2010

Class Notes for Math 915: Homological Algebra, Instructor Tom Marley

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Topics covered are: Complexes, homology, direct and inverse limits, Tor, Ext, and homological dimensions. Also, Koszul homology and cohomology.

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August 2010

Homological Algebra gained popularity in Commutative Algebra in the 1950s when the following open problems were solved:

Definition *Let* (R, m) *be a Noetherian commutative local ring. Then* R *is* ***regular*** *if* $m = (x_1, ..., x_d)$ *where* $d = \dim R$.

Solved Open Problems

1. If R is regular, is R_p regular for all prime ideals p of R? (proved by Serre-Auslander-Buchsbaum, '57)
2. If R is regular, is R a UFD? (proved by Auslander-Buchsbaum, '59)

1 Direct Limits

Definition 1.1. *Let* $\mathcal{C}$ *be a category and* I *a poset. A* ***direct system*** *in* $\mathcal{C}$ *indexed by* I *is a family of objects* $\{A_i\}_{i\in I}$ *in* $\mathcal{C}$ *such that*

1. *For* $i \leq j$*, there is a morphism* $\phi^i_j : A_i \to A_j$ *in* $\mathcal{C}$.
2. *For all* $i \in I$*, we have* $\phi^i_i = 1_{A_i}$.
3. *For all* $i \leq j \leq k$*, the diagram below commutes.*

$$\begin{array}{ccc} A_i & \xrightarrow{\phi^i_j} & A_j \\ {\scriptstyle \phi^i_k}\downarrow & \swarrow {\scriptstyle \phi^j_k} & \\ A_k & & \end{array}$$

With this notation, we say $\{A_i, \phi^i_j\}_{i,j\in I}$ *is a direct system.*

Examples.

1. Let I be any set and give I the trivial order, that is $i \leq j$ for $i, j \in I$ if and only if $i = j$. Then any family of objects of $\mathcal{C}$ is automatically a direct system over I.
2. Suppose $I = \mathbb{N}, \{A_i\}_{i\in\mathbb{N}}$ is a set of objects, and $\phi_i : A_i \to A_{i+1}$ are morphisms. This defines a direct system where $\phi^i_j : A_i \xrightarrow{\phi_i} A_{i+1} \xrightarrow{\phi_{i+1}} \cdots \xrightarrow{\phi_{j-1}} A_j$.
   Special Case. Let R be a ring, M an $R-$module, $x \in Z(R)$. Let $M_i = M$ for all i. Then $\phi^i_j : M_i \xrightarrow{x} M_{i+1} \xrightarrow{x} \cdots \xrightarrow{x} M_j$ (multiplication by x^{j-i}) yields a direct system.
3. Let $\mathcal{C} =<< R - \text{mod} >>, M$ a left $R-$module, $I \neq \emptyset$ a set of $R-$submodules of M. Note that I is a poset where the order is containment. For $A, B \in I$ with $A \subseteq B$, let $\phi^A_B : A \to B$ be the inclusion map. Then $\{A, \phi^A_B\}_{A,B\in I}$ is a direct system in $\mathcal{C}$.
   Special Case. One way to deal with a non-finitely generated module, is to consider the above situation where I is the set of all finitely generated submodules of M. Since every element of M is contained in some finitely generated submodule, the direct limit would have to be M.

Definition 1.2. *Let* $\{A_i, \phi^i_j\}$ *be a direct system in* $\mathcal{C}$*. A direct limit of the system is an object* X *of* $\mathcal{C}$ *together with morphisms* $\alpha_i : A_i \to X$ *such that for all* $i \leq j$*, the following diagram commutes*

$$\begin{array}{ccc} A_i & \xrightarrow{\alpha_i} & X \\ {\scriptstyle \phi^i_j}\downarrow & \nearrow {\scriptstyle \alpha_j} & \\ A_j & & \end{array}$$

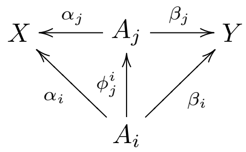
and with the following universal property: If there exists $\beta_i : A_i \to Y$ *for all* $i \in I$ *such that* $\beta_j\phi^i_j = \beta_i$*, then there exists a unique morphism* $\gamma : X \to Y$ *such that* $\gamma\alpha_i = \beta_i$*, that is, the following diagram commutes.*

$$\begin{array}{ccc} A_i & \xrightarrow{\beta_i} & Y \\ {\scriptstyle \alpha_i}\downarrow & \nearrow {\scriptstyle \gamma} & \\ X & & \end{array}$$

With the above notation, we say $X = \varinjlim_{i \in I} A_i$.

Exercise. If the direct limit exists, then it is unique up to isomorphism.

Proof. Let $\{A_i, \phi^i_j\}$ be a direct system and suppose $(X, \alpha_i), (Y, \beta_i)$ are direct limits. Then we have the following commutative diagram:



By definition, this gives us maps $\gamma : X \to Y$ and $\sigma : Y \to X$ such that $\beta_i = \gamma\alpha_i$ and $\alpha_i = \sigma\beta_i$. Then $\alpha_i = \sigma\gamma\alpha_i$ and $\beta_i = \gamma\sigma\beta_i$. By the uniqueness of the direct limit maps, since $1_X : X \to X$ and $\sigma\gamma : X \to X$ with $\alpha_i = 1_X\alpha_i$ and $\alpha_i = \sigma\gamma\alpha_i$, we see $\sigma\gamma = 1_X$ and similarly $\gamma\sigma = 1_Y$. Thus $X \cong Y$. □

Definition 1.3. *A poset I is **directed** if for all $i, j \in I$ there exists $k \in I$ such that $i \leq k, j \leq k$.*

Proposition 1.4. *Let M be a left $R-$module, I a directed set, ordered by containment, of submodules (that is, given $A, B \in I$, there exists $C \in I$ such that $A \cup B \subseteq C$). Then $\varinjlim_{A \in I} A = \cup_{A \in I} A$.*

Proof. First notice that $\cup_{A\in I} A$ is an $R-$submodule of M as I is directed. Define $\alpha_A : A \to \cup_{A \in I} A$ to be inclusion. Then $\alpha_A\phi^A_B = \alpha_B$ for all $A \subseteq B$ where $\phi^A_B : A \to B$ is also the inclusion map. So it is just left to show the universal property of direct limits holds. Suppose there exists $\beta_A : A \to Y$ for all $A \in I$ such that $\beta_A\phi^A_B = \beta_B$ for all $A \subseteq B$. Define $\gamma : \cup_{A \in I} A \to Y$ as follows: Let $x \in \cup A$. Then $x \in A$ for some $A \in I$ and so define $\gamma(x) = \beta_A(x)$. It is easy to see this is well defined and an $R-$module homomorphism (by the directed property of I). Thus $\varinjlim A = \cup_{A \in I} A$. □

Corollary 1.5. *For any $R-$module M, $M = \varinjlim_{N \in I} N$ where $I = \{N \subseteq M | N$ is finitely generated$\}$.*

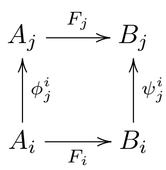
Example. Let $R \subseteq S$ be commutative domains, $\overline{R}$ the integral closure of R in S. In general, $\overline{R}$ is not a finitely generated $R-$module (and thus we can not say $\overline{R}$ is Noetherian when R is). To get around this, notice that $\overline{R} = \cup_{T \in I} T = \varinjlim_{T \in I} T$, where $I = \{T | R \subseteq T \subseteq S, T$ is a ring, T is a finitely generated $R-$module$\}$. Here, T is Noetherian when R is.

Exercise. Let R be a ring, M an $R-$module, $x \in Z(R)$. Recall the direct system given by $M_i = M$ and $\phi^i_j : M_i \to M_j$ defined by multiplication by x^{j-i} for $i \leq j$. Then $\varinjlim M \cong M_x$.

Proof. First note that ϕ^i_j is an $R-$module homomorphism as $x \in Z(R)$. Define $\alpha_i : M_i \to M_x$ by $m \mapsto \frac{m}{x^i}$. This is an $R-$module homomorphism and clearly $\alpha_j\phi^i_j = \alpha_i$ as $\alpha_j\phi^i_j(m) = \frac{x^{j-i}m}{x^j} = \frac{m}{x^i} = \alpha_i(m)$. To show the universal property holds, suppose there exists Y and $\beta_i : M_i \to Y$ such that for $i \leq j$ we have $\beta_j\phi^i_j = \beta_i$. Define $\gamma : M_x \to Y$ by $\frac{m}{x^i} \mapsto \beta_i(m)$. Then

- γ is well-defined: Suppose $\frac{a}{x^i} = \frac{b}{x^j}$ in M_x. Then there exists k such that $x^k(x^j a - x^i b) = 0$ which implies $x^{k+j}a = x^{k+i}b$. Then $\gamma(\frac{a}{x^i}) = \beta_i(a) = \beta_{j+k+i}\phi^i_{j+k+i}(a) = \beta_{j+k+i}(x^{j+k}a) = \beta_{j+k+i}(x^{k+i}b) = \beta_j(b) = \gamma(\frac{b}{x^j})$.
- γ is clearly an $R-$module homomorphism.
- $\gamma\alpha_i = \beta_i$ as for $a \in M_i$, we see $\gamma(\alpha_i(a)) = \gamma(\frac{a}{x^i}) = \beta_i(a)$.
- γ is unique: Suppose there exists $\lambda : M_x \to Y$ such that $\lambda\alpha_i = \beta_i$. For $\frac{m}{x^i} \in M_x$, we see $\lambda(\frac{m}{x^i}) = \lambda(\alpha_i(m)) = \beta_i(m) = \gamma(\alpha_i(m)) = \gamma(\frac{m}{x^i})$. Thus $\lambda = \gamma$. □

Remark. Let $\mathcal{C}$ be a category, I a poset. A morphism $F : \{A_i, \phi^i_j\} \to \{B_i, \psi^i_j\}$ of direct systems in $\mathcal{C}$ (with index I) is a set of morphisms $F_i : A_i \to B_i$ for all $i \in I$ such that for all $i \leq j$ the following diagram commutes:



One easily checks that this makes the direct systems in $\mathcal{C}$ over I a category, denoted by $Dir_{\mathcal{C}}(I)$.

"Definition" 1.6. *A category $\mathcal{C}$ is called* ***abelian*** *if*

- *the Hom sets of any two objects are abelian groups,*
- *there exists a zero object, denoted 0 (i.e., an object that is initial and terminal - for all objects C, there exist unique morphisms $0 \to C$ and $C \to 0$),*
- *every morphism in $\mathcal{C}$ has a kernel and cokernel in $\mathcal{C}$,*
- *the concept of "exact" makes sense, and*
- *finite products exist (i.e., if $A, B \in Obj\mathcal{C}$, then $A \times B$ is).*

[For a more precise definition, see Wiebel's Appendix]

Examples. $<< R - \text{mod} >>$ and $Dir_{<<R-\text{mod}>>}(I)$ are abelian categories.

Most of our examples are **concrete categories** where the objects are sets and morphisms are defined pointwise. In that situation, we can use our notions of kernel, cokernel, and exact for the above definition. In general, it is more complicated and technical.

Theorem 1.7. *Let $\mathcal{C}$ be an abelian category such that arbitrary sums exist. Then any direct system in $\mathcal{C}$ has a direct limit.*

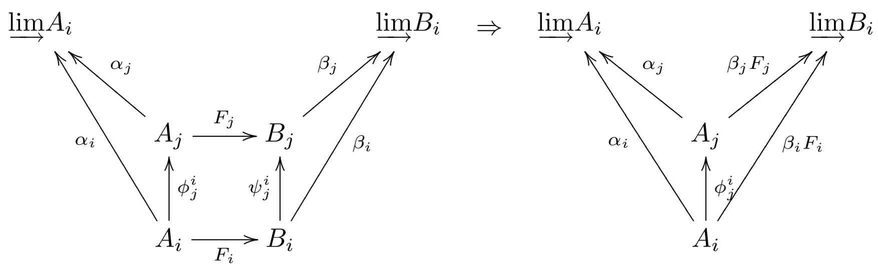
Proof. We will prove the theorem in the case that $\mathcal{C} =<< R - \text{mod} >>$. Let $\{A_i, \phi_j^i\}_{i,j \in I}$ be a direct system in $\mathcal{C}$. Let $F = \oplus_{i \in I} A_i$. Define $\lambda_i : A_i \to F$ to be the canonical injection. Let $S = \{\lambda_i(a) - \lambda_j \phi_j^i(a) | a \in A_i, i \le j\}$. Let N be the $R-$submodule of F generated by S.

Claim. $\varinjlim A_i = F/N$, where $\alpha_i : A_i \to F/N$ is defined by $a \mapsto \lambda_i(a) + N$.

Proof. By construction, $\lambda_j \phi_j^i = \lambda_i$ in F/N. So suppose there exist Y and $\beta_i : A_i \to Y$ for all i with $\beta_j \phi_j^i = \beta_i$. Define $\widetilde{\gamma} : F \to Y$ by $(a_i) \mapsto \sum \beta_i(a_i)$. Let $u = \lambda_i(a_i) - \lambda_j \phi_j^i(a_i)$. Then $\widetilde{\gamma}(u) = \beta_i(a_i) - \beta_j(\phi_j^i(a_i)) = 0$ by commutativity. Thus $\widetilde{\gamma}(N) = 0$. Thus we get the induced map $\gamma : F/N \to Y$. One can show γ is unique and $\gamma \alpha_i = \beta_i$. □

Corollary 1.8. *Let I be a trivially ordered poset and $\{A_i\}_{i \in I}$ a family of $R-$modules. Then $\varinjlim A_i = \oplus_{i \in I} A_i$.*

Remark. The direct limit is actually a covariant functor from $Dir_{\mathcal{C}}(I) \to \mathcal{C}$. Suppose $F : \{A_i, \phi_j^i\} \to \{B_i, \psi_j^i\}$ is a morphism in $Dir_{\mathcal{C}}(I)$. Consider the diagrams below.



By the definition of direct limit, there exists a unique $\gamma : \varinjlim A_i \to \varinjlim B_i$. Notationally, we will write $\gamma = \varinjlim F_i$. Thus morphisms go to morphisms.

Example. Let $\mathcal{C}$ be a category, $A \in Obj\mathcal{C}$. Let I be an index set. Define the constant direct system, denoted $|A| = \{A_i, \phi_j^i\}$, by $A_i = A$ for all i and $\phi_j^i = 1_A$ for all $i \le j$. This is clearly a direct system over I. Now, given $f : A \to B$ in $\mathcal{C}$, let $|f| : |A| \to |B|$ be defined by $f_i := f : A_i \to B_i$ for all $i \in I$. This makes $|\cdot|$ into a covariant functor from $\mathcal{C} \to Dir_{\mathcal{C}}(I)$. Note that if $\mathcal{C}$ is an additive category (i.e., the Hom sets are abelian groups), then $|\cdot|$ is an additive functor.

Exercise. If I is a directed set and $A \in Obj\mathcal{C}$, then $\varinjlim |A| = A$. However, this need not be true if I is not directed.

Proof. Define $A_i = A$ and $\alpha_i : A_i \to A$ the identity map. As ϕ_j^i was also defined to be the identity map, it is clear that $\alpha_j \phi_j^i = \alpha_i$. So suppose there exists Y and $\beta_i : A_i \to Y$ such that $\beta_j \phi_j^i = \beta_i$ for $i \le j$. Since $\phi_j^i = 1$, this just says $\beta_j = \beta_i$ for all $i \le j$. As I is a directed set, this says $\beta_i = \beta_j$ for all $i, j \in I$. Define $\gamma : A \to Y$ by $a \mapsto \beta_i(a)$. Clearly, $\gamma \alpha_i = \beta_i$ as α_i is the identity map.

Note that I needs to be directed. For example, if I has the trivial order and $|I| > 1$, we've seen $\varinjlim A_i = \oplus_{i \in I} A_i$. Of course, $\oplus_{i \in I} A_i$ is generally not equal to A (take A to be a field, for example). □

Recall. Let $f : A_1 \to A_2$ be a morphism in $\mathcal{C}$. For any object C of $\mathcal{C}$, we define $f_* : Hom_{\mathcal{C}}(C, A_1) \to Hom_{\mathcal{C}}(C, A_2)$ by $h \mapsto fh$ and $f^* : Hom_{\mathcal{C}}(A_2, C) \to Hom_{\mathcal{C}}(A_1, C)$ by $h \mapsto hf$.

Definition 1.9. *Let $\mathcal{C}$ and $\mathcal{D}$ be categories, $L : \mathcal{C} \to \mathcal{D}$ and $R : \mathcal{D} \to \mathcal{C}$ be covariant functors. We say (L, R) is an **adjoint pair** if for all $A \in Obj\mathcal{C}$ and all $B \in Obj\mathcal{D}$, there is a bijection $\tau_{AB} : Hom_{\mathcal{D}}(L(A), B) \to Hom_{\mathcal{C}}(A, R(B))$ such that the naturality condition holds, i.e., for all morphisms $f : A_1 \to A_2$ in $\mathcal{C}$ and $f : B_1 \to B_2$ in $\mathcal{D}$, we have the following commutative diagram.*

$$\begin{array}{ccccc} Hom_{\mathcal{D}}(L(A_2), B_1) & \xrightarrow{L(f)^*} & Hom_{\mathcal{D}}(L(A_1), B_1) & \xrightarrow{g_*} & Hom_{\mathcal{D}}(L(A_1), B_2) \\ \downarrow{\scriptstyle \tau_{A_2B_1}} & & \downarrow{\scriptstyle \tau_{A_1B_1}} & & \downarrow{\scriptstyle \tau_{A_1B_2}} \\ Hom_{\mathcal{C}}(A_2, R(B_1)) & \xrightarrow{f^*} & Hom_{\mathcal{C}}(A_1, R(B_1)) & \xrightarrow{R(g)_*} & Hom_{\mathcal{C}}(A_1, R(B_2)) \end{array}$$

Protypical Example. $Hom - \otimes$: Let R, S be rings and A an $S - R-$bimodule. Then $A \otimes_R - :<< R - mod >> \to << S - mod >>$ defined by $B \mapsto A \otimes_R B$ and $Hom_S(A, -) :<< S - mod >> \to << R - mod >>$ defined by $B \mapsto Hom_S(A, B)$ are covariant functors. We proved in 902 that $(A \otimes_R -, Hom_S(A, -))$ is an adjoint pair.

Definition 1.10. *If (L, R) is an adjoint pair, then L is called a **left adjoint** and R is called a **right adjoint**.*

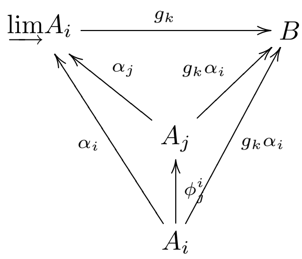
Fact. If L is a left adjoint, then it is right exact. Similarly, if R is a right adjoint, then it is left exact.

Theorem 1.11. *Let $\mathcal{C}$ be a category, I a poset such that direct limits in $\mathcal{C}$ over I exist. Then $\left(\varinjlim_{i \in I}, |\cdot|_I \right)$ is an adjoint pair.*

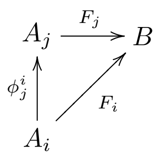
Proof. Let $\{A_i, \phi^i_j\}_{i,j \in I}$ be a direct system over I. Let $\alpha_i : A_i \to \varinjlim A_i$ be the direct limit maps. Let $g \in Hom_{\mathcal{C}}(\varinjlim A_i, B)$. Define $\overline{g} : \{A_i, \phi^i_j\} \to |B|$ to be the map formed by $g\alpha_i : A_j \to B$ (note that $g\alpha_j\phi^i_j = g\alpha_i$). This is a morphism of direct systems. Now, for all direct systems $\{A_i, \phi^i_j\}$ and $B \in Obj\mathcal{C}$, define $\tau : Hom_{\mathcal{C}}(\varinjlim A, B) \to Hom_{Dirc(I)}(\{A_i, \phi^i_j\}, |B|)$. We need to show τ is bijective and natural. We leave the naturality as an exercise.

Claim. τ is bijective.

Proof. Suppose $g_1, g_2 : \varinjlim A_i \to B$ and $\overline{g_1} = \overline{g_2}$. Then $g_1\alpha_i = g_2\alpha_i$ for all $i \in I$. Consider the following commutative diagram, where g_k represents either g_1 or g_2.



Note that both g_1 and g_2 make the diagram commute. So by the uniqueness of the direct limit map, $g_1 = g_2$. To show it is onto, let $f : \{A_i, \phi^i_j\} \to |B|$ be a morphism of direct systems. So for $i \le j$, we have the following diagram commutes:



By the definition of direct limit, there exists $g : \varinjlim A_i \to B$ such that $F_j = g\alpha_j$ for all j. Then $F = \overline{g} = \tau(g)$. □

Exercise. Let R be a ring, S a mcs with $1 \in S$. Put a relation $\le$ on S as follows: for $s, t \in S$, say $s \le t$ if and only if s is a unit in R_t. This relation is reflexive and transitive (but not antisymmetric) and is directed. Define a direct system with index set S by $A_s = R_s$ (the s for A_s denotes an index, but the s for R_s denotes localization) for $s \in S$ and $\phi^s_t : A_s \to A_t$ where $\frac{r}{s^n} \mapsto \frac{r}{s^n}$ for $s \le t$. Then $\varinjlim_{t \in S} R_t = R_S$.

Proof. For each $t \in S$, define $\alpha_t : R_t \to R_S$ by $\frac{r}{t^n} \mapsto \frac{r}{t^n}$. Then for $s \le t$, $\alpha_t\phi^s_t = \alpha_s$ as for ϕ^s_t and α_s are just the natural injection maps. To show the universal property, suppose there exists Y and $\beta_t : R_t \to Y$ such that $\beta_t\phi^s_t = \beta_s$. Define $\gamma : R_S \to Y$ by $\frac{r}{s} \mapsto \beta_s(\frac{r}{s})$. Then

- γ is well-defined: Suppose $\frac{r}{t} = \frac{a}{s}$ in R_s. Then there exists $u \in S$ such that $urs = uat$. Then $\gamma(\frac{r}{t}) = \beta_t(\frac{r}{t}) = \beta_{ust}(\phi^t_{ust})(\frac{r}{t}) = \beta_{ust}(\frac{r}{t}) = \beta_{ust}(\frac{usr}{ust}) = \beta_{ust}(\frac{uat}{ust}) = \beta_{ust}(\frac{a}{s}) = \beta_{ust}(\phi^s_{ust}(\frac{a}{s})) = \beta_s(\frac{a}{s}) = \gamma(\frac{a}{s})$.

- γ is a $R-$module homomorphism: For $\frac{r}{t}, \frac{a}{s} \in R_S$, we see $\gamma(\frac{r}{t} + \frac{a}{s}) = \gamma(\frac{rs+at}{ts}) = \beta_{ts}(\frac{rs+at}{ts}) = \beta_{ts}(\frac{rs}{ts}) + \beta_{ts}(\frac{at}{ts}) = \beta_{ts}(\frac{r}{t}) + \beta_{ts}(\frac{a}{s}) = \beta_t(\frac{r}{t}) + \beta_s(\frac{a}{s}) = \gamma(\frac{r}{t}) + \gamma(\frac{a}{s})$.

- $\gamma\alpha_t = \beta_t$ as for $\frac{r}{t^n} \in R_t$, we see $\gamma(\alpha_t(\frac{r}{t^n})) = \gamma(\frac{r}{t^n}) = \beta_{t^n}(\frac{r}{t^n}) = \beta_{t^n}(\phi^t_{t^n}(\frac{r}{t^n})) = \beta_t(\frac{r}{t^n})$.

