,1},\mu_{1,2},\mu_{2,2}}(A_1,A_2,A_1,A_2)\\
&= \sum_{\substack{i_1+j_1=m_1\\ i_2+j_2=m_2}} \frac{m_1!}{i_1!j_1!}\frac{m_2!}{i_2!j_2!} P^{i_1,i_2,j_1,j_2}_{\mu_{1,1},\mu_{2,1},\mu_{1,2},\mu_{2,2}}(A_1,A_2,A_1,A_2). \qquad (4.82)
\end{aligned}$$

We then apply theorem Theorem 4.3.10 to arrive at Equation (4.81).

We continue our discussion of the relationship of concatenation operation and the merge operation with the following theorem:

Theorem 4.3.11. *If $P_1, P_2, \ldots, P_n$ are pairwise disjoint, finite sets, and $\sigma_1 \in \mathcal{O}_{P_1}$, $\sigma_2 \in \mathcal{O}_{P_2}$, …, $\sigma_n \in \mathcal{O}_{P_n}$, then $\sigma_1.\sigma_2.\cdots.\sigma_n \in \{\sigma_1\} \odot \{\sigma_2\} \odot \cdots \odot \{\sigma_n\}$. (For the case*

$n = 1$ we will interpret $\{\sigma_1\} \odot \{\sigma_2\} \odot \cdots \odot \{\sigma_n\}$ to equal $\{\sigma_1\}$.)

Proof. We will prove this by induction. The case $n = 1$ is immediate, since it merely claims $\sigma_1 \in \{\sigma_1\}$. The case $n = 2$, is immediate from Theorem 4.3.5 (using (iii)⇒(i), letting $\tau_1 := \sigma_2, k = 1$). To prove the rest, we suppose that the conclusion of the theorem holds for some $n = k \geq 2$; that is, $\sigma_1.\sigma_2.\cdots.\sigma_k \in \{\sigma_1\} \odot \{\sigma_2\} \odot \cdots \{\sigma_k\}$. Then

$$\begin{aligned} \sigma_1.\sigma_2.\cdots.\sigma_k.\sigma_{k+1} &= (\sigma_1.\sigma_2.\cdots.\sigma_k).(\sigma_{k+1}) && (4.83)\\ &\in \{\sigma_1.\sigma_2.\cdots.\sigma_k\} \odot \{\sigma_{k+1}\} && (4.84)\\ &\subseteq (\{\sigma_1\} \odot \{\sigma_2\} \odot \cdots \odot \{\sigma_k\}) \odot \{\sigma_{k+1}\} && (4.85) \end{aligned}$$

by the case $n = 2$, by the induction hypothesis, and by Theorem 4.2.6. Thus $\sigma_1.\sigma_2.\cdots.\sigma_{k+1} \in \{\sigma_1\} \odot \{\sigma_2\} \odot \cdots \odot \{\sigma_{k+1}\}$, and the theorem follows by induction. □

Part of the usefulness of the merge and concatenation operations is that they enable us to express a set of orderings in two or more ways. In particular, if we use a set of orderings as the index set for a summation, then we can express the summation in more than one way. We are therefore interested in stating a few theorems which equate sets of orderings.

Theorem 4.3.12 (Set relations and concatenation)**.** *If P and Q are disjoint, finite sets, and if $\mathcal{U} \subseteq \mathcal{O}_P$ and $\mathcal{V}, \mathcal{W} \subseteq \mathcal{O}_Q$, then*

(i) $\mathcal{U}.(\mathcal{V} \cup \mathcal{W}) = (\mathcal{U}.\mathcal{V}) \cup (\mathcal{U}.\mathcal{W})$,

(ii) $\mathcal{U}.(\mathcal{V} \cap \mathcal{W}) = (\mathcal{U}.\mathcal{V}) \cap (\mathcal{U}.\mathcal{W})$,

(iii) $\mathcal{U}.(\mathcal{V} \setminus \mathcal{W}) = (\mathcal{U}.\mathcal{V}) \setminus (\mathcal{U}.\mathcal{W})$ *and* $(\mathcal{V} \setminus \mathcal{W}).\mathcal{U} = (\mathcal{V}.\mathcal{U}) \setminus (\mathcal{W}.\mathcal{U})$, *and*

(iv) if $\mathcal{V} \cap \mathcal{W} = \emptyset$*, then* $\mathcal{U}.(\mathcal{V} \dot{\cup} \mathcal{W}) = (\mathcal{U}.\mathcal{V}) \dot{\cup} (\mathcal{U}.\mathcal{W})$.

Proof. (i) Claim: $\mathcal{U}.(\mathcal{V} \cup \mathcal{W}) = (\mathcal{U}.\mathcal{V}) \cup (\mathcal{U}.\mathcal{W})$.

(Proof of $\subseteq$.) Let $\pi \in \mathcal{U}.(\mathcal{V} \cup \mathcal{W})$. Then there exist $\sigma \in \mathcal{U}, \tau \in \mathcal{V} \cup \mathcal{W}$ with $\pi = \sigma.\tau$. But then $\tau \in \mathcal{V}$ or $\tau \in \mathcal{W}$, so $\pi \in \mathcal{U}.\mathcal{V}$ or $\pi \in \mathcal{U}.W$. Thus $\pi \in (\mathcal{U}.\mathcal{V}) \cup (\mathcal{U}.\mathcal{W})$.

(Proof of $\supseteq$.) Since $\mathcal{V}, \mathcal{W} \subseteq \mathcal{V} \cup \mathcal{W}$, we have $\mathcal{U}.\mathcal{V} \subseteq \mathcal{U}.(\mathcal{V} \cup \mathcal{W})$ and $\mathcal{U}.\mathcal{W} \subseteq \mathcal{U}.(\mathcal{V} \cup \mathcal{W})$. Therefore, $(\mathcal{U}.\mathcal{V}) \cup (\mathcal{U}.\mathcal{W}) \subseteq \mathcal{U}.(\mathcal{V} \cup \mathcal{W})$.

(ii) Claim: $\mathcal{U}.(\mathcal{V} \cap \mathcal{W}) = (\mathcal{U}.\mathcal{V}) \cap (\mathcal{U}.\mathcal{W})$.

(Proof of $\subseteq$.) Since $\mathcal{V} \cap \mathcal{W} \subseteq \mathcal{V}, \mathcal{W}$ we have $\mathcal{U}.(\mathcal{V} \cap \mathcal{W}) \subseteq \mathcal{U}.\mathcal{V}$ and $\mathcal{U}.(\mathcal{V} \cap \mathcal{W}) \subseteq \mathcal{U}.\mathcal{W}$. Therefore, $\mathcal{U}.(\mathcal{V} \cap \mathcal{W}) \subseteq (\mathcal{U}.\mathcal{V}) \cap (\mathcal{U}.\mathcal{W})$.

(Proof of $\supseteq$.) Let $\pi \in (\mathcal{U}.\mathcal{V}) \cap (\mathcal{U}.\mathcal{W})$. Then $\pi \in \mathcal{U}.\mathcal{V}$ and $\pi \in \mathcal{U}.\mathcal{W}$, so there exist $\sigma \in \mathcal{U}, \tau \in \mathcal{V}$ with $\pi = \sigma.\tau$, and there exist $\sigma' \in \mathcal{U}, \tau' \in \mathcal{W}$ with $\pi = \sigma'.\tau'$. But then $\pi = [\sigma(1), \ldots, \sigma(\mathrm{card}(P)), \tau(1), \ldots, \tau(\mathrm{card}(Q))] = [\sigma'(1), \ldots, \sigma'(\mathrm{card}(P)), \tau'(1), \ldots, \tau'(\mathrm{card}(Q))]$, and equating these expressions term-by-term gives $\sigma = \sigma'$ and $\tau = \tau'$, so $\tau \in \mathcal{V} \cap \mathcal{W}$. Therefore, $\pi = \sigma.\tau \in \mathcal{U}.(\mathcal{V} \cap \mathcal{W})$, and hence $(\mathcal{U}.\mathcal{V}) \cap (\mathcal{U}.\mathcal{W}) \subseteq \mathcal{U}.(\mathcal{V} \cap \mathcal{W})$.

(iii) Claim: $\mathcal{U}.(\mathcal{V} \smallsetminus \mathcal{W}) = (\mathcal{U}.\mathcal{V}) \smallsetminus (\mathcal{U}.\mathcal{W})$.

(Proof of $\subseteq$.) Let $\pi \in \mathcal{U}.(\mathcal{V} \smallsetminus \mathcal{W})$. Then there exist $\sigma \in \mathcal{U}$, $\tau \in \mathcal{V} \smallsetminus \mathcal{W}$ such that $\pi = \sigma.\tau$. Then $\tau \in \mathcal{V}$ and $\tau \notin \mathcal{W}$. It is clear that $\pi = \sigma.\tau \in \mathcal{U}.\mathcal{V}$, and we claim that $\pi \notin \mathcal{U}.\mathcal{W}$. Supposing on the contrary that $\pi \in \mathcal{U}.\mathcal{W}$ implies that there are $\sigma' \in \mathcal{U}$, $\tau' \in \mathcal{W}$ with $\pi = \sigma'.\tau'$. But then $\sigma.\tau = \sigma'.\tau'$, so $\sigma = \sigma'$, $\tau = \tau'$, and then $\tau = \tau' \in \mathcal{W}$, which is a contradiction. Therefore, $\pi \notin \mathcal{U}.\mathcal{W}$, and therefore $\pi \in (\mathcal{U}.\mathcal{V}) \smallsetminus (\mathcal{U}.\mathcal{W})$. Thus $\mathcal{U}.(\mathcal{V} \smallsetminus \mathcal{W}) \subseteq (\mathcal{U}.\mathcal{V}) \smallsetminus (\mathcal{U}.\mathcal{W})$.

(Proof of $\supseteq$.) Conversely, suppose that $\pi \in (\mathcal{U}.\mathcal{V}) \smallsetminus (\mathcal{U}.\mathcal{W})$. Then $\pi \in \mathcal{U}.\mathcal{V}$, but $\pi \notin \mathcal{U}.\mathcal{W}$. For $\pi \in \mathcal{U}.\mathcal{V}$, write $\pi = \sigma.\tau$ for some $\sigma \in \mathcal{U}$, $\tau \in \mathcal{V}$. We claim that

$\tau \notin \mathcal{W}$. If on the contrary we suppose that $\tau \in \mathcal{W}$, then $\pi = \sigma.\tau \in \mathcal{U}.\mathcal{W}$, a contradiction. Therefore $\tau \notin \mathcal{W}$, and it follows that $\pi = \sigma.\tau \in \mathcal{U}.(\mathcal{V} \setminus \mathcal{W})$. We conclude $(\mathcal{U}.\mathcal{V}) \setminus (\mathcal{U}.\mathcal{W}) \subseteq \mathcal{U}.(\mathcal{V} \setminus \mathcal{W})$.

The proof that $(\mathcal{V} \setminus \mathcal{W}).\mathcal{U} = (\mathcal{V}.\mathcal{U}) \setminus (\mathcal{W}.\mathcal{U})$ is completely analogous.

(iv) Claim: If $\mathcal{V} \cap \mathcal{W} = \emptyset$, then $\mathcal{U}.(\mathcal{V} \dot\cup \mathcal{W}) = (\mathcal{U}.\mathcal{V}) \dot\cup (\mathcal{U}.\mathcal{W})$.

Let $\mathcal{V} \cap \mathcal{W} = \emptyset$. Then by (i) we have that $\mathcal{U}.(\mathcal{V} \dot\cup \mathcal{W}) = (\mathcal{U}.\mathcal{V}) \cup (\mathcal{U}.\mathcal{W})$, so all we need to show is that the union on the right-hand side is disjoint. But in fact, by (ii) we have that $(\mathcal{U}.\mathcal{V}) \cap (\mathcal{U}.\mathcal{W}) = \mathcal{U}.(\mathcal{V} \cap \mathcal{W}) = \mathcal{U}.\emptyset = \emptyset$, so the union on the right-hand side is a disjoint union.

□

Theorem 4.3.13. *If $\sigma \in \mathcal{O}_P$ and $\tau \in \mathcal{O}_Q$ for disjoint, finite sets P and Q with $m = \mathrm{card}(P)$, $n = \mathrm{card}(Q)$, then*

$$
\begin{aligned}
\{\sigma\} \odot \{\tau\} &= \dot{\bigcup_{\tau_1.\tau_2.\cdots.\tau_{m+1}=\tau}} \{\tau_1.[\sigma(1)].\tau_2.[\sigma(2)].\cdots.\tau_m.[\sigma(m)].\tau_{m+1}\} \\
&= \dot{\bigcup_{\substack{j_1+\cdots+j_{m+1}=n \\ j_1,\ldots,j_{m+1}\geq 0}}} \{[\tau(1), \tau(2), \ldots, \tau(j_1), \sigma(1), \\
&\qquad \tau(j_1+1), \tau(j_1+2), \ldots, \tau(j_1+j_2), \sigma(2), \\
&\qquad \tau(j_1+j_2+1), \tau(j_1+j_2+2), \ldots, \tau(j_1+j_2+j_3), \sigma(3), \\
&\qquad \ldots \quad \tau(j_1+\cdots+j_m), \sigma(m), \\
&\qquad \tau(j_1+\cdots+j_m+1), \ldots, \tau(n)]\},
\end{aligned} \tag{4.86}
$$

where '$\dot\cup$' indicates a disjoint union, and where $\tau_1, \ldots, \tau_{m+1}$ are disjoint orderings of finite subsets of subsets of Q.

Proof. Theorem 4.3.5 parts (i) and (ii) immediately give

$$\{\sigma\} \odot \{\tau\} = \bigcup_{\tau_1.\tau_2.\cdots.\tau_{m+1}=\tau} \{\tau_1.[\sigma(1)].\tau_2.[\sigma(2)].\cdots.\tau_m.[\sigma(m)].\tau_{m+1}\}, \tag{4.87}$$

so for the first equality it remains to show only that the union is disjoint.

Suppose that

$$\tau_1.[\sigma(1)].\tau_2.[\sigma(2)].\cdots.\tau_m.[\sigma(m)].\tau_{m+1} = \tau_1'.[\sigma(1)].\tau_2'.[\sigma(2)].\cdots.\tau_m'.[\sigma(m)].\tau_{m+1}' \tag{4.88}$$

for some $\tau_1.\tau_2.\cdots.\tau_{m+1} = \tau_1'.\tau_2'.\cdots.\tau_{m+1}' = \tau$. The two orderings on the two sides of Equation (4.88) must match entry-by-entry. Since $\sigma(1)$ on one side equals $\sigma(1)$ on the other, $\sigma(1)$ must appear at the same position in the two orderings, so we must have that length(τ_1) = length(τ_1'), and hence, $\tau_1 = \tau_1'$. Similarly $\tau_2 = \tau_2'$, $\tau_3 = \tau_3'$, $\ldots$, $\tau_{m+1} = \tau_{m+1}'$. But then as indices for the union, $\tau_1.\tau_2.\cdots.\tau_{m+1} = \tau$ and $\tau_1'.\tau_2'.\cdots.\tau_{m+1}' = \tau$ are the same index; they correspond to the same term of the union. Therefore, distinct indices correspond to distinct terms of the union, which are therefore disjoint sets (since the sets are singletons). Thus the union is disjoint.

The second equality in (4.86) holds because

$$\dot{\bigcup_{\tau_1.\tau_2.\cdots.\tau_{m+1}=\tau}} \{\tau_1.[\sigma(1)].\tau_2.[\sigma(2)].\cdots.\tau_m.[\sigma(m)].\tau_{m+1}\}$$

$$
\begin{aligned}
= \biguplus_{\substack{\tau_1.\tau_2.\cdots.\tau_{m+1}=\tau \\ j_1:=\mathrm{length}(\tau_1) \\ j_2:=\mathrm{length}(\tau_2) \\ \cdots \\ j_{m+1}:=\mathrm{length}(\tau_{m+1})}} \quad & \{[\tau(1),\tau(2),\dots,\tau(j_1)].[\sigma(1)] \\
& .[\tau(j_1+1),\tau(j_1+2),\dots,\tau(j_1+j_2)].[\sigma(2)] \\
& .[\tau(j_1+j_2+1),\tau(j_1+j_2+2),\dots,\tau(j_1+j_2+j_3)].[\sigma(3)] \\
& \qquad \dots \qquad .[\dots, \tau(j_1+\cdots+j_m)].[\sigma(m)] \\
& .[\tau(j_1+\cdots+j_m+1),\dots,\tau(n)]\} \\
= \biguplus_{\substack{j_1+\cdots+j_{m+1}=n \\ j_1,\dots,j_{m+1}\geq 0}} \quad & \{[\tau(1),\tau(2),\dots,\tau(j_1),\, \sigma(1), \\
& \tau(j_1+1),\tau(j_1+2),\dots,\tau(j_1+j_2),\sigma(2), \\
& \tau(j_1+j_2+1),\tau(j_1+j_2+2),\dots,\tau(j_1+j_2+j_3),\sigma(3), \\
& \qquad \dots \qquad \tau(j_1+\cdots+j_m),\sigma(m), \\
& \tau(j_1+\cdots+j_m+1),\dots,\tau(n)]\},
\end{aligned} \tag{4.89}
$$

where in the second step we have the unions applying to the same terms, so we merely have to note that the index sets are the same. That is true because on the left side of the equation we use the orderings $\tau_1,\dots,\tau_{m+1}$ only to find their lengths to determine the values of $j_1,\dots,j_{m+1}$ (they do not appear elsewhere in the expression), which must therefore take all possible nonnegative values having a sum of $\mathrm{length}(\tau)=n$, and that is exactly the index set specified on the right side of the equation. $\square$

Example 16. Let $\sigma := [1,7]$ ($P=\{1,7\}$) and $\tau := [8,9]$ ($Q=\{8,9\}$). Using Theorem 4.3.13, we note that $m=2, n=2$, so the second union in the theorem is taken over sums $j_1+j_2+j_3=2$, where j_1,j_2,j_3 are nonnegative integers. The only possibilities are $(j_1,j_2,j_3)=(2,0,0),(0,2,0),(0,0,2),(1,1,0),(1,0,1)$, and $(0,1,1)$. Each choice gives us one term of the union shown in the theorem. For example, $(1,0,1)$ gives us the ordering $[\tau(1),\sigma(1),\sigma(2),\tau(2)]=[8,1,7,9]$.

Another way to look at this is that we are splitting τ into a concatenation of $m+1=2+1=3$ pieces in all possible ways and inserting the entries of σ in between, one at a time. (That is essentially what the first union in the theorem does.) The choice of indices $(j_1, j_2, j_3) = (1, 0, 1)$ means τ is a concatenation of orderings of lengths $1, 0$, and 1, in that order, thus $\tau := [8, 9] = [8].\emptyset.[9]$. This contributes the ordering $[8].[\sigma(1)].\emptyset.[\sigma(2)].[9] = [8].[1].\emptyset.[7].[9] = [8, 1, 7, 9]$ to the union. Similarly, $(j_1, j_2, j_3) = (2, 0, 0)$ contributes the ordering $[8, 9].[1].\emptyset.[7].\emptyset = [8, 9, 1, 7]$.

Continuing with the other choices of indices we obtain the set of orderings

$$\{[8, 9, 1, 7], [1, 8, 9, 7], [1, 7, 8, 9], [8, 1, 9, 7], [8, 1, 7, 9], [1, 8, 7, 9]\},$$

which we can verify is exactly $\{\sigma\} \odot \{\tau\}$.

Theorem 4.3.14. *Let P and Q be finite, disjoint sets with $m = \mathrm{card}(P)$. If $\sigma \in \mathcal{O}_P$, then*

$$\{\sigma\} \odot \mathcal{O}_Q = \bigcup^{\bullet}_{Q_1 \uplus \cdots \uplus Q_{m+1} = Q} \mathcal{O}_{Q_1}.\{[\sigma(1)]\}.\mathcal{O}_{Q_2}.\{[\sigma(2)]\}.\cdots.\{[\sigma(m)]\}.\mathcal{O}_{Q_{m+1}}. \tag{4.90}$$

Proof. By Theorem 4.3.13 we have

$$\begin{aligned} \{\sigma\} \odot \mathcal{O}_Q &= \bigcup^{\bullet}_{\tau \in \mathcal{O}_Q} \{\sigma\} \odot \{\tau\} \\ &= \bigcup^{\bullet}_{\tau \in \mathcal{O}_Q} \bigcup^{\bullet}_{\tau_1.\tau_2.\cdots.\tau_{m+1} = \tau} \{\tau_1.[\sigma(1)].\tau_2.[\sigma(2)].\cdots.\tau_m.[\sigma(m)].\tau_{m+1}\} \end{aligned} \tag{4.91}$$

where $\tau_1, \ldots, \tau_{m+1}$ are disjoint orderings of subsets of Q.

We examine the index set for the last union. Given any choice of $\tau \in \mathcal{O}_Q$ and concatenation of disjoint orderings $\tau_1.\tau_2.\cdots.\tau_{m+1} = \tau$, we may assign the disjoint subsets of Q ordered by $\tau_1, \ldots, \tau_{m+1}$ the names $Q_1, \ldots, Q_{m+1}$, respectively. Note that this yields exactly one possible ordered choice of the sets $Q_1, \ldots, Q_{m+1}$, since each τ_j is an ordering of exactly one set. This implies $\tau = \tau_1.\tau_1.\cdots.\tau_{m+1} \in \mathcal{O}_{Q_1 \dot\cup \cdots \dot\cup Q_{m+1}}$, and since τ orders Q, this implies $Q = Q_1 \dot\cup \cdots \dot\cup Q_{m+1}$. On the other hand, given any partition $Q = Q_1 \dot\cup \cdots \dot\cup Q_{m+1}$ and choices $\tau_1 \in \mathcal{O}_{Q_1}, \ldots, \tau_{m+1} \in \mathcal{O}_{Q_{m+1}}$, there is a unique ordering $\tau = \tau_1.\tau_2.\cdots.\tau_{m+1}$; note $\tau \in \mathcal{O}_{Q_1 \dot\cup \cdots \dot\cup Q_{m+1}} = \mathcal{O}_Q$. We therefore have a bijection between the set of all $(m+2)$-tuples of the form $(\tau, \tau_1, \ldots, \tau_{m+1})$ with $\tau \in \mathcal{O}_Q$ and $\tau_1.\tau_2.\cdots.\tau_{m+1} = \tau$ (where $\tau_1, \ldots, \tau_{m+1}$ are disjoint) and the set of all $(2m+2)$-tuples of the form $(Q_1, \ldots, Q_{m+1}, \tau_1, \ldots, \tau_{m+1})$ with $Q = Q_1 \dot\cup \cdots \dot\cup Q_{m+1}$ and $\tau_1 \in \mathcal{O}_{Q_1}, \ldots, \tau_{m+1} \in \mathcal{O}_{Q_{m+1}}$. Consequently, we may re-index the union:

$$\{\sigma\} \odot \mathcal{O}_Q = \dot\bigcup_{Q_1 \dot\cup \cdots \dot\cup Q_{m+1} = Q} \; \dot\bigcup_{\substack{\tau_1 \in \mathcal{O}_{Q_1} \\ \tau_2 \in \mathcal{O}_{Q_2} \\ \cdots \\ \tau_{m+1} \in \mathcal{O}_{Q_{m+1}}}} \{\tau_1.[\sigma(1)].\tau_2.[\sigma(2)].\cdots.\tau_m.[\sigma(m)].\tau_{m+1}\}$$

$$= \dot\bigcup_{Q_1 \dot\cup \cdots \dot\cup Q_{m+1} = Q} \mathcal{O}_{Q_1}.\{[\sigma(1)]\}.\mathcal{O}_{Q_2}.\{[\sigma(2)]\}.\cdots.\{[\sigma(m)]\}.\mathcal{O}_{Q_{m+1}}, \qquad (4.92)$$

where the last step follows from the definition of concatenating sets. □

Corollary 4.3.15. *If P and Q are finite, disjoint sets and $m = \mathrm{card}(P)$, then*

$$\mathcal{O}_{P \dot\cup Q} = \mathcal{O}_P \odot \mathcal{O}_Q = \dot\bigcup_{\substack{\sigma \in \mathcal{O}_P \\ Q_1 \dot\cup \cdots \dot\cup Q_{m+1} = Q}} \mathcal{O}_{Q_1}.\{[\sigma(1)]\}.\mathcal{O}_{Q_2}.\{[\sigma(2)]\}.\cdots.\{[\sigma(m)]\}.\mathcal{O}_{Q_{m+1}}. \qquad (4.93)$$

Proof. The first equality was established earlier (Theorem 4.2.11), and it is included here only to draw attention to the relationship between the expression $\mathcal{O}_P \odot \mathcal{O}_Q$ and the right-hand side. The proof below will be of the second equality. Using the definition of merging and Theorem 4.3.14,

$$\begin{aligned}
\mathcal{O}_{P\uplus Q} &= \mathcal{O}_P \odot \mathcal{O}_Q \\
&= \dot{\bigcup_{\sigma\in\mathcal{O}_P}} \dot{\bigcup_{\tau\in\mathcal{O}_Q}} \{\sigma\} \odot \{\tau\} \\
&= \dot{\bigcup_{\sigma\in\mathcal{O}_P}} \{\sigma\} \odot \mathcal{O}_Q \\
&= \dot{\bigcup_{\sigma\in\mathcal{O}_P}} \dot{\bigcup_{Q_1\uplus\cdots\uplus Q_{m+1}=Q}} \mathcal{O}_{Q_1}.\{[\sigma(1)]\}.\mathcal{O}_{Q_2}.\{[\sigma(2)]\}.\cdots.\{[\sigma(m)]\}.\mathcal{O}_{Q_{m+1}} \\
&= \dot{\bigcup_{\substack{\sigma\in\mathcal{O}_P \\ Q_1\uplus\cdots\uplus Q_{m+1}=Q}}} \mathcal{O}_{Q_1}.\{[\sigma(1)]\}.\mathcal{O}_{Q_2}.\{[\sigma(2)]\}.\cdots.\{[\sigma(m)]\}.\mathcal{O}_{Q_{m+1}}. \qquad (4.94)
\end{aligned}$$

□

Example 17. (This example is motivated by Section 19.4 of [24].) Consider the same operators $A_1, \ldots, A_n, C_1, \ldots, C_m$, nonnegative integers $m_1, \ldots, m_n$, and measures $\mu_1, \ldots, \mu_n$ as in Example 11 above, together with an operator $B \in \mathcal{L}(X)$ associated with the finite, continuous Borel measure ν on $[0, 1]$ and a nonnegative integer k. We want to find an expression for $P^{m_1,\ldots,m_n,k}_{\mu_1,\ldots,\mu_n,\nu}(A_1, \ldots, A_n, B)$. Let $C_{m+1} := C_{m+2} := \cdots := C_{m+k} := B$. Then by the definition of the disentangling map,

$$\begin{aligned}
&P^{m_1,\ldots,m_n,k}_{\mu_1,\ldots,\mu_n,\nu}(A_1, \ldots, A_n, B) \\
&\quad = \sum_{\pi\in S_{m+k}} \int_{\{s_{\pi(m+k)}>\cdots>s_{\pi(1)}\}} C_{\pi(m+k)} \cdots C_{\pi(1)} (\mu_1^{m_1} \times \cdots \times \mu_n^{m_n} \times \nu^k)(ds_1, \ldots, ds_{m+k}). \qquad (4.95)
\end{aligned}$$

Using the second equality in the conclusion of Theorem 4.3.13, the summation is over

$$
\begin{aligned}
S_{m+k} = \mathcal{O}_{\{1,\ldots,m\}\cup\{m+1,\ldots,m+k\}} &= \mathcal{O}_{\{1,\ldots,m\}} \odot \mathcal{O}_{\{m+1,\ldots,m+k\}} \\
&= \biguplus_{\substack{\sigma\in\mathcal{O}_{\{1,\ldots,m\}} \\ \tau\in\mathcal{O}_{\{m+1,\ldots,m+k\}}}} \{\sigma\} \odot \{\tau\} \\
&= \biguplus_{\substack{\sigma\in\mathcal{O}_{\{1,\ldots,m\}} \\ \tau\in\mathcal{O}_{\{m+1,\ldots,m+k\}}}} \biguplus_{\substack{q_1+q_2+\cdots+q_{m+1}=k \\ q_1,\ldots,q_{m+1}\geq 0}} \{[\tau(1),\ldots,\tau(q_1),\sigma(1),\tau(q_1+1),\ldots,\tau(q_1+q_2),\sigma(2), \\
&\qquad \ldots,\tau(q_1+\cdots+q_m),\sigma(m),\tau(q_1+\cdots+q_m+1),\ldots,\tau(k)]\}. \qquad (4.96)
\end{aligned}
$$

With (4.96), the disentangling becomes

$$
\begin{aligned}
&P^{m_1,\ldots,m_n,k}_{\mu_1,\ldots,\mu_n,\nu}(A_1,\ldots,A_n,B) \\
&= \sum_{\sigma\in\mathcal{O}_{\{1,\ldots,m\}}} \sum_{\substack{\tau\in\mathcal{O}_{\{m+1,\ldots,m+k\}} \\ q_1+q_2+\cdots+q_{m+1}=k \\ q_1,\ldots,q_{m+1}\geq 0}} \int_{\{s_{\tau(k)}>\cdots>s_{\tau(q_1+\cdots+q_m+1)}>s_{\sigma(m)}>\cdots>s_{\tau(q_1+1)}>s_{\sigma(1)}>s_{\tau(q_1)}>\cdots>s_{\tau(1)}\}} C_{\tau(k)}\cdots C_{\tau(q_1+\cdots+q_m+1)}C_{\sigma(m)}\cdots C_{\tau(q_1+1)}C_{\sigma(1)}C_{\tau(q_1)}\cdots C_{\tau(1)} \\
&\qquad \times (\mu_1^{m_1}\times\cdots\times\mu_n^{m_n}\times\nu^k)(ds_1,\ldots,ds_{m+k}) \\
&= \sum_{\sigma\in\mathcal{O}_{\{1,\ldots,m\}}} \sum_{\substack{\tau\in\mathcal{O}_{\{m+1,\ldots,m+k\}} \\ q_1+q_2+\cdots+q_{m+1}=k \\ q_1,\ldots,q_{m+1}\geq 0}} \int_{\{s_{\tau(k)}>\cdots>s_{\tau(q_1+\cdots+q_m+1)}>s_{\sigma(m)}>\cdots>s_{\tau(q_1+1)}>s_{\sigma(1)}>s_{\tau(q_1)}>\cdots>s_{\tau(1)}\}} B^{q_{m+1}}C_{\sigma(m)}B^{q_m}\cdots C_{\sigma(2)}B^{q_2}C_{\sigma(1)}B^{q_1} \\
&\qquad \times (\mu_1^{m_1}\times\cdots\times\mu_n^{m_n}\times\nu^k)(ds_1,\ldots,ds_{m+k}), \qquad (4.97)
\end{aligned}
$$

since every image of τ is in $\{m+1,\ldots,m+k\}$, and $C_{m+1} = C_{m+2} = \cdots = C_{m+k} = B$.

As it happens, any choice of τ in the summation index will produce the same summand, since every $s_{\tau(j)}$ corresponds to the same operator B and to the same measure ν. We can therefore rewrite the summation by choosing a single ordering τ and multiplying by the number of choices for τ, which is $k!$. We will choose the ordering $\tau = [m+1, m+2, \ldots, m+k]$. (This means that the symbol τ will disappear from the expression, leaving that specific ordering in is place.)

$$\begin{aligned}
&P^{m_1,\ldots,m_n,k}_{\mu_1,\ldots,\mu_n,\nu}(A_1, \ldots, A_n, B) \\
&= k! \sum_{\sigma \in S_m} \sum_{\substack{q_1+q_2+\cdots+q_{m+1}=k \\ q_1,\ldots,q_{m+1}\geq 0}} \int_{\{s_{m+k}>\cdots>s_{m+q_1+\cdots+q_m+1}>s_{\sigma(m)}>\cdots>s_{m+q_1+1}>s_{\sigma(1)}>s_{m+q_1}>\cdots>s_{m+1}\}} B^{q_{m+1}} C_{\sigma(m)} \cdots C_{\sigma(2)} B^{q_2} C_{\sigma(1)} B^{q_1} \\
&\qquad \times (\mu_1^{m_1} \times \cdots \times \mu_n^{m_n} \times \nu^k)(ds_1, \ldots, ds_{m+k}).
\end{aligned}$$

Using the method of Example 11, we can express this as

$$\begin{aligned}
&P^{m_1,\ldots,m_n,k}_{\mu_1,\ldots,\mu_n,\nu}(A_1, \ldots, A_n, B) \\
&= m_1!\, m_2! \cdots m_n!\, k! \sum_{\substack{\sigma \in \mathcal{P}_{m_1,\ldots,m_n} \\ q_1+q_2+\cdots+q_{m+1}=k \\ q_1,\ldots,q_{m+1}\geq 0}} \int_{\substack{\{s_{m+k}>\cdots>s_{m+q_1+\cdots+q_m+1}>s_{\sigma(m)}>\cdots \\ >s_{m+q_1+1}>s_{\sigma(1)}>s_{m+q_1}>\cdots>s_{m+1}\}}} B^{q_{m+1}} C_{\sigma(m)} \cdots C_{\sigma(2)} B^{q_2} C_{\sigma(1)} B^{q_1} \\
&\qquad \times (\mu_1^{m_1} \times \cdots \times \mu_n^{m_n} \times \nu^k)(ds_1, \ldots, ds_{m+k}).
\end{aligned}$$

Theorem 4.3.16. *Given any finite set P, we have $\mathcal{O}_P = \dot{\bigcup}_{x\in P} \mathcal{O}_{P\smallsetminus\{x\}}.\{[x]\}$.*

Proof. (Proof of $\subseteq$.) Let $\pi \in \mathcal{O}_P$, let $m := \mathrm{card}(P)$, and let $x := \pi(m)$. Then $\pi = [\pi(1), \ldots, \pi(m-1)].[x]$ with $[\pi(1), \ldots, \pi(m-1)] \in \mathcal{O}_{P\smallsetminus\{x\}}$. Hence, $\pi \in \mathcal{O}_{P\smallsetminus\{x\}}.\{[x]\}$, and $\mathcal{O}_P \subseteq \dot{\bigcup}_{x\in P} \mathcal{O}_{P\smallsetminus\{x\}}.\{[x]\}$. (That the union is disjoint for distinct choices of x is clear.)

(Proof of $\supseteq$.) Conversely, let $\pi \in \dot{\bigcup}_{x\in P} \mathcal{O}_{P\smallsetminus\{x\}}.\{[x]\}$. Then $\pi = \sigma.\tau$ for some $\sigma \in \mathcal{O}_{P\smallsetminus\{x\}}, \tau \in \{[x]\} = \mathcal{O}_{\{x\}}$. But then by the definition of concatenation, $\pi = \sigma.\tau \in \mathcal{O}_{(P\smallsetminus\{x\})\dot{\cup}\{x\}} = \mathcal{O}_P$. Hence, $\dot{\bigcup}_{x\in P} \mathcal{O}_{P\smallsetminus\{x\}}.\{[x]\} \subseteq \mathcal{O}_P$, so $\mathcal{O}_P = \dot{\bigcup}_{x\in P} \mathcal{O}_{P\smallsetminus\{x\}}.\{[x]\}$.

$\square$

Example 18. We have previously defined the disentangling map $\mathcal{T}_{\mu_1,\ldots,\mu_n} : \mathbb{D} = \mathbb{D}(A_1,\ldots,A_n) \to \mathcal{L}(X)$ for finite, continuous Borel measures $\mu_1,\ldots,\mu_n$ defined on the interval $[0,1]$. Strictly for purposes of this example, we will define a similar disentangling map $\mathcal{T}^t_{\mu_1,\ldots,\mu_n} : \mathbb{D} \to \mathcal{L}(X)$ for finite, continuous Borel measures $\mu_1,\ldots,\mu_n$ defined on the interval $[0,T]$, where $t \in [0,T]$. (This has been done more generally in [19] on time-dependent operators; we continue to restrict our attention to time-independent operators.) As before, we associate the measures $\mu_1,\ldots,\mu_n$ with the operators $A_1,\ldots,A_n \in \mathcal{L}(X)$, respectively; $m_1,\ldots,m_n$ are nonnegative integers, with $m := \sum_j m_j$; we have blocks of integers $\mathrm{Bl}(1) := \{1,2,\ldots,m_1\}, \mathrm{Bl}(2) := \{m_1+1,\ldots,m_1+m_2\},\ldots,\mathrm{Bl}(n) := \{m_1+\cdots+m_{n-1}+1,\ldots,m\}$; as well as indeterminates $\tilde{C}_k := \tilde{A}_j$ and operators $C_k := A_j$ and measures $\nu_k := \mu_j$ for $k \in \mathrm{Bl}(j)$, where $j = 1,\ldots,n;\ k = 1,\ldots,m$. We further define, for any permutation $\pi \in S_m$, the set

$$\Delta^t_m(\pi) := \{(s_1,\ldots,s_m) \in [0,t]^m :\ 0 < s_{\pi(1)} < s_{\pi(2)} < \cdots < s_{\pi(m)} < t\}. \tag{4.98}$$

Then the disentangling map applied to a monomial is defined by

$$\begin{aligned}
&\mathcal{T}^t_{\mu_1,\ldots,\mu_n}[P^{m_1,\ldots,m_n}(\tilde{A}_1,\ldots,\tilde{A}_n)] \\
&:= \sum_{\pi\in S_m} \int_{\Delta^t_m(\pi)} C_{\pi(m)}\cdots C_{\pi(1)}\,(\mu_1^{m_1}\times\cdots\times\mu_n^{m_n})(ds_1,\ldots,ds_m).
\end{aligned} \tag{4.99}$$

With these definitions, we have, for $t \in (0,T)$,

$$\mathcal{T}^t_{\mu_1,\ldots,\mu_n}[P^{m_1,\ldots,m_n}(\tilde{A}_1,\ldots,\tilde{A}_n)] \\ = \sum_{i=1}^n \int_{[0,t]} A_i\, \mathcal{T}^s_{\mu_1,\ldots,\mu_n}\left[\frac{\partial}{\partial \tilde{A}_i} P^{m_1,\ldots,m_n}(\tilde{A}_1,\ldots,\tilde{A}_n)\right]\mu_i(ds). \quad (4.100)$$

To verify (4.100), we note that the left-hand is

$$\mathcal{T}^t_{\mu_1,\ldots,\mu_n}[P^{m_1,\ldots,m_n}(\tilde{A}_1,\ldots,\tilde{A}_n)] \\ := \sum_{\pi\in S_m} \int_{\Delta^t_m(\pi)} C_{\pi(m)}\cdots C_{\pi(1)}\,(\mu_1^{m_1}\times\cdots\times\mu_n^{m_n})(ds_1,\ldots,ds_m). \quad (4.101)$$

Using Theorem 4.3.16, the summation is over

$$S_m = \mathcal{O}_{\{1,\ldots,m\}} = \dot{\bigcup_{j=1}^m} \mathcal{O}_{\{1,\ldots,j-1,j+1,\ldots,m\}}.\{[j]\} = \dot{\bigcup_{j=1}^m} \dot{\bigcup_{\sigma\in\mathcal{O}_{\{1,\ldots,j-1,j+1,\ldots,m\}}}} \{\sigma.[j]\},$$

which tells us that

$$\mathcal{T}^t_{\mu_1,\ldots,\mu_n}[P^{m_1,\ldots,m_n}(\tilde{A}_1,\ldots,\tilde{A}_n)] \\ = \sum_{j=1}^m \sum_{\sigma\in\mathcal{O}_{\{1,\ldots,j-1,j+1,\ldots,m\}}} \int_{\{t>s_j>s_{\sigma(m-1)}>\cdots>s_{\sigma(1)}>0\}} C_j\, C_{\sigma(m-1)}\cdots C_{\sigma(1)}(\nu_1\times\cdots\times\nu_m)(ds_1,\ldots,ds_m) \\ = \sum_{j=1}^m \sum_{\sigma\in\mathcal{O}_{\{1,\ldots,j-1,j+1,\ldots,m\}}} \int_{\{t>s_j>s_{\sigma(m-1)}>\cdots>s_{\sigma(1)}>0\}} C_j\, C_{\sigma(m-1)}\cdots C_{\sigma(1)} \\ \times(\nu_1\times\cdots\times\nu_{j-1}\times\nu_{j+1}\times\cdots\times\nu_m\times\nu_j)(ds_1,\ldots,ds_{j-1},ds_{j+1},\ldots,ds_m,ds_j), \quad (4.102)$$

by Corollary 3.1.3. Next we change the region of integration into a characteristic function and factor the characteristic function.

$$
\begin{aligned}
&\mathcal{T}^t_{\mu_1,\ldots,\mu_n}[P^{m_1,\ldots,m_n}(\tilde{A}_1,\ldots,\tilde{A}_n)] \\
&= \sum_{j=1}^{m} \sum_{\sigma\in\mathcal{O}_{\{1,\ldots,j-1,j+1,\ldots,m\}}} \int_{(0,t)^m} C_j\, C_{\sigma(m-1)}\cdots C_{\sigma(1)} \\
&\quad \times \chi_{\{t>s_j>s_{\sigma(m-1)}>\cdots>s_{\sigma(1)}>0\}}(s_1,\ldots,s_{j-1},s_{j+1},\ldots,s_m,s_j) \\
&\quad \times (\nu_1\times\cdots\times\nu_{j-1}\times\nu_{j+1}\times\cdots\times\nu_m\times\nu_j)(ds_1,\ldots,ds_{j-1},ds_{j+1},\ldots,ds_m,ds_j) \\
&= \sum_{j=1}^{m} \sum_{\sigma\in\mathcal{O}_{\{1,\ldots,j-1,j+1,\ldots,m\}}} \int_{(0,t)^m} C_j\, C_{\sigma(m-1)}\cdots C_{\sigma(1)} \\
&\quad \times \chi_{\{t>s_j>s_{\sigma(m-1)}>\cdots>s_{\sigma(1)}>0\}}(s_1,\ldots,s_{j-1},s_{j+1},\ldots,s_m)\chi_{\{t>s_j>0\}}(s_j) \\
&\quad \times (\nu_1\times\cdots\times\nu_{j-1}\times\nu_{j+1}\times\cdots\times\nu_m\times\nu_j)(ds_1,\ldots,ds_{j-1},ds_{j+1},\ldots,ds_m,ds_j).
\end{aligned}
$$

We are then able to apply the Fubini-Tonelli Theorem, which allows us to change the integral into an iterated integral and guarantees that the inner integral is a measurable function with respect to the measure used in the outer integration. This yields

$$
\begin{aligned}
&\mathcal{T}^t_{\mu_1,\ldots,\mu_n}[P^{m_1,\ldots,m_n}(\tilde{A}_1,\ldots,\tilde{A}_n)] \\
&= \sum_{j=1}^{m} \sum_{\sigma\in\mathcal{O}_{\{1,\ldots,j-1,j+1,\ldots,m\}}} \int_{(0,t)} \bigg[\int_{(0,t)^{m-1}} C_j\, C_{\sigma(m-1)}\cdots C_{\sigma(1)} \\
&\times \chi_{\{s_j>s_{\sigma(m-1)}>\cdots>s_{\sigma(1)}>0\}}(s_1,\ldots,s_{j-1},s_{j+1},\ldots,s_m) \\
&\times (\nu_1\times\cdots\times\nu_{j-1}\times\nu_{j+1}\times\cdots\times\nu_m)(ds_1,\ldots,ds_{j-1},ds_{j+1},\ldots,ds_m)\bigg]\chi_{\{t>s_j>0\}}(s_j)\nu_j(ds_j) \\
&= \sum_{j=1}^{m} \sum_{\sigma\in\mathcal{O}_{\{1,\ldots,j-1,j+1,\ldots,m\}}} \int_{\{t>s_j>0\}} C_j \bigg[\int_{\{s_j>s_{\sigma(m-1)}>\cdots>s_{\sigma(1)}>0\}} C_{\sigma(m-1)}\cdots C_{\sigma(1)} \\
&\quad \times (\nu_1\times\cdots\times\nu_{j-1}\times\nu_{j+1}\times\cdots\times\nu_m)(ds_1,\ldots,ds_{j-1},ds_{j+1},\ldots,ds_m)\bigg]\nu_j(ds_j).
\end{aligned}
\tag{4.103}
$$

We now rename s_j as s, we change the outer region of integration from $(0,t)$ to $[0,t]$ (which we may do since ν_j is a continuous measure), we move the inner summation into the outer integral, and we apply the definition of the disentangling map:

$$
\begin{aligned}
&\mathcal{T}^t_{\mu_1,\ldots,\mu_n}[P^{m_1,\ldots,m_n}(\tilde{A}_1,\ldots,\tilde{A}_n)] \\
&= \sum_{j=1}^{m} \int_{[0,t]} C_j \left[\sum_{\sigma\in\mathcal{O}_{\{1,\ldots,j-1,j+1,\ldots,m\}}} \int_{\{s>s_{\sigma(m-1)}>\cdots>s_{\sigma(1)}>0\}} C_{\sigma(m-1)}\cdots C_{\sigma(1)} \right. \\
&\qquad \left. \times (\nu_1\times\cdots\times\nu_{j-1}\times\nu_{j+1}\times\cdots\times\nu_m)(ds_1,\ldots,ds_{j-1},ds_{j+1},\ldots,ds_m) \right] \nu_j(ds) \\
&= \sum_{j=1}^{m} \int_{[0,t]} C_j\, \mathcal{T}^s_{\nu_1,\ldots,\nu_{j-1},\nu_{j+1},\ldots,\nu_m}[P^{1,\ldots,1}(\tilde{C}_1,\ldots,\tilde{C}_{j-1},\tilde{C}_{j+1},\ldots,\tilde{C}_m)]\,\nu_j(ds). \quad (4.104)
\end{aligned}
$$

In the last expression, the m_1 terms of the sum for which $C_j = A_1$ are all identical, as are the m_2 terms having $C_j = A_2$, etc. For each of these, the disentangling expression in the integrand has one less factor of, respectively, A_1, A_2, etc. We may therefore rewrite the expression as

$$
\begin{aligned}
&\mathcal{T}^t_{\mu_1,\ldots,\mu_n}[P^{m_1,\ldots,m_n}(\tilde{A}_1,\ldots,\tilde{A}_n)] \\
&= \sum_{i=1}^{n} m_i \int_{[0,t]} A_i\, \mathcal{T}^s_{\mu_1,\ldots,\mu_n}[P^{m_1,\ldots,m_{j-1},m_j-1,m_{j+1},\ldots,m_n}(\tilde{A}_1,\ldots,\tilde{A}_n)]\,\mu_i(ds) \\
&= \sum_{i=1}^{n} \int_{[0,t]} A_i\, \mathcal{T}^s_{\mu_1,\ldots,\mu_n}[m_i\, P^{m_1,\ldots,m_{j-1},m_j-1,m_{j+1},\ldots,m_n}(\tilde{A}_1,\ldots,\tilde{A}_n)]\,\mu_i(ds) \\
&= \sum_{i=1}^{n} \int_{[0,t]} A_i\, \mathcal{T}^s_{\mu_1,\ldots,\mu_n}\left[\frac{\partial}{\partial\tilde{A}_i} P^{m_1,\ldots,m_n}(\tilde{A}_1,\ldots,\tilde{A}_n)\right] \mu_i(ds). \quad (4.105)
\end{aligned}
$$

Linearity of partial derivatives, linear operators, integrals, finite sums, and the disentangling map give that a similar property will hold if $P^{m_1,\ldots,m_n}(\tilde{A}_1,\ldots,\tilde{A}_n)$ is replaced by an arbitrary polynomial $f(\tilde{A}_1,\ldots,\tilde{A}_n)$.

4.4 The excerption operation

So far in this chapter we have defined orderings and two operations on orderings, namely the merge and concatenation operations. Concatenation of two orderings places them side-by-side to give another ordering, whereas merging two orderings, or rather merging two sets that consist of one ordering each, produces longer orderings that mix the two orderings together (while preserving the relative order of the elements of each). Both operations yield longer orderings. Next we consider the 'excerption' operation, which takes orderings and yields shorter orderings, one might say 'suborderings'. In a sense, the excerption operation can recover items that have been merged, and therefore plays a role something like an inverse of the merge operation (but not exactly), as we will discuss. First we will show that what we will define as an excerption from an ordering actually exists.

