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Fan Cohomology and Its Application to Equivariant K-Theory of Toric Varieties

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FAN COHOMOLOGY AND ITS APPLICATION TO EQUIVARIANT K-THEORY OF TORIC VARIETIES

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FAN COHOMOLOGY AND ITS APPLICATION TO EQUIVARIANT K-THEORY OF TORIC VARIETIES

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Mu-Wan Huang, Mark Walker and I established an explicit formula for the equivariant K-groups of affine toric varieties. We also recovered a result due to Vezzosi and Vistoli, which expresses the equivariant K-groups of a smooth toric variety in terms of the K-groups of its maximal open affine toric subvarieties. This dissertation investigates the situation when the toric variety X is neither affine nor smooth. In many cases, we compute the Čech cohomology groups of the presheaf K_q^T on X endowed with a topology. Using these calculations and Walker's Localization Theorem for equivariant K-theory, we give explicit formulas for the equivariant K-groups of toric varieties associated to all two dimensional fans and certain three dimensional fans.

To my aunt, Khanh Au,
who instilled in me an intellectual curiosity and inspired me to be a teacher.

ACKNOWLEDGMENTS

To my advisor, Mark Walker, I owe an enormous debt of gratitude for guiding me through my graduate career and teaching me much of what I know about my research area. I am fortunate to have studied at the University of Nebraska-Lincoln Mathematics Department, which has provided me with a better environment for graduate study than any student could hope for. Here, I have benefited from the advice of many outstanding and devoted professors. In particular, I would like to thank Brian Harbourne and Srikanth Iyengar for their thoughtful comments on this dissertation, and Jim Lewis for sharing with me his knowledge about teaching. I also want to express my appreciation to the many friends and colleagues who have greatly enriched my life in Lincoln. A special thanks goes to Mu-Wan Huang, who has been an invaluable research partner. Thank you to Herman Batelaan for his friendship and encouragement through the years. As always, I am deeply grateful for the love and support of my parents, Vinh and Phi Au, my sisters, Galant Chan and Gretchen Daniel, and my brothers-in-law, Edward Chan and Jim Daniel. Finally, I want to thank Fatemeh Parsa for being my rock and my wings depending on what is needed at the moment.

Contents

Chapter 1

Introduction

A torus in the theory of toric varieties is an algebraic group of the form $T := \mathrm{Spec}(k[x_1^{\pm 1}, \cdots, x_n^{\pm 1}])$, for some $n \in \mathbb{N}$, where k is an arbitrary fixed field. A toric variety X is defined to be a normal variety, that contains a torus T as a dense open subvariety, together with an action of T on X that extends the action of T on itself. It turns out that toric varieties arise from geometric objects called strongly convex rational polyhedral cones and fans of such things [8]. Most of the properties of toric varieties correspond to properties of the associated fans. For example, the singularities of a toric variety are determined by the combinatorial data of the fans. This correspondence between properties of the varieties and those of the associated fans makes toric varieties good examples for many phenomena in algebraic geometry.

Given an exact category $\mathcal{A}$, Quillen constructed a "K-theory space", $K\mathcal{A}$, and defined the K-group $K_n\mathcal{A}$ of $\mathcal{A}$ to be its homotopy group $\pi_n K\mathcal{A}$ [15]. The K-groups of an algebraic variety X are abelian groups, $K_q(X)$, for $q \geq 0$, defined in this fashion from the exact category of vector bundles over X. $K_0(X)$ can be concretely described in terms of generators and relations. It is generated by isomorphism classes of vector bundles over

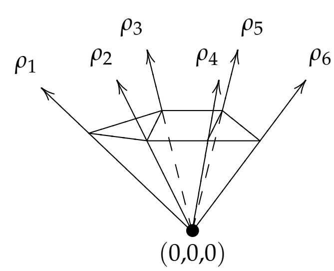
X modulo relations given by short exact sequences of vector bundles. If an algebraic group G over a field k acts on a given variety X, then we can study the $G-$equivariant vector bundles over X. The K-groups of the exact category of $G-$equivariant vector bundles over X are known as the $G-$equivariant K-groups of X, denoted by $K_q^G(X)$, for $q \geq 0$. In general, K-groups are very difficult to understand. However, due to the concreteness of the fan structures, the problem of computing the K-groups of toric varieties becomes more tractable. This dissertation is concerned with computing $T-$equivariant K-groups of toric varieties. The next paragraph contains a rough description of some of our prior results, which are the result of joint work with Mu-wan Huang and Mark Walker.

An affine toric variety U_σ is determined by a single cone σ and a fixed field k. The K-group $K_0^T(U_\sigma)$ is a free abelian group whose generators are given by the combinatorial data of σ (see Theorem 3.1), and the higher equivariant K-groups of U_σ are the higher K-groups of k tensored with $K_0^T(U_\sigma)$ over $\mathbb{Z}$. A general toric variety X is constructed by patching the open affine toric subvarieties along their intersections. In the case that X is smooth, Vezzosi and Vistoli proved that $K_q^T(X)$ is determined by the equivariant K-groups of the maximal open affine toric subvarieties and their intersections. (See Theorem 4.4.) Thus, the equivariant K-groups of affine toric varieties and smooth toric varieties are well understood. The main results of this thesis are about equivariant K-groups of toric varieties that are not necessarily affine or smooth. The rest of this chapter is an overview of the thesis.

1.1 Detailed Overview

A *strongly convex rational polyhedral cone* is the collection of non-negative real linear combinations of a set of vectors in $\mathbb{Q}^n$ that does not contain a non-zero subspace of $\mathbb{R}^n$.

This is the type of cones we will be working with, unless otherwise stated. Given a cone σ and a field k, we construct a monoid ring $k[\sigma^\vee \cap \mathbb{Z}^n]$ associated to the lattice points in the dual cone $\sigma^\vee$. The *affine toric variety* U_σ *associated to* σ is defined to be the spectrum of $k[\sigma^\vee \cap \mathbb{Z}^n]$. The intersection of σ with a set of the form $\{v \in \mathbb{R}^n | u \cdot v = 0\}$ for some $u \in \sigma^\vee$ is called a *face* of σ. If we have a finite set of cones Δ such that every face of a cone in Δ is also a cone in Δ and the intersection of two cones in Δ is a face of each, then we call Δ a *fan*. For example, the fan below consists of eighteen cones – $\sigma_1, \sigma_2, \sigma_3$ and all their faces, where σ_1 is generated by ρ_1, ρ_2, ρ_3; σ_2 is generated by $\rho_2, \cdots, \rho_5$, and σ_3 is generated by ρ_4, ρ_5, ρ_6.



The *toric variety* $X(\Delta)$ corresponding to Δ is obtained by patching affine toric varieties associated to the maximal cones in Δ along open subvarieties given by the intersections of maximal cones in Δ.

Let Δ be a fan in $\mathbb{R}^n$. The variety $X(\Delta)$ contains the torus $T := \mathrm{Spec}(k[x_1^{\pm 1}, \cdots, x_n^{\pm 1}])$ as an open subvariety, hence the name toric variety. Moreover, there is an action of T on $X(\Delta)$ that extends the action of T on itself. An *equivariant vector bundle over* $X(\Delta)$ is a vector bundle with a torus action by T that commutes with the action of T on $X(\Delta)$. The group $K_0^T(X(\Delta))$ is the group completion of the abelian monoid of isomorphism classes of equivariant vector bundles on $X(\Delta)$ modulo relations coming from short exact sequences. Since an equivariant bundle over the affine variety U_σ is given by a $\mathbb{Z}^n-$graded projective module over $k[\sigma^\vee \cap \mathbb{Z}^n]$, $K_0^T(U_\sigma)$ is just the group completion of the abelian monoid of isomorphism classes of projective graded modules over $k[\sigma^\vee \cap \mathbb{Z}^n]$. The higher equivariant K-groups $K_q^T(X(\Delta))$ are the K-groups of the exact category of equivariant vector bundles over $X(\Delta)$.

In [1] , Mu-Wan Huang, Mark Walker and I proved the following theorem for the equivariant K-theory of affine toric varieties. (See Chapter 3.)

Theorem (Au, Huang, Walker)**.** *For all strongly convex rational polyhedral cones σ in $\mathbb{R}^n$ and integers $q \geq 0$, there is a natural isomorphism*

$$K_q^T(U_\sigma) \cong \mathbb{Z}[M_\sigma] \otimes_{\mathbb{Z}} K_q(k),$$

where $M_\sigma = \mathbb{Z}^n/(\sigma^\perp \cap \mathbb{Z}^n)$ and $\sigma^\perp = \{u \in \mathbb{R}^n | u \cdot v = 0 \text{ for all } v \in \sigma\}$. In particular, $K_0^T(U_\sigma) \cong \mathbb{Z}[M_\sigma]$.

Notice that a fan Δ determines a poset whose elements are the cones in Δ and the order relation $\prec$ is given by face containment. Therefore, one can put the "poset topology" on Δ, meaning that $\Lambda \subseteq \Delta$ is open if and only if whenever $\sigma \in \Lambda$ and $\tau \prec \sigma$, we have $\tau \in \Lambda$. In other words, the open sets of Δ are subfans of Δ [2], [3]. In this finite topological space, the smallest open set containing a cone σ is the subfan $\langle \sigma \rangle$ consisting of σ and all of its faces. Thus, for all sheaves $\mathcal{F}$ on Δ, $\mathcal{F}(\langle \sigma \rangle) = \mathcal{F}_\sigma$, so a sheaf $\mathcal{F}$ on Δ is determined by its stalks and the maps between them. Since a containment of subfans $\Lambda \subseteq \Lambda'$ in Δ induces a homomorphism $K_q^T(X(\Lambda')) \to K_q^T(X(\Lambda))$, for all $q \geq 0$, the map $\Lambda \mapsto K_q^T(X(\Lambda))$ is a presheaf on Δ. In [1], we also recovered a theorem of Vezzosi and Vistoli [17, Theorem 6.2] concerning this presheaf on smooth fans Δ. (See Chapter 4.)

Theorem ([1]) **.** *Let Δ be a smooth fan. The presheaf $\Lambda \mapsto K_q^T(X(\Lambda))$ on Δ is a flasque sheaf. Moreover, there is a natural isomorphism*

$$K_q^T(X(\Delta)) \cong H^0\left(\Delta, \widetilde{K_0^T}\right) \otimes K_q(k). \tag{1.1}$$

And,

$$K_0^T(X(\Delta)) \cong H^0\left(\Delta, \widetilde{K_0^T}\right),$$

where $\widetilde{K_0^T}$ denotes the sheafification of the presheaf K_0^T on Δ, i.e. the sheaf whose stalk at $\sigma \in \Delta$ is $K_0^T(U_\sigma) \cong \mathbb{Z}[M_\sigma]$.

We will refer to this sheaf cohomology $H^\bullet\left(\Delta, \widetilde{K_0^T}\right)$ of $\widetilde{K_0^T}$ as "*fan cohomology*". The proof of the theorem above requires a spectral sequence by R. W. Thomason [16]. Namely, if $\mathcal{V}$ is an equivariant open cover of a smooth variety X equipped with an action by T, then there is a convergent spectral sequence

$$\check{H}^p(\mathcal{V}, K_q^T) \Longrightarrow K_{q-p}^T(X), \tag{1.2}$$

where $\check{H}^p(\mathcal{V}, K_q^T)$ are the Čech cohomology groups of K_q^T with respect to the cover $\mathcal{V}$.

As a result of (1.1), the equivariant K-groups of smooth toric varieties can be expressed in terms of $K_q(k)$ and $H^0\left(\Delta, \widetilde{K_0^T}\right)$. The lower K-groups of a field are well understood. For example, $K_0(k) = \mathbb{Z}, K_1(k) = k^\times$ and $K_2(k)$ is the quotient of $k^\times \otimes k^\times$ by the subgroup generated by the elements $x \otimes (1-x)$ for $x \in k \backslash \{0, 1\}$ [9]. When the field is finite, Quillen gave the following formulas for the K-groups [14, Theorem 8]:

$$K_q(\mathbb{F}_{p^e}) = \begin{cases} \mathbb{Z}, & q = 0 \\ 0, & q = 2i, \text{ for } i \in \mathbb{N} \\ \mathbb{Z}/(p^{ei}-1), & q = 2i-1, \text{ for } i \in \mathbb{N} \end{cases}$$

The group $H^0\left(\Delta, \widetilde{K_0^T}\right)$ is also relatively well understood, because Δ is a finite topological space and the sheaf $\widetilde{K_0^T}$ is determined by the stalks $\left(\widetilde{K_0^T}\right)_\sigma = \mathbb{Z}[M_\sigma]$, for $\sigma \in \Delta$. (For instance, see Example 4.3.)

This thesis is motivated by the isomorphism (1.1), and we want to know if it remains true for a general toric variety. In the cases where the isomorphism (1.1) does not hold, we want to measure how far off $K_q^T(X(\Delta))$ is from $H^0\left(\Delta, \widetilde{K_0^T}\right) \otimes K_q(k)$. In our investigation, we will use the non-smooth version of the spectral sequence (1.2) due to Walker [20]. If Δ is a fan whose associated toric variety $X(\Delta)$ is quasi-projective, then there is a convergent spectral sequence

$$\check{H}^p(\mathcal{V}, K_q^T) \Longrightarrow K_{q-p}^T(X(\Delta)), \tag{1.3}$$

where $\mathcal{V}$ is the equivariant open cover $\{U_\sigma | \sigma \text{ is a maximal cone in } \Delta\}$ of $X(\Delta)$. Since $\check{H}^p(\mathcal{V}, K_q^T) \cong H^p\left(\Delta, \widetilde{K_q^T}\right)$ (see isomorphism (2.15) on P. 22), one way to approach the problem of computing the equivariant K-groups of a general toric variety is to study its fan cohomology. If $H^{p+1}\left(\Delta, \widetilde{K_0^T}\right)$ is torsion free, the isomorphism (1.1) along with [21, Theorem 5.6.4] yield the following isomorphism.

$$H^p\left(\Delta, \widetilde{K_q^T}\right) \cong H^p\left(\Delta, \widetilde{K_0^T}\right) \otimes_{\mathbb{Z}} K_q(k)$$

This implies we really only need to consider the case when $q = 0$, if $H^p\left(\Delta, \widetilde{K_0^T}\right)$ is torsion free. Note that we know of no example of a fan Δ such that $H^p(\Delta, \widetilde{K_0^T})$ has torsion.

In this thesis, we prove the following results about $H^p\left(\Delta, \widetilde{K_0^T}\right)$. In Section 5.1, we show in Corollary 5.7 that if Δ is a fan, then $H^p\left(\Delta, \widetilde{K_0^T}\right) = 0$ for all $p \geq d$, where $d = \max\{\dim \sigma | \sigma \in \Delta\}$. Hence, for each q, only finitely many E_2 terms of the spectral sequence (1.3) are non-zero. We also verify in Lemma 5.14 that a large class of fans has the property that all higher fan cohomology groups vanish. In Sections 5.3 and 5.2, we compute the $(n-1)^{st}$ fan cohomology for non-complete and complete fans in $\mathbb{R}^n$ as stated in the following two theorems. (A fan Δ in $\mathbb{R}^n$ is complete if $\bigcup_{\sigma\in\Delta}\sigma = \mathbb{R}^n$.)

Theorem. *If Δ is a non-complete fan in $\mathbb{R}^n$ with $n > 1$, then $H^{n-1}\left(\Delta, \widetilde{K_0^T}\right) = 0$.*

Theorem. *Let Δ be an n-dimensional complete fan, where $n > 1$. Suppose Δ has s one-dimensional cones $\rho_1, \cdots, \rho_s$, and let v_i be the minimal lattice point of ρ_i, for $i = 1, \cdots, s$. Then $H^{n-1}\left(\Delta, \widetilde{K_0^T}\right) \cong \mathbb{Z}^{g-1}$, where g is the greatest common divisor of the set of all 2×2 minors of the matrix $\begin{bmatrix} v_1 \\ \vdots \\ v_s \end{bmatrix}$.*

Notice that if Δ contains a single smooth two dimensional cone, then g is 1, and thus $H^{n-1}(\Delta, \widetilde{K_0^T}) = 0$. So, in a sense, the number g as defined in the theorem above measures singularities of the fan. Finally, in Section 5.4, we demonstrate a large class of non-complete 3-dimensional fans Δ having the property that $H^1\left(\Delta, \widetilde{K_0^T}\right)$ is torsion free. As mentioned earlier, we know of no example of a fan Δ such that $H^p(\Delta, \widetilde{K_0^T})$ has torsion.