- γ is unique: Suppose there exists $\lambda : R_S \to Y$ with $\lambda\alpha_t = \beta_t$. Let $t \in S$. Then $\lambda(\frac{r}{t}) = \lambda\alpha_t(\frac{r}{t}) = \beta_t(\frac{r}{t}) = \gamma(\frac{r}{t})$. □

Remarks.

1. Combining this and the earlier exercise, we see $R_S = \varinjlim_{t\in S}\left(\varinjlim_{t\in S}(R \xrightarrow{t} R \xrightarrow{t} \cdots)\right)$. Furthermore, we can rewrite this with a single index set, which says $R_S = \varinjlim R$.

2. Note that we did not use the full power of a poset here (as our index set was NOT a poset). In general, we do not need the antisymmetry property of a poset to define a direct system or direct limit. So to define a direct system over I, we need only that I is reflexive, transitive (and sometimes directed).

Exercise. Let R be a commutative ring and x a non-zero-divisors of R. Define $A_i = R/(x^i)$ and $\phi^i_j : A_i \to A_j$ by $\overline{r} \mapsto \overline{rx^{j-i}}$ for $i \leq j$. Then $\varinjlim A_i \cong R_x/R$.

Proof. Define $\alpha_i : A_i \to R_x/R$ by $\overline{r} \mapsto \overline{\frac{r}{x^i}}$. This is well-defined as if $r + (x^i) = s + (x^i)$, then $r - s \in (x^i)$, that is, $r - sax^i$ for some $a \in R$. Then $\overline{\frac{r-s}{x^i}} = \overline{\frac{a}{1}} = 0$ as $\frac{a}{1} \in R$. So $\overline{\frac{r}{x^i}} = \overline{\frac{s}{x^i}}$. The α_i's are clearly $R-$module homomorphisms and $\alpha_j\phi^i_j = \alpha_i$ as $\alpha_j\phi^i_j(\overline{r}) = \alpha_j(\overline{rx^{j-i}}) = \overline{\frac{rx^{j-i}}{x^j}} = \overline{\frac{r}{x^i}} = \alpha_i(\overline{r})$. Thus it is only left to show that the universal property holds. So suppose there exists Y and $\beta_i : A_i \to Y$ such that $\beta_j\phi^i_j = \beta_i$. Define $\gamma : R_x/R \to Y$ by $\overline{\frac{r}{x^i}} \mapsto \beta_i(\overline{r})$. Then

- γ is well-defined: Suppose $\frac{r}{x^i} + R = \frac{s}{x^j} + R$. Then $\frac{r}{x^i} - \frac{s}{x^j} \in R$, that is, $\frac{rx^j - sx^i}{x^{i+j}} = \frac{a}{1}$ for some $a \in R$. So, there exists $k \in \mathbb{N}$ such that $(rx^j - sx^i)x^k = ax^k$. Since x is a non zero divisor, we must in fact have that $rx^j - sx^i = a$. So
$$\gamma\left(\overline{\frac{r}{x^i}}\right) = \beta_i(\overline{r}) = \beta_{j+i}\phi^i_{j+i}(\overline{r}) = \beta_{j+i}(\overline{rx^j}) = \beta_{j+i}(\overline{a}) + \beta_{j+i}(\overline{sx^i}) = \beta_{j+1}(\overline{0}) + \beta_{j+i}\phi^i_{j+i}(\overline{s}) = \beta_j(\overline{s}) = \gamma\left(\overline{\frac{s}{x^j}}\right).$$

- γ is an $R-$module homomorphism: This is clear as β_i is an $R-$module homomorphism.

- $\gamma\alpha_i = \beta_i$: This too is clear by our definition of gamma for $\gamma\alpha_i(\overline{r}) = \gamma(\overline{\frac{r}{x^i}}) = \beta_i(\overline{r})$.

- γ is unique: Suppose there exists $\delta : R_x/R \to Y$ such that $\delta\alpha_i = \beta_i$. Then $\delta(\overline{\frac{r}{x^i}}) = \delta\alpha_i(\overline{r}) = \beta_i(\overline{r}) = \gamma(\overline{\frac{r}{x^i}})$.

Thus there exists a unique morphism $\gamma : R_x/R \to Y$, which says the universal property holds. Thus $\varinjlim_{i\in\mathbb{N}} R/(x^i) \cong R_x/R$. □

Remark. The above example is one of a "local cohomology module." Also, the statement is true when x is a zero-divisor, as long as we replace R_x/R with $R_x/\phi(R)$ where $\phi : R \to R_x$ is defined by $r \mapsto \frac{r}{1}$. We assumed R was a non-zero-divisor, as in that case ϕ is injective, and thus $R = \phi(R)$.

Definition 1.12. *Let $\mathcal{C}$ be a category. Define the category $\mathcal{C}^{op}$ by $Obj\mathcal{C}^{op} = Obj\mathcal{C}$ and for all $A, B \in Obj\mathcal{C}$, there exists a bijection $Hom_{\mathcal{C}}(A, B) \leftrightarrow Hom_{\mathcal{C}^{op}}(B, A)$ defined by $(f : A \to B) \leftrightarrow (f^{op} : B \to A)$ such that whenever $A \xrightarrow{f} B \xrightarrow{g} X$ is in $\mathcal{C}$, we have $(gf)^{op} = f^{op}g^{op}$. This is a contravariant functor.*

Remarks.

1. $\mathcal{C}$ is abelian if and only if $\mathcal{C}^{op}$ is abelian. In particular, $A \xrightarrow{f} B \xrightarrow{g} C$ is exact in $\mathcal{C}$ if and only if $C \xrightarrow{g^{op}} B \xrightarrow{f^{op}} A$ is exact in $\mathcal{C}^{op}$.

   However, if $\mathcal{C} =<< R - mod >>$, $\mathcal{C}^{op} \neq<< S - mod >>$ or $<< mod - S >>$ for any ring S.

2. $(\mathcal{C}^{op})^{op} = \mathcal{C}$.

3. Given a covariant functor $F : \mathcal{C} \to \mathcal{D}$, define $F^{op} : \mathcal{C}^{op} \to \mathcal{D}^{op}$ by $F^{op}(A) = F(A)$ for all $A \in Obj\mathcal{C}$ and given $f^{op} : B \to A$, let $F^{op}(f^{op}) = F(f)^{op}$. One can check $F^{op}(1_A) = 1_{F^{op}(A)}$ for all objects A in $\mathcal{C}$ and $F^{op}(fg) = F^{op}(f)F^{op}(g)$. Thus F^{op} is a covariant functor. If $\mathcal{C}$ is abelian, then F^{op} is additive. Furthermore, F is left (resp. right) exact if and only if F^{op} is right (resp. left) exact (by Remark 1 and the fact that $F^{op}(f^{op}) = F(f)^{op}$).

4. Let $L : \mathcal{C} \to \mathcal{D}$ and $R : \mathcal{D} \to \mathcal{C}$. Then (L, R) is an adjoint pair if and only if (R^{op}, L^{op}) is an adjoint pair.

Lemma 1.13. *Let $\mathcal{C}$ be an abelian category. Then $A \xrightarrow{\alpha} B \xrightarrow{\beta} C$ is exact in $\mathcal{C}$ if for all objects M of $\mathcal{C}$, we have $Hom_{\mathcal{C}}(M, A) \xrightarrow{\alpha_*} Hom_{\mathcal{C}}(M, B) \xrightarrow{\beta_*} Hom_{\mathcal{C}}(M, C)$ is exact.*

Proof. Assume $\mathcal{C} =<< R - mod >>$. Let $M = A$. Then $\beta_*\alpha_* = 0$ by exactness. In particular, $\beta\alpha 1_A = 0$ which says $\beta\alpha = 0$. So $im\alpha \subseteq \ker\beta$. Let $M = \ker\beta$ and $i : \ker\beta \to \beta$ be the inclusion map. So $\beta_* i = \beta i = 0$, that is, $i \in im\alpha_*$. So there exists $h : \ker\beta \to A$ such that $\alpha h = i$. Then $\ker\beta = i(\ker\beta) = \alpha(h(\ker\beta)) \subseteq im\alpha$. □

Theorem 1.14. *Let $L : \mathcal{C} \to \mathcal{D}$ and $R : \mathcal{D} \to \mathcal{C}$ be covariant functors such that (L, R) is an adjoint pair. Then L is right exact and R is left exact.*

Proof. By using $\mathcal{C}^{op}$, it suffices to prove R is left exact. So suppose $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C$ is exact in $\mathcal{D}$. We want to show $0 \to R(A) \xrightarrow{R(\alpha)} R(B) \xrightarrow{R(\beta)} R(C)$ is exact in $\mathcal{C}$. By the lemma, it is enough to prove that for all objects M of $\mathcal{C}$, we have $0 \to Hom_{\mathcal{C}}(M, R(A)) \to Hom_{\mathcal{C}}(M, R(B)) \to Hom_{\mathcal{C}}(M, R(C))$ is exact. Note that by $Hom - \otimes$ adjointness, we have the following commutative diagram

$$\begin{array}{ccccccc} 0 \longrightarrow & Hom_{\mathcal{C}}(M, R(A)) & \longrightarrow & Hom_{\mathcal{C}}(M, R(B)) & \longrightarrow & Hom_{\mathcal{C}}(M, R(C)) \\ & \updownarrow & & \updownarrow & & \updownarrow \\ & Hom_{\mathcal{D}}(L(M), A) & \longrightarrow & Hom_{\mathcal{D}}(L(M), B) & \longrightarrow & Hom_{\mathcal{D}}(L(M), C) \end{array}$$

where the bottom row is exact as $Hom_{\mathcal{D}}(L(M), -)$ is left exact. By commutativity, this gives us that the top row is exact. □

Corollary 1.15. *For any $S - R$ bimodule A, we have $A \otimes_R -$ is right exact.*

Corollary 1.16. $\varinjlim : Dir_{\mathcal{C}}(I) \to \mathcal{C}$ *is right exact.*

Theorem 1.17. *Let (F, G) be an adjoint pair, where $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$. Let $\{A_i, \phi^i_j\}_{i,j \in I}$ be a direct system in $\mathcal{C}$. Then $\{F(A_i), F(\phi^i_j)\}$ is a direct system in $\mathcal{D}$ and $\varinjlim F(A_i) \cong F(\varinjlim A_i)$, that is, left adjoints preserve direct limits.*

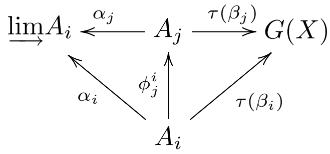
Proof. We will show that $F(\varinjlim A_i)$ has the desired universal property. Let $\alpha_i : A_i \to \varinjlim A_i$ be given as in the definition. Then, we get the following commutative diagram in $\mathcal{D}$:

$$\begin{array}{ccccc} F(\varinjlim A_i) & \xleftarrow{F(\alpha_j)} & F(A_j) & \xrightarrow{\beta_j} & X \\ & {\scriptstyle F(\alpha_i)} \nwarrow & \uparrow {\scriptstyle F(\phi^i_j)} & \nearrow {\scriptstyle \beta_i} & \\ & & F(A_i) & & \end{array}$$

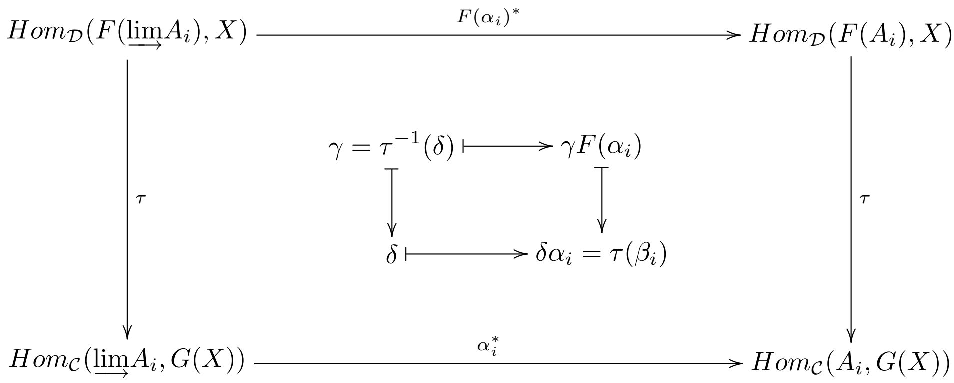
We want to show there exists $\gamma : F(\varinjlim A_i) \to X$ making the diagram below commute. By $Hom - \otimes$ adjointness, consider the following diagram

$$\begin{array}{ccc} Hom_{\mathcal{D}}(F(A_j), X) & \xrightarrow{\quad F(\phi^i_j)^* \quad} & Hom_{\mathcal{D}}(F(A_i), X) \\ & \beta_j \longmapsto \beta_j F(\phi^i_j) = \beta_i & \\ \downarrow \tau & \downarrow \qquad\qquad \downarrow & \downarrow \tau \\ & \tau(\beta_j) \longmapsto \tau(\beta_j)\phi^i_j & \\ Hom_{\mathcal{C}}(A_j, G(X)) & \xrightarrow{\quad (\phi^i_j)^* \quad} & Hom_{\mathcal{C}}(A_i, G(X)) \end{array}$$

Since $\beta_i \mapsto \tau(\beta_i)$, the above diagram shows $\tau(\beta_j)\phi^i_j = \tau(\beta_i)$. This says that the diagram below commutes:

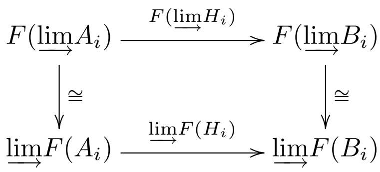


By the universal property, there exists a unique $\delta : \varinjlim A_i \to G(X)$ such that the diagram above commutes, that is, $\delta\alpha_j = \tau(\beta_j)$ for all j. Define $\gamma := \tau^{-1}(\delta) : F(\varinjlim A_i) \to X$. We want to show that $\gamma(F(\alpha_i)) = \beta_i$ so that our original diagram commutes. To do this, consider the following commutative diagram



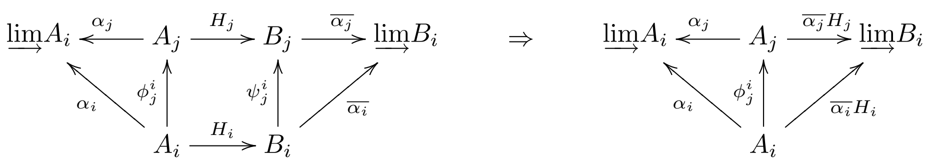
This says $\gamma F(\alpha_i) \mapsto \tau(\beta_i)$. Of course, τ is an isomorphism and $\beta_i \mapsto \tau(\beta_i)$. Thus $\gamma(F(\alpha_i)) = \beta_i$, giving γ the desired commutative property. To show that γ is unique, suppose there exists $\gamma' : F(\varinjlim A_i) \to X$ such that $\gamma' F(\alpha_i) = \beta_i$ for all $i \in I$. As above, this would yield $\tau(\gamma')\alpha_i = \tau(\beta_i)$ for all i. Of course $\delta = \tau(\gamma)$ was chosen to be the unique map such that $\delta\alpha_i = \tau(\beta_i)$. Thus $\tau(\gamma') = \delta = \tau(\gamma)$, and thus $\gamma' = \gamma$ as τ is injective. □

Remark. The above isomorphism is indeed "natural," that is, suppose $H : \{A_i, \phi^i_j\} \to \{B_i, \psi^i_j\}$ is a morphism of direct systems. Then the diagram below commutes.

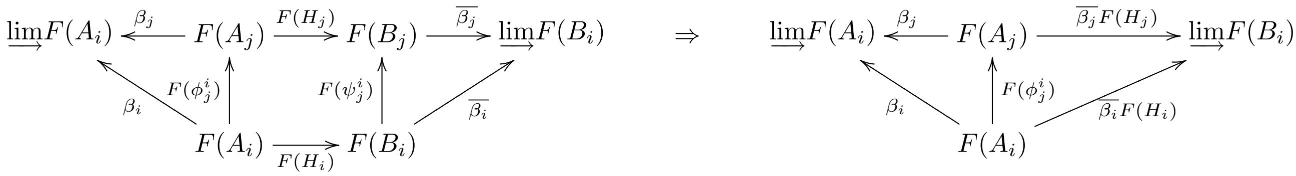


Proof. As in the theorem, let $\alpha_i : A_i \to \varinjlim A_i, \beta_i : F(A_i) \to \varinjlim F(A_i)$, and $\gamma : F(\varinjlim A_i) \to \varinjlim F(A_i)$ be the unique map such that $\gamma F(\alpha_i) = \beta_i$. Now, define $\overline{\alpha}_i, \overline{\beta}_i$, and $\overline{\gamma}_i$ to be the corresponding maps for $\{B_i, \psi^i_j\}$. Then, we want to show $\overline{\gamma} F(\varinjlim H_i) = (\varinjlim F(H_i))\gamma$.

Consider the following commutative diagrams for $i \leq j$.



By the universal property of direct limits, there exists a unique $h := \varinjlim H_i : \varinjlim A_i \to \varinjlim B_i$ such that $h\alpha_j = \overline{\alpha}_j H_j$ for $j \in I$. Now, apply the functor F to the above diagrams:



Again, by the universal property, there exists a unique $h' := \varinjlim F(H_i) : \varinjlim F(A_i) \to \varinjlim F(B_i)$ such that $h'\beta_j = \overline{\beta_j}F(H_j)$. Now, we have the following diagram,

$$\begin{array}{ccc} F(\varinjlim A_i) & \xrightarrow{F(h)} & F(\varinjlim B_i) \\ \big\updownarrow{\scriptstyle \gamma} & & \big\updownarrow{\scriptstyle \overline{\gamma}} \\ \varinjlim F(A_i) & \xrightarrow{h'} & \varinjlim F(B_i) \end{array}$$

which says $\overline{\gamma}F(h)\gamma^{-1} \in Hom_{\mathcal{D}}(\varinjlim F(A_i), \varinjlim F(B_i))$. As stated above, we want to show that this diagram commutes. To do so, consider the following two commutative diagrams:

$$\begin{array}{ccccccc} \varinjlim F(A_i) & \xrightarrow{\gamma^{-1}} & F(\varinjlim A_j) & \xrightarrow{F(h)} & F(\varinjlim B_j) & \xrightarrow{\overline{\gamma}} & \varinjlim F(B_i) \\ & {\scriptstyle \beta_i}\nwarrow & \uparrow{\scriptstyle F(\alpha_i)} & & \uparrow{\scriptstyle F(\overline{\alpha_i})} & \nearrow{\scriptstyle \overline{\beta_i}} & \\ & & F(A_i) & \xrightarrow[F(H_i)]{} & F(B_i) & & \end{array} \quad \Rightarrow \quad \begin{array}{ccc} \varinjlim F(A_i) & \xrightarrow{\overline{\gamma}F(h)\gamma^{-1}} & \varinjlim F(B_i) \\ \uparrow{\scriptstyle \beta_i} & \nearrow{\scriptstyle \overline{\beta_i}F(H_i)} & \\ F(A_i) & & \end{array}$$

Of course, $h' : \varinjlim F(A_i) \to \varinjlim F(\beta_i)$ is the unique such map. Thus, $h' = \overline{\gamma}F(h)\gamma^{-1}$, that is, $(\varinjlim F(H_i))\gamma = \overline{\gamma}F(\varinjlim H_i)$. ☐

Corollary 1.18. *Let A be an $S-R$ bimodule. Then the functor $A \otimes_R -$ preserves direct limits, that is, if $\{B_i, \phi^i_j\}_{i,j\in I}$ is a direct system of left $R-$modules, then $\{A \otimes_R B_i, 1 \otimes \phi^i_j\}$ is a direct system of $S-$modules and $\varinjlim A \otimes_R B_i \cong A \otimes_R \varinjlim B_i$ as left $S-$modules in a natural way.*

Corollary 1.19. *Let $\mathcal{C}$ be a category in which direct limits exist, I an index set. Then $\varinjlim_I : Dir_{\mathcal{C}}(I) \to \mathcal{C}$ is a left adjoint and thus preserves direct limits over any index set J of systems in $Dir_{\mathcal{C}}(I)$, that is, if $\{D_j\}_{j\in J}$ is a direct system in $Dir_{\mathcal{C}}(I)$ with index set J (an object in $Dir_{Dir_{\mathcal{C}}(I)}(J)$), then $\varinjlim_J \varinjlim_I D_j \cong \varinjlim_I \varinjlim_J D_j$. Thus any two direct limits commute.*

Corollary 1.20. *Let $\{A_j\}_{j\in J}$ be a family of objects in $Dir_{\mathcal{C}(I)}$. Then $\varinjlim_I(\oplus_{j\in J}A_j) \cong \oplus_{j\in J}(\varinjlim_I A_j)$.*

Exercise. Let (L, R) be an adjoint pair, $L : \mathcal{C} \to \mathcal{D}, R : \mathcal{D} \to \mathcal{C}$. If R is exact, then L preserves projectives. If L is exact, then R preserves injectives.

Proof. Note that by considering $\mathcal{D}^{op}$, it suffices to show if R is exact, then L preserves projectives. Suppose R is exact. Let $0 \to A \to B \to C \to 0$ be exact in $\mathcal{D}$ and let $M \in Obj\mathcal{C}$ be projective. Since R is exact, we know $0 \to R(A) \to R(B) \to R(C) \to 0$ is exact, and as M is projective, we have the top row of the following commutative diagram is exact

$$\begin{array}{ccccccc} 0 & \longrightarrow & Hom_{\mathcal{C}}(M, R(A)) & \longrightarrow & Hom_{\mathcal{C}}(M, R(B)) & \longrightarrow & Hom_{\mathcal{C}}(M, R(C)) \\ & & \big\updownarrow & & \big\updownarrow & & \big\updownarrow \\ 0 & \longrightarrow & Hom_{\mathcal{D}}(L(M), A) & \longrightarrow & Hom_{\mathcal{D}}(L(M), B) & \longrightarrow & Hom_{\mathcal{D}}(L(M), C) \end{array}$$

and thus the bottom row is exact. Therefore L preserves projectives. ☐

Example. Recall $(A \otimes_R -, Hom_S(A, -))$ is an adjoint pair. Now, $Hom_S(A, -)$ is exact if A is projective. So, by the above exercise, if A is projective, then $A \otimes_R -$ preserves projectives, i.e., if B is a projective left $R-$module, then $A \otimes_R B$ is a projective left $S-$module. Also, $A \otimes_R -$ is exact if A is flat. So, if A is flat as a right $R-$module, then $Hom_S(A, -)$ preserves injectives, i.e., if B is an injective left $S-$module, then $Hom_S(A, B)$ is an injective left $R-$module.

Exercise. Let M be a **finitely presented** left $R-$module (i.e., there exists an exact sequence $R^m \to R^n \to M \to 0$). Let $\{A_i, \phi^i_j\}$ be a direct system of $R-$modules over a directed index set I. Prove $Hom_R(M, \varinjlim A_i) \cong \varinjlim Hom_R(M, A_i)$.