Theorem 4.4.1. *Let P be a finite set, let $\sigma \in \mathcal{O}_P$, and let $Q \subseteq P$. Then there exists a unique ordering $\pi \in \mathcal{O}_Q$ that satisfies each of the following properties, which are equivalent:*

(i) $\pi^{-1}(x) < \pi^{-1}(y)$ *if and only if* $\sigma^{-1}(x) < \sigma^{-1}(y)$ *for all* $x, y \in Q$.

(ii) *If* $\pi^{-1}(x) < \pi^{-1}(y)$ *then* $\sigma^{-1}(x) < \sigma^{-1}(y)$ *for all* $x, y \in Q$.

(iii) *If* $\sigma^{-1}(x) < \sigma^{-1}(y)$ *then* $\pi^{-1}(x) < \pi^{-1}(y)$ *for all* $x, y \in Q$.

(iv) $\sigma^{-1}(\pi(1)) < \sigma^{-1}(\pi(2)) < \cdots < \sigma^{-1}(\pi(\mathrm{card}(Q)))$.

Proof. First we will show the existence and uniqueness of an ordering satisfying (i), and then we will show that (i)–(iv) are equivalent.

The first case to consider is if $\mathrm{card}(Q) = 0$, that is, $Q = \emptyset$. In that case, the only possible ordering $\pi \in \mathcal{O}_Q = \mathcal{O}_\emptyset = \{\emptyset\}$ is $\pi = \emptyset$, giving us the uniqueness of π. There are no $x, y \in Q$, so property (i) is in fact vacuously satisfied for the ordering $\pi = \emptyset$, giving us existence.

Second, we consider the case when $\mathrm{card}(Q) = 1$; say $Q = \{a\}$. Then there is only one ordering $\pi \in \mathcal{O}_Q = \mathcal{O}_{\{a\}} = \{[a]\}$, namely $\pi = [a]$, giving us the uniqueness of π. Since there is only one element $a \in Q$, we cannot have $x, y \in Q$ with $\sigma^{-1}(x) < \sigma^{-1}(y)$, so property (i) is vacuously satisfied for the ordering $\pi = [a]$, giving us existence.

If $\mathrm{card}(Q) > 1$, let $m := \mathrm{card}(P), n := \mathrm{card}(Q)$, so for $\sigma \in \mathcal{O}_P$ we have $\sigma = [\sigma(1), \sigma(2), \ldots, \sigma(m)]$. Since Q is a subset of $P = \{\sigma(1), \sigma(2), \ldots, \sigma(m)\}$, we may choose $i_1, i_2, \ldots, i_n \in \{1, 2, \ldots, m\}$ with

$$\{\sigma(i_1), \sigma(i_2), \ldots, \sigma(i_n)\} = Q,$$

and without loss of generality, $i_1 < i_2 < \cdots < i_n$. Let $\pi := [\sigma(i_1), \sigma(i_2), \ldots, \sigma(i_n)]$; that is, $\pi(j) = \sigma(i_j)$ for $j = 1, 2, \ldots, n$.

It follows that $\pi \in \mathcal{O}_Q$. Let $x, y \in Q$ with $\pi^{-1}(x) < \pi^{-1}(y)$, and let $j := \pi^{-1}(x), k := \pi^{-1}(y)$, so $j < k$. Hence

$$\sigma^{-1}(x) = \sigma^{-1}(\pi(j)) = \sigma^{-1}(\sigma(i_j)) = i_j < i_k = \sigma^{-1}(\sigma(i_k)) = \sigma^{-1}(\pi(k)) = \sigma^{-1}(y).$$

If, on the other hand, $x, y \in Q$ are such that $\pi^{-1}(x) = \pi^{-1}(y)$ then $\sigma^{-1}(x) = \sigma^{-1}(y)$, or if $x, y \in Q$ with $\pi^{-1}(x) > \pi^{-1}(y)$ then (by what was just done), $\sigma^{-1}(x) > \sigma^{-1}(y)$. Therefore, for any $x, y \in Q$ we have that $\pi^{-1}(x) < \pi^{-1}(y)$ if and only if $\sigma^{-1}(x) < \sigma^{-1}(y)$, which is property (i).

For uniqueness, consider a possibly different $\pi' \in \mathcal{O}_Q$ with the property (i) that if $x, y \in Q$, then $\pi'^{-1}(x) < \pi'^{-1}(y)$ if and only if $\sigma^{-1}(x) < \sigma^{-1}(y)$. We already know that $\sigma^{-1}(x) < \sigma^{-1}(y)$ if and only if $\pi^{-1}(x) < \pi^{-1}(y)$, and together these imply that

$\pi'^{-1}(x) < \pi'^{-1}(y)$ if and only if $\pi^{-1}(x) < \pi^{-1}(y)$. But then since clearly

$$\pi^{-1}(\pi(1)) < \pi^{-1}(\pi(2)) < \cdots < \pi^{-1}(\pi(n)),$$

we must have

$$\pi'^{-1}(\pi(1)) < \pi'^{-1}(\pi(2)) < \cdots < \pi'^{-1}(\pi(n)),$$

and then

$$1 = \pi'^{-1}(\pi(1)) \;<\; 2 = \pi'^{-1}(\pi(2)) \;<\; \cdots \;<\; n = \pi'^{-1}(\pi(n)).$$

Therefore, $\pi = \pi'$, and we have established uniqueness.

As for the equivalence of (i)–(iv), we will show that (i) is equivalent to each of the others. If $\text{card}(Q) = 0$ or $\text{card}(Q) = 1$, then the statements (i)–(iv) are all vacuously satisfied for the ordering π we have identified above, and thus they are equivalent. (It may not be as clear that property (iv) is vacuously satisfied, but we may regard that as equivalent to the statement that $\sigma^{-1}(\pi(j)) < \sigma^{-1}(\pi(j+1))$ for all $j, j+1 \in \{1, 2, \ldots, \text{card}(Q)\}$, and that statement is vacuously satisfied in these cases.)

For the remaining case, if $\text{card}(Q) > 1$, suppose that (i) holds. Then clearly (ii) and (iii) hold. It is also clear that

$$\pi^{-1}(\pi(1)) < \pi^{-1}(\pi(2)) < \cdots < \pi^{-1}(\pi(\text{card}(Q))),$$

so (i) implies that

$$\sigma^{-1}(\pi(1)) < \sigma^{-1}(\pi(2)) < \cdots < \sigma^{-1}(\pi(\text{card}(Q))),$$

which means that (iv) holds.

Conversely, assume (ii); i.e., $\pi^{-1}(x) < \pi^{-1}(y)$ implies that $\sigma^{-1}(x) < \sigma^{-1}(y)$ for all $x, y \in Q$. If we take $x, y \in Q$ with $\pi^{-1}(x) = \pi^{-1}(y)$, we obtain $x = y$, so that $\sigma^{-1}(x) = \sigma^{-1}(y)$. If we take $x, y \in Q$ with $\pi^{-1}(x) > \pi^{-1}(y)$, then we get by (ii) that $\sigma^{-1}(x) < \sigma^{-1}(y)$. Therefore, $\pi^{-1}(x) < \pi^{-1}(y)$ if and only if $\sigma^{-1}(x) < \sigma^{-1}(y)$ for all $x, y \in Q$, which is property (iii). Similarly, letting (iii) hold instead also implies (i).

Now suppose that (iv) holds: $\sigma^{-1}(\pi(1)) < \sigma^{-1}(\pi(2)) < \cdots < \sigma^{-1}(\pi(\mathrm{card}(Q)))$. Let $x, y \in Q$. Then we can find $j, k \in \{1, 2, \ldots, \mathrm{card}(Q)\}$ with $\pi(j) = x$, $\pi(k) = y$. If $\pi^{-1}(x) < \pi^{-1}(y)$, then $j < k$, so (iv) gives that $\sigma^{-1}(\pi(j)) < \sigma^{-1}(\pi(k))$, which is to say $\sigma^{-1}(x) < \sigma^{-1}(y)$. Thus (ii) holds, so (i) holds. Therefore, (i)–(iv) are equivalent. $\square$

We are now prepared to define the excerption operation on orderings:

Definition 4.4.2 (The excerption operation)**.** Let P be a finite set, and let Q be any set. Given any ordering $\sigma \in \mathcal{O}_P$, we define $\sigma\wr_Q$ ('σ **excerpt** Q') to be the unique ordering $\pi \in \mathcal{O}_{P\cap Q}$ that satisfies the following property:

(i) $\pi^{-1}(x) < \pi^{-1}(y)$ if and only if $\sigma^{-1}(x) < \sigma^{-1}(y)$ for all $x, y \in P \cap Q$.

We call $\sigma\wr_Q$ the **excerption of the set** Q **from the ordering** σ. (Note that if $P \cap Q$ is the empty set, then $\sigma\wr_Q$ is the null ordering, and if $P \cap Q$ has only one element, $P \cap Q = \{a\}$, then $\sigma\wr_Q = [a]$.)

We define a related map on a *set* of orderings: Given any set $\mathcal{U} \subseteq \mathcal{O}_P$ and any set Q, we define $\mathcal{U}\wr_Q$, the **excerption of the set** Q **from the set of orderings** $\mathcal{U}$ to be the set

$$\mathcal{U}\wr_Q := \bigcup_{\sigma\in\mathcal{U}} \{\sigma\wr_Q\}.$$

(Note that if $\mathcal{U}$ is empty, then $\mathcal{U}\wr_Q$ is empty, whereas if $\mathcal{U}$ is nonempty and $P \cap Q = \emptyset$, then $\mathcal{U}\wr_Q = \{\emptyset\}$, the set consisting of only the null ordering.)

[Although the symbol "$\wr$" is borrowed from algebra, where it indicates a "wreath product," the usage here is entirely unrelated to that. Here it is meant to suggest something like a restriction map.]

The excerption of a set Q from an ordering $\sigma \in \mathcal{O}_P$ is thought of as pulling the elements of Q out of σ and keeping those elements, preserving their order.

Remark 17. As established by Theorem 4.4.1, we may replace property (i) in Definition 4.4.2 by one of the following three statements:

(ii) If $\pi^{-1}(x) < \pi^{-1}(y)$ then $\sigma^{-1}(x) < \sigma^{-1}(y)$ for all $x, y \in P \cap Q$.

(iii) If $\sigma^{-1}(x) < \sigma^{-1}(y)$ then $\pi^{-1}(x) < \pi^{-1}(y)$ for all $x, y \in P \cap Q$.

(iv) $\sigma^{-1}(\pi(1)) < \sigma^{-1}(\pi(2)) < \cdots < \sigma^{-1}(\pi(\mathrm{card}(P \cap Q)))$.

Remark 18. If $\sigma \in \mathcal{O}_P$ for some finite set P, then excerpting the empty set from the ordering σ gives us $\sigma\wr_\emptyset = \emptyset$ (the null ordering). As for sets of orderings, we have $\emptyset\wr_Q = \emptyset$ (empty set excerpt Q is the empty set) for any set Q, since the union over the empty set is empty. If $\mathcal{U} \subseteq \mathcal{O}_P$ for a finite set P with $\mathcal{U} \neq \emptyset$, then $\mathcal{U}\wr_\emptyset = \{\emptyset\}$ (the set consisting of the null ordering). [The reader might again have reason for concern here, since the statement '$\emptyset\wr_Q = \emptyset$' could be a statement about null orderings or a statement about empty sets. But again, happily, the statement is true in both interpretations. Context should indicate which is intended.]

It is immediate from Definition 4.4.2 that if P is a finite set, if Q is any set, and if $\sigma \in \mathcal{O}_P$, then $\sigma\wr_Q = \sigma\wr_{P\cap Q}$, since $P \cap Q = P \cap (P \cap Q)$.

It also follows readily from the definition and the preceding theorem that for any finite set P and ordering $\sigma \in \mathcal{O}_P$ we can say that $\sigma\wr_P = \sigma$ (because certainly $\sigma^{-1}(\sigma(1)) < \sigma^{-1}(\sigma(2)) < \cdots < \sigma^{-1}(\sigma(\mathrm{card}(P)))$), and from that we can say that if

$\mathcal{U} \subseteq \mathcal{O}_P$ then $\mathcal{U}\wr_P = \mathcal{U}$. In fact, if $\mathcal{U} \subseteq \mathcal{O}_P$, then $\mathcal{U}\wr_R = \mathcal{U}$ for any set R with $P \subseteq R$ (because $\mathcal{U}\wr_R = \mathcal{U}\wr_{P\cap R} = \mathcal{U}\wr_P = \mathcal{U}$).

Example 19. Let $P := \{1,2,3,4,5\}$, $Q := \{2,4,5,7,8\}$, $\sigma := [3,4,1,5,2] \in \mathcal{O}_P$, and $\mathcal{U} := \{[3,5,4,1,2],[2,3,4,5,1],[3,2,4,1,5]\} \subseteq \mathcal{O}_P$. Then $P \cap Q = \{2,4,5\}$, so $\sigma\wr_Q = [4,5,2]$, and $\mathcal{U}\wr_Q = \{[5,4,2],[2,4,5]\}$.

Theorem 4.4.3 (Relationship between excerption and merging)**.** *Let P, Q be disjoint, finite sets. Then the following hold:*

(i) If $\sigma \in \mathcal{O}_P$ and $\tau \in \mathcal{O}_Q$, then $\pi \in \{\sigma\} \odot \{\tau\}$ if and only if $\pi \in \mathcal{O}_{P\dot\cup Q}$ and $\sigma = \pi\wr_P$ and $\tau = \pi\wr_Q$. (In particular, given any $\pi \in \mathcal{O}_{P\dot\cup Q}$, we have $\pi \in \{\pi\wr_P\} \odot \{\pi\wr_Q\}$.)

(ii) If $\mathcal{U} \subseteq \mathcal{O}_{P\dot\cup Q}$, then $\mathcal{U} \subseteq \mathcal{U}\wr_P \odot \mathcal{U}\wr_Q$. Consequently, $\mathcal{U}\wr_P \odot \mathcal{U}\wr_Q = \mathcal{U}$ if and only if $\mathrm{card}(\mathcal{U}\wr_P \odot \mathcal{U}\wr_Q) = \mathrm{card}(\mathcal{U})$.

(iii) If $\mathcal{U} \subseteq \mathcal{O}_{P\dot\cup Q}$, and both $\mathcal{V} \subseteq \mathcal{O}_P$ and $\mathcal{W} \subseteq \mathcal{O}_Q$ are nonempty, then $\mathcal{V} \odot \mathcal{W} = \mathcal{U}$ if and only if $\mathcal{V} = \mathcal{U}\wr_P$, $\mathcal{W} = \mathcal{U}\wr_Q$, and $\mathrm{card}(\mathcal{V} \odot \mathcal{W}) = \mathrm{card}(\mathcal{U})$.

Proof. To prove claim (i), let $\sigma \in \mathcal{O}_P$ and $\tau \in \mathcal{O}_Q$, and let $\pi \in \{\sigma\} \odot \{\tau\}$. By the definition of the merge operation (Definition 4.2.1) we have $\pi \in \mathcal{O}_{P\dot\cup Q}$ with the properties that $\pi^{-1}(x) < \pi^{-1}(y)$ if and only if $\sigma^{-1}(x) < \sigma^{-1}(y)$ for all $x, y \in P$, and that $\pi^{-1}(x) < \pi^{-1}(y)$ if and only if $\tau^{-1}(x) < \tau^{-1}(y)$ for all $x, y \in Q$. But then by definition of excerption, $\sigma = \pi\wr_P$ and $\tau = \pi\wr_Q$.

Conversely, let $\sigma \in \mathcal{O}_P$ and $\tau \in \mathcal{O}_Q$, let $\pi \in \mathcal{O}_{P\dot\cup Q}$, and let $\sigma = \pi\wr_P$ and $\tau = \pi\wr_Q$. Then for every $x, y \in P$ with $\pi^{-1}(x) < \pi^{-1}(y)$ we have $\sigma^{-1}(x) < \sigma^{-1}(y)$, and for every $x, y \in Q$ with $\pi^{-1}(x) < \pi^{-1}(y)$ we have $\tau^{-1}(x) < \tau^{-1}(y)$. The definition of merging yields that $\pi \in \{\sigma\} \odot \{\tau\}$.

To prove claim (ii), let $\mathcal{U} \subseteq \mathcal{O}_{P\dot\cup Q}$. By what was just established, given any $\pi \in \mathcal{U}$ we have $\pi \in \{\pi\wr_P\} \odot \{\pi\wr_Q\} \subseteq \mathcal{U}\wr_P \odot \mathcal{U}\wr_Q$, so $\mathcal{U} \subseteq \mathcal{U}\wr_P \odot \mathcal{U}\wr_Q$. Because $\mathcal{U} \subseteq \mathcal{U}\wr_P \odot \mathcal{U}\wr_P$

and because $\mathcal{U}$ and $\mathcal{U}\wr_P \odot \mathcal{U}\wr_P$ are finite, we have equality if and only if the cardinalities are the same. Thus, we have established the desired result.

The proof of (ii) established the "if" part of claim (iii). To prove the "only if" part of (iii), let $\mathcal{U} \subseteq \mathcal{O}_{P \cup Q}$, $\mathcal{V} \subseteq \mathcal{O}_P$, and $\mathcal{W} \subseteq \mathcal{O}_Q$, with $\mathcal{V}, \mathcal{W}$ nonempty, and suppose that $\mathcal{V} \odot \mathcal{W} = \mathcal{U}$. Then $\mathrm{card}(\mathcal{V} \odot \mathcal{W}) = \mathrm{card}(\mathcal{U})$. All we need to show, then, is that $\mathcal{V} = \mathcal{U}\wr_P$ and $\mathcal{W} = \mathcal{U}\wr_Q$.

To show that $\mathcal{V} = \mathcal{U}\wr_P$, we begin by showing that $\mathcal{V} \subseteq \mathcal{U}\wr_P$. Consider an arbitrary $\sigma \in \mathcal{V}$. Since $\mathcal{W} \neq \emptyset$, we can find an ordering $\tau \in \mathcal{W}$. We now select any $\pi \in \{\sigma\} \odot \{\tau\} \subseteq \mathcal{V} \odot \mathcal{W} = \mathcal{U}$, and then we have by (i) that $\sigma = \pi\wr_P \in \mathcal{U}\wr_P$. Therefore, $\mathcal{V} \subseteq \mathcal{U}\wr_P$. Similarly, $\mathcal{W} \subseteq \mathcal{U}\wr_Q$.

Next we show that $\mathcal{V} \supseteq \mathcal{U}\wr_P$. Consider an arbitrary element $\sigma' \in \mathcal{U}\wr_P$. Then $\sigma' = \pi'\wr_P$ for some $\pi' \in \mathcal{U} = \mathcal{V} \odot \mathcal{W}$. But then $\pi' \in \{\sigma''\} \odot \{\tau''\}$ for some $\sigma'' \in \mathcal{V}$ and $\tau'' \in \mathcal{W}$. By (i) this implies that $\sigma'' = \pi'\wr_P$. Therefore, $\sigma' = \pi'\wr_P = \sigma'' \in \mathcal{V}$, and hence, $\mathcal{U}\wr_P \subseteq \mathcal{V}$. Similarly, $\mathcal{U}\wr_Q \subseteq \mathcal{W}$. Therefore, $\mathcal{V} = \mathcal{U}\wr_P$ and $\mathcal{W} = \mathcal{U}\wr_Q$. $\square$

Example 20. Let $A_1, A_2, A_3 \in \mathcal{L}(X)$, and associate measures μ_1, μ_2, μ_3 to these operators, respectively. Let m_1, m_2, m_3 be nonnegative integers with $m := m_1 + m_2 + m_3$. The definitions of blocks of integers $\mathrm{Bl}(1), \mathrm{Bl}(2), \mathrm{Bl}(3)$ and operators $C_1, \ldots, C_m$ are, as before, given by

$$\begin{aligned} \mathrm{Bl}(1) &:= \{1, 2, \ldots, m_1\} \\ \mathrm{Bl}(2) &:= \{m_1 + 1, m_1 + 2, \ldots, m_1\} \\ \mathrm{Bl}(3) &:= \{m_1 + m_2 + 1, \ldots, m\}, \end{aligned} \qquad C_k := \begin{cases} A_1 & \text{if} \quad k \in \mathrm{Bl}(1) \\ A_2 & \text{if} \quad k \in \mathrm{Bl}(2) \\ A_3 & \text{if} \quad k \in \mathrm{Bl}(3). \end{cases} \tag{4.106}$$

The disentangling of the monomial $P^{m_1,m_2,m_3}(\tilde{A}_1, \tilde{A}_2, \tilde{A}_3)$ is

$$P^{m_1,m_2,m_3}_{\mu_1,\mu_2,\mu_3}(A_1, A_2, A_3) = \sum_{\pi \in S_m} \int_{\Delta_m(\pi)} C_{\pi(m)} \cdots C_{\pi(1)} (\mu_1^{m_1} \times \mu_2^{m_2} \times \mu_3^{m_3})(ds_1, \ldots, ds_m). \tag{4.107}$$

Let us suppose we are dealing with a situation in which we know that every operator A_1 occurs before every A_2. Thus for each

$$\pi \in S_m = \mathcal{O}_{\{1,2,\ldots,m\}} = \mathcal{O}_{\mathrm{Bl}(1) \dot\cup \mathrm{Bl}(2) \dot\cup \mathrm{Bl}(3)} = \mathcal{O}_{\mathrm{Bl}(1)} \odot \mathcal{O}_{\mathrm{Bl}(2)} \odot \mathcal{O}_{\mathrm{Bl}(3)} \tag{4.108}$$

we have $\pi^{-1}(x) < \pi^{-1}(y)$ for all $x \in \mathrm{Bl}(1), y \in \mathrm{Bl}(2)$. If we excerpt the set $\mathrm{Bl}(1) \dot\cup \mathrm{Bl}(2)$ from π we then have $(\pi \wr_{\mathrm{Bl}(1) \dot\cup \mathrm{Bl}(2)})^{-1}(x) < (\pi \wr_{\mathrm{Bl}(1) \dot\cup \mathrm{Bl}(2)})^{-1}(y)$ for all $x \in \mathrm{Bl}(1), y \in \mathrm{Bl}(2)$, and therefore since $\pi \wr_{\mathrm{Bl}(1) \dot\cup \mathrm{Bl}(2)} \in \mathcal{O}_{\mathrm{Bl}(1)} \odot \mathcal{O}_{\mathrm{Bl}(2)}$, we have $\pi \wr_{\mathrm{Bl}(1) \dot\cup \mathrm{Bl}(2)} \in \mathcal{O}_{\mathrm{Bl}(1)} . \mathcal{O}_{\mathrm{Bl}(2)}$. Consequently,

$$\pi \in \{\pi \wr_{\mathrm{Bl}(1) \dot\cup \mathrm{Bl}(2)}\} \odot \{\pi \wr_{\mathrm{Bl}(3)}\} \subseteq (O_{\mathrm{Bl}(1)} . \mathcal{O}_{\mathrm{Bl}(2)}) \odot \mathcal{O}_{\mathrm{Bl}(3)}.$$

Since we have not placed any further restrictions on the choice of π, we may therefore rewrite the sum as the sum over all such terms:

$$\begin{aligned} &P^{m_1,m_2,m_3}_{\mu_1,\mu_2,\mu_3}(A_1, A_2, A_3) \\ &\quad = \sum_{\pi \in (O_{\mathrm{Bl}(1)} . \mathcal{O}_{\mathrm{Bl}(2)}) \odot \mathcal{O}_{\mathrm{Bl}(3)}} \int_{\Delta_m(\pi)} C_{\pi(m)} \cdots C_{\pi(1)} (\mu_1^{m_1} \times \mu_2^{m_2} \times \mu_3^{m_3})(ds_1, \ldots, ds_m). \end{aligned} \tag{4.109}$$

We have therefore expressed the sum with fewer terms, using both the merge and concatenation operations in the summation index.

Corollary 4.4.4 (Excerption recovers merged sets). *Let P, Q be disjoint, finite sets.*

If $\mathcal{U} \subseteq \mathcal{O}_P$ *and* $\mathcal{V} \subseteq \mathcal{O}_Q$ *are nonempty, then* $\mathcal{U} = (\mathcal{U} \odot \mathcal{V})\wr_P$ *and* $\mathcal{V} = (\mathcal{U} \odot \mathcal{V})\wr_Q$.

Proof. This follows immediately from Theorem 4.4.3 part (iii). □

Corollary 4.4.4 enables us to go from the merging of two sets of orderings down to one of the two sets that are 'factors' of the merge operation. This suggests that we can use excerption in settings where we might want a kind of 'inverse' of the merge operation, and this was a primary motivation for defining the excerption operation. One place this use of excerption is especially apparent is in Theorem 4.4.12, below.

Example 21. Let $P := \{2,3\}$, $Q := \{5,6\}$. Let $\sigma := [2,3]$, $\tau := [6,5]$, $\pi := [6,2,5,3]$. Then we can see both that $\pi \in \{\sigma\} \odot \{\tau\}$ (that is, $[6,2,5,3] \in \{[2,3]\} \odot \{[6,5]\}$) and that $\sigma = \pi\wr_P, \tau = \pi\wr_Q$ (that is, $[2,3] = [6,2,5,3]\wr_{\{2,3\}}$ and $[6,5] = [6,2,5,3]\wr_{\{5,6\}}$, respectively).

Example 22. Let $P := \{3,7\}, Q := \{2,4\}$, and let $\mathcal{U} \subseteq \mathcal{O}_{P \cup Q}$ be the set

$$\mathcal{U} := \{[3,7,4,2], [4,2,3,7], [3,4,2,7], [3,4,7,2], [4,3,7,2]\}. \tag{4.110}$$

Then $\mathcal{U}\wr_P = \{[3,7]\}$ and $\mathcal{U}\wr_Q = \{[4,2]\}$. Does $\mathcal{U} = \mathcal{U}\wr_P \odot \mathcal{U}\wr_Q$? Certainly $\mathcal{U} \subseteq \mathcal{U}\wr_P \odot \mathcal{U}\wr_Q$, but they are not equal, because $\operatorname{card}(\mathcal{U}) = 5$, whereas

$$\operatorname{card}(\mathcal{U}\wr_P \odot \mathcal{U}\wr_Q) = \frac{[\operatorname{card}(P) + \operatorname{card}(Q)]!}{\operatorname{card}(P)! \operatorname{card}(Q)!} \operatorname{card}(\mathcal{U}\wr_P) \operatorname{card}(\mathcal{U}\wr_Q) = \frac{(2+2)!}{2!\,2!}(1)(1) = 6. \tag{4.111}$$

The missing ordering is $[4,3,2,7]$.

Theorem 4.4.5. *If* P *and* Q *are sets with* $P \subseteq Q$, *if* R *is a finite set, and if* $\sigma \in \mathcal{O}_R$, *then* $\sigma\wr_P = (\sigma\wr_Q)\wr_P$.

Proof. Let $\pi = \sigma\wr_Q$. Then $\pi \in \mathcal{O}_{R\cap Q}$ and

$$\sigma^{-1}(\pi(1)) < \sigma^{-1}(\pi(2)) < \cdots < \sigma^{-1}(\pi(\mathrm{card}(R\cap Q))). \tag{4.112}$$

Also, $(\sigma\wr_Q)\wr_P = \pi\wr_P \in \mathcal{O}_{(R\cap Q)\cap P} = \mathcal{O}_{R\cap P}$. By the definition of the excerption operation,

$$1 \leq \pi^{-1}(\pi\wr_P(1)) < \pi^{-1}(\pi\wr_P(2)) < \cdots < \pi^{-1}(\pi\wr_P(\mathrm{card}(R\cap P))) \leq \mathrm{card}(R\cap Q), \tag{4.113}$$

and combining the information in (4.112) and (4.113) (applying the first to the elements of the second), we have

$$\sigma^{-1}(\pi(\pi^{-1}(\pi\wr_P(1)))) < \sigma^{-1}(\pi(\pi^{-1}(\pi\wr_P(2)))) < \cdots < \sigma^{-1}(\pi(\pi^{-1}(\pi\wr_P(\mathrm{card}(R\cap P)))));$$

that is,

$$\sigma^{-1}(\pi\wr_P(1)) < \sigma^{-1}(\pi\wr_P(2)) < \cdots < \sigma^{-1}(\pi\wr_P(\mathrm{card}(R\cap P))).$$

Hence by the definition of the excerption operation (of $\sigma\wr_P$), $\sigma\wr_P = \pi\wr_P = (\sigma\wr_Q)\wr_P$. $\square$

Corollary 4.4.6. *Let P be a finite set, and let Q be any set. Then $\mathcal{O}_{P\cap Q} = (\mathcal{O}_P)\wr_Q$.*

Proof. Observe that $\mathcal{O}_P = \mathcal{O}_{P\cap Q} \odot \mathcal{O}_{P\setminus Q}$, with both $\mathcal{O}_{P\cap Q}$ and $\mathcal{O}_{P\setminus Q}$ nonempty. By Corollary 4.4.4, $\mathcal{O}_{P\cap Q} = (\mathcal{O}_{P\cap Q} \odot \mathcal{O}_{P\setminus Q})\wr_{P\cap Q} = (\mathcal{O}_P)\wr_{P\cap Q} = (\mathcal{O}_P)\wr_Q$. The very last step there comes from a careful reading of the definition of excerption, where we see that excerpting a set Q from an ordering of elements of P results in an ordering of the elements of their intersection. Since $P\cap(P\cap Q) = (P\cap Q)$, there is no distinction between excerpting Q from a set of orderings of P and excerpting $P\cap Q$ from a set of orderings of P. $\square$

The excerption and concatenation operations interact well, in that excerption distributes over concatenation, as the next theorem and its corollary state.

Theorem 4.4.7 (Distribution of excerption over ordering concatenation)**.** *Let P, Q be disjoint, finite sets, and let R be any set. If $\sigma \in \mathcal{O}_P$ and $\tau \in \mathcal{O}_Q$, then $(\sigma.\tau)\wr_R = \sigma\wr_R.\tau\wr_R$.*

Remark 19. To clarify the order of operations, we will stipulate that excerption is performed prior to concatenating or merging.

Proof. Let P, Q be disjoint, finite sets; let R be any set, and let $\sigma \in \mathcal{O}_P, \tau \in \mathcal{O}_Q$. Let $\pi := (\sigma.\tau)\wr_R$. We wish to show that $\pi = \sigma\wr_R.\tau\wr_R$.

We start by noting that π orders the correct objects: Since $\sigma.\tau \in \mathcal{O}_{P \dot\cup Q}$, we have $\pi = (\sigma.\tau)\wr_R \in \mathcal{O}_{(P \dot\cup Q) \cap R} = \mathcal{O}_{(P\cap R)\dot\cup(Q\cap R)}$, while $\sigma\wr_R \in \mathcal{O}_{P\cap R}$ and $\tau\wr_R \in \mathcal{O}_{Q\cap R}$, so $\sigma\wr_R.\tau\wr_R \in \mathcal{O}_{(P\cap R)\dot\cup(Q\cap R)}$.

Moreover, $\pi \in \mathcal{O}_{(P\cap R)\dot\cup(Q\cap R)} = \mathcal{O}_{P\cap R} \odot \mathcal{O}_{Q\cap R}$, and so there exist orderings $\mu \in \mathcal{O}_{P\cap R}$, $\nu \in \mathcal{O}_{Q\cap R}$ for which $\pi \in \{\mu\} \odot \{\nu\}$. We would like to show that $\pi = \mu.\nu$ and that $\mu = \sigma\wr_R$ and $\nu = \tau\wr_R$.

To show that $\pi = \mu.\nu$, we will use Theorem 4.3.9. We already have $\pi \in \mathcal{O}_{P\cap R} \odot \mathcal{O}_{Q\cap R}$, so we only need to show that for all $a \in P \cap R$, $b \in Q \cap R$ we have $\pi^{-1}(a) < \pi^{-1}(b)$. In fact, we have $(\sigma.\tau)^{-1}(a) < (\sigma.\tau)^{-1}(b)$ by Theorem 4.3.9, which implies that $(\sigma.\tau)\wr_R^{-1}(a) < (\sigma.\tau)\wr_R^{-1}(b)$ by the definition of excerption, that is, $\pi^{-1}(a) < \pi^{-1}(b)$. Thus $\pi = \mu.\nu$.

It remains to show that $\mu = \sigma\wr_R$ and $\nu = \tau\wr_R$. To show the first of these using the definition of excerption (Definition 4.4.2), we already have that $\sigma \in \mathcal{O}_P$ and $\mu \in \mathcal{O}_{P\cap R}$, so it suffices to show that for every $x, y \in P \cap R$ we have $\mu^{-1}(x) < \mu^{-1}(y)$ if and only if $\sigma^{-1}(x) < \sigma^{-1}(y)$. Let $x, y \in P \cap R$. Then $\mu^{-1}(x) < \mu^{-1}(y)$ if and only if $\pi^{-1}(x) < \pi^{-1}(y)$ (by definition of $\pi \in \{\mu\} \odot \{\nu\}$), if and only if $(\sigma.\tau)\wr_R^{-1}(x) <$

$(\sigma.\tau)\wr_R^{-1}(y)$ (by how we defined π), if and only if $(\sigma.\tau)^{-1}(x) < (\sigma.\tau)^{-1}(y)$ (by definition of excerption), if and only if $\sigma^{-1}(x) < \sigma^{-1}(y)$ (by definition of the concatenation $\sigma.\tau$). Hence $\mu = \sigma\wr_R$, and similarly, $\nu = \tau\wr_R$, giving us that $\pi = \sigma\wr_R.\tau\wr_R$, and completing the proof. □

Corollary 4.4.8 (Distribution of excerption over set concatenation)**.** *Let P, Q be disjoint, finite sets, and let R be any set. If $\mathcal{U} \subseteq \mathcal{O}_P$ and $\mathcal{V} \subseteq \mathcal{O}_Q$, then $(\mathcal{U}.\mathcal{V})\wr_R = \mathcal{U}\wr_R.\mathcal{V}\wr_R$.*

Proof. By definition of excerption for a set of orderings, using Theorem 4.4.7,

$$\begin{aligned}(\mathcal{U}.\mathcal{V})\wr_R &= \bigcup_{\pi\in\mathcal{U}.\mathcal{V}} \{\pi\wr_R\} = \bigcup_{\sigma\in\mathcal{U},\tau\in\mathcal{V}} \{(\sigma.\tau)\wr_R\} \stackrel{\substack{\text{Thm.}\\ 4.4.7}}{=} \bigcup_{\sigma\in\mathcal{U},\tau\in\mathcal{V}} \{\sigma\wr_R.\tau\wr_R\} = \bigcup_{\sigma'\in\mathcal{U}\wr_R,\tau'\in\mathcal{V}\wr_R} \{\sigma'.\tau'\} \\ &= \mathcal{U}\wr_R.\mathcal{V}\wr_R.\end{aligned}$$

□

Distributive properties also hold for excerption over merging.

Theorem 4.4.9. *[Distribution of excerption over merged singletons] Let P, Q be disjoint, finite sets, let R be any set, and let $\sigma \in \mathcal{O}_P$, $\tau \in \mathcal{O}_Q$. Then*

$$(\{\sigma\} \odot \{\tau\})\wr_R = \{\sigma\wr_R\} \odot \{\tau\wr_R\}.$$

Proof. (Proof of $\subseteq$.) Let $\pi \in (\{\sigma\} \odot \{\tau\})\wr_R$. Then $\pi = \rho\wr_R$ for some $\rho \in \{\sigma\} \odot \{\tau\}$, and $\pi \in \mathcal{O}_{(P\uplus Q)\cap R} = \mathcal{O}_{(P\cap R)\uplus(Q\cap R)}$.

But by Theorem 4.4.3(i), $\sigma = \rho\wr_P$, and thus by Theorem 4.4.5, $\pi\wr_{P\cap R} = (\rho\wr_R)\wr_{P\cap R} = \rho\wr_{P\cap R} = (\rho\wr_P)\wr_{P\cap R} = \sigma\wr_{P\cap R} = \sigma\wr_R$, and similarly, $\pi\wr_{Q\cap R} = \tau\wr_{Q\cap R} = \tau\wr_R$. Theorem 4.4.3(i) then shows that $\pi \in \{\pi\wr_{P\cap R}\} \odot \{\pi\wr_{Q\cap R}\} = \{\sigma\wr_R\} \odot \{\tau\wr_R\}$. Hence $(\{\sigma\} \odot \{\tau\})\wr_R \subseteq \{\sigma\wr_R\} \odot \{\tau\wr_R\}$.

(Proof of $\supseteq$.) Let $\pi \in \{\sigma\wr_R\} \odot \{\tau\wr_R\}$. Our objective is to find an ordering $\rho \in \{\sigma\} \odot \{\tau\}$ with $\rho\wr_R = \pi$. To do so we will write π, σ, and τ as concatenations of several orderings in the style of Theorem 4.3.5, then piece them together to form ρ, and finally demonstrate that ρ has the desired properties.

First, we let

$$\alpha := \sigma\wr_R,\ \beta := \tau\wr_R,$$

so we can say $\pi \in \{\alpha\} \odot \{\beta\}$, where $\alpha \in \mathcal{O}_{P\cap R}$ and $\beta \in \mathcal{O}_{Q\cap R}$. We note that $\pi \in \mathcal{O}_{(P\cap R)\cup(Q\cap R)} = \mathcal{O}_{(P\cup Q)\cap R}$. By Theorem 4.3.5 we have

$$\pi = \alpha_1.\beta_1.\alpha_2.\beta_2.\cdots.\alpha_k.\beta_k$$

for some concatenations

$$\alpha = \alpha_1.\alpha_2.\cdots.\alpha_k, \quad \beta = \beta_1.\beta_2.\cdots.\beta_k \quad \text{with } k \geq 1.$$

Because α is an excerption from σ, all the elements of α appear in the ordering σ, with order preserved. In particular, the initial elements in each of the orderings $\alpha_1, \ldots, \alpha_k$ appear in α and thus in σ, in order. We may thus break up σ just before each of those elements, defining

$$\begin{aligned}
\sigma_0 &:= [\sigma(1),\ \sigma(2),\ \ldots,\ \sigma(\sigma^{-1}(\alpha_1(1)) - 1)],\\
\sigma_1 &:= [\sigma(\sigma^{-1}(\alpha_1(1))) = \alpha_1(1),\ \sigma(\sigma^{-1}(\alpha_1(1)) + 1),\ \ldots,\ \sigma(\sigma^{-1}(\alpha_2(1)) - 1)],\\
\sigma_2 &:= [\sigma(\sigma^{-1}(\alpha_2(1))) = \alpha_2(1),\ \sigma(\sigma^{-1}(\alpha_2(1)) + 1),\ \ldots,\ \sigma(\sigma^{-1}(\alpha_3(1)) - 1)],\\
&\ \ \vdots\\
\sigma_k &:= [\sigma(\sigma^{-1}(\alpha_k(1))) = \alpha_k(1),\ \sigma(\sigma^{-1}(\alpha_k(1)) + 1),\ \ldots,\ \sigma(\mathrm{card}(P))],
\end{aligned} \tag{4.114}$$

and similarly for β and τ,

$$\begin{aligned}
\tau_0 &:= [\tau(1),\ \tau(2),\ \ldots,\ \tau(\tau^{-1}(\beta_1(1)) - 1)],\\
\tau_1 &:= [\tau(\tau^{-1}(\beta_1(1)))=\beta_1(1),\ \tau(\tau^{-1}(\beta_1(1)) + 1),\ \ldots,\ \tau(\tau^{-1}(\beta_2(1)) - 1)],\\
\tau_2 &:= [\tau(\tau^{-1}(\beta_2(1)))=\beta_2(1),\ \tau(\tau^{-1}(\beta_2(1)) + 1),\ \ldots,\ \tau(\tau^{-1}(\beta_3(1)) - 1)],\\
&\ \vdots\\
\tau_k &:= [\tau(\tau^{-1}(\beta_k(1)))=\beta_k(1),\ \tau(\tau^{-1}(\beta_k(1)) + 1),\ \ldots,\ \tau(\mathrm{card}(Q))]. \qquad (4.115)
\end{aligned}$$

Then $\sigma = \sigma_0.\sigma_1.\cdots.\sigma_k$, $\tau = \tau_0, \tau_1.\cdots.\tau_k$. Define $\rho := \sigma_0.\tau_0.\sigma_1.\tau_1.\cdots.\sigma_k.\tau_k$.

We claim that $\pi = \rho\wr_R$. Note that by the definition of excerption (or rather, the equivalent statement (iv) following the definition) applied to $\alpha = \sigma\wr_R$ we have

$$\sigma^{-1}(\alpha(1)) < \sigma^{-1}(\alpha(2)) < \cdots < \sigma^{-1}(\alpha(\mathrm{card}(P \cap R))), \qquad (4.116)$$

and therefore (recalling that $\alpha = \alpha_1.\alpha_2.\cdots.\alpha_k$),

$$\begin{aligned}
&\sigma^{-1}(\alpha_1(1)) < \sigma^{-1}(\alpha_1(2)) < \cdots < \sigma^{-1}(\alpha_1(\mathrm{length}(\alpha_1)))\\
<&\sigma^{-1}(\alpha_2(1)) < \sigma^{-1}(\alpha_2(2)) < \cdots < \sigma^{-1}(\alpha_2(\mathrm{length}(\alpha_2)))\\
&\quad\vdots\\
<&\sigma^{-1}(\alpha_k(1)) < \sigma^{-1}(\alpha_k(2)) < \cdots < \sigma^{-1}(\alpha_k(\mathrm{length}(\alpha_k))). \qquad (4.117)
\end{aligned}$$

Now, for each σ_j, $j = 1, 2, \ldots, k$, we have by inspection that $\sigma_j(x) = \sigma(x + \gamma_j)$ for all x in the domain of σ_j, where $\gamma_j = \mathrm{length}(\sigma_0) + \mathrm{length}(\sigma_1) + \cdots + \mathrm{length}(\sigma_{j-1})$. Consequently, if $i = \sigma_j(x) = \sigma(x + \gamma_j)$, then $\sigma^{-1}(i) = x + \gamma_j = \sigma_j^{-1}(i) + \gamma_j$. Looking at each row $j = 1, 2, \ldots, k$ of the inequality (4.117) will then give us that

$$\sigma_j^{-1}(\alpha_j(1)) + \gamma_j < \sigma_j^{-1}(\alpha_j(2)) + \gamma_j < \cdots < \sigma_j^{-1}(\alpha_j(\text{length}(\alpha_j)) + \gamma_j, \tag{4.118}$$

and therefore canceling every γ_j,

$$\sigma_j^{-1}(\alpha_j(1)) < \sigma_j^{-1}(\alpha_j(2)) < \cdots < \sigma_j^{-1}(\alpha_j(\text{length}(\alpha_j)). \tag{4.119}$$

Hence, $\sigma_j\wr_R = \alpha_j$ for $j = 1, 2, \ldots, k$, and since there are no elements of R in the image of σ_0 we have $\sigma_0\wr_R = \emptyset$. (The first part of the previous sentence may not be so obvious. The orderings $\sigma_0, \sigma_1, \ldots, \sigma_k$ together order the entire set that σ orders, which is the set P. This includes the set $P \cap R$, which together $\alpha_1, \ldots, \alpha_k$ order. The ordering σ_j was specifically defined so that of those elements of $P \cap R$, it orders exactly those that α_j orders—not those of any other α_i. All other elements ordered by σ_j are still in P but outside of R. Thus the intersection of R with the set ordered by σ_j is the set ordered by α_j, so if we excerpt the set R from σ_j, we end up with an ordering that orders exactly what α_j orders. The string of inequalities then yields that $\sigma_j\wr_R = \alpha_j$.) Similarly, $\tau_0\wr_R = \emptyset$, $\tau_j\wr_R = \beta_j$ for $j = 1, 2, \ldots, k$.

Thus by Theorem 4.4.7, $\rho\wr_R = \sigma_0\wr_R.\tau_0\wr_R.\sigma_1\wr_R.\tau_1\wr_R.\cdots.\sigma_k\wr_R.\tau_k\wr_R = \emptyset.\emptyset.\alpha_1.\beta_1.\alpha_2.\beta_2.\cdots.\alpha_k.\beta_k = \pi$. Hence, $\pi = \rho\wr_R$ with $\rho \in \{\sigma\} \odot \{\tau\}$, so $\pi \in (\{\sigma\} \odot \{\tau\})\wr_R$. Therefore, $\{\sigma\wr_R\} \odot \{\tau\wr_R\} \subseteq (\{\sigma\} \odot \{\tau\})\wr_R$. $\square$

Corollary 4.4.10 (Distribution of excerption over merged sets)**.** *If P and Q are disjoint, finite sets, if $\mathcal{U} \subseteq \mathcal{O}_P$ and $\mathcal{V} \subseteq \mathcal{O}_Q$, and if R is any set, then*

$$(\mathcal{U} \odot \mathcal{V})\wr_R = \mathcal{U}\wr_R \odot \mathcal{V}\wr_R.$$

Proof. With the hypotheses as stated, using the definitions of excerption and merging and Theorem 4.4.9,

$$
\begin{aligned}
(\mathcal{U} \odot \mathcal{V})\wr_R &= \bigcup_{\pi \in \mathcal{U}\odot\mathcal{V}} \{\pi\wr_R\} = \bigcup_{\sigma\in\mathcal{U},\tau\in\mathcal{V}} \ \bigcup_{\pi\in\{\sigma\}\odot\{\tau\}} \{\pi\wr_R\} = \bigcup_{\sigma\in\mathcal{U},\tau\in\mathcal{V}} (\{\sigma\}\odot\{\tau\})\wr_R \\
&= \bigcup_{\sigma\in\mathcal{U},\tau\in\mathcal{V}} (\{\sigma\wr_R\}\odot\{\tau\wr_R\}) = \bigcup_{\sigma'\in\mathcal{U}\wr_R,\tau'\in\mathcal{V}\wr_R} (\{\sigma'\}\odot\{\tau'\}) \\
&= \mathcal{U}\wr_R \odot \mathcal{V}\wr_R.
\end{aligned}
$$

□

Viewing the merge operation as something like a multiplication, we might wonder whether it is possible to factor a given set of orderings nontrivially, and furthermore, whether there is some kind of canonical factorization for a set of orderings. With that in mind, to conclude this section, we begin by defining irreducibility of a set of orderings.

Definition 4.4.11 (Irreducible set of orderings)**.** Let P be a finite set. A set of orderings $\mathcal{U} \subseteq \mathcal{O}_P$ is called **irreducible** if $P \neq \emptyset$, $\mathcal{U} \neq \emptyset$, and there are no pairs of disjoint, nonempty subsets $Q, R \subseteq P$ with $Q \cup R = P$ and nonempty sets of orderings $\mathcal{V} \subseteq \mathcal{O}_Q$, $\mathcal{W} \subseteq \mathcal{O}_R$ for which $\mathcal{V} \odot \mathcal{W} = \mathcal{U}$. (Note that irreducibility of $\mathcal{U} \subseteq \mathcal{O}_P$ requires that P is nonempty, and hence $\mathcal{U}$ is not the set consisting of the null ordering.)

Remark 20. The negation of irreducibility is useful to know for proving the next theorem. If P is a finite set and $\mathcal{U} \subseteq \mathcal{O}_P$ is a nonempty set of orderings that is *not* irreducible, then this definition implies that there are disjoint, nonempty subsets $Q, R \subseteq P$ with $Q \cup R = P$ and nonempty sets of orderings $\mathcal{V} \subseteq \mathcal{O}_Q$, $\mathcal{W} \subseteq \mathcal{O}_R$ for which $\mathcal{V} \odot \mathcal{W} = \mathcal{U}$. By Theorem 4.4.3(iii) this implies that $\mathcal{V} = \mathcal{U}\wr_Q$ and $\mathcal{W} = \mathcal{U}\wr_R$.

Hence, $\mathcal{U} = \mathcal{U}\wr_Q \odot \mathcal{U}\wr_R$, with both factors nonempty and not consisting of just the null ordering.