At this point, we have a good understanding of $H^\bullet\left(\Delta, \widetilde{K_0^T}\right)$ for any two dimensional fan and some three dimensional fans. In Chapter 6, we express the $K-$groups $K_q^T(X(\Delta))$ of toric varieties arising from these fans in terms of $H^0\left(\Delta, \widetilde{K_0^T}\right)$ and the $K-$groups, $K_q(k)$, of the ground field k. The isomorphism (1.1) holds for all two dimensional non-complete fans. For two dimensional complete fans and those three dimensional fans with $H^1\left(\Delta, \widetilde{K_0^T}\right) = 0$, $K_q^T(X(\Delta))$ fits into the following short exact sequence:

$$0 \longrightarrow K_{q+1}(k)^{g-1} \longrightarrow K_q^T(X(\Delta)) \longrightarrow H^0\left(\Delta, \widetilde{K_0^T}\right) \otimes K_q(k) \longrightarrow 0$$

Also, notice that toric varieties can be built over any ground field. If the ground field k is finite, then many of the E_2 terms of the spectral sequence (1.3) vanish due to Quillen's result that $K_{2i}(k) = 0$ for all $i \in \mathbb{N}$ [14, Theorem 8]. Hence, computing the K-theory of toric varieties becomes manageable even without knowing some of the fan cohomology groups, $H^\bullet(\Delta, K_0^T)$. This is discussed in Section 6.3.

Chapter 2

Background

2.1 Cones and Fans

2.1.1 Cones

In this section, we introduce properties of convex polyhedral cones and fans, which can be found in several standard textbooks such as [4], [12] and [8].

Let N be an abelian group isomorphic to $\mathbb{Z}^n$. We refer to N as a *lattice* and write the real vector space $N \otimes_{\mathbb{Z}} \mathbb{R}$ as $N_{\mathbb{R}}$. A *convex polyhedral cone* σ is defined to be a subset of $N_{\mathbb{R}}$ of the form

$$\sigma = \langle v_1, \cdots, v_j \rangle = \{r_1 v_1 + \cdots + r_j v_j | r_i \in \mathbb{R}_{\geq 0}\}$$

for some $v_1, \cdots, v_j \in N_{\mathbb{R}}$. The vectors $v_1, \cdots, v_j$ are called *generators* of the cone σ. Here are some pictures of cones.

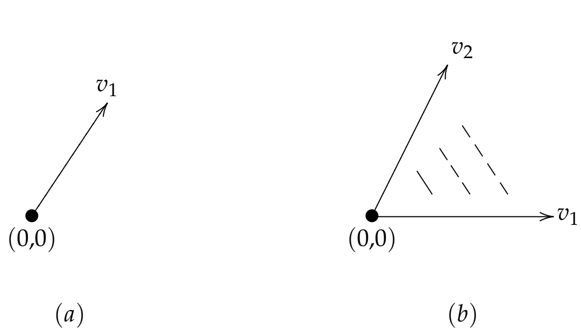


Figure 2.1: Convex Polyhedral Cones

A cone is said to be *simplicial* if its generators can be chosen to be linearly independent. In Figure 2.1, (a) and (b) are simplicial and (c) is not. The *dimension* of a cone σ is the dimension of the vector space spanned by σ, i.e.,

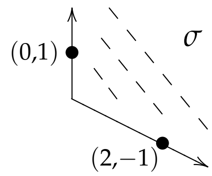
$$\dim(\sigma) = \dim(\mathbb{R}\sigma),$$

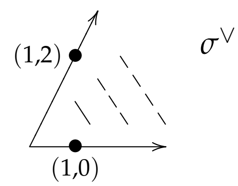
where $\mathbb{R}\sigma = \sigma + (-\sigma)$. Let M be the dual lattice, $Hom(N, \mathbb{Z})$, of N. For a cone, σ, its *dual* $\sigma^\vee \subseteq M_\mathbb{R} = M \otimes_\mathbb{Z} \mathbb{R}$ is given by

$$\sigma^\vee = \{u \in M_\mathbb{R} | u(v) \geq 0 \text{ for all } v \in \sigma\}.$$

Example 2.2. *Suppose $N = \mathbb{Z}^2$, then $N_\mathbb{R} \cong \mathbb{R}^2$ and $M_\mathbb{R} \cong \mathbb{R}^2$. If σ is the cone in $\mathbb{R}^2$ generated by e_1, then $\sigma^\vee$ is the right half-space.*

Example 2.3. *Let N be $\mathbb{Z}^2$ and σ be the cone generated by e_2 and $2e_1 - e_2$, then $\sigma^\vee$ is generated by e_1 and $e_1 + 2e_2$.*





The following statement is an important fact from the theory of convex sets:

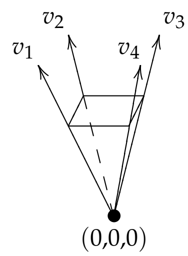
(*) If σ is a convex polyhedral cone and $v \in N_\mathbb{R} \backslash \sigma$, then there is some $u \in \sigma^\vee$ such that $u(v) < 0$.

From (*), one can deduce many properties of convex polyhedral cones, including the fact that the dual of a convex polyhedral cone is a convex polyhedral cone (known as Farkas' Theorem) and the dual of $\sigma^\vee$ is σ again.

Given a dual vector $u \in M_\mathbb{R}$, $u^\perp$ is the set $\{v \in N_\mathbb{R} | u(v) = 0\}$. A *face* τ of a cone σ is defined to be the intersection of any supporting hyperplane with the cone, i.e.

$$\tau = \sigma \cap u^\perp = \{v \in \sigma | u(v) = 0\}$$

for some $u \in \sigma^\vee$. It is clear that a cone is a face of itself with $u = 0$. A face of a cone σ is generated by the generators of σ which are in the kernel of u, and so σ has finitely many faces and each of them is a convex polyhedral cone. Faces of codimension one are called *facets* of the cone. For example, the following cone has one $3-$dimensional face, four facets, four $1-$dimensional faces (rays) and a $0-$dimensional face.



By "$\tau \preceq \sigma$", we mean τ is a face of the cone σ. One can check the following properties of faces by using (*):

1. An intersection of faces is also a face.

2. If $\delta \preceq \tau$ and $\tau \preceq \sigma$, then $\delta \preceq \sigma$.

3. A proper face is the intersection of all facets containing it.

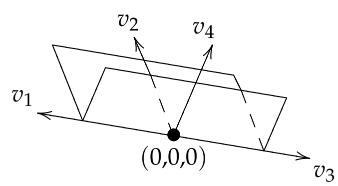
4. The topological boundary of a cone that spans $N_{\mathbb{R}}$ is the union of its facets.

5. There is a one-to-one correspondence between the faces of σ and the faces of $\sigma^\vee$. More specifically, if $\tau \preceq \sigma$, then $\tau^\perp \cap \sigma^\vee \preceq \sigma^\vee$ and

$$\dim(\tau) + \dim(\tau^\perp \cap \sigma^\vee) = \dim(N_{\mathbb{R}}),$$

where $\tau^\perp = \{w \in M_{\mathbb{R}} | w(v) = 0, \forall v \in \tau\}$. Another consequence of (*) is the Separation Lemma, which says that if α and β are convex polyhedral cones that share a common face τ, then there exists $u \in \alpha^\vee \cap (-\beta)^\vee$ such that $\tau = \alpha \cap u^\perp = \beta \cap u^\perp$.

A cone in $N_{\mathbb{R}}$ is said to be *rational* if all of its generators $v_1, \cdots, v_j$ may be taken to belong to N or, equivalently, $N_{\mathbb{Q}}$. The dual of a convex rational polyhedral cone is rational. For reasons that will be discussed later, we want the zero cone to be a face, and so we will be working with *strongly convex cones*, which are cones that do not contain any non-zero subspaces of $N_{\mathbb{R}}$. The cones in Figure 2.1.1 are all strongly convex. The "infinite trough" below is a cone that is not strongly convex.



If σ is strongly convex, then each element of a minimal generating set of σ generates a one dimensional face of σ, and this accounts for all the one dimensional faces of σ. From now on, when we say "cone", we usually mean strongly convex rational polyhedral cone, unless otherwise stated, because we only build toric varieties from this type of cone. A strongly convex polyhedral cone is *smooth* if the set of minimal lattice points along its rays may be extended to a $\mathbb{Z}-$basis of N. For example, the cone σ in example 2.3 is not smooth but the cone generated by e_1 and e_2 is.

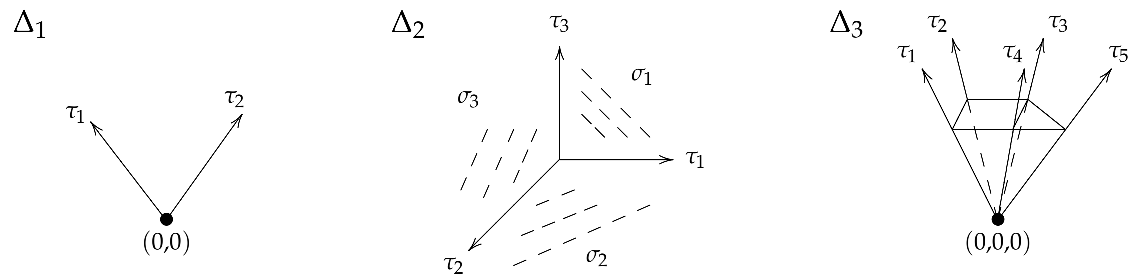
2.1.2 Fans

Certain cones can be "glued" together along their common faces to form a larger geometric object called a *fan*.

Definition 2.4. *A* fan *is a finite set Δ of cones in $N_{\mathbb{R}}$ such that*

- *every face of a cone in Δ is also a cone in Δ and*
- *the intersection of two cones in Δ is a face of each.*

Example 2.5. *The fans $\Delta_1 = \{\tau_1, \tau_2, 0\}$ and $\Delta_2 = \{\sigma_1, \sigma_2, \sigma_3, \tau_1, \tau_2, \tau_3, 0\}$ below have two and three maximal cones respectively. The fan Δ_3 consists of two 3-dimensional cones, six 2-dimensional cones, five 1-dimensional cones and the zero cone.*



A fan Δ in $N_{\mathbb{R}} \cong \mathbb{R}^n$ is *complete* if its support $|\Delta| := \bigcup_{\sigma \in \Delta} \sigma$ is all of $\mathbb{R}^n$. The fan Δ_2 above is complete, while Δ_1 and Δ_3 are not. The *dimension* of a fan $\Delta, \dim \Delta$, is defined to be the dimension of the real vector space spanned by the generators of the cones in Δ — i.e., the smallest subspace of $N_{\mathbb{R}}$ containing every cone in Δ.

A fan is smooth if each of its cones is smooth. We call the fan consisting of a single cone σ and its faces *the fan spanned by* σ, and we write it as $\langle \sigma \rangle$.

2.2 Toric Varieties

Suppose σ is a convex rational polyhedral cone in $N_{\mathbb{R}}$. Gordon's Lemma says that $\sigma^\vee \cap M$ is a finitely generated abelian monoid under addition.

Example 2.6. *Let $N = \mathbb{Z}^2$ and σ be the cone in $N_{\mathbb{R}}$ generated by e_2 and $2e_1 - e_2$, then $\sigma^\vee \cap M \cong \sigma^\vee \cap \mathbb{Z}^2$ is the abelian monoid generated by $e_1, e_1 + e_2$ and $e_1 + 2e_2$.*

Now, we fix a ground field k. Then $k[\sigma^\vee \cap M]$ is a monoid ring and a $k-$algebra. A typical element in $k[\sigma^\vee \cap M]$ has the form $\sum_{i=1}^{j} a_i \chi^{m_i}$, for some $j \in \mathbb{N}, a_1, \cdots, a_j \in k$ and $m_1, \cdots, m_j \in \sigma^\vee \cap M$. Here, χ is just an arbitrary symbol used to convey the fact that multiplication in $k[\sigma^\vee \cap M]$ is determined by the addition in $\sigma^\vee \cap M$, i.e., $\chi^m \chi^{m'} = \chi^{m+m'}$ for all $m, m' \in \sigma^\vee \cap M$.

Definition 2.7. *Let σ be a convex rational polyhedral cone in $N_\mathbb{R}$, then the affine toric variety associated to σ is defined to be*

$$U_\sigma := \mathrm{Spec}(k[\sigma^\vee \cap M]).$$

Many references for toric varieties assume k to be algebraically closed, and some such as [4] and [12] simply take k to be $\mathbb{C}$. However, toric varieties can be built over any ground field, and more importantly the properties of varieties discussed in this thesis do not depend on the properties of k.

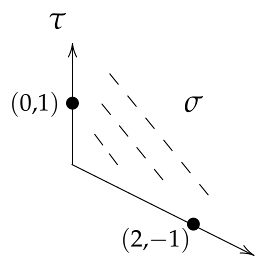
Example 2.8. *Consider the cone σ in example 2.6 above generated by e_2 and $2e_1 - e_2$. The abelian monoid $\sigma^\vee \cap \mathbb{Z}^2$ is generated by $e_1, e_1 + e_2$ and $e_1 + 2e_2$. Let $s = \chi^{e_1}$ and $t = \chi^{e_2}$, then $\chi^{e_1+e_2} = st$ and $\chi^{e_1+2e_2} = st^2$. So, $k[\sigma^\vee \cap \mathbb{Z}^2] \cong k[s, st, st^2]$. Therefore, the associated affine toric variety is*

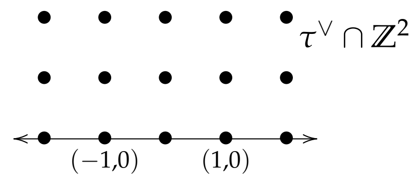
$$U_\sigma = \mathrm{Spec}(k[s, st, st^2]) \cong \mathrm{Spec}(k[x, y, z]/(y^2 - xz)).$$

Suppose $\varphi : N \to N'$ is a homomorphism of lattices that maps a cone α in $N_\mathbb{R}$ into a cone β in $N'_\mathbb{R}$. Then φ induces a map from M' to M which maps $\beta^\vee$ into $\alpha^\vee$. Hence, φ determines a ring map $k[\beta^\vee \cap M'] \to k[\alpha^\vee \cap M]$, which corresponds to a morphism $U_\alpha \to U_\beta$ of affine toric varieties. In particular, if τ is a face of σ, then $\sigma^\vee \cap M$ is a sub-monoid of $\tau^\vee \cap M$. Since $\tau^\vee$ is given by $\sigma^\vee + \mathbb{R}_{\geq 0} \cdot (-u)$, where $\tau = \sigma \cap u^\perp$, the ring

$k[\tau^\vee \cap M]$ is the localization of $k[\sigma^\vee \cap M]$ obtained by inverting χ^u. That implies that U_τ is an open subset of U_σ.

Example 2.9. *Let σ be the cone in $\mathbb{R}^2$ generated by e_2 and $2e_1 - e_2$ as in example 2.6, and let τ be the face generated by e_2.*





The ring $k[\tau^\vee \cap \mathbb{Z}^2]$ is isomorphic to $k[\sigma^\vee \cap \mathbb{Z}^2]_{\chi^{e_1}}$. Because $k[\sigma^\vee \cap \mathbb{Z}^2] \cong k[s, st, st^2]$,

$$k[\tau^\vee \cap \mathbb{Z}^2] \cong k[s, st, st^2]_{(s)} \cong k[s^{\pm 1}, t]$$

and

$$U_\tau = \operatorname{Spec}(k[s^{\pm 1}, t]) \hookrightarrow \operatorname{Spec}(k[s, st, st^2]) = U_\sigma.$$

From the picture of $\tau^\vee \cap \mathbb{Z}^2$, we can see that U_τ is indeed $\operatorname{Spec}(k[s^{\pm 1}, t])$.

Let σ be the zero cone in $N_{\mathbb{R}} \cong \mathbb{R}^n$, then $\sigma^\vee$ is $M_{\mathbb{R}}$. Suppose $\{e_1, \cdots, e_n\}$ is a basis for M. For $i = 1, \cdots, n$, let $x_i = \chi^{e_i}$. Then the monoid ring $k[\sigma^\vee \cap M] = k[M]$ is the Laurent polynomial ring in n variables, $k[x_1^{\pm 1}, \cdots, x_n^{\pm 1}]$. Thus, the affine toric variety associated to the zero cone is

$$T_N := U_{\{0\}} \cong \operatorname{Spec}(k[x_1^{\pm 1}, \cdots, x_n^{\pm 1}])$$

We call T_N the *n-dimensional torus*. Since every cone has the zero cone as a face, every affine toric variety contains the torus as an open subvariety, hence the name *toric variety*.