Proof. First, note that

$$\begin{array}{rcl} \varinjlim Hom_R(R^n, A_i) & = & \varinjlim Hom_R(\oplus^n R, A_i) \\ & = & \varinjlim \oplus^n Hom_R(R, A_i) \\ & = & \varinjlim \oplus^n A_i \\ & = & \oplus^n \varinjlim A_i \text{ as direct limits commute} \\ & = & \oplus^n Hom_R(R, \varinjlim A_i) \\ & = & Hom_R(R^n, \varinjlim A_i) \end{array}$$

As M is finitely presented, we have $R^m \to R^n \to M \to 0$ is exact for some m, n. Since $Hom_R(-, A_i)$ is left exact, we see $0 \to Hom_R(M, A_i) \to Hom_R(R^n, A_i) \to Hom_R(R^m, A_i)$ is exact $(*)$. This gives us the following commutative diagram.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & 0 & \longrightarrow & Hom_R(M, \varinjlim A_i) & \xrightarrow{\alpha} & Hom_R(R^n, \varinjlim A_i)) & \longrightarrow & Hom_R(R^m, \varinjlim A_i) \\
\cong \uparrow & & \cong \uparrow & & \uparrow & & \cong \uparrow g & & \cong \uparrow \\
0 & \longrightarrow & 0 & \longrightarrow & \varinjlim Hom_R(M, A_i) & \xrightarrow{\beta} & \varinjlim Hom_R(R^n, A_i) & \longrightarrow & \varinjlim Hom_R(R^m, A_i)
\end{array}$$

Note that the top row is exact as $Hom_R(-, A)$ is left exact and the bottom row is exact by $(*)$ as I is directed. We want to find a map $f : \varinjlim Hom_R(M, A_i) \to Hom_R(M, \varinjlim A_i)$. So, let $x \in \varinjlim Hom_R(M, A_i)$. Then $g\beta(x) \in Hom_R(R^n, \varinjlim A_i)$. Since the rows are exact, α is injective. So there exists a unique y such that $\alpha(y) = g\beta(x)$. So, define $f(x) = y$. That gives us our morphism f and keeps the diagram commutative. Now, by the five-lemma, we must have that f is in fact an isomorphism. □

Example. The above result fails when M is not finitely presented. For example, take $R_i = R$ and consider the direct system $R_1 \xrightarrow{x} R_2 \xrightarrow{x} R_3 \cdots$. We've seen $\varinjlim R_i \cong R_x$. Notice $Hom_R(R_x, \varinjlim R_i) = Hom_R(R_x, R_x) \neq 0$ as it contains at least the identity map. On the other hand, $\varinjlim Hom_R(R_x, R_i) = 0$ by Krull's Intersection Theorem [Take $f \in Hom_R(R_x, R)$. Let $a = f(\frac{1}{x^m})$. Then for all n, $a = f(\frac{x^n}{x^{n+m}}) = x^n(f(\frac{1}{x^{n+m}})) \in (x^n)$. Thus $a \in \cap_{n \geq 1}(x^n) = 0$. So $f(\frac{1}{x^m}) = 0$ for all m, which says $f = 0$.]

Exercise. Let $\{M_i, \phi_j^i\}$ be a direct system of left $R-$modules over a directed index set. Suppose $\lambda_R(M_i) \leq n$ for all $i \in I$. Then $\lambda_R(\varinjlim M_i) \leq n$.

Proof. Suppose not. Then, we have a sequence of submodules of $\varinjlim M_i$ of length $> n$. Say $\varinjlim M_i \supsetneq N_m \supseteq N_{m-1} \supsetneq \cdots \supsetneq N_0 = (0)$ where $m > n$. Choose $b_j \in N_j \setminus N_{j-1}$ for $j \geq 1$. This yields a chain $(b_1, ..., b_m) \supsetneq (b_1, ...b_{m-1}) \supsetneq \cdots \supsetneq (b_1) \supsetneq (0)$. For each j, there exists k_j such that $b_j = \alpha_{k_j}(a_{k_j})$, where $a_{k_j} \in M_{k_j}$ by the previous theorem. Then $\alpha_{k_j}(a_{k_j}) = \alpha_t \phi_t^{k_j}(a_{k_j})$. Let $c_j = \phi_t^{k_j}(a_{k_j}) \in M_t$. So we get yet another chain $(\alpha_t(c_1), ..., \alpha_t(c_m)) \supsetneq \cdots \supsetneq (\alpha_t(c_1)) \supsetneq (0)$. Then $(c_1, ..., c_m) \supset \cdots \supset (c_1) \supset (0)$ is chain of submodules of M_t, which has length $\leq n$. Thus there exists j such that $(c_1, ..., c_j) = (c_1, ..., c_{j-1})$, that is, $c_j = r_1c_1 + ... + r_{j-1}c_{j-1}$. Then $\alpha_t(c_j) = r_1\alpha_t(c_1) + ... + r_{j-1}\alpha_t(c_{j-1})$, a contradiction. □

Exercise. Let (R, m) be a commutative local domain of $\dim R > 1$. Can we write the field of fractions as $Q = \varinjlim_{i \in J} R_i$ where $R_i = R$? Note: This is true if $\dim R = 1$ as then $R_y = Q$ for some $y \in m \setminus \{0\}$.

Theorem 1.21. *Let $\{A_i, \phi_j^i\}_{i,j \in I}$ be a direct system of left $R-$modules. Assume I is directed. Let $\alpha_i : A_i \to \varinjlim A_i$. Then*

1. $\varinjlim A_i = \{\alpha_i(a_i) | a_i \in A_i, i \in I\}$.
2. *For $i \in I$, $\alpha_i(a_i) = 0$ if and only if there exists $t \in I$ with $t \geq i$ such that $\phi_t^i(a_i) = 0$.*
3. $\varinjlim A_i = 0$ *if and only if for all $i \in I$ and $a_i \in A_i$, there exists $t \geq i$ such that $\phi_t^i(a_i) = 0$.*

Proof. First note that 1 and 2 imply 3. To prove 1, represent $\varinjlim A_i$ as $\oplus_{i \in I} A_i / N$, where N is the submodule of $\oplus A_i$ generated by $\{\lambda_j \phi_j^i(a_i) - \lambda_i(a_i) | a_i \in A_i, i \leq j \in I\}$ where $\lambda_i : A_i \to \oplus A_i$ are the natural injections. Under this representation, $\alpha_i : A_i \to \varinjlim A_i$ is defined by $a_i \mapsto \lambda(a_i) + N$. Let $x \in \varinjlim A_i$. Then $x = \sum_{j \in S} \lambda_j(a_j) + N$ where S is some finite set. Choose $t \in I$ such that $t \geq j$ for all $j \in S$. Let $b_t = \sum_{j \in S} \phi_t^j(a_j) \in A_t$. Then $x = \lambda_t(b) + N = \alpha_t(b)$.

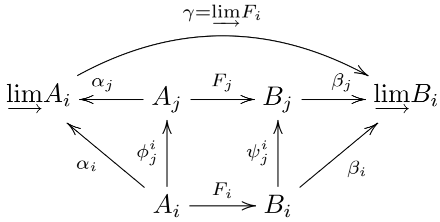
To prove 2, suppose $\phi_t^j(a_i) = 0$. Then $\alpha_i(a_i) = \alpha_t \phi_t^i(a_i) = \alpha_t(0) = -$. For the other direction, suppose $\alpha_i(a_i) = 0$. Then $\lambda_i(a_i) \in N$, which implies $\lambda_i(a_i) = \sum_{j \in T} \lambda_{k_j} \phi_{k_j}^j(b_j) - \lambda_j(b_j)$ for some finite set T. Choose $t \in I$ such that $t \geq i$ and $t \geq k_j$ for all $j \in T$. Then

$$\lambda_t \phi_t^i a_i = \lambda_t \phi_t^i a_i - \lambda_i a_i + \lambda_i a_i = \lambda_t \phi_t^i a_i - \lambda_i a_i + \sum_{j \in T} \lambda_{k_j} \phi_{k_j}^j(b_j) - \lambda_j b_j.$$

Since $k_j \leq t$ for all j, we have $\lambda_{k_j} \phi_{k_j}^j b_j - \lambda_j b_j = \lambda_t \phi_t^j b_j - \lambda_j b_j + \lambda_t \phi_t^{k_j}(-\phi_{k_j}^j(b_j)) - \lambda_{k_j}(-\phi_{k_j}^j(b_j))$. Resetting notation, we have $\lambda_t \phi_t^i(a_i) = \sum_\ell \lambda_t \phi_t^\ell(c_\ell) - \lambda_\ell(c_\ell)$. Assume the ℓ's are distinct (if not, group them). Let $\pi_j : \oplus A_i \to A_j$ be the natural projection. Then $\pi_j(\lambda_t \phi_t^j(a_i)) = 0$ if $j \neq t$ and $\pi_j(\sum_\ell \lambda_t \phi_t^\ell(c_\ell) - \lambda_\ell(c_\ell)) = -c_j$ if $j \neq t$. So $c_j = 0$ for all $j \neq t$, which implies $\lambda_t \phi_t^\ell(c_\ell) - \lambda_\ell c_\ell = 0$ if $\ell \neq t$. So $\lambda_t \phi_t^i(a_i) = \lambda_t \phi_t^t(c_t) - \lambda_t(c_t) = 0$. Since λ_t is injective, we have $\phi_t^i(a_i) = 0$. □

Theorem 1.22. *Let $\mathcal{C} =<< R - mod >>$, I a direct index set. Then $\varinjlim : Dir_{\mathcal{C}}(I) \to \mathcal{C}$ is exact.*

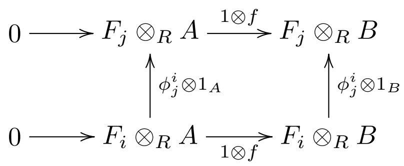
Proof. As $\varinjlim$ is a left adjoint, it is right exact. So it is enough to show that it preserves injections. Suppose $0 \to \{A_i, \phi^i_j\} \xrightarrow{F} \{B_i, \psi^i_j\}$ is an exact sequence of direct systems, i.e. $F_i : A_i \to B_i$ is injective for all $i \in I$.



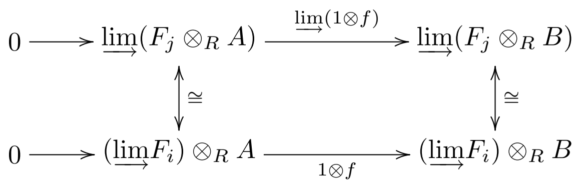
Suppose $\gamma(x) = 0$ for some $x \in \varinjlim A_i$. Then $x = \alpha_i(a_i)$ for some $a_i \in A_i$. So $\beta_i(F_i(a_i)) = 0$. By the previous theorem, there exists $j \geq i$ such that $\psi^i_j(F_i(a_i)) = 0$. Thus $F_j(\phi^i_j(a_i)) = 0$ which implies $\phi^i_j(a_i) = 0$ as F_j is injective. Thus $x = \alpha_i(a_i) = 0$. □

Corollary 1.23. *Let $\{F_i, \phi^i_j\}_{i,j \in I}$ be a direct system of right $R-$modules over a directed index set. Suppose each F_i is flat. Then $\varinjlim F_i$ is a flat $R-$module.*

Proof. As $\otimes$ is right exact, it is enough to show $\varinjlim F_i \otimes -$ preserves injections. So suppose $0 \to A \xrightarrow{f} B$ is an exact sequence of left $R-$modules. This gives rise to a morphism of direct systems $\{F_i \otimes_R A_j, \phi^i_j \otimes 1_A\} \to \{F_i \otimes_R B, \phi^i_j \otimes 1_B\}$. Thus we have the commutative diagram



where $(1 \otimes f)_i$ is injective as F_i is flat. Since $\varinjlim$ is an exact functor, we have the following commutative diagram where the top row is exact.



By commutativity, since the top is exact, we have that the bottom row is exact. Thus $\varinjlim F_i$ is flat. □

1.1 Inverse Limits

Definition 1.24. *Let $\mathcal{C}$ be a category and I and index set. An **inverse system** in $\mathcal{C}$ over I is a family $\{A_i\}_{i \in I}$ of objects in $\mathcal{C}$ and morphisms $\psi^j_i : A_j \to A_i$ whenever $i \leq j$ such that $\psi^i_i = 1_{A_i}$ and if $i \leq j \leq k$, then $\psi^j_i \psi^k_j = \psi^k_i$. An **inverse limit** $\varprojlim A_i$ is an object in $\mathcal{C}$ with maps $\alpha_i : \varprojlim A_i \to A_i$ which commute with ψ^j_i and if $\beta_i : X \to A_i$ also commutes with ψ^j_i, then there exists a unique $\delta : X \to \varprojlim A_i$ such that $\alpha_i \gamma = \beta_i$ for all i.*

Remark. $\{A_i, \psi^j_i\}$ is an inverse system in $\mathcal{C}$ if and only if $\{A_i, (\psi^j_i)^{op}\}$ is a direct system in $\mathcal{C}^{op}$.

Proposition 1.25. *Let $\mathcal{C} = << R - mod >>$. Then any inverse system in $\mathcal{C}$ has an inverse limit.*

Proof. Let $F = \prod_{i \in I} A_i$ and $\pi_j : F \to A_j$ be the natural projective. Let $Y = \{(a_i) = x \in F | \text{ for all } i \leq j, \pi_i(x) = \psi^j_i(\pi_j(x))\}$. Define $\alpha_i : Y \to A_i$ via projection. Then $Y = \varprojlim A_i$. □

Examples.

1. Let $A \in Obj\mathcal{C}$. Then $|A|$ is the constant inverse system over I. So $A_i = A$ for all $i \in I$ and $\psi^j_i : A_j \to A_i$ is the identity. If I is directed, then $\varprojlim |A| = A$.

2. If I has the trivial order, then $\varprojlim_{i \in I} A_i = \prod A_i$.

3. Let M be a left $R-$module and I a left ideal of R. Then the sequence $M/IM \leftarrow M/I^2M \leftarrow M/I^3M \leftarrow \cdots$ defined by $x+IM \leftarrow x+I^2M \leftarrow x+I^3M \cdots$ is an inverse system. The inverse limit $\varprojlim M/I^nM$ is called that **I-adic completion** if M.

Proposition 1.26. *$(|\cdot|, \varprojlim)$ is an adjoint pair.*

Proof. We've seen $(\varinjlim, |\cdot|)$ is an adjoint pair in $\mathcal{C}^{op}$. □

Corollary 1.27. *$\varprojlim$ is left exact.*

Corollary 1.28. *Right adjoints preserve inverse limits.*

Corollary 1.29. *The inverse limit of an inverse limit exists.*

Corollary 1.30. *If A is an $S-R$ bimodule, then $Hom_S(A, \varprojlim B_i) \cong \varprojlim Hom_S(A, B_i)$ for any inverse system $\{B_i\}$ of left $S-$modules.*

Caution. $\varprojlim$ is not generally exact, even over directed index sets. Thus, the direct limit is far more useful in Commutative Algebra.

Terminology. We refer to $\prod A_i$ as a product and $\oplus A_i$ or $\coprod A_i$ as a coproduct. As a result, the inverse limit is often called the limit and the direct limit is often called the colimit.

2 Chain Complexes

Definition 2.1. *Let R be a ring. A **chain complex** of left $R-$modules is a family of $R-$modules $\{C_i\}_{i\in\mathbb{Z}}$ and $R-$module homomorphisms $d_i : C_i \to C_{i-1}$ such that $d_i d_{i+1} = 0$ for all i. The d_i are called **differentials**. We denote a chain complex by $(C., d.)$. Often, we suppress the indices.*

Definition 2.2. *A chain complex (C, d) is **bounded on the right** if $C_i = 0$ for all $i \leq n$ for some n. Similarly, one can define **bounded on the left** and **bounded**.*

Examples.

1. Any exact sequence with an indexing is a chain complex.

2. Let M be an $R-$module. Then $0 \xrightarrow{0} 0 \xrightarrow{0} M \xrightarrow{0} 0 \xrightarrow{0} 0 \cdots$ is a chain complex where M is said to be in the 0^{th} spot.

3. Let $R = \mathbb{Z}/(r)$. Then $\cdots \xrightarrow{\bar{2}} \underset{3}{R} \xrightarrow{\bar{2}} \underset{2}{R} \xrightarrow{\bar{2}} \underset{1}{R} \xrightarrow{\bar{2}} \underset{0}{0} \xrightarrow{\bar{2}} \underset{-1}{0} \cdots$ is a chain complex.

4. Let M be an $R-$module, $x \in R$. Then $\cdots \to 0 \to \underset{1}{M} \xrightarrow{x} \underset{0}{M} \to 0$ is a chain complex.

Definition 2.3. *Let (C, d) be a complex of $R-$modules. The **module of** $n-$**cycles** is defined to be $Z_nC := \ker d_n$. The $n-$**boundaries** of C are the elements of $B_nC := d_{n+1}C_{n+1} = imd_{n+1}$. Since $d^2 = 0$, we see $dC_{n+1} \subseteq Z_nC$. Define the n^{th} **homology** of C to be the module $H_n(C) = \frac{Z_nC}{dC_{n+1}} = \frac{\ker d_n}{imd_{n+1}}$. We say a complex C is **exact** if $H_n(C) = 0$ for all n.*

Examples. Considering the examples above, once again.

1. $H_n(C) = 0$ for all n for any exact sequence (i.e., an exact sequence is an exact complex).

2. $H_0(C) = M, H_i(C) = 0$ for all $i \neq 0$.

3. $H_i(C) = 0$ for all $i \neq 1$ and $H_1(C) = \mathbb{Z}/(2)$.

4. $H_0(C) = M/xM$, $H_1(C) = \{m \in M | xm = 0\} = (0 :_M x)$.

Definition 2.4. *Let (C, d) and (D, d') be chain complexes. A **chain map** $\phi : C \to D$ is a family of $R-$module homomorphisms $\phi_i : C_i \to D_i$ for all $i \in \mathbb{Z}$ such that $d'_i\phi_i = \phi_{i-1}d_i$ (or, with suppressed indices, we usually just says $d\phi = \phi d$).*

Remark. Let $Ch(R-mod)$ denote the category of chain complexes of $R-$modules and chain maps. In fact, it is an abelian category with sums and products. Thus $\varinjlim$ and $\varprojlim$ exist.

Exercise. Let $\{C_i, \phi_j^i\}$ be a direct system of chain complexes over a directed index set. Then for $i \leq j$, we have the following commutative diagram

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & C_{i,n+1} & \xrightarrow{d} & C_{i,n} & \xrightarrow{d} & C_{i,n-1} & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle (\phi_j^i)_{n+1}} & & \downarrow{\scriptstyle (\phi_j^i)_n} & & \downarrow{\scriptstyle (\phi_j^i)_{n-1}} & & \\
\cdots & \longrightarrow & C_{j,n+1} & \xrightarrow{d} & C_{j,n} & \xrightarrow{d} & C_{j,n-1} & \longrightarrow & \cdots
\end{array}$$

Fixing n, we see $\{C_{i,n}, (\phi_j^i)_n\}$ is a direct system of $R-$modules. Show that $\varinjlim C_i$ is isomorphic to the chain complex $\cdots \to \varinjlim C_{i,n} \to \varinjlim C_{i,n-1}$. Then, find $\varinjlim K_i$ in the following commutative diagram where K_i represents the complex formed by the i^{th} row and the first column of $R's$ represents index 1 and the second index 0:

$$\begin{array}{cccccccc}
K_0: & 0 & \longrightarrow & R & \xrightarrow{x} & R & \longrightarrow & 0 \\
 & & & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle x} & & \\
K_1: & 0 & \longrightarrow & R & \xrightarrow{x^2} & R & \longrightarrow & 0 \\
 & & & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle x} & & \\
K_2: & 0 & \longrightarrow & R & \xrightarrow{x^3} & R & \longrightarrow & 0 \\
 & & & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle x} & & \\
 & & & \vdots & & \vdots & &
\end{array}$$

Proof. Let $X_n = \varinjlim C_{i,n}, \alpha_{i,n} : C_{i,n} \to X$ where $\alpha_{j,n}\phi_{j,n}^i = \alpha_{i,n}$ and $d_n = \varinjlim d_{i,n}$. Then, we want to show $\varinjlim C_i = (X, \hat{\alpha}_i)$ where $\hat{\alpha}_i : C_i \to X$ is defined by $\hat{\alpha}_{i,n} = \alpha_{i,n}$. We have

1. X is a chain complex. Notice $d_{n-1}d_n = \varinjlim d_{i,n-1} \varinjlim d_{i,n} = \varinjlim d_{i,n-1}d_{i,n} = \varinjlim 0 = 0$.

2. $\hat{\alpha}_i$ are chain maps such that $\hat{\alpha}_j\phi_j^i = \hat{\alpha}_i$ (since $\alpha_{j,n}\phi_{j,n}^i = \alpha_{i,n}$ for all n), $\hat{\alpha}_{i,n}$ are $R-$module homomorphisms and for all n, we have the following diagram commutes (by the universal property of the direct limit):

$$\begin{array}{ccc}
C_{i,n} & \xrightarrow{\alpha_{i,n}} & X_n \\
\downarrow{\scriptstyle d_{i,n}} & & \downarrow{\scriptstyle d_n} \\
C_{i,n-1} & \xrightarrow{\alpha_{i,n-1}} & X_{n-1}
\end{array}$$

3. The universal property of direct limits holds: Suppose $\hat{\beta}_i : C_i \to Y$ are chain maps such that $\hat{\beta}_j\phi_j^i = \hat{\beta}_i$. Then, for all n we see $\beta_{i,n} : C_{i,n} \to Y_n$ are such that $\beta_{j,n}\phi_{j,n}^i = \beta_{i,n}$. Of course, then there exists a unique $\gamma_n : X_n \to Y_n$ such that $\gamma_n\alpha_{i,n} = \beta_{i,n}$. Define $\hat{\gamma} : X \to Y$ by $\hat{\gamma}_n = \gamma_n$. Uniqueness follows. Also, $\hat{\gamma}$ is a chain map as we have the following commutative diagrams:

$$\begin{array}{ccccc}
 & & \overset{\gamma_n}{\frown} & & \\
X_n & \xleftarrow{\alpha_{i,n}} & C_{i,n} & \xrightarrow{\beta_{i,n}} & Y_n \\
\downarrow{\scriptstyle d_n} & & \downarrow{\scriptstyle d_{i,n}} & & \downarrow{\scriptstyle d'_n} \\
X_{n-1} & \xleftarrow{\alpha_{i,n-1}} & C_{i,n-1} & \xrightarrow{\beta_{i,n-1}} & Y_{n-1} \\
 & & \underset{\gamma_{n-1}}{\smile} & &
\end{array}
\quad \Rightarrow \quad
\begin{array}{ccc}
X_n & \xrightarrow{\gamma} & Y_n \\
\downarrow{\scriptstyle d_n} & & \downarrow{\scriptstyle d'_n} \\
X_{n-1} & \xrightarrow{\gamma_{n-1}} & Y_{n-1}
\end{array}$$

By the above, we see we can, in a sense, commute direct limits with chain complexes. Thus to find $\varinjlim K_i$, we simply need to take the direct limits of each column. We've seen the direct limits of these sequences before. So $\varinjlim K_i = \cdots \to 0 \to R \to R_x \to 0 \to \cdots$ where $R \to R_x$ is defined by $r \mapsto \frac{r}{1}$ and $\alpha_i : R \to R_x$ are defined by $r \mapsto \frac{r}{x^i}$. $\square$

Remark. Let $\phi : C \to D$ be a chain map of complexes. Then

1. $\phi_n(Z_n(C)) \subseteq Z_n(D)$ for all n
2. $\phi_n(B_n(C)) \subseteq B_n(D)$ for all n

Therefore, there exists an induced map on the homology: $(\phi_*)_n : H_n(C) \to H_n(D)$ defined by $\overline{x} \mapsto \phi_n(x)$.

Proof. Recall that we have the following commutative diagram

$$\begin{array}{ccccc} C_{n+1} & \xrightarrow{d} & C_n & \xrightarrow{d} & C_{n-1} \\ \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \phi} \\ D_{n+1} & \xrightarrow{d} & D_n & \xrightarrow{d} & D_{n-1} \end{array}$$

Let $x \in Z_n(C)$. So $d(x) = 0$, which implies $0 = \phi d(x) = d\phi(x)$ and thus $\phi(x) \in Z_n(D)$. Let $y \in B_n(C)$. So $y = d(t)$, which implies $\phi(y) = \phi d(t) = d\phi(t)$ and thus $\phi(y) \in B_n(D)$. $\square$

If C is a complex, one can view the homology as a complex $\cdots \xrightarrow{0} H_n(C) \xrightarrow{0} H_{n-1}(C) \xrightarrow{0} H_{n-2}(C) \xrightarrow{0} \cdots =: H_*(C)$. We let the maps be the zero maps so that the homology of this complex at a given spot is still $H_n(C)$. In this context, $\phi_* : H_*(C) \to H_*(D)$ is a chain map.