Theorem 4.4.12. *[Unique merge factorization into irreducibles] If P is a finite, nonempty set, and there is a nonempty set of orderings $\mathcal{W} \subseteq \mathcal{O}_P$, then there exist nonempty, pairwise disjoint subsets $P_1, P_2, \ldots, P_k \subseteq P$ with $\dot{\bigcup}_{i=1}^{k} P_i = P$, and there exist sets of orderings $\mathcal{W}_1 \subseteq \mathcal{O}_{P_1}$, $\mathcal{W}_2 \subseteq \mathcal{O}_{P_2}$, $\ldots$, $\mathcal{W}_k \subseteq \mathcal{O}_{P_k}$, for which $\mathcal{W}_i$ is irreducible for each $i = 1, 2, \ldots, k$, and $\mathcal{W} = \mathcal{W}_1 \odot \mathcal{W}_2 \odot \cdots \odot \mathcal{W}_k$. Moreover, this factorization into irreducibles is unique up to order of the factors.*

Proof. We will prove both statements of the theorem using induction. For the first we use strong induction on the cardinality of the set P. For the anchor step, let P be a finite, nonempty set, let $\mathcal{W} \subseteq \mathcal{O}_P$ be nonempty, and suppose $\mathrm{card}(P) = 1$, say, $P = \{a\}$. The fact that $\mathcal{W} \subseteq \mathcal{O}_P$ is nonempty tells us that $\mathcal{W} = \{[a]\}$. But then P cannot be expressed as a disjoint union of nonempty sets, and therefore $\mathcal{W}$ is irreducible. Our factorization is $\mathcal{W} = \mathcal{W}$.

For the induction step, let P be a finite, nonempty set with cardinality $\mathrm{card}(P) > 1$, and suppose that factorization into irreducibles is possible for every nonempty set $\mathcal{W}'$ of orderings of nonempty finite sets P' having $\mathrm{card}(P') < \mathrm{card}(P)$. Let $\mathcal{W} \subseteq \mathcal{O}_P$ be nonempty. If $\mathcal{W}$ is irreducible, then again we have the factorization we sought, $\mathcal{W} = \mathcal{W}$. If, on the other hand, $\mathcal{W}$ is not irreducible, then there exist nonempty, disjoint subsets $Q, R \subseteq P$ with $Q \dot{\cup} R = P$ and nonempty sets of orderings $\mathcal{U} \subseteq \mathcal{O}_Q$, $\mathcal{V} \subseteq \mathcal{O}_R$ with $\mathcal{U} \odot \mathcal{V} = \mathcal{W}$. Since Q and R are nonempty, and their union is P, their cardinalities must be strictly less than $\mathrm{card}(P)$, which implies by the induction hypothesis that $\mathcal{U}$ and $\mathcal{V}$ can be factored into irreducibles.

We therefore have nonempty, pairwise disjoint subsets $Q_1, Q_2, \ldots, Q_m \subseteq Q$ with $\dot{\bigcup}_{i=1}^{m} Q_i = Q$, and sets of orderings $\mathcal{U}_1 \subseteq \mathcal{O}_{Q_1}$, $\mathcal{U}_2 \subseteq \mathcal{O}_{Q_2}$, $\ldots$, $\mathcal{U}_m \subseteq \mathcal{O}_{Q_m}$, for which

$\mathcal{U}_i$ is irreducible for each $i = 1, 2, \ldots, m$, and $\mathcal{U} = \mathcal{U}_1 \odot \mathcal{U}_2 \odot \cdots \odot \mathcal{U}_m$. We also have nonempty, pairwise disjoint subsets $R_1, R_2, \ldots, R_n \subseteq R$ with $\dot{\bigcup}_{i=1}^{n} R_i = R$, and sets of orderings $\mathcal{V}_1 \subseteq \mathcal{O}_{R_1}$, $\mathcal{V}_2 \subseteq \mathcal{O}_{R_2}$, $\ldots$, $\mathcal{V}_n \subseteq \mathcal{O}_{R_n}$, for which $\mathcal{V}_i$ is irreducible for each $i = 1, 2, \ldots, n$, and $\mathcal{V} = \mathcal{V}_1 \odot \mathcal{V}_2 \odot \cdots \odot \mathcal{V}_n$. But then since Q and R are disjoint, the entire collection $Q_1, \ldots, Q_m, R_1, \ldots, R_n$ is pairwise disjoint, and $\mathcal{W} = \mathcal{U} \odot \mathcal{V} = \mathcal{U}_1 \odot \cdots \odot \mathcal{U}_m \odot \mathcal{V}_1 \odot \cdots \odot \mathcal{V}_n$, so we have found a factorization of $\mathcal{W}$ into irreducibles.

Therefore, by strong mathematical induction, we know that any nonempty $\mathcal{W} \subseteq \mathcal{O}_P$ for a finite, nonempty set P can be factored into irreducibles as described. It remains to show that the factorization into irreducibles is unique up to the order of the factors.

Suppose for this $\mathcal{W}$ that

$$\mathcal{W} = \mathcal{U}_1 \odot \cdots \odot \mathcal{U}_m = \mathcal{V}_1 \odot \cdots \odot \mathcal{V}_n, \tag{4.120}$$

where $\mathcal{U}_1, \ldots, \mathcal{U}_m, \mathcal{V}_1, \ldots, \mathcal{V}_n$ are irreducible, with $\mathcal{U}_1 \subseteq \mathcal{O}_{Q_1}$, $\ldots$, $\mathcal{U}_m \subseteq \mathcal{O}_{Q_m}$, $\mathcal{V}_1 \subseteq \mathcal{O}_{R_1}$, $\ldots$, $\mathcal{V}_n \subseteq \mathcal{O}_{R_n}$, where $Q_1, \ldots, Q_m$ are pairwise disjoint with $\dot{\bigcup}_{i=1}^{m} Q_i = P$, and $R_1, \ldots, R_n$ are pairwise disjoint with $\dot{\bigcup}_{i=1}^{n} R_i = P$. (Note that except for P and $\mathcal{W}$, the names we are using here represent completely different entities than they did in the proof a moment ago that $\mathcal{W}$ factors into irreducibles.)

Consider $\mathcal{W}\wr_{Q_1}$. By Corollary 4.4.10 we have $\mathcal{W}\wr_{Q_1} = \mathcal{U}_1\wr_{Q_1} \odot \cdots \odot \mathcal{U}_m\wr_{Q_1} = \mathcal{U}_1 \odot \{\emptyset\} \odot \cdots \odot \{\emptyset\} = \mathcal{U}_1$. But then

$$\mathcal{U}_1 = \mathcal{W}\wr_{Q_1} = \mathcal{V}_1\wr_{Q_1} \odot \cdots \odot \mathcal{V}_n\wr_{Q_1}. \tag{4.121}$$

Note that for all $j = 1, \ldots, n$ we know $V_j\wr_{Q_1}$ orders the set $Q_1 \cap R_j$. If $Q_1 \cap R_j$ is nonempty for at least two values of $j \in \{1, 2, \ldots, n\}$, then the right-hand side gives a nontrivial factorization for $\mathcal{U}_1$, which cannot be the case, since $\mathcal{U}_1$ is irreducible. So at most one R_j has nonempty intersection with Q_1. But at least one R_j has nonempty intersection with Q_1 (since $\emptyset \neq Q_1 \subseteq P$ and $R_1 \dot\cup \cdots \dot\cup R_n = P$), so we can say that exactly one has; without loss of generality, $Q_1 \cap R_1 \neq \emptyset$, and in fact $Q_1 \subseteq R_1$. We can use the same reasoning (looking at $\mathcal{V}_1$) to show that R_1 has nonempty intersection with exactly one Q_j, where $j \in \{1, 2, \ldots, m\}$, and we know in particular that must be Q_1. So $R_1 \subseteq Q_1$. Therefore, $Q_1 = R_1$. But then

$$\mathcal{U}_1 = \mathcal{W}\wr_{Q_1} = \mathcal{W}\wr_{R_1} = \mathcal{V}_1\wr_{R_1} \odot \cdots \odot \mathcal{V}_n\wr_{R_1} = \mathcal{V}_1 \odot \{\emptyset\} \odot \cdots \odot \{\emptyset\} = \mathcal{V}_1. \tag{4.122}$$

By similar reasoning we can say without loss of generality that $\mathcal{U}_2 = \mathcal{V}_2$. (We know that $\mathcal{U}_2$ must equal one of the factors on the right-hand expression in Equation (4.120), and that it orders a different set from what $\mathcal{V}_1$ orders.) Continuing the process tells us that the middle expression and right-hand expression in Equation (4.120) must have an equal number of factors, and that those two sets of factors are identical. Therefore, factorization is unique up to the order of the factors. $\square$

Chapter 5

Further properties of the merge, concatenation, and excerption operations

5.1 Set relationships and excerption

We would like to express more relationships involving the merging and concatenating of sets, but to do so, we first need to prove a few more facts involving excerption, such as set relations.

Theorem 5.1.1 (Set relations and excerption). *If P is a finite set, if $\mathcal{U}, \mathcal{V} \subseteq \mathcal{O}_P$, and if R is any set, then the following hold:*

(i) $(\mathcal{U} \cup \mathcal{V})\wr_R = \mathcal{U}\wr_R \cup \mathcal{V}\wr_R$.

(ii) $(\mathcal{U} \cap \mathcal{V})\wr_R \subseteq \mathcal{U}\wr_R \cap \mathcal{V}\wr_R$. *In particular, if* $\mathcal{U}\wr_R \cap \mathcal{V}\wr_R = \emptyset$, *then* $(\mathcal{U} \cap \mathcal{V})\wr_R = \emptyset$, *and therefore,* $\mathcal{U} \cap \mathcal{V} = \emptyset$.

(iii) $(\mathcal{U} \setminus \mathcal{V})\wr_R \supseteq \mathcal{U}\wr_R \setminus \mathcal{V}\wr_R$.

Proof.

(i) Claim: $(\mathcal{U} \cup \mathcal{V})\wr_R = \mathcal{U}\wr_R \cup \mathcal{V}\wr_R$.

We prove this using a string of equivalences: We have $\pi \in (\mathcal{U} \cup \mathcal{V})\wr_R$ if and only if $\pi = \sigma\wr_R$ for some $\sigma \in \mathcal{U}$ or $\sigma \in \mathcal{V}$, which in turn is true if and only if $\pi \in \mathcal{U}\wr_R$ or $\pi \in \mathcal{V}\wr_R$, which is to say $\pi \in \mathcal{U}\wr_R \cup \mathcal{V}\wr_R$.

(ii) Claim: $(\mathcal{U} \cap \mathcal{V})\wr_R \subseteq \mathcal{U}\wr_R \cap \mathcal{V}\wr_R$.

Let $\pi \in (\mathcal{U} \cap \mathcal{V})\wr_R$. Then $\pi = \sigma\wr_R$ for some $\sigma \in \mathcal{U} \cap \mathcal{V}$. But then $\sigma \in \mathcal{U}$ and $\sigma \in \mathcal{V}$, so $\pi \in \mathcal{U}\wr_R$ and $\pi \in \mathcal{V}\wr_R$; that is, $\pi \in \mathcal{U}\wr_R \cap \mathcal{V}\wr_R$.

(The proof of the "in particular" statement is in the statement itself. The last part is due to the contrapositive of the existence statement given in Theorem 4.4.1.)

(iii) Claim: $(\mathcal{U} \smallsetminus \mathcal{V})\wr_R \supseteq \mathcal{U}\wr_R \smallsetminus \mathcal{V}\wr_R$.

Let $\pi \in \mathcal{U}\wr_R \smallsetminus \mathcal{V}\wr_R$. Then $\pi \in \mathcal{U}\wr_R$, so there exists $\sigma \in \mathcal{U}$ with $\pi = \sigma\wr_R$. If we suppose that $\sigma \in \mathcal{V}$ also, then $\pi \in \mathcal{V}\wr_R$, which is a contradiction. Therefore, $\sigma \notin \mathcal{V}$, implying $\sigma \in \mathcal{U} \smallsetminus \mathcal{V}$. Hence, $\pi \in (\mathcal{U} \smallsetminus \mathcal{V})\wr_R$. $\square$

Incidentally, it is easy to show that statements (ii) and (iii) cannot be changed to equality. For (ii), if $P := \{1, 2\}$ is ordered by the sets $\mathcal{U} := \{[1, 2]\}$ and $\mathcal{V} := \{[2, 1]\}$, and if $R := \{1\}$, then $(\mathcal{U} \cap \mathcal{V})\wr_R = \emptyset\wr_R = \emptyset$, while $\mathcal{U}\wr_R \cap \mathcal{V}\wr_R = \{[1]\} \cap \{[1]\} = \{[1]\}$. For (iii), using the same sets, $(\mathcal{U} \smallsetminus \mathcal{V})\wr_R = \{[1, 2]\}\wr_R = \{[1]\}$, whereas $\mathcal{U}\wr_R \smallsetminus \mathcal{V}\wr_R = \{[1]\} \smallsetminus \{[1]\} = \emptyset$.

For some of the manipulations we will be doing in the next section, the following theorem will be useful:

Theorem 5.1.2. *If P and Q are disjoint, finite sets, if $\mathcal{U} \subseteq \mathcal{O}_P$ and $\mathcal{V} \subseteq \mathcal{O}_Q$, and if R is any set, then*

$$[(\mathcal{U} \odot \mathcal{V}) \smallsetminus (\mathcal{U}.\mathcal{V})]\wr_R \supseteq (U\wr_R \odot \mathcal{V}\wr_R) \smallsetminus (U\wr_R.\mathcal{V}\wr_R). \tag{5.1}$$

Proof. By Theorem 5.1.1 we have

$$\begin{aligned}[(\mathcal{U} \odot \mathcal{V}) \setminus (\mathcal{U}.\mathcal{V})]\wr_R &\supseteq (\mathcal{U} \odot \mathcal{V})\wr_R \setminus (\mathcal{U}.\mathcal{V})\wr_R \\ &= (U\wr_R \odot \mathcal{V}\wr_R) \setminus (U\wr_R.\mathcal{V}\wr_R). \end{aligned} \tag{5.2}$$

□

Of these two theorems, Theorems 5.1.1 and 5.1.2, the most useful properties for us will be parts (i) and (ii) of Theorem 5.1.1. Another useful fact occurs in a case when equality holds for Equation (5.1), as follows:

Theorem 5.1.3. *Let $L, P, Q,$ and R be pairwise disjoint, finite sets, and let $\mathcal{U} \subseteq \mathcal{O}_L, \mathcal{V} \subseteq \mathcal{O}_P, \mathcal{W} \subseteq \mathcal{O}_Q$, and $\mathcal{Z} \subseteq \mathcal{O}_R$. Then*

$$\Big([(\mathcal{U}.\mathcal{V}) \odot (\mathcal{W}.\mathcal{Z})] \setminus [(\mathcal{U}.\mathcal{V}).(\mathcal{W}.\mathcal{Z})]\Big)\wr_{P\cup Q} = (\mathcal{V} \odot \mathcal{W}) \setminus (\mathcal{V}.\mathcal{W}). \tag{5.3}$$

Proof. By Theorem 5.1.2,

$$\begin{aligned}&\Big([(\mathcal{U}.\mathcal{V}) \odot (\mathcal{W}.\mathcal{Z})] \setminus [(\mathcal{U}.\mathcal{V}).(\mathcal{W}.\mathcal{Z})]\Big)\wr_{P\cup Q} \\ &\qquad \supseteq [(\mathcal{U}.\mathcal{V})\wr_{P\cup Q} \odot (\mathcal{W}.\mathcal{Z})\wr_{P\cup Q}] \setminus [(\mathcal{U}.\mathcal{V})\wr_{P\cup Q}.(\mathcal{W}.\mathcal{Z})\wr_{P\cup Q}] \\ &\qquad = [(\{\emptyset\}.\mathcal{V}) \odot (\mathcal{W}.\{\emptyset\})] \setminus [(\{\emptyset\}.\mathcal{V}).(\mathcal{W}.\{\emptyset\})] = (\mathcal{V} \odot \mathcal{W}) \setminus (\mathcal{V}.\mathcal{W}). \end{aligned} \tag{5.4}$$

For inclusion in the forward direction, let $l := \mathrm{card}(L)$, $p := \mathrm{card}(P)$, $q := \mathrm{card}(Q)$, $r := \mathrm{card}(R)$, and let

$$\pi \in \Big([(\mathcal{U}.\mathcal{V}) \odot (\mathcal{W}.\mathcal{Z})] \setminus [(\mathcal{U}.\mathcal{V}).(\mathcal{W}.\mathcal{Z})]\Big)\wr_{P\cup Q}. \tag{5.5}$$

By definition of excerption, $\pi = \rho\wr_{P\cup Q}$ for some ordering $\rho \in [(\mathcal{U}.\mathcal{V}) \odot (\mathcal{W}.\mathcal{Z})] \setminus$

$[(\mathcal{U}.\mathcal{V}).(\mathcal{W}.\mathcal{Z})]$. Since $\rho \in (\mathcal{U}.\mathcal{V}) \odot (\mathcal{W}.\mathcal{Z})$, but $\rho \notin (\mathcal{U}.\mathcal{V}).(\mathcal{W}.\mathcal{Z})$, there must be an $x \in L \cup P$ and $y \in Q \cup R$ with $\rho^{-1}(x) \not< \rho^{-1}(y)$ by Theorem 4.3.9, so $\rho^{-1}(y) < \rho^{-1}(x)$ (since $x \neq y$).

Let $\sigma := \rho \wr_{L \cup P}$, $\tau := \rho \wr_{Q \cup R}$, so σ orders $L \cup P$, and τ orders $Q \cup R$, and

$$\sigma = [\sigma(1), \sigma(2), \ldots, \sigma(l+p)],\ \tau = [\tau(1), \tau(2), \ldots, \tau(q+r)]. \tag{5.6}$$

But then in this explicit representation for σ we have $\sigma(l+p)$ appearing to the right of every other element of $L \cup P$, and therefore $\sigma^{-1}(x) \leq \sigma^{-1}(\sigma(l+p))$, and similarly, $\tau^{-1}(\tau(1)) \leq \tau^{-1}(y)$, where $\sigma(l+p) \in P$ and $\tau(1) \in Q$. By the definition of the excerptions $\sigma = \rho \wr_{L \cup P}$, $\tau = \rho \wr_{Q \cup R}$, this tells us that $\rho^{-1}(x) \leq \rho^{-1}(\sigma(l+p))$ and $\rho^{-1}(\tau(1)) \leq \rho^{-1}(y)$; hence

$$\rho^{-1}(\tau(1)) \leq \rho^{-1}(y) < \rho^{-1}(x) \leq \rho^{-1}(\sigma(l+p)). \tag{5.7}$$

The definition of the excerption $\rho \wr_{P \cup Q}$ and the facts $\sigma(l+p) \in P$ and $\tau(1) \in Q$ then say that

$$\rho \wr_{P \cup Q}^{-1}(\tau(1)) \leq \rho \wr_{P \cup Q}^{-1}(\sigma(l+p)). \tag{5.8}$$

The fact that $\rho \in (\mathcal{U}.\mathcal{V}) \odot (\mathcal{W}.\mathcal{Z})$ implies that

$$\begin{aligned}
\rho \wr_{P \cup Q} &\in [(\mathcal{U}.\mathcal{V}) \odot (\mathcal{W}.\mathcal{Z})] \wr_{P \cup Q} \\
&= (\mathcal{U} \wr_{P \cup Q} . \mathcal{V} \wr_{P \cup Q}) \odot (\mathcal{W} \wr_{P \cup Q} . \mathcal{Z} \wr_{P \cup Q}) \\
&= (\{\emptyset\}.\mathcal{V}) \odot (\mathcal{W}.\{\emptyset\}) \\
&= \mathcal{V} \odot \mathcal{W}.
\end{aligned} \tag{5.9}$$

But we just said that $\rho \wr_{P \cup Q}^{-1}(\tau(1)) \leq \rho \wr_{P \cup Q}^{-1}(\sigma(l+p))$ with $\sigma(l+p) \in P$ and $\tau(1) \in Q$, so by Theorem 4.3.9, $\rho\wr_{P\cup Q} \notin \mathcal{V}.\mathcal{W}$. Therefore, $\pi = \rho\wr_{P\cup Q} \in (V \odot \mathcal{W}) \smallsetminus (V.\mathcal{W})$. This establishes Equation (5.3). $\square$

5.2 Combining sets of orderings

Next we offer two theorems about the intersection of sets formed by both merging and concatenating.

Theorem 5.2.1. *If P, Q and R are pairwise disjoint, finite sets, with sets of orderings $\mathcal{U} \subseteq \mathcal{O}_P$, $\mathcal{V} \subseteq \mathcal{O}_Q$, and $\mathcal{W} \subseteq \mathcal{O}_R$, then*

$$\mathcal{U}.(\mathcal{V} \odot \mathcal{W}) = [(\mathcal{U}.\mathcal{V}) \odot \mathcal{W}] \cap [(\mathcal{U}.\mathcal{W}) \odot \mathcal{V}]. \tag{5.10}$$

Proof. We note first that if any of $\mathcal{U}, \mathcal{V}$, or $\mathcal{W}$ is empty, then both sides of (5.10) are the empty set, and thus they are equal. For the remainder of the proof we will therefore assume the sets to be nonempty.

(Proof of $\subseteq$.) Since $\mathcal{U}.(\mathcal{V} \odot \mathcal{W}) \subseteq \mathcal{O}_{(P\cup Q)\cup R}$, we have by Theorem 4.4.3(ii) that

$$\begin{aligned}
\mathcal{U}.(\mathcal{V} \odot \mathcal{W}) &\subseteq [\mathcal{U}.(\mathcal{V} \odot \mathcal{W})]\wr_{P\cup Q} \odot [\mathcal{U}.(\mathcal{V} \odot \mathcal{W})]\wr_R \\
&= [\mathcal{U}\wr_{P\cup Q}.(\mathcal{V}\wr_{P\cup Q} \odot \mathcal{W}\wr_{P\cup Q})] \odot [\mathcal{U}\wr_R.(\mathcal{V}\wr_R \odot \mathcal{W}\wr_R)] \\
&= [\mathcal{U}.(\mathcal{V} \odot \{\emptyset\})] \odot [\{\emptyset\}.(\{\emptyset\} \odot \mathcal{W})] \\
&= (\mathcal{U}.\mathcal{V}) \odot \mathcal{W}.
\end{aligned} \tag{5.11}$$

Similarly (excerpting by $P\cup R$ and by Q), we have $\mathcal{U}.(\mathcal{V}\odot\mathcal{W}) \subseteq (\mathcal{U}.\mathcal{W})\odot\mathcal{V}$. Therefore,

$$\mathcal{U}.(\mathcal{V} \odot \mathcal{W}) \subseteq [(\mathcal{U}.\mathcal{V}) \odot \mathcal{W}] \cap [(\mathcal{U}.\mathcal{W}) \odot \mathcal{V}]. \tag{5.12}$$

(Proof of $\supseteq$.) Using this technique again, and applying Theorem 5.1.1,

$$
\begin{aligned}
&[(\mathcal{U}.\mathcal{V}) \odot \mathcal{W}] \cap [(\mathcal{U}.\mathcal{W}) \odot \mathcal{V}] \\
&\subseteq \Big([(\mathcal{U}.\mathcal{V}) \odot \mathcal{W}] \cap [(\mathcal{U}.\mathcal{W}) \odot \mathcal{V}]\Big)\wr_P \odot \Big([(\mathcal{U}.\mathcal{V}) \odot \mathcal{W}] \cap [(\mathcal{U}.\mathcal{W}) \odot \mathcal{V}]\Big)\wr_{Q\cup R} \\
&\subseteq \Big([(\mathcal{U}.\mathcal{V}) \odot \mathcal{W}]\wr_P \cap [(\mathcal{U}.\mathcal{W}) \odot \mathcal{V}]\wr_P\Big) \odot \Big([(\mathcal{U}.\mathcal{V}) \odot \mathcal{W}]\wr_{Q\cup R} \cap [(\mathcal{U}.\mathcal{W}) \odot \mathcal{V}]\wr_{Q\cup R}\Big) \\
&= (\mathcal{U} \cap \mathcal{U}) \odot [(\mathcal{V} \odot \mathcal{W}) \cap (\mathcal{W} \odot \mathcal{V})] \\
&= \mathcal{U} \odot (\mathcal{V} \odot \mathcal{W}).
\end{aligned} \tag{5.13}
$$

Furthermore, for any $\pi \in [(\mathcal{U}.\mathcal{V}) \odot \mathcal{W}] \cap [(\mathcal{U}.\mathcal{W}) \odot \mathcal{V}]$ we have that $\pi\wr_{P\cup Q} \in \mathcal{U}.\mathcal{V}$, so for any $x \in P$, $y \in Q$ we have $\pi\wr_{P\cup Q}^{-1}(x) < \pi\wr_{P\cup Q}^{-1}(y)$, so $\pi^{-1}(x) < \pi^{-1}(y)$. Similarly, for any $x \in P$, $y \in R$ we have $\pi^{-1}(x) < \pi^{-1}(y)$. Therefore, for any $x \in P$, $y \in Q\cup R$ we have $\pi^{-1}(x) < \pi^{-1}(y)$. Combining this with the fact that $\pi \in \mathcal{U} \odot (\mathcal{V} \odot \mathcal{W})$ tells us by Theorem 4.3.9 that

$$\pi \in \mathcal{U}.(\mathcal{V} \odot \mathcal{W}). \tag{5.14}$$

□

Roughly speaking, Theorem 5.2.1 says that if $\mathcal{U}$ precedes $\mathcal{V}$ and $\mathcal{W}$, then $\mathcal{U}$ precedes $\mathcal{V}$, and $\mathcal{U}$ precedes $\mathcal{W}$.

Theorem 5.2.2. *If P, Q and R are pairwise disjoint, finite sets, with sets of orderings $\mathcal{U} \subseteq \mathcal{O}_P$, $\mathcal{V} \subseteq \mathcal{O}_Q$, and $\mathcal{W} \subseteq \mathcal{O}_R$, then*

$$[(\mathcal{U}.\mathcal{V}) \odot \mathcal{W}] \cap [\mathcal{U} \odot (\mathcal{V}.\mathcal{W})] = \mathcal{U}.\mathcal{V}.\mathcal{W} \tag{5.15}$$

holds.

Proof. Starting from the left-hand side,

$$
\begin{aligned}
&[(\mathcal{U}.\mathcal{V}) \odot \mathcal{W}] \cap [\mathcal{U} \odot (\mathcal{V}.\mathcal{W})] \\
&\quad = \Big([(\mathcal{U}.\mathcal{V}).\mathcal{W}] \dot{\cup} \big[((\mathcal{U}.\mathcal{V}) \odot \mathcal{W}) \smallsetminus ((\mathcal{U}.\mathcal{V}).\mathcal{W}) \big] \Big) \cap \ [\mathcal{U} \odot (\mathcal{V}.\mathcal{W})] \\
&\quad = \Big((\mathcal{U}.\mathcal{V}.\mathcal{W}) \cap [\mathcal{U} \odot (\mathcal{V}.\mathcal{W})] \Big) \dot{\cup} \Big(\big[((\mathcal{U}.\mathcal{V}) \odot \mathcal{W}) \smallsetminus ((\mathcal{U}.\mathcal{V}).\mathcal{W}) \big] \cap [\mathcal{U} \odot (\mathcal{V}.\mathcal{W})] \Big).
\end{aligned}
\tag{5.16}
$$

Applying excerption properties (including Theorem 5.1.3) to the right-hand term, we have

$$
\begin{aligned}
&\Big(\big[((\mathcal{U}.\mathcal{V}) \odot \mathcal{W}) \smallsetminus ((\mathcal{U}.\mathcal{V}).\mathcal{W}) \big] \cap [\mathcal{U} \odot (\mathcal{V}.\mathcal{W})] \Big) \wr_{Q \dot{\cup} R} \\
&\qquad \subseteq \big[((\mathcal{U}.\mathcal{V}) \odot \mathcal{W}) \smallsetminus ((\mathcal{U}.\mathcal{V}).\mathcal{W}) \big] \wr_{Q \dot{\cup} R} \cap [\mathcal{U} \odot (\mathcal{V}.\mathcal{W})] \wr_{Q \dot{\cup} R} \\
&\qquad = [(\mathcal{V} \odot \mathcal{W}) \smallsetminus (\mathcal{V}.\mathcal{W})] \cap (\mathcal{V}.\mathcal{W}) \\
&\qquad = \emptyset.
\end{aligned}
\tag{5.17}
$$

Consequently,

$$
\big[((\mathcal{U}.\mathcal{V}) \odot \mathcal{W}) \smallsetminus ((\mathcal{U}.\mathcal{V}).\mathcal{W}) \big] \cap [\mathcal{U} \odot (\mathcal{V}.\mathcal{W})] = \emptyset. \tag{5.18}
$$

That process eliminates the last parenthetical expression in Equation (5.16), leaving

$$
\begin{aligned}
&[(\mathcal{U}.\mathcal{V}) \odot \mathcal{W}] \cap [\mathcal{U} \odot (\mathcal{V}.\mathcal{W})] \\
&\qquad = (\mathcal{U}.\mathcal{V}.\mathcal{W}) \cap [\mathcal{U} \odot (\mathcal{V}.\mathcal{W})] \\
&\qquad = (\mathcal{U}.\mathcal{V}.\mathcal{W}) \cap \Big([\mathcal{U}.(\mathcal{V}.\mathcal{W})] \dot{\cup} [(\mathcal{U} \odot (\mathcal{V}.\mathcal{W})) \smallsetminus (\mathcal{U}.(\mathcal{V}.\mathcal{W}))] \Big)
\end{aligned}
$$

$$= [(\mathcal{U}.\mathcal{V}.\mathcal{W}) \cap (\mathcal{U}.\mathcal{V}.\mathcal{W})] \cup \Big((\mathcal{U}.\mathcal{V}.\mathcal{W}) \cap [(\mathcal{U} \odot (\mathcal{V}.\mathcal{W})) \smallsetminus (\mathcal{U}.\mathcal{V}.\mathcal{W})] \Big)$$

$$= (\mathcal{U}.\mathcal{V}.\mathcal{W}) \cup \emptyset$$

$$= \mathcal{U}.\mathcal{V}.\mathcal{W}. \tag{5.19}$$

□

Roughly speaking, the statement of the theorem is that if $\mathcal{U}$ precedes $\mathcal{V}$, and $\mathcal{V}$ precedes $\mathcal{W}$, then $\mathcal{U}$ precedes $\mathcal{V}$ precedes $\mathcal{W}$. The interpretation of the theorem is that while merging several sets of orderings produces a larger set of orderings, concatenating produces fewer orderings from that same larger set, so if we take a number of sets of orderings that are combined by merging or concatenating or both, and intersect that result with a different combination of mergings or concatenatings of the same sets where the sets appear in the same order, then the intersection will be a combination of the same sets that tends to preserve the concatenations. (To express a general theorem of this sort is beyond the scope of this dissertation, but the process used in proving the last theorem can be applied in more general situations.)

Chapter 6

Disentangling through an intermediate space

6.1 Issues that arise in different approaches to disentangling

We would like to consider another approach to Feynman's Operational Calculi and to the disentangling map $\mathcal{T}_{\mu_1,\ldots,\mu_n} : \mathbb{D} \to \mathcal{L}(X)$ defined by Jefferies and Johnson ([13], [15], [16], [14]; see Chapter 2 above). In particular, we would like to consider how the process of evaluating the map might be simplified by a further use of commutativity.

Before we do so, let us first review the disentangling process as Feynman described it (but we will adjust his notation slightly) and then compare his process to the Jefferies-Johnson system. Consider noncommuting, time-independent operators A and B. In Feynman's paper introducing his operational calculus [10], as we have discussed earlier, he created a notation for describing the operator product AB by means of indices attached to the operators to show which operator operates first. For example, $A(1)B(0)$ in his notation means that B operates before A, because $0 < 1$. (The indices can be viewed as indicating the "time" of operation, with the lower time happening first.) This gives $A(1)B(0) = AB$. That product can also be written as $B(0)A(1) = AB$; we are allowed to exchange the order of $A(1)$ and $B(0)$ on the page because the indices tell us in which order they operate. Therefore, although $A(1)B(0)$ equals the product of two operators A and B that do not commute with each other, we can manipulate $A(1)$ and $B(0)$ as though they did in fact commute. We might say that the notation commutes, even though the operators do not.

Feynman was interested in certain formulas involving products of noncommuting operators, with the products potentially including the operator factors in all possible orders. He would begin with a product expressed in his time-indexed notation and then rearrange it until he could write it in the conventional right-to-left notation (the operator on the right acting first), and this he called 'disentangling.'

Let us again summarize Feynman's 'rules' as we did in Chapter 1:

(1) Express the order of operation of a product of noncommuting operators not by means of conventional right-to-left order of operation, but instead by attaching time indices to the operators (an earlier time means earlier operation).

(2) Form functions of the operators, with the indices attached, and then manipulate the operators as though they were commuting.

(3) Finally, 'disentangle' the resulting expressions; that is, restore the conventional ordering of the operators.

For example, Feynman would start with operators A and B and attach indices as just described, then express their product as

$$\int_0^1 A(s)\,ds \int_0^1 B(s)\,ds, \tag{6.1}$$

which equals

$$\int_0^1 \int_0^1 A(s)B(t)\,dt\,ds, \tag{6.2}$$

where $A(s)B(t) := AB$ if $t < s$ and $A(s)B(t) := BA$ if $s < t$. (The expression '$A(s)B(t)$' is undefined for $s = t$). The next step is to break the integration into the region where $t < s$ and the region where $s < t$ (ignoring regions where $s = t$ since those together have zero measure). That gives

$$\int_0^1 A(s)\,ds \int_0^1 B(s)\,ds = \int_0^1 \int_0^s A(s)B(t)\,dt\,ds + \int_0^1 \int_s^1 B(t)A(s)\,dt\,ds. \tag{6.3}$$

The terms on the right-hand side are time-ordered ($t < s$ in the first integral, and $s < t$ in the second), so now he returns to conventional notation, yielding

$$\begin{aligned}\int_0^1 \int_0^s AB\,dt\,ds + \int_0^1 \int_s^1 BA\,dt\,ds &= AB \int_0^1 \int_0^s dt\,ds + BA \int_0^1 \int_s^1 dt\,ds \\ &= \frac{1}{2}AB + \frac{1}{2}BA. \end{aligned} \tag{6.4}$$

We want to notice two things here. First of all, the process begins with a somewhat vague notion of a 'product of two operators A and B', not specifying whether the product is AB or BA or something else, from which it then it jumps to the expression $\int_0^1 A(s)\,ds \int_0^1 B(s)\,ds$. Secondly, as Feynman stated, the two factors of this expression are not to be evaluated independently. Even though $\int_0^1 A(s)\,ds = A$ and $\int_0^1 B(s)\,ds = B$, making those replacements in the expression $\int_0^1 A(s)\,ds \int_0^1 B(s)\,ds$ is not allowed; that would yield AB, which we see does not equal the final result, $\frac{1}{2}AB + \frac{1}{2}BA$.

This also holds for the simpler expression $A(s)B(t)$, where $A(s) = A$ and $B(t) = B$; substitution is not allowed unless it is known whether $s < t$ or $t < s$ and unless the factors have been put in their proper order. These two facts about Feynman's approach might be considered weaknesses from a mathematical standpoint—a vague definition and the failure of substitution.

A benefit of the approach of Jefferies and Johnson, which we have been using throughout this thesis, is that they do not run into those difficulties. The reason is that in their approach, they perform commutative operations in a space ($\mathbb{D}$) that is different from the noncommutative space of operators ($\mathcal{L}(X)$). The disentangled, noncommutative expression at the end of the process is not equal to the original, commutative expression; it is instead the image of a map from the commutative space into the noncommutative space. (Feynman was aware that he had not made his notation completely rigorous, but it is not clear from his article whether he viewed the particular issues mentioned here as being genuinely problematic, as far as the present author can tell.)

In the Jefferies-Johnson approach, recall, they map from the 'disentangling algebra' $\mathbb{D}$, which is a commutative space of complex functions of complex indeterminates, to the generally noncommutative space $\mathcal{L}(X)$ of bounded, linear operators on the Banach space X. Taking $A, B \in \mathcal{L}(X)$ to be noncommuting, time-independent operators, they associate to them complex indeterminates called $\tilde{A}, \tilde{B} \in \mathbb{D}$, respectively (these could just as well be called z_1 and z_2, but using the same letters A and B serves as a reminder of the association between the indeterminates and the operators). They use those indeterminates to express the function they want to disentangle. Continuing the example from the previous paragraphs, the function to be disentangled is the product $\tilde{A}\tilde{B} \in \mathbb{D}$. Since the indeterminates $\tilde{A}$ and $\tilde{B}$ commute, their product can be expressed without implying which operator operates first; this overcomes the first difficulty in Feynman's system of how to refer to a product of A and B in some sense without having to commit to one of the two orders AB or BA.

To perform the disentangling process for this example in the Jefferies and Johnson approach—still reflecting the three 'rules' that Feynman used but did not state—they would first attach time indices s_1 and s_2, using Lebesgue measure l on $[0,1]$, to the two indeterminates $\tilde{A}$ and $\tilde{B}$, giving $\tilde{A}(s_1) \equiv \tilde{A}$ and $\tilde{B}(s_2) \equiv \tilde{B}$ (and similarly for the operators, $A(s_1) \equiv A$ and $B(s_2) \equiv B \in \mathcal{L}(X)$). (Recall, in general the Jefferies-Johnson approach can use other measures besides Lebesgue measure, and the operators $A(s_1), B(s_2)$ are viewed as operating or not operating depending on whether s_1 and s_2 lie within the supports of their respective measures—so the time indices are said to be "attached using measures.") Following the second rule, to disentangle $\tilde{A}\tilde{B}$, it is first noted—because Lebesgue measure is a probability measure on $[0,1]$—that

$$\begin{aligned}
\tilde{A}\tilde{B} &= \int_0^1 \tilde{A}(s_1)\,ds_1 \int_0^1 \tilde{B}(s_2)\,ds_2 \\
&= \int_0^1 \int_0^{s_1} \tilde{A}(s_1)\tilde{B}(s_2)\,ds_2\,ds_1 + \int_0^1 \int_{s_1}^1 \tilde{A}(s_1)\tilde{B}(s_2)\,ds_2\,ds_1 \\
&= \int_{\{1>s_1>s_2>0\}} \tilde{A}(s_1)\tilde{B}(s_2)\,(l\times l)(ds_1, ds_2) + \int_{\{1>s_2>s_1>0\}} \tilde{B}(s_2)\tilde{A}(s_1)\,(l\times l)(ds_1, ds_2), \qquad (6.5)
\end{aligned}$$

in which commutativity has been used to help separate the terms where $s_1 > s_2$ from those where $s_1 < s_2$. Note here that the second issue of Feynman's notation, the failure of direct substitution, is resolved; $\tilde{A}$ may be freely substituted for $\int_0^1 \tilde{A}(s_1)\,ds_1$ and $\tilde{B}$ for $\int_0^1 \tilde{B}(s_2)\,ds_2$ in the product $\int_0^1 \tilde{A}(s_1)\,ds_1 \int_0^1 \tilde{B}(s_2)\,ds_2 = \tilde{A}\tilde{B}$, because the work is done in a commutative space, namely $\mathbb{D}$.

Based on the form of the last expression in Equation (6.5), the Jefferies-Johnson definition of the disentangling of the product $\tilde{A}\tilde{B}$ under the disentangling map $\mathcal{T}_{l,l}$ is a reasonable one. They define it to be

$$\begin{aligned}
\mathcal{T}_{l,l}[\tilde{A}\tilde{B}] &= \mathcal{T}_{l,l}\left[\int_{\{1>s_1>s_2>0\}} \tilde{A}(s_1)\tilde{B}(s_2)\,(l\times l)(ds_1, ds_2) + \int_{\{1>s_2>s_1>0\}} \tilde{B}(s_2)\tilde{A}(s_1)\,(l\times l)(ds_1, ds_2)\right] \\
&:= \int_{\{1>s_1>s_2>0\}} A(s_1)B(s_2)\,(l\times l)(ds_1, ds_2) + \int_{\{1>s_2>s_1>0\}} B(s_2)A(s_1)\,(l\times l)(ds_1, ds_2) \\
&= AB\int_{\{1>s_1>s_2>0\}} (l\times l)(ds_1, ds_2) + BA\int_{\{1>s_2>s_1>0\}} (l\times l)(ds_1, ds_2) \\
&= \frac{1}{2}AB + \frac{1}{2}BA, \qquad (6.6)
\end{aligned}$$

where the disentangling map seems to have the effect of simply removing the tildes. This map thereby reflects the third rule, returning a time-ordered expression to conventional operator notation. Be careful to note that in these expressions in the Jefferies-Johnson approach, a product of time-indexed operators does not have the same meaning as in the Feynman system, so here '$A(s_1)B(s_2)$' means only AB—never BA—whereas for Feynman '$A(s_1)B(s_2)$' means AB if $s_1 > s_2$ or BA if $s_1 < s_2$.

As shown, then, the approach of Jefferies and Johnson improves on the two perceived weaknesses of the Feynman notation by defining the commutative space $\mathbb{D}$, which the disentangling map $\mathcal{T}$ maps into the noncommutative space $\mathcal{L}(X)$. Now we would like to take an additional step, motivated by a somewhat different interpretation of Feynman's 'rules' than the one used in the Jefferies-Johnson approach. The effect will be that while we continue to use the map defined by Jefferies and Johnson, we will consider another process by which the image of the map may be calculated, a process in which commutativity may be further exploited.

To motivate the new process, we return to a claim made in Chapter 2 above (shortly after Theorem 2.0.4), namely that though the definition of the disentangling map takes a sum of terms involving indeterminates to a sum of terms in the same form but involving the corresponding operators,

$$\mathcal{T}_{\mu_1,\ldots,\mu_n}\left[\sum_{\pi\in S_m}\int_{\Delta_m(\pi)}\tilde{C}_{\pi(m)}(s_{\pi(m)})\cdots\tilde{C}_{\pi(1)}(s_{\pi(1)})\,(\mu_1^{m_1}\times\cdots\times\mu_n^{m_n})(ds_1,\ldots,ds_m)\right]$$
$$=\sum_{\pi\in S_m}\int_{\Delta_m(\pi)}C_{\pi(m)}\cdots C_{\pi(1)}\,(\mu_1^{m_1}\times\cdots\times\mu_n^{m_n})(ds_1,\ldots,ds_m) \quad (6.7)$$

in the case of probability measures $\mu_1,\ldots,\mu_n\in\mathcal{M}_{cb}[0,1]$ (shown earlier as Equation (2.17)), the corresponding terms of this sum do not necessarily map to each other. We will do an example to show this.

Let noncommuting operators $A,B\in\mathcal{L}(X)$ each be associated with Lebesgue measure l on $[0,1]$, and let $\tilde{A}(s)\equiv\tilde{A}$ and $\tilde{B}(s)\equiv\tilde{B}$ on $[0,1]$. We then have

$$\mathcal{T}_{l,l}\left[\tilde{A}\tilde{B}\right]=\mathcal{T}_{l,l}\left[\int_{\{t>s\}}\tilde{B}(t)\tilde{A}(s)\,(l\times l)(ds,dt)+\int_{\{s>t\}}\tilde{A}(s)\tilde{B}(t)\,(l\times l)(ds,dt)\right]$$
$$=\int_{\{t>s\}}BA\,(l\times l)(ds,dt)+\int_{\{s>t\}}AB\,(l\times l)(ds,dt). \quad (6.8)$$

However, we claim,

$$\mathcal{T}_{l,l}\left[\int_{\{s>t\}}\tilde{A}(s)\tilde{B}(t)\ (l\times l)(ds,dt)\right]\neq\int_{\{s>t\}}AB\ (l\times l)(ds,dt), \quad (6.9)$$

and

$$\mathcal{T}_{l,l}\left[\int_{\{t>s\}}\tilde{B}(t)\tilde{A}(s)\ (l\times l)(ds,dt)\right]\neq\int_{\{t>s\}}BA\ (l\times l)(ds,dt). \quad (6.10)$$

We will show this for inequality (6.9) by calculating both sides. On the left-hand side we have

$$\mathcal{T}_{l,\,l}\left[\int_{\{1>s>t>0\}}\tilde{A}(s)\tilde{B}(t)\,(l\times l)(ds,dt)\right]=\mathcal{T}_{l,\,l}\left[\tilde{A}\tilde{B}\int_{\{1>s>t>0\}}(l\times l)(ds,dt)\right]$$

$$
\begin{aligned}
&= \mathcal{T}_{l,l}\left[\frac{1}{2}\tilde{A}\tilde{B}\right] \\
&= \frac{1}{2}\mathcal{T}_{l,l}\left[\tilde{A}\tilde{B}\right] \\
&= \frac{1}{2}\left(\frac{1}{2}AB + \frac{1}{2}BA\right) \\
&= \frac{1}{4}AB + \frac{1}{4}BA. \qquad (6.11)
\end{aligned}
$$

However, on the right-hand side we have

$$\int_{\{1>s>t>0\}} A(s)B(t)\,(l \times l)(ds, dt) = AB \int_{\{1>s>t>0\}} (l \times l)(ds, dt) = \frac{1}{2}AB, \qquad (6.12)$$

and these do not agree since A and B do not commute. Therefore, the individual summands on the left side of Equation (6.7) to which the disentangling map is applied do not necessarily map to the corresponding summands (without the tildes) on the right side of the equation, even though the entire sum does map to the entire sum.

This suggests that we need to be careful to recognize that in applying the third of Feynman's 'rules', moving from a commutative expression to an expression of the same form in conventional operator notation, the Jefferies-Johnson approach has defined this in the case of a monomial $P^{m_1,\ldots,m_n}(\tilde{A}_1, \ldots, \tilde{A}_n)$ only if that monomial is expressed in the particular time-ordered form

$$\sum_{\pi \in S_m} \int_{\Delta_m(\pi)} \tilde{C}_{\pi(m)}(s_{\pi(m)}) \cdots \tilde{C}_{\pi(1)}(s_{\pi(1)})\,(\mu_1^{m_1} \times \cdots \times \mu_n^{m_n})(ds_1, \ldots, ds_m); \qquad (6.13)$$

only then can the tildes be freely erased. If monomials are written in a different form, even if the form has the appearance of being time-ordered, one might not be at liberty to erase the tildes. For example, suppose we have (as we had a moment ago) noncommuting operators $A, B \in \mathcal{L}(X)$, each associated with Lebesgue measure l on

$[0,1]$, and suppose $\tilde{A}(s) \equiv \tilde{A}$ and $\tilde{B}(s) \equiv \tilde{B}$ on $[0,1]$. Then $\tilde{A}\tilde{B} \in \mathbb{D}$, and it is valid in the space $\mathbb{D}$ to write

$$\tilde{A}\tilde{B} = 2\int_{\{s>t\}} \tilde{A}(s)\tilde{B}(t)(l \times l)(ds, dt), \tag{6.14}$$

but this does *not* map under $\mathcal{T}_{l,l}$ to an expression in the same form as the right-hand side without the tildes, which is

$$2\int_{\{s>t\}} A(s)B(t)(l \times l)(ds, dt) = 2\int_{\{s>t\}} (l \times l)(ds, dt)A(s)B(t) = 2\left(\frac{1}{2}\right) AB = AB; \tag{6.15}$$

instead it maps to $\mathcal{T}_{l,l}[\tilde{A}\tilde{B}] = \frac{1}{2}AB + \frac{1}{2}BA$.

We might find it beneficial, if we are able, to develop a space (modeled partly after $\mathbb{D}$) in which not only are indeterminates (associated with operators) commutative, but also in which converting indeterminates to operators ('erasing the tildes') from *any* valid form of an element of the space will yield the element's image under the disentangling map. For now, let us call that space $\mathbb{E}$. If we had such a space, this might allow us to apply a somewhat different interpretation of Feynman's 'rules': Beginning with a function f of indeterminates in the space $\mathbb{D}$, we would first attach time indices (as before) and then form an expression g in the space $\mathbb{E}$. Second, we would freely manipulate g according to the rules of the space $\mathbb{E}$, including commutativity, until g is in whichever time-ordered form we desire. Third (if we have properly defined the space $\mathbb{E}$), we would then be able to convert the indeterminates in that form of g to operators ('erase the tildes') and yield the element of $\mathcal{L}(X)$ that is the image of f under the disentangling map.