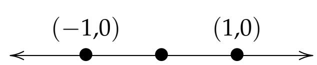
Given a fan Δ, the *toric variety* $X(\Delta)$ corresponding to Δ is obtained by patching affine toric varieties associated to the maximal cones in Δ along open subvarieties given by the intersections of those maximal cones, i.e.,

$$X(\Delta) = \varinjlim_{\sigma \in \Delta} U_\sigma.$$

For any two cones α and β in Δ, the intersection of U_α and U_β is the open subvariety $U_{\alpha\cap\beta}$ given by their common face $\alpha \cap \beta$ due to the following identities:

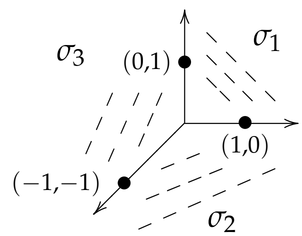
$$\mathrm{Spec}(k[\alpha^\vee \cap M]) \cap \mathrm{Spec}(k[\beta^\vee \cap M]) = \mathrm{Spec}(k[(\alpha^\vee \cap M) + (\beta^\vee \cap M)]) = \mathrm{Spec}(k[(\alpha \cap \beta)^\vee \cap M])$$

Example 2.10. *Suppose $N = \mathbb{Z}$ and Δ is the 1-dimensional complete fan with two maximal cones generated by e_1 and $-e_1$ respectively.*



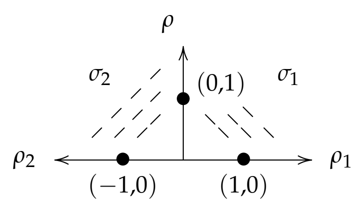
These maximal cones correspond to the rings $k[x]$ and $k[x^{-1}]$. Patching $\mathrm{Spec}(k[x])$ and $\mathrm{Spec}(k[x^{-1}])$ along $\mathrm{Spec}(k[x^{\pm}])$ gives the variety $X(\Delta) = \mathbb{P}^1_k$, projective 1-space.

Example 2.11. *Consider the following 2-dimensional complete fan Δ with three maximal cones, σ_1, σ_2 and σ_3.*



The cones σ_1, σ_2 and σ_3 correspond to the rings $k[x,y], k[y^{-1}, xy^{-1}]$ and $k[x^{-1}, x^{-1}y]$ respectively; patching gives the variety $X(\Delta) = \mathbb{P}^2_k$, projective 2-space.

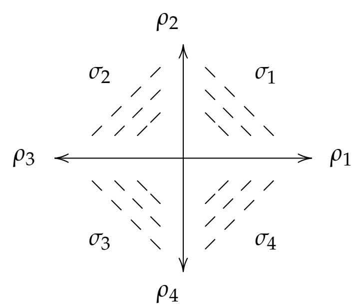
Example 2.12. *Let Δ be the following fan in $\mathbb{R}^2$ with two maximal cones σ_1 and σ_2.*



$k[\sigma_1^\vee \cap \mathbb{Z}^2] \cong k[x,y]$, $k[\sigma_2^\vee \cap \mathbb{Z}^2] \cong k[x^{-1}, y]$ and $k[\rho^\vee \cap \mathbb{Z}^2] \cong k[x^\pm, y]$. Patching $U_{\sigma_1} \cong \mathbb{A}^2_k$ and $U_{\sigma_2} \cong \mathbb{A}^2_k$ along $U_\rho \cong \mathrm{Spec}(k[x^{\pm 1}]) \times \mathbb{A}^1_k$ yields the variety $X(\Delta) = \mathbb{P}^1_k \times \mathbb{A}^1_k$.

Notice that if we consider $\Delta_1 := \{\rho_1, \rho_2, 0\}$ and $\Delta_2 := \{\rho, 0\}$ to be fans in $\mathbb{R}$, then $X(\Delta_1) = \mathbb{P}^1_k$ and $X(\Delta_2) = \mathbb{A}^1_k$. The fan Δ from Example 2.12 is like the "cartesian product" of Δ_1 and Δ_2, and so $X(\Delta) = \mathbb{P}^1_k \times \mathbb{A}^1_k$. In fact, this is true in general. Suppose N and N' are lattices. If Δ and Δ' are fans in $N_\mathbb{R}$ and $N'_\mathbb{R}$ respectively, then the set of cones, $\Delta \times \Delta' := \{\sigma \times \sigma' | \sigma \in \Delta, \sigma' \in \Delta'\}$, is a fan in $(N \oplus N')_\mathbb{R}$, and its associated toric variety $X(\Delta \times \Delta')$ is isomorphic to $X(\Delta) \times X(\Delta')$. Here, $\sigma \times \sigma'$ denotes the set $\{(u,w) \in (N \oplus N')_\mathbb{R} | u \in \sigma, w \in \sigma'\}$.

Example 2.13. *The following two dimensional complete fan gives rise to the toric variety $\mathbb{P}^1_k \times \mathbb{P}^1_k$, because $X(\Delta_1) = \mathbb{P}^1_k = X(\Delta_2)$, where $\Delta_1 = \{\rho_1, \rho_3, 0\}$ and $\Delta_2 = \{\rho_2, \rho_4, 0\}$.*



One of the reasons toric varieties are good sources of examples in the study of algebraic geometry is that many properties of toric varieties correspond to properties of the associated fans. For instance, if $X(\Delta)$ is an affine toric variety, then Δ is affine, meaning that Δ is spanned by a single cone. A toric variety is smooth if and only if the associated fan is smooth, and it is complete if and only if its associated fan is complete. Projective toric varieties are built from complete fans with a special property described below.

Recall that $|\Delta|$ is the support $\bigcup_{\sigma \in \Delta} \sigma$ of the fan Δ. A function $h : |\Delta| \to \mathbb{R}$ is a $\Delta-$*linear support function* if for each $\sigma \in \Delta$, there exists $u_\sigma \in M$ with $h(v) = u_\sigma(v)$ for all $v \in \sigma$. By the definition of M, it is clear that for all $v \in N \cap |\Delta|$, $h(v) \in \mathbb{Z}$. Since h is a function, $u_\sigma(w) = u_\tau(w)$ for all $w \in \tau \preceq \sigma$. That is to say, a $\Delta-$linear support function is a real-valued piece-wise linear function on the support $|\Delta|$ such that the domains of the pieces are the maximal cones of Δ. Every element m of M can be considered as a $\Delta-$linear support function by taking $u_\sigma = m$ for all $\sigma \in \Delta$. A $\Delta-$linear support function h is said to *strictly upper convex with respect to* Δ, if for all $\sigma \in \Delta$ and $v \in N_{\mathbb{R}}$, we have $u_\sigma(v) \geq h(v)$ with equality holding whenever $v \in \sigma$. A complete toric variety $X(\Delta)$ is projective if and only if there exists a $\Delta-$linear support function $h : |\Delta| \to \mathbb{R}$ that is strictly upper convex with respect to Δ [12, Corollary 2.16]. It turns out that this condition is equivalent to the existence of a convex polytope $Q \subset N_{\mathbb{R}}$ containing the origin in its interior such that $\Delta = \{\mathbb{R}_{\geq 0}Q' | Q' \text{ is a proper face of } Q\}$ [12, Proposition 2.19] [6, 7.9.2]. Observe that every complete fan of dimension one or two determines a projective toric variety, but there are examples of non-projective complete toric varieties of dimension three [4, Page 71].

2.3 Equivariant K-groups of Toric Varieties

Let Δ be a fan in $N_{\mathbb{R}}$, then the toric variety $X(\Delta)$ contains the torus T_N as an open subvariety. There is also an action of T_N on $X(\Delta)$. Locally, for $\sigma \in \Delta$, T_N acts on U_σ as follows:

The map $T_N \times U_\sigma \to U_\sigma$ is given by the map of algebras,

$$k[\sigma^\vee \cap M] \to k[M] \otimes k[\sigma^\vee \cap M],$$

sending χ^u to $\chi^u \otimes \chi^u$, for all $u \in \sigma^\vee \cap M$.

Furthermore, this action of T_N on U_σ is compatible with inclusions of open subsets associated to faces of σ. In other words, if τ is a face of σ, the following diagram commutes:

$$\begin{tikzcd} T \times U_\tau \arrow[r] \arrow[d, "id \times i"] & U_\tau \arrow[d, "i"] \\ T \times U_\sigma \arrow[r] & U_\sigma \end{tikzcd}$$

where i is the open embedding of U_τ into U_σ. This compatibility means that there is an action of T_N on $X(\Delta)$ extending the usual product in T_N.

An *equivariant vector bundle over* $X(\Delta)$ is a vector bundle $V \xrightarrow{\phi} X(\Delta)$ with a torus action by T_N that commutes with the action of T_N on $X(\Delta)$, i.e., the following diagram commutes:

$$\begin{tikzcd} T_N \times V \arrow[r] \arrow[d, "id\times\phi"] & V \arrow[d, "\phi"] \\ T_N \times X(\Delta) \arrow[r] & X(\Delta) \end{tikzcd}$$

The *equivariant K-group* $K_0^T(X(\Delta))$ is the group completion of the abelian monoid of isomorphism classes of equivariant vector bundles on $X(\Delta)$ modulo a relation $[V_2] = [V_1] + [V_3]$ for every short exact sequence

$$0 \longrightarrow V_1 \longrightarrow V_2 \longrightarrow V_3 \longrightarrow 0$$

of equivariant vector bundles, where the maps in the sequence are T_N-equivariant.

Notice that we can put an M grading on the ring $k[\sigma^\vee \cap M]$ by declaring elements in k to be of degree 0 and setting $\deg(\chi^m)$ as m for all m in $\sigma^\vee \cap M$. If an $M-$graded $k[\sigma^\vee \cap M]-$module is projective in the ungraded sense, then it is projective as an $M-$graded module over $k[\sigma^\vee \cap M]$ [11, Corollary 2.3.2]. Now, a vector bundle over the affine variety U_σ corresponds to a finitely generated projective module over $k[\sigma^\vee \cap M]$ [5], and the torus action on the vector bundle gives an $M-$grading of the corresponding module. Thus, an equivariant bundle over U_σ is given by a finitely generated projective $M-$graded module over $k[\sigma^\vee \cap M]$. Hence, $K_0^T(U_\sigma)$ is simply the group completion of the abelian monoid of isomorphism classes of finitely generated projective $M-$graded modules over $k[\sigma^\vee \cap M]$. The higher equivariant K-groups $K_q^T(X(\Delta))$ are the K-groups of the exact category of equivariant vector bundles over $X(\Delta)$. If $U \cup V$ is an equivariant open cover for the toric variety $X(\Delta)$, then the K-groups fit into the following long exact sequence:

$$\cdots \longrightarrow K_q^T(X(\Delta)) \longrightarrow K_q^T(U) \oplus K_q^T(V) \longrightarrow K_q^T(U \cap V) \longrightarrow K_{q-1}^T(X(\Delta)) \longrightarrow \cdots,$$

where q ranges over all integers [20]. The negative equivariant K-groups are defined in the same manner as the negative K-groups of rings.

2.4 Fan Cohomology

A fan Δ is a poset whose order relation is defined as $\tau \preceq \sigma$ whenever τ is a face of σ. A poset in turn determines a category whose objects are elements of the poset and where there is a unique morphism from τ to σ if and only if $\tau \preceq \sigma$. So, we will treat a fan as a category as needed and write it as $Cat(\Delta)$. On the other hand, we can define a topology on a fan Δ by declaring the subfans of Δ to be open sets [2], [3]. This is a finite topological space, and because the smallest open set containing a cone $\sigma \in \Delta$ is the subfan $\langle\sigma\rangle$ spanned by σ, for all sheaves $\mathcal{F}$, the value of $\mathcal{F}(\langle\sigma\rangle)$ is the stalk $\mathcal{F}_\sigma$. Let $\mathcal{C}$ be the category of sheaves of abelian groups on Δ, and let $\mathcal{D}$ be the category of contravariant functors from $Cat(\Delta)$ to the category Ab of abelian groups. There is a natural equivalance Φ between $\mathcal{C}$ and $\mathcal{D}$. More specifically, for all sheaves $\mathcal{F} \in \mathcal{C}$, $\Phi(\mathcal{F})$ is the contravariant functor that sends $\sigma \in \Delta$ to $\mathcal{F}_\sigma$ and a morphism $\tau \preceq \sigma$ to the map $\mathcal{F}_\sigma \to \mathcal{F}_\tau$ given by identifying $\mathcal{F}_\sigma$ with $\mathcal{F}(\langle\sigma\rangle)$ and using that $\langle\tau\rangle$ is an open subset of $\langle\sigma\rangle$. The inverse of Φ sends $F \in \mathcal{D}$ to $\mathcal{F} \in \mathcal{C}$ which is defined as follows:

$$\mathcal{F}(\Lambda) = \varprojlim_{\sigma \in \Lambda} F(\sigma), \text{ for all subfans } \Lambda \subseteq \Delta.$$

As a result, a sheaf $\mathcal{F}$ on Δ is determined by its values on stalks and the maps between them.

Proposition 2.14. *For all presheaves $\mathcal{F}$ on Δ and $p \geq 0$, the Čech cohomology groups $\check{H}^p(\mathcal{U}, \mathcal{F})$ of $\mathcal{F}$ with respect to the open cover $\mathcal{U} := \{\langle\sigma\rangle | \sigma$ is a maximal cone in $\Delta\}$ of Δ is isomorphic to the sheaf cohomology groups $H^p(\Delta, \widetilde{\mathcal{F}})$, where $\widetilde{\mathcal{F}}$ is the sheafification of $\mathcal{F}$.*

Proof. Observe that $\langle\sigma_1\rangle \cap \cdots \cap \langle\sigma_t\rangle = \langle\sigma_1 \cap \cdots \cap \sigma_t\rangle$ for any fixed $t \in \mathbb{N}$, and both $\widetilde{\mathcal{F}}(\langle\sigma_1 \cap \cdots \cap \sigma_t\rangle)$ and $\mathcal{F}(\langle\sigma_1 \cap \cdots \cap \sigma_t\rangle)$ are equal to the stalk of $\widetilde{\mathcal{F}}$ at $\sigma_1 \cap \cdots \cap \sigma_t$. So, $\check{H}^p(\mathcal{U}, \mathcal{F}) = \check{H}^p(\mathcal{U}, \widetilde{\mathcal{F}})$. By a standard argument, it suffices to prove that for any finite intersection $\sigma_1 \cap \cdots \cap \sigma_t$ of maximal cones in Δ, $H^p(\langle\sigma_1\rangle \cap \cdots \cap \langle\sigma_t\rangle, \widetilde{\mathcal{F}}) = 0$ for all $p > 0$. Since taking stalks is an exact functor, by the observation above, $H^p(\langle\sigma_1\rangle \cap \cdots \cap \langle\sigma_t\rangle, \widetilde{\mathcal{F}}) = 0$ for all $p > 0$. $\square$

We can define a topology on the toric variety $X(\Delta)$ by declaring the equivariant open subvarieties to be the only open sets. For each $q \geq 0$, $K_q^T(-)$ is a presheaf on $X(\Delta)$, sending an equivariant open subvariety to its equivariant K-group and a morphism between open subvarieties to the induced map between K-groups. We will use the same name $K_q^T(-)$ for the presheaf on Δ that sends a subfan Λ to $K_q^T(X(\Lambda))$. Let $\mathcal{V}$ be the open cover $\{U_\sigma | \sigma$ is a maximal cone in $\Delta\}$ of $X(\Delta)$ and $\mathcal{U}$ be defined as in Proposition 2.14. Then it is clear that $\check{H}^p(\mathcal{V}, K_q^T) \cong \check{H}^p(\mathcal{U}, K_q^T)$. Thus, by Proposition 2.14

$$\check{H}^p(\mathcal{V}, K_q^T) \cong \check{H}^p(\mathcal{U}, K_q^T) \cong H^p\left(\Delta, \widetilde{K_q^T}\right), \tag{2.15}$$

where $\widetilde{K_q^T}$ is the sheafification of the presheaf K_q^T on Δ. We will refer to the sheaf cohomology of $\widetilde{K_q^T}$ on Δ as "*fan cohomology*". Other than Thomason's and Walker's spectral sequences discussed in the introduction, the theory of fan cohomology will be the main tool we use in the computation of equivariant K-groups of non-affine toric varieties.

But first, we will discuss a formula for equivariant K-groups of affine toric varieties in the next chapter.

Chapter 3

Equivariant K-groups of Affine Toric Varieties

In the paper with Mark Walker and Mu-Wan Huang [1], we give a formula for the equivariant K-groups of an affine toric variety in terms of the associated cone and the K-groups of the ground field. The full statement is as follows:

Theorem 3.1 (Au, Huang, Walker)**.** *For all strongly convex rational polyhedral cones σ in $\mathbb{R}^n$, there is a natural isomorphism*

$$K_q^T(U_\sigma) \cong \mathbb{Z}[M_\sigma] \otimes_{\mathbb{Z}} K_q(k), \tag{3.2}$$

where $M_\sigma = \mathbb{Z}^n/(\sigma^\perp \cap \mathbb{Z}^n)$. In particular, $K_0^T(U_\sigma) \cong \mathbb{Z}[M_\sigma]$.

The isomorphism (3.2) is natural with respect to the inclusion of a face τ into σ, i.e., for $\tau \preceq \sigma$, the diagram

$$\begin{tikzcd} K_q^T(U_\sigma) \arrow[r, "\cong"] \arrow[d, "f"] & \mathbb{Z}[M_\sigma] \otimes_{\mathbb{Z}} K_q(k) \arrow[d, "\pi\otimes id"] \\ K_q^T(U_\tau) \arrow[r, "\cong"] & \mathbb{Z}[M_\tau] \otimes_{\mathbb{Z}} K_q(k) \end{tikzcd}$$

commutes. The map π is the canonical surjection, while f is the induced map between K-groups.