Definition 2.5. *A chain map $\phi : C \to D$ is called a* ***quasi-isomorphism*** *(q.i.) if ϕ_* is an isomorphism, that is, ϕ induces an isomorphism $H_n(C) \to H_n(D)$ for all n.*

Example. Let C be a chain complex, 0 the zero chain complex. Then there is a unique chain map $\phi : 0 \to C$. Also, ϕ is a q.i. if and only if C is exact.

Snake Lemma. *Consider the following commutative diagram of $R-$modules, where the rows are exact:*

$$\begin{array}{ccccccc} & & A & \xrightarrow{\alpha} & B & \xrightarrow{\beta} & C & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} & & \\ 0 & \longrightarrow & D & \xrightarrow{\delta} & E & \xrightarrow{\epsilon} & F & & \end{array}$$

Then, the sequence

$$\ker f \xrightarrow{\alpha'} \ker g \xrightarrow{\beta'} \ker h \xrightarrow{\partial} cokerf \xrightarrow{\overline{\delta}} cokerg \xrightarrow{\overline{\epsilon}} cokerh$$

is exact. Furthermore, if α is injective, then α' is and if ϵ is surjective, then $\overline{\epsilon}$ is.

Proof. Define $\alpha' : \ker f \to \ker g$ by $\alpha' = \alpha|_{\ker f}$. Note that the image really is $\ker g$ by commutativity of our diagram. We can similarly define β'. Now, recall that $cokerf = D/imf$ and $cokerg = E/img$. So we can get a map from $D \to E \to E/img$, which induces the map $\overline{\delta} : D/imf \to E/img$. Similarly, we can define $\overline{\epsilon}$. Thus, we need only to define ∂. To do so, first note that we have the following commutative diagram:

$$\begin{array}{ccccccccc} & & & & & & 0 & & \\ & & & & & & \downarrow & & \\ & & & & & & \ker h & & \\ & & & & & & \downarrow & & \\ & & A & \xrightarrow{\alpha} & B & \xrightarrow{\beta} & C & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} & & \\ 0 & \longrightarrow & D & \xrightarrow{\delta} & E & \xrightarrow{\epsilon} & F & & \\ & & \downarrow & & & & & & \\ & & cokerf & & & & & & \\ & & \downarrow & & & & & & \\ & & 0 & & & & & & \end{array}$$

Let $c \in \ker h$. Choose $b \in B$ such that $\beta(b) = c$. Then $\epsilon(g(b)) = h(\beta(b)) = h(c) = 0$. So $g(b) \in \ker \epsilon = im\delta$. As δ is injective, there exists a unique $d \in D$ such that $\delta(d) = g(b)$. Define $\partial(c) = \overline{d} = d + imf \in cokerf$. We need to show ∂ is well-defined. Since everything is linear, it is enough to show in the case that $c = 0$. So suppose $c = 0$. Then $b \in \ker \beta = im\alpha$. So $b = \alpha(a)$ for some $a \in A$. Then $g(b) = \delta(f(a))$ which implies $d = f(a)$. So $\partial(c) = \overline{f(a)} = 0$.

Now, we need to show the sequence is exact.

It's exact at $cokerf$: Let $\overline{d} \in cokerf$ and suppose $\overline{d} \in im\partial$. Then $\overline{d} = \overline{y}$ where $y \in D$ such that $\delta(y) \in img$. Thus $\overline{\delta}(\overline{d}) = \overline{\delta}(\overline{y}) = \overline{\delta(y)} = 0$ as $\delta(y) \in img$. So $im\partial \subseteq \ker \overline{\delta}$. Now suppose $\overline{d} \in \ker \overline{\delta}$. Then $\delta(d)\ inimg$, which implies $\delta(d) = g(b)$. Note $h\beta(b) = \epsilon g(b) = \epsilon\delta(d) = 0$ by exactness. Thus $c = \beta(b) \in \ker h$. By definition of ∂, we have $\partial(c) = \overline{d}$.

It's exact at $\ker h$: Let $x \in \ker h$ such that $\partial x = 0$. Then there exists $b \in B$ such that $\beta b = x$ and $g(b) \in \ker \epsilon$, since $\epsilon g(b) = h\beta b = hx = 0$. By exactness, there is a unique $d \in D$ with $\delta(d) = g(b)$. Since $\partial(x) = \overline{d}$, we see $\overline{d} = 0$, that is, $d \in imf$. So there exists $a \in A$ such that $f(a) = d$. Take $b - \alpha(a) \in B$. Then $g(b - \alpha(a)) = g(b) - g\alpha(a) = g(b) - \delta f(a) = g(b) - g(b) = 0$. So $b - \alpha(a) \in \ker g$. Note $\beta'(b - \alpha(a)) = \beta(b) - \beta\alpha(a) = \beta(b) = x$. So $\ker \partial \subseteq im\beta'$. Now, let $b \in \ker g$. Then $c := \beta(b) \in \ker h$. To define ∂c, note there exists $b' \in B$ such that $\beta(b') = c$ as $h\beta(b') = 0$ implies $g(b') \in im\delta$. As δ is injective, there exists $d \in D$ with $g(b') = \delta(d)$. So $\partial c = \overline{d}$. By well-definedness, choose $b' = b$. Then $g(b') = 0$, which says $d = 0$ and thus $im\beta' \subseteq \ker \partial$. □

Note. The Snake Lemma is in fact true in any abelian category.

Exercise. Let M be a finitely presented $R-$module. Let $f : N \to M$ be a surjective homomorphism where N is finitely generated. Prove $\ker f$ is finitely generated.

Proof. As M is finitely presented, we have $R^m \xrightarrow{\alpha'} R^n \xrightarrow{\beta} M \to 0$ is an exact sequence. Let $L = \ker \beta = im\alpha'$. Then $0 \to L \xrightarrow{\alpha} R^n \xrightarrow{\beta} M \to 0$ is an exact sequence, and moreover, L is finitely generated as it is the surjective image of R^m, which is finitely generated. Thus we have the commutative diagram with exact rows below:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & L & \xrightarrow{\alpha} & R^n & \xrightarrow{\beta} & M & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow 1_M & & \\
0 & \longrightarrow & \ker f & \xrightarrow{i} & N & \xrightarrow{f} & M & \longrightarrow & 0
\end{array}$$

where i is the natural injection map. As R^n is projective and we have a map $\beta : R^n \to M$, we can define a map $\epsilon : R^n \to N$ such that the above diagram still commutes. Now, we need to define a map $\delta : L \to \ker f$. Let $\ell \in L$. Then $\epsilon\alpha(\ell) \in N$ and in particular, $f\epsilon\alpha(\ell) = \beta\alpha(\ell) = 0$. Thus $\epsilon\alpha(\ell) \in \ker f$. So we may define $\delta = \epsilon\alpha$. Thus we have the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
 & & & & & & 0 & & \\
 & & & & & & \downarrow & & \\
0 & \longrightarrow & L & \xrightarrow{\alpha} & R^n & \xrightarrow{\beta} & M & \longrightarrow & 0 \\
 & & \downarrow \delta & & \downarrow \epsilon & & \downarrow 1_M & & \\
0 & \longrightarrow & \ker f & \xrightarrow{i} & N & \xrightarrow{f} & M & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & coker\delta & & coker\epsilon & & 0 & &
\end{array}$$

By the Snake Lemma, we have $0 \to coker\delta \to coker\epsilon \to 0$ is an exact sequence. So $coker\delta \cong coker\epsilon$. Note that $coker\epsilon$ is finitely generated as it is the surjective image of N, a finitely generated $R-$module. Thus $coker\delta$ is finitely generated. Now we have $0 \to \delta(L) \to \ker f \to coker\delta \to 0$ is an exact sequence where both $\delta(L)$ and $coker\delta$ are finitely generated. Thus $\ker f$ is finitely generated. □

Proposition 2.6. *Let $0 \to A \xrightarrow{\phi} B \xrightarrow{\psi} C \to 0$ be a short exact sequence of chain complexes of $R-$modules (or objects in any abelian category). Then there exists a long exact sequence $\cdots \to H_n(A) \xrightarrow{(\phi_*)_n} H_n(B) \xrightarrow{(\psi_*)_n} H_n(C) \xrightarrow{\partial_n} H_{n-1}(A) \xrightarrow{(\phi_*)_{n-1}} \cdots$.*

Proof. Consider the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A_n & \xrightarrow{\phi_n} & B_n & \xrightarrow{\psi_n} & C_n & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle d} & & \downarrow{\scriptstyle d} & & \downarrow{\scriptstyle d} & & \\ 0 & \longrightarrow & A_{n-1} & \xrightarrow[\phi_{n-1}]{} & B_{n-1} & \xrightarrow[\psi_{n-1}]{} & C_{n-1} & \longrightarrow & 0 \end{array}$$

By the Snake Lemma, we have $0 \to Z_nA \xrightarrow{\phi} Z_nB \to Z_nC$ is exact and $A_{n-1}/dA_n \to B_{n-1}/dB_n \to C_{n-1}/dC_n \to 0$ is exact. So consider the following diagram, which commutes as the above diagram did.

$$\begin{array}{ccccccccc} & & A_n/dA_{n+1} & \xrightarrow{\overline{\phi}_n} & B_n/dB_{n+1} & \xrightarrow{\overline{\psi}_n} & C_n/dC_{n+1} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \overline{d}_n} & & \downarrow{\scriptstyle \overline{d}_n} & & \downarrow{\scriptstyle \overline{d}_n} & & \\ 0 & \longrightarrow & Z_{n-1}(A) & \xrightarrow[\phi_{n-1}]{} & Z_{n-1}(B) & \xrightarrow[\psi_{n-1}]{} & Z_{n-1}(C) & & \end{array}$$

Note that $\ker(\overline{d}_n) = \frac{\ker d_n}{dA_{n+1}} = H_n(A)$ and $coker(\overline{d}_n) = \frac{Z_{n-1}(A)}{im\overline{d}_n} = \frac{Z_{n-1}(A)}{imd_n} = H_{n-1}(A)$. By the Snake Lemma, we're done. ☐

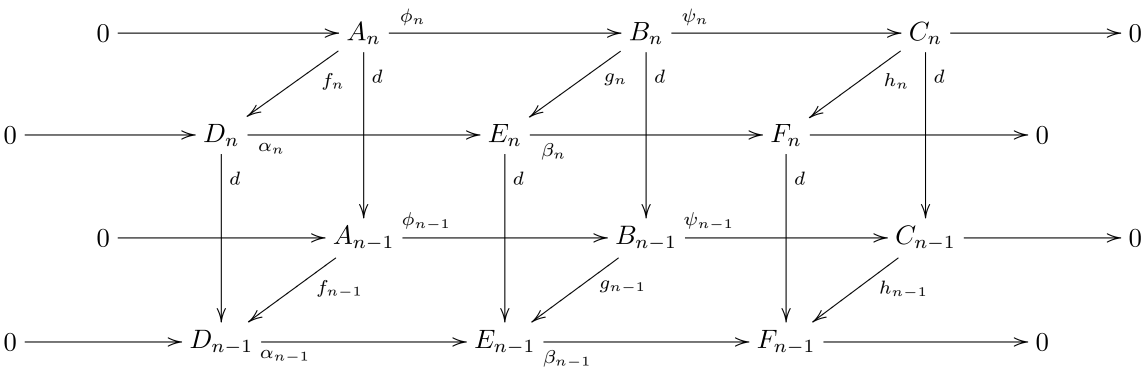
Proposition 2.7. *The long exact sequence on homology is natural, that is, if we have the following commutative diagram of chain maps with exact rows,*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{\phi} & B & \xrightarrow{\psi} & C & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} & & \\ 0 & \longrightarrow & D & \xrightarrow[\alpha]{} & E & \xrightarrow[\beta]{} & F & \longrightarrow & 0 \end{array}$$

then there exists the following commutative diagram of long exact sequences:

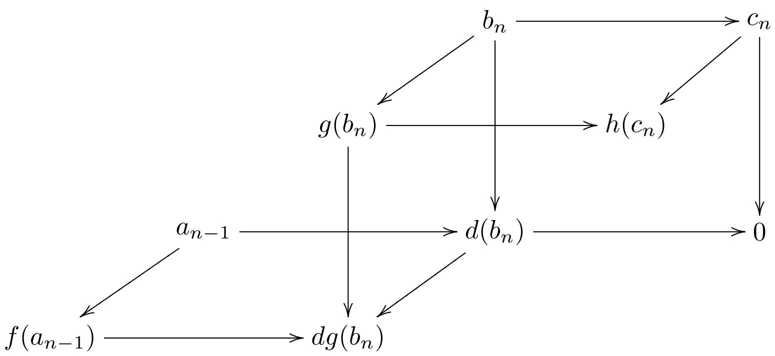
$$\begin{array}{ccccccccc} H_n(A) & \xrightarrow{\phi_*} & H_n(B) & \xrightarrow{\psi_*} & H_n(C) & \xrightarrow{\partial_n} & H_{n-1}(A) & \longrightarrow & \cdots \\ \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle g_*} & & \downarrow{\scriptstyle h_*} & & \downarrow{\scriptstyle f_*} & & \\ H_n(D) & \xrightarrow[\alpha_*]{} & H_n(E) & \xrightarrow[\beta_*]{} & H_n(F) & \xrightarrow[\partial_n']{} & H_{n-1}(D) & \longrightarrow & \cdots \end{array}$$

Proof. Note that the first two squares are easily seen to be commutative. So we need only show the third square is, that is, we wish to show $\partial_n' h_* = f_*\partial_n$.



Let $\overline{c_n} \in H_n(C)$. We can lift $\overline{c_n}$ to some $c_n \in Z_n(C) \subseteq C_n$. By surjectivity, we can choose $b_n \in B_n$ such that $\psi(b_n) = c_n$. Then, we can push b_n down into B_{n-1} to get $d(b_n)$. Note here that since $c_n \in Z_n(C)$, we have $\psi_{n-1}d(b_n) = d\psi_n(b_n) = dc_n = 0$. Thus $b_n \in \ker\psi_{n-1} = im\phi_{n-1}$. So we can lift $d(b_n)$ to $a_{n-1} \in A_{n-1}$, and thus we see $f_*\partial_n(\overline{c_n}) = \overline{a_{n-1}}$. On the other hand, we could first push c_n to $h(c_n) \in F_n$. Then, by commutativity, we know we can lift $h(c_n)$ to $g(b_n) \in E_n$. Next, we push $g(b_n)$ down to $dg(b_n) \in E_{n-1}$ and again by commutativity, we lift it to $f(a_{n-1}) \in D_{n-1}$. Thus $\partial_n' h_*(\overline{c_n}) = \overline{f(a_{n-1})} = f_*(\overline{a_{n-1}})$.

Thus $\partial'_n h_* - f_*\partial_n$, and the statement is proven. The diagram below illustrates the diagram chase of elements:



□

Definition 2.8. *Let $f : C \to D$ be a chain map of chain complexes. We say f is* ***null homotopic*** *if for all n there exist maps $s_n : C_n \to D_{n+1}$ such that $f_n = ds_n + s_{n-1}d$. The collection $\{s_n\}$ is called the* ***chain contraction****. Two chain maps $f, g : C \to D$ are called* ***chain homotopic*** *if $f - g$ is* ***null homotopic****.*

Remarks.

1. $f_* - g_* = (f - g)_*$.

2. If f, g are chain homotopic, then $f_* = g_* : H_*(C) \to H_*(D)$.

   Proof. Let $\overline{u} \in H_n(C)$, where $u \in Z_n(C)$. To show $f_*(\overline{u}) = g_*(\overline{u})$, it is enough to show $(f - g)_*(\overline{u}) = 0$. It suffices to prove if f is null-homotopic, then $f_* = 0$. Note $f_*(\overline{u}) = \overline{f(u)} = \overline{sd(u) + ds(u)} = s(0) = 0$ ($\overline{ds(u)} = 0$ as it is in the boundary and $d(u) = 0$ as $u \in \ker d$). □

Note. Chain homotopy is an equivalence relation. This is different, however, from the following notion:

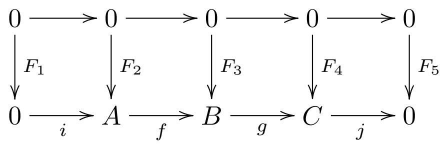
Definition 2.9. *A chain map $f : C \to D$ is called a* ***chain homotopy equivalence*** *if there exists a chain map $g : D \to C$ such that fg is chain homotopic to 1_D and gf to 1_C.*

Note. This means, in particular, that $f_*g_* = 1_*$ on $H_*(D)$ and $g_*f_* = 1_*$ on $H_*(C)$. Thus chain homotopy equivalence induces isomorphisms on homology.

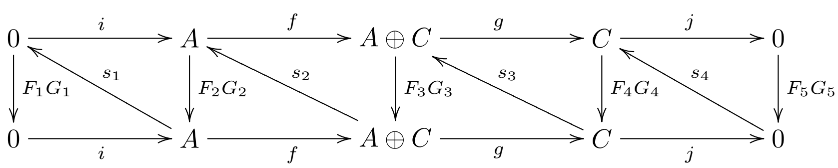
Remark. Every chain homotopy equivalence is a quasi-isomorphism of chain complexes, however, the converse is not true. Recall $0 \to \mathcal{C}$ is q.i. if and only if C is exact.

Exercise: Let $\mathcal{C}$ be a short exact sequence. Then $0 \to \mathcal{C}$ is a chain homotopy equivalence if and only if $\mathcal{C}$ is split exact.

Proof. Let $\mathcal{C}$ be the short exact sequence $0 \to A \to B \to C \to 0$. First note by $0 \xrightarrow{F} \mathcal{C}$, we mean we have the following diagram



We will first suppose $\mathcal{C}$ is split exact, that is, $B = A \oplus C$. Define $G : \mathcal{C} \to 0$ by $G_i = 0$. Note $GF = Id_0$ and so it is chain homotopic to Id_0. Recall, as $\mathcal{C}$ is split exact, that there exists $\phi : B \to A$ such that $\phi f = 1_A$ and $\psi : C \to B$ such that $g\psi = 1_C$. Furthermore, we can choose ϕ and ψ such that $Id_B = f\phi + \psi g$. To show FG is chain homotopic to $Id_{\mathcal{C}}$, we want to show $Id_{\mathcal{C}} - FG$ is null homotopic. Define $s_1 = s_4 = 0, s_2 = \phi, s_3 = \psi$. This gives us the following diagram



Now, as G_i is the zero map, $F_iG_i = 0$ for all i. Thus, we have $Id_A - F_2G_2 = Id_A = \phi f = is_1 + s_2 f, Id_{A\oplus c} - F_3G_3 = Id_{A\oplus C} = f\phi + \psi g = fs_2 + s_3 g, Id_C - F_4G_4 = Id_C = g\psi = fs_3 + s_4 g$. Thus FG is chain homotopic equivalent to $1_{\mathcal{C}}$ which says $0 \xrightarrow{F} \mathcal{C}$ is a chain homotopy equivalence.

For the other direction, choose G_i such that we have chain homotopy equivalence. Then $GF = Id_0$ and $FG = Id_0$ (as we must have that G_i is the zero map). Note that we have the following diagram

$$\begin{array}{ccccccccc}
0 & \xrightarrow{i} & A & \longrightarrow & A\oplus C & \longrightarrow & C & \xrightarrow{j} & 0 \\
\downarrow{\scriptstyle 0} & \nwarrow^{s_1} & \downarrow{\scriptstyle 0} & \nwarrow^{s_2} & \downarrow{\scriptstyle 0} & \nwarrow^{s_3} & \downarrow{\scriptstyle 0} & \nwarrow^{s_4} & \downarrow{\scriptstyle 0} \\
0 & \xrightarrow[i]{} & A & \xrightarrow[f]{} & B & \xrightarrow[g]{} & C & \xrightarrow[j]{} & 0
\end{array}$$

By definition, we see $Id_A = Id_A - F_2G_2 = is_1 + s_2 f = s_2 f$. Thus $\mathcal{C}$ splits. □

Comparison Theorem for Projective Resolutions. *Let $\mathcal{C}$ be an abelian category and consider the following diagram in $\mathcal{C}$:*

$$\begin{array}{ccccccccccccc}
\longrightarrow & P_i & \xrightarrow{d} & P_{i-1} & \xrightarrow{d} \cdots \xrightarrow{d} & P_1 & \xrightarrow{d} & P_0 & \xrightarrow{d} & M & \longrightarrow & 0 \\
 & & & & & & & & & \downarrow{\scriptstyle f=f_{-1}} & & \\
\longrightarrow & Q_i & \xrightarrow{d'} & Q_{i-1} & \xrightarrow{d'} \cdots \xrightarrow{d'} & Q_1 & \xrightarrow{d'} & Q_0 & \xrightarrow{d'} & N & \longrightarrow & 0
\end{array}$$

Suppose the top row is a complex, P_i is projective for all i, and the bottom row is exact. Then there exists $f_i : P_i \to Q_i$ for all i such that $f_i d_{i+1} = d'_{i+1} f_{i+1}$ for all $i \geq -1$, that is, there exists a chain map which "lifts" f. Furthermore, any two such liftings are chain homotopic.

Proof. Induct on n to show there exists $f_i : P_i \to Q_i$ for $i \leq n$ such that $f_i d_{i+1} = d'_{i+1} f_{i+1}$ for all $i \leq n-1$. For $n \leq -1$, this is trivially true (let $f_i = 0$ for all $i \leq -2$ and $f_{-1} = f$). Assume we have $\{f_i\}_{i=-1}^n$ which work. So, we have the following commutative diagram:

$$\begin{array}{ccccccc}
0 & \longrightarrow & Z_n(P) & \longrightarrow & P_n & \xrightarrow{d_n} & P_{n-1} \\
 & & & & \downarrow{\scriptstyle f_n} & & \downarrow{\scriptstyle f_{n-1}} \\
0 & \longrightarrow & Z_n(Q) & \longrightarrow & Q_n & \xrightarrow{d'_n} & Q_{n-1}
\end{array}$$

As before, there exists $\widetilde{f_n} : Z_n(P) \to Z_n(Q)$ (take $\widetilde{f_n} = f_n|_{Z_n(P)}$). Since $dP_{n+1} \subseteq Z_n(P)$ and $Z_n(Q) = dQ_{n+1}$ (by exactness), we get $\widetilde{\widetilde{f_n}} : dP_{n+1} \to dQ_{n+1}$ where $\widetilde{\widetilde{f_n}} = \widetilde{f_n}|_{dP_{n+1}}$. Now, we have the diagram below, where the bottom row is exact:

$$\begin{array}{ccccc}
P_{n+1} & \xrightarrow{d_{n+1}} & dP_{n+1} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \widetilde{\widetilde{f_n}}} & & \\
Q_{n+1} & \xrightarrow{d'_{n+1}} & dQ_{n+1} & \longrightarrow & 0
\end{array}$$

As P_{n+1} is projective, we get a map $f_{n+1} : P_{n+1} \to Q_{n+1}$ such that $\widetilde{\widetilde{f_n}} d_{n+1} = d'_{n+1} f_{n+1}$. It is easily seen that this implies $f_n d_{n+1} = d'_{n+1} f_{n+1}$.