Our main objective of this chapter is therefore to define a space $\mathbb{E}$ as described, which we will call the 'intermediate disentangling space' for the disentangling map.

When we have done so, we will be able to obtain various results from existing FOCi work, but some can be obtained more easily. In some cases we can do so much more easily, for example, when obtaining decomposing disentangling formulas (see Examples 28 and 36), especially when the number of measures is three or more.

6.2 An intermediate set for the disentangling map

Let us suppose we have a monomial in $\mathbb{D}$. As we have said, we would like to manipulate its form, and then use that form to find the image of the disentangling map in $\mathcal{L}(X)$. As we have also said, Feynman did something like that process; he began with the notion of a product of operators A and B, then he manipulated the form of that product using an unconventional operator notation (which we can summarize by saying '$A(s)B(t)$' represents the function of (s,t) given by $\chi_{\{s>t\}}(s,t)AB + \chi_{\{t>s\}}(s,t)BA$), and once he reached a form in which the operators were in proper time order, he returned to conventional operator notation. In a sense, then, he moved from a product to a space of forms to a space of operators. We will do similarly, by mapping a monomial in $\mathbb{D}$ into a space of expressions we will call $\mathbb{E}$ (so called because, for one thing, it follows $\mathbb{D}$) that includes various forms in which the disentangled monomial may be expressed, and then map from there to the disentangled operator $\mathcal{L}(X)$ that has a corresponding form.

In fact, Feynman's Operational Calculi in the Jefferies-Johnson approach already has something like that intermediate space of forms, namely, the set of expressions of the form '$P^{m_1,\ldots,m_n}_{\mu_1,\ldots,\mu_n}(A_1,\ldots,A_n)$', which will serve as our starting point.

The reasons the set of expressions of the form $P^{m_1,\ldots,m_n}_{\mu_1,\ldots,\mu_n}(A_1,\ldots,A_n)$ can be said to be like an intermediate space of forms are, firstly, in the disentangling process it does appear intermediate between a monomial in $\mathbb{D}$ and its disentangled image in $\mathcal{L}(X)$. Specifically, given operators $A_1, \ldots, A_n \in \mathcal{L}(X)$ associated with measures $\mu_1, \ldots, \mu_n \in \mathcal{M}_{cb}[0,1]$, respectively, along with nonnegative integers $m_1, \ldots, m_n$, we have corresponding indeterminates $\tilde{A}_1, \ldots, \tilde{A}_n \in \mathbb{D}$, and we make the usual assignments of the operator names $C_1, \ldots, C_m$, where $m = m_1 + \cdots + m_n$. Since $\mathbb{D}$ is an algebra, $\tilde{A}_1^{m_1} \cdots \tilde{A}_n^{m_n} \in \mathbb{D}$, and

$$\begin{aligned}\mathcal{T}_{\mu_1,\ldots,\mu_n}[\tilde{A}_1^{m_1} \cdots \tilde{A}_n^{m_n}] &= P^{m_1,\ldots,m_n}_{\mu_1,\ldots,\mu_n}(A_1, \ldots, A_n) \\ &= \sum_{\pi \in S_m} \int_{\Delta_m(\pi)} C_{\pi(m)} \cdots C_{\pi(1)} (\mu_1^{m_1} \times \cdots \times \mu_n^{m_n})(ds_1, \ldots, ds_m) \in \mathcal{L}(X). \end{aligned} \tag{6.16}$$

Secondly, the form of $P^{m_1,\ldots,m_n}_{\mu_1,\ldots,\mu_n}(A_1, \ldots, A_n)$ can be changed in certain ways without changing its value as an element of $\mathcal{L}(X)$. For example, by Corollary 3.1.9 we have that for any permutation $\sigma \in S_n$,

$$P^{m_1,\ldots,m_n}_{\mu_1,\ldots,\mu_n}(A_1, \ldots, A_n) = P^{m_{\sigma(1)},\ldots,m_{\sigma(n)}}_{\mu_{\sigma(1)},\ldots,\mu_{\sigma(n)}}(A_{\sigma(1)}, \ldots, A_{\sigma(n)}). \tag{6.17}$$

That is essentially a kind of commutativity, where we commute the operators, measures, and exponents in a consistent way dictated by the permutation σ. Third, although the expression $P^{m_1,\ldots,m_n}_{\mu_1,\ldots,\mu_n}(A_1, \ldots, A_n)$ represents an element of $\mathcal{L}(X)$, not elements of another space, in a sense it can be considered to be a map P into $\mathcal{L}(X)$ from a space of $3n$-tuples of operators, measures, and exponents, each of which we could write as $(A_1, \ldots, A_n; \mu_1, \ldots, \mu_n; m_1, \ldots, m_n)$. We will imitate this type of map in our definition of an intermediate space for the disentangling map.

Before we define the 'intermediate disentangling space' that we will call $\mathbb{E}$, we will first define a set $\mathcal{G}'$ which contains (as a proper subset) the set of generators for a vector space $\mathcal{V}$, of which the space $\mathbb{E}$ is a quotient space. The set $\mathcal{G}'$ together with a map $\mathcal{T}_{\mathcal{G}'}$ can also be made to play an intermediate role in the disentangling process, as we will see. But to define $\mathcal{G}'$, we first need to define a certain notation for the arguments of a function.

In undergraduate mathematics when dealing with functions, instructors are sometimes careful to emphasize for their students the distinction between a function f and the value $f(x, y)$ of the function at a point (x, y). At times 'dot' notation $f(\cdot, \cdot)$ is used to emphasize that f is a function of two arguments, without having to name the arguments. There are, however, times when it would be helpful to be able to both refer to a function as a function (not a function value) and name its arguments at the same time. For this purpose we will, in a manner of speaking, "name the dots" by putting variable names above the dots, or rather, dots under variable names. For example we will refer to the function $f(\cdot, \cdot)$ as $f(\underset{\cdot}{x}, \underset{\cdot}{y})$ to indicate that in the immediate context the names of the arguments of the function f will be 'x' and 'y'. This allows us, for example, to refer to the *function* $g(\underset{\cdot}{x}) = 3\underset{\cdot}{x} + 2$ or the *function* $(\underset{\cdot}{x} - 1)^2$, in contrast to the *values* $g(x) = 3x + 2$ and $(x - 1)^2$. (Incidentally, one place this could make useful distinctions is with partial derivatives; given a function $F(\underset{\cdot}{x}, \underset{\cdot}{y}, \underset{\cdot}{z})$ with $\underset{\cdot}{z} = \underset{\cdot}{x} + \underset{\cdot}{y}$, we could distinguish between two partial derivatives with respect to x, namely the derivative of the function $F(\underset{\cdot}{x}, y, x + y)$ and the derivative of the function $F(\underset{\cdot}{x}, y, \underset{\cdot}{x} + y)$. But we will not do any of that in this thesis.) Given a function f on a set $X \times Y$, the statement '$f(\underset{\cdot}{x}, \underset{\cdot}{y}) = g(\underset{\cdot}{y}, \underset{\cdot}{x})$' implies that g is a function on $Y \times X$ and that $f(x, y) = g(y, x)$ for all values of $x \in X$ and $y \in Y$.

We now define the set $\mathcal{G}'$. ('$\mathcal{G}'$' is for 'generator', though technically a subset $\mathcal{G}$ will be used to generate $\mathcal{V}$, of which $\mathbb{E}$ is a quotient space. It must be admitted that it is therefore not entirely necessary to define all of $\mathcal{G}'$, but we do so because it is a good context in which to define certain notations and to become familiar with them before going on.) Its elements will involve characteristic functions, and often it may be helpful to think of those as playing a role something like integral symbols.

Definition 6.2.1 (The set $\mathcal{G}'$)**.** Let $\mathcal{G}'$ be the set of all functions on $[0,1]^m$ (for all $m \geq 0$) of the form

$$\chi_E(\underset{\cdot}{s}_1, \ldots, \underset{\cdot}{s}_m)\tilde{C}_1(s_1)\cdots\tilde{C}_m(s_m)d\nu_1(s_1)\cdots d\nu_m(s_m) \tag{6.18}$$

(which we will refer to as a 'monomial'), where all of the following hold:

1. $E \subseteq [0,1]^m$ is a Borel set.

   a) If $E' \subseteq [0,1]^m$ is a Borel set differing from E by a set of $\nu_1\times\cdots\times\nu_m$-measure zero, then $\chi_{E'}(\underset{\cdot}{s}_1, \ldots, \underset{\cdot}{s}_m)\tilde{C}_1(s_1)\cdots\tilde{C}_m(s_m)d\nu_1(s_1)\cdots d\nu_m(s_m)$ is considered to be the same as $\chi_E(\underset{\cdot}{s}_1, \ldots, \underset{\cdot}{s}_m)\tilde{C}_1(s_1)\cdots\tilde{C}_m(s_m)d\nu_1(s_1)\cdots d\nu_m(s_m)$ as elements of $\mathcal{G}'$.

   b) If $\sigma \in S_m$ is any permutation, then

   $$\begin{aligned}\chi_{E^\sigma}&(\underset{\cdot}{s}_1, \ldots, \underset{\cdot}{s}_m)^\sigma\tilde{C}_1(s_1)\cdots\tilde{C}_m(s_m)d\nu_1(s_1)\cdots d\nu_m(s_m)\\ &= \chi_{E^\sigma}(\underset{\cdot}{s}_{\sigma(1)}, \ldots, \underset{\cdot}{s}_{\sigma(m)})\tilde{C}_1(s_1)\cdots\tilde{C}_m(s_m)d\nu_1(s_1)\cdots d\nu_m(s_m)\end{aligned} \tag{6.19}$$

   is considered to be the same element of $\mathcal{G}'$ as $\chi_E(\underset{\cdot}{s}_1, \ldots, \underset{\cdot}{s}_m)\tilde{C}_1(s_1)\cdots$ $\tilde{C}_m(s_m)d\nu_1(s_1)\cdots d\nu_m(s_m)$.

2. $\tilde{C}_1, \ldots \tilde{C}_m, d\nu_1, \ldots, d\nu_m$ are $2m$ (not necessarily distinct) commuting complex indeterminates that are associated, respectively, with nonzero operators $C_1, \ldots,$ $C_m \in \mathcal{L}(X)$ and measures $\nu_1, \ldots, \nu_m \in \mathcal{M}_{cb}[0,1]$. (Recall, $\mathcal{M}_{cb}[0,1]$ is the set of finite, continuous Borel measures on $[0,1]$.)

3. The use of the index 's_j' in the expression $\tilde{C}_j(s_j)$ is merely a label to indicate that the indeterminate $\tilde{C}_j$ is associated with the (dotted) argument $\underset{\cdot}{s}_j$ of the function χ_E. It is not a value. (In general, we can think of $\tilde{C}(s)$ as the ordered pair $(\tilde{C}, \text{'}s\text{'})$.) Similarly, the '$s_j$' in the expression $d\nu_j(s_j)$ is a label to indicate that the indeterminate $d\nu_j$ is associated with the argument $\underset{\cdot}{s}_j$ of the function χ_E.

4. When $m = 0$, we will regard the expression $\chi_E(\underset{\cdot}{s}_1, \ldots, \underset{\cdot}{s}_m)\tilde{C}_1(s_1)\cdots\tilde{C}_m(s_m)$ $d\nu_1(s_1)\cdots d\nu_m(s_m) \in \mathcal{G}'$ as being equal to 1. (One might attempt to picture the constant function 1 as being like a degenerate characteristic function on a single point we could call $[0,1]^0$ and having no arguments, but we will just say the expression equals 1.)

It may very well be that it would be clear enough most of the time to write elements of the set $\mathcal{G}'$ without dots under the arguments of the characteristic functions, with the reader understanding that a function is intended and not just a function value; however, for the sake of clarity in this thesis we will keep the dots throughout.

Remark 21. At times we may choose to give the indeterminates more conventional complex variable names such as z_j and w_j, and then we would have expressions such as

$$\chi_E(\underset{\cdot}{s}_1, \ldots, \underset{\cdot}{s}_m)z_1(s_1)\cdots z_m(s_m)w_1(s_1)\cdots w_m(s_m) \in \mathcal{G}'. \tag{6.20}$$

Remark 22. In what follows, if we make a declaration of the form

$$\chi_E(\underset{\cdot}{s}_1, \ldots, \underset{\cdot}{s}_m)\tilde{C}_1(s_1)\cdots\tilde{C}_m(s_m)d\nu_1(s_1)\cdots d\nu_m(s_m) \in \mathcal{G}', \tag{6.21}$$

it will be understood that the various elements are what they should be as stipulated in the definition of $\mathcal{G}'$: $m \geq 0$; $E \subseteq [0,1]^m$ is a Borel set; the operators are represented by capital letters $C_1, \ldots, C_m \in \mathcal{L}(X)$; the measures are represented by lowercase Greek letters $\nu_1, \ldots, \nu_m \in \mathcal{M}_{cb}[0,1]$; etc.

Remark 23. Since $\mathcal{G}'$ is only a set, it does not include operations. For example, we cannot add two elements in the set $\mathcal{G}'$. In particular, even if we have disjoint Borel sets $E_1, E_2 \subseteq [0,1]$ with $\chi_{E_1 \cup E_2}(\underset{\cdot}{s})\tilde{A}(s)d\mu(s) \in \mathcal{G}'$, which we can express as $[\chi_{E_1} + \chi_{E_2}](\underset{\cdot}{s})\tilde{A}(s)d\mu(s)$ or even as $[\chi_{E_1}(\underset{\cdot}{s}) + \chi_{E_2}(\underset{\cdot}{s})]\tilde{A}(s)d\mu(s)$, we will avoid writing $\chi_{E_1 \cup E_2}(\underset{\cdot}{s})\tilde{A}(s)d\mu(s)$ as $\chi_{E_1}(\underset{\cdot}{s})\tilde{A}(s)d\mu(s) + \chi_{E_2}(\underset{\cdot}{s})\tilde{A}(s)d\mu(s)$ as an element of $\mathcal{G}'$ (even though that would be a sum of products of real-valued functions and complex indeterminates, which has a natural pointwise definition). However, later when we form the space $\mathbb{E}$ from elements of $\mathcal{G}'$, we will introduce an addition operation.

In what follows we will generally consider only nonzero operators.

Example 23. The sets $(0,1)^2$ and $\{(s,t) : 0 < s < t < 1\}$ are Borel sets in $[0,1]^2$, so for operators $A, B \in \mathcal{L}(X)$ and measures μ, ν we have

$$\chi_{(0,1)^2}(\underset{\cdot}{s}, \underset{\cdot}{t})\tilde{A}(s)\tilde{B}(t)d\mu(s)d\nu(t) \in \mathcal{G}', \tag{6.22}$$

and

$$\begin{aligned}
&\chi_{\{(s,t):\, 0<s<t<1\}}(\underset{\cdot}{s}, \underset{\cdot}{t})\tilde{A}(s)\tilde{B}(t)d\mu(s)d\nu(t) \\
&=: \chi_{\{s<t\}}(\underset{\cdot}{s}, \underset{\cdot}{t})\tilde{A}(s)\tilde{B}(t)d\mu(s)d\nu(t) \\
&= \chi_{\{(t,s):\, 0<s<t<1\}}(\underset{\cdot}{t}, \underset{\cdot}{s})\tilde{A}(s)\tilde{B}(t)d\mu(s)d\nu(t) \\
&=: \chi_{\{s<t\}}(\underset{\cdot}{t}, \underset{\cdot}{s})\tilde{A}(s)\tilde{B}(t)d\mu(s)d\nu(t) \in \mathcal{G}'.
\end{aligned} \tag{6.23}$$

(By commutativity, we could just as well write the last expression as $d\nu(t)d\mu(s)\tilde{B}(t)$

$\tilde{A}(s)\chi_{\{s<t\}}(\underset{\cdot}{t}, \underset{\cdot}{s})$, but usually we will not.) In fact, we can write

$$\begin{aligned} &\chi_{(0,1)^2}(\underset{\cdot}{s}, \underset{\cdot}{t})\tilde{A}(s)\tilde{B}(t)d\mu(s)d\nu(t) \\ &= [\chi_{\{s<t\}} + \chi_{\{s>t\}}](\underset{\cdot}{s}, \underset{\cdot}{t})\tilde{A}(s)\tilde{B}(t)d\mu(s)d\nu(t) \in \mathcal{G}'. \end{aligned} \tag{6.24}$$

Before we state two more notational conventions in $\mathcal{G}'$, we will define a map $\mathcal{T}_{\mathcal{G}'}$ from $\mathcal{G}'$ into $\mathcal{L}(X)$ that will play a role similar to that played by the disentangling map $\mathcal{T}_{\mu_1,\ldots,\mu_n}$ on $\mathbb{D}$.

Definition 6.2.2 (The intermediate disentangling map $\mathcal{T}_{\mathcal{G}'}$)**.** Given the definitions above, define $\mathcal{T}_{\mathcal{G}'} : \mathcal{G}' \to \mathcal{L}(X)$ by $\mathcal{T}_{\mathcal{G}'}[1] = I$, the identity operator in $\mathcal{L}(X)$, and

$$\begin{aligned} &\mathcal{T}_{\mathcal{G}'}[\chi_E(\underset{\cdot}{s}_1, \ldots, \underset{\cdot}{s}_m)\tilde{C}_1(s_1)\cdots\tilde{C}_m(s_m)d\nu_1(s_1)\cdots d\nu_m(s_m)] \\ &\quad := \sum_{\pi \in S_m} \int_{\Delta_m(\pi)} \chi_E(s_1, \ldots, s_m)C_{\pi(m)}\cdots C_{\pi(1)}(\nu_1 \times \cdots \times \nu_m)(ds_1, \ldots, ds_m) \end{aligned} \tag{6.25}$$

for arbitrary $\chi_E(\underset{\cdot}{s}_1, \ldots, \underset{\cdot}{s}_m)\tilde{C}_1(s_1)\cdots\tilde{C}_m(s_m)d\nu_1(s_1)\cdots d\nu_m(s_m) \in \mathcal{G}'$, $m \geq 1$.

Remark 24. We need to show that the map $\mathcal{T}_{\mathcal{G}'}$ is well-defined with regard to the commuting of indeterminates and with regard to Definition 6.2.1 parts 1a and 1b.

First, the fact that the indeterminates of $\chi_E(\underset{\cdot}{s}_1, \ldots, \underset{\cdot}{s}_m)\tilde{C}_1(s_1)\cdots\tilde{C}_m(s_m)d\nu_1(s_1)$ $\cdots d\nu_m(s_m)$ commute is not really an issue, since once we have identified the coefficient function of the monomial as $\chi_E(\underset{\cdot}{s}_1, \ldots, \underset{\cdot}{s}_m)$, the association between the arguments of χ_E and the indeterminates and the choice of permutation $\pi \in S_m$ will dictate uniquely the order that the operators and measures will appear in the right-hand expression in Equation (6.25).

Second, it is clear that if the set E is changed by a set of $\nu_1 \times \cdots \times \nu_m$-measure zero, the value of the integral defining $\mathcal{T}_{\mathcal{G}'}$ will remain unchanged.

More of an issue is the fact that there are several ways that the coefficient function can be written without changing the element of $\mathcal{G}'$ to which we are referring. Suppose, for example, we choose any $\sigma \in S_m$ and rewrite

$$\chi_E(\underset{\cdot}{s}_1, \ldots, \underset{\cdot}{s}_m)\tilde{C}_1(s_1)\cdots\tilde{C}_m(s_m)d\nu_1(s_1)\cdots d\nu_m(s_m) \tag{6.26}$$

as

$$\chi_{E^\sigma}(\underset{\cdot}{s}_{\sigma(1)}, \ldots, \underset{\cdot}{s}_{\sigma(m)})\tilde{C}_1(s_1)\cdots\tilde{C}_m(s_m)d\nu_1(s_1)\cdots d\nu_m(s_m). \tag{6.27}$$

By commutativity of the indeterminates in $\mathcal{G}'$ we may also write this as

$$\chi_{E^\sigma}(\underset{\cdot}{s}_{\sigma(1)}, \ldots, \underset{\cdot}{s}_{\sigma(m)})\tilde{C}_{\sigma(1)}(s_{\sigma(1)})\cdots\tilde{C}_{\sigma(m)}(s_{\sigma(m)})d\nu_{\sigma(1)}(s_{\sigma(1)})\cdots d\nu_{\sigma(m)}(s_{\sigma(m)}). \tag{6.28}$$

Then by definition of $\mathcal{T}_{\mathcal{G}'}$ (making use of Theorem 3.1.7 and Corollary 3.1.5, and an argument similar to one used to prove Corollary 3.1.9) we have

$$\begin{aligned}
&\mathcal{T}_{\mathcal{G}'}[\chi_{E^\sigma}(\underset{\cdot}{s}_{\sigma(1)}, \ldots, \underset{\cdot}{s}_{\sigma(m)})\tilde{C}_1(s_1)\cdots\tilde{C}_m(s_m)d\nu_1(s_1)\cdots d\nu_m(s_m)] \\
&= \mathcal{T}_{\mathcal{G}'}[\chi_{E^\sigma}(\underset{\cdot}{s}_{\sigma(1)}, \ldots, \underset{\cdot}{s}_{\sigma(m)})\tilde{C}_{\sigma(1)}(s_{\sigma(1)})\cdots\tilde{C}_{\sigma(m)}(s_{\sigma(m)})d\nu_{\sigma(1)}(s_{\sigma(1)})\cdots d\nu_{\sigma(m)}(s_{\sigma(m)})] \\
&= \sum_{\pi\in S_m}\int_{\Delta_m(\pi)}\chi_{E^\sigma}(s_{\sigma(1)}, \ldots, s_{\sigma(m)})C_{\sigma\pi(m)}\cdots C_{\sigma\pi(1)}(\nu_{\sigma(1)}\times\cdots\times\nu_{\sigma(m)})(ds_{\sigma(1)}, \ldots, ds_{\sigma(m)}) \\
&= \sum_{\pi\in S_m}\int_{[\Delta_m(\pi)]^{\sigma^{-1}}}\chi_E(s_1, \ldots, s_m)C_{\sigma\pi(m)}\cdots C_{\sigma\pi(1)}(\nu_{\sigma(1)}\times\cdots\times\nu_{\sigma(m)})^{\sigma^{-1}}(ds_{\sigma(1)}, \ldots, ds_{\sigma(m)})^{\sigma^{-1}} \\
&= \sum_{\pi\in S_m}\int_{\Delta_m(\sigma\pi)}\chi_E(s_1, \ldots, s_m)C_{\sigma\pi(m)}\cdots C_{\sigma\pi(1)}(\nu_1\times\cdots\times\nu_m)(ds_1, \ldots, ds_m).
\end{aligned} \tag{6.29}$$

Changing the index of summation to $\sigma\pi \in S_m$ and letting $\rho := \sigma\pi$ gives

$$\mathcal{T}_{\mathcal{G}'}[\chi_{E^\sigma}(\underset{\cdot}{s}_{\sigma(1)}, \ldots, \underset{\cdot}{s}_{\sigma(m)})\tilde{C}_1(s_1)\cdots\tilde{C}_m(s_m)d\nu_1(s_1)\cdots d\nu_m(s_m)]$$

$$= \sum_{\rho \in S_m} \int_{\Delta_m(\rho)} \chi_E(s_1, \ldots, s_m) C_{\rho(m)} \cdots C_{\rho(1)} (\nu_1 \times \cdots \times \nu_m)(ds_1, \ldots, ds_m)$$

$$= \mathcal{T}_{\mathcal{G}'}[\chi_E(\underset{\cdot}{s}_1, \ldots, \underset{\cdot}{s}_m) \tilde{C}_1(s_1) \cdots \tilde{C}_m(s_m) d\nu_1(s_1) \cdots d\nu_m(s_m)]. \tag{6.30}$$

Consequently, changing the coefficient function using σ as shown has no effect on the image of $\mathcal{T}_{\mathcal{G}'}$. Thus $\mathcal{T}_{\mathcal{G}'}$ is well-defined.

Example 24. We will apply $\mathcal{T}_{\mathcal{G}'}$ to the elements of $\mathcal{G}'$ presented in the previous example. For the first we have (letting $A(s) \equiv A, B(t) \equiv B$)

$$\begin{aligned}
&\mathcal{T}_{\mathcal{G}'}[\chi_{(0,1)^2}(\underset{\cdot}{s}, \underset{\cdot}{t}) \tilde{A}(s) \tilde{B}(t) d\mu(s) d\nu(t)] \\
&= \int_{\{t>s\}} \chi_{(0,1)^2}(s,t) B(t) A(s) \, (\mu \times \nu)(ds, dt) + \int_{\{s>t\}} \chi_{(0,1)^2}(s,t) A(s) B(t) \, (\mu \times \nu)(ds, dt) \\
&= \int_{\{t>s\}} B(t) A(s) \, (\mu \times \nu)(ds, dt) + \int_{\{s>t\}} A(s) B(t) \, (\mu \times \nu)(ds, dt) \\
&= P^{1,1}_{\mu,\nu}(A, B) \\
&= (\mu \times \nu)(\{(s,t) : 0 < s < t < 1\}) BA + (\mu \times \nu)(\{(s,t) : 0 < t < s < 1\}) AB.
\end{aligned} \tag{6.31}$$

For the second we have

$$\begin{aligned}
&\mathcal{T}_{\mathcal{G}'}[\chi_{\{0<s<t<1\}}(\underset{\cdot}{s}, \underset{\cdot}{t}) \tilde{A}(s) \tilde{B}(t) d\mu(s) d\nu(t)] \\
&= \int_{\{t>s\}} \chi_{\{s<t\}}(s,t) B(t) A(s) \, (\mu \times \nu)(ds, dt) + \int_{\{s>t\}} \chi_{\{s<t\}}(s,t) A(s) B(t) \, (\mu \times \nu)(ds, dt) \\
&= \int_{\{t>s\}} B(t) A(s) \, (\mu \times \nu)(ds, dt) \qquad + \qquad 0 \\
&= (\mu \times \nu)(\{(s,t) : 0 < s < t < 1\}) BA.
\end{aligned} \tag{6.32}$$

In particular, if μ, ν are both Lebesgue measure on $[0,1]$, then the former result equals $\frac{1}{2}AB + \frac{1}{2}BA$, and the latter result equals $\frac{1}{2}BA$.

In fact, the above example, especially the first of the two parts, is the starting point for much of what follows, which is to relate the map $\mathcal{T}_{\mathcal{G}'}$ on $\mathcal{G}'$ to the disentangling map $\mathcal{T}_{\mu_1,\ldots,\mu_n}$ on $\mathbb{D}$ by way of the monomial disentangling $P^{m_1,\ldots,m_n}_{\mu_1,\ldots,\mu_n}(A_1,\ldots,A_n)$. We begin with a theorem that relates the two directly.

Theorem 6.2.3 (Disentangling a monomial of first-power factors)**.** *Given operators $C_1,\ldots,C_m \in \mathcal{L}(X)$ and measures $\nu_1,\ldots,\nu_m \in \mathcal{M}_{cb}[0,1]$, we have*

$$\mathcal{T}_{\mathcal{G}'}[\chi_{(0,1)^m}(\underset{\cdot}{s}_1,\ldots,\underset{\cdot}{s}_m)\tilde{C}_1(s_1)\cdots\tilde{C}_m(s_m)d\nu_1(s_1)\cdots d\nu_m(s_m)] = P^{1,\ldots,1}_{\nu_1,\ldots,\nu_m}(C_1,\ldots,C_m). \tag{6.33}$$

Proof. The result follows immediately from the definitions of both expressions. (For the degenerate case $m=0$, both sides equal the identity operator $I \in \mathcal{L}(X)$.) □

In fact, we can state a more general result.

Theorem 6.2.4 (Disentangling over restricted measures)**.** *Given operators $C_1,\ldots,$ $C_m \in \mathcal{L}(X)$, given measures $\nu_1,\ldots,\nu_m \in \mathcal{M}_{cb}[0,1]$, and given Borel sets $E_1,\ldots,E_m$ $\subseteq [0,1]$, we have*

$$\begin{aligned}\mathcal{T}_{\mathcal{G}'}[\chi_{E_1\times\cdots\times E_m}(\underset{\cdot}{s}_1,\ldots,\underset{\cdot}{s}_m)\tilde{C}_1(s_1)\cdots\tilde{C}_m(s_m)d\nu_1(s_1)\cdots d\nu_m(s_m)]&\\ = P^{1,\ldots,1}_{\nu_1|_{E_1},\ldots,\nu_m|_{E_m}}(C_1,\ldots,C_m).&\end{aligned} \tag{6.34}$$

Proof. Starting from the right-hand side, we have

$$\begin{aligned}&P^{1,\ldots,1}_{\nu_1|_{E_1},\ldots,\nu_m|_{E_m}}(C_1,\ldots,C_m)\\ &= \sum_{\pi\in S_m}\int_{\Delta_m(\pi)} C_{\pi(m)}\cdots C_{\pi(1)}(\nu_1|_{E_1}\times\cdots\times\nu_m|_{E_m})(ds_1,\ldots,ds_m)\\ &= \sum_{\pi\in S_m}\int_{\Delta_m(\pi)} \chi_{E_1\times\cdots\times E_m}(s_1,\ldots,s_m)C_{\pi(m)}\cdots C_{\pi(1)}(\nu_1\times\cdots\times\nu_m)(ds_1,\ldots,ds_m)\end{aligned}$$

$$= \mathcal{T}_{\mathcal{G}'}[\chi_{E_1\times\cdots\times E_m}(\underset{\cdot}{s}_1,\ldots,\underset{\cdot}{s}_m)\tilde{C}_1(s_1)\cdots\tilde{C}_m(s_m)d\nu_1(s_1)\cdots d\nu_m(s_m)], \tag{6.35}$$

which is the left-hand side. □

We see, then, a relationship between the map $\mathcal{T}_{\mathcal{G}'}$ and the disentangling of a monomial whose exponents all equal 1. We would like to address a general monomial $P^{m_1,\ldots,m_n}(\tilde{A}_1,\ldots,\tilde{A}_n)$.

To find $P^{m_1,\ldots,m_n}_{\mu_1,\ldots,\mu_n}(A_1,\ldots,A_n)$, we need to be able to express exponentiation of an element of $\mathcal{G}'$. For example, if $\chi_E(\underset{\cdot}{s})\tilde{A}(s)d\mu(s)\in\mathcal{G}'$, we would like to be able to, in effect, multiply it by itself. However, we cannot write that as

$$\text{‘}[\chi_E(s)\tilde{A}(s)d\mu(s)]^2 = \chi_E(\underset{\cdot}{s})\tilde{A}(s)d\mu(s)\chi_E(\underset{\cdot}{s})\tilde{A}(s)d\mu(s)\text{’},$$

because the expression on the right has more than one characteristic function (and that is not yet defined in $\mathcal{G}'$); moreover, it is unclear on the right whether each label ‘s’ is associated with the first argument $\underset{\cdot}{s}$ or the second. Our solution will be to establish two notational conventions, both of which will make manipulating characteristic functions similar to manipulating integral symbols.

Our first convention will be that characteristic functions may be factored into other characteristic functions in whatever way this can ordinarily be done with characteristic functions. Thus, for example, if $E_1\subseteq[0,1]^j$ and $E_2\subseteq[0,1]^k$ are Borel sets, if $C_1,\ldots,C_{j+k}\in\mathcal{L}(X)$, and if $\nu_1,\ldots,\nu_{j+k}\in\mathcal{M}_{cb}[0,1]$, then we can write

$$\begin{aligned}&\chi_{E_1\times E_2}(\underset{\cdot}{s}_1,\ldots,\underset{\cdot}{s}_{j+k})\tilde{C}_1(s_1)\cdots\tilde{C}_{j+k}(s_{j+k})d\nu_1(s_1)\cdots d\nu_{j+k}(s_{j+k})\\&\quad=\chi_{E_1}(\underset{\cdot}{s}_1,\ldots,\underset{\cdot}{s}_j)\chi_{E_2}(\underset{\cdot}{s}_{j+1},\ldots,\underset{\cdot}{s}_{j+k})\tilde{C}_1(s_1)\cdots\tilde{C}_{j+k}(s_{j+k})d\nu_1(s_1)\cdots d\nu_{j+k}(s_{j+k}).\end{aligned} \tag{6.36}$$

As a second example of this, if $A, B \in \mathcal{L}(X)$ and $\mu, \nu \in \mathcal{M}_{cb}[0,1]$, then we may write

$$\chi_{\{0<\underset{\cdot}{s}<\underset{\cdot}{t}<1\}}(\underset{\cdot}{s},\underset{\cdot}{t})\tilde{A}(s)\tilde{B}(t)d\mu(s)d\nu(t) = \chi_{(0,1)}(\underset{\cdot}{t})\chi_{(0,\underset{\cdot}{t})}(\underset{\cdot}{s})\tilde{A}(s)\tilde{B}(t)d\mu(s)d\nu(t). \tag{6.37}$$

We will express this more generally by means of the next definition. We make use here of the 'named dots' notation in order to represent a function that is the section of another function. (As an example of the section of a function, if $f(\underset{\cdot}{x},\underset{\cdot}{y},\underset{\cdot}{z})$ is a function of three variables, then the x-section of f is the function on two variables that results from setting a fixed value for x. We represent the x-section of f as '$f(x,\underset{\cdot}{y},\underset{\cdot}{z})$' and the y-section of f as '$f(\underset{\cdot}{x},y,\underset{\cdot}{z})$', etc.)

Definition 6.2.5. Let $\chi_E(\underset{\cdot}{s}_1,\ldots,\underset{\cdot}{s}_m)\tilde{C}_1(s_1)\cdots\tilde{C}_m(s_m)d\nu_1(s_1)\cdots d\nu_m(s_m) \in \mathcal{G}'$. Suppose $m_1,\ldots,m_n$ are nonnegative integers with $m_1+\cdots+m_n = m$, and suppose $E_1,\ldots,E_n \subseteq [0,1]^m$ are Borel sets with $\bigcap_{j=1}^n E_j = E$ (and consequently we have $\chi_{E_1}\chi_{E_2}\cdots\chi_{E_n} = \chi_E$). Then we define

$$\begin{aligned}
&\chi_{E_1}(\underset{\cdot}{s}_1,\ldots,\underset{\cdot}{s}_{m_1},s_{m_1+1},\ldots,s_m)\chi_{E_2}(s_1,\ldots,s_{m_1},\underset{\cdot}{s}_{m_1+1},\ldots,\underset{\cdot}{s}_{m_1+m_2},s_{m_1+m_2+1},\ldots,s_m)\\
&\quad\cdots\chi_{E_n}(s_1,\ldots,s_{m_1+\cdots+m_{n-1}},\underset{\cdot}{s}_{m_1+\cdots+m_{n-1}+1},\ldots,\underset{\cdot}{s}_m)\tilde{C}_1(s_1)\cdots\tilde{C}_m(s_m)d\nu_1(s_1)\cdots d\nu_m(s_m)\\
&\qquad := \chi_E(\underset{\cdot}{s}_1,\ldots,\underset{\cdot}{s}_m)\tilde{C}_1(s_1)\cdots\tilde{C}_m(s_m)d\nu_1(s_1)\cdots d\nu_m(s_m).
\end{aligned} \tag{6.38}$$

Moreover, each characteristic function in the left-hand expression is allowed to commute with the others and with the labeled indeterminates $\tilde{C}_1(s_1),\ldots,\tilde{C}_m(s_m), d\nu_1(s_1), \ldots, d\nu_m(s_m)$. (For the degenerate case, the number 1 may also be placed as a factor in the expression and allowed to commute with the other factors.)

Furthermore, we are free to replace any of the characteristic functions in (6.38) with an equivalent expression. For example, if for some $j \in \{1,\ldots,n\}$ we can write $E_j = [0,1]^{m_1+\cdots+m_{j-1}} \times F_j \times [0,1]^{m-m_1-\cdots-m_j}$ for some Borel set $F_j \subseteq [0,1]^{m_j}$, then we may replace

$$\chi_{E_j}(s_1,\ldots,s_{m_1+\cdots+m_{j-1}},\underset{\cdot}{s}_{m_1+\cdots+m_{j-1}+1},\ldots,\underset{\cdot}{s}_{m_1+\cdots+m_j},s_{m_1+\cdots+m_j+1},\ldots,s_m) \quad (6.39)$$

with

$$\chi_{E_j}(\underset{\cdot}{s}_{m_1+\cdots+m_{j-1}+1},\ldots,\underset{\cdot}{s}_{m_1+\cdots+m_j}), \quad (6.40)$$

since these are equal as functions of $\underset{\cdot}{s}_{m_1+\cdots+m_{j-1}+1},\ldots,\underset{\cdot}{s}_{m_1+\cdots+m_j}$.

To interpret Definition 6.2.5, notice mainly that the arguments on the left, with dots, match those on the right. The idea is this: We begin with a product of characteristic functions; for all $(s_1,\ldots,s_m)\in(0,1)^m$ we have

$$\chi_{E_1}(s_1,\ldots,s_m)\chi_{E_2}(s_1,\ldots,s_m)\cdots\chi_{E_n}(s_1,\ldots,s_m)=\chi_E(s_1,\ldots,s_m). \quad (6.41)$$

For the characteristic function over E on the right-hand side, we view $s_1,\ldots,s_m$ as arguments (putting dots under them). For each of the characteristic functions on the left, we view some of $s_1,\ldots,s_m$ as arguments and some as parameters, so each becomes a characteristic function with a smaller number of arguments, and together the arguments include each of $s_1,\ldots,s_m$ appearing exactly once. If the two sides are then multiplied by the product of indeterminates $\tilde{C}_1(s_1)\cdots\tilde{C}_m(s_m)d\nu_1(s_1)\cdots d\nu_m(s_m)$, the result is regarded as the same element of $\mathcal{G}'$. We will look at a specific example in Example 25 below.

Remark 25. It might raise concern that the foregoing definition allows us to change a characteristic function on one domain to a characteristic function on another domain (the domain is a product, and we are exchanging factors, yielding a possibly different domain), meaning it is then a different function. For example, if $E_1,E_2\subseteq[0,1]$ are

Borel sets, if $A, B \in \mathcal{L}(X)$, and if $\mu, \nu \in \mathcal{M}_{cb}[0,1]$, then we are able to write

$$\begin{aligned}
\chi_{E_1 \times E_2}(\underset{\cdot}{s}, \underset{\cdot}{t})\tilde{A}(s)\tilde{B}(t)d\mu(s)d\nu(t) &= \chi_{E_1}(\underset{\cdot}{s})\chi_{E_2}(\underset{\cdot}{t})\tilde{A}(s)\tilde{B}(t)d\mu(s)d\nu(t) \\
&= \chi_{E_2}(\underset{\cdot}{t})\chi_{E_1}(\underset{\cdot}{s})\tilde{A}(s)\tilde{B}(t)d\mu(s)d\nu(t) \\
&= \chi_{E_2 \times E_1}(\underset{\cdot}{t}, \underset{\cdot}{s})\tilde{A}(s)\tilde{B}(t)d\mu(s)d\nu(t). \qquad (6.42)
\end{aligned}$$

Because in Equation (6.42) the characteristic function has changed from $\chi_{E_1 \times E_2}$ to $\chi_{E_2 \times E_1}$, one might be concerned about whether the last expression is a different element of $\mathcal{G}'$ than the first—but in fact, the two elements are the same. By Definition 6.2.1 part 1b, an element of $\mathcal{G}'$ is unchanged if we permute the arguments of its characteristic function and we correspondingly permute the coordinates of its characteristic function set; in particular,

$$\begin{aligned}
\chi_{E_1 \times E_2}(\underset{\cdot}{s}, \underset{\cdot}{t})\tilde{A}(s)\tilde{B}(t)d\mu(s)d\nu(t) &= \chi_{\{(s,t)\colon s \in E_1,\, t \in E_2\}}(\underset{\cdot}{s}, \underset{\cdot}{t})\tilde{A}(s)\tilde{B}(t)d\mu(s)d\nu(t) \\
&= \chi_{\{(t,s)\colon s \in E_1,\, t \in E_2\}}(\underset{\cdot}{t}, \underset{\cdot}{s})\tilde{A}(s)\tilde{B}(t)d\mu(s)d\nu(t) \\
&= \chi_{E_2 \times E_1}(\underset{\cdot}{t}, \underset{\cdot}{s})\tilde{A}(s)\tilde{B}(t)d\mu(s)d\nu(t). \qquad (6.43)
\end{aligned}$$

By similar reasoning we can see that commuting characteristic functions in Equation (6.38) is valid, and so Definition 6.2.5 is well-defined.

Example 25. Although the left-hand expression in Equation (6.38) is rather involved, it does not always need to be so in practice. Consider $\chi_{\{s<t\}}(\underset{\cdot}{s}, \underset{\cdot}{t})\tilde{A}(s)\tilde{B}(t)d\mu(s)d\nu(t) \in \mathcal{G}'$. The set $E := \{s < t\} = \{(s,t) \in (0,1)^2 : s < t\}$ equals the intersection of the set $E_1 := \{(s,t) \in (0,1)^2\} = (0,1)^2$ with the set $E_2 := \{(s,t) \in (0,1)^2 : s < t\} = E$.

We could take sections of the characteristic functions χ_{E_1} and χ_{E_2} in two different ways. We will take the s-section of χ_{E_1} and the t-section of χ_{E_2}. For all $(s,t) \in (0,1)^2$, we have

$$\chi_{E_1}(s,t) = \chi_{\{(s,t)\in(0,1)^2\}}(s,t) = \chi_{\{t\in(0,1)\}}(t) = \chi_{(0,1)}(t), \tag{6.44}$$

so for any $s \in (0,1)$ we can write

$$\chi_{E_1}(s,\underset{\cdot}{t}) = \chi_{(0,1)}(\underset{\cdot}{t}). \tag{6.45}$$

For all $(s,t) \in (0,1)^2$, we have

$$\chi_{E_2}(s,t) = \chi_{\{(s,t)\in(0,1)^2\colon\, s<t\}}(s,t) = \chi_{\{s\in(0,1)\colon\, s<t\}}(s) = \chi_{\{s<t\}}(s) = \chi_{(0,t)}(s), \tag{6.46}$$

so for any $t \in (0,1)$ we can write

$$\chi_{E_2}(\underset{\cdot}{s},t) = \chi_{\{s<t\}}(\underset{\cdot}{s}) = \chi_{(0,t)}(\underset{\cdot}{s}). \tag{6.47}$$

We may therefore write

$$\begin{aligned} &\chi_{\{s<t\}}(\underset{\cdot}{s},\underset{\cdot}{t})\tilde{A}(s)\tilde{B}(t)d\mu(s)d\nu(t) \\ &\qquad = \chi_{(0,1)}(\underset{\cdot}{t})\chi_{(0,t)}(\underset{\cdot}{s})\tilde{A}(s)\tilde{B}(t)d\mu(s)d\nu(t) \\ &\qquad = \chi_{(0,1)}(\underset{\cdot}{t})\tilde{B}(t)\chi_{(0,t)}(\underset{\cdot}{s})\tilde{A}(s)d\mu(s)d\nu(t). \end{aligned} \tag{6.48}$$

The first convention, just defined, for dealing with exponentiation addresses the issue of having multiple characteristic functions in the expression for one element of $\mathcal{G}'$. Our second convention will address the problem of trying to match the labels on the indeterminates to the arguments of characteristic functions when there is more than one. The solution is merely a matter of defining the 'scope' of each variable. (Here we borrow a computer programming concept; the 'scope' of a variable refers to the broadest context in which a variable name has a particular meaning. A variable name may be used in more than one context to mean different things, so these contexts need to be clearly specified. In mathematics we see something similar when dealing with 'dummy variables' in integrals; two integrals multiplied by each other may use the same integration variable name, such as t in $\int_X f(t)dt \int_Y g(t)dt$, but the variables in the two integrals have meanings that are independent of each other.) The scope rules we choose will be like those used with integrals.

Definition 6.2.6 (Scope rules and associated notation in $\mathcal{G}'$)**.** Given an element

$$\chi_E(\underset{\cdot}{s}_1, \ldots, \underset{\cdot}{s}_m)\tilde{C}_1(s_1) \cdots \tilde{C}_m(s_m) d\nu_1(s_1) \cdots d\nu_m(s_m) \in \mathcal{G}', \tag{6.49}$$

we will sometimes write it as

$$\chi_E \, \tilde{C}_1(s_1) \cdots \tilde{C}_m(s_m)(d\nu_1 \times \cdots \times d\nu_m)(s_1, \ldots, s_m), \tag{6.50}$$

where the changes here are that the arguments have been taken off of the characteristic function, and the measure-related indeterminates have been combined with product signs. This notation may also be applied to characteristic functions and their corresponding indeterminates within a larger expression in $\mathcal{G}'$. In all cases the following rules shall apply:

- The same notation must be used for all characteristic functions (and associated measure-related indeterminates) in the expression.
- A characteristic function and its associated measure-related indeterminates act like left and right parentheses, in the sense that the characteristic function must appear to the left of the measure-related indeterminate(s) that correspond(s)

to it, and given any two pairs of characteristic functions and their associated measure-related indeterminates, either the first pair is entirely inside (between) the second pair, or else neither part of the first pair is inside the second pair (that is, they nest like parentheses).

- Finally, if a variable name somewhere in the expression refers to an argument of a given characteristic function (as an argument or as a label, for example), then that use of the variable name must occur between the characteristic function and the labeled measure-related indeterminate to which it is associated, inclusive; uses outside of that are independent of uses inside.

Example 26. Drawing from our previous example, we can write

$$\chi_{(0,1)}(\d{t})\chi_{\{s<t\}}(\d{s})\tilde{A}(s)\tilde{B}(t)d\mu(s)d\nu(t) \tag{6.51}$$

as

$$\chi_{(0,1)}\chi_{\{s<t\}}\tilde{A}(s)\tilde{B}(t)d\mu(s)d\nu(t) \tag{6.52}$$

or as

$$\chi_{(0,1)}\tilde{B}(t)\chi_{\{s<t\}}\tilde{A}(s)d\mu(s)d\nu(t). \tag{6.53}$$

In both cases, $\chi_{(0,1)}$ is then understood to be a function having argument $\d{t}$, while $\chi_{\{s<t\}}$ has argument $\d{s}$. Alternatively, we could write the same expression as

$$\chi_{\{s<t\}}\tilde{A}(s)\tilde{B}(t)(d\mu \times d\nu)(s,t). \tag{6.54}$$

Notice the similarity between the above expression and its image under the map $\mathcal{T}_{\mathcal{G}'}$, where for purposes of illustration we will use $A(s) \equiv A$ and $B(t) \equiv B$:

$$\begin{aligned}
&\mathcal{T}_{\mathcal{G}'}[\chi_{\{s<t\}}\tilde{A}(s)\tilde{B}(t)(d\mu \times d\nu)(s,t)] \\
&= \int_{\{s<t\}} \chi_{\{s<t\}} B(t)A(s)(\mu \times \nu)(ds,dt) + \int_{\{s>t\}} \chi_{\{s>t\}} A(s)B(t)(\mu \times \nu)(ds,dt) \\
&= \int_{\{s<t\}} B(t)A(s)(\mu \times \nu)(ds,dt). \qquad (6.55)
\end{aligned}$$

We see that the characteristic function has been replaced by an integral sign in its image, while the operators have been time-ordered and the tildes have been deleted.

Furthermore, we may write

$$\left[\chi_{(0,1)}\tilde{A}(s)d\mu(s)\right]^2 = \left[\chi_{(0,1)}\tilde{A}(s)d\mu(s)\right]\left[\chi_{(0,1)}\tilde{A}(s)d\mu(s)\right] \in \mathcal{G}' \qquad (6.56)$$

to refer to

$$\chi_{(0,1)^2}(s_1, s_2)\tilde{A}(s_1)\tilde{A}(s_2)d\mu(s_1)d\mu(s_2), \qquad (6.57)$$

because the scope rules make the label s in the earlier expression a 'dummy' variable; the use in the first factor $\chi_{(0,1)}\tilde{A}(s)d\mu(s)$ is independent of its use in the second factor. When we change back to the original notation, we need to name those independent uses with different names, here s_1 and s_2.

The convention we have just defined therefore allows us to write several factors with different argument names as a single factor with an exponent, so now we may work with exponentiation of elements of $\mathcal{G}'$. (To be precise, we have not really defined a product of elements of $\mathcal{G}'$—which is still a set without operations—but we can write some of those elements in ways that suggest products and exponentiation. In the space $\mathbb{E}$ described below that is developed from part of $\mathcal{G}'$, however, we will define a product operation.) Besides what we have just defined, we will dictate that any element of $\mathcal{G}'$

taken to the zeroth power is 1.