If σ is smooth, Theorem 3.1 follows from basic properties of equivariant K-theory of smooth toric varieties [10] . In order to establish this same result for an affine toric variety that is not necessarily smooth, we proved the following statement regarding the K-theory of graded projective modules:

Theorem 3.3 (Au, Huang, Walker)**.** *If R is a commutative ring, M an abelian group, and A a sub-monoid of M, then for all $q \geq 0$, we have*

$$K_q^M(R[A]) \cong \mathbb{Z}[M/U] \otimes_{\mathbb{Z}} K_q(R),$$

where U is the subgroup of units of A and $K_q^M(R[A])$ is the K-theory of the exact category, $\mathcal{P}^M(R[A])$, of finitely generated $M-$graded projective $R[A]-$modules.

A consequence of this theorem is the isomorphism

$$K_q^T(U_\sigma \times_k \operatorname{Spec} R) \cong \mathbb{Z}[M_\sigma] \otimes_{\mathbb{Z}} K_q(R), \tag{3.4}$$

where R is an arbitrary $k-$algebra on which T_N acts trivially. The isomorphism (3.2) is the special case of (3.4) when R is taken to be k itself.

A main ingredient of the proof for Theorem 3.3 is that there is an equivalence of categories between $\bigoplus_S \mathcal{P}(R)$ and $\mathcal{P}^M(R[U])$, which gives the isomorphisms

$$K_q^M(R[U]) \cong \bigoplus_S K_q(R) \cong \mathbb{Z}[M/U] \otimes_{\mathbb{Z}} K_q(R).$$

Here, S is a fixed set of coset representatives for the subgroup U of M, and $\mathcal{P}(R)$ is the category of finitely generated projective $R-$modules. Theorem 3.3 is obtained by proving the exact functor

$$\mathcal{P}^M(R[U]) \longrightarrow \mathcal{P}^M(R[A]),$$

induced by extension of scalars induces a homotopy equivalence on K-theory spaces.

Example 3.5. *Let N be an abelian group isomorphic to $\mathbb{Z}^n$, then for all cones σ in $N_{\mathbb{R}}$ such that* $\dim(\sigma) = n$, *we have*

$$K_q^T(U_\sigma) \cong \begin{cases} \mathbb{Z}[M], & q = 0 \\ \mathbb{Z}[M] \otimes K_q(k), & q > 0 \end{cases}$$

In the subsequent chapters, we will discuss the computation of K-groups of non-affine toric varieties.

Chapter 4

Equivariant K-groups of Smooth Toric Varieties

Given a smooth toric variety $X(\Delta)$ (possibly non-affine), a theorem of Vezzosi and Vistoli on arbitrary actions by diagonalizable groups gives a calculation of $K_q^T(X(\Delta))$, for all $q \geq 0$ [17] [18]. When applied to the toric variety $X(\Delta)$, the result says that the following sequence is exact for all $q \geq 0$:

$$0 \longrightarrow K_q^T(X(\Delta)) \longrightarrow \bigoplus_{\sigma} K_q^T(U_\sigma) \longrightarrow \bigoplus_{\delta<\tau} K_q^T(U_{\delta\cap\tau}) \longrightarrow \bigoplus_{\delta<\tau} K_q^T(U_{\delta\cap\tau}) \longrightarrow \bigoplus_{\delta<\tau<\epsilon} K_q^T(U_{\delta\cap\tau\cap\epsilon}) \longrightarrow \cdots, \qquad (4.1)$$

where the direct sums are indexed by the set of maximal cones in Δ and $<$ is an arbitrary fixed ordering of the maximal cones. In other words, if $\mathcal{V}$ is the equivariant open cover, $\{U_\sigma | \sigma \text{ is a maximal cone in } \Delta\}$, of $X(\Delta)$, then

$$K_q^T(X(\Delta)) \cong \check{H}^0(\mathcal{V}, K_q^T)$$

and

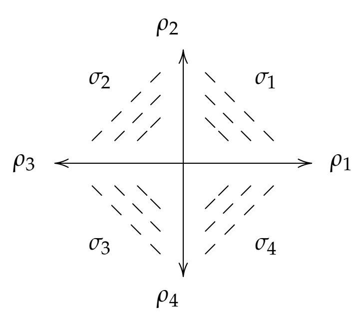
$$\check{H}^p(\mathcal{V}, K_q^T) \cong 0,$$

for all $p > 0$. Observe that exactness of (4.1) implies that K_q^T is a sheaf on $X(\Delta)$ endowed with the topology whose only open sets are equivariant open subvarieties. Also, the higher Čech cohomology of K_q^T with respect to $\mathcal{V}$ vanishes. By (3.2), we obtain the long exact sequence

$$0 \longrightarrow K_q^T(X(\Delta)) \longrightarrow \bigoplus_{\sigma} \mathbb{Z}[M_\sigma] \otimes_{\mathbb{Z}} K_q(k) \longrightarrow \bigoplus_{\delta<\tau} \mathbb{Z}[M_{\delta\cap\tau}] \otimes_{\mathbb{Z}} K_q(k) \longrightarrow \bigoplus_{\delta<\tau<\epsilon} \mathbb{Z}[M_{\delta\cap\tau\cap\epsilon}] \otimes_{\mathbb{Z}} K_q(k) \longrightarrow \cdots, \tag{4.2}$$

which is useful for computing equivariant K-groups of smooth toric varieties.

Example 4.3. *Let Δ be the following two dimensional complete fan, then $X(\Delta) = \mathbb{P}_k^1 \times \mathbb{P}_k^1$.*



Since $M_{\sigma_i} = M$, for $i = 1, \cdots, 4$, by applying (4.2), we see that

$$\begin{aligned} K_0^T(X(\Delta)) &\cong ker\left(\bigoplus_{i}^{4} \mathbb{Z}[M] \longrightarrow \mathbb{Z}[M_{\rho_1}] \oplus \mathbb{Z}[M_{\rho_2}] \oplus \mathbb{Z}[M_{\rho_3}] \oplus \mathbb{Z}[M_{\rho_4}]\right) \\ &\cong \left\{(f_i)_{i=1}^4 | f_i \text{ has the form } \sum_{t=1}^{r} s_t \chi^{(a_t,b_t)} \text{ with } f_i|_{\sigma_i\cap\sigma_j} = f_j|_{\sigma_i\cap\sigma_j}, \forall i \neq j\right\}, \end{aligned}$$

where $s_t, a_t, b_t \in \mathbb{Z}$ and χ is an arbitrary symbol.

In our paper [1], we demonstrated a new proof of Vezzosi and Vistoli's Theorem [17] for the special case of toric varieties . More specifically, we proved:

Theorem 4.4. *([1]) Assume $X = X(\Delta)$ is a smooth toric variety over the field k. Then the presheaf K_q^T on Δ is a flasque sheaf. Moreover, there is an isomorphism*

$$K_q^T(X) \cong K_0^T(X) \otimes K_q(k).$$

Sketch of Proof. For $q \geq 0$, let $\mathcal{A}_q$ be the sheaf on Δ whose stalk at σ is $\mathbb{Z}[M_\sigma] \otimes_{\mathbb{Z}} K_q(k)$ for all $\sigma \in \Delta$. Because Δ is smooth, one can prove that $\mathcal{A}_0$ is flasque. $\mathcal{A}_0$ is a sheaf of free abelian groups with trivial higher Čech cohomology, and so $\mathcal{A}_0 \otimes_{\mathbb{Z}} K_q(k)$ is a flasque sheaf. Therefore, $\mathcal{A}_q$ must be the sheaf $\mathcal{A}_0 \otimes_{\mathbb{Z}} K_q(k)$. Since for all $\sigma \in \Delta$, we have natural isomorphisms

$$\begin{aligned} \bigoplus_\sigma \mathcal{A}_q(\langle\sigma\rangle) &\cong \bigoplus_\sigma \mathcal{A}_0(\langle\sigma\rangle) \otimes_{\mathbb{Z}} K_q(k) \\ &\cong \bigoplus_\sigma K_0^T(U_\sigma) \otimes_{\mathbb{Z}} K_q(k) \quad , \\ &\cong \bigoplus_\sigma K_q^T(U_\sigma) \end{aligned}$$

we have for all $p > 0$,

$$\check{H}^p(\mathcal{V}, K_q^T) \cong \check{H}^p(\mathcal{U}, \mathcal{A}_q) = 0. \tag{4.5}$$

where $\mathcal{V} := \{U_\sigma | \sigma \text{ is a maximal cone in } \Delta\}$ and $\mathcal{U} := \{\langle\sigma\rangle | \sigma \text{ is a maximal cone in } \Delta\}$. Now, there is a convergent spectral sequence due to Thomason [16]:

$$\check{H}^p(\mathcal{V}, K_q^T) \Longrightarrow K_{q-p}^T(X(\Delta))$$

Because of (4.5), the spectral sequence collapses and yields the isomorphism

$$K_q^T(X(\Delta)) \cong \check{H}^0(\mathcal{V}, K_q^T). \quad \square$$

As a result of Theorem 4.4, the equivariant K-theory of smooth toric varieties is well understood. The rest of this thesis will concern computing $K_q^T(X(\Delta))$ when $X(\Delta)$ is neither affine nor smooth.

Chapter 5

Calculations of Fan Cohomology Groups

Given a quasi-projective toric variety $X(\Delta)$, i.e., an equivariant open subvariety of a projective toric variety, there is a convergent spectral sequence

$$\check{H}^p(\mathcal{V}, K_q^T) \Longrightarrow K_{q-p}^T(X(\Delta))$$

[20], where $\mathcal{V}$ is the equivariant open cover $\{U_\sigma | \sigma \text{ is a maximal cone in } \Delta\}$ of $X(\Delta)$. Hence, one could try to understand the equivariant K-groups of $X(\Delta)$ by studying the fan cohomology groups, $\check{H}^p(\mathcal{V}, K_q^T)$. Consider the Čech complex $P^\bullet$ of the presheaf K_0^T with respect to $\mathcal{V}$:

$$P^\bullet : \bigoplus_{\sigma} K_0^T(U_\sigma) \longrightarrow \bigoplus_{\delta<\tau} K_0^T(U_{\delta\cap\tau}) \longrightarrow \bigoplus_{\delta<\tau} K_0^T(U_{\delta\cap\tau}) \longrightarrow \bigoplus_{\delta<\tau<\epsilon} K_0^T(U_{\delta\cap\tau\cap\epsilon}) \longrightarrow \cdots$$

It is a chain complex of free abelian groups by Theorem 3.1. By applying the isomorphism from the theorem,

$$K_q^T(U_\sigma) \cong K_0^T(U_\sigma) \otimes_{\mathbb{Z}} K_q(k),$$

we see that tensoring $P^\bullet$ with $K_q(k)$ gives the Čech complex for the presheaf K_q^T for all $q > 0$. Thus, for all $p, q \geq 0$, using [21, 3.6.2]

$$\begin{aligned} \check{H}^p(\mathcal{V}, K_q^T) &\cong \check{H}^p(\mathcal{V}, K_0^T \otimes_{\mathbb{Z}} K_q(k)) \\ &\cong \check{H}^p(\mathcal{V}, K_0^T) \otimes_{\mathbb{Z}} K_q(k) \oplus \mathrm{Tor}_1^{\mathbb{Z}}(\check{H}^{p+1}(\mathcal{V}, K_0^T), K_q(k)). \end{aligned}$$

If $\check{H}^{p+1}(\mathcal{V}, K_0^T)$ is torsion free, then we have the isomorphism

$$\check{H}^p(\mathcal{V}, K_q^T) \cong \check{H}^p(\mathcal{V}, K_0^T) \otimes_{\mathbb{Z}} K_q(k)$$

whenever $p, q \geq 0$ [21, 3.2.1]. This implies we really only need to consider the case when $q = 0$, if $\check{H}^\bullet(\mathcal{V}, K_0^T)$ is torsion free. In the rest of this chapter, we will compute some of the fan cohomology groups, $\check{H}^p(\mathcal{V}, K_0^T)$, for non-smooth toric varieties.

5.1 Higher Fan Cohomology Groups

Definition 5.1. *Given a sheaf $\mathcal{F}$ on a topological space X, let $\mathcal{G}^0 = \prod_{x \in X} i_{x*}(\mathcal{F}_x)$ and φ_0 be the inclusion $\mathcal{F} \hookrightarrow \mathcal{G}^0$. For $i \geq 1$, define $\mathcal{G}^i = \prod_{x \in X} i_{x*}((cok\ \varphi_{i-1})_x)$ and φ_i as the composition of $\mathcal{G}^{i-1} \to cok\ \varphi_{i-1}$ and $cok\ \varphi_{i-1} \hookrightarrow \mathcal{G}^i$. The resulting exact sequence of sheaves*

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G}^0 \longrightarrow \mathcal{G}^1 \longrightarrow \cdots$$

is called the Godement resolution of $\mathcal{F}$.

Remark 5.2. *For all sheaves $\mathcal{F}$, $\prod_{x \in X} i_{x*}(\mathcal{F}_x)$ is flasque, so the Godement resolution of $\mathcal{F}$ is a flasque resolution. $\Gamma(X, \mathcal{G}^\bullet)$ is a complex of groups which in degree i is $\prod_{x \in X}(cok\ \varphi_{i-1})_x$, and $H^i\Gamma(X, \mathcal{G}^\bullet)$ is isomorphic to $H^i(X, \mathcal{F})$, the sheaf cohomology of $\mathcal{F}$ [19, Sec. 4.3.1].*

Recall from Proposition 2.14 that if $\mathcal{U}$ is the open cover of a fan Δ consisting of subfans spanned by maximal cones in Δ, then for all sheaves $\mathcal{F}$ and for all $p \geq 0, \check{H}^p(\mathcal{U}, \mathcal{F}) \cong H^p(\Delta, \mathcal{F})$. The next lemma gives us one more way to compute fan cohomology.

Lemma 5.3. *Let $\mathcal{F}$ be a sheaf on a fan Δ and $\mathcal{U}$ be the open cover $\{\langle\sigma\rangle | \sigma$ is a maximal cone in $\Delta\}$, then for all $p \geq 0, \check{H}^p(\mathcal{U}, \mathcal{F}) = H^p(A^\bullet)$, where $A^\bullet$ is the complex,*

$$0 \longrightarrow \bigoplus_{\sigma_0 \in \Delta} \mathcal{F}(\langle\sigma_0\rangle) \xrightarrow{\partial_0} \bigoplus_{\sigma_0 \prec \sigma_1} \mathcal{F}(\langle\sigma_0\rangle) \xrightarrow{\partial_1} \bigoplus_{\sigma_0 \prec \sigma_1 \prec \sigma_2} \mathcal{F}(\langle\sigma_0\rangle) \longrightarrow \cdots,$$

with the direct sums being taken over strict chains of cones in Δ and the map ∂_k defined as

$$(\alpha_{\sigma_0 \prec \cdots \prec \sigma_k})_{\sigma_0 \prec \cdots \prec \sigma_k} \mapsto \left(\sum_{i=0}^{k+1} (-1)^i \alpha_{\tau_0 \prec \cdots \prec \tau_{i-1} \prec \tau_{i+1} \prec \cdots \prec \tau_{k+1}} \right)_{\tau_0 \prec \cdots \prec \tau_{k+1}}.$$

Proof. Let $\mathcal{G}^\bullet$ be the Godement resolution for $\mathcal{F}$. Consider the following bi-complex $B^\bullet$:

$$\begin{array}{ccccccc}
\bigoplus_{\sigma_0 \in \Delta} \mathcal{G}^0(\langle\sigma_0\rangle) & \longrightarrow & \bigoplus_{\sigma_0 \in \Delta} \mathcal{G}^1(\langle\sigma_0\rangle) & \longrightarrow & \bigoplus_{\sigma_0 \in \Delta} \mathcal{G}^2(\langle\sigma_0\rangle) & \longrightarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \\
\bigoplus_{\sigma_0 \prec \sigma_1} \mathcal{G}^0(\langle\sigma_0\rangle) & \longrightarrow & \bigoplus_{\sigma_0 \prec \sigma_1} \mathcal{G}^1(\langle\sigma_0\rangle) & \longrightarrow & \bigoplus_{\sigma_0 \prec \sigma_1} \mathcal{G}^2(\langle\sigma_0\rangle) & \longrightarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \\
\bigoplus_{\sigma_0 \prec \sigma_1 \prec \sigma_2} \mathcal{G}^0(\langle\sigma_0\rangle) & \longrightarrow & \bigoplus_{\sigma_0 \prec \sigma_1 \prec \sigma_2} \mathcal{G}^1(\langle\sigma_0\rangle) & \longrightarrow & \bigoplus_{\sigma_0 \prec \sigma_1 \prec \sigma_2} \mathcal{G}^2(\langle\sigma_0\rangle) & \longrightarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \\
\vdots & & \vdots & & \vdots & &
\end{array}$$

The following sequence is exact for all $j \geq 0$, because taking stalks is an exact functor.

$$0 \longrightarrow \bigoplus_{\sigma_0 \prec \cdots \prec \sigma_j} \mathcal{F}(\langle \sigma_0 \rangle) \longrightarrow \bigoplus_{\sigma_0 \prec \cdots \prec \sigma_j} \mathcal{G}^0(\langle \sigma_0 \rangle) \longrightarrow \bigoplus_{\sigma_0 \prec \cdots \prec \sigma_j} \mathcal{G}^1(\langle \sigma_0 \rangle) \longrightarrow \cdots$$

So, $A^\bullet$ maps to the total complex $Tot^{\oplus}(B^\bullet)$ via a quasi-isomorphism, since the columns of $B^\bullet$ have finitely many non-zero terms. (The n^{th} term of the cochain complex $Tot^{\oplus}(B^\bullet)$ is $\bigoplus_{i+j=n} B^{i,j}$, and the differentials ∂ are defined by the formula $\partial = \partial^v + \partial^d$, where ∂^v and ∂^d are the vertical and horizontal differentials respectively.) We claim that for all $i \geq 0$

$$\mathcal{G}^i(\Delta) \longrightarrow \bigoplus_{\sigma_0 \in \Delta} \mathcal{G}^i(\langle \sigma_0 \rangle) \longrightarrow \bigoplus_{\sigma_0 \prec \sigma_1} \mathcal{G}^i(\langle \sigma_0 \rangle) \longrightarrow \cdots \tag{5.4}$$

is exact. Granting this, $\mathcal{G}^\bullet(\Delta)$ is quasi-isomorphic to $Tot^{\oplus}(B^\bullet)$. Thus, the cohomology groups of $\mathcal{G}^\bullet(\Delta)$ are isomorphic to those of $A^\bullet$, and the result follows from Remark 5.2 and the isomorphism $H^p(\Delta, \mathcal{F}) \cong \check{H}^p(\mathcal{U}, \mathcal{F})$ in Proposition 2.14.