To prove that any two liftings are chain homotopic, suppose both $f.$ and $g.$ lift f. Then $f. - g.$ lifts $0 : M \to N$. So it is enough to show if $f = 0$, then any lifting of f is null-homotopic. We use induction on n to show there exist maps $s_i : P_i \to Q_{i+1}$ for $i \leq n$ such that $f_i = d_{i+1}s_i + s_{i-1}d_i$. Let $s_i = 0$ for $i \leq -1$. Clearly, $f_{-1} = 0 = ds + sd$. So assume we have done this for $i \leq n$. Then $f_n = s_{n-1}d + ds_n$, and so $ds_n = f_n - s_{n-1}d$. Now $d(f_{n+1} - s_n d) = df_{n+1} - ds_n d = df_{n+1} - f_n d + s_{n-1}dd = df_{n+1} - f_n d = 0$. So $im(f_{n+1} - s_n d) \subseteq \ker d_{n+1} = im d_{n+2}$. So we have the following commutative diagram

$$\begin{array}{ccccc}
 & & P_{n+1} & & \\
 & & \downarrow{\scriptstyle f_{n+1} - s_n d} & & \\
Q_{n+2} & \longrightarrow & dQ_{n+2} & \longrightarrow & 0 \text{ exact}
\end{array}$$

and as P_{n+1} is projective, there exists $s_{n+1} : P_{n+1} \to Q_{n+2}$ such that $d_{n+2}s_{n+1} = f_{n+1} - s_n d$. □

Definition 2.10. *Let $\mathcal{C}$ be an abelian category and $M \in Obj\mathcal{C}$. A* ***projective resolution*** *of M in $\mathcal{C}$ is a chain complex $P.$ such that*

1. *$P_i = 0$ for all $i < 0$*
2. *P_i is projective*
3. *$H_i(P.) = \begin{cases} M, & \text{if } i = 0, \\ 0, & \text{otherwise.} \end{cases}$*

Write $P. : \cdots \to P_i \to P_{i-1} \to \cdots \to P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} 0$ where $H_0(P_0) = M = P_0/imd_1 = cokerd_1$. Equivalently, $\cdots \to P_2 \to P_1 \to P_0 \xrightarrow{\epsilon} M \to 0$ is exact where ϵ is called the ***augmentation map****.*

Definition 2.11. *An abelian category $\mathcal{C}$ is said to have* ***enough projectives*** *if for all objects A of $\mathcal{C}$, there exists a surjective morphism $P \twoheadrightarrow A$, where P is projective.*

Example. $\mathcal{C} =<< R - mod >>$. Every $R-$module is the quotient of a free module, which is projective. Thus there are enough projectives in $\mathcal{C}$.

Remark. If $\mathcal{C}$ is an abelian category with enough projectives, then every object of $\mathcal{C}$ has a projective resolution.

Proof. Let M be an object and fine $P_0 \xrightarrow{\epsilon} M$ where P_0 is projective. Next, find $P_1 \xrightarrow{\delta_0} \ker \epsilon \xrightarrow{i_0} P_0$ where P_1 is projective. Let $d_1 = i_0\delta_0$. Continuing, we get a chain as follows:

$$\begin{array}{ccccccccc} \cdots \longrightarrow & P_2 & \xrightarrow{d_2} & P_1 & \xrightarrow{d_1} & P_0 & \xrightarrow{\epsilon} & M & \longrightarrow 0 \\ & \downarrow{\scriptstyle \delta_1} & \nearrow{\scriptstyle i_1} & \downarrow{\scriptstyle \delta_0} & \nearrow{\scriptstyle i_0} & & & & \\ & \ker d_1 & & \ker \epsilon & & & & & \end{array}$$

□

Examples.

1. Let R be a commutative ring, $x \in R$ a non-zero-divisor. Then $0 \to 0 \to \underset{1}{R} \xrightarrow{x} \underset{0}{R} \to 0$ is a projective resolution of $R/(x) = cokerx$.
2. Let P be projective. Then $\cdots \to 0 \to \underset{0}{P} \to 0$ is a projective resolution of P.
3. Let $R = \mathbb{Z}/(4)$. Then $\cdots \xrightarrow{2} \underset{1}{R} \xrightarrow{2} \underset{0}{R} \to 0$ is a projective resolution of $coker2 = R/(2) \cong \mathbb{Z}/(2)$ as an $R-$module.

Note. Projective resolutions are not unique. For example $0 \to \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{f} \mathbb{Z} \oplus \mathbb{Z} \to 0$ defined by $f(r, s) = (2r, s)$ and $0 \to \mathbb{Z} \xrightarrow{2} \mathbb{Z} \to 0$ are projective resolutions of $\mathbb{Z}/(2)$ as $\mathbb{Z}-$modules.

Proposition 2.12. *Let $\mathcal{C}$ be an abelian category, $M \in Obj\mathcal{C}$, and suppose $P.$ and $Q.$ are projective resolutions of M. Then there exists a chain homotopy equivalence $f. : P. \to Q.$. That is, projective resolutions are unique up to chain homotopy equivalence.*

Proof. Suppose we are given the following diagram with exact rows

$$\begin{array}{ccccc} P. & \xrightarrow{\epsilon} & M & \longrightarrow & 0 \\ \downarrow & & \downarrow{\scriptstyle 1_M} & & \\ Q. & \xrightarrow{\delta} & M & \longrightarrow & 0 \\ \downarrow & & \downarrow{\scriptstyle 1_M} & & \\ P. & \xrightarrow{\epsilon} & M & \longrightarrow & 0 \\ \downarrow & & \downarrow{\scriptstyle 1_M} & & \\ Q. & \xrightarrow{\delta} & M & \longrightarrow & 0 \end{array}$$

By the comparison theorem, there exist chain maps $g. : P. \to Q.$ and $h. : Q. \to P.$ by lifting 1_M. Then $h.g. : P. \to P.$ is a lifting of 1_M, as is $1_{P.} : P. \to P.$. Thus $h.g.$ is chain homotopic to $1_{P.}$. Similarly, $g.h.$ is chain homotopic to $1_{Q.}$. □

Remark. Suppose $f : C \to D$ is a chain homotopy equivalence, where C and D are chain complexes in some category $\mathcal{C}$. Let F be a covariant additive functor. Then $F(f) : F(C) \to F(D)$ is a chain homotopy equivalence.

Proof. First note that if $f : C \to D$ is null homotopic, so is $F(f)$ (as if $s_n : C_n \to D_{n+1}$ is such that $f = ds + sd$, then $F(f) = F(d)F(s) + F(s)F(d)$). In general, if $fg - 1_D$ is null homotopic, then $F(f)F(g) - 1_{F(D)}$ is null homotopic. Similarly, if $gf - 1_C$ is, then so is $F(g)F(f) - 1_{F(C)}$. □

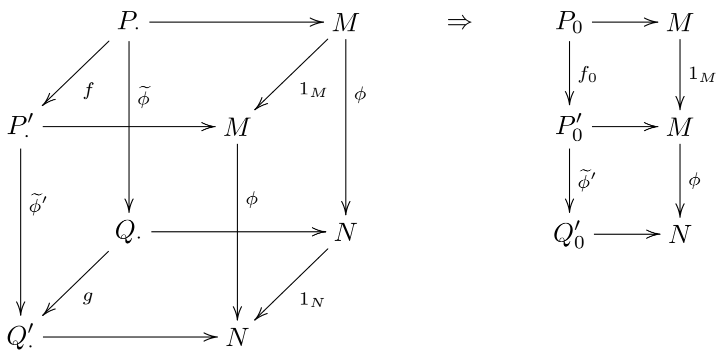
2.1 Left Derived Functors

Definition 2.13. *Let $F : \mathcal{C} \to \mathcal{D}$ be an additive, covariant, right exact functor on abelian categories, where $\mathcal{C}$ has enough projectives. For $i \geq 0$, define the i^{th}* ***left derived functor*** *L_iF of F as follows: Let $M \in Obj\mathcal{C}$ and $P.$ be a projective resolution of M. Then $(L_iF)(M) := H_i(F(P.))$.*

- *This is well-defined: Suppose $Q.$ is another projective resolution. Then there exists a chain homotopy equivalence $f. : P. \to Q.$. Hence, $F(f.) : F(P.) \to F(Q.)$ is a chain homotopy equivalence, which induces an isomorphism on homology. Thus $F(f_*) : H_*(F(P.)) \to H_*(F(Q.))$ is an isomorphism.*

Now, suppose $\phi : M \to N$ is a morphism in $\mathcal{C}$. Let $P.$ be a projective resolution of M, and $Q.$ of N. By the comparison theorem, there exists a chain map $\widetilde{\phi}$ lifting ϕ. Then $F(\widetilde{\phi}) : F(P.) \to F(Q.)$ is still a chain map. Define $(L_iF)(\phi) = F(\widetilde{\phi}_*) : H_i(F(P.)) \to H_i(F(Q.))$.

- *This is well-defined: Suppose we had $\widetilde{\phi} : P. \to Q.$ and $\widetilde{\phi}' : P'. \to Q'.$. Note then, by the comparison theorem, that we have maps $f : P. \to P'.$ and $g : Q. \to Q'.$ which lift 1_M giving us the following diagrams where the second is commutative:*



Compacting the second diagram, we see that $\widetilde{\phi}' f_0 = \phi$ and, in general, $\widetilde{\phi}' f$ lifts ϕ. Similarly, by considering a different portion of the cube, we get $g\widetilde{\phi}$ lifts ϕ and thus $\widetilde{\phi}' f$ and $g\widetilde{\phi}$ are chain homotopic. Thus $F(\widetilde{\phi}' f)$ and $F(g\widetilde{\phi})$ are chain homotopic, which says $F(\widetilde{\phi}' f)_* = F(g\widetilde{\phi})_*$. Thus we see $F(\widetilde{\phi}')_* F(f)_* = f(g)_* F(\widetilde{\phi})_*$, which says the following diagram is commutative:

$$\begin{array}{ccc} H_i(F(P.)) & \xrightarrow{F(\widetilde{\phi})_*} & H_i(F(Q.)) \\ \downarrow{\scriptstyle F(f)_*} & & \downarrow{\scriptstyle F(g)_*} \\ H_i(F(P'.)) & \xrightarrow{F(\widetilde{\phi}')_*} & H_i(F(Q'.)) \end{array}$$

Note that $F(f)_*$ and $F(g)_*$ are isomorphisms as f and g lifted the identity maps.

Exercise. Show $L_0F \cong F$.

Proof. Let $M \in Obj\mathcal{C}$ and $\cdots \to P_1 \to P_0 \to 0$ be a projective resolution for M. Then $F(P_1) \xrightarrow{\alpha} F(P_0) \to F(M) \to 0$ is exact, as F is right exact. So $F(M) \cong F(P_0)/im\alpha$. Thus, when we consider the sequence $F(P_1) \xrightarrow{\alpha} F(P_0) \xrightarrow{0} 0$, we see $H_0(F(P.)) = \ker 0/im\alpha = F(P_0)/im\alpha \cong F(M)$. Thus $(L_0F)(M) := H_0(F(P_0)) \cong F(M)$. To show $L_0F \cong F$, we also need to check that $F(f) = L_0F(f)$ for $f : M \to N$. Of course, $L_0F(f) : L_0F(M) \to L_0F(N)$ is defined by $L_0F(f) = F(\widetilde{f_0}) : H_0F(P.) \to H_0(F(Q.))$. Now, note that $\widetilde{f_0} = f_0$. Thus $L_0F(f) = F(f)$ and so $L_0F \cong F$. □

Special Case. Let R, S be rings, M an $S-R$ bimodule. Then $M \otimes_R - :<< R-mod >> \to << S-mod >>$ and we denote $L_i(M \otimes_R -)$ as $Tor_i^R(M, -)$.

Remarks.

1. If N is a left $R-$module and $P.$ is a projective resolution for N, then $Tor_i^R(M,N) = H_i(M \otimes_R P.)$.

2. If M is an $S-R$ bimodule, then $F_M = M \otimes_R - :<< R-mod >> \to << S-mod >>$ is covariant, right exact, and additive. So, by the above exercise, $Tor_0^R(M,-) \cong M \otimes_R -$.

Examples.

1. Compute $Tor_i^{\mathbb{Z}}(\mathbb{Z}/(2), \mathbb{Z}/(2))$ for all i.

   - Note that $P: 0 \to \underset{1}{\mathbb{Z}} \xrightarrow{2} \underset{0}{\mathbb{Z}} \to 0$ is a projective resolution of $\mathbb{Z}/(2)$. Now, apply our functor $\mathbb{Z}/(2) \otimes_{\mathbb{Z}} -$ to get:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathbb{Z}/(2) \otimes_{\mathbb{Z}} \mathbb{Z} & \xrightarrow{1\otimes 2} & \mathbb{Z}/(2) \otimes_{\mathbb{Z}} \mathbb{Z} & \longrightarrow & 0 \\ & & \downarrow \cong & & \downarrow \cong & & \\ 0 & \longrightarrow & \mathbb{Z}/(2) & \xrightarrow{\overline{2}=0} & \mathbb{Z}/(2) & \longrightarrow & 0 \end{array}$$

   Thus $Tor_i^{\mathbb{Z}}(\mathbb{Z}/(2), \mathbb{Z}/(2)) = \begin{cases} \mathbb{Z}/(2), & \text{if } i = 0,1 \\ 0, & \text{otherwise.} \end{cases}$

2. Compute $Tor_i^{\mathbb{Z}/(2)}(\mathbb{Z}/(2), \mathbb{Z}/(2))$ for all i.

   - Note that $P: 0 \to \underset{0}{\mathbb{Z}/(2)} \to 0$ is a projective resolution of $\mathbb{Z}/(2)$. Now, apply our functor $\mathbb{Z}/(2) \otimes_{\mathbb{Z}/(2)} -$ to get:

$$\begin{array}{ccccc} 0 & \longrightarrow & \mathbb{Z}/(2) \otimes_{\mathbb{Z}/(2)} \mathbb{Z}/(2) & \longrightarrow & 0 \\ & & \downarrow \cong & & \\ 0 & \longrightarrow & \mathbb{Z}/(2) & \longrightarrow & 0 \end{array}$$

   Thus $Tor_i^{\mathbb{Z}/(2)}(\mathbb{Z}/(2), \mathbb{Z}/(2)) = \begin{cases} \mathbb{Z}/(2), & \text{if } i = 0 \\ 0, & \text{otherwise.} \end{cases}$

3. Compute $Tor_i^{\mathbb{Z}/(4)}(\mathbb{Z}/(2), \mathbb{Z}/(2))$ for all i.

   - Note that $P: \xrightarrow{2} \underset{1}{\mathbb{Z}/(4)} \xrightarrow{2} \underset{0}{\mathbb{Z}/(4)} \to 0$ is a projective resolution of $\mathbb{Z}/(2)$. Now, apply our functor $\mathbb{Z}/(2) \otimes_{\mathbb{Z}/(4)} -$ to get:

$$\begin{array}{cccccc} \longrightarrow & \mathbb{Z}/(2) \otimes_{\mathbb{Z}/(4)} \mathbb{Z}/(4) & \xrightarrow{1\otimes 2} & \mathbb{Z}/(2) \otimes_{\mathbb{Z}/(4)} \mathbb{Z}/(4) & \longrightarrow & 0 \\ & \downarrow \cong & & \downarrow \cong & & \\ \xrightarrow{0} & \mathbb{Z}/(2) & \xrightarrow{0} & \mathbb{Z}/(2) & \longrightarrow & 0 \end{array}$$

   Thus $Tor_i^{\mathbb{Z}/(4)}(\mathbb{Z}/(2), \mathbb{Z}/(2)) = \mathbb{Z}/(2)$ for all $i \geq 0$.

Remark. Let N be a left $R-$module. Then $G: - \otimes_R N :<< S-R\ bimod >> \to << S-mod >>$ is right exact, covariant, and additive. So one could construct $L_iG := \underline{Tor}_i^R(-,N)$.

Q: For M an $S-R$ bimodule and N a left $R-$module, is $Tor_i^R(M,N) \cong \underline{Tor}_i^R(M,N)$?

A: Yes! In the case where M is a projective module, note that $0 \to M \to 0$ is a projective resolution. Then, applying the functor $- \otimes_R N$, we see $\underline{Tor}_i^R(M,N) = \begin{cases} M \otimes_R N, & \text{if } i = 0 \\ 0, & \text{otherwise.} \end{cases}$ On the other hand, if $P.$ is a projective resolution for N. Applying $M \otimes_R -$ keeps the sequence $\cdots \to P_1 \to P_0 \to N \to 0$ exact as M is projective (and thus flat). So $Tor_i^R(M,N) = \begin{cases} M \otimes_R N, & \text{if } i = 0, \\ 0, & \text{otherwise.} \end{cases}$ Thus $Tor = \underline{Tor}$. This is true when M is not projective, however we need more machinery to prove it.

Lemma 2.14. *Let $\mathcal{C}$ be an abelian category and consider the diagram*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & P & \xrightarrow{\delta} & Q & \xrightarrow{\epsilon} & R & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f} & & & & \downarrow{\scriptstyle h} & & \\ 0 & \longrightarrow & L & \xrightarrow{\ell} & M & \xrightarrow{m} & N & \longrightarrow & 0 \end{array}$$

Suppose R is projective. Then there exists $g : Q \to M$ making the diagram commute.

Proof. As R is projective, the top sequence splits. Let $i : Q \to P$ be a splitting map such that $i\delta = 1_P$. Also, as R is projective, there exists $j : R \to M$ such that $mj = h$. Let $g = \ell f i + j\epsilon$. Then the diagram commutes as $g\delta = \ell f i\delta + j\epsilon\delta = \ell f$ as $\epsilon\delta = 0$ and $i\delta = 1$ and also $mg = m\ell f i + mj\epsilon = mj\epsilon = h\epsilon$. ☐

Horseshoe Lemma. *Let $\mathcal{C}$ be an abelian category, $0 \to A \to B \to C \to 0$ a short exact sequence of objects in $\mathcal{C}$, and $P., R.$ projective resolutions of A and C, respectively. Then there exists a projective resolution $Q.$ of B and chain maps $i : P. \to Q.$ and $\pi : Q. \to R.$ such that $0 \to P. \to Q. \to R. \to 0$ is a short exact sequence of complexes.*

Proof. For each n, let $Q_n = P_n \oplus R_n$, $i_n : P_n \to Q_n$ be the canonical injection and $\pi_n : Q_n \to R_n$ the canonical projective. Clearly, $0 \to P_n \to Q_n \to R_n \to 0$ is exact for all n. Let d, d' denote the differentials for $P., R.$. We will define $d''_n : Q_n \to Q_{n-1}$ inductively. Let $d''_0 = 0$. So we have the following diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & P_0 & \xrightarrow{i_0} & Q_0 & \xrightarrow{\pi_0} & R_0 & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \epsilon} & & \downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle \delta} & & \\ 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \end{array}$$

By the lemma, there exists $\gamma : Q_0 \to B$ making the diagram commute. Furthermore, γ is surjective by the snake lemma as $0 = coker\epsilon \to coker\gamma \to coker\delta = 0$ implies $coker\gamma = 0$. Moreover, the snake lemma gives us that $0 \to \ker\epsilon \xrightarrow{i_0} \ker\gamma \xrightarrow{\pi_0} \ker\delta \to 0$ is exact. Now, since $imd_i \subseteq \ker d_i$, we have

$$\begin{array}{ccccccccc} 0 & \longrightarrow & P_1 & \longrightarrow & Q_1 & \longrightarrow & R_1 & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle d_1} & & \downarrow & & \downarrow{\scriptstyle d_2} & & \\ 0 & \longrightarrow & \ker\epsilon & \longrightarrow & \ker\gamma & \longrightarrow & \ker\delta & \longrightarrow & 0 \\ & & \downarrow & & & & \downarrow & & \\ & & 0 & & & & 0 & & \end{array}$$

Again, there exists $d''_1 : Q_1 \to \ker\gamma \hookrightarrow Q_0$ making the diagram commute. Also, $imd''_1 = \ker\gamma$ by the Snake Lemma and $0 \to \ker d_1 \to \ker d''_1 \to \ker d'_1 \to 0$ is exact. Continue inductively. ☐

Corollary 2.15. *Let $\mathcal{C}, \mathcal{D}$ be abelian categories with enough projectives. Let $F : \mathcal{C} \to \mathcal{D}$ be a covariant, right exact, additive functor. Given any short exact sequence $0 \to L \xrightarrow{f} M \xrightarrow{N} 0$ in $\mathcal{C}$, there exists a long exact sequence $\cdots \to L_iF(L) \xrightarrow{L_iF(f)} L_iF(M) \xrightarrow{L_iF(g)} L_iF(N) \xrightarrow{\partial_i} L_{i-1}F(L) \xrightarrow{L_{i-1}F(f)} \cdots$.*

Proof. By the Horseshoe Lemma, there exists a short exact sequence of complexes $0 \to P. \to Q. \to R. \to 0$ where P, Q, R are projective resolutions of L, M, N respectively. For each n, we see $0 \to P_n \to Q_n \to R_n \to 0$ is split exact, which implies $0 \to F(P_n) \to F(Q_n) \to F(R_n) \to 0$ is split exact. Thus $0 \to F(P.) \to F(Q.) \to F(R.) \to 0$ is a short exact sequence of chain complexes in D. Thus $\cdots \to H_i(F(P.)) \to H_i(F(Q.)) \to H_i(F(R.)) \to H_{i-1}(F(P.)) \to \cdots$ is a long exact sequence. ☐

Example. Let R be commutative, M, N $R-$modules, and $x \in R$ a non-zero-divisor on M. Then $0 \to M \xrightarrow{x} M \to M/xM \to 0$ is a short exact sequence of $R-$modules. Thus there exists a long exact sequence $\cdots \to Tor_i^R(M,N) \to Tor_i^R(M,N) \to Tor_i^R(M/xM,N) \to \cdots$.

Exercise. Let $\mathcal{A}$ be an abelian category, $f, g : C \to D$ chain maps of chain complexes. Say $f \simeq g$ if and only if f and g are chain homotopic.

1. Prove $\simeq$ is an equivalence relation.

Proof. (a) Reflexive: Say $f. : C. \to D.$ is a chain map. Define $s_n = 0$. Then $d'_{n+1}s_n + s_{n-1}d_n = 0 = f_n - f_n$. Thus $f \simeq f$.

(b) Symmetry: Suppose $f. \simeq g.$. Then there exists $s_n : C_n \to D_{n+1}$ such that $f_n - g_n = d'_{n+1}s_n + s_{n+1}d_n$. Then $g_n - f_n = d_{n+1}(-s_n) + (-s_{n+1})d_n$, which says $g. \simeq f.$.

(c) Transitivity: Say $f. \simeq g.$ and $g. \simeq h.$. Then there exists $s_n : C_n \to D_{n+1}$ and $t_n : C_n \to D_{n+1}$ such that $f_n - g_n = d'_{n+1}s_n + s_{n+1}d_n$ and $g_n - h_n = d'_{n+1}t_n + t_{n+1}d_n$. This says $f_n - h_n = (f_n - g_n) + (g_n - h_n) = d'_{n+1}(s+t)_n + (s+t)_{n+1}d_n$. □

2. Let $Hom_{K(\mathcal{A})}(C, D)$ denote the set of equivalence classes of chain maps from $\mathcal{C}$ to $\mathcal{D}$ (that is, $Hom_{K(\mathcal{A})}(C, D) \cong Hom_{Ch(\mathcal{A})}(C, D)/ \simeq$). Define $+$ on $Hom_{K(\mathcal{A})}(C, D)$ by $[f] + [g] = [f + g]$. Prove that this is well defined.