Now we may state Theorem 6.2.3 more generally, expressing the disentangling of a monomial $P^{m_1,\ldots,m_n}(\tilde{A}_1,\ldots,\tilde{A}_n)$ in terms of the intermediate disentangling map $\mathcal{T}_{\mathcal{G}'}$.

Theorem 6.2.7. *Let operators $A_1,\ldots,A_n \in \mathcal{L}(X)$ be associated with, respectively, measures $\mu_1,\ldots,\mu_n \in \mathcal{M}_{cb}[0,1]$, and let $m_1,\ldots,m_n$ be nonnegative integers. Then*

$$\begin{aligned}\mathcal{T}_{\mu_1,\ldots,\mu_n}[\tilde{A}_1^{m_1}\cdots\tilde{A}_n^{m_n}] &= P^{m_1,\ldots,m_n}_{\mu_1,\ldots,\mu_n}(A_1,\ldots,A_n)\\ &= \mathcal{T}_{\mathcal{G}'}\left[\Big(\chi_{(0,1)}\tilde{A}(s_1)d\mu(s_1)\Big)^{m_1}\cdots\Big(\chi_{(0,1)}\tilde{A}(s_n)d\mu(s_n)\Big)^{m_n}\right]. \end{aligned}\tag{6.58}$$

Proof. We let $m = m_1 + \cdots + m_n$ and assign names of blocks $\mathrm{Bl}(1),\ldots,\mathrm{Bl}(n)$, operators $C_1,\ldots,C_m$, and measures $\nu_1,\ldots,\nu_m$ as usual. Then by Theorems 6.2.3 and 2.0.4 we have

$$\begin{aligned}&\mathcal{T}_{\mathcal{G}'}\left[\Big(\chi_{(0,1)}\tilde{A}(s_1)d\mu(s_1)\Big)^{m_1}\cdots\Big(\chi_{(0,1)}\tilde{A}(s_n)d\mu(s_n)\Big)^{m_n}\right]\\ &\quad= \mathcal{T}_{\mathcal{G}'}\left[\Big(\chi_{(0,1)}(\underset{\cdot}{t}_1)\tilde{C}(t_1)d\mu(t_1)\Big)\cdots\Big(\chi_{(0,1)}(\underset{\cdot}{t}_m)\tilde{C}(t_m)d\mu(t_m)\Big)\right]\\ &\quad= \mathcal{T}_{\mathcal{G}'}\left[\chi_{(0,1)^m}(\underset{\cdot}{t}_1,\ldots,\underset{\cdot}{t}_m)\tilde{C}(t_1)d\mu(t_1)\cdots\tilde{C}(t_m)d\mu(t_m)\right]\\ &\quad= P^{1,\ldots,1}_{\nu_1,\ldots,\nu_m}(C_1,\ldots,C_m)\\ &\quad= P^{m_1,\ldots,m_n}_{\mu_1,\ldots,\mu_n}(A_1,\ldots,A_n). \end{aligned}\tag{6.59}$$

Note that this also holds when $m = 0$, in which case the disentangling map and $\mathcal{T}_{\mathcal{G}'}$ both yield the identity operator $I \in \mathcal{L}(X)$. $\square$

The map $\mathcal{T}_{\mathcal{G}'}$, especially Theorem 6.2.7, will be our focus in the next section. After that we will build the space $\mathbb{E}$ using the set $\mathcal{G}'$.

6.3 Using the intermediate set $\mathcal{G}'$

The main results we wish to address now, while still working with the set $\mathcal{G}'$ and the map $\mathcal{T}_{\mathcal{G}'}$, are a distributive property and a binomial theorem related to sums of characteristic functions, together with the three most important theorems for our work with intermediate disentangling maps, which deal with applying the map $\mathcal{T}_{\mathcal{G}'}$ to time-ordered or partially time-ordered expressions in $\mathcal{G}'$. First we look at the distributive property and binomial theorem.

Theorem 6.3.1 ($\mathcal{T}_{\mathcal{G}'}$ distributes over certain sums of characteristic functions)**.** *Given integer $m \geq 1$, $C_1, \ldots, C_m \in \mathcal{L}(X)$ and $\nu_1, \ldots, \nu_m \in \mathcal{M}_{cb}[0,1]$, and given disjoint Borel sets $E_1, E_2 \subseteq [0,1]^m$, we have*

$$\begin{aligned}
&\mathcal{T}_{\mathcal{G}'}[(\chi_{E_1} + \chi_{E_2})(\underset{\cdot}{s}_1, \ldots, \underset{\cdot}{s}_m)\tilde{C}_1(s_1)\cdots\tilde{C}_m(s_m)d\nu_1(s_1)\cdots d\nu_m(s_m)] \\
&\qquad = \mathcal{T}_{\mathcal{G}'}[\chi_{E_1}(\underset{\cdot}{s}_1, \ldots, \underset{\cdot}{s}_m)\tilde{C}_1(s_1)\cdots\tilde{C}_m(s_m)d\nu_1(s_1)\cdots d\nu_m(s_m)] \\
&\qquad\quad + \mathcal{T}_{\mathcal{G}'}[\chi_{E_2}(\underset{\cdot}{s}_1, \ldots, \underset{\cdot}{s}_m)\tilde{C}_1(s_1)\cdots\tilde{C}_m(s_m)d\nu_1(s_1)\cdots d\nu_m(s_m)].
\end{aligned} \tag{6.60}$$

Proof. We will omit the proof, which involves simply applying the definition of the map $\mathcal{T}_{\mathcal{G}'}$ and splitting the sum inside the integral. $\square$

Theorem 6.3.2 (A binomial theorem for certain sums of characteristic functions)**.** *Let $m \in \mathbb{N} \cup \{0\}$; $A, C_1, C_2, \ldots, C_m \in \mathcal{L}(X)$; and $\mu, \nu_1, \nu_2, \ldots, \nu_m \in \mathcal{M}_{cb}[0,1]$. Let $E_1, E_2 \subseteq [0,1]$ be disjoint Borel sets, and $F \subseteq [0,1]^m$ be a Borel set. Then for any $n \in \mathbb{N} \cup \{0\}$, we have*

$$\mathcal{T}_{\mathcal{G}'}\left[\Big((\chi_{E_1} + \chi_{E_2})\tilde{A}(s)d\mu(s)\Big)^n \chi_F \tilde{C}_1(t_1)\cdots\tilde{C}_m(t_m)(d\nu_1 \times \cdots \times d\nu_m)(t_1, \ldots, t_m)\right]$$

$$= \sum_{k=0}^{n} \binom{n}{k} \mathcal{T}_{\mathcal{G}'} \left[\left(\chi_{E_1} \tilde{A}(s) d\mu(s) \right)^k \left(\chi_{E_2} \tilde{A}(s) d\mu(s) \right)^{n-k} \right.$$
$$\left. \cdot \chi_F \tilde{C}_1(t_1) \cdots \tilde{C}_m(t_m)(d\nu_1 \times \cdots \times d\nu_m)(t_1, \ldots, t_m) \right]. \tag{6.61}$$

Proof. (Note that the theorem is expressed using notation as in Definition 6.2.6, so arguments of characteristic functions are omitted.) If $n = 0$, then the zeroth-power factors on both sides of the expression above equal 1, and the result is immediate. Suppose, then, that $n \geq 1$ (in which case we may use Theorem 6.3.1). We observe first that given any $s_1, \ldots, s_n, t_1, \ldots, t_m \in [0, 1]$, we have

$$\begin{aligned}
&\chi_{(E_1 \cup E_2)^n \times F}(s_1, \ldots, s_n, t_1, \ldots, t_m) \\
&\qquad = (\chi_{E_1} + \chi_{E_2})(s_1) \cdots (\chi_{E_1} + \chi_{E_2})(s_n) \chi_F(t_1, \ldots, t_m) \\
&\qquad = [\chi_{E_1}(s_1) + \chi_{E_2}(s_1)] \cdots [\chi_{E_1}(s_n) + \chi_{E_2}(s_n)] \chi_F(t_1, \ldots, t_m) \\
&\qquad = \sum_{q_1, \ldots, q_n = 1}^{2} \chi_{E_{q_1}}(s_1) \chi_{E_{q_2}}(s_2) \cdots \chi_{E_{q_n}}(s_n) \chi_F(t_1, \ldots, t_m) \\
&\qquad = \sum_{q_1, \ldots, q_n = 1}^{2} \chi_{E_{q_1} \times E_{q_2} \times \cdots \times E_{q_n} \times F}(s_1, s_2, \ldots, s_n, t_1, \ldots, t_m),
\end{aligned} \tag{6.62}$$

where the sets $E_{q_1} \times E_{q_2} \times \cdots \times E_{q_n} \times F$ in the last summation are pairwise disjoint for different terms in the sum.

Consequently, by Theorem 6.3.1,

$$\begin{aligned}
&\mathcal{T}_{\mathcal{G}'} \left[\left((\chi_{E_1} + \chi_{E_2}) \tilde{A}(s) d\mu(s) \right)^n \chi_F \tilde{C}_1(t_1) \cdots \tilde{C}_m(t_m)(d\nu_1 \times \cdots \times d\nu_m)(t_1, \ldots, t_m) \right] \\
&= \mathcal{T}_{\mathcal{G}'} \Bigg[\left((\chi_{E_1} + \chi_{E_2})(\underset{\cdot}{s}_1) \tilde{A}(s_1) d\mu(s_1) \right) \cdots \left((\chi_{E_1} + \chi_{E_2})(\underset{\cdot}{s}_n) \tilde{A}(s_n) d\mu(s_n) \right) \\
&\qquad\qquad \cdot \chi_F(\underset{\cdot}{t}_1, \ldots, \underset{\cdot}{t}_m) \tilde{C}_1(t_1) \cdots \tilde{C}_m(t_m) d\nu_1(t_1) \cdots d\nu_m(t_m) \Bigg]
\end{aligned}$$

$$
\begin{aligned}
&= \mathcal{T}_{\mathcal{G}'}\Big[(\chi_{E_1}+\chi_{E_2})(\underset{\cdot}{s}_1)\cdots(\chi_{E_1}+\chi_{E_2})(\underset{\cdot}{s}_n)\chi_F(\underset{\cdot}{t}_1,\ldots,\underset{\cdot}{t}_m) \\
&\qquad\qquad \cdot \tilde{A}(s_1)d\mu(s_1)\cdots\tilde{A}(s_n)d\mu(s_n)\tilde{C}_1(t_1)\cdots\tilde{C}_m(t_m)d\nu_1(t_1)\cdots d\nu_m(t_m)\Big] \\
&= \sum_{q_1,\ldots,q_n=1}^{2}\mathcal{T}_{\mathcal{G}'}\Big[\chi_{E_{q_1}\times E_{q_2}\times\cdots\times E_{q_n}\times F}(\underset{\cdot}{s}_1,\underset{\cdot}{s}_2,\ldots,\underset{\cdot}{s}_n,\underset{\cdot}{t}_1,\ldots,\underset{\cdot}{t}_m) \\
&\qquad\qquad \cdot \tilde{A}(s_1)d\mu(s_1)\cdots\tilde{A}(s_n)d\mu(s_n)\tilde{C}_1(t_1)\cdots\tilde{C}_m(t_m)d\nu_1(t_1)\cdots d\nu_m(t_m)\Big] \\
&= \sum_{q_1,\ldots,q_n=1}^{2}\mathcal{T}_{\mathcal{G}'}\Big[\Big(\chi_{E_{q_1}}(\underset{\cdot}{s}_1)\tilde{A}(s_1)d\mu(s_1)\Big)\cdots\Big(\chi_{E_{q_n}}(\underset{\cdot}{s}_n)\tilde{A}(s_n)d\mu(s_n)\Big) \\
&\qquad\qquad \cdot \chi_F(\underset{\cdot}{t}_1,\ldots,\underset{\cdot}{t}_m)\tilde{C}_1(t_1)\cdots\tilde{C}_m(t_m)d\nu_1(t_1)\cdots d\nu_m(t_m)\Big] \\
&= \sum_{q_1,\ldots,q_n=1}^{2}\mathcal{T}_{\mathcal{G}'}\Big[\Big(\chi_{E_{q_1}}\tilde{A}(s)d\mu(s)\Big)\cdots\Big(\chi_{E_{q_n}}\tilde{A}(s)d\mu(s)\Big) \\
&\qquad\qquad \cdot \chi_F\tilde{C}_1(t_1)\cdots\tilde{C}_m(t_m)(d\nu_1\times\cdots\times d\nu_m)(t_1,\ldots,t_m)\Big].
\end{aligned}
\tag{6.63}
$$

We then group the factors of $\Big(\chi_{E_{q_1}}\tilde{A}(s)d\mu(s)\Big)\cdots\Big(\chi_{E_{q_n}}\tilde{A}(s)d\mu(s)\Big)$ according to whether they include χ_{E_1} or χ_{E_2}. Note that these factors do commute, because the factors are made up of commuting indeterminates and characteristic functions, which by Definition 6.2.5 are allowed to commute with each other and with the indeterminates. (Be aware, however, that we have not defined a multiplication on $\mathcal{G}'$; the factors of

$$
\begin{aligned}
&\Big(\chi_{E_{q_1}}\tilde{A}(s)d\mu(s)\Big)\cdots\Big(\chi_{E_{q_n}}\tilde{A}(s)d\mu(s)\Big) \\
&\qquad = \Big(\chi_{E_{q_1}}(\underset{\cdot}{s}_1)\tilde{A}(s_1)d\mu(s_1)\Big)\cdots\Big(\chi_{E_{q_n}}(\underset{\cdot}{s}_n)\tilde{A}(s_n)d\mu(s_n)\Big)
\end{aligned}
\tag{6.64}
$$

are not being viewed as separate elements of $\mathcal{G}'$ that are joined by multiplication;

they are viewed collectively as a product of the characteristic function

$$\chi_{\{(s_1,\ldots,s_n):\, s_1\in E_{q_1},\,\ldots,\, s_n\in E_{q_n}\}}(\underset{\cdot}{s}_1,\ldots,\underset{\cdot}{s}_n) \tag{6.65}$$

and the commuting indeterminates $\tilde{A}(s_1),\ldots,\tilde{A}(s_n), d\mu(s_1),\ldots d\mu(s_n)$, whose product forms an element of $\mathcal{G}'$.) Counting terms that are identical, we note that for each integer $k=0,\ldots,n$ there are $\binom{n}{k}$ terms in the sum that have k factors of χ_{E_1} and $n-k$ factors of χ_{E_2}. (We enumerate those by considering the n labels $q_1,\ldots,q_n$ and asking how many ways we can choose k of them to equal 1 and $n-k$ of them to equal 2, hence $\binom{n}{k}$.) The result is

$$\begin{aligned}
&\mathcal{T}_{\mathcal{G}'}\left[\Big((\chi_{E_1}+\chi_{E_2})\tilde{A}(s)d\mu(s)\Big)^n \chi_F\tilde{C}_1(t_1)\cdots\tilde{C}_m(t_m)(d\nu_1\times\cdots\times d\nu_m)(t_1,\ldots,t_m)\right]\\
&=\sum_{k=0}^{n}\binom{n}{k}\mathcal{T}_{\mathcal{G}'}\Big[\Big(\chi_{E_1}\tilde{A}(s)d\mu(s)\Big)^k\Big(\chi_{E_2}\tilde{A}(s)d\mu(s)\Big)^{n-k}\\
&\qquad\qquad\cdot\chi_F\tilde{C}_1(t_1)\cdots\tilde{C}_m(t_m)(d\nu_1\times\cdots\times d\nu_m)(t_1,\ldots,t_m)\Big],
\end{aligned} \tag{6.66}$$

as we claimed. □

Recalling our discussion above that motivated the use of an intermediate disentangling space $\mathbb{E}$—or for right now, the use of the set $\mathcal{G}'$—between $\mathbb{D}$ and $\mathcal{L}(X)$, we hope to be able to exploit commutativity in the intermediate space, manipulating an expression there until it is in the desired form, before mapping it finally into $\mathcal{L}(X)$. Specifically, we want to perform the final mapping after the expression is in a form which we consider to be time-ordered. We will now consider three theorems to that effect, one in which the expression in $\mathcal{G}'$ is time-ordered, and two in which it is partly time-ordered. The most fundamental of the three theorems is the first.

Theorem 6.3.3 (Applying $\mathcal{T}_{\mathcal{G}'}$ to a time-ordered expression)**.** *Let $C_1, \ldots, C_m \in \mathcal{L}(X)$, let $\nu_1, \ldots, \nu_m \in \mathcal{M}_{cb}[0,1]$, and let $E \subseteq [0,1]^m$ be a Borel set with $E \subseteq \Delta_m(\sigma)$ for some fixed $\sigma \in S_m$, $m \geq 1$. Then*

$$\mathcal{T}_{\mathcal{G}'}\left[\chi_E(\underset{\cdot}{s}_1, \ldots, \underset{\cdot}{s}_m)\tilde{C}_{\sigma(m)}(s_{\sigma(m)}) \cdots \tilde{C}_{\sigma(1)}(s_{\sigma(1)}) d\nu_1(s_1) \cdots d\nu_m(s_m)\right] \\ = \int_E C_{\sigma(m)} \cdots C_{\sigma(1)} (\nu_1 \times \cdots \times \nu_m)(ds_1, \ldots, ds_m). \quad (6.67)$$

Proof. By definition of $\mathcal{T}_{\mathcal{G}'}$ (and using Theorem 3.1.7) we have

$$\begin{aligned} &\mathcal{T}_{\mathcal{G}'}\left[\chi_E(\underset{\cdot}{s}_1, \ldots, \underset{\cdot}{s}_m)\tilde{C}_{\sigma(m)}(s_{\sigma(m)}) \cdots \tilde{C}_{\sigma(1)}(s_{\sigma(1)}) d\nu_1(s_1) \cdots d\nu_m(s_m)\right] \\ &= \mathcal{T}_{\mathcal{G}'}\left[\chi_E(\underset{\cdot}{s}_1, \ldots, \underset{\cdot}{s}_m)\tilde{C}_m(s_m) \cdots \tilde{C}_1(s_1) d\nu_1(s_1) \cdots d\nu_m(s_m)\right] \\ &= \sum_{\pi \in S_m} \int_{\Delta_m(\pi)} \chi_E(s_1, \ldots, s_m) C_{\pi(m)} \cdots C_{\pi(1)} (\nu_1 \times \cdots \times \nu_m)(ds_1, \ldots, ds_m). \qquad (6.68) \end{aligned}$$

Since $E \subseteq \Delta_m(\sigma)$, and since the sets $\Delta_m(\pi)$ are pairwise disjoint, the integrals will integrate to zero except possibly when $\pi = \sigma$, yielding

$$\begin{aligned} &\mathcal{T}_{\mathcal{G}'}\left[\chi_E(\underset{\cdot}{s}_1, \ldots, \underset{\cdot}{s}_m)\tilde{C}_{\sigma(m)}(s_{\sigma(m)}) \cdots \tilde{C}_{\sigma(1)}(s_{\sigma(1)}) d\nu_1(s_1) \cdots d\nu_m(s_m)\right] \\ &= \int_{\Delta_m(\sigma)} \chi_E(s_1, \ldots, s_m) C_{\sigma(m)} \cdots C_{\sigma(1)} (\nu_1 \times \cdots \times \nu_m)(ds_1, \ldots, ds_m) \\ &= \int_E C_{\sigma(m)} \cdots C_{\sigma(1)} (\nu_1 \times \cdots \times \nu_m)(ds_1, \ldots, ds_m). \qquad (6.69) \end{aligned}$$

□

Example 27. Let $\chi_{\{0<r<s<t<1\}}(\underset{\cdot}{r}, \underset{\cdot}{s}, \underset{\cdot}{t})\tilde{A}(r)\tilde{B}(s)\tilde{C}(t) d\mu(r) d\nu(s) d\eta(t) \in \mathcal{G}'$. Then using commutativity of indeterminates and applying Theorem 6.3.3, we have that

$$\mathcal{T}_{\mathcal{G}'}\left[\chi_{\{0<r<s<t<1\}}(\underset{\cdot}{r}, \underset{\cdot}{s}, \underset{\cdot}{t})\tilde{A}(r)\tilde{B}(s)\tilde{C}(t) d\mu(r) d\nu(s) d\eta(t)\right]$$

$$= \mathcal{T}_{\mathcal{G}'}\left[\chi_{\{0<r<s<t<1\}}(\underset{\cdot}{r},\underset{\cdot}{s},\underset{\cdot}{t})\tilde{C}(t)\tilde{B}(s)\tilde{A}(r)d\mu(r)d\nu(s)d\eta(t)\right]$$
$$= \int_{\{1>t>s>r>0\}} CBA\ (\mu\times\nu\times\eta)(dr,ds,dt). \tag{6.70}$$

The second time-ordering theorem involves two characteristic functions, one over a set whose elements are smaller than a given fixed value, and the other over a set whose elements are larger than that value.

Theorem 6.3.4 (Applying $\mathcal{T}_{\mathcal{G}'}$ to an expression with two sets time-ordered relative to each other)**.** *Let integer* $m \geq 0$*, let* $C_1, \cdots C_m \in \mathcal{L}(X)$*, let* $\nu_1,\ldots,\nu_m \in \mathcal{M}_{cb}[0,1]$*, let* $a\in(0,1)$ *be fixed, and let* $k\in\{1,2,\ldots,m-1\}$*. If* $E_1 \subseteq (0,a)^k$ *and* $E_2 \subseteq (a,1)^{m-k}$ *are Borel sets, then*

$$\begin{aligned}
&\mathcal{T}_{\mathcal{G}'}\big[\chi_{E_2}(\underset{\cdot}{s}_{k+1},\ldots,\underset{\cdot}{s}_m)\tilde{C}_{k+1}(s_{k+1})\cdots\tilde{C}_m(s_m)d\nu_{k+1}(s_{k+1})\cdots d\nu_m(s_m)\\
&\qquad\qquad\cdot\chi_{E_1}(\underset{\cdot}{s}_1,\ldots,\underset{\cdot}{s}_k)\tilde{C}_1(s_1)\cdots\tilde{C}_k(s_k)d\nu_1(s_1)\cdots d\nu_k(s_k)\big]\\
&\quad=\mathcal{T}_{\mathcal{G}'}\big[\chi_{E_2}(\underset{\cdot}{s}_{k+1},\ldots,\underset{\cdot}{s}_m)\tilde{C}_{k+1}(s_{k+1})\cdots\tilde{C}_m(s_m)d\nu_{k+1}(s_{k+1})\cdots d\nu_m(s_m)\big]\\
&\qquad\qquad\times\mathcal{T}_{\mathcal{G}'}\big[\chi_{E_1}(\underset{\cdot}{s}_1,\ldots,\underset{\cdot}{s}_k)\tilde{C}_1(s_1)\cdots\tilde{C}_k(s_k)d\nu_1(s_1)\cdots d\nu_k(s_k)\big].
\end{aligned}\tag{6.71}$$

(Note that the statement (6.71) *also holds if either* $\chi_{E_1}(\underset{\cdot}{s}_1,\ldots,\underset{\cdot}{s}_k)\tilde{C}_1(s_1)\cdots\tilde{C}_k(s_k)$ $d\nu_1(s_1)\cdots d\nu_k(s_k)$ *or* $\chi_{E_2}(\underset{\cdot}{s}_{k+1},\ldots,\underset{\cdot}{s}_m)\tilde{C}_{k+1}(s_{k+1})\cdots\tilde{C}_m(s_m)d\nu_{k+1}(s_{k+1})\cdots d\nu_m(s_m)$ *is replaced by the element* $1\in\mathcal{G}'$*, so the statement holds in that sense for* $k=0$ *or* $k=m$*, provided the corresponding hypothesis,* $E_1\subseteq(0,a)^k$ *or* $E_2\subseteq(a,1)^{m-k}$*, or both if* $m=0$*, is omitted.)*

Proof. The method of proof is to partition the sets E_1 and E_2 (up to sets of measure zero). For $\tau\in\mathcal{O}_{\{k+1,\ldots,m\}}$ we define $\Delta_{k+1,m}(\tau) := \{(s_{k+1},\ldots,s_m) :\ 0 < s_{\tau(1)} < \ldots <$

$s_{\tau(m-k)} < 1\}$. Then under the given assumptions we have by Theorem 6.3.1 that

$$
\begin{aligned}
&\mathcal{T}_{\mathcal{G}'}\big[\chi_{E_2}(\underset{\cdot}{s}_{k+1},\ldots,\underset{\cdot}{s}_m)\tilde{C}_{k+1}(s_{k+1})\cdots\tilde{C}_m(s_m)d\nu_{k+1}(s_{k+1})\cdots d\nu_m(s_m)\\
&\qquad\qquad\qquad \cdot\chi_{E_1}(\underset{\cdot}{s}_1,\ldots,\underset{\cdot}{s}_k)\tilde{C}_1(s_1)\cdots\tilde{C}_k(s_k)d\nu_1(s_1)\cdots d\nu_k(s_k)\big]\\
&= \mathcal{T}_{\mathcal{G}'}\big[\chi_{E_1\times E_2}(\underset{\cdot}{s}_1,\ldots,\underset{\cdot}{s}_k,\underset{\cdot}{s}_{k+1},\ldots,\underset{\cdot}{s}_m)\tilde{C}_{k+1}(s_{k+1})\cdots\tilde{C}_m(s_m)\tilde{C}_1(s_1)\cdots\tilde{C}_k(s_k)\\
&\qquad\qquad\qquad\qquad\qquad \cdot d\nu_1(s_1)\cdots d\nu_k(s_k)d\nu_{k+1}(s_{k+1})\cdots d\nu_m(s_m)\big]\\
&= \sum_{\substack{\sigma\in\mathcal{O}_{\{1,\ldots,k\}}\\ \tau\in\mathcal{O}_{\{k+1,\ldots,m\}}}} \mathcal{T}_{\mathcal{G}'}\big[\chi_{[E_1\cap\Delta_k(\sigma)]\times[E_2\cap\Delta_{k+1,m}(\tau)]}(\underset{\cdot}{s}_1,\ldots,\underset{\cdot}{s}_k,\underset{\cdot}{s}_{k+1},\ldots,\underset{\cdot}{s}_m)\\
&\cdot\tilde{C}_{k+1}(s_{k+1})\cdots\tilde{C}_m(s_m)\tilde{C}_1(s_1)\cdots\tilde{C}_k(s_k)d\nu_1(s_1)\cdots d\nu_k(s_k)d\nu_{k+1}(s_{k+1})\cdots d\nu_m(s_m)\big]\\
&= \sum_{\substack{\sigma\in\mathcal{O}_{\{1,\ldots,k\}}\\ \tau\in\mathcal{O}_{\{k+1,\ldots,m\}}}} \mathcal{T}_{\mathcal{G}'}\big[\chi_{[E_1\cap\Delta_k(\sigma)]\times[E_2\cap\Delta_{k+1,m}(\tau)]}(\underset{\cdot}{s}_1,\ldots,\underset{\cdot}{s}_k,\underset{\cdot}{s}_{k+1},\ldots,\underset{\cdot}{s}_m)\\
&\qquad\qquad \cdot\tilde{C}_{\tau(m-k)}(s_{\tau(m-k)})\cdots\tilde{C}_{\tau(1)}(s_{\tau(1)})\tilde{C}_{\sigma(k)}(s_{\sigma(k)})\cdots\tilde{C}_{\sigma(1)}(s_{\sigma(1)})\\
&\qquad\qquad \cdot d\nu_1(s_1)\cdots d\nu_k(s_k)d\nu_{k+1}(s_{k+1})\cdots d\nu_m(s_m)\big].
\end{aligned}
\tag{6.72}
$$

Here we may apply Theorem 6.3.3, since $[E_1\cap\Delta_k(\sigma)]\times[E_2\cap\Delta_{k+1,m}(\tau)] \subseteq \{(s_1,\ldots,s_k, s_{k+1},\ldots,s_m):\ 0 < s_{\sigma(1)} < \cdots < s_{\sigma(k)} < a < s_{\tau(1)} < \cdots < s_{\tau(m-k)} < 1\}$. The result is that

$$
\begin{aligned}
&\mathcal{T}_{\mathcal{G}'}\big[\chi_{E_2}(\underset{\cdot}{s}_{k+1},\ldots,\underset{\cdot}{s}_m)\tilde{C}_{k+1}(s_{k+1})\cdots\tilde{C}_m(s_m)d\nu_{k+1}(s_{k+1})\cdots d\nu_m(s_m)\\
&\qquad\qquad\qquad\qquad \cdot\chi_{E_1}(\underset{\cdot}{s}_1,\ldots,\underset{\cdot}{s}_k)\tilde{C}_1(s_1)\cdots\tilde{C}_k(s_k)d\nu_1(s_1)\cdots d\nu_k(s_k)\big]\\
&= \sum_{\substack{\sigma\in\mathcal{O}_{\{1,\ldots,k\}}\\ \tau\in\mathcal{O}_{\{k+1,\ldots,m\}}}} \int_{[E_1\cap\Delta_k(\sigma)]\times[E_2\cap\Delta_{k+1,m}(\tau)]} C_{\tau(m-k)}\cdots C_{\tau(1)}C_{\sigma(k)}\cdots C_{\sigma(1)}\\
&\qquad\qquad \times(\nu_1\times\cdots\times\nu_k\times\nu_{k+1}\times\cdots\times\nu_m)(ds_1,\ldots,ds_k,ds_{k+1},\ldots,ds_m),
\end{aligned}
\tag{6.73}
$$

which by Fubini-Tonelli is

$$
\begin{aligned}
&\sum_{\substack{\sigma\in\mathcal{O}_{\{1,\dots,k\}}\\ \tau\in\mathcal{O}_{\{k+1,\dots,m\}}}} \int_{E_2\cap\Delta_{k+1,m}(\tau)} \Bigg(\int_{E_1\cap\Delta_k(\sigma)} C_{\tau(m-k)}\cdots C_{\tau(1)} C_{\sigma(k)}\cdots C_{\sigma(1)} \\
&\qquad \times (\nu_1\times\cdots\times\nu_k)(ds_1,\dots,ds_k)\Bigg) \times (\nu_{k+1}\times\cdots\times\nu_m)(ds_{k+1},\dots,ds_m) \\
&= \left(\sum_{\tau\in\mathcal{O}_{\{k+1,\dots,m\}}} \int_{E_2\cap\Delta_{k+1,m}(\tau)} C_{\tau(m-k)}\cdots C_{\tau(1)} (\nu_{k+1}\times\cdots\times\nu_m)(ds_{k+1},\dots,ds_m) \right) \\
&\qquad \times \left(\sum_{\sigma\in\mathcal{O}_{\{1,\dots,k\}}} \int_{E_1\cap\Delta_k(\sigma)} C_{\sigma(k)}\cdots C_{\sigma(1)} (\nu_1\times\cdots\times\nu_k)(ds_1,\dots,ds_k) \right) \\
&= \Bigg(\sum_{\tau\in\mathcal{O}_{\{k+1,\dots,m\}}} \int_{\Delta_{k+1,m}(\tau)} \chi_{E_2}(s_{k+1},\dots,s_m) C_{\tau(m-k)}\cdots C_{\tau(1)} \\
&\qquad\qquad\qquad\qquad \times (\nu_{k+1}\times\cdots\times\nu_m)(ds_{k+1},\dots,ds_m) \Bigg) \\
&\qquad \times \left(\sum_{\sigma\in\mathcal{O}_{\{1,\dots,k\}}} \int_{\Delta_k(\sigma)} \chi_{E_1}(s_1,\dots,s_k) C_{\sigma(k)}\cdots C_{\sigma(1)} (\nu_1\times\cdots\times\nu_k)(ds_1,\dots,ds_k) \right) \\
&= \mathcal{T}_{\mathcal{G}'}\left[\chi_{E_2}(\underset{\cdot}{s}_{k+1},\dots,\underset{\cdot}{s}_m) \tilde{C}_{k+1}(s_{k+1})\cdots\tilde{C}_m(s_m) d\nu_{k+1}(s_{k+1})\cdots d\nu_m(s_m) \right] \\
&\qquad\qquad \times \mathcal{T}_{\mathcal{G}'}\left[\chi_{E_1}(\underset{\cdot}{s}_1,\dots,\underset{\cdot}{s}_k) \tilde{C}_1(s_1)\cdots\tilde{C}_k(s_k) d\nu_1(s_1)\cdots d\nu_k(s_k) \right].
\end{aligned}
\tag{6.74}
$$

□

Example 28. Now we are able to work with decomposing disentanglings (discussed earlier, Example 15) using the intermediate set $\mathcal{G}'$. This example is based on the heuristic 'derivation' of decomposing disentanglings of a monomial in two indeterminates found in [17, p. 4]. The referenced article also includes a proof of the formula; we will provide a different proof here using the set $\mathcal{G}'$, and in Example 36 below we will extend the formula to a third measure.

Let $A, B \in \mathcal{L}(X)$, let $\mu, \nu \in \mathcal{M}_{cb}[0,1]$ be probability measures, let m_1, m_2 be nonnegative integers, and let $a \in (0,1)$. Then by Theorems 6.3.2 and 6.3.4 we have

$$
\begin{aligned}
&P^{m_1,m_2}_{\mu,\nu}(A,B)\\
&\qquad = \mathcal{T}_{\mathcal{G}'}\left[\left(\chi_{(0,1)}\tilde{A}(r)d\mu(r)\right)^{m_1}\left(\chi_{(0,1)}\tilde{B}(s)d\nu(s)\right)^{m_2}\right]\\
&\qquad = \mathcal{T}_{\mathcal{G}'}\left[\left([\chi_{(0,a)}+\chi_{(a,1)}]\tilde{A}(r)d\mu(r)\right)^{m_1}\left([\chi_{(0,a)}+\chi_{(a,1)}]\tilde{B}(s)d\nu(s)\right)^{m_2}\right]\\
&\qquad = \sum_{\substack{i_1+j_1=m_1\\ i_2+j_2=m_2}} \frac{m_1!}{i_1!j_1!}\frac{m_2!}{i_2!j_2!}\mathcal{T}_{\mathcal{G}'}\Big[\left(\chi_{(0,a)}\tilde{A}(r)d\mu(r)\right)^{i_1}\left(\chi_{(a,1)}\tilde{A}(r)d\mu(r)\right)^{j_1}\\
&\qquad\qquad\qquad\qquad \cdot\left(\chi_{(0,a)}\tilde{B}(s)d\nu(s)\right)^{i_2}\left(\chi_{(a,1)}\tilde{B}(s)d\nu(s)\right)^{j_2}\Big]\\
&\qquad = \sum_{\substack{i_1+j_1=m_1\\ i_2+j_2=m_2}} \frac{m_1!}{i_1!j_1!}\frac{m_2!}{i_2!j_2!}\mathcal{T}_{\mathcal{G}'}\Big[\left(\chi_{(a,1)}\tilde{A}(r)d\mu(r)\right)^{j_1}\left(\chi_{(a,1)}\tilde{B}(s)d\nu(s)\right)^{j_2}\\
&\qquad\qquad\qquad\qquad \cdot\left(\chi_{(0,a)}\tilde{A}(r)d\mu(r)\right)^{i_1}\left(\chi_{(0,a)}\tilde{B}(s)d\nu(s)\right)^{i_2}\Big]\\
&\qquad = \sum_{\substack{i_1+j_1=m_1\\ i_2+j_2=m_2}} \frac{m_1!}{i_1!j_1!}\frac{m_2!}{i_2!j_2!}\mathcal{T}_{\mathcal{G}'}\left[\left(\chi_{(a,1)}\tilde{A}(r)d\mu(r)\right)^{j_1}\left(\chi_{(a,1)}\tilde{B}(s)d\nu(s)\right)^{j_2}\right]\\
&\qquad\qquad\qquad\qquad \times\mathcal{T}_{\mathcal{G}'}\left[\left(\chi_{(0,a)}\tilde{A}(r)d\mu(r)\right)^{i_1}\left(\chi_{(0,a)}\tilde{B}(s)d\nu(s)\right)^{i_2}\right]\\
&\qquad = \sum_{\substack{i_1+j_1=m_1\\ i_2+j_2=m_2}} \frac{m_1!}{i_1!j_1!}\frac{m_2!}{i_2!j_2!}P^{j_1,j_2}_{\mu|_{(a,1)},\nu|_{(a,1)}}(A,B)P^{i_1,i_2}_{\mu|_{(0,a)},\nu|_{(0,a)}}(A,B).
\end{aligned}
\tag{6.75}
$$

We see here that the set $\mathcal{G}'$ and the map $\mathcal{T}_{\mathcal{G}'}$ enable us to perform certain calculations much as we would do with the ordinary binomial theorem.

The third time-ordering theorem involves characteristic function sets that depend on the argument of another characteristic function. (The sets are $E_1(t)$ and $E_2(t)$, which depend on the argument t of the characteristic function $\chi_F(\underset{\cdot}{t})$.) It is similar to Theorem 6.3.4, in which the image of the map $\mathcal{T}_{\mathcal{G}'}$ is split into two factors to the left and right of a fixed value $a \in (0,1)$; here, however, the split occurs at a variable t, which is associated with an operator. In the final expression, that operator is between the split factors. (This theorem is closely related to a special case of [17, Theorem 3.6, p. 15].)

Theorem 6.3.5 (Third time-ordering theorem in $\mathcal{G}'$)**.** *Let $B, C_1, C_2, \ldots, C_m \in \mathcal{L}(X)$ be operators associated with measures $\mu, \nu_1, \nu_2, \ldots, \nu_m \in \mathcal{M}_{cb}[0,1]$ for an integer $m \geq 0$, let $F \subseteq (0,1)$ be a Borel set, and let $k \in \{1, 2, \ldots, m-1\}$. Further, for all $t \in [0,1]$ let $E_1(t) \subseteq (0,t)^k, E_2(t) \subseteq (t,1)^{m-k}$ be Borel sets. Then*

$$\begin{aligned}
&\mathcal{T}_{\mathcal{G}'}\Big[\chi_F(\underset{\cdot}{t})\Big(\chi_{E_2(t)}(\underset{\cdot}{s}_{k+1}, \ldots, \underset{\cdot}{s}_m)\tilde{C}_{k+1}(s_{k+1}) \cdots \tilde{C}_m(s_m) d\nu_{k+1}(s_{k+1}) \cdots d\nu_m(s_m)\Big)\tilde{B}(t) \\
&\quad \cdot \Big(\chi_{E_1(t)}(\underset{\cdot}{s}_1, \ldots, \underset{\cdot}{s}_k)\tilde{C}_1(s_1) \cdots \tilde{C}_k(s_k) d\nu_1(s_1) \cdots d\nu_k(s_k)\Big) d\mu(t)\Big] \\
&= \int_F \mathcal{T}_{\mathcal{G}'}\Big[\chi_{E_2(t)}(\underset{\cdot}{s}_{k+1}, \ldots, \underset{\cdot}{s}_m)\tilde{C}_{k+1}(s_{k+1}) \cdots \tilde{C}_m(s_m) d\nu_{k+1}(s_{k+1}) \cdots d\nu_m(s_m)\Big] B(t) \\
&\quad \times \mathcal{T}_{\mathcal{G}'}\Big[\chi_{E_1(t)}(\underset{\cdot}{s}_1, \ldots, \underset{\cdot}{s}_k)\tilde{C}_1(s_1) \cdots \tilde{C}_k(s_k) d\nu_1(s_1) \cdots d\nu_k(s_k)\Big]\mu(dt). \qquad (6.76)
\end{aligned}$$

(The statement will hold as well if $k = 0$ and no set $E_1(t)$ is hypothesized, or if $k = m$ and no set $E_2(t)$ is hypothesized, or both.)

Proof. The method of proof is similar to that of Theorem 6.3.4. Under the given assumptions we let $G := \{(s_1, \ldots, s_k, s_{k+1}, \ldots, s_m, t) \in (0,1)^{m+1} : (s_1, \ldots, s_k) \in E_1(t), (s_{k+1}, \ldots, s_m) \in E_2(t), \text{ and } t \in F\}$. Then

$$\begin{aligned}
&\mathcal{T}_{\mathcal{G}'}\Big[\chi_F(\underset{\cdot}{t})\Big(\chi_{E_2(t)}(\underset{\cdot}{s}_{k+1}, \ldots, \underset{\cdot}{s}_m)\tilde{C}_{k+1}(s_{k+1}) \cdots \tilde{C}_m(s_m) d\nu_{k+1}(s_{k+1}) \cdots d\nu_m(s_m)\Big)\tilde{B}(t) \\
&\qquad \cdot \Big(\chi_{E_1(t)}(\underset{\cdot}{s}_1, \ldots, \underset{\cdot}{s}_k)\tilde{C}_1(s_1) \cdots \tilde{C}_k(s_k) d\nu_1(s_1) \cdots d\nu_k(s_k)\Big) d\mu(t)\Big]
\end{aligned}$$

$$= \mathcal{T}_{\mathcal{G}'}\Big[\chi_G(\underset{\cdot}{s}_1,\ldots,\underset{\cdot}{s}_k,\underset{\cdot}{s}_{k+1},\ldots,\underset{\cdot}{s}_m,\underset{\cdot}{t})\tilde{C}_{k+1}(s_{k+1})\cdots\tilde{C}_m(s_m)\tilde{B}(t)\tilde{C}_1(s_1)\cdots\tilde{C}_k(s_k)$$
$$\cdot\, d\nu_1(s_1)\cdots d\nu_k(s_k)d\nu_{k+1}(s_{k+1})\cdots d\nu_m(s_m)d\mu(t)\Big]. \tag{6.77}$$

Given any $\tau \in \mathcal{O}_{\{k+1,\ldots,m\}}$, define as before $\Delta_{k+1,m}(\tau) := \{(s_{k+1},\ldots,s_m) : 0 < s_{\tau(1)} < \ldots < s_{\tau(m-k)} < 1\}$. Then up to a set of $\nu_1\times\cdots\times\nu_k\times\nu_{k+1}\times\cdots\times\nu_m\times\mu$-measure zero we have

$$G = G\cap\big[(0,1)^{m+1}\big] = G\cap\left[\left(\dot{\bigcup_{\sigma\in\mathcal{O}\{1,\ldots,k\}}}\Delta_k(\sigma)\right)\times\left(\dot{\bigcup_{\tau\in\mathcal{O}_{\{k+1,\ldots,m\}}}}\Delta_{k+1,m}(\tau)\right)\times(0,1)\right]$$
$$= \dot{\bigcup_{\substack{\sigma\in\mathcal{O}\{1,\ldots,k\}\\ \tau\in\mathcal{O}_{\{k+1,\ldots,m\}}}}}\Big[G\cap\Big(\Delta_k(\sigma)\times\Delta_{k+1,m}(\tau)\times(0,1)\Big)\Big]. \tag{6.78}$$

Consequently, by Theorem 6.3.1 we have

$$\mathcal{T}_{\mathcal{G}'}\Big[\chi_F(\underset{\cdot}{t})\left(\chi_{E_2(\underset{\cdot}{t})}(\underset{\cdot}{s}_{k+1},\ldots,\underset{\cdot}{s}_m)\tilde{C}_{k+1}(s_{k+1})\cdots\tilde{C}_m(s_m)d\nu_{k+1}(s_{k+1})\cdots d\nu_m(s_m)\right)\tilde{B}(t)$$
$$\cdot\left(\chi_{E_1(\underset{\cdot}{t})}(\underset{\cdot}{s}_1,\ldots,\underset{\cdot}{s}_k)\tilde{C}_1(s_1)\cdots\tilde{C}_k(s_k)d\nu_1(s_1)\cdots d\nu_k(s_k)\right)d\mu(t)\Big]$$
$$= \sum_{\substack{\sigma\in\mathcal{O}\{1,\ldots,k\}\\ \tau\in\mathcal{O}_{\{k+1,\ldots,m\}}}}\mathcal{T}_{\mathcal{G}'}\Big[\chi_{G\cap[\Delta_k(\sigma)\times\Delta_{k+1,m}(\tau)\times(0,1)]}(\underset{\cdot}{s}_1,\ldots,\underset{\cdot}{s}_k,\underset{\cdot}{s}_{k+1},\ldots,\underset{\cdot}{s}_m,\underset{\cdot}{t})$$
$$\cdot\,\tilde{C}_{k+1}(s_{k+1})\cdots\tilde{C}_m(s_m)\tilde{B}(t)\tilde{C}_1(s_1)\cdots\tilde{C}_k(s_k)$$
$$\cdot\, d\nu_1(s_1)\cdots d\nu_k(s_k)d\nu_{k+1}(s_{k+1})\cdots d\nu_m(s_m)d\mu(t)\Big]$$
$$= \sum_{\substack{\sigma\in\mathcal{O}\{1,\ldots,k\}\\ \tau\in\mathcal{O}_{\{k+1,\ldots,m\}}}}\mathcal{T}_{\mathcal{G}'}\Big[\chi_{G\cap[\Delta_k(\sigma)\times\Delta_{k+1,m}(\tau)\times(0,1)]}(\underset{\cdot}{s}_1,\ldots,\underset{\cdot}{s}_k,\underset{\cdot}{s}_{k+1},\ldots,\underset{\cdot}{s}_m,\underset{\cdot}{t})$$
$$\cdot\,\tilde{C}_{\tau(m-k)}(s_{\tau(m-k)})\cdots\tilde{C}_{\tau(1)}(s_{\tau(1)})\tilde{B}(t)\tilde{C}_{\sigma(k)}(s_{\sigma(k)})\cdots\tilde{C}_{\sigma(1)}(s_{\sigma(1)})$$
$$\cdot\, d\nu_1(s_1)\cdots d\nu_k(s_k)d\nu_{k+1}(s_{k+1})\cdots d\nu_m(s_m)d\mu(t)\Big]. \tag{6.79}$$

Now, if $(s_1, \ldots, s_k, s_{k+1}, \ldots, s_m, t) \in G \cap [\Delta_k(\sigma) \times \Delta_{k+1,m}(\tau) \times (0,1)]$ for some $\sigma \in \mathcal{O}_{\{1,\ldots,k\}}, \tau \in \mathcal{O}_{\{k+1,\ldots,m\}}$, then by definition of G,

$$(s_1, \ldots, s_k) \in E_1(t) \subseteq (0,t), \quad \text{and} \quad (s_{k+1}, \ldots, s_m) \in E_2(t) \subseteq (t,1); \tag{6.80}$$

thus $s_i < t$ for $i = 1, \ldots, k$, and $t < s_i$ for $i = k+1, \ldots, m$. Furthermore, $0 < s_{\sigma(1)} < \cdots < s_{\sigma(k)} < 1$ and $0 < s_{\tau(1)} < \cdots < s_{\tau(m-k)} < 1$. Together these yield that $0 < s_{\sigma(1)} < \cdots < s_{\sigma(k)} < t < s_{\tau(1)} < \cdots < s_{\tau(m-k)} < 1$. Hence we may apply Theorem 6.3.3 to obtain that

$$\begin{aligned}
&\mathcal{T}_{\mathcal{G}'}\Big[\chi_F(\underset{\cdot}{t})\left(\chi_{E_2(\underset{\cdot}{t})}(\underset{\cdot}{s}_{k+1}, \ldots, \underset{\cdot}{s}_m)\tilde{C}_{k+1}(s_{k+1}) \cdots \tilde{C}_m(s_m) d\nu_{k+1}(s_{k+1}) \cdots d\nu_m(s_m)\right) \tilde{B}(t) \\
&\quad \cdot \left(\chi_{E_1(\underset{\cdot}{t})}(\underset{\cdot}{s}_1, \ldots, \underset{\cdot}{s}_k)\tilde{C}_1(s_1) \cdots \tilde{C}_k(s_k) d\nu_1(s_1) \cdots d\nu_k(s_k)\right) d\mu(t)\Big] \\
&= \sum_{\substack{\sigma \in \mathcal{O}\{1,\ldots,k\} \\ \tau \in \mathcal{O}_{\{k+1,\ldots,m\}}}} \int_{G \cap [\Delta_k(\sigma) \times \Delta_{k+1,m}(\tau) \times (0,1)]} C_{\tau(m-k)} \cdots C_{\tau(1)} B C_{\sigma(k)} \cdots C_{\sigma(1)} \\
&\quad \times (\nu_1 \times \cdots \times \nu_k \times \nu_{k+1} \times \cdots \times \nu_m \times \mu)(ds_1, \ldots, ds_k, ds_{k+1}, \ldots, ds_m, dt) \\
&= \sum_{\substack{\sigma \in \mathcal{O}\{1,\ldots,k\} \\ \tau \in \mathcal{O}_{\{k+1,\ldots,m\}}}} \int_{\Delta_k(\sigma) \times \Delta_{k+1,m}(\tau) \times (0,1)} \chi_G(s_1, \ldots, s_k, s_{k+1}, \ldots, s_m, t) C_{\tau(m-k)} \cdots C_{\tau(1)} B C_{\sigma(k)} \cdots C_{\sigma(1)} \\
&\quad \times (\nu_1 \times \cdots \times \nu_k \times \nu_{k+1} \times \cdots \times \nu_m \times \mu)(ds_1, \ldots, ds_k, ds_{k+1}, \ldots, ds_m, dt) \\
&= \sum_{\substack{\sigma \in \mathcal{O}\{1,\ldots,k\} \\ \tau \in \mathcal{O}_{\{k+1,\ldots,m\}}}} \int_{\Delta_k(\sigma) \times \Delta_{k+1,m}(\tau) \times (0,1)} \chi_{E_1(t)}(s_1, \ldots, s_k) \chi_{E_2(t)}(s_{k+1}, \ldots, s_m) \chi_F(t) \\
&\quad \times C_{\tau(m-k)} \cdots C_{\tau(1)} B C_{\sigma(k)} \cdots C_{\sigma(1)} \\
&\quad \times (\nu_1 \times \cdots \times \nu_k \times \nu_{k+1} \times \cdots \times \nu_m \times \mu)(ds_1, \ldots, ds_k, ds_{k+1}, \ldots, ds_m, dt).
\end{aligned} \tag{6.81}$$