We will prove that (5.4) is exact for a fixed i. $\mathcal{G}^i$ is a direct product of skyscraper sheaves indexed by Δ. Let $\mathcal{H}_\tau$ denote the component of $\mathcal{G}^i$ with the index $\tau \in \Delta$. Then, (5.4) is rewritten as:

$$0 \longrightarrow \prod_{\tau \in \Delta} \mathcal{H}_\tau(\Delta) \longrightarrow \bigoplus_{\sigma_0 \in \Delta} \prod_{\tau \in \Delta} \mathcal{H}_\tau(\langle \sigma_0 \rangle) \longrightarrow \bigoplus_{\sigma_0 \prec \sigma_1} \prod_{\tau \in \Delta} \mathcal{H}_\tau(\langle \sigma_0 \rangle) \longrightarrow \bigoplus_{\sigma_0 \prec \sigma_1 \prec \sigma_2} \prod_{\tau \in \Delta} \mathcal{H}_\tau(\langle \sigma_0 \rangle) \longrightarrow \cdots$$

In fact, each component (of the finite product) of the complex is exact, i.e. for a fixed $\tau \in \Delta$, the sequence

$$0 \longrightarrow \mathcal{H}_\tau(\Delta) \longrightarrow \bigoplus_{\sigma_0 \in \Delta} \mathcal{H}_\tau(\langle \sigma_0 \rangle) \longrightarrow \bigoplus_{\sigma_0 \prec \sigma_1} \mathcal{H}_\tau(\langle \sigma_0 \rangle) \longrightarrow \bigoplus_{\sigma_0 \prec \sigma_1 \prec \sigma_2} \mathcal{H}_\tau(\langle \sigma_0 \rangle) \longrightarrow \cdots$$

is exact, because, for all $\sigma \in \Delta$,

$$\mathcal{H}_\tau(\langle\sigma\rangle) = \begin{cases} \mathcal{H}_\tau, & \tau \preceq \sigma \\ 0, & \text{else} \end{cases},$$

and the sequence

$$0 \longrightarrow \mathcal{H}_\tau \longrightarrow \bigoplus_{\tau \preceq \sigma_0} \mathcal{H}_\tau \longrightarrow \bigoplus_{\tau \preceq \sigma_0 \prec \sigma_1} \mathcal{H}_\tau \longrightarrow \bigoplus_{\tau \preceq \sigma_0 \prec \sigma_1 \prec \sigma_2} \mathcal{H}_\tau \longrightarrow \cdots$$

is split exact, where the splittings are given by

$$(\alpha_{\tau \preceq \tau \preceq \sigma_0 \prec \cdots \prec \sigma_{k-1}})_{\tau \preceq \sigma_0 \prec \cdots \prec \sigma_{k-1}} \longleftarrow (\alpha_{\tau \preceq \sigma_0 \prec \cdots \prec \sigma_k})_{\tau \preceq \sigma_0 \prec \cdots \prec \sigma_k}. \quad \square$$

Recall that the stalk of $\widetilde{K_0^T}$ at $\sigma \in \Delta$ is $K_0^T(U_\sigma) \cong \mathbb{Z}[M_\sigma]$. Let $\widetilde{\mathbb{Z}}[M_\sigma]$ denote the kernel of the map $\mathbb{Z}[M_\sigma] \to \mathbb{Z}$ sending χ^m to 1, for all m in M_σ, and let $\mathcal{F}'$ be the sheaf on Δ such that $\mathcal{F}'_\sigma = \widetilde{\mathbb{Z}}[M_\sigma], \forall \sigma \in \Delta$. We have a split exact sequence of sheaves:

$$(*)\ 0 \longrightarrow \mathcal{F}' \longrightarrow \widetilde{K_0^T} \underset{\psi}{\overset{}{\rightleftarrows}} \mathbb{Z} \longrightarrow 0,$$

where the splitting ψ is given by the evident inclusion $\mathbb{Z} \subseteq \mathbb{Z}[M_\sigma]$. Since $\widetilde{\mathbb{Z}}[M_0] = 0$, in many cases it is easier to compute $H^p(\Delta, \mathcal{F}')$ than $H^p\left(\Delta, \widetilde{K_0^T}\right)$. The following lemma allows us to relate the two groups.

Lemma 5.5. $H^p\left(\Delta, \widetilde{K_0^T}\right) \cong \begin{cases} H^p(\Delta, \mathcal{F}') \oplus \mathbb{Z}, & p = 0 \\ H^p(\Delta, \mathcal{F}'), & p > 0 \end{cases}$

Proof. Since $(*)$ is split exact, we see that $\widetilde{K_0^T} \cong \mathcal{F}' \oplus \mathbb{Z}$, which implies

$$H^p\left(\Delta, \widetilde{K_0^T}\right) \cong H^p(\Delta, \mathcal{F}') \oplus H^p(\Delta, \mathbb{Z})$$

for all $p \geq 0$. Because constant sheaves are flasque, $H^p(\Delta, \mathbb{Z}) = 0$, for all $p > 0$ [5, Thm III.2.5].

□

Definition 5.6. *The Krull dimension of a fan* Δ *is defined to be* $\max\{\dim \sigma | \sigma \in \Delta\}$.

An immediate consequence of Lemma 5.3 is that the p^{th} cohomology group of any sheaf on Δ vanishes for all p greater than the Krull dimension of Δ. In the case when the sheaf is $\widetilde{K_0^T}$, more is true.

Corollary 5.7. *For any fan* Δ *with Krull dimension* $d > 0$, $H^p\left(\Delta, \widetilde{K_0^T}\right) = 0$ *for all* $p \geq d$.

Proof. Consider the following chain complex $C^\bullet$:

$$0 \longrightarrow \bigoplus_{\sigma_0 \in \Delta} \widetilde{\mathbb{Z}}[M_{\sigma_0}] \longrightarrow \bigoplus_{\sigma_0 \prec \sigma_1} \widetilde{\mathbb{Z}}[M_{\sigma_0}] \longrightarrow \bigoplus_{\sigma_0 \prec \sigma_1 \prec \sigma_2} \widetilde{\mathbb{Z}}[M_{\sigma_0}] \longrightarrow \cdots$$

$\widetilde{\mathbb{Z}}[M_0] = 0$ implies $H^p(C^\bullet) = 0$ for all p greater than or equal to the Krull dimension of Δ. The result follows from Lemma 5.5.

□

Definition 5.8. *For a collection* $\mathcal{C}$ *of fans in* $\mathbb{R}^n$, *define* $\overline{\mathcal{C}}$ *to be the smallest collection of fans in* $\mathbb{R}^n$ *containing* $\mathcal{C}$ *that satisfies the following condition:*

(*) *Suppose Δ is the union of two subfans Δ' and Δ'' with $\Delta' \cap \Delta''$ consisting of a single cone and its faces. If Δ' and Δ'' are in $\overline{\mathcal{C}}$, then so is Δ.*

Remark 5.9. *Given a collection of fans $\mathcal{C}$, the intersection of all collections of fans containing $\mathcal{C}$ and satisfying the condition (*) is the unique smallest collection we call $\overline{\mathcal{C}}$.*

Example 5.10. *If $\mathcal{C}$ consists of two fans Δ' and Δ'' in $\mathbb{R}^n$ such that*

- *$\Delta' \cap \Delta'' = \langle\sigma\rangle$ for some cone σ in Δ' and Δ'' and*
- *for all $\alpha \in \Delta'$ and $\beta \in \Delta''$, $\alpha \cap \beta$ is either $\{0\}$ or a face of α and β,*

then $\overline{\mathcal{C}}$ is the collection of three fans $\{\Delta', \Delta'', \Delta' \cup \Delta''\}$.

Example 5.11. *If $\mathcal{C}$ is a set of fans such that for all Δ' and Δ'' in $\mathcal{C}$, $\Delta' \cap \Delta'' \neq \langle\sigma\rangle$ for any non-zero cone σ in Δ' and Δ'', then $\overline{\mathcal{C}} = \mathcal{C}$.*

Remark 5.12. *For each fan Δ in $\overline{\mathcal{C}}$, there exist $\Delta' \in \mathcal{C}$ such that $\Delta = \Delta' \cup \Delta''$ and $\Delta' \cap \Delta'' = \langle\tau\rangle$ for some cone τ in Δ' and Δ'', where Δ'' is a fan in $\overline{\mathcal{C}} \backslash \{\Delta'\}$.*

Definition 5.13. *Let $\mathcal{G}$ be a sheaf on a fan Δ. We say Δ is* acyclic *with respect to a sheaf $\mathcal{G}$, if $H^p(\Delta, \mathcal{G}) = 0$, for all $p > 0$.*

For every cone σ, the affine fan $\langle\sigma\rangle$ is acyclic with respect to every sheaf as in the proof of Proposition 2.14. A smooth fan Δ is acyclic with respect to the sheaf $\widetilde{K_0^T}$ by Theorem 4.4.

Lemma 5.14. *Let $\mathcal{C}$ be a collection of fans acyclic with respect to $\widetilde{K_0^T}$, then for all Δ in $\overline{\mathcal{C}}$, Δ is acyclic with respect to $\widetilde{K_0^T}$.*

Proof. Let $\mathcal{D}$ be the collection of all fans acyclic with respect to $\widetilde{K_0^T}$. We will show that $\overline{\mathcal{C}} \subseteq \mathcal{D}$ by verifying that $\mathcal{D}$ contains $\mathcal{C}$ and satisfies the condition (*) in Definition 5.8. Indeed, suppose Δ is the union of two subfans Δ' and Δ'', which are in $\mathcal{D}$, with

$\Delta' \cap \Delta'' = \langle\tau\rangle$ for some cone τ in Δ' and Δ''. Since $\Delta', \Delta'' \in \mathcal{D}$ and the sheaf cohomology groups vanish for affine spaces, $H^p\left(\Delta', \widetilde{K_0^T}\right) \oplus H^p\left(\Delta'', \widetilde{K_0^T}\right) = 0$ and $H^p\left(\langle\tau\rangle, \widetilde{K_0^T}\right) = 0$, for all $p > 0$. Therefore, we have the long exact sequence:

$$H^0\left(\Delta', \widetilde{K_0^T}\right) \oplus H^0\left(\Delta'', \widetilde{K_0^T}\right) \xrightarrow{h} H^0\left(\langle\tau\rangle, \widetilde{K_0^T}\right) \longrightarrow H^1\left(\Delta, \widetilde{K_0^T}\right) \longrightarrow 0 \longrightarrow 0 \longrightarrow H^2\left(\Delta, \widetilde{K_0^T}\right) \longrightarrow \cdots$$

Now, $H^0\left(\langle\tau\rangle, \widetilde{K_0^T}\right) \cong \mathbb{Z}[M_\tau]$. Consider the following diagram:

$$\begin{tikzcd}
\mathbb{Z}[M] \arrow[d, "g"] \arrow[dr, "\pi"] & \\
H^0\left(\Delta', \widetilde{K_0^T}\right) \arrow[r, "f"] & \mathbb{Z}[M_\tau]
\end{tikzcd}$$

$H^0\left(\Delta', \widetilde{K_0^T}\right)$ is contained in $\bigoplus_{\sigma \in Max(\Delta')} \mathbb{Z}[M_\sigma]$, and g is the diagonal map. The composite map $g \circ f$ sends an element a in $\mathbb{Z}[M]$ to the image of a under the canonical surjection π. So, the diagram above commutes. Since π is surjective, f is surjective, and therefore so is h. That implies Δ is acyclic with respect to $\widetilde{K_0^T}$ and belongs in $\mathcal{D}$. Hence, $\overline{\mathcal{C}} \subseteq \mathcal{D}$.

$\square$

Remark 5.15. *If $\mathcal{C}$ is a collection of affine fans, i.e. fans spanned by a single cone, then every fan in $\overline{\mathcal{C}}$ is acyclic with respect to $\widetilde{K_0^T}$.*

Example 5.16. *The figure 5.17 below shows the intersection of each non-zero cone in a certain $3-$dimensional fan Δ with S^2, the unit $2-$sphere centered at the origin. Δ consists of five maximal $3-$dimensional cones $\sigma_1, \cdots, \sigma_5$. Let $\mathcal{C}$ be the collection $\{\langle\sigma_1\rangle, \cdots, \langle\sigma_5\rangle\}$ of affine fans. Then Δ is a fan in $\overline{\mathcal{C}}$ and it is acyclic with respect to $\widetilde{K_0^T}$.*

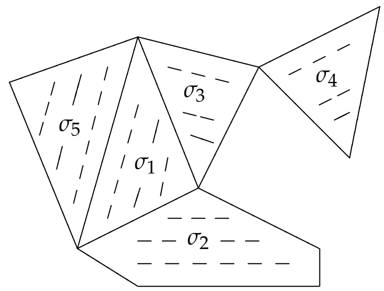


Figure 5.17: Intersection of S^2 and Δ

5.2 Complete Fans

In this section, we will prove that when Δ is an $n-$dimensional complete fan, the $(n-1)^{st}$ cohomology group of $\widetilde{K_0^T}$ is a free abelian group of finite rank. Furthermore, the rank depends only on the coordinates of the rays of Δ. First, we need a lemma.

Lemma 5.18. *Suppose $v_1, \cdots, v_s$ are vectors in $\mathbb{Z}^n$. For $i = 1, \cdots, s$, let K_i be the kernel of $\varphi_i : \mathbb{Z}^n \to \mathbb{Z}$, which is defined as $\varphi_i(u) = v_i \cdot u$, for all $u \in \mathbb{Z}^n$. If each φ_i is a surjection, then $\mathbb{Z}^n / \sum_{i=1}^s K_i \cong \mathbb{Z}/g\mathbb{Z}$, where g is the greatest common divisor of the set of all 2×2 minors of the matrix* $\begin{bmatrix} v_1 \\ \vdots \\ v_s \end{bmatrix}$.

Proof. Each v_i is unimodular, since φ_i is onto. Without lost of generality, let $v_1 = e_1$. Then, g is the greatest common divisor of the entries in the matrix $\begin{bmatrix} w_2 \\ \vdots \\ w_s \end{bmatrix}$, where $w_i \in \mathbb{Z}^{n-1}$ is defined by omitting the first component of v_i, for $i = 2 \cdots, s$.