Proof. Suppose $f. \simeq h.$ and $g. \simeq i.$. Then, there exists s, t such that $f - h = ds + sd$ and $g - i = dt + td$. So $(f+g)-(h+i) = (f-h)+(g-i) = d(s+t)-(s+t)d$. Thus, $f+g \simeq h+i$, and so $[f]+[g] = [f+g] = [h+i] = [h]+[i]$. □

Note that this makes $Hom_{K(\mathcal{A})}(C, D)$ into an abelian group.

3. For $[f] \in Hom_{K(\mathcal{A})}(C, D)$ and $[g] \in Hom_{K(\mathcal{A})}(D, E)$, define $[g] \circ [f] = [gf]$. Show $\circ$ is well defined.

Proof. Say $f_1 \simeq f_2$ and $g_1 \simeq g_2$ where $f_i : C \to D$ and $g_i : D \to E$. Then there exists $s_n : C_n \to D_{n+1}$ and $t_n : D_n \to E_{n+1}$ such that $f_1 - f_2 = sd + ds$ and $g_1 - g_2 = td + dt$. Define $u_n : C_n \to E_{n+1}$ by $u_n = g_{2,n+1}s_n + t_n f_{1,n}$. Then

$$\begin{aligned} u_{n-1}d + du_n &= g_{2,n}s_{n-1}d + t_{n-1}f_{1,n-1}d + dg_{2,n+1}s_n + dt_n f_{1,n} \\ &= g_{2,n}s_{n-1}d + g_{2,n}ds_n + t_{n-1}df_{1,n} + dt_n f_{1,n} \\ &= g_{2,n}(s_{n-1}d + ds_n) + (t_{n-1}d + dt_n)f_{1,n} \\ &= g_{2,n}(f_{1,n} - f_{2,n}) + (g_{1,n} - g_{2,n})f_{1,n} \\ &= g_{1,n}f_{1,n} - g_{2,n}f_{2,n}. \end{aligned}$$

Thus $g_1f_1 \simeq g_2f_2$ and thus $[g_1][f_1] = [g_1f_1] = [g_2f_2] = [g_2][f_2]$. □

Definition 2.16. *Let $\mathcal{A}$ be an abelian category. The category $K(\mathcal{A})$ is defined as follows: $Obj K(\mathcal{A})$ is the class of chain complexes in $\mathcal{A}$ and for $C, D \in Obj(\mathcal{A})$, let $Hom_{K(\mathcal{A})}(C, D)$ and composition be defined as above. Then $K(\mathcal{A})$ is called the* ***chain homotopy category*** *of $\mathcal{A}$.*

Definition 2.17. *Let $\mathcal{A}$ be an additive category. Let $f : C \to D$ be a morphism. A* ***kernel*** *of f is an object K and a morphism $i : K \to C$ such that $f_i = 0$ and if $g : A \to C$ is a morphism such that $fg = 0$, then there exists a unique morphism $g :': A \to L$ such that $g = ig'$.*

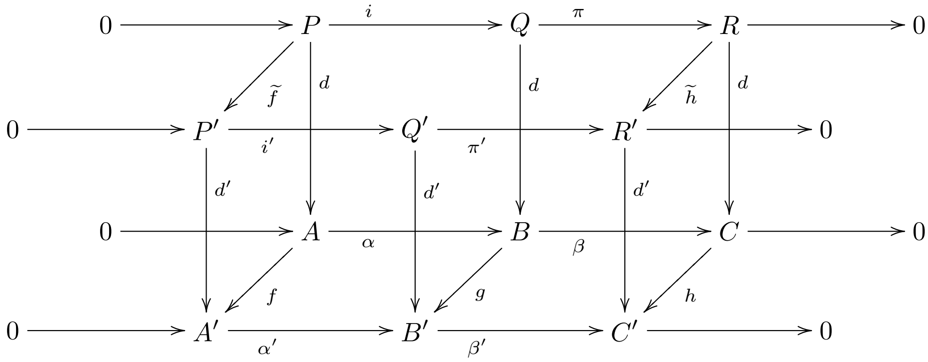
Note.

1. $K(\mathcal{A})$ is an additive category (so Hom sets are abelian groups and there is a 0 object), but it is not abelian (as the kernel/cokernel does not always exist).

   Example. Let $C. : 0 \to \underset{1}{\mathbb{Z} \oplus \mathbb{Z}} \xrightarrow{\alpha} \underset{0}{\mathbb{Z}/(2)} \to 0$ where $\beta(a, b) = \overline{a + b}$ and $D. : 0 \to \underset{1}{\mathbb{Z}/(2) \oplus \mathbb{Z}/(2)} \to 0$. Then, we can define $f : C. \to D.$ by $f_1(a, b) = (\overline{a}, \overline{b})$ and $f_i = 0$ for all $i \neq 1$. Then $f \in Hom_{Ch(\mathcal{A})}(C, D)$ and so $[f] \in Hom_{K(\mathcal{A})}(C, D)$. It can be shown that ker$[f]$ does not exist in $K(\mathcal{A})$.

2. Often, the derived category comes up in Homological Algebra. The **derived category** is $K(\mathcal{A})$ "localized" at the set of quasi-isomorphisms.

Lemma 2.18. *Let $\mathcal{A}$ be an abelian category and consider the following commutative diagram in $\mathcal{A}$, where the rows are exact, R, R' are projective, and $d' : P' \to A'$ is onto.*

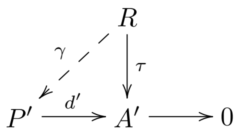


Then there exists $\widetilde{g} : Q \to Q'$ making the diagram commute.

Proof. As R, R' are projective, the top two rows split. Define $\rho : R \to Q, \rho' : R' \to Q', \phi : Q \to P, \phi' : Q' \to P'$ such that $\pi\rho = 1_R, \phi i = 1_P, 1_Q = i\phi + \rho\pi$ and similarly for the primed maps. Observe

$$\begin{aligned}
\beta' g d\rho &= h\beta d\rho \text{ as } \beta' g = h\beta \\
&= hd\pi\rho \text{ as } \beta d = d\pi \\
&= hd \text{ as } \pi\rho = 1 \\
&= d'\widetilde{h} \text{ by commutativity} \\
&= d'\pi'\rho'\widetilde{h} \text{ as } \pi'\rho' = 1 \\
&= \beta' d\rho'\widetilde{h} \text{ as } d'\pi' = \beta' d
\end{aligned}$$

Thus $im(gd\rho - d'\rho\widetilde{h}) \subseteq \ker\beta' = im\alpha'$. So we can define a map $\tau : R \to A'$ by $\tau = (\alpha')^{-1}(gd\rho - d'\rho'\widetilde{h})$ (as α' is injective). Now we have,



As the bottom row is exact and R is projective, there exists $\gamma : R \to P'$ such that $d'\gamma = \tau$. Define $\widetilde{g} : Q \to Q'$ by $\widetilde{g} = i'\widetilde{f}\phi + i'\gamma\pi + \rho'\widetilde{h}\pi$. To show this makes our original diagram commute, note

- $\widetilde{g}i = i'\widetilde{f}\phi i + i'\gamma\pi i + \rho'\widetilde{h}\pi i = i'\widetilde{f}$ as $\phi i = 1$ and $\pi i = 0$.
- $\pi'\widetilde{g} = \pi i'\widetilde{f}\phi + \pi' i'\gamma\pi + \pi\rho'\widetilde{h}\pi = \pi\rho'\widetilde{h}\pi = \widetilde{h}\pi$ as $\pi' i' = 0$.
- $$\begin{aligned}
d'\widetilde{g} = d'i'\widetilde{f}\phi + d'i\gamma\pi + d'\rho'\widetilde{h}\pi &= \alpha' d'\widetilde{f}\phi + \alpha' d'\gamma\pi + d'\rho'\widetilde{h}\pi \text{ as } d'i = \alpha' d' \\
&= \alpha' d'\widetilde{f}\phi + (gd\rho - d'\rho'\widetilde{h})\pi + d'\rho'\widetilde{h}\pi \text{ as } d'\gamma = \tau = (\alpha')^{-1}(gd\rho - d'\rho'\widetilde{h}) \\
&= \alpha' d'\widetilde{f}\phi + gd\rho\pi \\
&= \alpha' f d\phi + gd\rho\pi \text{ as } d'\widetilde{f} = fd \\
&= g\alpha d\phi + gd\rho\pi \text{ as } \alpha' f = g\alpha \\
&= gdi\phi + gd\rho\pi \text{ as } \alpha d = di \\
&= gd \text{ as } i\phi + \rho\pi = 1.
\end{aligned}$$

□

Lemma 2.19. *Let $\mathcal{A}$ be an abelian category and consider the following diagram in $Ch(\mathcal{A})$.*

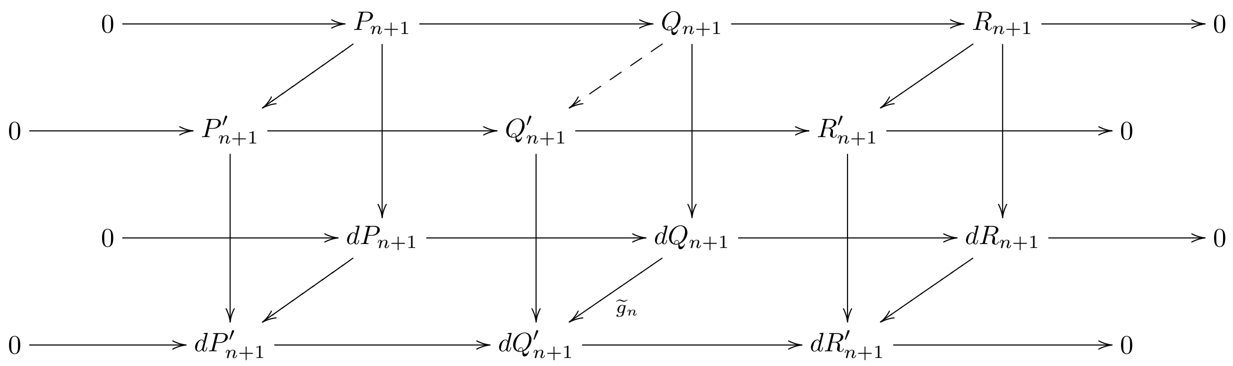
$$\begin{array}{ccccccccc}
0 & \longrightarrow & P & \xrightarrow{i} & Q & \xrightarrow{\pi} & R & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \widetilde{f}} & & & & \downarrow{\scriptstyle \widetilde{h}} & & \\
0 & \longrightarrow & P' & \xrightarrow{i'} & Q' & \xrightarrow{\pi'} & R' & \longrightarrow & 0
\end{array}$$

Suppose that the rows are exact, all complexes are 0 in negative indices, R, R' are complexes of projective modules, and there exists $g : H_0(Q) \to H_0(Q')$ such that the following diagram commutes.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & P_0 & \longrightarrow & Q_0 & \longrightarrow & R & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & H_0(P) & \longrightarrow & H_0(Q) & \longrightarrow & H_0(R) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \widetilde{f}_*} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle \widetilde{h}_*} & & \\ 0 & \longrightarrow & H_0(P') & \longrightarrow & H_0(Q') & \longrightarrow & H_0(R') & \longrightarrow & 0 \end{array}$$

Then there exists $\widetilde{g} : Q \to Q'$ making the initial diagram commute and lifting g.

Proof. Define $\widetilde{g}_n : Q_n \to Q_{n-1}$ inductively. Let $\widetilde{g}_i = 0$ for all $i < 0$. Assume $\widetilde{g}_i$ is defined for $i \leq n$. Define $\widetilde{g}_{n+1}$:



(This is commutative). By assumption, the top 2 rows are exact and by induction and the Snake Lemma, the bottom two rows are exact. As R_{n+1}, R'_{n+1} are projective and $d' : P'_{n+1} \to d'P'_{n+1}$ is surjective, we are done by the previous lemma. $\square$

Note. A projective resolution (or even a chain of projectives) is not a projective object in the category of chain complexes. Otherwise, the above two results would be trivial.

Example. Let $\mathcal{A} =<< \mathbb{Z} - mod >>$ and consider the chain complex $P. : 0 \to \underset{1}{\mathbb{Z}} \xrightarrow{d_P} \underset{0}{\mathbb{Z}} \to 0$, where d_P is defined as multiplication by 2. This is a chain of free (and thus projective) modules. However, $P.$ is not a projective object in $Ch(\mathcal{A})$.

Proof. We want to show that there exist chain complexes A and B in $Ch(A)$ such that $A \to B \to 0$ is exact and we have a map $P \to B$, but that there does not exist a map $P \to A$ making the diagram commute. Define $A. : 0 \to 0 \xrightarrow{d_A} \underset{0}{\mathbb{Z}} \to 0$ and $B. : 0 \to 0 \to \mathbb{Z}/(2) \to 0$. Define $g. : A. \to B.$ by $g_0(a) = \overline{a}$ and $g_i = 0$ for all $i \neq 0$. Also define $h. : P. \to B.$ by $h_0(a) = \overline{a}$ and $h_i = 0$ for all $i \neq 0$. Then we have the following diagram, where the bottom row is exact.

$$\begin{array}{ccccc} & & P. & & \\ & & \downarrow{\scriptstyle h.} & & \\ A. & \xrightarrow{g.} & B. & \longrightarrow & 0 \end{array}$$

Now, suppose there exists $f. : P. \to A.$. Clearly, $f_1 = 0$. So $f_0 d_P = d_A f_1 = 0$. Say $f_0 : \mathbb{Z} \to \mathbb{Z}$ is defined by $1 \mapsto m$. Then $0 = f_0 d_P(1) = f_0(2) = 2m = 0$. Then $m = 0$, which says $f_0 = 0$. Thus $f. = 0$, but $h = gf = 0$, a contradiction as $h \neq 0$. $\square$

Theorem 2.20. *Let $F : \mathcal{C} \to \mathcal{D}$ be a covariant, right exact, additive functor on abelian categories, where $\mathcal{C}$ has enough projectives. Consider the following commutative diagram:*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{\alpha} & B & \xrightarrow{\beta} & C & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} & & \\ 0 & \longrightarrow & A' & \xrightarrow{\alpha'} & B' & \xrightarrow{\beta'} & C' & \longrightarrow & 0 \end{array}$$

Then the corresponding diagram of long exact sequences on L_iF commutes, that is, LF is a covariant functor from $<<\text{ SES in }\mathcal{C}>>\to<<\text{ LES in }\mathcal{D}>>$.

Proof. Let $P_., R_., P'_., R'_.$ be projective resolutions for A, C, A', C' respectively. By the Horseshoe Lemma, there exists projective resolutions $Q_., Q'_.$ of B, B' such that $0 \to P \to Q \to R \to 0$ and $0 \to P' \to Q' \to R' \to 0$ are exact. By the Comparison Theorem, there exists $\widetilde{f}, \widetilde{h}$ that lift f, h. By the above lemma, there exists $\widetilde{g} : Q \to Q'$ lifting g and making the diagram commute. Now, apply F :

$$\begin{array}{ccccccccc}
0 & \longrightarrow & F(P) & \longrightarrow & F(Q) & \longrightarrow & F(R) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & F(P') & \longrightarrow & F(Q') & \longrightarrow & F(R') & \longrightarrow & 0
\end{array}$$

Note that the rows are exact as R is projective (and F preserves split exact sequences). The diagram of long exact sequences of L_iF commutes by naturality of the connecting homomorphism in the long exact sequence on homology. ☐

Definition 2.21. *Let C be a chain complex in $\mathcal{A}$. Let $p \in \mathbb{Z}$. Define a chain complex $C[p]$ by $C[p]_n := C_{p+n}$ and $d[p]_n : C[p]_n \to C[p]_{n-1}$ by $(-1)^p d_{p+n}$.*

Example. Suppose C is the complex $\underset{n}{C_n} \xrightarrow{d} \underset{n-1}{C_{n-1}} \xrightarrow{d} \underset{n-2}{C_{n-2}} \xrightarrow{d} \cdots$. Then, $C[-1]$ is the complex $\underset{n}{C_{n-1}} \xrightarrow{-d} \underset{n-1}{C_{n-2}} \xrightarrow{-d} \underset{n-2}{C_{n-3}} \xrightarrow{-d} \cdots$, that is, the complex C shifted to the left by one.

Remark. $H_n(C[p]) \cong H_{n+p}(C)$ for all n.

Definition 2.22. *Let $f : B \to C$ be a chain map of complexes. Define the **mapping cone** of f, denoted $cone(f)$, to be the chain complex such that $cone(f)_n = B_{n-1} \oplus C_n$ for all n and $d_n : B_{n-1} \oplus C_n \to B_{n-2} \oplus C_{n-1}$ is defined as $(b, c) \mapsto (-d(b), d(c) - f(b))$. [This is a chain complex as $d_{n-1}d_n((b, c)) = d_{n-1}(-d(b), d(c) - f(b)) = (d^2(b) - d^2(c) - df(b) + fd(b)) = 0$.]*

Now, define $g : C \to cone(f)$ where $g_n : C_n \to B_{n-1} \oplus C_n$ is defined by $c \mapsto (0, c)$. This is an injective chain map. Similarly, define $h : cone(f) \to B[-1]$ by $h_n : B_{n-1} \oplus C_n \to B[-1]_n = B_{n-1}$ where $(b, c) \mapsto -b$. This too is a chain map. Note that $0 \to C_n \xrightarrow{g} B_{n-1} \oplus C_n \xrightarrow{h} B_{n-1} \to 0$ is exact. Thus $0 \to C \to cone(f) \to B[-1] \to 0$ is a short exact sequence of complexes. Hence, we get a long exact sequence $\cdots \to H_n(C) \to H_n(cone(f)) \to H_n(B[-1]) \xrightarrow{\partial_n} H_{n-1}(C) \to \cdots$.

Claim. $\partial_n = (f_*)_{n-1} : H_n(B) \to H_{n-1}(C)$.

Proof. First consider the following diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & C_n & \longrightarrow & B_{n-1} \oplus C_n & \longrightarrow & B_{n-1} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle -d} & & \downarrow & & \downarrow{\scriptstyle -d} & & \\
0 & \longrightarrow & C_{n-1} & \longrightarrow & B_{n-2} \oplus C_{n-1} & \longrightarrow & B_{n-2} & \longrightarrow & 0
\end{array}$$

Let $b \in Z_{n-1}(B)$. Lift b to $(-b, 0)$ in $B_{n-1} \oplus C_n$. Now, push to $(-d(-b), d(0) - f_{n-1}(-b)) = (0, f_{n-1}(b))$. This lifts uniquely to $f_{n-1}(b) \in C_{n-1}$. In $H_{n-1}(C)$, the image is $d_n(\overline{b}) = \overline{f_{n-1}(b)} = (f_{n-1})_*(\overline{b})$. Thus $\partial_{n-1} = (f_{n-1})_*$.

Exercise. Let $f : C \to D$ be a chain map in an abelian category. Let $i : C \to cone(1_C) = C_{n-1} \oplus C_n$ be the natural injection, that is, $i_n : C_n \to C_{n-1} \oplus C_n$ where $c \mapsto (0, c)$. This is a chain map. Prove that f is null-homotopic if and only if there exists a chain map $g : cone(1_C) \to D$ such that $gi = f$.

Proof. Let $c : C_n \to C_{n-1}, d : D_n \to D_{n-1}, e : cone(1_C)_n \to cone(1_C)_{n-1}$ be the differential maps for each of the complexes. First, suppose f is null-homotopic. Then there exists $s_n : C_n \to D_{n+1}$ such that $f_n = ds_n + s_{n-1}c$. Define $g_n : C_{n-1} \oplus C_n \to D_n$ by $(x, y) \mapsto f_n(y) - s_{n-1}(x)$. Then $g_n i_n(y) = g_n(0, y) = f_n(y) - s_{n-1}(0) = f_n(y)$. To show g is a chain map, note that

$$\begin{aligned}
g_{n-1}e(x, y) &= g_{n-1}(-c(x), c(y) - x) \\
&= f_{n-1}(c(y) - x) - s_{n-2}(-c(x)) \\
&= f_{n-1}(c(y)) - f_{n-1}(x) + s_{n-2}(c(x)) \\
&= df_n(y) - ds_{n-1}(x) = d(f_n(y) - s_{n-1}(x)) = dg_n(x, y).
\end{aligned}$$

To prove the other implication, suppose $g : cone(1_C) \to D$ is defined such that $gi = f$ and g is a chain map (so that $g_{n-1}e = dg_n$). Define $s_n : C_n \to D_{n+1}$ by $x \mapsto g_{n+1}(x, 0)$. Then

$$\begin{aligned} ds_n(x) + s_{n-1}c(x) &= dg_{n+1}(x) + g_n(c(x), 0) \\ &= g_n e(x, 0) + g_n(c(x), 0) \\ &= g_n(-c(x), x) + g_n(c(x), 0) \\ &= g_n(0, x) = g_n i(x) = f_n(x) \end{aligned}$$

□

2.2 The Koszul Complex and regular sequences

Definition 2.23. *Let R be a commutative ring and C a chain complex of $R-$modules. Given $x \in R$, there is an induced chain map $\hat{x} : C \to C$ where $\hat{x}_n : C_n \to C_n$ is defined by $c \mapsto xc$ for all n. This is in fact a chain map as for $c \in C_n$, we have $d(xc) = xd(c)$ as d is $R-$linear.*

Definition 2.24. *Let $x_1, ..., x_n \in R$. Define the Koszul complex $K.(x_1, ..., x_n)$ inductively as follows: For $n = 1$, let $K.(x_1)$ be the chain complex $0 \to \underset{1}{R} \xrightarrow{x_1} \underset{0}{R} \to 0$. Given $K.(x_1, ..., x_{n-1})$, let $K.(x_1, .., x_n)$ be the mapping cone of $\hat{x}_n : K(x_1, ..., x_{n-1}) \xrightarrow{x_n} K(x_1, ..., x_{n-1})$.*

Example. We will compute $K.(x_1, x_2)$. Consider the following diagram, where $R_i = R_i' = R$ for all i.

$$\begin{array}{ccccccc} B: 0 & \longrightarrow & R_1 & \xrightarrow{x_1} & R_0 & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle x_2} & & \downarrow{\scriptstyle x_2} & & \\ C: 0 & \longrightarrow & R_1' & \xrightarrow{x_1} & R_0' & \longrightarrow & 0 \end{array}$$

Then, $K.(x_1, x_2) : 0 \to \underset{2}{R_1} \xrightarrow{\alpha} \underset{1}{R_0 \oplus R_1'} \xrightarrow{\beta} \underset{0}{R_0'} \to 0$ where $\alpha(1) = (-d(1), -f(1)) = (-x_1, -x_2), \beta(1, 0) = -x_2$, and $\beta(0, 1) = x_1$. Compacting the indices (since everything is just R), we see $K.(x_1, x_2) : 0 \to R \xrightarrow{\alpha} R^2 \xrightarrow{\beta} R \to 0$ where α is multiplication by $(-x_1, -x_2)$ and β is multiplication by the matrix $\begin{bmatrix} -x_2 \\ x_1 \end{bmatrix}$.

Exercise. Find $K.(x_1, x_2, x_3)$.