Applying Fubini-Tonelli gives

$$\mathcal{T}_{\mathcal{G}'}\Big[\chi_F(\underset{\cdot}{t})\left(\chi_{E_2(t)}(\underset{\cdot}{s}_{k+1}, \ldots, \underset{\cdot}{s}_m)\tilde{C}_{k+1}(s_{k+1}) \cdots \tilde{C}_m(s_m) d\nu_{k+1}(s_{k+1}) \cdots d\nu_m(s_m)\right) \tilde{B}(t)$$

$$\cdot \Big(\chi_{E_1(t)}(\underset{\cdot}{s}_1,\ldots,\underset{\cdot}{s}_k)\tilde{C}_1(s_1)\cdots\tilde{C}_k(s_k)d\nu_1(s_1)\cdots d\nu_k(s_k)\Big)\, d\mu(t)\Big]$$

$$= \sum_{\substack{\sigma\in\mathcal{O}\{1,\ldots,k\}\\ \tau\in\mathcal{O}_{\{k+1,\ldots,m\}}}} \int_{(0,1)}\int_{\Delta_{k+1,m}(\tau)}\int_{\Delta_k(\sigma)} \chi_{E_1(t)}(s_1,\ldots,s_k)\chi_{E_2(t)}(s_{k+1},\ldots,s_m)\chi_F(t)$$

$$\times\, C_{\tau(m-k)}\cdots C_{\tau(1)}BC_{\sigma(k)}\cdots C_{\sigma(1)}$$

$$\times\,(\nu_1\times\cdots\times\nu_k)(ds_1,\ldots,ds_k)(\nu_{k+1}\times\cdots\times\nu_m)(ds_{k+1},\ldots,ds_m)\mu(dt)$$

$$= \sum_{\substack{\sigma\in\mathcal{O}\{1,\ldots,k\}\\ \tau\in\mathcal{O}_{\{k+1,\ldots,m\}}}} \int_{(0,1)}\chi_F(t)\int_{\Delta_{k+1,m}(\tau)} \chi_{E_2(t)}(s_{k+1},\ldots,s_m)C_{\tau(m-k)}\cdots C_{\tau(1)}B$$

$$\times\int_{\Delta_k(\sigma)}\chi_{E_1(t)}(s_1,\ldots,s_k)C_{\sigma(k)}\cdots C_{\sigma(1)}$$

$$\times\,(\nu_1\times\cdots\times\nu_k)(ds_1,\ldots,ds_k)(\nu_{k+1}\times\cdots\times\nu_m)(ds_{k+1},\ldots,ds_m)\mu(dt)$$

$$= \int_F\Bigg(\sum_{\tau\in\mathcal{O}_{\{k+1,\ldots,m\}}}\int_{\Delta_{k+1,m}(\tau)}\chi_{E_2(t)}(s_{k+1},\ldots,s_m)C_{\tau(m-k)}\cdots C_{\tau(1)}$$

$$\times\,(\nu_{k+1}\times\cdots\times\nu_m)(ds_{k+1},\ldots,ds_m)\Bigg)B$$

$$\times\Bigg(\sum_{\sigma\in\mathcal{O}\{1,\ldots,k\}}\int_{\Delta_k(\sigma)}\chi_{E_1(t)}(s_1,\ldots,s_k)C_{\sigma(k)}\cdots C_{\sigma(1)}$$

$$\times\,(\nu_1\times\cdots\times\nu_k)(ds_1,\ldots,ds_k)\Bigg)\mu(dt)$$

$$= \int_F \mathcal{T}_{\mathcal{G}'}\Big[\chi_{E_2(t)}(\underset{\cdot}{s}_{k+1},\ldots,\underset{\cdot}{s}_m)\tilde{C}_{k+1}(s_{k+1})\cdots\tilde{C}_m(s_m)d\nu_{k+1}(s_{k+1})\cdots d\nu_m(s_m)\Big]B$$

$$\times\mathcal{T}_{\mathcal{G}'}\Big[\chi_{E_1(t)}(\underset{\cdot}{s}_1,\ldots,\underset{\cdot}{s}_k)\tilde{C}_1(s_1)\cdots\tilde{C}_k(s_k)d\nu_1(s_1)\cdots d\nu_k(s_k)\Big]\mu(dt), \tag{6.82}$$

concluding the proof for $1\le k\le m$. The cases $k=0$ and $k=m$ can be proved by a similar treatment. □

Example 29. We will consider a simple example using Theorem 6.3.5. (We will achieve the same effect by a somewhat different means in Example 38 below.) Consider operators $A, B\in\mathcal{L}(X)$ and measures $\mu,\nu\in\mathcal{M}_{cb}[0,1]$. We are interested in calculating

$$P^{2,1}_{\mu,\nu}(A,B) = \mathcal{T}_{\mathcal{G}'}\left[\left(\chi_{(0,1)}\tilde{A}(s)d\mu(s)\right)^2 \chi_{(0,1)}\tilde{B}(t)d\nu(t)\right]$$
$$= \mathcal{T}_{\mathcal{G}'}\left[\chi_{(0,1)^3}(\underset{\cdot}{r},\underset{\cdot}{s},\underset{\cdot}{t})\tilde{A}(r)\tilde{A}(s)\tilde{B}(t)d\mu(r)d\mu(s)d\nu(t)\right]. \tag{6.83}$$

We note that

$$\chi_{(0,1)^3} \overset{\text{a.e.-}\mu^2\times\nu}{=} \chi_{\{(r,s,t)\in(0,1):\, r\in(0,t)\cup(t,1) \text{ and } s\in(0,t)\cup(t,1)\}}$$
$$= \chi_{\{(r,s,t)\in(0,1):\, r\in(0,t) \text{ and } s\in(0,t)\}} + \chi_{\{(r,s,t)\in(0,1):\, r\in(0,t) \text{ and } s\in(t,1)\}}$$
$$+ \chi_{\{(r,s,t)\in(0,1):\, r\in(t,1) \text{ and } s\in(0,t)\}} + \chi_{\{(r,s,t)\in(0,1):\, r\in(t,1) \text{ and } s\in(t,1)\}}, \tag{6.84}$$

so for $(r,s,t)\in(0,1)^3$ a.e.-$\mu^2\times\nu$,

$$\chi_{(0,1)^3}(r,s,t) = \chi_{(0,t)}(r)\chi_{(0,t)}(s)\chi_{(0,1)}(t) + \chi_{(0,t)}(r)\chi_{(t,1)}(s)\chi_{(0,1)}(t)$$
$$+ \chi_{(t,1)}(r)\chi_{(0,t)}(s)\chi_{(0,1)}(t) + \chi_{(t,1)}(r)\chi_{(t,1)}(s)\chi_{(0,1)}(t). \tag{6.85}$$

Hence by Theorems 6.3.1 and 6.3.5,

$$P^{2,1}_{\mu,\nu}(A,B) = \mathcal{T}_{\mathcal{G}'}\left[\chi_{(0,t)}(\underset{\cdot}{r})\chi_{(0,t)}(\underset{\cdot}{s})\chi_{(0,1)}(\underset{\cdot}{t})\tilde{A}(r)\tilde{A}(s)\tilde{B}(t)d\mu(r)d\mu(s)d\nu(t)\right]$$
$$+ \mathcal{T}_{\mathcal{G}'}\left[\chi_{(0,t)}(\underset{\cdot}{r})\chi_{(t,1)}(\underset{\cdot}{s})\chi_{(0,1)}(\underset{\cdot}{t})\tilde{A}(r)\tilde{A}(s)\tilde{B}(t)d\mu(r)d\mu(s)d\nu(t)\right]$$
$$+ \mathcal{T}_{\mathcal{G}'}\left[\chi_{(t,1)}(\underset{\cdot}{r})\chi_{(0,t)}(\underset{\cdot}{s})\chi_{(0,1)}(\underset{\cdot}{t})\tilde{A}(r)\tilde{A}(s)\tilde{B}(t)d\mu(r)d\mu(s)d\nu(t)\right]$$
$$+ \mathcal{T}_{\mathcal{G}'}\left[\chi_{(t,1)}(\underset{\cdot}{r})\chi_{(t,1)}(\underset{\cdot}{s})\chi_{(0,1)}(\underset{\cdot}{t})\tilde{A}(r)\tilde{A}(s)\tilde{B}(t)d\mu(r)d\mu(s)d\nu(t)\right]$$

$$= \mathcal{T}_{\mathcal{G}'}\left[\chi_{(0,1)}(\underset{\cdot}{t})\tilde{B}(t)\left(\chi_{(0,t)^2}(\underset{\cdot}{r},\underset{\cdot}{s})\tilde{A}(r)\tilde{A}(s)d\mu(r)d\mu(s)\right)d\nu(t)\right]$$
$$+ \mathcal{T}_{\mathcal{G}'}\left[\chi_{(0,1)}(\underset{\cdot}{t})\left(\chi_{(t,1)}(\underset{\cdot}{s})\tilde{A}(s)d\mu(s)\right)\tilde{B}(t)\left(\chi_{(0,t)}(\underset{\cdot}{r})\tilde{A}(r)d\mu(r)\right)d\nu(t)\right]$$

$$+ \mathcal{T}_{\mathcal{G}'} \left[\chi_{(0,1)}(\underset{\cdot}{t}) \left(\chi_{(t,1)}(\underset{\cdot}{r}) \tilde{A}(r) d\mu(r) \right) \tilde{B}(t) \left(\chi_{(0,t)}(\underset{\cdot}{s}) \tilde{A}(s) d\mu(s) \right) d\nu(t) \right]$$
$$+ \mathcal{T}_{\mathcal{G}'} \left[\chi_{(0,1)}(\underset{\cdot}{t}) \left(\chi_{(t,1)^2}(\underset{\cdot}{r}, \underset{\cdot}{s}) \tilde{A}(r) \tilde{A}(s) d\mu(r) d\mu(s) \right) \tilde{B}(t) d\nu(t) \right]$$

$$\begin{aligned}
= &\int_{(0,1)} B\, \mathcal{T}_{\mathcal{G}'} \left[\chi_{(0,t)^2}(\underset{\cdot}{r}, \underset{\cdot}{s}) \tilde{A}(r) \tilde{A}(s) d\mu(r) d\mu(s) \right] \nu(dt) \\
&+ \int_{(0,1)} \mathcal{T}_{\mathcal{G}'} \left[\chi_{(t,1)}(\underset{\cdot}{s}) \tilde{A}(s) d\mu(s) \right] B\, \mathcal{T}_{\mathcal{G}'} \left[\chi_{(0,t)}(\underset{\cdot}{r}) \tilde{A}(r) d\mu(r) \right] \nu(dt) \\
&+ \int_{(0,1)} \mathcal{T}_{\mathcal{G}'} \left[\chi_{(t,1)}(\underset{\cdot}{r}) \tilde{A}(r) d\mu(r) \right] B\, \mathcal{T}_{\mathcal{G}'} \left[\chi_{(0,t)}(\underset{\cdot}{s}) \tilde{A}(s) d\mu(s) \right] \nu(dt) \\
&+ \int_{(0,1)} \mathcal{T}_{\mathcal{G}'} \left[\chi_{(t,1)^2}(\underset{\cdot}{r}, \underset{\cdot}{s}) \tilde{A}(r) \tilde{A}(s) d\mu(r) d\mu(s) \right] B\, \nu(dt). \qquad (6.86)
\end{aligned}$$

6.4 The intermediate disentangling space $\mathbb{E}$

We are now almost prepared to use elements of the set $\mathcal{G}'$ to define the space $\mathbb{E}$, after which we will define the intermediate disentangling map $\mathcal{T}_{\mathbb{E}}$ on $\mathbb{E}$ that corresponds to the map $\mathcal{T}_{\mathcal{G}'}$ on $\mathcal{G}'$. However, we will define $\mathbb{E}$ as a quotient space, so we need to first define the vector space $\mathcal{V}$ and the subspace $\mathcal{V}'$ that will form the quotient. Also, we will define $\mathcal{V}$ using only some, not all, of the elements of $\mathcal{G}'$.

Definition 6.4.1 (The set $\mathcal{G}$ and the space $\mathcal{V}$)**.** Let $\mathcal{G} \subseteq \mathcal{G}'$ be the set of elements of $\mathcal{G}'$ of the form

$$\chi_F(\underset{\cdot}{s}_1, \ldots, \underset{\cdot}{s}_m) \tilde{C}_1(s_1) \cdots \tilde{C}_m(s_m) d\nu_1(s_1) \cdots d\nu_m(s_m) \qquad (6.87)$$

for all nonnegative integers m, and all selections of nonzero operators $C_1, \ldots, C_m \in \mathcal{L}(X)$ and measures $\nu_1, \ldots, \nu_m \in \mathcal{M}_{cb}[0,1]$, for which F is a Borel set with $F \subseteq \Delta_m(e_m)$, where '$e_m$' here (and in what follows) refers to the identity permutation $e_m \in S_m$ (so $\Delta_m(e_m) = \{(s_1, \ldots, s_m) : 0 < s_1 < \cdots < s_m < 1\}$). (Again, for $m = 0$

the expression (6.87) is taken to equal 1.)

We then define $\mathcal{V}$ to be the free module over $\mathbb{C}$ (and hence free vector space) having basis $\mathcal{G}$. (For the definition and introductory concepts of free modules, see [32, p. 135].)

Remark 26. When dealing with the set $\mathcal{G}'$, we have avoided expressing elements of $\mathcal{G}'$ as sums of other elements, since $\mathcal{G}'$ is a set and does not have any operations defined on it. We did that even though we might have been inclined to think of certain elements as sums. For example, because we can split up a characteristic function and write $\chi_{(0,1)}(\underset{.}{s})\tilde{A}(s)d\mu(s) \in \mathcal{G}'$ as $[\chi_{(0,\frac{1}{2})} + \chi_{(\frac{1}{2},0)}](\underset{.}{s})\tilde{A}(s)d\mu(s) \in \mathcal{G}'$, we might have wanted to write that element as $\chi_{(0,\frac{1}{2})}(\underset{.}{s})\tilde{A}(s)d\mu(s) + \chi_{(\frac{1}{2},0)}(\underset{.}{s})\tilde{A}(s)d\mu(s)$, but we have avoided doing so. The reason is that we wanted to put off speaking of the sum of two elements until we have a vector space in which a sum is defined. Now that we have the vector space $\mathcal{V}$, if we write a sum of two elements, then we will always interpret it as the vector space sum. Consequently, the expression $\chi_{(0,\frac{1}{2})}(\underset{.}{s})\tilde{A}(s)d\mu(s)+$ $\chi_{(\frac{1}{2},0)}(\underset{.}{s})\tilde{A}(s)d\mu(s) \in \mathcal{V}$ does not refer to the single vector $\chi_{(0,1)}(\underset{.}{s})\tilde{A}(s)d\mu(s) \in \mathcal{V}$; it refers to a sum of two other vectors. (Eventually we will want to consider those to be equal in the space $\mathbb{E}$, which we will accomplish by defining certain equivalence classes of elements of $\mathcal{V}$ as cosets of a subspace $\mathcal{V}'$, and defining $\mathbb{E}$ to be the quotient space $\mathcal{V}/\mathcal{V}'$.)

A second important thing to note about the sum of two vectors, or more generally about a linear combination of vectors in $\mathcal{V}$, is that the arguments of variables and the corresponding labels attached to indeterminates are independent from one term to the next. Thus, for example, if we write

$$2\chi_{(0,1)}(\underset{.}{t})\tilde{A}(t)d\mu(t) + 3\chi_{(0,1)}(\underset{.}{t})\tilde{B}(t)d\nu(t), \tag{6.88}$$

then the occurrences of 't' and '$\underset{\cdot}{t}$' in the first term are unrelated to the occurrences in the second term.

Also using this example we observe that as we have defined the elements of $\mathcal{V}$, the scalar coefficients 2 and 3 in the expression (6.88) are multiplied by the entire element of $\mathcal{G}$, not just by the respective characteristic functions (since $\mathcal{V}$ is a space of linear combinations of elements of $\mathcal{G}$). However, it is easy to make sense of a scalar multiple of a characteristic function, and later our definition of $\mathbb{E}$ will be designed in a way that makes the two such scalar multiplications equivalent.

Although $\mathcal{V}$ has been defined so that the characteristic functions sets are subsets of $\Delta_m(e_m)$ for some $m \geq 0$, since $\mathcal{V}$ was generated by $\mathcal{G} \subseteq \mathcal{G}'$, we may still permute arguments as before, so that if

$$\chi_F(\underset{\cdot}{s}_1, \ldots, \underset{\cdot}{s}_m)\tilde{C}_1(s_1) \cdots \tilde{C}_m(s_m) d\nu_1(s_1) \cdots d\nu_m(s_m) \in \mathcal{G} \subseteq \mathcal{V} \tag{6.89}$$

with $F \subseteq \Delta_m(e_m)$, then we may still say for any $\sigma \in S_m$ that

$$\begin{aligned} &\chi_F(\underset{\cdot}{s}_1, \ldots, \underset{\cdot}{s}_m)\tilde{C}_1(s_1) \cdots \tilde{C}_m(s_m) d\nu_1(s_1) \cdots d\nu_m(s_m) \\ &\qquad = \chi_{F^\sigma}(\underset{\cdot}{s}_{\sigma(1)}, \ldots, \underset{\cdot}{s}_{\sigma(m)})\tilde{C}_1(s_1) \cdots \tilde{C}_m(s_m) d\nu_1(s_1) \cdots d\nu_m(s_m) \in \mathcal{G} \subseteq \mathcal{V}. \end{aligned} \tag{6.90}$$

This means that the characteristic function sets for elements of $\mathcal{V}$ will always be subsets of $\Delta_m(\sigma)$ for some $\sigma \in S_m$. We would like to expand these expressions to include characteristic function sets that are arbitrary Borel subsets of $[0, 1]^m$ for any $m \geq 0$. (This does not add elements to $\mathcal{V}$; it just adds to the expressions allowed for representing elements of $\mathcal{V}$.)

Definition 6.4.2 (Denoting elements of $\mathcal{V}$ using more general characteristic functions)**.** Let $E \subseteq [0, 1]^m$ be a Borel set for some $m \geq 0$, let $C_1, \ldots, C_m \in \mathcal{L}(X)$, and let $\nu_1, \ldots, \nu_m \in \mathcal{M}_{cb}[0,1]$. Then we define

$$\chi_E(\underset{\cdot}{s}_1, \ldots, \underset{\cdot}{s}_m)\tilde{C}_1(s_1)\cdots\tilde{C}_m(s_m)d\nu_1(s_1)\cdots d\nu_m(s_m) \in \mathcal{V} \tag{6.91}$$

by

$$\begin{aligned}
&\chi_E(\underset{\cdot}{s}_1, \ldots, \underset{\cdot}{s}_m)\tilde{C}_1(s_1)\cdots\tilde{C}_m(s_m)d\nu_1(s_1)\cdots d\nu_m(s_m) \\
&\quad := \sum_{\pi \in S_m} \left[\chi_{E^\pi \cap \Delta_m(e_m)}(\underset{\cdot}{s}_1, \ldots, \underset{\cdot}{s}_m)^\pi \tilde{C}_1(s_1)\cdots\tilde{C}_m(s_m)d\nu_1(s_1)\cdots d\nu_m(s_m)\right] \\
&\quad = \sum_{\pi \in S_m} \left[\chi_{E^\pi \cap \Delta_m(e_m)}(\underset{\cdot}{s}_{\pi(1)}, \ldots, \underset{\cdot}{s}_{\pi(m)})\tilde{C}_1(s_1)\cdots\tilde{C}_m(s_m)d\nu_1(s_1)\cdots d\nu_m(s_m)\right].
\end{aligned} \tag{6.92}$$

(For $m = 0$ there is a single summand, regarded as equal to 1.)

Remark 27. We need to verify that the above is well-defined, where our concern is that the arguments of the characteristic function could be permuted by some $\sigma \in S_m$ without changing the expression on the left, and we want to make sure it doesn't change the expression on the right. In fact, using Definition 6.4.2 in that case, we would have in $\mathcal{V}$ that

$$\begin{aligned}
&\chi_{E^\sigma}(\underset{\cdot}{s}_{\sigma(1)}, \ldots, \underset{\cdot}{s}_{\sigma(m)})\tilde{C}_1(s_1)\cdots\tilde{C}_m(s_m)d\nu_1(s_1)\cdots d\nu_m(s_m) \\
&\quad = \sum_{\pi \in S_m} \left[\chi_{(E^\sigma)^\pi \cap \Delta_m(e_m)}(\underset{\cdot}{s}_{\sigma(1)}, \ldots, \underset{\cdot}{s}_{\sigma(m)})^\pi \tilde{C}_1(s_1)\cdots\tilde{C}_m(s_m)d\nu_1(s_1)\cdots d\nu_m(s_m)\right] \\
&\quad = \sum_{\sigma\pi \in S_m} \left[\chi_{E^{\sigma\pi} \cap \Delta_m(e_m)}(\underset{\cdot}{s}_{\sigma\pi(1)}, \ldots, \underset{\cdot}{s}_{\sigma\pi(m)})\tilde{C}_1(s_1)\cdots\tilde{C}_m(s_m)d\nu_1(s_1)\cdots d\nu_m(s_m)\right] \\
&\quad = \sum_{\rho \in S_m} \left[\chi_{E^\rho \cap \Delta_m(e_m)}(\underset{\cdot}{s}_{\rho(1)}, \ldots, \underset{\cdot}{s}_{\rho(m)})\tilde{C}_1(s_1)\cdots\tilde{C}_m(s_m)d\nu_1(s_1)\cdots d\nu_m(s_m)\right] \\
&\quad = \chi_E(\underset{\cdot}{s}_1, \ldots, \underset{\cdot}{s}_m)\tilde{C}_1(s_1)\cdots\tilde{C}_m(s_m)d\nu_1(s_1)\cdots d\nu_m(s_m).
\end{aligned} \tag{6.93}$$

Hence the defined expression is well-defined.

As a consequence of this definition, we are able to use all of the valid expressions in $\mathcal{G}'$ as valid expressions for single terms in $\mathcal{V}$, and they are equal in $\mathcal{G}'$ if and only if they are equal in $\mathcal{V}$. (That is true because the allowed differences in expression between single terms that are equal, in either space, consist of changing a characteristic function set by a set of measure zero or permuting indices. So equivalent expressions in one are equivalent expressions in the other.)

Example 30. Let $A, B \in \mathcal{L}(X)$, and let $\mu, \nu \in \mathcal{M}_{cb}[0,1]$. then

$$
\begin{aligned}
&\chi_{(0,1)^2}(\underset{\cdot}{s}, \underset{\cdot}{t})\tilde{A}(s)\tilde{B}(t)d\mu(s)d\nu(t) \\
&\quad = \chi_{\{0<s<t<1\}}(\underset{\cdot}{s}, \underset{\cdot}{t})\tilde{A}(s)\tilde{B}(t)d\mu(s)d\nu(t) + \chi_{\{0<t<s<1\}}(\underset{\cdot}{t}, \underset{\cdot}{s})\tilde{A}(s)\tilde{B}(t)d\mu(s)d\nu(t) \in \mathcal{V}.
\end{aligned}
\tag{6.94}
$$

Note that in the case where $E \subseteq \Delta_m(\sigma)$ for some $\sigma \in S_m$, the definition is consistent with itself, saying nothing new.

Before we go on, we will define a multiplication on $\mathcal{V}$ (because we need a multiplication on $\mathbb{E}$, and it is easier to define here first). This will make $\mathcal{V}$ an algebra.

Definition 6.4.3 (Multiplication in $\mathcal{V}$)**.** Let $v, w \in \mathcal{V}$. In the case that v, w are basis vectors, that is, when $v, w \in \mathcal{G}$, let us say that

$$
\begin{aligned}
v &= \chi_{F_1}(\underset{\cdot}{s}_1, \dots, \underset{\cdot}{s}_m)\tilde{A}(s_1)\cdots\tilde{A}(s_m)d\mu_1(s_1)\cdots d\mu_m(s_m), \\
w &= \chi_{F_2}(\underset{\cdot}{t}_1, \dots, \underset{\cdot}{t}_n)\tilde{B}(t_1)\cdots\tilde{B}(t_m)d\nu_1(t_1)\cdots d\nu_n(t_n),
\end{aligned}
\tag{6.95}
$$

where $A_1, \dots, A_m, B_1, \dots, B_n \in \mathcal{L}(X)$ are operators, $\mu_1, \dots, \mu_m, \nu_1, \dots, \nu_n \in \mathcal{M}_{cb}[0,1]$ are measures, and $F_1 \subseteq \Delta_m(e_m)$, $F_2 \subseteq \Delta_n(e_n)$ are Borel sets. Then we define the product $vw \in \mathcal{V}$ to be

$$\begin{aligned} vw &:= \chi_{F_1}(\underset{\cdot}{s}_1, \ldots, \underset{\cdot}{s}_m)\tilde{A}(s_1)\cdots\tilde{A}(s_m)d\mu_1(s_1)\cdots d\mu_m(s_m) \\ &\qquad\qquad\qquad \cdot \chi_{F_2}(\underset{\cdot}{t}_1, \ldots, \underset{\cdot}{t}_n)B(t_1)\cdots\tilde{B}(t_m)d\nu_1(t_1)\cdots d\nu_n(t_n) \\ &= \chi_{F_1\times F_2}(\underset{\cdot}{s}_1, \ldots, \underset{\cdot}{s}_m, \underset{\cdot}{t}_1, \ldots, \underset{\cdot}{t}_n)\tilde{A}(s_1)\cdots\tilde{A}(s_m)B(t_1)\cdots\tilde{B}(t_m) \\ &\qquad\qquad\qquad \cdot d\mu_1(s_1)\cdots d\mu_m(s_m)d\nu_1(t_1)\cdots d\nu_n(t_n). \end{aligned} \tag{6.96}$$

In particular, the element $1 \in \mathcal{V}$ acts as a multiplicative identity.

On the other hand, in the case v and w are not necessarily basis vectors, then say they are expressed in terms of the basis vectors $\mathcal{G}$ of $\mathcal{V}$ as

$$v = \sum_{i=1}^{M} a_i g_i, \qquad w = \sum_{j=1}^{N} b_j h_j \tag{6.97}$$

where $a_1, \ldots, a_M, b_1, \ldots, b_N \in \mathbb{C}$, where $g_1, \ldots, g_M \in \mathcal{G}$ are distinct, and where $h_1, \ldots, h_N \in \mathcal{G}$ are distinct. Then we define the product vw to be

$$vw := \sum_{i=1}^{M}\sum_{j=1}^{N} a_i b_j (g_i h_j). \tag{6.98}$$

Remark 28. It is necessary to show that multiplication in $\mathcal{V}$ is well-defined. In fact, the only ambiguity here, in the second part of the definition, is in how v and w are expressed as linear combinations of basis vectors, because we have not ruled out the possibility that some coefficients a_i or b_j might equal zero. But in fact, if any did equal zero, then the terms they multiply in the double summation would simply drop out, meaning they have no effect on the value assigned to vw. Thus multiplication in $\mathcal{V}$ is well-defined.

Theorem 6.4.4. *Multiplication in $\mathcal{V}$ is commutative.*

Proof. Given $v, w \in \mathcal{V}$, we want to show $vw = wv$. Let us first take the case where v and w are single terms ($v, w \in \mathcal{G}$), say

$$
\begin{aligned}
v &= \chi_{F_1}(\underset{\cdot}{s}_1, \ldots, \underset{\cdot}{s}_m)\tilde{A}(s_1)\cdots\tilde{A}(s_m)d\mu_1(s_1)\cdots d\mu_m(s_m), \\
w &= \chi_{F_2}(\underset{\cdot}{t}_1, \ldots, \underset{\cdot}{t}_n)\tilde{B}(t_1)\cdots\tilde{B}(t_m)d\nu_1(t_1)\cdots d\nu_n(t_n),
\end{aligned} \tag{6.99}
$$

as in the definition (Definition 6.4.3) above (with $F_1 \subseteq \Delta_m(e_m)$ and $F_2 \subseteq \Delta_n(e_n)$). Then by that definition and by the fact that we may commute characteristic functions and indeterminates in expressions in $\mathcal{G} \subseteq \mathcal{G}'$, we have

$$
\begin{aligned}
vw := \chi_{F_1}&(\underset{\cdot}{s}_1, \ldots, \underset{\cdot}{s}_m)\tilde{A}(s_1)\cdots\tilde{A}(s_m)d\mu_1(s_1)\cdots d\mu_m(s_m) \\
&\qquad\qquad\cdot \chi_{F_2}(\underset{\cdot}{t}_1, \ldots, \underset{\cdot}{t}_n)B(t_1)\cdots\tilde{B}(t_m)d\nu_1(t_1)\cdots d\nu_n(t_n) \\
= \chi_{F_2}&(\underset{\cdot}{t}_1, \ldots, \underset{\cdot}{t}_n)B(t_1)\cdots\tilde{B}(t_m)d\nu_1(t_1)\cdots d\nu_n(t_n) \\
&\qquad\qquad\cdot \chi_{F_1}(\underset{\cdot}{s}_1, \ldots, \underset{\cdot}{s}_m)\tilde{A}(s_1)\cdots\tilde{A}(s_m)d\mu_1(s_1)\cdots d\mu_m(s_m) \\
= wv&.
\end{aligned} \tag{6.100}
$$

For the general case, where arbitrary $v, w \in \mathcal{V}$ are expressed in terms of the basis vectors $\mathcal{G}$ of $\mathcal{V}$ as

$$
v = \sum_{i=1}^{M} a_i g_i, \qquad w = \sum_{j=1}^{N} b_j h_j \tag{6.101}
$$

where $a_1, \ldots, a_M, b_1, \ldots, b_N \in \mathbb{C}$, where $g_1, \ldots, g_M \in \mathcal{G}$ are distinct, and where $h_1, \ldots, h_N \in \mathcal{G}$ are distinct, we then have

$$
vw := \sum_{i=1}^{M}\sum_{j=1}^{N} a_i b_j (g_i h_j) = \sum_{j=1}^{N}\sum_{i=1}^{M} b_j a_i (h_j g_i) = wv. \tag{6.102}
$$

$\square$

Theorem 6.4.5. *In $\mathcal{V}$, scalar multiplication associates with vector multiplication, and vector multiplication distributes over addition; that is: Given $c \in \mathbb{C}$ and $v, w \in \mathcal{V}$, we have $c(vw) = (cv)w$. Given $u, v, w \in \mathcal{V}$, we have $u(v + w) = uv + uw$.*

Proof. For associativity of scalar multiplication with vector multiplication, let $c \in \mathbb{C}$ and $v, w \in \mathcal{V}$. Say $v = \sum_{i=1}^{M} a_i g_i$ and $w = \sum_{j=1}^{N} b_j h_j$, where $g_1, \ldots, g_m \in \mathcal{G}$ are distinct and $h_1, \ldots, h_n \in \mathcal{G}$ are distinct. Then by the definition of multiplication in $\mathcal{V}$ (twice) we have

$$\begin{aligned} c(vw) &= c\left[\left(\sum_{i=1}^{M} a_i g_i\right)\left(\sum_{j=1}^{N} b_j h_j\right)\right] = c\left[\sum_{i=1}^{M}\sum_{j=1}^{N} a_i b_j (g_i h_j)\right] \\ &= \sum_{i=1}^{M}\sum_{j=1}^{N} c\, a_i b_j (g_i h_j) = \left(\sum_{i=1}^{M} c\, a_i g_i\right)\left(\sum_{j=1}^{N} b_j h_j\right) \\ &= (cv)w. \end{aligned} \tag{6.103}$$

We may therefore write $c(vw) = (cv)w =: cvw$. By commutativity we may also say $v(cw) = (cw)v = c(wv) = c(vw) = cvw$.

For the distributive law, let $u, v, w \in \mathcal{V}$. Let us say $u = \sum_{i=1}^{M} a_i g_i$, and without loss of generality we may say $v = \sum_{j=1}^{N} b_j h_j$ and $w = \sum_{j=1}^{N} c_j h_j$, where $g_1, \ldots, g_m \in \mathcal{G}$ are distinct and $h_1, \ldots, h_n \in \mathcal{G}$ are distinct. Then applying vector space properties and the definition of multiplication in $\mathcal{V}$ (twice again) we have

$$\begin{aligned} u(v + w) &= \left(\sum_{i=1}^{M} a_i g_i\right)\left(\sum_{j=1}^{N} b_j h_j + \sum_{j=1}^{N} c_j h_j\right) = \left(\sum_{i=1}^{M} a_i g_i\right)\left[\sum_{j=1}^{N} (b_j + c_j) h_j\right] \\ &= \sum_{i=1}^{M}\sum_{j=1}^{N} a_i (b_j + c_j) g_i h_j = \sum_{i=1}^{M}\sum_{j=1}^{N} a_i b_j g_i h_j + \sum_{i=1}^{M}\sum_{j=1}^{N} a_i c_j g_i h_j \\ &= \left(\sum_{i=1}^{M} a_i g_i\right)\left(\sum_{j=1}^{N} b_j h_j\right) + \left(\sum_{i=1}^{M} a_i g_i\right)\left(\sum_{j=1}^{N} c_j h_j\right) \end{aligned}$$

$$= uv + uw. \tag{6.104}$$

□

We insert here an additional result about multiplication in $\mathcal{V}$ to make it more useful in the context in which we will be working, namely, when characteristic functions are arbitrary Borel subsets of $[0,1]^m$ for some $m \geq 0$.

Theorem 6.4.6 (Multiplication in $\mathcal{V}$ involving more general characteristic functions)**.** *Let $v, w \in \mathcal{V}$ be given by*

$$\begin{aligned} v &= \chi_{E_1}(\underset{\cdot}{s}_1, \ldots, \underset{\cdot}{s}_m)\tilde{A}_1(s_1)\cdots\tilde{A}_m(s_m)d\mu_1(s_1)\cdots d\mu_m(s_m), \\ w &= \chi_{E_2}(\underset{\cdot}{t}_1, \ldots, \underset{\cdot}{t}_n)\tilde{B}_1(t_1)\cdots\tilde{B}_n(t_n)d\nu_1(t_1)\cdots d\nu_n(t_n), \end{aligned} \tag{6.105}$$

where $E_1 \subseteq [0,1]^m$ and $E_2 \subseteq [0,1]^n$ are Borel sets, $A_1, \ldots, A_m, B_1, \ldots, B_n \in \mathcal{L}(X)$ are operators, and $\mu_1, \ldots, \mu_m, \nu_1, \ldots, \nu_n \in \mathcal{M}_{cb}[0,1]$ are measures. Then

$$\begin{aligned} vw &= \chi_{E_1}(\underset{\cdot}{s}_1, \ldots, \underset{\cdot}{s}_m)\tilde{A}_1(s_1)\cdots\tilde{A}_m(s_m)d\mu_1(s_1)\cdots d\mu_m(s_m) \\ &\qquad \cdot \chi_{E_2}(\underset{\cdot}{t}_1, \ldots, \underset{\cdot}{t}_n)\tilde{B}_1(t_1)\cdots\tilde{B}_n(t_n)d\nu_1(t_1)\cdots d\nu_n(t_n) \\ &= \chi_{E_1\times E_2}(\underset{\cdot}{s}_1, \ldots, \underset{\cdot}{s}_m, \underset{\cdot}{t}_1, \ldots, \underset{\cdot}{t}_n)\tilde{A}_1(s_1)\cdots\tilde{A}_m(s_m)\tilde{B}_1(t_1)\cdots\tilde{B}_n(t_n) \\ &\qquad \cdot d\mu_1(s_1)\cdots d\mu_m(s_m)d\nu_1(t_1)\cdots d\nu_n(t_n). \end{aligned} \tag{6.106}$$

Proof. To avoid a little confusion, we will temporarily represent the multiplication operation in $\mathcal{V}$ by an asterisk $(*)$. Given the hypotheses as stated, we have that

$$\begin{aligned} v * w = &\Big[\chi_{E_1}(\underset{\cdot}{s}_1, \ldots, \underset{\cdot}{s}_m)\tilde{A}_1(s_1)\cdots\tilde{A}_m(s_m)d\mu_1(s_1)\cdots d\mu_m(s_m)\Big] \\ &\qquad * \Big[\chi_{E_2}(\underset{\cdot}{t}_1, \ldots, \underset{\cdot}{t}_n)\tilde{B}_1(t_1)\cdots\tilde{B}_n(t_n)d\nu_1(t_1)\cdots d\nu_n(t_n)\Big] \end{aligned}$$

$$
\begin{aligned}
&= \left[\sum_{\sigma \in S_m} \chi_{E_1^\sigma \cap \Delta_m(e_m)}(\underset{\cdot}{s}_{\sigma(1)}, \ldots, \underset{\cdot}{s}_{\sigma(m)}) \tilde{A}_1(s_1) \cdots \tilde{A}_m(s_m) d\mu_1(s_1) \cdots d\mu_m(s_m) \right] \\
&\qquad * \left[\sum_{\tau \in S_n} \chi_{E_2^\tau \cap \Delta_n(e_n)}(\underset{\cdot}{t}_{\tau(1)}, \ldots, \underset{\cdot}{t}_{\tau(n)}) \tilde{B}_1(t_1) \cdots \tilde{B}_n(t_n) d\nu_1(t_1) \cdots d\nu_n(t_n) \right].
\end{aligned} \tag{6.107}
$$

Applying Definition 6.4.3 yields

$$
\begin{aligned}
v * w = &\sum_{\sigma \in S_m} \sum_{\tau \in S_n} \Big[\chi_{(E_1^\sigma \cap \Delta_m(e_m)) \times (E_2^\tau \cap \Delta_n(e_n))}(\underset{\cdot}{s}_{\sigma(1)}, \ldots, \underset{\cdot}{s}_{\sigma(m)}, \underset{\cdot}{t}_{\tau(1)}, \ldots, \underset{\cdot}{t}_{\tau(n)}) \\
&\quad \cdot \tilde{A}_1(s_1) \cdots \tilde{A}_m(s_m) \tilde{B}_1(t_1) \cdots \tilde{B}_n(t_n) d\mu_1(s_1) \cdots d\mu_m(s_m) d\nu_1(t_1) \cdots d\nu_n(t_n) \Big] \\
= &\sum_{\sigma \in S_m} \sum_{\tau \in S_n} \Big[\chi_{(E_1 \cap \Delta_m(\sigma))^\sigma \times (E_2 \cap \Delta_n(\tau))^\tau}(\underset{\cdot}{s}_{\sigma(1)}, \ldots, \underset{\cdot}{s}_{\sigma(m)}, \underset{\cdot}{t}_{\tau(1)}, \ldots, \underset{\cdot}{t}_{\tau(n)}) \\
&\quad \cdot \tilde{A}_1(s_1) \cdots \tilde{A}_m(s_m) \tilde{B}_1(t_1) \cdots \tilde{B}_n(t_n) d\mu_1(s_1) \cdots d\mu_m(s_m) d\nu_1(t_1) \cdots d\nu_n(t_n) \Big] \\
= &\sum_{\sigma \in S_m} \sum_{\tau \in S_n} \Big[\chi_{(E_1 \cap \Delta_m(\sigma)) \times (E_2 \cap \Delta_n(\tau))}(\underset{\cdot}{s}_1, \ldots, \underset{\cdot}{s}_m, \underset{\cdot}{t}_1, \ldots, \underset{\cdot}{t}_n) \\
&\quad \cdot \tilde{A}_1(s_1) \cdots \tilde{A}_m(s_m) \tilde{B}_1(t_1) \cdots \tilde{B}_n(t_n) d\mu_1(s_1) \cdots d\mu_m(s_m) d\nu_1(t_1) \cdots d\nu_n(t_n) \Big] \\
= &\sum_{\sigma \in S_m} \sum_{\tau \in S_n} \Big[\chi_{(E_1 \times E_2) \cap (\Delta_m(\sigma) \times \Delta_n(\tau))}(\underset{\cdot}{s}_1, \ldots, \underset{\cdot}{s}_m, \underset{\cdot}{t}_1, \ldots, \underset{\cdot}{t}_n) \\
&\quad \cdot \tilde{A}_1(s_1) \cdots \tilde{A}_m(s_m) \tilde{B}_1(t_1) \cdots \tilde{B}_n(t_n) d\mu_1(s_1) \cdots d\mu_m(s_m) d\nu_1(t_1) \cdots d\nu_n(t_n) \Big].
\end{aligned} \tag{6.108}
$$

We rename the variables $t_1, \ldots, t_n$ as $s_{m+1}, \ldots, s_{m+n}$, respectively, rewrite the index sets, and apply Definition 6.4.2, and the expression becomes

$$
\begin{aligned}
v * w = \sum_{\sigma \in \mathcal{O}_{\{1,\ldots,m\}}} \sum_{\rho \in \mathcal{O}_{\{m+1,\ldots,m+n\}}} &\Big[\chi_{(E_1 \times E_2) \cap (\Delta_m(\sigma) \times \Delta_{m+1,m+n}(\rho))}(\underset{\cdot}{s}_1, \ldots, \underset{\cdot}{s}_{m+n}) \\
&\cdot \tilde{A}_1(s_1) \cdots \tilde{A}_m(s_m) \tilde{B}_1(s_{m+1}) \cdots \tilde{B}_n(s_{m+n}) \\
&\cdot d\mu_1(s_1) \cdots d\mu_m(s_m) d\nu_1(s_{m+1}) \cdots d\nu_n(s_{m+n}) \Big]
\end{aligned}
$$

$$= \sum_{\substack{\sigma\in\mathcal{O}_{\{1,\ldots,m\}}\\ \rho\in\mathcal{O}_{\{m+1,\ldots,m+n\}}}} \sum_{\pi\in S_{m+n}} \Big[\chi_{[(E_1\times E_2)\cap(\Delta_m(\sigma)\times\Delta_{m+1,m+n}(\rho))]^\pi\cap\Delta_{m+n}(e_{m+n})}(s_1,\ldots,s_{m+n})^\pi$$

$$\cdot \tilde{A}_1(s_1)\cdots\tilde{A}_m(s_m)\tilde{B}_1(s_{m+1})\cdots\tilde{B}_n(s_{m+n})$$

$$\cdot\, d\mu_1(s_1)\cdots d\mu_m(s_m)d\nu_1(s_{m+1})\cdots d\nu_n(s_{m+n})\Big], \qquad (6.109)$$

where $\Delta_{m+1,m+n}(\rho) := \{(s_{m+1},\ldots,s_{m+n}) : 0 < s_{\rho(1)} < \ldots < s_{\rho(n)} < 1\}$. But then we notice that for all $\sigma\in\mathcal{O}_{\{1,\ldots,m\}}, \rho\in\mathcal{O}_{\{m+1,\ldots,m+n\}}$, up to a set of $\mu_1\times\cdots\times\mu_m\times\nu_1\times\cdots\times\nu_n$-measure zero,

$$\Delta_m(\sigma)\times\Delta_{m+1,m+n}(\rho)$$

$$= \{(s_1,\ldots,s_{m+n}) :\ 0 < s_{\sigma(1)} < \cdots < s_{\sigma(m)} < 1 \text{ and } 0 < s_{\rho(1)} < \cdots < s_{\rho(n)} < 1\}$$

$$= \dot{\bigcup_{\eta\in\{\sigma\}\odot\{\rho\}}} \{(s_1,\ldots,s_{m+n}) :\ 0 < s_{\eta(1)} < \cdots < s_{\eta(m+n)} < 1\}$$

$$= \dot{\bigcup_{\eta\in\{\sigma\}\odot\{\rho\}}} \Delta_{m+n}(\eta). \qquad (6.110)$$

This implies that

$$[\Delta_m(\sigma)\times\Delta_{m+1,m+n}(\rho)]^\pi\cap\Delta_{m+n}(e_{m+n}) = [\Delta_m(\sigma)\times\Delta_{m+1,m+n}(\rho)\cap\Delta_{m+n}(\pi)]^\pi$$

$$= \left[\left(\dot{\bigcup_{\eta\in\{\sigma\}\odot\{\rho\}}} \Delta_{m+n}(\eta)\right)\cap\Delta_{m+n}(\pi)\right]^\pi$$

$$= \begin{cases} [\Delta_{m+n}(\pi)]^{\pi} \text{ if } \pi \in \{\sigma\} \odot \{\rho\}, \\ \emptyset \quad \text{(the empty set) otherwise.} \end{cases} \tag{6.111}$$

Hence,

$$\begin{aligned} v * w = &\sum_{\substack{\sigma \in \mathcal{O}_{\{1,\ldots,m\}} \\ \rho \in \mathcal{O}_{\{m+1,\ldots,m+n\}}}} \sum_{\pi \in \{\sigma\} \odot \{\rho\}} \Big[\chi_{[(E_1 \times E_2) \cap \Delta_{m+n}(\pi)]^{\pi}} (\underset{\cdot}{s}_1, \ldots, \underset{\cdot}{s}_{m+n})^{\pi} \tilde{A}_1(s_1) \cdots \tilde{A}_m(s_m) \\ &\qquad \cdot \tilde{B}_1(s_{m+1}) \cdots \tilde{B}_n(s_{m+n}) d\mu_1(s_1) \cdots d\mu_m(s_m) d\nu_1(s_{m+1}) \cdots d\nu_n(s_{m+n}) \Big] \\ = &\sum_{\pi \in S_{m+n}} \Big[\chi_{(E_1 \times E_2)^{\pi} \cap \Delta_{m+n}(e_{m+n})} (\underset{\cdot}{s}_1, \ldots, \underset{\cdot}{s}_{m+n})^{\pi} \tilde{A}_1(s_1) \cdots \tilde{A}_m(s_m) \\ &\qquad \cdot \tilde{B}_1(s_{m+1}) \cdots \tilde{B}_n(s_{m+n}) d\mu_1(s_1) \cdots d\mu_m(s_m) d\nu_1(s_{m+1}) \cdots d\nu_n(s_{m+n}) \Big] \\ = &\chi_{E_1 \times E_2}(\underset{\cdot}{s}_1, \ldots, \underset{\cdot}{s}_{m+n}) \tilde{A}_1(s_1) \cdots \tilde{A}_m(s_m) \tilde{B}_1(s_{m+1}) \cdots \tilde{B}_n(s_{m+n}) \\ &\qquad \cdot d\mu_1(s_1) \cdots d\mu_m(s_m) d\nu_1(s_{m+1}) \cdots d\nu_n(s_{m+n}) \\ = &\chi_{E_1 \times E_2}(\underset{\cdot}{s}_1, \ldots, \underset{\cdot}{s}_m, \underset{\cdot}{t}_1, \ldots, \underset{\cdot}{t}_n) \tilde{A}_1(s_1) \cdots \tilde{A}_m(s_m) \tilde{B}_1(t_1) \cdots \tilde{B}_n(t_n) \\ &\qquad \cdot d\mu_1(s_1) \cdots d\mu_m(s_m) d\nu_1(t_1) \cdots d\nu_n(t_n) \\ = &\chi_{E_1}(\underset{\cdot}{s}_1, \ldots, \underset{\cdot}{s}_m) \tilde{A}_1(s_1) \cdots \tilde{A}_m(s_m) d\mu_1(s_1) \cdots d\mu_m(s_m) \\ &\qquad \cdot \chi_{E_2}(\underset{\cdot}{t}_1, \ldots, \underset{\cdot}{t}_n) \tilde{B}_1(t_1) \cdots \tilde{B}_n(t_n) d\nu_1(t_1) \cdots d\nu_n(t_n). \end{aligned} \tag{6.112}$$

$\square$

Theorem 6.4.7. *Multiplication in $\mathcal{V}$ is associative.*

Proof. Let $u, v, w \in \mathcal{V}$. First we consider the case where each is a basis vector,

$u, v, w \in \mathcal{G}$, say

$$
\begin{aligned}
u &= \chi_{F_1}(r_1, \ldots, r_m)\tilde{A}_1(r_1)\cdots\tilde{A}_m(r_m)d\mu_1(r_1)\cdots d\mu_m(r_m),\\
v &= \chi_{F_2}(s_1, \ldots, s_n)\tilde{B}_1(s_1)\cdots\tilde{B}_n(s_n)d\nu_1(s_1)\cdots d\nu_n(s_n),\\
w &= \chi_{F_3}(t_1, \ldots, t_p)\tilde{C}_1(t_1)\cdots\tilde{C}_p(t_p)d\eta_1(t_1)\cdots d\eta_p(t_p),
\end{aligned} \tag{6.113}
$$

where $m, n, p \geq 0$, $F_1 \subseteq \Delta_m(e_m), F_2 \subseteq \Delta_n(e_n), F_3 \subseteq \Delta_p(e_p)$ are Borel sets, $A_1, \ldots, A_m, B_1, \ldots, B_n, C_1, \ldots, C_p \in \mathcal{L}(X)$ are operators, and $\mu_1, \ldots, \mu_m, \nu_1, \ldots, \nu_n, \eta_1, \ldots, \eta_p \in \mathcal{M}_{cb}[0,1]$ are measures. Then by Theorem 6.4.6 we have (again temporarily using an asterisk '$*$' to represent multiplication in $\mathcal{V}$):

$$
\begin{aligned}
(u * v) * w = {} & \Big[\Big(\chi_{F_1}(r_1, \ldots, r_m)\tilde{A}_1(r_1)\cdots\tilde{A}_m(r_m)d\mu_1(r_1)\cdots d\mu_m(r_m)\Big)\\
& * \Big(\chi_{F_2}(s_1, \ldots, s_n)\tilde{B}_1(s_1)\cdots\tilde{B}_n(s_n)d\nu_1(s_1)\cdots d\nu_n(s_n)\Big)\Big]\\
& * \Big(\chi_{F_3}(t_1, \ldots, t_p)\tilde{C}_1(t_1)\cdots\tilde{C}_p(t_p)d\eta_1(t_1)\cdots d\eta_p(t_p)\Big)\\
= {} & \Big(\chi_{F_1\times F_2}(r_1, \ldots, r_m, s_1, \ldots, s_n)\tilde{A}_1(r_1)\cdots\tilde{A}_m(r_m)\\
& \cdot \tilde{B}_1(s_1)\cdots\tilde{B}_n(s_n)d\mu_1(r_1)\cdots d\mu_m(r_m)d\nu_1(s_1)\cdots d\nu_n(s_n)\Big)\\
& * \Big(\chi_{F_3}(t_1, \ldots, t_p)\tilde{C}_1(t_1)\cdots\tilde{C}_p(t_p)d\eta_1(t_1)\cdots d\eta_p(t_p)\Big)\\
= {} & \chi_{F_1\times F_2\times F_3}(r_1, \ldots, r_m, s_1, \ldots, s_n, t_1, \ldots, t_p)\\
& \cdot \tilde{A}_1(r_1)\cdots\tilde{A}_m(r_m)\tilde{B}_1(s_1)\cdots\tilde{B}_n(s_n)\tilde{C}_1(t_1)\cdots\tilde{C}_p(t_p)\\
& \cdot d\mu_1(r_1)\cdots d\mu_m(r_m)d\nu_1(s_1)\cdots d\nu_n(s_n)d\eta_1(t_1)\cdots d\eta_p(t_p)\\
= {} & \Big(\chi_{F_1}(r_1, \ldots, r_m)\tilde{A}_1(r_1)\cdots\tilde{A}_m(r_m)d\mu_1(r_1)\cdots d\mu_m(r_m)\Big)\\
& * \Big(\chi_{F_2\times F_3}(s_1, \ldots, s_n, t_1, \ldots, t_p)\tilde{B}_1(s_1)\cdots\tilde{B}_n(s_n)\\
& \tilde{C}_1(t_1)\cdots\tilde{C}_p(t_p)d\nu_1(s_1)\cdots d\nu_n(s_n)d\eta_1(t_1)\cdots d\eta_p(t_p)\Big)\\
= {} & \Big(\chi_{F_1}(r_1, \ldots, r_m)\tilde{A}_1(r_1)\cdots\tilde{A}_m(r_m)d\mu_1(r_1)\cdots d\mu_m(r_m)\Big)
\end{aligned}
$$

$$
\begin{aligned}
&* \left[\left(\chi_{F_2}(s_1, \ldots, s_n) \tilde{B}_1(s_1) \cdots \tilde{B}_n(s_n) d\nu_1(s_1) \cdots d\nu_n(s_n) \right) \right. \\
&\qquad \left. * \left(\chi_{F_3}(t_1, \ldots, t_p) \tilde{C}_1(t_1) \cdots \tilde{C}_p(t_p) d\eta_1(t_1) \cdots d\eta_p(t_p) \right) \right] \\
&= u * (v * w). \qquad (6.114)
\end{aligned}
$$

Therefore, we have associativity for the case of single terms, and we may write $(uv)w = u(vw) =: uvw$.