For $2 \leq j \leq s$, let $\theta_j : K_j \to \mathbb{Z}^n/K_1$ be the composition of the canonical surjection $K_j \to \dfrac{K_j + K_1}{K_1}$ and the inclusion $\dfrac{K_j + K_1}{K_1} \hookrightarrow \dfrac{\mathbb{Z}^n}{K_1}$. Then $\theta_j = \varphi_1 \circ \iota_j$, where ι_j is the inclusion of K_j into $\mathbb{Z}^n$. Say $v_j = [a_1, \cdots, a_n]$. Since φ_j is surjective, there exists $b_1, \cdots, b_n \in \mathbb{Z}$ such that $a_1 b_1 + \cdots + a_n b_n = 1$. Consider the split exact sequence in the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & K_j \arrow[r, "\iota_j"'] \arrow[dr, "\theta_j"'] & \mathbb{Z}^n \arrow[l, bend right, "\pi_j"'] \arrow[r, "\varphi_j"'] \arrow[d, "\varphi_1"] & \mathbb{Z} \arrow[l, bend right, "\phi_j"'] \arrow[r] & 0 \\
& & \mathbb{Z} \cong \mathbb{Z}^n/K_1 & &
\end{tikzcd}$$

Here, the splitting ϕ_j is given by $1 \mapsto \hat{v}_j := \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$ and $\iota_j \circ \pi_j + \phi_j \circ \varphi_j$ is the identity. The map $\theta_j \circ \pi_j = \varphi_1 \circ \iota_j \circ \pi_j$ is given by multiplication by the matrix

$$e_1(I_n - \hat{v}_j v_j) = [1 - b_1 a_1, -b_1 a_2, \cdots, -b_1 a_n].$$

It is straight forward to check that the ideals $(1 - b_1 a_1, -b_1 a_2, \cdots, -b_1 a_n)$ and $(a_2, \cdots, a_n)$ in $\mathbb{Z}$ are the same. Hence, for $i = 2, \cdots, s$, we have

$$im(\psi_i) = im(\theta_i \circ \pi_i) = im(\theta_i),$$

where $\psi_i : \mathbb{Z}^{n-1} \to \mathbb{Z}$ is the map defined by $\psi_i(u) = w_i \cdot u$ for all $u \in \mathbb{Z}^{n-1}$.

Now, let $\gamma : \dfrac{\sum_{i=1}^{s} K_i}{K_1} \hookrightarrow \dfrac{\mathbb{Z}^n}{K_1}$ be the inclusion, and let $\xi : \sum\limits_{i=1}^{s} K_i \to \dfrac{\mathbb{Z}^n}{K_1}$ be the composition of γ and the surjection $\sum\limits_{i=1}^{s} K_i \to \dfrac{\sum_{i=1}^{s} K_i}{K_1}$. Then, the following isomorphisms hold:

$$\begin{aligned}
\frac{\mathbb{Z}^n}{\sum_{i=1}^{s} K_i} &\cong coker(\gamma) \\
&\cong coker(\xi) \\
&\cong \frac{\mathbb{Z}}{\sum_{i=2}^{s} im(\theta_i)} \quad \text{, since } \mathbb{Z} \cong \frac{\mathbb{Z}^n}{K_1} \text{ and } \theta_i = \varphi_1 \circ \iota_i \\
&\cong \frac{\mathbb{Z}}{\sum_{i=2}^{s} im(\psi_i)} \\
&\cong \frac{\mathbb{Z}}{g\mathbb{Z}} \quad \text{, by definition of } \psi_i
\end{aligned}$$

□

Theorem 5.19. *Let Δ be an n-dimensional complete fan, where $n > 1$. Suppose Δ has s one dimensional cones $\rho_1, \cdots, \rho_s$, and let v_i be the minimal lattice point of ρ_i, for $i = 1, \cdots, s$. Then $H^{n-1}\left(\Delta, \widetilde{K_0^T}\right) \cong \mathbb{Z}^{g-1}$, where g is the greatest common divisor of the set of all 2×2 minors of the matrix $\begin{bmatrix} v_1 \\ \vdots \\ v_s \end{bmatrix}$.*

Proof. Let I_σ denote the kernel of the canonical surjection from $\mathbb{Z}[M]$ to $\mathbb{Z}[M_\sigma]$. Let $\mathcal{F}'$ and $\mathcal{F}''$ be sheaves on Δ such that for all $\sigma \in \Delta, \mathcal{F}'_\sigma = \mathbb{Z}[M]$ and $\mathcal{F}''_\sigma = I_\sigma$. Then we have the following short exact sequence of sheaves:

$$0 \longrightarrow \mathcal{F}'' \longrightarrow \mathcal{F}' \longrightarrow \widetilde{K_0^T} \longrightarrow 0,$$

which induces the long exact sequence of sheaf cohomology groups:

$$\cdots \longrightarrow H^{n-1}(\Delta, \mathcal{F}') \longrightarrow H^{n-1}\left(\Delta, \widetilde{K_0^T}\right) \xrightarrow{h} H^n(\Delta, \mathcal{F}'') \longrightarrow H^n(\Delta, \mathcal{F}') \longrightarrow \cdots$$

$\mathcal{F}'$ is a constant sheaf and therefore flasque. That implies the higher cohomology groups $H^p(\Delta, \mathcal{F}')$ vanish, and so for $n \geq 2$, the map h is an isomorphism. Since Δ is complete, the following cellular complex $C^\bullet$ for Δ with respect to $\mathcal{F}''$ computes the Čech cohomology of $\mathcal{F}''$. See [3, Proposition 3.5] and [7].

$$\bigoplus_{dim\sigma=n} \mathcal{F}''(\langle\sigma\rangle) \longrightarrow \bigoplus_{dim\sigma=n-1} \mathcal{F}''(\langle\sigma\rangle) \longrightarrow \cdots \longrightarrow \bigoplus_{dim\sigma=2} \mathcal{F}''(\langle\sigma\rangle) \xrightarrow{\tilde{i}-\tilde{t}} \bigoplus_{dim\rho=1} \mathcal{F}''(\langle\rho\rangle) \longrightarrow \mathcal{F}''(\{0\}),$$

where $\tilde{i}((\alpha_\sigma)_\sigma) = \left(\sum_{\substack{dim\sigma=2 \\ i(\sigma)=\rho}} \alpha_\sigma|_\rho \right)_\rho$ and $\tilde{t}((\alpha_\sigma)_\sigma) = \left(\sum_{\substack{dim\sigma=2 \\ t(\sigma)=\rho}} \alpha_\sigma|_\rho \right)_\rho$, for all $(\alpha_\sigma)_\sigma \in \bigoplus_{dim\sigma=2} \mathcal{F}''(\langle\sigma\rangle)$.

So, $H^n(\Delta, \mathcal{F}'')$ is the cokernel of the map, $\bigoplus_{\dim\rho=1} I_\rho \longrightarrow I_0$, sending each tuple to the sum of its coordinates. Thus,

$$\begin{aligned}
\check{H}^{n-1}\left(\Delta, \widetilde{K_0^T}\right) &\cong H^{n-1}\left(\Delta, \widetilde{K_0^T}\right) \\
&\cong H^n(\Delta, \mathcal{F}'') \\
&\cong \frac{I_0}{\sum_{i=1}^s I_{\rho_i}} \\
&\subseteq \frac{\mathbb{Z}[M]}{\sum_{i=1}^s I_{\rho_i}} \\
&\cong \mathbb{Z}\left[\frac{M}{\sum_{i=1}^s \rho_i^\perp \cap M}\right]
\end{aligned}$$

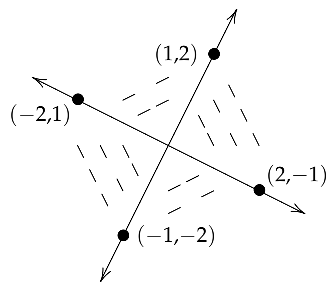
The last isomorphism is clear, since the kernel of the canonical surjection from $\mathbb{Z}[M]$ to $\mathbb{Z}\left[\frac{M}{\sum_{i=1}^s \rho_i^\perp \cap M}\right]$ is $\sum_{i=1}^s I_{\rho_i}$. Now, for $i = 1, \cdots, s$, $\rho_i^\perp \cap M$ is the kernel of the map $\varphi_i : M \to \mathbb{Z}$ sending $u \in M$ to $v_i \cdot u$. Therefore, by Lemma 5.18,

$$\frac{M}{\sum_{i=1}^s \rho_i^\perp \cap M} \cong \frac{\mathbb{Z}}{g\mathbb{Z}}.$$

That implies $\frac{\mathbb{Z}[M]}{\sum_{i=1}^s I_{\rho_i}} \cong \mathbb{Z}^g$. Finally, because $\frac{\mathbb{Z}[M]}{I_0} \cong \mathbb{Z}$, we have $\frac{I_0}{\sum_{i=1}^s I_{\rho_i}} \cong \mathbb{Z}^{g-1}$.

$\square$

Example 5.20. *Suppose Δ is the following 2 dimensional complete fan with four maximal cones.*



The gcd of the set of all 2×2 minors of the matrix $\begin{bmatrix} 1 & 2 \\ 2 & -1 \\ -1 & -2 \\ -2 & 1 \end{bmatrix}$ *is 5, and therefore $H^1(\Delta, K_0^T)$ is isomorphic to $\mathbb{Z}^4$.*

Remark 5.21. *Given a complete fan Δ in $N_{\mathbb{R}}$, Theorem 5.19 implies that we have $H^{n-1}(\Delta, \widetilde{K_0^T}) = 0$ whenever the minimal lattice points on two of its rays may be extended to a $\mathbb{Z}-$basis of N. In particular, if at least one of the two dimensional cones of Δ is smooth, then $H^{n-1}(\Delta, \widetilde{K_0^T}) = 0$.*

5.3 Non-Complete Fans

Definition 5.22. *Let Δ be a fan with Krull dimension d. For $d \geq 2$, we define the* $(d-1)-$boundary of Δ *to be the union of the $(d-1)-$dimensional faces that are facets of exactly one cone. We say* Δ has a $(d-1)-$boundary *if this union is not empty.*

Example 5.23. *An $n-$dimensional complete fan does not have an $(n-1)-$boundary.*

Example 5.24. *For $d \geq 2$, the $d-$skeleton of a complete fan does not have a $(d-1)-$boundary.*

Example 5.25. *For, $n \geq 3$, the boundary of an n-dimensional cone does not have an $(n-2)-$boundary.*

Remark 5.26. *A fan of Krull dimension d has a $(d-1)-$boundary if and only if there exists a d-dimensional cone σ such that $\langle\sigma\rangle \cap \langle\Delta\backslash\langle\sigma\rangle\rangle \subsetneq \partial\sigma$, where $\langle\Delta\backslash\langle\sigma\rangle\rangle$ is the fan generated by the maximal cones of Δ that are not equal to σ.*

Lemma 5.27. *If Δ is a fan with Krull dimension $d \geq 2$ such that every subfan of Krull dimension d has a $(d-1)-$boundary, then $H^{d-1}\left(\Delta, \widetilde{K_0^T}\right) = 0$.*

Proof. There exists a d-dimensional cone σ such that $\langle\sigma\rangle \cap \langle\Delta\backslash\langle\sigma\rangle\rangle \subsetneq \partial\sigma$. Let $\Delta' = \langle\Delta\backslash\langle\sigma\rangle\rangle$. Induct on d and the number l of cones of dimension d. Consider the following long exact sequence of cohomology groups:

$$\cdots \longrightarrow H^{d-2}\left(\langle\sigma\rangle, \widetilde{K_0^T}\right) \oplus H^{d-2}\left(\Delta', \widetilde{K_0^T}\right) \xrightarrow{h} H^{d-2}\left(\langle\sigma\rangle \cap \Delta', \widetilde{K_0^T}\right) \longrightarrow H^{d-1}\left(\Delta, \widetilde{K_0^T}\right) \longrightarrow H^{d-1}\left(\langle\sigma\rangle, \widetilde{K_0^T}\right) \oplus H^{d-1}\left(\Delta', \widetilde{K_0^T}\right) \longrightarrow \cdots$$

Because $d \geq 2$ and $\langle\sigma\rangle$ is affine, $H^{d-1}\left(\langle\sigma\rangle, \widetilde{K_0^T}\right) = 0$. We will first check that the lemma holds for $d = 2$. In the case of $l = 1$, we see that h is surjective, because $\langle\sigma\rangle \cap \Delta'$ is $\{0\}$. Δ' has no 2-dimensional cones, and so $H^1\left(\Delta', \widetilde{K_0^T}\right) = 0$. When $l > 1$, h is still surjective, because $\langle\sigma\rangle \cap \Delta'$ is a ray in the boundary of σ. Δ' has Krull dimension 2 and every one of its subfans of Krull dimension 2 has an 1-boundary. By induction on l, $H^1\left(\Delta', \widetilde{K_0^T}\right) = 0$. Therefore, $H^1\left(\Delta, \widetilde{K_0^T}\right) = 0$.

Now, consider the case of $d > 2$ and $l \geq 1$. When $l = 1$, Δ' has Krull dimension d, and by Corollary 5.7, we have $H^{d-1}\left(\Delta', \widetilde{K_0^T}\right) = 0$. For $l > 1$, every subfan of Δ of Krull dimension d has an $(d-1)-$boundary, so $H^{d-1}\left(\Delta', \widetilde{K_0^T}\right) = 0$ by induction on l. What is left to show is that $H^{d-2}\left(\langle\sigma\rangle \cap \Delta', \widetilde{K_0^T}\right) = 0$. Let $\Lambda = \langle\sigma\rangle \cap \Delta'$. If the Krull dimension of Λ is strictly less than $d-1$, then $H^{d-2}\left(\Lambda, \widetilde{K_0^T}\right) = 0$. Assume Λ has Krull dimension $d-1$, we will prove that every subfan of Λ of Krull dimension $d-1$ has an $(d-2)-$boundary. Then by induction on d, the proof will be complete.

Let Λ' be a subfan of Λ with Krull dimension $d-1$. Consider the dual $\sigma^\vee$ of σ in $\mathbb{R}\sigma = \mathbb{R}^d$. By the one-to-one correspondence between the faces of σ and those of $\sigma^\vee$, $\Lambda' \subseteq \Lambda \subsetneq \partial\sigma$ implies $D := \{\tau^\perp \cap \sigma^\vee | \tau \text{ is a facet of } \Lambda'\}$ is a proper subset of the set of rays of $\sigma^\vee$. Since the intersection X of $\sigma^\vee$ and the $(d-1)$-sphere is a connected CW-complex, its $1-$skeleton is connected [13, Thm 3.32]. Let G be the graph whose vertices represent the $0-$cells of X (or rays of $\sigma^\vee$) and edges represent the 1-cells (or 2-dim faces). Then G is connected. That means there are rays $\rho_1 \in D$, $\rho_2 \notin D$ such that ρ_1 and ρ_2 form the boundary of a 2-dimensional face of $\sigma^\vee$. Thus, there exists $(d-1)$-dimensional cones $\tau_1 \in \Lambda'$ and $\tau_2 \notin \Lambda'$ such that $\tau_1 \cap \tau_2$ is a facet of both of them. Since every codimension 2 face of a cone is the intersection of exactly two facets [4, P. 10], $\tau_1 \cap \tau_2$ is not a face of any other maximal cone of Λ'. Therefore, Λ' has a $(d-2)-$boundary.

□

Theorem 5.28. *Let Δ be a non-complete fan in $\mathbb{R}^n$ with $n \geq 2$. Then $H^{n-1}\left(\Delta, \widetilde{K_0^T}\right) = 0$.*

Proof. If the Krull dimension of Δ is at most $n-1$, then by Lemma 5.3, $H^{n-1}\left(\Delta, \widetilde{K_0^T}\right)$ is 0. Assume Δ had Krull dimension n. By Lemma 5.27, it suffices to prove that every $n-$dimensional subfan of Δ has an $(n-1)-$boundary. Let σ be a cone in Δ with dimension n and p be a point in the interior of σ. Define B to the set

$$\{q \in \mathbb{R}^n \backslash |\Delta| : \overleftrightarrow{pq} \cap \mathbb{R}\tau \neq \emptyset, \text{ for some } \tau \in \Delta \text{ with } \dim\tau \leq n-2\},$$

where $\overleftrightarrow{pq}$ is the line through p and q. We claim that

$$B \subsetneq \mathbb{R}^n \backslash |\Delta|. \tag{5.29}$$

To prove (5.29), let τ be a cone in Δ of dimension at most $n-2$, define

$$B_\tau = \{q \in \mathbb{R}^n : \overleftrightarrow{pq} \cap \mathbb{R}_\tau \neq \emptyset\}.$$

For all points $r \in B_\tau$, there exists $w \in \overleftrightarrow{pr} \cap \mathbb{R}_\tau$. Since $w \neq p, r \in \operatorname{span}(\mathbb{R}_\tau, p)$. So, B_τ is a subset of $\operatorname{span}(\mathbb{R}_\tau, p)$, which is a real vector space of dimension at most $n-1$. Now, B is the intersection of $\mathbb{R}^n \backslash |\Delta|$ and the finite union $U := \bigcup\limits_{\dim \tau \leq n-2} B_\tau$. Suppose B is all of $\mathbb{R}^n \backslash |\Delta|$, then $U \cup |\Delta| = \mathbb{R}^n$. But, the hypervolume of the intersection of S^{n-1} and $U \cup |\Delta|$ is strictly less than that of S^{n-1}, because $U \cup |\Delta|$ is contained in a finite union of proper closed subsets of $\mathbb{R}^n$. So, we have a contraction; therefore, B is a proper subset of $\mathbb{R}^n \backslash |\Delta|$.