Proof. Using $K.(x_1, x_2)$ above, we construct $K.(x_1, x_2, x_3)$:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & R_1 & \longrightarrow & R^2 & \longrightarrow & R_2 & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle x_3} & & \downarrow{\scriptstyle x^3} & & \downarrow{\scriptstyle x_2} & & \\ 0 & \longrightarrow & R_1' & \longrightarrow & R_0^2 & \longrightarrow & R_2' & \longrightarrow & 0 \end{array}$$

where the maps on the rows are defined as above. Then, $K.(x_1, x_2, x_3)$ is the chain complex $0 \to R_1 \oplus 0 \xrightarrow{\alpha} R^2 \oplus R_1' \xrightarrow{\beta} R_2 \oplus R_0^2 \xrightarrow{\gamma} 0 \oplus R_2' \to 0$, where the maps are defined as follows:

$$\alpha(1, 0) = (-d(1), d(0) - f(1)) = (x_1, x_2, -x_3)$$

$$\begin{aligned} \beta(1, 0, 0) &= (-d(1, 0), d(0) - f(1, 0)) &= (x_2, -x_3, 0) \\ \beta(0, 1, 0) &= (-d(0, 1), d(0) - f(0, 1)) &= (-x_1, 0, x_3) \\ \beta(0, 0, 1) &= (-d(0, 0), d(1) - f(0, 0)) &= (0, -x_1, -x_2) \end{aligned}$$

$$\begin{aligned} \gamma(1, 0, 0) &= (-d(1), d(0, 0) - f(1)) &= (0, -x_3) \\ \gamma(0, 1, 0) &= (-d(0), d(1, 0) - f(0)) &= (0, -x_2) \\ \gamma(0, 0, 1) &= (-d(0), d(0, 1) - f(0)) &= (0, x_1) \end{aligned}$$

Finally, compacting the indices, we see $K.(x_1, x_2, x_3)$ is the chain complex $0 \to R \xrightarrow{\alpha} R^3 \xrightarrow{\beta} R^3 \xrightarrow{\gamma} R \to 0$ where $\alpha(1) = (x_1, x_2, -x_3)$, β is multiplication by the matrix $\begin{bmatrix} x_2 & -x_3 & 0 \\ -x_1 & 0 & -x_3 \\ 0 & -x_1 & -x_2 \end{bmatrix}$, and γ is multiplication by the matrix $\begin{bmatrix} -x_3 \\ -x_2 \\ x_1 \end{bmatrix}$.

□

Remark. If $\hat{x} : C \to C$ is as above, then $(\hat{x})_* : H_*(C) \to H_*(C)$ is also multiplication by x.

Definition 2.25. *Let $x_1, ..., x_n \in R$. Then the i^{th} **Koszul Homology** of $x_1, .., x_n$, denoted $H_i(x_1, ..., x_n)$ is the i^{th} homology of the Koszul complex, that is, $H_i(K(x_1, ..., x_n))$.*

Proposition 2.26. *Let $x_1, ..., x_n \in R$. Then there exists a long exact sequence on Koszul Homology $\cdots \to H_{i+1}(x_1, ..., x_n) \to H_i(x_1, ..., x_{n-1}) \xrightarrow{x_n} H_i(x_1, ..., x_{n-1}) \to H_i(x_1, ..., x_n) \to \cdots$.*

Proof. Let $\hat{x}_n : K.(x_1, ..., x_{n-1}) \to K.(x_1, ..., x_{n-1})$. Then $K.(x_1, ..., x_n) = cone(\hat{x}_n)$. Thus there exists a short exact sequence of complexes $0 \to K.(x_1, ..., x_{n-1}) \to cone(\hat{x}_n) \to K.(x_1, ..., x_{n-1})[-1] \to 0$, which says there is a long exact sequence on homology, namely the one above. □

Definition 2.27. *Let R be a commutative ring, $x_1, ..., x_n \in R$. We say $x_1, ..., x_n$ is a **regular sequence** if $(x_1, ..., x_n)R \neq R$ and for $i = 1, .., n$, $\overline{x_i}$ is a non-zero-divisor in $R/(x_1, ..., x_{i-1})$.*

Exercise. Show $K_i(x_1, ..., x_n) = R^{\binom{n}{i}}$ for $0 \le i \le n$ and $K_i(x_1, ..., x_n) = 0$ otherwise. Also, show $H_n(x_1, ..., x_n) = ann_R(x_1, ..., x_n) = (0 :_R (x_1, ..., x_n))$.

Proof. We will induct on n. For $n = 1$, we have $0 \to \underset{1}{R} \xrightarrow{x_1} \underset{0}{R} \to 0$. Clearly, $K(x_1)_i = \begin{cases} R = R^{\binom{1}{0}}, & \text{if } i = 0, \\ R = R^{\binom{1}{1}}, & \text{if } i = 1, \\ 0, & \text{otherwise} \end{cases}$. So suppose the claim holds for $n - 1$. Then $K.(x_1, ..., x_n)$ is the mapping cone of $\hat{x}_n : K.(x_1, ..., x_{n-1}) \to K.(x_1, ..., x_{n-1})$ and so $K(x_1, ..., x_n)_i = K(x_1, ..., x_{n-1})_{i-1} \oplus K(x_1, ..., x_{n-1})_i = \begin{cases} R^{\binom{n-1}{i-1}} \oplus R^{\binom{n-1}{i}} = R^{\binom{n-1}{i-1}+\binom{n-1}{i}}, & \text{if } 1 \le i \le n-1, \\ R^0 \oplus R^{\binom{n-1}{0}} = R^{\binom{n}{0}} = R, & \text{if } i = 0, \\ R^{\binom{n-1}{n-1}} \oplus 0 = R^{\binom{n}{n}} = R, & \text{if } i = n, \\ 0, & \text{otherwise.} \end{cases}$. Since $\binom{n-1}{i-1} + \binom{n-1}{i} = \binom{n}{i}$, we are done.

For the second claim, we will also induct on n. For $n = 1$, $H_1(K.(x_1)) = \ker x_1 / im 0 = ann_R(x_1)$. So suppose the claim is true for $n - 1$. Then $H_{n-1}(x_1, ..., x_{n-1}) = ann_R(x_1, ..., x_{n-1})$. Consider the long exact sequence

$$\cdots \to \underbrace{H_n(x_1, ..., x_{n-1})}_{=0} \to H_n(x_1, ..., x_n) \xrightarrow{g} \underbrace{H_{n-1}(x_1, ..., x_{n-1})}_{=ann_R(x_1, ..., x_n)} \xrightarrow{\overline{x_n}} \underbrace{H_{n-1}(x_1, ..., x_{n-1})}_{=ann_R(x_1, ..., x_n)} \to \cdots.$$

Now, g is injective, and thus $H_n(x_1, ..., x_n) \cong img = \ker \overline{x_n} = \{r \in ann_R(x_1, ..., x_{n-1}) | rx_n = 0\} = ann_R(x_1, ..., x_n)$.

□

Proposition 2.28. *Suppose $x_1, ..., x_n$ is a regular sequence. Then $H_i(x_1, ..., x_n) = \begin{cases} R/(x_1, ..., x_n), & \text{if } i = 0 \\ 0, & \text{otherwise.} \end{cases}$*

Proof. We will use induction on n. When $n = 1$, note that x_1 is a non-zero-divisor on R. Recall $K(x_1) : 0 \to \underset{1}{R} \xrightarrow{x_1} \underset{0}{R} \to 0$. Note that multiplication by x_1 is injective as x_1 is a non-zero-divisor. So $H_1(x_1) = 0$. Also $H_0(x_1) = \ker 0 / im(x_1) = R/(x_1)$. So let $n > 1$ and assume the hypothesis holds for $n - 1$. From the long exact sequence on Koszul Homology,

$$\cdots \to \underbrace{H_i(x_1, ..., x_{n-1})}_{=0 \ if \ i>0} \to H_i(x_1, ..., x_n) \to \underbrace{H_{i-1}(x_1, .., x_{n-1})}_{=0 \ if \ i>1} \xrightarrow{x_n} H_{i-1}(x_1, ..., x_{n-1}) \to \cdots$$

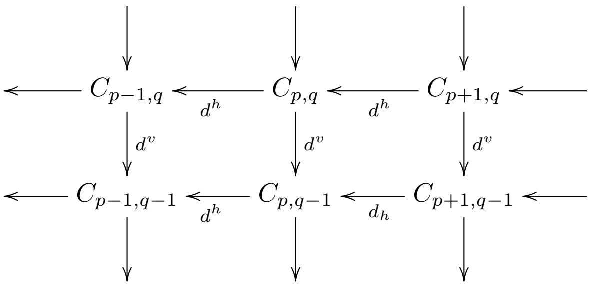
we see $H_i(x_1, .., x_n) = 0$ for $i > 1$. For $i = 1$, we have

$$0 \to H_1(x_1, ..., x_n) \to R/(x_1, ..., x_{n-1}) \xrightarrow{\overline{x}_{n-1}} R/(x_1, ..., x_{n-1}) \to H_0(x_1, ..., x_n) \to 0.$$

Note again that multiplication by $\overline{x}_n$ is injective as $\overline{x}_n$ is a non-zero-divisor. Thus $H_1(x_1, ..., x_n) \cong \ker \overline{x}_n = 0$. Now, $H_0(x_1, ..., x_{n-1}) = coker(\overline{x}_n) = R/(x_1, ..., x_n)$. □

Remark. If $x_1, ..., x_n$ is a regular sequence, the Koszul complex is a projective resolution of $R/(x_1, ..., x_n)$.

Definition 2.29. *A **double chain complex** $C_{.,.}$ in an abelian category $\mathcal{A}$ is a family of objects $\{C_{p,q}\}_{p,q\in\mathbb{Z}}$ and morphisms $d^v_{p,q} : C_{p,q} \to C_{p,q-1}$ and $d^h_{p,q} : C_{p,q} \to C_{p-1,q}$ for all p, q where $(d^v)^2 = (d^h)^2 = 0$ and $d^v d^h + d^h d^v = 0$ (that is, the squares **anticommute**). The diagram of $C_{.,.}$ looks like*



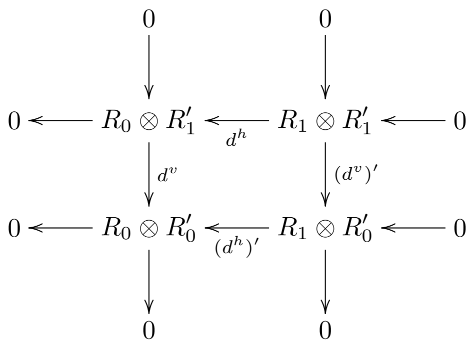
Example. Let $C., D.$ be chain complexes. Define a double chain complex $T_{.,.}$ by $T_{p,q} = C_p \otimes D_q$ for all $p, q \in \mathbb{Z}$ where $d^h_{p,q} : C_p \otimes D_q \to C_{p-1} \otimes D_q$ is the map $d_C \otimes 1$ and $d^v_{p,q} : C_p \otimes D_q \to C_p \otimes D_{q-1}$ is the map $(-1)^p \otimes d_D$.

Definition 2.30. *Let T be a double complex. Define a chain complex $Tot^{\prod}(T)$ by $Tot^{\prod}(T)_n = \prod_{i+j=n} T_{i,j}$ and $d_n : Tot^{\prod}(T)_n \to Tot^{\prod}(T)_{n-1}$ by $c \in T_{i,j} \mapsto d^v(c) + d^h(c)$, that is, $d_{tot} = d^V_T + d^h_T$. Note $d^2 = (d^v + d^h)^2 = 0$ by anticommutativity. So this is a chain complex. Similarly, define $Tot^{\oplus}(T)$ by $Tot^{\oplus}(T)_n = \oplus_{i+j=n} T_{i,j}$ and $d_{tot} = d^v_T + d^h_T$.*

Definition 2.31. *Let $C., D.$ be chain complexes. Then the **tensor product of two chain complexes** $C \otimes D$ is the chain complex $Tot^{\oplus}(T)$, where T is the double complex $\{C_p \otimes D_q\}$ defined above, that is $(C \otimes D)_n = \bigoplus_{i+j=n} C_i \otimes D_j$ and $d : (C \otimes D)_n \to (C \otimes D)_{n-1}$ is defined by $c_i \otimes d_j \mapsto d(c_i) \otimes d_j + (-1)^i c_i \otimes d(d_j)$ for $c_i \in C_i, d_j \in D_j$.*

Example. Find $K.(x_1) \otimes K.(x_2)$.

Proof. Let $C. : 0 \to R_1 \xrightarrow{x_1} R_0 \to 0$ and $D. : 0 \to R'_1 \xrightarrow{x_2} R'_2 \to 0$. Then we have



where $d^h(1\otimes 1) = x_1 \otimes 1, d^v(1 \otimes 1) = 1 \otimes x_2, (d^v)'(1 \otimes 1) = -1 \otimes x_2$, and $(d^h)'(1 \otimes 1) = x_1 \otimes 1$. Then, we get $0 \to R_1 \otimes R'_1 \xrightarrow{\alpha} (R_0 \otimes R'_1) \oplus (R_1 \otimes R'_0) \xrightarrow{\beta} R_0 \otimes R'_0 \to 0$ where α is defined by $\alpha(1 \otimes 1) = (x_1 \otimes 1, -1 \otimes x_2)$ and β is defined by $\beta(1 \otimes 1, 0) = 1 \otimes x_2$ and $\beta(0, 1 \otimes 1) = x_1 \otimes 1$. Compacting, we see $0 \to R \to R^2 \to R \to 0$ where $1 \mapsto (x_1, -x_2)$, $(1, 0) \mapsto x_2$, and $(0, 1) \mapsto x_1$. This gives an alternated construction of the Koszul complex, that is, $K.(x_1, ..., x_n) = K(x_1, ..., x_{n-1}) \otimes K(x_n) = \otimes^n_{i=1} K.(x_i)$. □

Definition 2.32. *A double complex T is said to be **first quadrant** (or **second quadrant, upper half plane**, etc) if $T_{i,j} = 0$ when (i, j) is outside of the first quadrant (or second quadrant, upper half plane, etc). T is said to be **bounded** if for all $n \in \mathbb{Z}$ there exists only finitely many nonzero terms $T_{i,j}$ such that $i + j = n$.*

Example. First and Third Quadrant complexes are bounded.

Remark. If T is bounded, then $Tot^{\prod}(T) = Tot^{\oplus}(T)$ and we simply write them as $Tot(T)$.

Acyclic Assembly Lemma. *Let $\mathcal{C}$ be a bounded double complex such that either all of the rows are exact or all of the columns are exact. Then $Tot(C)$ is exact.*

Proof. By interchanging rows and columns, it suffices to prove in the case when all the columns are exact. By shifting the double complex to the left or right, it is enough to show $H_0(Tot(C)) = 0$. By shifting C along $i + j = 0$, we can assume

$C_{i,-i} = 0$ for $i < 0$ and, since bounded, for $i > n$ as well for some n. Thus, any element of $Tot(C)_0$ can be represented as $(c_0, ..., c_n)$ where $c_i \in C_{i,-i}$. Let $c = (c_0, ..., c_n) \in Z_0(Tot(C))$. Then $d^v(c_{i-1}) + d^h(c_i) = 0$ for all $i = 0, ..., n$. We want to show there exists $b = (b_0, ..., b_n) \in Tot(C)$ where $b_i \in C_{i,-i+1}$ such that $d(b) = c$, that is $d^v(b_i) + d^h(b_{i+1}) = c_i$ for $i = 0, ..., n$. Define b_i inductively. Let $b_{n+1} = 0$. As c is a cycle, $d^v(c_n) = 0$. As columns are exact, there exists $b_n \in C_{n,-n+1}$ such that $d^v(b_n) = c_n$. Thus $d^v(b_n) + d^n(b_{n+1}) = c_n$. So suppose there exists $b_n, b_{n-1}..., b_{j+1}$ such that $d^v(b_i) + d^h(b_{i+1}) = c_i$ for $j + 1 \le i \le n$. Notice

$$\begin{aligned} d^v(c_j - d^h(b_{j+1})) &= d^v(c_j) - d^v d^h(b_{j+1}) \\ &= d^v(c_j) + d^h d^v(b_{j+1}) \\ &= d^v(c_j) + d^h(c_{j+1} - d^h(b_{j+2})) \\ &= d^v(c_j) + d^h(c_{j+1}) = 0 \end{aligned}$$

As the columns are exact, there exists $b_j \in C_{j,-j+1}$ such that $d^v(b_j) = c_j - d^h(b_{j+1})$. Note $d^h(b_0) = 0$, thus the induction must end. □

Exercise.

1. Let R be a commutative ring, M, N $R-$modules. Prove $Ann_R Tor_i^R(M, N) \supseteq Ann_R M + Ann_R N$ for all i.

   Proof. We want to show $Ann_R M, Ann_R N \subseteq Ann_R Tor_i^R(M, N)$. Let $P.$ be a projective resolution for N. Then $Tor_i^R(M, N) = H_i(M \otimes_R P.) = \ker d_{i+1}/im d_i$. Thus, elements are of the form $\sum \overline{m \otimes p}$ where $m \in M, p \in P_i$. Let $r \in Ann_R M$. Then $r(m \otimes p) = (rm) \otimes p = 0 \otimes p = 0$. Thus $r \in Ann_R Tor_i^R(M, N)$. Similarly, as $Tor_i^R(M, N) = H_i(Q. \otimes_R N)$ where $Q.$ is a projective resolution of M, we see $Ann_R N \subseteq Ann_R Tor_i^R(M, N)$. □

2. Let $F : \mathcal{A} \to \mathcal{D}$ be an exact covariant functor on abelian categories. Prove that for any chain complex C in $\mathcal{A}$, $H_i(F(C)) \cong F(H_i(C))$ for all i.

   Proof. Suppose we have that the sequence $0 \to \ker f \xrightarrow{i} A \xrightarrow{f} B \xrightarrow{\pi} coker f \to 0$ is exact. Then, as F is exact, the top row of the following diagram is exact.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & F(\ker f) & \xrightarrow{F(i)} & F(A) & \xrightarrow{F(f)} & F(B) & \xrightarrow{F(\pi)} & coker f & \longrightarrow & 0 \\ & & & & \downarrow \cong & & \downarrow \cong & & & & \\ 0 & \longrightarrow & \ker F(f) & \xrightarrow{\tilde{i}} & F(A) & \xrightarrow{F(f)} & F(B) & \xrightarrow{\tilde{\pi}} & coker F(f) & \longrightarrow & 0 \end{array}$$

   By universality of the kernel and cokernel, we get induced maps $\tau : F(\ker f) \to \ker F(f)$ and $\sigma : coker F(f) \to F(coker f)$. By the Five Lemma, they must in fact be isomorphisms.

   Now, note that $0 \to \ker d_{n+1} \to im d_n \to H_n(C) \to 0$ is exact, and thus the following rows are exact:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & F(\ker d_{n+1}) & \longrightarrow & F(im d_n) & \longrightarrow & F(H_n(C)) & \longrightarrow & 0 \\ & & \downarrow \cong & & & & \downarrow \cong & & \\ 0 & \longrightarrow & \ker F(d_{n+1}) & \longrightarrow & im F(d_n) & \longrightarrow & H_n(F(C)) & \longrightarrow & 0 \end{array}$$

   By exactness (lift, push, push), we get $\alpha : F(H_n(C)) \to H_n(F(C))$. By the Five Lemma (one can show that the diagram above commutes as the isomorphisms are natural), α must be an isomorphism. □

3. Let $\{M_i\}_{i \in I}$ be a direct system of right $R-$modules over a directed index set I. Prove that for all left $R-$modules, $Tor_j^R\left(\varinjlim_{i \in I} M_i, N\right) \cong \varinjlim_{i \in I} Tor_j^R(M_i, N)$ for all j.

   Proof. Let $P.$ be a projective resolution of N. Then $\varinjlim(M_i \otimes_R P.)$ is isomorphic to the chain complex $\cdots \to \varinjlim M_i \otimes P_k \to$

$\varinjlim M_i \otimes_R P_{k-1} \to \cdots$. Of course, the following diagram commutes:

$$\begin{array}{ccc} \varinjlim(M_i \otimes P_k) & \longrightarrow & \varinjlim(M_i \otimes P_{k-1}) \\ \downarrow \cong & & \downarrow \\ (\varinjlim M_i) \otimes P_k & \longrightarrow & (\varinjlim M_i) \otimes P_{k-1} \end{array}$$

As it commutes in a natural way, we see $H_j(\varinjlim(M_i \otimes_R P.)) \cong H_j((\varinjlim M_i \otimes_R P)$.

Now, as I is directed, $\varinjlim$ is an exact covariant functor. Thus $H_j(\varinjlim M_i \otimes_R P) \cong \varinjlim H_j(M_i \otimes_R P)$. Hence,

$$\begin{aligned} Tor_i^R((\varinjlim M_i, N) &= H_j((\varinjlim M_i) \otimes_R P.) \\ &= H_j(\varinjlim(M_i \otimes P.)) \\ &= \varinjlim Tor_i^R(M_i \otimes P.) \end{aligned}$$

□

4. Let R be a PID, M, N $R-$modules. Prove that $Tor_i^R(M, N) = 0$ for all $i \geq 2$.

Proof. First suppose M is finitely generated. Find $f : R^n \to M$ which is onto. Then $0 \to \ker f \to R^n \xrightarrow{f} M \to 0$ is exact. Since R is a PID, submodules of free modules are free. Thus $\ker f$ is free and we have a projective resolution. Now, tensor our projective resolution with N to get $0 \to \ker f \otimes_R N \to R^n \otimes_R N \to M \otimes_R N \to 0$, which says $Tor_i^R(M, N) = 0$ for all $i \geq 2$. Now, suppose M is not finitely generated. Then $M = \varinjlim M_j$, where $\{M_j\}_{j\in J}$ is the set of finitely generated submodules of M and J is directed. So $Tor_i^R(M, N) = Tor_i^R\left(\varinjlim_{j\in J} M_j, N\right) = \varinjlim_{j\in J} Tor_i^R(M_j, N) = 0$ for all $i \geq 2$. □

5. Let R be commutative, M, N $R-$modules. Prove $Tor_i^R(M, N) \cong Tor_i^R(N, M)$ for all i.

Proof. Recall that $P. \otimes_R N \cong N \otimes_R P.$. Thus $Tor_i^R(M, N) = H_i(P. \otimes_R N) \cong H_i(P. \otimes_R N) \cong H_i(N \otimes_R P.) = Tor_i^R(N, M)$. □

6. Let R be a commutative domain, Q its field of fractions. Prove that for all $R-$modules M, $Tor_1^R(Q/R, M) = T(M)$, the torsion submodule of M.

Proof. Note that $0 \to R \xrightarrow{i} Q \xrightarrow{\pi} Q/R \to 0$ is exact. Since $Tor_i^R(-, M) = H_i(- \otimes_R P.)$ is a left derived functor, we have the following sequence is exact

$$\cdots \to \underbrace{Tor_1^R(Q, M)}_{=0 \ as \ Q \ is \ flat} \to Tor_1^R(Q/R, M) \to \underbrace{Tor_0^R(R, M)}_{=R\otimes_R M = M} \to \underbrace{Tor_0^R(Q, M)}_{=Q\otimes_R M = M_{(0)}} \to \cdots$$

(as $Q = R_{(0)}$). Thus we have the exact sequence $0 \to Tor_1^R(Q/R, M) \xrightarrow{\alpha} M \xrightarrow{\beta} M_0$. This says α is injective, and thus $Tor_1^R(Q/R, M) \cong im(\alpha) = \ker \beta$. Now, $m \in \ker \beta$ if and only if $\frac{m}{1} = \frac{0}{1}$ which is if and only if there exists $r \in R \setminus \{0\}$ such that $rm = 0$ (i.e., $m \in T(M)$). Thus $\ker \beta = T(M)$ and $Tor_1^R(Q/R, M) \cong T(M)$. □

Two Homology Filtrations on a double complex

Let C be a double complex, $Z_{p,q}^v(C) = \ker d_{p,q}^v$ and $B_{p,q}^v(C) = imd_{p,q+1}^v$. By anticommutativity, $d^h(Z_{p,q}^v) \subseteq Z_{p-1,q}^v$ and $d^h(B_{p,q}^v) \subseteq V_{p-1,q}^v$. Similarly, if we let $Z^h(C) = \ker d_{p,q}^h$ and $B_{p,q}^h(C) = imd_{p+1,q}^h$, then $d^v(Z_{p,q}^h) \subseteq Z_{p,q-1}^h$ and $d^v(B_{p,1}^h) \subseteq B_{p,q-1}^h$. Thus we get an induced map $\overline{d^h} : H_{p,q}^v(C) = Z_{p,q}^v(C)/B_{p,q}^v(C) \to H_{p-1,q}^v(C)$. So for each q, we get a chain complex $\cdots \xleftarrow{\overline{d^h}} H_{p-1,q}^v(C) \xleftarrow{\overline{d^h}} H_{p,q}^v(C) \xleftarrow{\overline{d^h}} \cdots$. Denote this complex as $H_Q^v(C)$. Let $H_p^h H_q^v(C) = H_p(H_q^v(C)) = \ker \overline{d_{p,q}^h}/\overline{imd_{p+1,q}^h}$.