Now consider arbitrary $u, v, w \in \mathcal{V}$; let us say that they are given by

$$
u = \sum_{i=1}^{L} a_i u_i, \qquad v = \sum_{j=1}^{M} b_j v_j, \qquad \text{and} \qquad w = \sum_{k=1}^{N} c_k w_k, \qquad (6.115)
$$

where $a_1, \ldots, a_L, b_1, \ldots, b_M, c_1, \ldots, c_N \in \mathbb{C}$ and $u_1, \ldots, u_L, v_1, \ldots, v_M, w_1, \ldots, w_N \in \mathcal{G}$. Then by Theorem 6.4.5 and by the associativity for elements of $\mathcal{G}$ that we have just established, we have

$$
\begin{aligned}
(uv)w &= \left[\left(\sum_{i=1}^{L} a_i u_i \right) \left(\sum_{j=1}^{M} b_j v_j \right) \right] \left(\sum_{k=1}^{N} c_k w_k \right) = \left(\sum_{i=1}^{L} \sum_{j=1}^{M} a_i b_j u_i v_j \right) \left(\sum_{k=1}^{N} c_k w_k \right) \\
&= \sum_{i=1}^{L} \sum_{j=1}^{M} \sum_{k=1}^{N} a_i b_j c_k u_i v_j w_k = \left(\sum_{i=1}^{L} a_i u_i \right) \left(\sum_{j=1}^{M} \sum_{k=1}^{N} b_j c_k v_j w_k \right) \\
&= \left(\sum_{i=1}^{L} a_i u_i \right) \left[\left(\sum_{j=1}^{M} b_j v_j \right) \left(\sum_{k=1}^{N} c_k w_k \right) \right] \\
&= u(vw). \qquad (6.116)
\end{aligned}
$$

□

We now know all that we need to know about the vector space $\mathcal{V}$. After briefly defining the subspace $\mathcal{V}' \subseteq \mathcal{V}$, we will be able to define $\mathbb{E}$ as the quotient of the two. There are two effects we would like to accomplish in $\mathbb{E}$ by means of our definition of $\mathcal{V}'$. Roughly speaking, the first is that for vectors in $\mathcal{V}$ having all the same indeterminates (with time indices in the same order), we would like to relate their characteristic functions 'linearly' in the way one would expect (for example, if two of the characteristic functions add to a third, then we want the three corresponding vectors to have the same relationship). The second is that we would like a vector in $\mathcal{V}$ to represent the zero vector in $\mathbb{E}$ if the measures associated with the vector give measure zero when applied to the characteristic function set as a product measure. We can accomplish this in one step. We will define $\mathcal{V}'$ by means of a generating set $\mathcal{U}$.

Theorem 6.4.8. *Define a subset $\mathcal{U} \subseteq \mathcal{V}$ to be the set of all linear combinations*

$$\sum_{i=1}^{n} a_i \left[\chi_{F_i}(\underset{\cdot}{s}_1, \ldots, \underset{\cdot}{s}_m)\tilde{C}_1(s_1)\cdots\tilde{C}_m(s_m)d\nu_1(s_1)\cdots d\nu_m(s_m)\right] \tag{6.117}$$

for all possible integers $m, n \geq 1$, all complex numbers $a_1, \ldots, a_n$, all Borel sets $F_1, \ldots, F_n \subseteq \Delta_m(e_m)$, all nonzero operators $C_1, \ldots, C_m \in \mathcal{L}(X)$, and all measures $\nu_1, \ldots, \nu_m \in \mathcal{M}_{cb}[0,1]$ for which $\sum_{i=1}^n a_i\chi_{F_i} \equiv 0$ up to a set of $\nu_1 \times \cdots \times \nu_m$-measure zero.

Then the set $\mathcal{V}'$ of all (finite) linear combinations of elements from $\mathcal{U}$, that is,

$$\mathcal{V}' := \left\{ v \in \mathcal{V} : \, v = \sum_{j=1}^{N} c_j u_j, \text{ where } N \in \mathbb{N}, \, c_1, \ldots, c_N \in \mathbb{C}, \text{ and } u_1, \ldots, u_N \in \mathcal{U} \right\}, \tag{6.118}$$

is a subspace of $\mathcal{V}$.

Remark 29. Note that under the definition of $\mathcal{U}$ stated in Theorem 6.4.8, given $m \geq 1$, Borel set $F \subseteq \Delta_m(e_m)$, nonzero operators $C_1, \ldots, C_m \in \mathcal{L}(X)$ and measures $\nu_1, \ldots, \nu_m \in \mathcal{M}_{cb}[0,1]$ with $(\nu_1 \times \cdots \times \nu_m)(F) = 0$ we have that $\chi_F \equiv 0$ up to a set of measure zero, and thus $\chi_F(\underset{\cdot}{s}_1, \ldots, \underset{\cdot}{s}_m)\tilde{C}_1(s_1)\cdots\tilde{C}_m(s_m)d\nu_1(s_1)\cdots d\nu_m(s_m) \in \mathcal{U}$.

Proof of Theorem 6.4.8. The set of linear combinations of elements of a subset of a vector space always form a subspace (algebraically). (See [32, p. 129] on submodules.) $\square$

Example 31. Let $A, B \in \mathcal{L}(X)$ and $\mu, \nu \in \mathcal{M}_{cb}[0,1]$. Then

$$\begin{aligned}\chi_{\{0<s<t<1\}}(\underset{\cdot}{s}, \underset{\cdot}{t})\tilde{A}(s)\tilde{B}(t)d\mu(s)d\nu(t) - \chi_{\{0<1-t<s<t<1\}}(\underset{\cdot}{s}, \underset{\cdot}{t})\tilde{A}(s)\tilde{B}(t)d\mu(s)d\nu(t)& \\ - \chi_{\{0<s<t<1-s<1\}}(\underset{\cdot}{s}, \underset{\cdot}{t})\tilde{A}(s)\tilde{B}(t)d\mu(s)d\nu(t) \in \mathcal{V}',& \quad (6.119)\end{aligned}$$

since $\chi_{\{0<s<t<1\}}(\underset{\cdot}{s}, \underset{\cdot}{t}) - \chi_{\{\{0<1-t<s<t<1\}\}}(\underset{\cdot}{s}, \underset{\cdot}{t}) - \chi_{\{\{0<s<t<1-s<1\}\}}(\underset{\cdot}{s}, \underset{\cdot}{t}) = 0 \quad (\mu \times \nu)$-a.e.

Definition 6.4.9 (The intermediate disentangling space $\mathbb{E}$)**.** Using $\mathcal{V}'$ as in Theorem 6.4.8, we now define $\mathbb{E}$ to be the quotient vector space $\mathbb{E} := \mathcal{V}/\mathcal{V}'$. (In general, taking the quotient of a vector space with respect to a subspace yields a vector space, as module quotients by submodules yield modules, see [7, p. 452].) As usual for quotient spaces, in that $\mathbb{E}$ is a space of cosets of $\mathcal{V}'$ in $\mathcal{V}$, elements of $\mathcal{V}$ are representatives of those cosets, and we will therefore often use elements of $\mathcal{V}$ to represent elements of $\mathbb{E}$.

Example 32. Let $A, B \in \mathcal{L}(X)$ and $\mu, \nu \in \mathcal{M}_{cb}[0,1]$. As noted in the previous example,

$$\begin{aligned}\chi_{\{0<s<t<1\}}(\underset{\cdot}{s}, \underset{\cdot}{t})\tilde{A}(s)\tilde{B}(t)d\mu(s)d\nu(t) - \chi_{\{0<1-t<s<t<1\}}(\underset{\cdot}{s}, \underset{\cdot}{t})\tilde{A}(s)\tilde{B}(t)d\mu(s)d\nu(t)& \\ - \chi_{\{0<s<t<1-s<1\}}(\underset{\cdot}{s}, \underset{\cdot}{t})\tilde{A}(s)\tilde{B}(t)d\mu(s)d\nu(t) \in \mathcal{V}'.& \quad (6.120)\end{aligned}$$

Therefore, in the space $\mathbb{E}$ we have

$$\chi_{\{0<s<t<1\}}(\underset{\cdot}{s}, \underset{\cdot}{t})\tilde{A}(s)\tilde{B}(t)d\mu(s)d\nu(t)$$

$$= \chi_{\{0<1-t<s<t<1\}}(\underset{\cdot}{s}, \underset{\cdot}{t})\tilde{A}(s)\tilde{B}(t)d\mu(s)d\nu(t) + \chi_{\{0<s<t<1-s<1\}}(\underset{\cdot}{s}, \underset{\cdot}{t})\tilde{A}(s)\tilde{B}(t)d\mu(s)d\nu(t), \tag{6.121}$$

since the right-hand side of Equation (6.121) differs from the left-hand side by the element of $\mathcal{V}'$ shown in the previous expression, (6.120).

Although we have defined all the elements of $\mathbb{E}$ as well as its addition and scalar multiplication (which are implied by its definition as a quotient vector space), we need to show how multiplication in $\mathbb{E}$ can be carried out using coset representatives from $\mathcal{V}$.

Definition 6.4.10 (Multiplication in $\mathbb{E}$)**.** Let $x, y \in \mathbb{E}$. Then given any representatives $v, w \in \mathcal{V}$, respectively, so $x = v + \mathcal{V}'$ and $y = w + \mathcal{V}'$, we define the product of x and y in $\mathbb{E}$ by $xy := vw + \mathcal{V}'$, where the expression 'vw' on the right is the product of v and w in $\mathcal{V}$.

Remark 30. It is necessary for us to show that this product for $\mathbb{E}$ is well-defined. For purposes of this demonstration, we will use '$*_{\mathbb{E}}$' to mean multiplication in $\mathbb{E}$ and '$*_{\mathcal{V}}$' to mean multiplication in $\mathcal{V}$.

As stated in the definition, let $x, y \in \mathbb{E}$, and take any representatives $v, w \in \mathcal{V}$, respectively, so $x = v + \mathcal{V}'$ and $y = w + \mathcal{V}'$. Now consider any other representative of x, which we may write as $v + v'$ for some $v' \in \mathcal{V}'$, so we have $x = v + v' + \mathcal{V}'$. Then the definition of multiplication in $\mathbb{E}$ gives that both $x *_{\mathbb{E}} y = v *_{\mathcal{V}} w + \mathcal{V}'$ and $x *_{\mathbb{E}} y = (v + v') *_{\mathcal{V}} w + \mathcal{V}'$; we claim that those are equal. It suffices to show that $(v + v') *_{\mathcal{V}} w - v *_{\mathcal{V}} w \in \mathcal{V}'$; that is, that $v' *_{\mathcal{V}} w \in \mathcal{V}'$.

Since $v' \in \mathcal{V}'$, we may write $v' = \sum_{j=1}^{N} b_j u_j$, where $N \in \mathbb{N}$, $b_1, \dots, b_N \in \mathbb{C}$, and $u_1, \dots, u_N \in \mathcal{U}$, with $\mathcal{U}$ defined as in Theorem 6.4.8. Since $w \in \mathcal{V}$, we may write

$w = \sum_{i=1}^{M} a_i g_i$ with $a_1, \ldots, a_M \in \mathbb{C}$ and $g_1, \ldots, g_M \in \mathcal{G}$. Therefore, we claim

$$v' *_{\mathcal{V}} w = \left(\sum_{j=1}^{N} b_j u_j \right) *_{\mathcal{V}} \left(\sum_{i=1}^{M} a_i g_i \right) = \sum_{j=1}^{N} \sum_{i=1}^{M} b_j a_i (u_j *_{\mathcal{V}} g_i) \in \mathcal{V}'. \tag{6.122}$$

Since $\mathcal{V}'$ is a vector space, it will suffice to show that each $u_j *_{\mathcal{V}} g_i \in \mathcal{V}'$, or in general, that $u *_{\mathcal{V}} g \in \mathcal{V}'$ for all $u \in \mathcal{U}$, $g \in \mathcal{G}$.

Let $u \in \mathcal{U}$. Then

$$u = \sum_{i=1}^{p} \alpha_i \left[\chi_{E_i}(\underset{\cdot}{s}_1, \ldots, \underset{\cdot}{s}_m) \tilde{A}_1(s_1) \cdots \tilde{A}_m(s_m) d\mu_1(s_1) \cdots d\mu_m(s_m) \right] \in \mathcal{V} \tag{6.123}$$

with $m, p \geq 1$, $\alpha_1, \ldots, \alpha_m \in \mathbb{C}$, Borel sets $E_1, \ldots, E_p \subseteq \Delta_m(e_m)$, operators $A_1, \ldots, A_m \in \mathcal{L}(X)$, and measures $\mu_1, \ldots, \mu_m \in \mathcal{M}_{cb}[0,1]$, having $\sum_{i=1}^{p} \alpha_i \chi_{E_i} \equiv 0$ up to a set of $\mu_1 \times \cdots \times \mu_m$-measure zero. Let $g \in \mathcal{G}$; say

$$g = \chi_F(\underset{\cdot}{t}_1, \ldots, \underset{\cdot}{t}_n) \tilde{B}_1(t_1) \cdots \tilde{B}_n(t_n) d\nu_1(t_1) \cdots d\nu_n(t_n) \tag{6.124}$$

with $n \geq 0$, with operators $B_1, \ldots, B_n \in \mathcal{L}(X)$, with measures $\nu_1, \ldots, \nu_n \in \mathcal{M}_{cb}[0,1]$, and with $F \subseteq \Delta_n(e_n)$ a Borel set. Then

$$\begin{aligned}
&u *_{\mathcal{V}} g \\
&= \left[\sum_{i=1}^{p} \alpha_i \left(\chi_{E_i}(\underset{\cdot}{s}_1, \ldots, \underset{\cdot}{s}_m) \tilde{A}_1(s_1) \cdots \tilde{A}_m(s_m) d\mu_1(s_1) \cdots d\mu_m(s_m) \right) \right] \\
&\qquad\qquad *_{\mathcal{V}} \left[\chi_F(\underset{\cdot}{t}_1, \ldots, \underset{\cdot}{t}_n) \tilde{B}_1(t_1) \cdots \tilde{B}_n(t_n) d\nu_1(t_1) \cdots d\nu_n(t_n) \right] \\
&= \sum_{i=1}^{p} \alpha_i \Big[\chi_{E_i \times F}(\underset{\cdot}{s}_1, \ldots, \underset{\cdot}{s}_m, \underset{\cdot}{t}_1, \ldots, \underset{\cdot}{t}_n) \tilde{A}_1(s_1) \cdots \tilde{A}_m(s_m) \tilde{B}_1(t_1) \cdots \tilde{B}_n(t_n) \\
&\qquad\qquad \cdot d\mu_1(s_1) \cdots d\mu_m(s_m) d\nu_1(t_1) \cdots d\nu_n(t_n) \Big]
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=1}^{p} \alpha_i \Big[\chi_{E_i \times F}(s_1, \ldots, s_m, s_{m+1}, \ldots, s_{m+n}) \tilde{A}_1(s_1) \cdots \tilde{A}_m(s_m) \\
&\qquad \cdot \tilde{B}_1(s_{m+1}) \cdots \tilde{B}_n(s_{m+n}) d\mu_1(s_1) \cdots d\mu_m(s_m) d\nu_1(s_{m+1}) \cdots d\nu_n(s_{m+n}) \Big] \\
&= \sum_{i=1}^{p} \alpha_i \sum_{\pi \in S_{m+n}} \Big[\chi_{(E_i \times F)^\pi \cap \Delta_{m+n}(e_{m+n})}(s_1, \ldots, s_{m+n})^\pi \tilde{A}_1(s_1) \cdots \tilde{A}_m(s_m) \\
&\qquad \cdot \tilde{B}_1(s_{m+1}) \cdots \tilde{B}_n(s_{m+n}) d\mu_1(s_1) \cdots d\mu_m(s_m) d\nu_1(s_{m+1}) \cdots d\nu_n(s_{m+n}) \Big] \\
&= \sum_{\pi \in S_{m+n}} \sum_{i=1}^{p} \alpha_i \Big[\chi_{(E_i \times F)^\pi \cap \Delta_{m+n}(e_{m+n})}(s_1, \ldots, s_{m+n})^\pi \tilde{A}_1(s_1) \cdots \tilde{A}_m(s_m) \\
&\qquad \cdot \tilde{B}_1(s_{m+1}) \cdots \tilde{B}_n(s_{m+n}) d\mu_1(s_1) \cdots d\mu_m(s_m) d\nu_1(s_{m+1}) \cdots d\nu_n(s_{m+n}) \Big].
\end{aligned}
\tag{6.125}
$$

To show that the above expression is in $\mathcal{V}'$, it suffices to show that it is a sum of elements from $\mathcal{U}$, which we will do by demonstrating that for each fixed $\pi \in S_{m+n}$,

$$
\sum_{i=1}^{p} \alpha_i \chi_{(E_i \times F)^\pi \cap \Delta_{m+n}(e_{m+n})} \equiv 0 \tag{6.126}
$$

up to a set of $(\mu_1 \times \cdots \times \mu_m \times \nu_1 \times \cdots \times \nu_n)^\pi$-measure zero. But in fact, for fixed $\pi \in S_{m+n}$, for all values of $s_1, \ldots, s_{m+n}$, we have

$$
\begin{aligned}
&\sum_{i=1}^{p} \alpha_i \chi_{(E_i \times F)^\pi \cap \Delta_{m+n}(e_{m+n})}(s_1, \ldots, s_{m+n})^\pi \\
&= \sum_{i=1}^{p} \alpha_i \chi_{[(E_i \times F) \cap \Delta_{m+n}(\pi)]^\pi}(s_1, \ldots, s_{m+n})^\pi \\
&= \left[\sum_{i=1}^{p} \alpha_i \chi_{(E_i \times F) \cap \Delta_{m+n}(\pi)}(s_1, \ldots, s_{m+n}) \right]^\pi \\
&= \left[\sum_{i=1}^{p} \alpha_i \chi_{E_i}(s_1, \ldots, s_m) \chi_F(s_{m+1}, \ldots, s_{m+n}) \chi_{\Delta_{m+n}(\pi)}(s_1, \ldots, s_{m+n}) \right]^\pi \\
&= \left[\chi_F(s_{m+1}, \ldots, s_{m+n}) \chi_{\Delta_{m+n}(\pi)}(s_1, \ldots, s_{m+n}) \sum_{i=1}^{p} \alpha_i \chi_{E_i}(s_1, \ldots, s_m) \right]^\pi
\end{aligned}
$$

$$= 0 \tag{6.127}$$

a.e.-$(\mu_1 \times \cdots \times \mu_m \times \nu_1 \times \cdots \times \nu_n)^\pi$ as a function of $(s_1, \ldots, s_{m+n})^\pi$, since $\sum_{i=1}^p \alpha_i \chi_{E_i} = 0$ a.e.-$\mu_1 \times \cdots \times \mu_m$ (as function of $(s_1, \ldots, s_m)$). This gives us the condition (6.126) we sought, meaning that $u *_{\mathcal{V}} g \in \mathcal{V}'$.

Therefore, using a different representative for x will yield the same product in $\mathbb{E}$. By commutativity in $\mathcal{V}$, the same fact will hold for choosing a different representative for the other factor y in the product as well (or changing representatives ofboth x and y). Hence the product in $\mathbb{E}$ is well-defined, making $\mathbb{E}$ an algebra.

We should note as a result that multiplication in $\mathbb{E}$ inherits commutative, associative, and distributive properties from $\mathcal{V}$: Let $c \in \mathbb{C}$, $x, y, z \in \mathbb{E}$, and $u, v, w \in \mathcal{V}$ with $x = u + \mathcal{V}', y = v + \mathcal{V}', z = w + \mathcal{V}'$. Then

$$\begin{aligned}
xy &= (u + \mathcal{V}')(v + \mathcal{V}') = (uv + \mathcal{V}') = (vu + \mathcal{V}') = (v + \mathcal{V}')(u + \mathcal{V}') = yx, \\
c(xy) &= c[(u + \mathcal{V}')(v + \mathcal{V}')] = c(uv + \mathcal{V}') = c(uv) + \mathcal{V}' = (cu)v + \mathcal{V}' \\
&= (cu + \mathcal{V}')(v + \mathcal{V}') = (cx)y, \\
(xy)z &= [(u + \mathcal{V}')(v + \mathcal{V}')](w + \mathcal{V}') = (uv + \mathcal{V}')(w + \mathcal{V}') = (uv)w + \mathcal{V}' = u(vw) + \mathcal{V}' \\
&= (u + \mathcal{V}')(vw + \mathcal{V}') = (u + \mathcal{V}')[(v + \mathcal{V}')(w + \mathcal{V}')] = x(yz),
\end{aligned} \tag{6.128}$$

and

$$\begin{aligned}
x(y + z) &= (u + \mathcal{V}')[(v + \mathcal{V}') + (w + \mathcal{V}')] = (u + \mathcal{V}')[(v + w) + \mathcal{V}'] = u(v + w) + \mathcal{V}' \\
&= (uv + uw) + \mathcal{V}' = (uv + \mathcal{V}') + (uw + \mathcal{V}') \\
&= (u + \mathcal{V}')(v + \mathcal{V}') + (u + \mathcal{V}')(w + \mathcal{V}') = xy + xz.
\end{aligned} \tag{6.129}$$

Although we now have the space $\mathbb{E}$ entirely in hand, with all its operations, we cannot yet use it as freely as we would like. For example, we would like to replace the characteristic functions that appear in expressions with (general) simple functions (linear combinations of measurable characteristic functions). With that in mind, we will not change the space $\mathbb{E}$, but we will expand the notation used.

Remark 31. At times a characteristic function can be written as a linear combination of other characteristic functions. Consider one such, say $\sum_{i=1}^n a_i\chi_{E_i} = \chi_{E_0}$ for constants $a_1, \ldots, a_n \in \mathbb{C}$ and Borel sets $E_0, E_1, \ldots, E_n \subseteq [0,1]^m$ for some $m, n \geq 1$. In this case, given operators $C_1, \ldots, C_m \in \mathcal{L}(X)$ and measures $\nu_1, \ldots, \nu_m \in \mathcal{M}_{cb}[0,1]$, we can say that in $\mathbb{E}$ (as we will show momentarily),

$$\begin{aligned}
&\left[\sum_{i=1}^n a_i\chi_{E_i}\right](\underset{\cdot}{s}_1, \ldots, \underset{\cdot}{s}_m)\tilde{C}_1(s_1)\cdots\tilde{C}_m(s_m)d\nu_1(s_1)\cdots d\nu_m(s_m) \\
&\qquad = \sum_{i=1}^n a_i\left[\chi_{E_i}(\underset{\cdot}{s}_1, \ldots, \underset{\cdot}{s}_m)\tilde{C}_1(s_1)\cdots\tilde{C}_m(s_m)d\nu_1(s_1)\cdots d\nu_m(s_m)\right]. \quad (6.130)
\end{aligned}$$

The rationale is as follows: By hypothesis, the left-hand side is defined to equal

$$\chi_{E_0}(\underset{\cdot}{s}_1, \ldots, \underset{\cdot}{s}_m)\tilde{C}_1(s_1)\cdots\tilde{C}_m(s_m)d\nu_1(s_1)\cdots d\nu_m(s_m). \quad (6.131)$$

Subtracting this from the right-hand side and letting $a_0 := -1$ gives the expression

$$\begin{aligned}
&\sum_{i=0}^n a_i\left[\chi_{E_i}(\underset{\cdot}{s}_1, \ldots, \underset{\cdot}{s}_m)\tilde{C}_1(s_1)\cdots\tilde{C}_m(s_m)d\nu_1(s_1)\cdots d\nu_m(s_m)\right] \\
&= \sum_{i=0}^n a_i \sum_{\pi\in S_m}\left[\chi_{E_i^\pi\cap\Delta_m(e_m)}(\underset{\cdot}{s}_1, \ldots, \underset{\cdot}{s}_m)^\pi\tilde{C}_1(s_1)\cdots\tilde{C}_m(s_m)d\nu_1(s_1)\cdots d\nu_m(s_m)\right] \\
&= \sum_{\pi\in S_m}\left(\sum_{i=0}^n a_i\left[\chi_{E_i^\pi\cap\Delta_m(e_m)}(\underset{\cdot}{s}_1, \ldots, \underset{\cdot}{s}_m)^\pi\tilde{C}_1(s_1)\cdots\tilde{C}_m(s_m)d\nu_1(s_1)\cdots d\nu_m(s_m)\right]\right),
\end{aligned} \quad (6.132)$$

and we would like to show that this equals the zero vector in $\mathbb{E}$, which is to say, as a vector in $\mathcal{V}$ we want to show it is in the subspace $\mathcal{V}'$.

By hypothesis, $\sum_{i=1}^{n} a_i \chi_{E_i} = \chi_{E_0}$, or equivalently, $\sum_{i=0}^{n} a_i \chi_{E_i} \equiv 0$. But then for any $\pi \in S_m$ we have that

$$0 \equiv \left(\sum_{i=0}^{n} a_i \chi_{E_i} \right)^{\pi} = \sum_{i=0}^{n} a_i \chi_{E_i^{\pi}}, \tag{6.133}$$

so

$$0 \equiv \left(\sum_{i=0}^{n} a_i \chi_{E_i^{\pi}} \right) \chi_{\Delta_m(e_m)} = \sum_{i=0}^{n} a_i \left(\chi_{E_i^{\pi}} \chi_{\Delta_m(e_m)} \right) = \sum_{i=0}^{n} a_i \left(\chi_{E_i^{\pi} \cap \Delta_m(e_m)} \right). \tag{6.134}$$

Consequently, the expression (6.132) is a linear combination of elements of $\mathcal{U}$ and therefore is an element of $\mathcal{V}'$, establishing our claim.

Equation (6.130) is therefore true in the case when $\sum\limits_{i=1} a_i \chi_{E_i}$ is a characteristic function on $[0,1]^m$. If $\sum\limits_{i=1} a_i \chi_{E_i}$ is not a characteristic function, then the left-hand expression in (6.130) is not defined. We would therefore like to define it, in such a way that the equation will hold in that case also.

Definition 6.4.11. Given any simple function $\sum_{j=1}^{N} b_j \chi_{E_j}$ on $[0,1]^m$ for $m \geq 1$ an integer, where $b_1, \ldots, b_N \in \mathbb{C}$, and $E_1, \ldots, E_N \subseteq [0,1]^m$ are Borel sets, and given any operators $C_1, \ldots, C_m \in \mathcal{L}(X)$ and measures $\nu_1, \ldots, \nu_m \in \mathcal{M}_{cb}[0,1]$, we define

$$\left[\sum_{j=1}^{N} b_j \chi_{E_j} \right] (\underset{\cdot}{s}_1, \ldots, \underset{\cdot}{s}_m) \tilde{C}_1(s_1) \cdots \tilde{C}_m(s_m) d\nu_1(s_1) \cdots d\nu_m(s_m) \in \mathbb{E} \tag{6.135}$$

by

$$\left[\sum_{j=1}^{N} b_j \chi_{E_j}\right](\underset{\cdot}{s}_1,\ldots,\underset{\cdot}{s}_m)\tilde{C}_1(s_1)\cdots\tilde{C}_m(s_m)d\nu_1(s_1)\cdots d\nu_m(s_m)$$
$$:=\sum_{j=1}^{N} b_j\left[\chi_{E_j}(\underset{\cdot}{s}_1,\ldots,\underset{\cdot}{s}_m)\tilde{C}_1(s_1)\cdots\tilde{C}_m(s_m)d\nu_1(s_1)\cdots d\nu_m(s_m)\right]. \quad (6.136)$$

Example 33. Let $A, B \in \mathcal{L}(X)$ and $\mu, \nu \in \mathcal{M}_{cb}[0,1]$. Then we have

$$\left[2\chi_{\{s<t\}} + 3\chi_{\{s>t\}}\right](\underset{\cdot}{s},\underset{\cdot}{t})\tilde{A}(s)\tilde{B}(t)d\mu(s)d\nu(t) \in \mathbb{E}, \quad (6.137)$$

and

$$\left[2\chi_{\{s<t\}} + 3\chi_{\{s>t\}}\right](\underset{\cdot}{s},\underset{\cdot}{t})\tilde{A}(s)\tilde{B}(t)d\mu(s)d\nu(t)$$
$$= 2\left[\chi_{\{s<t\}}(\underset{\cdot}{s},\underset{\cdot}{t})\tilde{A}(s)\tilde{B}(t)d\mu(s)d\nu(t)\right] + 3\left[\chi_{\{s>t\}}(\underset{\cdot}{s},\underset{\cdot}{t})\tilde{A}(s)\tilde{B}(t)d\mu(s)d\nu(t)\right]. \quad (6.138)$$

Remark 32. Note that the definition is consistent with what we said in the remark above in the case that $\sum_{j=1}^{N} b_j\chi_{E_j}$ equals a single characteristic function. However, we still need to show in general that the definition is well-defined. That is, if we have two linear combinations that are equal to each other almost everywhere, say $\sum_{j=1}^{N} b_j\chi_{E_j} = \sum_{i=1}^{n} a_i\chi_{F_i}$ a.e.-$\nu_1 \times \cdots \times \nu_m$ on $[0,1]^m$, with $a_1,\ldots,a_n \in \mathbb{C}$ and Borel sets $F_1,\ldots,F_n \subseteq [0,1]^m$, then we claim that the definition will produce the same member of the space $\mathbb{E}$. In fact, examining this we see that $\displaystyle\sum_{j=1}^{N} b_j\chi_{E_j} - \sum_{i=1}^{n} a_i\chi_{F_i}$ equals $\chi_\emptyset$ (the characteristic function that is identically zero) a.e.-$\nu_1 \times \cdots \times \nu_m$ on $[0,1]^m$. Thus in $\mathcal{V}$ we have

$$\left[\sum_{j=1}^{N} b_j\chi_{E_j} - \sum_{i=1}^{n} a_i\chi_{F_i}\right](\underset{\cdot}{s}_1,\ldots,\underset{\cdot}{s}_m)\tilde{C}_1(s_1)\cdots\tilde{C}_m(s_m)d\nu_1(s_1)\cdots d\nu_m(s_m) \in \mathcal{V}', \quad (6.139)$$

and then by Remark 31 we have in $\mathbb{E}$ that

$$
\begin{aligned}
0 &= \left[\sum_{j=1}^{N} b_j \chi_{E_j} - \sum_{i=1}^{n} a_i \chi_{F_i}\right] (\underset{\cdot}{s}_1, \ldots, \underset{\cdot}{s}_m)\tilde{C}_1(s_1)\cdots\tilde{C}_m(s_m)d\nu_1(s_1)\cdots d\nu_m(s_m) \\
&= \sum_{j=1}^{N} b_j \left[\chi_{E_j}(\underset{\cdot}{s}_1, \ldots, \underset{\cdot}{s}_m)\tilde{C}_1(s_1)\cdots\tilde{C}_m(s_m)d\nu_1(s_1)\cdots d\nu_m(s_m)\right] \\
&\qquad - \sum_{i=1}^{n} a_i \left[\chi_{F_i}(\underset{\cdot}{s}_1, \ldots, \underset{\cdot}{s}_m)\tilde{C}_1(s_1)\cdots\tilde{C}_m(s_m)d\nu_1(s_1)\cdots d\nu_m(s_m)\right], \qquad (6.140)
\end{aligned}
$$

and therefore

$$
\begin{aligned}
&\sum_{j=1}^{N} b_j \left[\chi_{E_j}(\underset{\cdot}{s}_1, \ldots, \underset{\cdot}{s}_m)\tilde{C}_1(s_1)\cdots\tilde{C}_m(s_m)d\nu_1(s_1)\cdots d\nu_m(s_m)\right] \\
&\qquad = \sum_{i=1}^{n} a_i \left[\chi_{F_i}(\underset{\cdot}{s}_1, \ldots, \underset{\cdot}{s}_m)\tilde{C}_1(s_1)\cdots\tilde{C}_m(s_m)d\nu_1(s_1)\cdots d\nu_m(s_m)\right]. \qquad (6.141)
\end{aligned}
$$

Hence Definition 6.4.11 is well-defined.

Consequently, we can be less careful about our use of parentheses; the expression

$$
\sum_{j=1}^{N} b_j \chi_{E_j}(\underset{\cdot}{s}_1, \ldots, \underset{\cdot}{s}_m)\tilde{C}_1(s_1)\cdots\tilde{C}_m(s_m)d\nu_1(s_1)\cdots d\nu_m(s_m) \qquad (6.142)
$$

could be interpreted with parentheses (or brackets) as on either side of Equation 6.136, or even as

$$
\sum_{j=1}^{N} \left[\left(b_j \chi_{E_j}\right)(\underset{\cdot}{s}_1, \ldots, \underset{\cdot}{s}_m)\tilde{C}_1(s_1)\cdots\tilde{C}_m(s_m)d\nu_1(s_1)\cdots d\nu_m(s_m)\right], \qquad (6.143)
$$

but the definition implies that these are all equal.

Having sufficiently defined the intermediate disentangling space $\mathbb{E}$, we come at last to the intermediate disentangling map $\mathcal{T}_{\mathbb{E}}$, in two steps.

Definition 6.4.12. We define $\phi : \mathcal{V} \to \mathcal{L}(X)$ to be the unique linear map given by

$$\phi(g) := \mathcal{T}_{\mathcal{G}'}[g] \tag{6.144}$$

for all $g \in \mathcal{G}$. (Note under this definition $\phi(1) = I$, the identity operator.)

The fact that $\mathcal{V}$ is a free vector space with basis $\mathcal{G}$ implies that there exists such a unique linear map ϕ ([32, p. 135, Theorem 4.1]).

Lemma 6.4.13. *Given any integer $m \geq 0$, any operators $C_1, \ldots, C_m \in \mathcal{L}(X)$ and measures $\nu_1, \ldots, \nu_m \in \mathcal{M}_{cb}[0,1]$, and, if $m > 0$, any Borel set $E \subseteq [0,1]^m$, we have*

$$\begin{aligned}\phi&\left(\chi_E(\underset{\cdot}{s}_1, \ldots, \underset{\cdot}{s}_m)\tilde{C}_1(s_1)\cdots\tilde{C}(s_m)d\nu_1(s_1)\cdots d\nu_m(s_m)\right)\\ &= \mathcal{T}_{\mathcal{G}'}\left[\chi_E(\underset{\cdot}{s}_1, \ldots, \underset{\cdot}{s}_m)\tilde{C}_1(s_1)\cdots\tilde{C}(s_m)d\nu_1(s_1)\cdots d\nu_m(s_m)\right].\end{aligned} \tag{6.145}$$

(Note that on the left, '$\chi_E(\underset{\cdot}{s}_1, \ldots, \underset{\cdot}{s}_m)\tilde{C}_1(s_1)\cdots\tilde{C}(s_m)d\nu_1(s_1)\cdots d\nu_m(s_m)$' is interpreted as an element of the space $\mathcal{V}$, whereas on the right it is interpreted as an element of the set $\mathcal{G}'$.)

Proof. For $m = 0$ we have $\phi(1) = I = \mathcal{T}_{\mathcal{G}'}[1]$. Suppose $m \geq 1$. By linearity of ϕ we have that

$$\begin{aligned}&\phi\left(\chi_E(\underset{\cdot}{s}_1, \ldots, \underset{\cdot}{s}_m)\tilde{C}_1(s_1)\cdots\tilde{C}(s_m)d\nu_1(s_1)\cdots d\nu_m(s_m)\right)\\ &= \phi\left(\sum_{\pi \in S_m} \chi_{E^\pi \cap \Delta_m(e_m)}(\underset{\cdot}{s}_1, \ldots, \underset{\cdot}{s}_m)^\pi \tilde{C}_1(s_1)\cdots\tilde{C}(s_m)d\nu_1(s_1)\cdots d\nu_m(s_m)\right)\\ &= \sum_{\pi \in S_m} \phi\left(\chi_{E^\pi \cap \Delta_m(e_m)}(\underset{\cdot}{s}_1, \ldots, \underset{\cdot}{s}_m)^\pi \tilde{C}_1(s_1)\cdots\tilde{C}(s_m)d\nu_1(s_1)\cdots d\nu_m(s_m)\right)\\ &= \sum_{\pi \in S_m} \mathcal{T}_{\mathcal{G}'}\left[\chi_{E^\pi \cap \Delta_m(e_m)}(\underset{\cdot}{s}_1, \ldots, \underset{\cdot}{s}_m)^\pi \tilde{C}_1(s_1)\cdots\tilde{C}(s_m)d\nu_1(s_1)\cdots d\nu_m(s_m)\right]\end{aligned}$$

$$
\begin{aligned}
&= \sum_{\pi \in S_m} \mathcal{T}_{\mathcal{G}'} \left[\chi_{E \cap \Delta_m(\pi)}(\underset{\cdot}{s}_1, \ldots, \underset{\cdot}{s}_m) \tilde{C}_1(s_1) \cdots \tilde{C}(s_m) d\nu_1(s_1) \cdots d\nu_m(s_m) \right] \\
&= \sum_{\pi \in S_m} \sum_{\rho \in S_m} \int_{\Delta_m(\rho)} \chi_{E \cap \Delta_m(\pi)}(s_1, \ldots, s_m) C_{\rho(m)} \cdots C_{\rho(1)} (\nu_1 \times \cdots \times \nu_m)(s_1, \ldots, s_m) \\
&= \sum_{\pi \in S_m} \int_{\Delta_m(\pi)} \chi_E(s_1, \ldots, s_m) C_{\pi(m)} \cdots C_{\pi(1)} (\nu_1 \times \cdots \times \nu_m)(s_1, \ldots, s_m) \\
&= \mathcal{T}_{\mathcal{G}'} \left[\chi_E(\underset{\cdot}{s}_1, \ldots, \underset{\cdot}{s}_m) \tilde{C}_1(s_1) \cdots \tilde{C}(s_m) d\nu_1(s_1) \cdots d\nu_m(s_m) \right]. \qquad (6.146)
\end{aligned}
$$

□

Remark 33. To be clear, we must note that even though we have used the expression $\chi_E(\underset{\cdot}{s}_1, \ldots, \underset{\cdot}{s}_m) \tilde{C}_1(s_1) \cdots \tilde{C}(s_m) d\nu_1(s_1) \cdots d\nu_m(s_m)$ for an element of $\mathcal{G}'$—all elements of $\mathcal{G}'$ are in this form—and we have used it for an element of $\mathcal{V}$, we have never established that $\mathcal{G}'$ is a subset of $\mathcal{V}$, nor will we (though we regard $\mathcal{G}$ as a subset of both). However, the expression is valid in both spaces, and the above lemma says that its images under the corresponding maps agree. We will say more about this in a later remark, after we have defined the map $\mathcal{T}_{\mathbb{E}}$.

Lemma 6.4.14. *Given the space $\mathcal{V}' \subseteq \mathcal{V}$ as defined in Theorem 6.4.8, we have $\mathcal{V}' \subseteq \ker(\phi)$.*

Proof. Recalling that we defined a subset $\mathcal{U} \subseteq \mathcal{V}$ in Theorem 6.4.8, and that $\mathcal{U}$ generates $\mathcal{V}'$, let $u \in \mathcal{U}$. Then

$$
u = \sum_{i=1}^{n} a_i \chi_{F_i}(\underset{\cdot}{s}_1, \ldots, \underset{\cdot}{s}_m) \tilde{C}_1(s_1) \cdots \tilde{C}_m(s_m) d\nu_1(s_1) \cdots d\nu_m(s_m) \qquad (6.147)
$$

for some $m, n \geq 1$, $a_1, \ldots, a_n \in \mathbb{C}$, Borel sets $F_1, \ldots, F_n \subseteq \Delta_m(e_m)$ ($e_m \in S_m$ is the identity permutation), and measures $\nu_1, \ldots, \nu_m \in \mathcal{M}_{cb}[0,1]$, and $\sum_{i=1}^n a_i \chi_{F_i} \equiv 0$ up to a set of $\nu_1 \times \cdots \times \nu_m$-measure zero. We apply the map ϕ to u (we may since $u \in \mathcal{V}$)

to get

$$\begin{aligned}
\phi(u) &= \sum_{i=1}^{n} a_i \mathcal{T}_{\mathcal{G}'} \left[\chi_{F_i}(\underset{\cdot}{s}_1, \ldots, \underset{\cdot}{s}_m) \tilde{C}_1(s_1) \cdots \tilde{C}_m(s_m) d\nu_1(s_1) \cdots d\nu_m(s_m) \right] \\
&= \sum_{i=1}^{n} a_i \sum_{\pi \in S_m} \int_{\Delta_m(\pi)} \chi_{F_i}(s_1, \ldots, s_m) C_{\pi(m)} \cdots C_{\pi(1)} (\nu_1 \times \cdots \times \nu_m)(ds_1, \ldots, ds_m) \\
&= \sum_{\pi \in S_m} \int_{\Delta_m(\pi)} \sum_{i=1}^{n} a_i \chi_{F_i}(s_1, \ldots, s_m) C_{\pi(m)} \cdots C_{\pi(1)} (\nu_1 \times \cdots \times \nu_m)(ds_1, \ldots, ds_m) \\
&= 0,
\end{aligned} \tag{6.148}$$

since $\sum_{i=1}^{n} a_i \chi_{F_i} = 0$ a.e. Thus $u \in \ker(\phi)$ and $\mathcal{U} \subseteq \ker(\phi)$. Moreover, since $\mathcal{V}'$ consists of linear combinations of elements of $\mathcal{U}$, and the map ϕ is linear on all of $\mathcal{V}$, we have that $\mathcal{V}' \subseteq \ker(\phi)$. $\square$

Theorem 6.4.15 (The map $\mathcal{T}_{\mathbb{E}}$)**.** *There exists a unique linear map $\mathcal{T}_{\mathbb{E}} : \mathbb{E} \to \mathcal{L}(X)$ given by $\mathcal{T}_{\mathbb{E}}[x] := \phi(v)$, where $v \in \mathcal{V}$ is any coset representative in $\mathcal{V}$ of $x \in \mathbb{E}$ (that is, $x = v + \mathcal{V}'$). Moreover, given any $\chi_E(\underset{\cdot}{s}_1, \ldots, \underset{\cdot}{s}_m) \tilde{C}_1(s_1) \cdots \tilde{C}(s_m) d\nu_1(s_1) \cdots d\nu_m(s_m) \in \mathcal{G}'$ (where $E \subseteq [0,1]^m$), $m \geq 0$, we have that*

$$\begin{aligned}
&\mathcal{T}_{\mathbb{E}} \left[\chi_E(\underset{\cdot}{s}_1, \ldots, \underset{\cdot}{s}_m) \tilde{C}_1(s_1) \cdots \tilde{C}(s_m) d\nu_1(s_1) \cdots d\nu_m(s_m) \right] \\
&\qquad = \mathcal{T}_{\mathcal{G}'} \left[\chi_E(\underset{\cdot}{s}_1, \ldots, \underset{\cdot}{s}_m) \tilde{C}_1(s_1) \cdots \tilde{C}(s_m) d\nu_1(s_1) \cdots d\nu_m(s_m) \right].
\end{aligned} \tag{6.149}$$

Proof. (The first statement of this theorem is a special case of the mapping property for homomorphisms of modules, see [7, p. 452], but we will go through a proof anyway.)