By (5.29), there exists $x \in \mathbb{R}^n \backslash |\Delta|$ such that $\overleftrightarrow{xp} \cap \mathbb{R}_\tau = \emptyset$ for all $\tau \in \Delta$ with dimension at most $n-2$. The intersection of the line segment $\overline{xp}$ and $|\Delta|$ is non-empty, because p is in the interior of σ. Let y be the point in $\overline{xp} \cap |\Delta|$ closest to x. Then y is an interior point of a facet τ of an n-dimensional cone α of Δ. We will prove that τ is not a facet of another maximal cone of Δ, and this will show that Δ has a boundary. Suppose τ is a facet of another maximal cone β, then $\tau = \alpha \cap \beta$. Let $w \in \alpha^\vee$ such that $\tau = w^\perp \cap \alpha$, then w is in the ray $\tau^\perp \cap \alpha^\vee$. Also, by the Separation Lemma (see Section 2.1.1), there exists $u \in \alpha^\vee \cap (-\beta)^\vee$ such that $\tau = u^\perp \cap \alpha = u^\perp \cap \beta$. So, u is in the ray $\tau^\perp \cap \alpha^\vee$ as well. That implies u is a positive multiple of w. Thus, $-w$ is in $\beta^\vee$ and $\tau = w^\perp \cap \beta = (-w)^\perp \cap \beta$. By linearity of $\overline{xp}$, the function w is negative on the interval $[x, y)$, zero at y and positive on $(y, p]$. $\tau = (-w)^\perp \cap \beta$ implies w is negative on the closed interval $\overline{xp} \cap \beta \subseteq [x, y)$, which contradicts the fact that y is the point in $\overline{xp} \cap |\Delta|$ closest to x.

$\square$

5.4 $H^1\left(\Delta, \widetilde{K_0^T}\right)$ of Some Three Dimensional Fans Δ

In this section, we give some examples of three dimensional fans Δ with the property that $H^1\left(\Delta, \widetilde{K_0^T}\right)$ is torsion free. First, we need to prove three statements about limits and colimits.

Lemma 5.30. *Let $\mathcal{I}$ be an indexing category and $i \mapsto M_i$ be a functor from $\mathcal{I}$ to the category Ab of abelian groups. Then $\varinjlim_{\mathcal{I}}{}^{Rings}\mathbb{Z}[M_i] \cong \mathbb{Z}\left[\varinjlim_{\mathcal{I}}{}^{Ab}(M_i)\right]$ as rings.*

Proof. Let $S = \varinjlim^{Rings} \mathbb{Z}[M_i]$ and $T = \mathbb{Z}\left[\varinjlim^{Ab}(M_i)\right]$. Notice that $\mathbb{Z}[-] : Ab \longrightarrow Rings$ and $(-)^{\times} : Rings \longrightarrow Ab$ are adjoint functors, and so for any ring R, we have

$$\begin{aligned} Hom_{Rings}(S, R) &\cong \varprojlim{}^{Ab} Hom_{Rings}(\mathbb{Z}[M_i], R) \\ &\cong \varprojlim{}^{Ab} Hom_{Ab}(M_i, R^{\times}) \\ &\cong Hom_{Ab}\left(\varinjlim{}^{Ab} M_i, R^{\times}\right) \\ &\cong Hom_{Rings}(T, R) \end{aligned}$$

By Yoneda's Lemma, $S \cong T$. $\square$

Remark 5.31. *$\mathbb{Z}[-] : Sets \longrightarrow Ab$ and the forgetful functor from Ab to $Sets$ form an adjoint pair. Thus, the same argument proves that $\varinjlim^{Ab} \mathbb{Z}[M_i]$ is isomorphic to $\mathbb{Z}\left[\varinjlim^{Sets}(M_i)\right]$ in the category of abelian groups.*

Lemma 5.32. *Let $\sigma_1, \cdots, \sigma_r$ be cones in a fan Δ. Suppose L is the limit of the diagram*

$$\begin{tikzcd}
M_{\sigma_1} \arrow[dr] & & M_{\sigma_2} \arrow[dl] \arrow[dr] & & \cdots & & M_{\sigma_r} \arrow[dl] \\
& M_{\sigma_1\cap\sigma_2} & & M_{\sigma_2\cap\sigma_3} & \cdots & M_{\sigma_{r-1}\cap\sigma_r} &
\end{tikzcd}$$

in the category of abelian groups, where the maps are canonical surjections. Let Q be the limit of the induced diagram

$$\begin{tikzcd}
\mathbb{Z}[M_{\sigma_1}] \arrow[dr] & & \mathbb{Z}[M_{\sigma_2}] \arrow[dl] \arrow[dr] & & \cdots & & \mathbb{Z}[M_{\sigma_r}] \arrow[dl] \\
& \mathbb{Z}[M_{\sigma_1\cap\sigma_2}] & & \mathbb{Z}[M_{\sigma_2\cap\sigma_3}] & \cdots & \mathbb{Z}[M_{\sigma_{r-1}\cap\sigma_r}] &
\end{tikzcd}$$

in the category of rings. Then the ring map from $\mathbb{Z}[L]$ to Q given by the universality of Q is surjective.

Proof. Let $f : A \to C$ and $g : B \to C$ be two arbitrary group homomorphisms and P be the fiber product of the induced ring maps $\hat{f} : \mathbb{Z}[A] \to \mathbb{Z}[C]$ and $\hat{g} : \mathbb{Z}[B] \to \mathbb{Z}[C]$. We claim that the ring map $\varphi : \mathbb{Z}[A \times_C B] \to P$ given by the universality of P is surjective. Indeed, given an element (α, β) in P, α and β have the forms $\sum_{i=1}^{s} n_i\chi^{a_i}$ and $\sum_{j=1}^{t} m_j\chi^{b_j}$ respectively, for some $a_i \in A, b_j \in B$ and $n_i, m_j \in \mathbb{Z}$. Fix $c \in C$ and define α_c to be $\sum_{f(a_i)=c} n_i\chi^{a_i}$. Let β_c be defined similarly. Without lost of generality, we may assume $\alpha = \alpha_c$ and $\beta = \beta_c$. Then, $\sum_{i=1}^{s} n_i = \sum_{j=1}^{t} m_j$ and $f(a_i) = c = g(b_j)$ for all $1 \le i \le s, 1 \le j \le t$. Notice that (a_i, b_j) is an element of $A \times_C B$, for all $1 \le i \le s, 1 \le j \le t$. Also, for $i = 1, \cdots, s$ and $j = 1, \cdots, t$, there exists $d_{i,j} \in \mathbb{Z}$ such that $\sum_{j=1}^{t} d_{i,j} = n_i$ and $\sum_{i=1}^{s} d_{i,j} = m_j$. (For example, we may choose $d_{i,j}$ to be 0, for all $1 \le i \le s-1$ and $1 \le j \le t-1$, $d_{i,t}$ to be n_i for $i = 1, \cdots, s-1$, $d_{s,j}$ to be m_j for $i = 1, \cdots, t-1$, and $d_{s,t}$ to be $m_t - \sum_{i=1}^{s-1} n_i$.) One can check that $\varphi\left(\sum_{i,j} d_{i,j}\chi^{(a_i,b_j)}\right) = (\alpha_c, \beta_c)$.

For $i = 1, \cdots, r$, define L_i to be the limit of the diagram

$$\begin{tikzcd}
M_{\sigma_1} \arrow[dr, two heads] & & M_{\sigma_2} \arrow[dl, two heads] \arrow[dr, two heads] & & \cdots & & M_{\sigma_i} \arrow[dl, two heads] \\
& M_{\sigma_1 \cap \sigma_2} & & M_{\sigma_2 \cap \sigma_3} & \cdots & M_{\sigma_{i-1} \cap \sigma_i} &
\end{tikzcd}$$

in the category of abelian groups and define Q_i to be the limit of the diagram

$$\begin{tikzcd}
\mathbb{Z}[M_{\sigma_1}] \arrow[dr, two heads] & & \mathbb{Z}[M_{\sigma_2}] \arrow[dl, two heads] \arrow[dr, two heads] & & \cdots & & \mathbb{Z}[M_{\sigma_i}] \arrow[dl, two heads] \\
& \mathbb{Z}[M_{\sigma_1 \cap \sigma_2}] & & \mathbb{Z}[M_{\sigma_2 \cap \sigma_3}] & \cdots & \mathbb{Z}[M_{\sigma_{i-1} \cap \sigma_i}] &
\end{tikzcd}$$

in the category of rings. Notice that L_r is the limit of the diagram:

$$\begin{tikzcd}
L_{r-1} \arrow[dr, two heads] & & M_{\sigma_r} \arrow[dl, two heads] \\
& M_{\sigma_{r-1} \cap \sigma_r} &
\end{tikzcd}$$

So, by the assertion above, the universal map ξ from $\mathbb{Z}[L_r]$ to the limit T of the following diagram is surjective.

$$\begin{tikzcd}
\mathbb{Z}[L_{r-1}] \arrow[dr, two heads] & & \mathbb{Z}[M_{\sigma_r}] \arrow[dl, two heads] \\
& \mathbb{Z}[M_{\sigma_{r-1} \cap \sigma_r}] &
\end{tikzcd}$$

Observe that Q_r is the limit of the diagram:

$$\begin{tikzcd}
Q_{r-1} \arrow[dr, two heads] & & \mathbb{Z}[M_{\sigma_r}] \arrow[dl, two heads] \\
& \mathbb{Z}[M_{\sigma_{r-1} \cap \sigma_r}] &
\end{tikzcd}$$

Therefore, T is the limit of the diagram:

$$\begin{tikzcd}
\mathbb{Z}[L_{r-1}] \arrow[dr, "\theta"'] & & Q_r \arrow[dl] \\
& Q_{r-1} &
\end{tikzcd}$$

By induction on r, θ is surjective, and so its pullback $\theta' : T \to Q_r$ is surjective. Hence, the composition $\theta' \circ \xi : \mathbb{Z}[L_r] \to Q_r$ is surjective, and it is universal.

□

Now, we are ready to show that some three dimensional fans Δ satisfy the property that $H^1\left(\Delta, \widetilde{K_0^T}\right)$ is torsion free.

Lemma 5.33. *Let Δ be the boundary of a three dimensional cone, then $H^1\left(\Delta, \widetilde{K_0^T}\right)$ is torsion free.*

Proof. Fix a two dimensional cone σ in the Δ. Let Δ' denote $\langle \Delta \backslash \langle \sigma \rangle \rangle$, and let ρ_1 and ρ_2 be the rays in the boundary of σ. Consider the following diagram:

$$\begin{tikzcd}
& H^0\left(\Delta', \widetilde{K_0^T}\right) \oplus H^0\left(\langle\sigma\rangle, \widetilde{K_0^T}\right) \arrow[d, "f"] \arrow[dr, "h"] & & & \\
0 \arrow[r] & H^0\left(\partial\sigma, \widetilde{K_0^T}\right) \arrow[r, "g"] \arrow[d] & \mathbb{Z}[M_{\rho_1}] \oplus \mathbb{Z}[M_{\rho_2}] \arrow[r] & \mathbb{Z} \arrow[r] & 0 \\
& H^1\left(\Delta, \widetilde{K_0^T}\right) \arrow[d] & & & \\
& H^1\left(\Delta', \widetilde{K_0^T}\right) \oplus H^1\left(\langle\sigma\rangle, \widetilde{K_0^T}\right) & & &
\end{tikzcd}$$

The row in the diagram is the Čech complex for $\widetilde{K_0^T}$ on $\partial\sigma$ with respect to the cover $\{\langle\rho_1\rangle, \langle\rho_2\rangle\}$, and it is exact. The map h is the composition of f and g. By Lemma 5.14, $H^1\left(\Delta', \widetilde{K_0^T}\right) \oplus H^1\left(\langle\sigma\rangle, \widetilde{K_0^T}\right) = 0$, so $H^1\left(\Delta, \widetilde{K_0^T}\right)$ is the cokernel of f. Thus, we obtain the induced short exact sequence:

$$0 \longrightarrow H^1\left(\Delta, \widetilde{K_0^T}\right) \longrightarrow coker(h) \xrightarrow{\zeta} \mathbb{Z} \longrightarrow 0 \tag{5.34}$$

We will prove that $coker(h)$ is free. Let $\sigma_1, \cdots, \sigma_r$ be the two dimensional cones in Δ', then $H^0\left(\Delta', \widetilde{K_0^T}\right)$ is the equalizer of

$$\bigoplus_{i=1}^{r} \mathbb{Z}[M_{\sigma_i}] \ \underset{\theta_2}{\overset{\theta_1}{\rightrightarrows}} \ \bigoplus_{1 \le i < j \le r} \mathbb{Z}[M_{\sigma_i \cap \sigma_j}]$$

in the category of rings, where θ_1 and θ_2 are defined as $(\alpha_i)_i \mapsto (\alpha_j|_{\sigma_i \cap \sigma_j})_{i<j}$ and $(\alpha_i)_i \mapsto (\alpha_i|_{\sigma_i \cap \sigma_j})_{i<j}$ respectively. Since for $i = 1, \cdots, r-1$, we have $\sigma_i \cap \sigma_j \neq 0$ if and only if $j = i+1$, $H^0\left(\Delta', \widetilde{K_0^T}\right)$ is the equalizer of

$$\bigoplus_{i=1}^{r} \mathbb{Z}[M_{\sigma_i}] \ \underset{\theta_2'}{\overset{\theta_1'}{\rightrightarrows}} \ \bigoplus_{i=1}^{r-1} \mathbb{Z}[M_{\sigma_i \cap \sigma_{i+1}}] \, ,$$

where θ_1' and θ_2' are defined as $(\alpha_i)_i \mapsto (\alpha_{i+1}|_{\sigma_i \cap \sigma_{i+1}})_i$ and $(\alpha_i)_i \mapsto (\alpha_i|_{\sigma_i \cap \sigma_{i+1}})_i$ respectively. Thus, $H^0\left(\Delta', \widetilde{K_0^T}\right)$ is the limit of the following diagram in the category of rings:

$$\begin{tikzcd}
\mathbb{Z}[M_{\sigma_1}] \arrow[dr] & & \mathbb{Z}[M_{\sigma_2}] \arrow[dl] \arrow[dr] & & \cdots & & \mathbb{Z}[M_{\sigma_r}] \arrow[dl] \\
& \mathbb{Z}[M_{\sigma_1 \cap \sigma_2}] & & \mathbb{Z}[M_{\sigma_2 \cap \sigma_3}] & \cdots & \mathbb{Z}[M_{\sigma_{r-1} \cap \sigma_r}] &
\end{tikzcd}$$

By Lemma 5.32, there is a surjection π from $\mathbb{Z}[L]$ to $H^0\left(\Delta', \widetilde{K_0^T}\right)$, where L is the limit of the following diagram in the category of abelian groups:

$$\begin{tikzcd}[column sep=small]
M_{\sigma_1} \arrow[dr, two heads] & & M_{\sigma_2} \arrow[dl, two heads] \arrow[dr, two heads] & & \cdots & & M_{\sigma_r} \arrow[dl, two heads] \\
& M_{\sigma_1\cap\sigma_2} & & M_{\sigma_2\cap\sigma_3} & \cdots & M_{\sigma_{r-1}\cap\sigma_r} &
\end{tikzcd}$$

Let B be the pushout of the canonical surjections $M_\sigma \to M_{\rho_1}$ and $M_\sigma \to M_{\rho_2}$. Because $H^0\left(\langle\sigma\rangle, \widetilde{K_0^T}\right) = \mathbb{Z}[M_\sigma]$, the sequence

$$H^0\left(\langle\sigma\rangle, \widetilde{K_0^T}\right) \longrightarrow \mathbb{Z}[M_{\rho_1}] \oplus \mathbb{Z}[M_{\rho_2}] \longrightarrow \mathbb{Z}[B] \longrightarrow 0$$

is exact, by Lemma 5.30. Consider the following diagram:

$$\begin{tikzcd}
& \mathbb{Z}[L] \arrow[d, "\pi"] \arrow[ddr, "\varphi"] & & \\
& H^0\left(\Delta', \widetilde{K_0^T}\right) \arrow[d, "h''"] \arrow[dr, "\psi"] & & \\
H^0\left(\langle\sigma\rangle, \widetilde{K_0^T}\right) \arrow[r, "h'"] & \mathbb{Z}[M_{\rho_1}] \oplus \mathbb{Z}[M_{\rho_2}] \arrow[r, "\kappa"] & \mathbb{Z}[B] \arrow[r] & 0
\end{tikzcd}$$

where $h = h' - h''$, $\psi = \kappa \circ h''$ and $\varphi = \kappa \circ h'' \circ \pi$. Now,

$$coker(h) \cong coker(h')/im(\psi) = coker(h')/im(\varphi) = coker(\varphi),$$

and $coker(\varphi)$ is the coequalizer of

$$\mathbb{Z}[L] \underset{\eta_2}{\overset{\eta_1}{\rightrightarrows}} \mathbb{Z}[B]$$

in the category of abelian groups, where η_1 and η_2 are the maps

$$\sum_i a_i \chi^{(m_1,\cdots,m_r)_i} \mapsto \sum_i a_i \chi^{\overline{((m_1)_{\rho_1},0)}}$$

and

$$\sum_i a_i \chi^{(m_1,\cdots,m_r)_i} \mapsto \sum_i a_i \chi^{\overline{(0,(m_r)_{\rho_2})}}.$$

Finally, by Remark 5.31, $coker(h)$ is isomorphic to $\mathbb{Z}[C]$, where C is the coequalizer of

$$L \underset{\xi_2}{\overset{\xi_1}{\rightrightarrows}} B$$

in the category of sets. The maps, ξ_1 and ξ_2, are given by

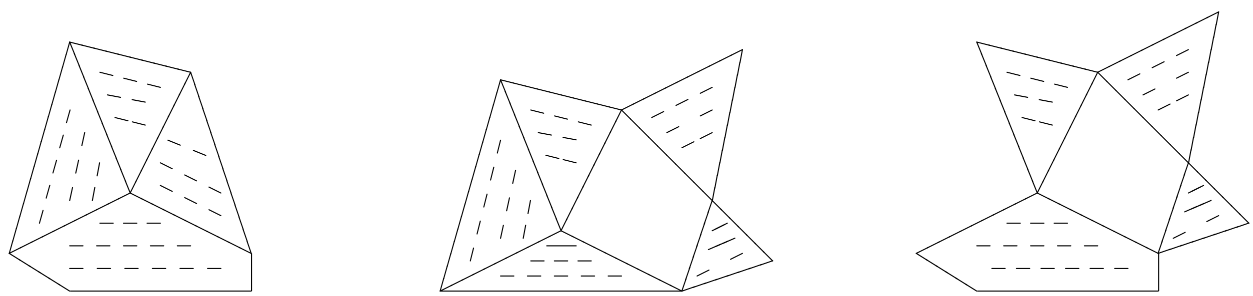
$$(m_1, \cdots, m_r) \mapsto \overline{((m_1)_{\rho_1}, 0)}$$

and

$$(m_1, \cdots, m_r) \mapsto \overline{(0, (m_r)_{\rho_2})}.$$

Therefore, $coker(h)$ is free. That means $H^1\left(\Delta, \widetilde{K_0^T}\right)$ is free as well because of (5.34) and the fact that ζ is the canonical surjection. $\square$

Remark 5.35. *The proof of Lemma 5.33 goes through for every* $3-$*dimensional fan* Δ *with the property* $H^1\left(\langle\Delta\backslash\sigma\rangle, \widetilde{K_0^T}\right) = 0$*, for some cone* $\sigma \in \Delta$ *such that* $\langle\Delta\backslash\sigma\rangle \cap \langle\sigma\rangle$ *has two maximal cones. The picture below illustrates some examples of such* $3-$*dimensional fans, and their cohomology groups* $H^1\left(\Delta, \widetilde{K_0^T}\right)$ *are torsion free. Each figure shows the intersections of the non-zero cones in each* $3-$*dimensional fan with* S^2*, the unit* $2-$*sphere centered at the origin.*



(a) (b) (c)

Figure 5.36: Intersections of S^2 and Δ

If $\mathcal{C}$ is a collection of fans Δ that satisfy the property that $H^1\left(\Delta, \widetilde{K_0^T}\right)$ is torsion free, then every fan in $\overline{\mathcal{C}}$ as defined in Section 5.1 has this property also. From this, we can see that there is a large family of three dimensional fans whose cohomology group $H^1\left(\Delta, \widetilde{K_0^T}\right)$ has no torsion. This property is helpful for the purpose of computing equivariant K-groups. Again, we do not know of an example of a fan Δ such that $H^1\left(\Delta, \widetilde{K_0^T}\right)$ has torsion.

Chapter 6

Applications to Equivariant K-Theory

From Chapter 4, we see that for a smooth fan Δ, the Čech complex of the presheaf K_q^T with respect to the equivariant open cover $\mathcal{V} = \{U_\sigma | \sigma \text{ is a maximal cone in } \Delta\}$,

$$0 \longrightarrow K_q^T(X(\Delta)) \longrightarrow \bigoplus_{\sigma} K_q^T(U_\sigma) \longrightarrow \bigoplus_{\delta<\tau} K_q^T(U_{\delta\cap\tau}) \longrightarrow \bigoplus_{\delta<\tau} K_q^T(U_{\delta\cap\tau}) \longrightarrow \bigoplus_{\delta<\tau<\epsilon} K_q^T(U_{\delta\cap\tau\cap\epsilon}) \longrightarrow \cdots,$$

is exact, and

$$K_q^T(X(\Delta)) \cong \check{H}^0(\mathcal{V}, K_0^T) \otimes_{\mathbb{Z}} K_q(k).$$

In this chapter, we investigate the question: If Δ is not smooth, how close is $K_q^T(X(\Delta))$ to the group $\check{H}^0(\mathcal{V}, K_0^T) \otimes_{\mathbb{Z}} K_q(k) \cong H^0\left(\Delta, \widetilde{K_0^T}\right) \otimes_{\mathbb{Z}} K_q(k)$? Due to the results in Chapter 5, we now have some understanding of fan cohomology for two and three dimensional fans, and so we can detect the difference between $K_q^T(X(\Delta))$ and the group $H^0\left(\Delta, \widetilde{K_0^T}\right) \otimes_{\mathbb{Z}} K_q(k)$ in these cases. When the ground field k is finite, we obtain even more explicit formulas for the K-groups.

6.1 Equivariant K-groups of Two Dimensional Fans

Recall that for a quasi-projective toric variety $X(\Delta)$ and the equivariant open cover $\mathcal{V} = \{U_\sigma | \sigma \text{ is a maximal cone in } \Delta\}$, there is a convergent spectral sequence [20]

$$\check{H}^p(\mathcal{V}, K_q^T) \Longrightarrow K_{q-p}^T(X(\Delta)), \tag{6.1}$$

and if $\check{H}^{p+1}(\mathcal{V}, K_0^T)$ is torsion free, we have the isomorphisms

$$\check{H}^p(\mathcal{V}, K_q^T) \cong \check{H}^p(\mathcal{V}, K_0^T) \otimes_{\mathbb{Z}} K_q(k),$$

[21, 3.2.1], and so by Proposition 2.14,

$$H^p\left(\Delta, \widetilde{K_q^T}\right) \cong H^p\left(\Delta, \widetilde{K_0^T}\right) \otimes_{\mathbb{Z}} K_q(k),$$

whenever $p, q \geq 0$.

Theorem 6.2. *Let Δ be a two dimensional fan with s one dimensional cones $\rho_1, \cdots, \rho_s$. If Δ is not complete, then for all $q \geq 0$,*

$$K_q^T(X(\Delta)) \cong H^0\left(\Delta, \widetilde{K_0^T}\right) \otimes_{\mathbb{Z}} K_q(k).$$

If Δ is complete, then $K_q^T(X(\Delta))$ is an extension of $H^0\left(\Delta, \widetilde{K_0^T}\right) \otimes_{\mathbb{Z}} K_q(k)$ by $K_{q+1}(k)^{g-1}$, where g is the greatest common divisor of the set of all 2×2 non-zero minors of the matrix $\begin{bmatrix} v_1 \\ \vdots \\ v_s \end{bmatrix}$, and v_i is the minimal lattice point of ρ_i for $i = 1, \cdots, s$.

Proof. By Corollary 5.7, $H^p\left(\Delta, \widetilde{K_0^T}\right) = 0$ for all $p \geq 2$; therefore, the spectral sequence

(6.1) gives the short exact sequence,

$$0 \longrightarrow H^1\left(\Delta, \widetilde{K^T_{q+1}}\right) \longrightarrow K^T_q(X(\Delta)) \longrightarrow H^0\left(\Delta, \widetilde{K^T_q}\right) \longrightarrow 0. \tag{6.3}$$

Since $H^2\left(\Delta, \widetilde{K^T_0}\right) = 0$,

$$H^1\left(\Delta, \widetilde{K^T_{q+1}}\right) \cong H^1\left(\Delta, \widetilde{K^T_0}\right) \otimes_{\mathbb{Z}} K_{q+1}(k) \tag{6.4}$$

In the case of Δ being non-complete, $H^1\left(\Delta, \widetilde{K^T_0}\right) = 0$ by Theorem 5.28. That means $H^1\left(\Delta, \widetilde{K^T_{q+1}}\right) \cong 0$ and

$$H^0\left(\Delta, \widetilde{K^T_q}\right) \cong H^0\left(\Delta, \widetilde{K^T_0}\right) \otimes_{\mathbb{Z}} K_q(k). \tag{6.5}$$

As a result, for all $q \geq 0$,

$$K^T_q(X(\Delta)) \cong H^0\left(\Delta, \widetilde{K^T_0}\right) \otimes_{\mathbb{Z}} K_q(k). \tag{6.6}$$

If Δ is complete, Theorem 5.19 says that $H^1\left(\Delta, \widetilde{K^T_0}\right) \cong \mathbb{Z}^{g-1}$ where g is defined as above. That implies

$$H^1\left(\Delta, \widetilde{K^T_{q+1}}\right) \cong K_{q+1}(k)^{g-1}$$

and

$$H^0\left(\Delta, \widetilde{K^T_q}\right) \cong H^0\left(\Delta, \widetilde{K^T_0}\right) \otimes_{\mathbb{Z}} K_q(k).$$

$\square$

6.2 Equivariant K-groups of Three Dimensional Fans

If Δ is a fan that is acyclic with respect to $\widetilde{K_0^T}$ (such as the one in Example 5.16), then the isomorphism (6.6) holds for all $q \geq 0$, as in the cases when Δ is smooth or Δ is a $2-$dimensional non-complete fan.

For three dimensional non-complete fans, we still have the short exact sequence (6.3) and the isomorphism (6.4), since $H^p\left(\Delta, \widetilde{K_0^T}\right) = 0$ for all $p \geq 2$. However, the isomorphism (6.5) no longer necessarily holds. Instead, we have

$$H^0\left(\Delta, \widetilde{K_q^T}\right) \cong H^0\left(\Delta, \widetilde{K_0^T}\right) \otimes_{\mathbb{Z}} K_q(k) \oplus \mathrm{Tor}_1^{\mathbb{Z}}\left(H^1\left(\Delta, \widetilde{K_0^T}\right), K_q(k)\right)$$

[21, 3.6.2]. In other words, up to an extension the difference between $K_q^T(X(\Delta))$ and $H^0\left(\Delta, \widetilde{K_0^T}\right) \otimes_{\mathbb{Z}} K_q(k)$ can be measured by the group $H^1\left(\Delta, \widetilde{K_0^T}\right)$. In the cases where $H^1\left(\Delta, \widetilde{K_0^T}\right)$ is torsion free (such as those fans discussed in Section 5.4), $K_q^T(X(\Delta))$ is an extension of $H^0\left(\Delta, \widetilde{K_0^T}\right) \otimes_{\mathbb{Z}} K_q(k)$ by a direct sum of $K_{q+1}(k)$, just like the situation with two dimensional complete fans.

6.3 Finite Fields

In the special case where the ground field k is finite, we can apply our results together with the following calculation of Quillen's [14, Theorem 8] to get explicit determinations of the equivariant K-groups of toric varieties:

For a finite field $k = \mathbb{F}_r$,

$$K_q(k) \cong \begin{cases} \mathbb{Z}, & q = 0 \\ \mathbb{Z}/(r^i - 1), & q = 2i - 1 \text{ for } i \in \mathbb{N} \\ 0, & q = 2i \text{ for } i \in \mathbb{N}. \end{cases}$$

Example 6.7. *Let Δ be a two dimensional non-complete fan in $\mathbb{R}^2$ and k be the finite field $\mathbb{F}_r$. Then by the isomorphism (6.6),*

$$K_q^T(X(\Delta)) \cong \begin{cases} H^0\left(\Delta, \widetilde{K_0^T}\right), & q = 0 \\ H^0\left(\Delta, \widetilde{K_0^T}\right) \otimes_{\mathbb{Z}} \mathbb{Z}/(r^i - 1), & q = 2i - 1 \textit{ for } i \in \mathbb{N} \\ 0, & q = 2i \textit{ for } i \in \mathbb{N}. \end{cases}$$

Example 6.8. *Let Δ be a two dimensional complete fan and k be the finite field $\mathbb{F}_r$. Then $K_0^T(X(\Delta))$ is an extension of $H^0\left(\Delta, \widetilde{K_0^T}\right)$ by $(\mathbb{Z}/(r-1))^{g-1}$, where g is as defined in Theorem 5.19. The higher equivariant K-groups are as follows:*

$$K_q^T(X(\Delta)) \cong \begin{cases} H^0\left(\Delta, \widetilde{K_0^T}\right) \otimes_{\mathbb{Z}} \mathbb{Z}/(r^i - 1), & q = 2i - 1 \textit{ for } i \in \mathbb{N} \\ \left(\mathbb{Z}/(r^i - 1)\right)^{g-1}, & q = 2i \textit{ for } i \in \mathbb{N} \end{cases}$$

Theorem 6.9. *If Δ is a three dimensional fan and k is a finite field, then we have*

$$K_{-2}^T(X(\Delta)) \cong \begin{cases} \mathbb{Z}^{g-1}, & \Delta \textit{ is complete} \\ 0, & \textit{otherwise}, \end{cases}$$

where g is as defined in Theorem 5.19.

Proof. Since Δ is three dimensional, $H^p\left(\Delta, \widetilde{K_q^T}\right) = 0$ for all $q \geq 0$ and $p \geq 3$. Also, for all $p \geq 0$ and $j \in \mathbb{N}$,

$$H^p\left(\Delta, \widetilde{K_{2j}^T}\right) \cong H^p\left(\Delta, \widetilde{K_0^T}\right) \otimes_{\mathbb{Z}} K_{2j}(k) = 0$$

So, the convergent spectral sequence (6.1) yields the isomorphism

$$K_{-2}^T(X(\Delta)) \cong H^2\left(\Delta, \widetilde{K_0^T}\right),$$

and the result follows from Theorem 5.19. □

Theorem 6.10. *Suppose Δ is a three dimensional complete fan and k is the finite field $\mathbb{F}_r$. Then for all $j \in \mathbb{N}$, $K_{2j-1}^T(X(\Delta))$ is an extension of $H^0\left(\Delta, \widetilde{K_{2j-1}^T}\right)$ and $\left(\mathbb{Z}/(r^{j+2}-1)\right)^{g-1}$, where g is as defined in Theorem 5.19.*

Proof. The spectral sequence (6.1) yields the following exact sequences for 3 dimensional fans:

$$0 \longrightarrow H^2\left(\Delta, \widetilde{K_{2j+1}^T}\right) \longrightarrow K_{2j-1}^T(X(\Delta)) \longrightarrow H^0\left(\Delta, \widetilde{K_{2j-1}^T}\right) \longrightarrow 0 \tag{6.11}$$

For all $j \in \mathbb{N}$, we have the isomorphisms:

$$\begin{aligned} H^2\left(\Delta, \widetilde{K_{2j+1}^T}\right) &\cong H^2\left(\Delta, \widetilde{K_0^T}\right) \otimes_{\mathbb{Z}} K_{2j+1}(k), \quad \text{since } H^3\left(\Delta, \widetilde{K_0^T}\right) = 0 \\ &\cong \mathbb{Z}^{g-1} \otimes_{\mathbb{Z}} \mathbb{Z}/(r^{j+2}-1) \\ &\cong \left(\mathbb{Z}/(r^{j+2}-1)\right)^{g-1} \end{aligned}$$

□

Example 6.12. *Suppose Δ is the boundary of a three dimensional cone and k is the finite field $\mathbb{F}_r$, then $H^2\left(\Delta, \widetilde{K_0^T}\right) = 0$ by Theorem 5.28, so for all $j \in \mathbb{N}$,*

$$\begin{aligned}
K^T_{2j-1}(X(\Delta)) &\cong H^0\left(\Delta, \widetilde{K^T_{2j-1}}\right), && \textit{by the short exact sequence (6.11)} \\
&\cong H^0\left(\Delta, \widetilde{K^T_0}\right) \otimes_{\mathbb{Z}} K_{2j-1}(k), && \textit{since } H^1\left(\Delta, \widetilde{K^T_0}\right) \textit{ is free by Lemma 5.33} \\
&\cong H^0\left(\Delta, \widetilde{K^T_0}\right) \otimes_{\mathbb{Z}} \mathbb{Z}/(r^j - 1).
\end{aligned}$$

and

$$\begin{aligned}
K^T_{2j}(X(\Delta)) &\cong H^1\left(\Delta, \widetilde{K^T_{2j+1}}\right) \\
&\cong H^1\left(\Delta, \widetilde{K^T_0}\right) \otimes_{\mathbb{Z}} K_{2j+1}(k), && \textit{since } H^2\left(\Delta, \widetilde{K^T_0}\right) = 0 \\
&\cong H^1\left(\Delta, \widetilde{K^T_0}\right) \otimes_{\mathbb{Z}} \mathbb{Z}/(r^{j+2} - 1), && \textit{where } k = \mathbb{F}_r.
\end{aligned}$$

$H^1\left(\Delta, \widetilde{K^T_0}\right)$ *is free of finite rank, and so* $K^T_{2j}(X(\Delta))$ *is a direct sum of finitely many copies of* $\mathbb{Z}/(r^{j+2} - 1)$.

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