First we need to show that $\mathcal{T}_{\mathbb{E}}$ given by $\mathcal{T}_{\mathbb{E}}[x] = \phi(v)$ is well-defined. Let $v, w \in \mathcal{V}$ be two representatives of $x \in \mathbb{E}$, so $x = v + \mathcal{V}' = w + \mathcal{V}'$. Then $w = v + v'$ for some $v' \in \mathcal{V}'$. Hence $\phi(w) = \phi(v + v') = \phi(v) + \phi(v') = \phi(v)$ since $\mathcal{V}' \subseteq \ker(\phi)$. Thus the map $\mathcal{T}_{\mathbb{E}}$ is well-defined.

To see that $\mathcal{T}_{\mathbb{E}}$ is linear, we consider any $x, y \in \mathbb{E}$ and any $a, b \in \mathbb{C}$. Let $v, w \in \mathcal{V}$ be coset representatives of x, y, respectively. Then

$$\begin{aligned}\mathcal{T}_{\mathbb{E}}[ax+by] &= \mathcal{T}_{\mathbb{E}}[a(v+\mathcal{V}')+b(w+\mathcal{V}')] = \mathcal{T}_{\mathbb{E}}[(av+\mathcal{V}')+(bw+\mathcal{V}')] \\ &= \mathcal{T}_{\mathbb{E}}[(av+bw)+\mathcal{V}'] = \phi(av+bw) = a\phi(v)+b\phi(w) = a\mathcal{T}_{\mathbb{E}}[x]+b\mathcal{T}_{\mathbb{E}}[y]. \quad (6.150)\end{aligned}$$

Therefore, $\mathcal{T}_{\mathbb{E}}$ is a linear map. The second statement in the theorem, (6.149), then follows by Lemma 6.4.13. □

Remark 34. As in Lemma 6.4.13, in Theorem 6.4.15 we have used the expression $\chi_E(\underset{\cdot}{s}_1, \ldots, \underset{\cdot}{s}_m)\tilde{C}_1(s_1)\cdots\tilde{C}(s_m)d\nu_1(s_1)\cdots d\nu_m(s_m)$ both as an element of $\mathcal{G}'$, and as an element of $\mathcal{V}$ representing an element of $\mathbb{E}$. Although the expression is used in both places, we do not regard it as representing the same entity, since we have never said that $\mathcal{G}'$ is a subset of $\mathcal{V}$ (though perhaps it is possible to make $\mathcal{G}'$ a subset of $\mathcal{V}$; the present author is unable to say at this time). Nevertheless, every element of $\mathcal{G}'$ has an expression that is also valid as an element of $\mathbb{E}$ (as a monomial), and the maps $\mathcal{T}_{\mathcal{G}'}$ and $\mathcal{T}_{\mathbb{E}}$ agree on those expressions. Furthermore, in $\mathcal{G}'$ we introduced other notations for elements (such as those used for exponentiation, involving scope rules for variables), and where there is no danger of confusion, we will apply the same kind of notation for terms in $\mathbb{E}$ as well.

Most importantly, the same time-ordering theorems that hold for the map $\mathcal{T}_{\mathcal{G}'}$ on $\mathcal{G}'$ hold for the map $\mathcal{T}_{\mathbb{E}}$ on monomials in $\mathbb{E}$.

Example 34. Let $x := \chi_{(0,1)^2}(\underset{\cdot}{s}, \underset{\cdot}{t})\tilde{A}(s)\tilde{B}(t)d\mu(s)d\nu(t) \in \mathbb{E}$, where $A, B \in \mathcal{L}(X)$ and $\mu, \nu \in \mathcal{M}_{cb}[0,1]$. Then

$$\mathcal{T}_{\mathbb{E}}[x] = \mathcal{T}_{\mathbb{E}}\left[\chi_{(0,1)^2}(\underset{\cdot}{s}, \underset{\cdot}{t})\tilde{A}(s)\tilde{B}(t)d\mu(s)d\nu(t)\right]$$

$$= \mathcal{T}_{\mathbb{E}}\left[\chi_{\{s<t\}}(\underset{\cdot}{s},\underset{\cdot}{t})\tilde{A}(s)\tilde{B}(t)d\mu(s)d\nu(t) + \chi_{\{t<s\}}(\underset{\cdot}{t},\underset{\cdot}{s})\tilde{A}(s)\tilde{B}(t)d\mu(s)d\nu(t)\right]$$
$$= \mathcal{T}_{\mathcal{G}'}\left[\chi_{\{s<t\}}(\underset{\cdot}{s},\underset{\cdot}{t})\tilde{A}(s)\tilde{B}(t)d\mu(s)d\nu(t)\right] + \mathcal{T}_{\mathcal{G}'}\left[\chi_{\{t<s\}}(\underset{\cdot}{t},\underset{\cdot}{s})\tilde{A}(s)\tilde{B}(t)d\mu(s)d\nu(t)\right]. \tag{6.151}$$

We may also write

$$\mathcal{T}_{\mathbb{E}}[x] = \mathcal{T}_{\mathbb{E}}\left[\chi_{(0,1)}\chi_{(0,t)}\tilde{B}(t)\tilde{A}(s)d\mu(s)d\nu(t) + \chi_{(0,1)}\chi_{(t,1)}\tilde{A}(s)\tilde{B}(t)d\mu(s)d\nu(t)\right]$$
$$= \int_{\{s<t\}} BA(\mu\times\nu)(ds,dt) + \int_{\{t<s\}} AB(\mu\times\nu)(ds,dt). \tag{6.152}$$

As if all that were not enough, we can also define a norm on $\mathbb{E}$, which is one reason we chose the set $\mathcal{U}$ as we did earlier.

Theorem 6.4.16 (The norm $\|\cdot\|_{\mathbb{E}}$)**.** *Let $\|\cdot\|_{\mathbb{E}} : \mathbb{E} \to [0,\infty)$ be the map defined as follows: Given any $x \in \mathbb{E}$, let $v \in \mathcal{V}$ be a coset representative of x, so $x = v + \mathcal{V}'$. Write v as*

$$v = \sum_{j=1}^{N}\sum_{k=1}^{n_j} a_{j,k}\Big[\chi_{F_{j,k}}(\underset{\cdot}{s}_{j,1},\ldots,\underset{\cdot}{s}_{j,m_j})\tilde{C}_{j,m_j}(s_{j,m_j})\cdots\tilde{C}_{j,1}(s_{j,1})$$
$$d\nu_{j,1}(s_{j,1})\cdots d\nu_{j,m_j}(s_{j,m_j})\Big], \tag{6.153}$$

for some $N \geq 1, n_j \geq 1$, $m_j \geq 0$, $a_{j,k} \in \mathbb{C}$, Borel sets $F_{j,k} \subseteq \Delta_{m_j}(e_{m_j})$, operators $C_{j,1},\ldots,C_{j,m_j} \in \mathcal{L}(X)$, and measures $\nu_{j,1},\ldots,\nu_{j,m_j} \in \mathcal{M}_{cb}[0,1]$ for all $j \in \{1,\ldots,N\}$ and all $k \in \{1,\ldots,n_j\}$, with the additional requirement that for any $i,j \in \{1,\ldots,N\}$ with $i \neq j$, the terms have $(\tilde{C}_{i,1},\ldots,\tilde{C}_{i,m_i}, d\nu_{i,1},\ldots,d\nu_{i,m_i}) \neq$

$(\tilde{C}_{j,1}, \ldots, \tilde{C}_{j,m_j}, d\nu_{j,1}, \ldots, d\nu_{j,m_j})$. *Define*

$$\|x\|_{\mathbb{E}} := \sum_{j=1}^{N} \left[\int_{\Delta_{m_j}(e_{m_j})} \left| \sum_{k=1}^{n_j} a_{j,k} \chi_{F_{j,k}}(s_{j,1}, \ldots, s_{j,m_j}) \right| \|C_{j,m_j}\| \cdots \|C_{j,1}\| \right.$$
$$\left. (\nu_{j,1} \times \cdots \times \nu_{j,m_j})(ds_{j,1}, \ldots, ds_{j,m_j}) \right]. \quad (6.154)$$

Then the map $\|\cdot\|_{\mathbb{E}}$ *so defined is a norm on the space* $\mathbb{E}$.

Example 35. Before we prove both that the map $\|\cdot\|_{\mathbb{E}}$ is well-defined and is a norm, it may be helpful to look at an example. The idea of the map is that given any $x \in \mathbb{E}$, we choose a representative $v \in \mathcal{V}$, giving $x = v + \mathcal{V}'$. Then we write v in terms of basis vectors in $\mathcal{V}$, which is to say, as a linear combination of expressions from $\mathcal{G}$. We group the terms according to whether they have the same sequence of indeterminates (with the indeterminates placed in decreasing time order, first for the operators, then for the measures, just so we can compare them). Then we replace the operator-related indeterminates with operator norms, and integrate over the absolute value of the linear combination of characteristic functions, grouped by matching indeterminate terms.

For a specific example, consider

$$x = \chi_{(0,1)^2} \tilde{A}(s)\tilde{B}(t)d\mu(s)d\nu(t) - 2\chi_{\{s<t\}}\tilde{A}(s)\tilde{B}(t)d\mu(s)d\nu(t) \in \mathbb{E}, \quad (6.155)$$

where $A, B \in \mathcal{L}(X)$ and $\mu, \nu \in \mathcal{M}_{cb}[0,1]$. We can write

$$\begin{aligned} x &= \chi_{\{t<s\}}\tilde{A}(s)\tilde{B}(t)d\mu(s)d\nu(t) + \chi_{\{s<t\}}\tilde{A}(s)\tilde{B}(t)d\mu(s)d\nu(t) \\ &\qquad - 2\chi_{\{s<t\}}\tilde{A}(s)\tilde{B}(t)d\mu(s)d\nu(t) \\ &= \chi_{\{t<s\}}\tilde{A}(s)\tilde{B}(t)d\mu(s)d\nu(t) - \chi_{\{s<t\}}\tilde{A}(s)\tilde{B}(t)d\mu(s)d\nu(t) \end{aligned}$$

$$= \chi_{\{t<s\}}\tilde{A}(s)\tilde{B}(t)d\nu(t)d\mu(s) - \chi_{\{s<t\}}\tilde{B}(t)\tilde{A}(s)d\mu(s)d\nu(t). \tag{6.156}$$

We note that the sequences of operator and measure indeterminates are different for the different terms, so we then have

$$\begin{aligned}\|x\|_{\mathbb{E}} &= (\nu\times\mu)(\{(t,s) :\ t<s\})\,\|A\|\,\|B\| + (\mu\times\nu)(\{(s,t) :\ s<t\})\,\|B\|\,\|A\| \\ &= (\mu\times\nu)\left([0,1]^2\right)\|A\|\,\|B\|\,. \end{aligned} \tag{6.157}$$

Proof. First we need to show that the map $\|\cdot\|_{\mathbb{E}}$ is well-defined. Let $x \in \mathbb{E}$, and let $v \in \mathcal{V}$ with $x = v + \mathcal{V}'$, expressible as in Equation (6.153). Since $\mathcal{V}$ is a free vector space with basis $\mathcal{G}$, there is, apart from the order of summation and terms with coefficients $a_{j,k}$ equal to zero, only one way of expressing v as shown (there is one way to express v as a linear combination of basis vectors, and then there is only one way to group those according to indeterminates). If any coefficients are zero, then they make no contribution to $\|x\|_{\mathbb{E}}$, so we do not need to be concerned with them.

The only other possible ambiguity for x is in the representative chosen for it from $\mathcal{V}$. Take any other representative, which we may write as $v + v'$, with $v' \in \mathcal{V}'$, so $x = v + \mathcal{V}' = v + v' + \mathcal{V}'$. We can write $v' = \sum_{i=1}^{n} c_i u_i$ for some $n \in \mathbb{N}$, where $c_1, \ldots, c_n \in \mathbb{C}$ and $u_1, \ldots, u_n \in \mathcal{U}$, so $x = v + c_1u_1 + \cdots + c_nu_n + \mathcal{V}'$. If we can show that expressing x as $v + c_1u_1 + \mathcal{V}'$ yields the same value for $\|x\|_{\mathbb{E}}$ as when we use $x = v+\mathcal{V}'$, then we can repeat the process to show that the expression $x = v+v'+\mathcal{V}'$ yields the same value also. In other words, without loss of generality it is enough to take only the case $v' = cu$ with $c \in \mathbb{C}$ and $u \in \mathcal{U}$.

Let $u \in \mathcal{U}$. Then we may without loss of generality say that

$$cu = \sum_{k=1}^{n_1} c\, b_{1,k} \Big[\chi_{F_{1,k}}(s_{1,1}, \ldots, s_{1,m_1}) \tilde{C}_{1,m_1}(s_{1,m_1}) \cdots \tilde{C}_{1,1}(s_{1,1})$$
$$d\nu_{1,1}(s_{1,1}) \cdots d\nu_{1,m_1}(s_{1,m_1}) \Big], \quad (6.158)$$

meaning that we are matching the term $j = 1$ in (6.153) except for the scalars. Consequently, when we group terms of $v + v' = v + cu$, it will affect only the $j = 1$ term, changing each $a_{1,k}$ to $a_{1,k} + cb_{1,k}$. The effect on the expression (6.154) is that

$$\sum_{k=1}^{n_1} a_{1,k} \chi_{F_{1,k}}(s_{1,1}, \ldots, s_{1,m_1}) \tag{6.159}$$

will change to

$$\sum_{k=1}^{n_1} a_{1,k} \chi_{F_{1,k}}(s_{1,1}, \ldots, s_{1,m_1}) + \sum_{k=1}^{n_1} c\, b_{1,k} \chi_{F_{1,k}}(s_{1,1}, \ldots, s_{1,m_1}). \tag{6.160}$$

However, since $u \in \mathcal{U}$, we have that $c \sum_{k=1}^{n_1} b_{1,k} \chi_{F_{1,k}} = 0$ a.e., so that change affects the value of (6.159) only on a set of measure zero, and changes its absolute value only on a set of measure zero, and that change has no effect on the corresponding integral in (6.154) in the definition of $\|x\|_{\mathbb{E}}$.

Therefore, the map $\|\cdot\|_{\mathbb{E}}$ is well-defined on $\mathbb{E}$. In addition, we can see that it genuinely is a function from $\mathbb{E}$ to $[0, \infty)$. We claim, further, that it is a norm. To establish that, we claim first that for any $a \in \mathbb{C}$ and any $x \in \mathbb{E}$ we have $\|ax\|_{\mathbb{E}} = |a| \, \|x\|_{\mathbb{E}}$. In fact that is not difficult to see from Equations (6.153) and (6.154) by factoring the a to the inside in the first and then factoring $|a|$ out of the absolute values in the second.

Second, we claim that the triangle inequality holds. Let $x, y \in \mathbb{E}$. Say they are represented, respectively, by $v, w \in \mathcal{V}$, so $x = v + \mathcal{V}'$, $y = w + \mathcal{V}'$. Without loss of generality we may write

$$v = \sum_{j=1}^{N} \sum_{k=1}^{n_j} a_{j,k} \Big[\chi_{F_{j,k}}(\underset{\cdot}{s}_{j,1}, \ldots, \underset{\cdot}{s}_{j,m_j}) \tilde{C}_{j,m_j}(s_{j,m_j}) \cdots \tilde{C}_{j,1}(s_{j,1}) \\ d\nu_{j,1}(s_{j,1}) \cdots d\nu_{j,m_j}(s_{j,m_j}) \Big] \quad (6.161)$$

and

$$w = \sum_{j=1}^{N} \sum_{k=1}^{n_j} b_{j,k} \Big[\chi_{F_{j,k}}(\underset{\cdot}{s}_{j,1}, \ldots, \underset{\cdot}{s}_{j,m_j}) \tilde{C}_{j,m_j}(s_{j,m_j}) \cdots \tilde{C}_{j,1}(s_{j,1}) \\ d\nu_{j,1}(s_{j,1}) \cdots d\nu_{j,m_j}(s_{j,m_j}) \Big], \quad (6.162)$$

with each $b_{j,k} \in \mathbb{C}$ and all other stipulations as for Equation (6.153) above. Then $x + y$ is represented by

$$v + w = \sum_{j=1}^{N} \sum_{k=1}^{n_j} (a_{j,k} + b_{j,k}) \Big[\chi_{F_{j,k}}(\underset{\cdot}{s}_{j,1}, \ldots, \underset{\cdot}{s}_{j,m_j}) \tilde{C}_{j,m_j}(s_{j,m_j}) \cdots \tilde{C}_{j,1}(s_{j,1}) \\ d\nu_{j,1}(s_{j,1}) \cdots d\nu_{j,m_j}(s_{j,m_j}) \Big], \quad (6.163)$$

so that

$$\begin{aligned} \|x + y\|_{\mathbb{E}} = \sum_{j=1}^{N} \Bigg[\int_{\Delta_{m_j}(e_{m_j})} & \left| \sum_{k=1}^{n_j} (a_{j,k} + b_{j,k}) \chi_{F_{j,k}}(s_{j,1}, \ldots, s_{j,m_j}) \right| \|C_{j,m_j}\| \cdots \|C_{j,1}\| \\ & (\nu_{j,1} \times \cdots \times \nu_{j,m_j})(ds_{j,1}, \ldots, ds_{j,m_j}) \Bigg] \\ \leq \sum_{j=1}^{N} \Bigg[\int_{\Delta_{m_j}(e_{m_j})} & \left| \sum_{k=1}^{n_j} a_{j,k} \chi_{F_{j,k}}(s_{j,1}, \ldots, s_{j,m_j}) \right| \|C_{j,m_j}\| \cdots \|C_{j,1}\| \end{aligned}$$

$$
\begin{aligned}
&\qquad\qquad (\nu_{j,1} \times \cdots \times \nu_{j,m_j})(ds_{j,1}, \ldots, ds_{j,m_j})\Bigg] \\
&+ \sum_{j=1}^{N} \left[\int_{\Delta_{m_j}(e_{m_j})} \left| \sum_{k=1}^{n_j} b_{j,k} \chi_{F_{j,k}}(s_{j,1}, \ldots, s_{j,m_j}) \right| \left\|C_{j,m_j}\right\| \cdots \left\|C_{j,1}\right\| \right. \\
&\qquad\qquad \left. (\nu_{j,1} \times \cdots \times \nu_{j,m_j})(ds_{j,1}, \ldots, ds_{j,m_j}) \right] \\
&= \|x\|_{\mathbb{E}} + \|y\|_{\mathbb{E}}\,.
\end{aligned} \tag{6.164}
$$

The remaining step is to show that if $x \in \mathbb{E}$ with $\|x\|_{\mathbb{E}} = 0$, then x is the zero vector in $\mathbb{E}$. (We already know by the statement $\|ax\|_{\mathbb{E}} = |a|\,\|x\|_{\mathbb{E}}$ for all $a \in \mathbb{C}, x \in \mathbb{E}$ that $\|0\|_{\mathbb{E}} = 0$.) If $x = v + \mathcal{V}'$ for $v \in \mathcal{V}$, then this is equivalent to saying that $v \in \mathcal{V}'$. Let $x \in \mathbb{E}$ and $v \in \mathcal{V}$ be as described in the theorem statement, and let $\|x\|_{\mathbb{E}} = 0$. Then $x = v + \mathcal{V}'$,

$$
\begin{aligned}
v = \sum_{j=1}^{N} \sum_{k=1}^{n_j} a_{j,k} \Big[& \chi_{F_{j,k}}(\underset{\cdot}{s}_{j,1}, \ldots, \underset{\cdot}{s}_{j,m_j}) \tilde{C}_{j,m_j}(s_{j,m_j}) \cdots \tilde{C}_{j,1}(s_{j,1}) \\
& d\nu_{j,1}(s_{j,1}) \cdots d\nu_{j,m_j}(s_{j,m_j}) \Big],
\end{aligned} \tag{6.165}
$$

and

$$
\begin{aligned}
\|x\|_{\mathbb{E}} = \sum_{j=1}^{N} \Bigg[& \int_{\Delta_{m_j}(e_{m_j})} \left| \sum_{k=1}^{n_j} a_{j,k} \chi_{F_{j,k}}(s_{j,1}, \ldots, s_{j,m_j}) \right| \left\|C_{j,m_j}\right\| \cdots \left\|C_{j,1}\right\| \\
& (\nu_{j,1} \times \cdots \times \nu_{j,m_j})(ds_{j,1}, \ldots, ds_{j,m_j}) \Bigg] = 0.
\end{aligned} \tag{6.166}
$$

Consequently, for any fixed $j \in \{1, \ldots, N\}$ we have that

$$\int_{\Delta_{m_j}(e_{m_j})} \left| \sum_{k=1}^{n_j} a_{j,k} \chi_{F_{j,k}}(s_{j,1}, \ldots, s_{j,m_j}) \right| \left\| C_{j,m_j} \right\| \cdots \left\| C_{j,1} \right\| \\ (\nu_{j,1} \times \cdots \times \nu_{j,m_j})(ds_{j,1}, \ldots, ds_{j,m_j}) = 0, \quad (6.167)$$

which implies that $\sum_{k=1}^{n_j} a_{j,k} \chi_{F_{j,k}}(s_{j,1}, \ldots, s_{j,m_j}) = 0$ almost everywhere (since we know the operators have nonzero norms). That implies for each j,

$$\sum_{k=1}^{n_j} a_{j,k} \left[\chi_{F_{j,k}}(\underset{\cdot}{s}_{j,1}, \ldots, \underset{\cdot}{s}_{j,m_j}) \tilde{C}_{j,m_j}(s_{j,m_j}) \cdots \tilde{C}_{j,1}(s_{j,1}) d\nu_{j,1}(s_{j,1}) \cdots d\nu_{j,m_j}(s_{j,m_j}) \right] \in \mathcal{U}, \quad (6.168)$$

and therefore $v \in \mathcal{V}'$, establishing the claim. Thus $\|\cdot\|_{\mathbb{E}}$ is a norm on $\mathbb{E}$. □

The natural progression at this point would be to extend $\mathbb{E}$ to a Banach algebra, if possible. It would be good, if possible, to be able to use the space to disentangle a function expressible as a power series (an exponential function, for example). However, for lack of time we will not deal with any of that, and will leave it to anyone else who is interested. Instead we will simply finish with examples.

Example 36. In Example 28 above we provided an alternate proof of a decomposing disentangling formula from [17, p. 4]. We did so using the set $\mathcal{G}'$ and the map $\mathcal{T}_{\mathcal{G}'}$. We will now extend that result to a third measure using the space $\mathbb{E}$ and the map $\mathcal{T}_{\mathbb{E}}$.

Let $A, B, C \in \mathcal{L}(X)$, let $\mu, \nu, \eta \in \mathcal{M}_{cb}[0,1]$ be probability measures, let m_1, m_2, m_3 be nonnegative integers, and let $a \in (0,1)$. Then, taking advantage of commutativity and the distributive law in $\mathbb{E}$ (which imply that a binomial theorem holds in $\mathbb{E}$), as well as linearity of the map $\mathcal{T}_{\mathbb{E}}$, we have

$$
\begin{aligned}
&P^{m_1,m_2,m_3}_{\mu,\nu,\eta}(A,B,C)\\
&=\mathcal{T}_{\mathbb{E}}\left[\left(\chi_{(0,1)}\tilde{A}(r)d\mu(r)\right)^{m_1}\left(\chi_{(0,1)}\tilde{B}(s)d\nu(s)\right)^{m_2}\left(\chi_{(0,1)}\tilde{C}(t)d\eta(t)\right)^{m_3}\right]\\
&=\mathcal{T}_{\mathbb{E}}\Big[\left(\chi_{(0,a)}\tilde{A}(r)d\mu(r)+\chi_{(a,1)}\tilde{A}(r)d\mu(r)\right)^{m_1}\left(\chi_{(0,a)}\tilde{B}(s)d\nu(s)+\chi_{(a,1)}\tilde{B}(s)d\nu(s)\right)^{m_2}\\
&\qquad\cdot\left(\chi_{(0,a)}\tilde{C}(t)d\eta(t)+\chi_{(a,1)}\tilde{C}(t)d\eta(t)\right)^{m_3}\Big]\\
&=\mathcal{T}_{\mathbb{E}}\Big[\sum_{\substack{i_1+j_1=m_1\\ i_2+j_2=m_2\\ i_3+j_3=m_3}}\frac{m_1!}{i_1!j_1!}\frac{m_2!}{i_2!j_2!}\frac{m_3!}{i_3!j_3!}\left(\chi_{(0,a)}\tilde{A}(r)d\mu(r)\right)^{i_1}\left(\chi_{(a,1)}\tilde{A}(r)d\mu(r)\right)^{j_1}\\
&\quad\cdot\left(\chi_{(0,a)}\tilde{B}(s)d\nu(s)\right)^{i_2}\left(\chi_{(a,1)}\tilde{B}(s)d\nu(s)\right)^{j_2}\left(\chi_{(0,a)}\tilde{C}(t)d\eta(t)\right)^{i_3}\left(\chi_{(a,1)}\tilde{C}(t)d\eta(t)\right)^{j_3}\Big]\\
&=\sum_{\substack{i_1+j_1=m_1\\ i_2+j_2=m_2\\ i_3+j_3=m_3}}\frac{m_1!}{i_1!j_1!}\frac{m_2!}{i_2!j_2!}\frac{m_3!}{i_3!j_3!}\mathcal{T}_{\mathbb{E}}\Big[\left(\chi_{(a,1)}\tilde{A}(r)d\mu(r)\right)^{j_1}\left(\chi_{(a,1)}\tilde{B}(s)d\nu(s)\right)^{j_2}\\
&\quad\cdot\left(\chi_{(a,1)}\tilde{C}(t)d\eta(t)\right)^{j_3}\left(\chi_{(0,a)}\tilde{A}(r)d\mu(r)\right)^{i_1}\left(\chi_{(0,a)}\tilde{B}(s)d\nu(s)\right)^{i_2}\left(\chi_{(0,a)}\tilde{C}(t)d\eta(t)\right)^{i_3}\Big].
\end{aligned}
\tag{6.169}
$$

We now apply Theorem 6.3.4, yielding

$$
\begin{aligned}
&P^{m_1,m_2,m_3}_{\mu,\nu,\eta}(A,B,C)\\
&=\sum_{\substack{i_1+j_1=m_1\\ i_2+j_2=m_2\\ i_3+j_3=m_3}}\frac{m_1!}{i_1!j_1!}\frac{m_2!}{i_2!j_2!}\frac{m_3!}{i_3!j_3!}\mathcal{T}_{\mathbb{E}}\Big[\left(\chi_{(a,1)}\tilde{A}(r)d\mu(r)\right)^{j_1}\left(\chi_{(a,1)}\tilde{B}(s)d\nu(s)\right)^{j_2}\\
&\quad\cdot\left(\chi_{(a,1)}\tilde{C}(t)d\eta(t)\right)^{j_3}\Big]\mathcal{T}_{\mathbb{E}}\Big[\left(\chi_{(0,a)}\tilde{A}(r)d\mu(r)\right)^{i_1}\left(\chi_{(0,a)}\tilde{B}(s)d\nu(s)\right)^{i_2}\left(\chi_{(0,a)}\tilde{C}(t)d\eta(t)\right)^{i_3}\Big]\\
&=\sum_{\substack{i_1+j_1=m_1\\ i_2+j_2=m_2\\ i_3+j_3=m_3}}\frac{m_1!}{i_1!j_1!}\frac{m_2!}{i_2!j_2!}\frac{m_3!}{i_3!j_3!}P^{j_1,j_2,j_3}_{\mu|_{(a,1)},\nu|_{(a,1)},\eta|_{(a,1)}}(A,B,C)P^{i_1,i_2,i_3}_{\mu|_{(0,a)},\nu|_{(0,a)},\eta|_{(0,a)}}(A,B,C).
\end{aligned}
\tag{6.170}
$$

Example 37. At the end of Section 3.2 we established Theorem 3.2.2, which we restate here (and renumber):

Theorem 6.4.17 (Disentangling a monomial that involves a sum of two measures)**.** *Given a Banach space X, together with operators $A_1, \ldots, A_n \in \mathcal{L}(X)$, non-negative integers $m_1, \ldots, m_n$, and finite, continuous Borel measures $\nu, \eta, \mu_2, \mu_3, \ldots, \mu_n$ on the interval $[0,1]$ associated with $A_1, A_1, A_2, A_3, \ldots, A_n$, respectively, we have that*

$$\begin{aligned} &P^{m_1,m_2,\ldots,m_n}_{\nu+\eta,\mu_2,\ldots,\mu_n}(A_1, A_2, A_3, \ldots, A_n) \\ &\qquad = \sum_{k=0}^{m_1} \begin{pmatrix} m_1 \\ k \end{pmatrix} P^{k,m_1-k,m_2,m_3,\ldots,m_n}_{\nu,\eta,\mu_2,\mu_3\ldots,\mu_n}(A_1, A_1, A_2, A_3, \ldots, A_n). \end{aligned} \tag{6.171}$$

Using this together with the relationship we have established (by way of the map $\mathcal{T}_{\mathcal{G}'}$) between the disentangling map and the map $\mathcal{T}_{\mathbb{E}}$ on $\mathbb{E}$, especially Theorem 6.2.7, we have under the conditions of Theorem 6.4.17 that

$$\begin{aligned} &\mathcal{T}_{\mathbb{E}}\left[\left(\chi_{(0,1)}\tilde{A}(s_1)d(\nu+\eta)(s_1)\right)^{m_1}\left(\chi_{(0,1)}\tilde{A}(s_2)d\mu(s_2)\right)^{m_2}\cdots\left(\chi_{(0,1)}\tilde{A}(s_n)d\mu(s_n)\right)^{m_n}\right] \\ &= P^{m_1,m_2,\ldots,m_n}_{\nu+\eta,\mu_2,\ldots,\mu_n}(A_1, A_2, A_3, \ldots, A_n) \\ &= \sum_{k=0}^{m_1} \begin{pmatrix} m_1 \\ k \end{pmatrix} P^{k,m_1-k,m_2,m_3,\ldots,m_n}_{\nu,\eta,\mu_2,\mu_3\ldots,\mu_n}(A_1, A_1, A_2, A_3, \ldots, A_n) \\ &= \sum_{k=0}^{m_1} \begin{pmatrix} m_1 \\ k \end{pmatrix} \mathcal{T}_{\mathbb{E}}\left[\left(\chi_{(0,1)}\tilde{A}(s_1)d\nu(s_1)\right)^{k}\left(\chi_{(0,1)}\tilde{A}(s_1)d\eta(s_1)\right)^{m_1}\cdots\left(\chi_{(0,1)}\tilde{A}(s_n)d\mu(s_n)\right)^{m_n}\right] \\ &= \mathcal{T}_{\mathbb{E}}\left[\sum_{k=0}^{m_1} \begin{pmatrix} m_1 \\ k \end{pmatrix}\left(\chi_{(0,1)}\tilde{A}(s_1)d\nu(s_1)\right)^{k}\left(\chi_{(0,1)}\tilde{A}(s_1)d\eta(s_1)\right)^{m_1}\cdots\left(\chi_{(0,1)}\tilde{A}(s_n)d\mu(s_n)\right)^{m_n}\right] \\ &= \mathcal{T}_{\mathbb{E}}\Bigg[\left(\chi_{(0,1)}\tilde{A}(s_1)d\nu(s_1) + \chi_{(0,1)}\tilde{A}(s_1)d\eta(s_1)\right)^{m_1}\left(\chi_{(0,1)}\tilde{A}(s_2)d\mu(s_2)\right)^{m_2} \end{aligned}$$

$$\cdots \left(\chi_{(0,1)}\tilde{A}(s_n)d\mu(s_n)\right)^{m_n}\Bigg]. \tag{6.172}$$

In effect, then, we have replaced the expression '$\chi_{(0,1)}\tilde{A}(s_1)d(\nu+\eta)(s_1)$' with the expression '$\chi_{(0,1)}\tilde{A}(s_1)d\nu(s_1)+\chi_{(0,1)}\tilde{A}(s_1)d\eta(s_1)$', where the measures have been split up. A similar result can be achieved for operators using Theorem 4.3.8.

Example 38. Let $A, B \in \mathcal{L}(X)$, and let $\mu, \nu \in \mathcal{M}_{cb}[0,1]$. Then

$$\left(\chi_{(0,1)}\tilde{A}(s)d\mu(s)\right)^2 \left(\chi_{(0,1)}\tilde{B}(t)d\nu(t)\right) \in \mathbb{E} \tag{6.173}$$

and we can write

$$\begin{aligned}
&\left(\chi_{(0,1)}\tilde{A}(s)d\mu(s)\right)^2 \chi_{(0,1)}\tilde{B}(t)d\nu(t)\\
&= \chi_{(0,1)^3}(\underset{\cdot}{r}, \underset{\cdot}{s}, \underset{\cdot}{t})\tilde{A}(r)\tilde{A}(s)\tilde{B}(t)d\mu(r)d\mu(s)d\nu(t)\\
&= \chi_{[(0,t)\cup(t,1)]\times[(0,t)\cup(t,1)]\times(0,1)}(\underset{\cdot}{r}, \underset{\cdot}{s}, \underset{\cdot}{t})\tilde{A}(r)\tilde{A}(s)\tilde{B}(t)d\mu(r)d\mu(s)d\nu(t)\\
&\\
&= \chi_{(0,t)\times(0,t)\times(0,1)}(\underset{\cdot}{r}, \underset{\cdot}{s}, \underset{\cdot}{t})\tilde{A}(r)\tilde{A}(s)\tilde{B}(t)d\mu(r)d\mu(s)d\nu(t)\\
&\quad + \chi_{(0,t)\times(t,1)\times(0,1)}(\underset{\cdot}{r}, \underset{\cdot}{s}, \underset{\cdot}{t})\tilde{A}(r)\tilde{A}(s)\tilde{B}(t)d\mu(r)d\mu(s)d\nu(t)\\
&\quad + \chi_{(t,1)\times(0,t)\times(0,1)}(\underset{\cdot}{r}, \underset{\cdot}{s}, \underset{\cdot}{t})\tilde{A}(r)\tilde{A}(s)\tilde{B}(t)d\mu(r)d\mu(s)d\nu(t)\\
&\quad + \chi_{(t,1)\times(t,1)\times(0,1)}(\underset{\cdot}{r}, \underset{\cdot}{s}, \underset{\cdot}{t})\tilde{A}(r)\tilde{A}(s)\tilde{B}(t)d\mu(r)d\mu(s)d\nu(t)\\
&\\
&= \chi_{(0,t)}(\underset{\cdot}{r})\tilde{A}(r)d\mu(r)\chi_{(0,t)}(\underset{\cdot}{s})\tilde{A}(s)d\mu(s)\chi_{(0,1)}(\underset{\cdot}{t})\tilde{B}(t)d\nu(t)\\
&\quad + \chi_{(0,t)}(\underset{\cdot}{r})\tilde{A}(r)d\mu(r)\chi_{(t,1)}(\underset{\cdot}{s})\tilde{A}(s)d\mu(s)\chi_{(0,1)}(\underset{\cdot}{t})\tilde{B}(t)d\nu(t)\\
&\quad + \chi_{(t,1)}(\underset{\cdot}{r})\tilde{A}(r)d\mu(r)\chi_{(0,t)}(\underset{\cdot}{s})\tilde{A}(s)d\mu(s)\chi_{(0,1)}(\underset{\cdot}{t})\tilde{B}(t)d\nu(t)
\end{aligned}$$

$$+ \chi_{(t,1)}(\underset{\cdot}{r})\tilde{A}(r)d\mu(r)\chi_{(t,1)}(\underset{\cdot}{s})\tilde{A}(s)d\mu(s)\chi_{(0,1)}(\underset{\cdot}{t})\tilde{B}(t)d\nu(t)$$

$$\begin{aligned}
&= \chi_{(0,1)}\tilde{B}(t)\left(\chi_{(0,t)}\tilde{A}(s)d\mu(s)\right)^2 d\nu(t) \\
&\quad + 2\,\chi_{(0,1)}\left(\chi_{(t,1)}\tilde{A}(r)d\mu(r)\right)\tilde{B}(t)\left(\chi_{(0,t)}\tilde{A}(s)d\mu(s)\right)d\nu(t) \\
&\quad + \chi_{(0,1)}\left(\chi_{(t,1)}\tilde{A}(s)d\mu(s)\right)^2\tilde{B}(t)d\nu(t)
\end{aligned}$$

$$= \sum_{k=0}^{2}\begin{pmatrix}2\\k\end{pmatrix}\chi_{(0,1)}\left(\chi_{(t,1)}\tilde{A}(s)d\mu(s)\right)^{m-k}\tilde{B}(t)\left(\chi_{(0,t)}\tilde{A}(s)d\mu(s)\right)^{k}d\nu(t). \tag{6.174}$$

Hence, using Theorems 6.3.5 and 6.2.4,

$$\begin{aligned}
&P^{m,1}_{\mu,\nu}(A,B) \\
&\quad = \mathcal{T}_{\mathbb{E}}\left[\left(\chi_{(0,1)}\tilde{A}(s)d\mu(s)\right)^2\left(\chi_{(0,1)}\tilde{B}(t)d\nu(t)\right)\right] \\
&\quad = \mathcal{T}_{\mathbb{E}}\left[\sum_{k=0}^{2}\begin{pmatrix}2\\k\end{pmatrix}\chi_{(0,1)}\left(\chi_{(t,1)}\tilde{A}(s)d\mu(s)\right)^{m-k}\tilde{B}(t)\left(\chi_{(0,t)}\tilde{A}(s)d\mu(s)\right)^{k}d\nu(t)\right] \\
&\quad = \sum_{k=0}^{2}\begin{pmatrix}2\\k\end{pmatrix}\int_{(0,1)}\mathcal{T}_{\mathbb{E}}\left[\left(\chi_{(t,1)}\tilde{A}(s)d\mu(s)\right)^{m-k}\right]B\,\mathcal{T}_{\mathbb{E}}\left[\left(\chi_{(0,t)}\tilde{A}(s)d\mu(s)\right)^{k}\right]\nu(dt) \\
&\quad = \sum_{k=0}^{2}\begin{pmatrix}2\\k\end{pmatrix}\int_{(0,1)}P^{m-k}_{\mu|_{(t,1)}}(A)B\,P^{k}_{\mu|_{(0,t)}}(A)\nu(dt).
\end{aligned} \tag{6.175}$$

In these calculations we see that

$$\left(\chi_{(0,1)}\tilde{A}(s)d\mu(s)\right)^2\left(\chi_{(0,1)}\tilde{B}(t)d\nu(t)\right) \tag{6.176}$$

became

$$\sum_{k=0}^{2} \begin{pmatrix} 2 \\ k \end{pmatrix} \left(\chi_{(t,1)}\tilde{A}(s)d\mu(s)\right)^{m-k} \left(\chi_{(0,t)}\tilde{A}(s)d\mu(s)\right)^{k} \chi_{(0,1)}\tilde{B}(t)d\nu(t). \tag{6.177}$$

This hints at the possibility that we might be able to write these two expressions as an intermediate expression

$$`\left(\chi_{(0,t)}\tilde{A}(s)d\mu(s) + \chi_{(t,1)}\tilde{A}(s)d\mu(s)\right)^{2} \left(\chi_{(0,1)}\tilde{B}(t)d\nu(t)\right)', \tag{6.178}$$

and then apply a binomial theorem. However, this expression has not been defined in $\mathbb{E}$, or even in $\mathcal{V}$; our definition of multiplication in $\mathbb{E}$, based on that of $\mathcal{V}$ (Definition 6.4.3), applies only to elements of $\mathcal{V}$ that have fixed characteristic function sets, namely F_1, F_2 in Equation (6.95). To define (6.178) it may be necessary to define multiplication in the case when the characteristic function sets depend on the arguments of another characteristic function. For example, F_1, F_2 in Equation (6.95) in that case might need to be changed to $F_1(t_1, \ldots, t_n), F_2(s_1, \ldots, s_m)$, and one would have to see whether it would be possible to make sense of Definition 6.4.3 then.

Example 39. We end with a few comments on how to approximately disentangle a product $\tilde{A}\tilde{B}$ if $\tilde{B}$ is associated with a finite, continuous Borel measure ν_K on a generalized Cantor set K, and $\tilde{A}$ is associated with a finite, continuous Borel measure μ_J on the complement of that set $J = [0,1] \setminus K$ in the interval $[0,1]$. (Because the set is complicated, we are not working with a general monomial in $\tilde{A}$ and $\tilde{B}$.) Although one could define a measure on the set J one interval at a time, it would be impossible to do the same with K, since generalized Cantor sets contain no intervals. Instead we will assume that μ and ν are finite, continuous Borel measures on the entire interval $[0,1]$ and that μ_J and ν_K are their restrictions, respectively, to J and K, respectively.

Let us suppose that K is formed in the usual manner of generalized Cantor sets by removing an open interval I_1 from the center of the interval $[0,1]$, then removing open intervals I_2 and I_3 from, respectively, the centers of the remaining left interval and right interval, then removing each of the intervals I_4, I_5, I_6, I_7 from, respectively, the centers of the remaining four subintervals (left to right again), etc. We then have $J = \dot{\bigcup}_{j=1}^{\infty} I_j$ and $K = [0,1] \setminus J$. Since the sequence of deleted intervals is infinite, and the intervals do not overlap, the measures of the deleted intervals must approach zero for both μ and ν.

We are going to consider unions of the intervals $I_1, I_2, I_3, \ldots$ in two groups. For $n = 1, 2, \ldots$ we define $L_n := \dot{\bigcup}_{j=1}^{n} I_j$ and $R_n := \dot{\bigcup}_{j=n+1}^{\infty} I_j$, so that $L_n \dot\cup R_n = J$. Then $K = [0,1] \setminus L_n \setminus R_n$. Disentangling the monomial $\tilde{A}\tilde{B}$ using the associated measures $\mu_J = \mu|_J, \nu_K = \nu|_K$ (associating μ_J with A and ν_K with B) gives (by Theorem 6.2.4)

$$
\begin{aligned}
&P^{1,1}_{\mu_J,\nu_K}(A,B) \\
&= \mathcal{T}_{\mathbb{E}}\left[\chi_{J\times K}(\underset{\cdot}{s},\underset{\cdot}{t})\tilde{A}(s)\tilde{B}(t)d\mu(s)d\nu(t)\right] \\
&= \mathcal{T}_{\mathbb{E}}\left[\chi_{L_n\dot\cup R_n}(\underset{\cdot}{s})\chi_{[0,1]\setminus L_n\setminus R_n}(\underset{\cdot}{t})\tilde{A}(s)\tilde{B}(t)d\mu(s)d\nu(t)\right] \\
&= \mathcal{T}_{\mathbb{E}}\left[\left(\chi_{L_n}+\chi_{R_n}\right)(\underset{\cdot}{s})\left(\chi_{[0,1]}-\chi_{L_n}-\chi_{R_n}\right)(\underset{\cdot}{t})\tilde{A}(s)\tilde{B}(t)d\mu(s)d\nu(t)\right] \\
&= \mathcal{T}_{\mathbb{E}}\left[\chi_{L_n}(\underset{\cdot}{s})\chi_{[0,1]}(\underset{\cdot}{t})\tilde{A}(s)\tilde{B}(t)d\mu(s)d\nu(t)\right] - \mathcal{T}_{\mathbb{E}}\left[\chi_{L_n}(\underset{\cdot}{s})\chi_{L_n}(\underset{\cdot}{t})\tilde{A}(s)\tilde{B}(t)d\mu(s)d\nu(t)\right] \\
&\quad - \mathcal{T}_{\mathbb{E}}\left[\chi_{L_n}(\underset{\cdot}{s})\chi_{R_n}(\underset{\cdot}{t})\tilde{A}(s)\tilde{B}(t)d\mu(s)d\nu(t)\right] \\
&\quad + \mathcal{T}_{\mathbb{E}}\left[\chi_{R_n}(\underset{\cdot}{s})\chi_{[0,1]\setminus L_n\setminus R_n}(\underset{\cdot}{t})\tilde{A}(s)\tilde{B}(t)d\mu(s)d\nu(t)\right] \\
&= P^{1,1}_{\mu|_{L_n},\nu}(A,B) - P^{1,1}_{\mu|_{L_n},\nu|_{L_n}}(A,B) - \mathcal{T}_{\mathbb{E}}\left[\chi_{L_n}(\underset{\cdot}{s})\chi_{R_n}(\underset{\cdot}{t})\tilde{A}(s)\tilde{B}(t)d\mu(s)d\nu(t)\right] \\
&\quad + \mathcal{T}_{\mathbb{E}}\left[\chi_{R_n}(\underset{\cdot}{s})\chi_{[0,1]\setminus L_n\setminus R_n}(\underset{\cdot}{t})\tilde{A}(s)\tilde{B}(t)d\mu(s)d\nu(t)\right].
\end{aligned}
\tag{6.179}
$$

Since the set L_n is a union of disjoint intervals, the first two terms in the last expression may be calculated by existing methods, and they serve as the estimate of $P^{1,1}_{\mu_J,\nu_K}(A,B)$. As for the other two terms, we can say of their operator norms in $\mathcal{L}(X)$ that

$$\begin{aligned}
&\left\| \mathcal{T}_{\mathbb{E}}\left[\chi_{L_n}(\underset{\cdot}{s})\chi_{R_n}(\underset{\cdot}{t})\tilde{A}(s)\tilde{B}(t)d\mu(s)d\nu(t)\right]\right\|_{\mathcal{L}(X)} \\
&= \left\| \int_{\{s>t\}} \chi_{L_n}(s)\chi_{R_n}(t)AB(\mu\times\nu)(ds,dt) + \int_{\{t>s\}} \chi_{L_n}(s)\chi_{R_n}(t)BA(\mu\times\nu)(ds,dt)\right\|_{\mathcal{L}(X)} \\
&\leq \|A\|_{\mathcal{L}(X)}\|B\|_{\mathcal{L}(X)}\left[\int_{\{s>t\}} \chi_{R_n}(t)(\mu\times\nu)(ds,dt) + \int_{\{t>s\}} \chi_{R_n}(t)(\mu\times\nu)(ds,dt)\right] \\
&= \|A\|_{\mathcal{L}(X)}\|B\|_{\mathcal{L}(X)}\,\mu([0,1])\nu(R_n). \qquad (6.180)
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\left\| \mathcal{T}_{\mathbb{E}}\left[\chi_{R_n}(\underset{\cdot}{s})\chi_{[0,1]\smallsetminus L_n\smallsetminus R_n}(\underset{\cdot}{t})\tilde{A}(s)\tilde{B}(t)d\mu(s)d\nu(t)\right]\right\|_{\mathcal{L}(X)} \\
&\qquad \leq \|A\|_{\mathcal{L}(X)}\|B\|_{\mathcal{L}(X)}\,\mu(R_n)\nu([0,1]). \qquad (6.181)
\end{aligned}$$

Combining these two gives us our error estimate, so by choice of sufficiently large $n\in\mathbb{N}$ we may force $\mu(R_n)$ and $\nu(R_n)$ to be small and get our estimated value of the operator $P^{1,1}_{\mu_J,\nu_K}(A,B)$ as close to the correct value as we would like. Disentangling a monomial exactly in the case of a generalized Cantor set and its complement might require a different approach—something recursive, perhaps.

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