Similarly, let $H_p^h(C)$ denote the complex

$$\begin{array}{c} \vdots \\ \downarrow \\ H^h_{p,q+1}(C) \\ \downarrow \overline{d^v} \\ H^h_{p,q}(C) \\ \downarrow \\ \vdots \end{array}$$

Define $H_q^v H_p^h(C) = H_q(H_p^h(C))$.

If C is a first quadrant double complex, then both "filtrations" $H_p^h H_q^v(C)$ and $H_q^v H_p^h(C)$ "converge" to $H_{p+q}(Tot(C))$. [To understand the meaning of converge, refer to spectral sequences in Wiebel.]

Theorem 2.33. *Let C be a first quadrant double complex of $R-$modules. Suppose $H^v_{p,q}(C) = 0$ for all $q > 0$ (that is, the columns of C are exact, except maybe at $q = 0$). Then $H_q^h H_0^v(C) \cong H_p(Tot(C))$. Similarly, if $H^h_{p,q}(C) = 0$ for all $p > 0$, then $H_q^v H_0^h(C) \cong H_q(Tot(C))$. Hence, if $H^v_{p,q} = 0$ for all $q > 0$ and $H^h_{p,q} = 0$ for all $p > 0$, then $H_p^h H_0^v(C) = H_p^v H_0^h(C)$.*

To prove this theorem, we need a few results. But first, we will consider the consequences of the theorem.

Recall. If M is a right $R-$module, N a left $R-$module, $P.$ a projective resolution of M and $Q.$ a projective resolution for N, then $Tor_i^R(M, N) = H_i(M \otimes_R Q.)$ and $\underline{Tor}_i^R = H_i(P. \otimes_R N)$.

Corollary 2.34. *For all i, $Tor_i^R(M, N) = \underline{Tor}_i^R(M, N)$.*

Proof. With the notation above, let T be the double complex $\{P_p \otimes Q_q\}_{p,q\in\mathbb{Z}}$. Then T is first quadrant. The q^{th} row of T is $P. \otimes_R Q.$. We know $P. \to M \to 0$ is exact. As Q_q is projective (and thus flat), $P. \otimes Q_q \to M \otimes Q_q \to 0$ is exact. Thus $H^h_{p,q}(T) = \begin{cases} 0, & \text{if } p > 0 \\ M \otimes_R Q_q, & \text{if } p = 0. \end{cases}$ Then the complex $H_0^h(T)$ is $\cdots \to M \otimes_R Q_{q+1} \to M \otimes_R Q_q \to M \otimes_R Q_{q-1} \to \cdots$, that is, $H_0^h(T) = M \otimes_R Q.$. Therefore, $H_q^v H_0^h(T) = H_q(M \otimes_R Q.) = Tor_q(M, N)$. Using the facts that $Q. \to N \to 0$ and $P.$ is flat, we get $H^v_{p,q}(T) = \begin{cases} 0, & \text{if } q > 0, \\ P_p \otimes_R N, & \text{if } q = 0. \end{cases}$ Similarly, $H_p^h H_0^v(T) = H_p(P \cdot \otimes_R N) = \underline{Tor}_p(P. \otimes_R N)$. ☐

Definition 2.35. *Let C, D be double complexes. A **morphism** $f : C \to D$ is a family of maps $\{f_{p,q} : C_{p,q} \to D_{p,q}\}_{p,q\in\mathbb{Z}}$ such that $fd^v = d^v f$ and $fd^h = d^h f$.*

Remark. If $f : C \to D$ is a morphism of double complexes, then it induces a chain map $\widetilde{f} : Tot(C) \to Tot(D)$ (define it componentwise). It is a chain map as $f(d^v + d^h) = (d^v + d^v)f$.

Lemma 2.36. *Let C be a first quadrant double complex and D a chain complex (consider it as a double complex with only one nonzero row). Suppose $f : C \to D$ is a map of double complexes. Then we get the induced map $\widetilde{f} : Tot(C) \to Tot(D) = D$. Let T be the double complex obtained by adjoining D to the $q-1$ row of C with the differential in D multiplied by -1 and let $d^v_{p,0} : T_{p,0} = C_{p,0} \to T_{p,-1} = D_{p,0}$ be the map $f_{p,0}$. The $cone(\widetilde{f}) = Tot(T)[-1]$.*

Proof. Note that $cone(\widetilde{f})_n = Tot(C)_{n-1} \oplus TotD_n = \bigoplus_{p+q=n-1} C_{p,q} \oplus D_n$ and $cone(\widetilde{f})_{n-1} = \bigoplus_{p+q=n-2} C_{p,q} \oplus D_{n-1}$. Note that we get maps $-d : \bigoplus_{p+q=n-1} C_{p,q} \to \bigoplus_{p+q=n-2} C_{p,q}$, $-\widetilde{f} : \bigoplus_{p+q=n-1} C_{p,q} \to D_{n-1}$, and $d : D_n \to D_{n-1}$. Then, our map from $cone(\widetilde{f})_n \to cone(\widetilde{f})_{n-1}$ is defined by $(c, d_n) \mapsto (-d(c), d(d_n) - \widetilde{f}(c)) = (-d(c), d(d_n) - f(c_{n-1,0}))$ where $c = (c_{0,n-1}, c_{1,n-2}, ..., c_{n-1,0}) \in Tot(C)_{n-1}$.

Similarly, we have $Tot(T)[-1]_n = Tot(T)_{n-1} = \bigoplus_{p+q=n-1} C_{p,q} \oplus D_n$ and $Tot(T)_{n-1} = \bigoplus_{p+q=n-2} C_{p,q} \oplus D_{n-1}$. Note that we now have maps $-d_{Tot(T)} : Tot(T)_{n-1} \to Tot(T)_{n-2}$, $-d : \bigoplus_{p+q=n-1} C_{p,q} \to \bigoplus_{p+q=n-2} C_{p,q}$, $-f : \oplus C_{p,q} \to D_{n-1}$ and $-d : D_n \to D_{n-1}$. One can see that $-d_{Tot(T)}$ will be defined above. Thus $cone(\widetilde{f}) = Tot(T)[-1]$. ☐

Proof. (Of Theorem) Suppose C is a first quadrant double complex and suppose the columns are exact except at $p = 0$. Let $f_{p,0} : C_{p,0} \to H^v_{p,0} = C_{p,0}/imd'_{p,1}$ be the natural surjection. Consider $H^v_0(C)$ as a double complex concentrated in row $q = 0$ and $f : C \to H^v_0(C)$ defined as above.

$$\begin{array}{ccccccc}
0 \leftarrow & C_{0,0} & \leftarrow & C_{1,0} & \xleftarrow{d^h} & C_{2,0} & \leftarrow \\
 & \downarrow & & \downarrow & & \downarrow & \\
0 \leftarrow & H_{0,0}(C) & \leftarrow & H_{1,0}(C) & \xleftarrow{\overline{d^h}} & H_{2,0}(C) & \leftarrow \\
 & \downarrow & & \downarrow & & \downarrow & \\
 & 0 & & 0 & & 0 &
\end{array}$$

This gives a chain map $\widetilde{f} : Tot(C) \to H^v_0(C)$. Recall $\widetilde{f}$ is a quasi-isomorphism if and only if $cone(\widetilde{f})$ is exact. Now, let T be the double complex obtained by putting $H^v_0(C)$ into row $q = -1$, multiply the differential by -1, and keeping C in the first quadrant, with $d^v_{p,0} = f_{p,0}$ for all p. Then T is a bounded complex where all of the columns are exact. By the acyclic assembly lemma, $Tot(T)$ is exact. Thus $Tot(T)[-1]$ is exact, and thus by the lemma, $cone(\widetilde{f})$ is exact. Therefore, $\widetilde{f}$ is a quasi-isomorphism. Thus $H_p(Tot(C)) \cong H^h_p H^v_0(C)$ for all p. □

Exercise. If R is a Noetherian, commutative ring, M, N finitely generated $R-$modules, then $Tor^R_i(M, N)$ is finitely generated for all i.

Recall. Let (R, m) be a commutative local Noetherian ring, M a finitely generated $R-$module. TFAE

1. $\lambda_R(M) < \infty$
2. M is Artinian and Noetherian
3. $\sqrt{Ann(M)} \supseteq m$
4. $R/Ann(M)$ is zero-dimensional
5. $M_p = 0$ for all $p \neq m$.

Proposition 2.37. *If $\lambda(Tor^R_0(M, N)) < \infty$, then $\lambda(Tor^R_i(M, N)) < \infty$ for all i.*

Proof. Suppose $\lambda(M \otimes N) = \lambda(Tor^R_0(M, N)) < \infty$. Then $m \subseteq \sqrt{Ann_R(M \otimes_R N)} = \sqrt{Ann_R M + Ann_R N}(*)$. By the above exercise, $Tor^R_i(M, N)$ is finitely generated for all $i \geq 0$. By exercise 1 above, $AnnTor^R_i(M, N) \supseteq AnnM + AnnN$. By $(*)$, we have $\sqrt{Ann_R Tor^R_i(M, N)} \supseteq m$, which implies $\lambda(Tor^R_i(M, N)) < \infty$ for all i. □

Note. If M or N has a projective resolution of finite length, then $Tor^R_i(M, N) = 0$ for $i >> 0$.

Definition 2.38. *Let (R, m) be a local commutative ring, M, N finitely generated $R-$modules such that $\lambda(M \otimes_R N) < \infty$. Suppose M or N has a projective resolution of finite length. Then the* ***intersection multiplicity*** *$\chi(M, N)$ is defined to be $\chi(M, N) = \sum_{i=0}^{\infty}(-1)^i \lambda(Tor^R_i(M, N))$.*

Conjectures.

1. Non-negativity: $\chi(M, N) \geq 0$
2. Vanishing: $\chi(M, N) = 0$ if $\dim_R M + \dim_R N < \dim R$ where $\dim_R M = \dim(R/AnnM)$.
3. Nonvanishing: $\chi(M, N) \neq 0$ if $\dim M + \dim N = \dim R$.

Serre proved the above three conjectures in the case that R is a regular local ring.

Proposition 2.39. *Let R be a ring, F a right $R-$module. TFAE*

1. *F is flat.*
2. *$Tor^R_i(F, N) = 0$ for all $i \geq 1$ and left $R-$modules N.*
3. *$Tor^R_i(F, N) = 0$ for all finitely generated $R-$modules.*

Proof. (1) ⇒ (2) ⇒ (3) is clear. So we shall just prove (3) ⇒ (1). For an arbitrary module N, note that $N = \varinjlim_{N_i \ f.g.} N_i$. So $Tor_i^R(F,N) = \varinjlim Tor_i^R(F,N_i) = 0$ (Exercise 3 above works both ways as $-\otimes_R N$ is also a left adjoint and thus $\varinjlim(M_i \otimes N) \cong (\varinjlim M_i) \otimes N$.) Now, it is enough to show $F \otimes_R -$ preserves short exact sequences. Let $0 \to A \to B \to C \to 0$ be a short exact sequence. Then applying $F\otimes_R -$, we get a long exact sequence on Tor: $0 = Tor_1^R(F,C) \to F\otimes_R A \to F \otimes_R B \to F \otimes_R C \to 0$. But then $0 \to F \otimes_R A \to F \otimes_R B \to F \otimes_R C \to 0$ is exact, which says F is flat. □

Proposition 2.40. *Let $f : R \to S$ be a ring homomorphism. Suppose R is commutative and S is a flat $R-$algebra. Then for all $R-$modules M, N we have $Tor_i^S(M \otimes_R S, N \otimes_R S) \cong (Tor_i^R(M,N)) \otimes_R S$.*

Proof. Let $P.$ be a projective resolution of M. Then $P. \otimes_R S$ is a projective resolution of $M \otimes_R S$ as an $S-$module. So $Tor_i^S(M \otimes_R S, N \otimes_R S) = H_i((P \otimes_R S) \otimes_S (N \otimes_R S)) = H_i((P \otimes_R N) \otimes_R S) = H_i(P \otimes_R N) \otimes S = Tor_i^R(M,N) \otimes_R S$ as $- \otimes_R S$ is an exact functor (as S is flat). □

Corollary 2.41. *Let R be commutative, W a multiplicatively closed subset of R, M, N $R-$modules. Then $Tor_i^{R_W}(M_W, N_W) = (Tor_i^R(M,N))_W$ for all $i \geq 0$.*

Exercise. Let R be a commutative ring and I, J ideals. Then $Tor_i^R(R/I, R/J) \cong I \cap J/IJ$.

Proof. As $0 \to I \to R \to R/I \to 0$ is exact, we get the following long exact sequence on Tor:

$$\cdots \to \underbrace{Tor_1^R(R,R/J)}_{=0 \ as \ R \ is \ flat} \to Tor_1^R(R/I,R/J) \to \underbrace{Tor_0^R(I,R/J)}_{\cong I \otimes_R R/J} \xrightarrow{i\otimes 1} \underbrace{Tor_0^R(R,R/J)}_{\cong R \otimes R/J \cong R/J} \to \cdots$$

which yields the exact sequence $0 \to Tor_1^R(R/I, R_j) \xrightarrow{\psi} I/IJ \xrightarrow{\phi} R/J$ where $\phi(i + IJ) = i + J$. Then, ψ is injective, and $Tor_1^R(R/I, R/J) \cong im\psi = \ker\phi = I \cap J/IJ$. □

As a result, since every $R-$module is projective and thus flat in a semisimple ring, we see $0 = Tor_1^R(R/I,R/J) = I \cap J/IJ$, which implies $I \cap J = IJ$ and $I = I^2$.

Exercise. Let $f : R \to S$ be a homomorphism of commutative rings. Let $x_1, ..., x_n \in R$. Prove $K.(x_1, .., x_n) \otimes_R S \cong K.(f(x_1), ..., f(x_n))$.

Proof. This is easily proven for the $n = 1, 2$ case, however it gets significantly more complicated after that. The statement, however, is true. □

Exercise. Let S be a commutative ring, $x_1, ..., x_n \in S$. Let $R = S[T_1, ..., T_n]$ be a polynomial ring in n variables over S. Define a ring homomorphism $f : R \to S$ by $T_i \mapsto x_i$. Prove $H_i(x_1, ..., x_n) \cong Tor_i^R(R/(T_1, ..., T_n), S)$.

Proof. By a previous exercise, we know $K.(T_1, ..., T_n) \otimes_S R \cong K.(x_1, ..., x_n)$. Note that $K.(T_1, ..., T_n)$ is a free resolution as $T_1, ..., T_n$ are variables. Also, T_i is a non-zero-divisor of $S/(T_1, ..., T_{i-1})$, which implies $T_1, ..., T_n$ is a regular sequence. Now, $Tor_i^R(S/(T_1, ..., T_n), R) = H_i(K(T_1, ..., T_n) \otimes R) = H_i(K.(x_1, ..., x_n)) = H_i(x_1, ..., x_n)$. □

This shows the following:

1. $H_i(x_1, ..., x_n) \cong H_i(x_{\sigma(1)}, ..., x_{\sigma(n)})$ for all $\sigma \in S_n$.

2. If R is Noetherian and $x_1, ..., x_n \in J(R)$ is a regular sequence, then $x_{\sigma(1)}, ..., x_{\sigma(n)}$ is a regular sequence.

Exercise. Let R be a commutative domain, which is not a field. Then $Q(R)$ is not projective.

Proof. Suppose Q is projective. Then there exists Q' such that $Q \oplus Q' = \bigoplus_{i \in I} R$. So we can find $i : Q \hookrightarrow \bigoplus_{i\in I} R$ which is nonzero. Then there exists a nonzero component, say $j \in I$ and so $\rho := \pi_j i : Q \to R$ is nonzero. Let $r = \rho(1)$. As ρ is nonzero, there exists some $\frac{a}{b} \in Q$ such that $\rho(\frac{a}{b}) \neq 0$. Then $\frac{1}{b}\rho(a) \neq 0$, which says $a\rho(1) = \rho(a) \neq 0$. Thus $\rho(1) \neq 0$. Now $r = n\rho(\frac{1}{n}) \in (n)$ for all $(n) \in R \setminus \{0\}$. Assume $r \neq 0$. Then $r \in (r^2)$, which says $r = sr^2$ for some $s \in R$. Then $(1 - sr)r = 0$, which says r is a unit as R is a domain. As R is not a field, there exists $v \in R$ which is not a unit. Then, $r \in (v) \neq R$, a contradiction. Thus Q is not projective. □

Theorem 2.42. *Let R be a Noetherian commutative ring, $x_1, ..., x_n \in J(R)$. Then $x_1, ..., x_n$ is a regular sequence if and only if $H_1(x_1, ..., x_n) = 0$.*

Proof. We have already proved the forward direction, so suppose $H_1(x_1, ..., x_n) = 0$ and induct on n. If $n = 1$, then $0 = H_1(x_1) \cong (0 : x_1)$. Thus x_1 is a non-zero-divisor on R, which says it is a regular sequence. So suppose true for $n-1$ elements. Consider the long exact sequence on Koszul homology: $\cdots \to H_1(x_1, ..., x_n) \xrightarrow{x_n} H_1(x_1, ..., x_{n-1}) \to H_1(x_1, ..., x_n) = 0$. This says $H_1(x_1, ..., x_{n-1}) = x_n H_1(x_1, ..., x_{n-1}) \subseteq J(R)H_1(x_1, ..., x_{n-1})$. Now, as R is Noetherian, $H_1(x_1, ..., x_{n-1})$ is finitely generated. By NAK, $H_1(x_1, ..., x_{n-1}) = 0$. Thus $x_1, ..., x_{n-1}$ is a regular sequence. Now, consider $0 = H_1(x_1, ..., x_n) \to \underbrace{R/(x_1, ..., x_{n-1}}_{=H_0(x_1,...,x_{n-1})} \xrightarrow{x_n} R/(x_1, ..., x_{n-1}) \to \cdots$. As this sequence is exact, x_n is a non-zero-divisor in $R/(x_1, ..., x_n)$. Hence, $\{x_1, ..., x_n\}$ is a regular sequence. ☐

Definition 2.43. *A **flat resolution** of an $R-$module M is a complex $F.$ such that*

1. *$F_i = 0$ for all $i < 0$*
2. *F_i is flat for all i*
3. *$H_i(F.) = \begin{cases} M, & \text{if } i = 0 \\ 0, & \text{if } i \neq 0. \end{cases}$. If $F_n \neq 0$ and $F_i = 0$ for all $i > n$, we say $F.$ has length n.*

Remark. Any projective resolution of M is a flat resolution. Hence flat resolutions always exist.

Lemma 2.44 (Dimension Shifting Lemma). *Let R be a ring and $0 \to C \to F_{n-1} \to \cdots \to F_0 \to M \to 0$ be exact, where F_i is flat for all i. Then, for all $i \geq 1$, $Tor_i^R(C, N) \cong Tor_{n+i}(M, N)$ for all $R-$modules N.*

Proof. We will induct on n. For $n = 0$, we have $0 \to C \to F_0 \to M \to 0$ is exact. From the long exact sequence on Tor, this gives us $\cdots \to \underbrace{Tor_{i+1}^R(F_0, N)}_{=0} \to Tor_{i+1}^R(M, N) \to Tor_i^R(C, N) \to \underbrace{Tor_i^R(F_0, M)}_{=0 \text{ if } i \geq 1} \to \cdots$, which says $Tor_{i+1}^R(M, N) \cong Tor_i^R(C, N)$. So suppose true for $n-1$. Let $D = \ker(F_{n-1} \to F_{n-2}) = coker(C \to F_{n-1})$. By induction, $Tor_{n+i-1}(M, N) \cong Tor_i^R(D, N)$ for all $i \geq 1$. By the $n = 1$ case, we have $Tor_{i+1}^R(D, N) \cong Tor_i^R(C, N)$ for all $i \geq 1$. ☐

Proposition 2.45 (Flat Resolution Lemma). *Let R be a ring, M a right $R-$module, N a left $R-$module. Let $F.$ be a flat resolution of M. Then $Tor_i^R(M, N) \cong H_i(F. \otimes_R N)$ for all $i \geq 0$.*

Proof. We will induct on i. If $i = 0$, then $\cdots \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0 \to M \to 0$ is exact. By the right exactness of tensor products, this says $F_1 \otimes_R N \to F_0 \otimes_R N \to M \otimes_R N$ is exact. Hence $H_0(F. \otimes_R N) \cong M \otimes_R N \cong Tor_0^R(M, N)$. Let K_1 be the kernel of the map $F_0 \to M$. By exactness, $K_1 = coker(F_2 \to F_1)$. So we have the exact sequences $0 \to K_1 \xrightarrow{i} F_0 \to M \to 0$ and $F_2 \xrightarrow{d_2} F_1 \to K_1 \to 0$. Tensor with N to get the long exact sequence $\underbrace{Tor_1^R(F_0, N)}_{=0} \to Tor_1^R(M, N) \to \underbrace{Tor_0^R(K_1, N)}_{=K_1 \otimes_R N} \to \underbrace{Tor_0^R(F_0, N)}_{=F_0 \otimes_R N}$. We also have that $F_2 \otimes_R N \xrightarrow{d_2 \otimes 1} F_1 \otimes_R N \to K_1 \otimes_R N \to 0$ is exact. So consider the commutative diagram

$$\begin{array}{ccccccc} & & F_2 \otimes_R N & \xrightarrow{d_2 \otimes 1} & F_1 \otimes_R N & \longrightarrow & K_1 \otimes_R N & \longrightarrow & 0 \\ & & \downarrow \phi & & \downarrow = & & \downarrow i \otimes 1 & & \\ 0 & \longrightarrow & \ker(d_1 \otimes 1) & \longrightarrow & F_1 \otimes_R N & \longrightarrow & F_0 \otimes_R N & & \end{array}$$

By the Snake Lemma, $Tor_1^R(M, N) \cong \ker(i \otimes 1) \cong coker\phi \cong H_1(F. \otimes_R N)$.

Now, suppose the theorem holds up to i. By dimension shifting, we know $Tor_{i+1}^R(M, N) \cong Tor_i^R(K_1, N)$. Note $F'_. := \cdots F_2 \xrightarrow{d_2} F_1 \to 0$ is a flat resolution of K_1. Thus $Tor_i^R(K_1, N) \cong H_i(F'_. \otimes_R N) \cong H_{i+1}(F. \otimes_R N)$ for $i \geq 1$. ☐

Note. A similar result holds if one takes a flat resolution of N.

Definition 2.46. *Let R be a ring and M an $R-$module. The **flat dimension** of M is*

$$fd_R M := inf\{n | M \text{ has a flat resolution of length } n\}.$$

Theorem 2.47. *Let M be an $R-$module. TFAE*

1. $fd_R M \leq n$

2. *For every exact sequence* $F_{n-1} \xrightarrow{d_{n-1}} F_{n-2} \to \cdots \to F_0 \to M \to 0$ *such that* F_i *is flat for all* i, $\ker d_{n-1}$ *is flat.*

3. $Tor^R_{n+1}(M,N) = 0$ *for all* $R-$*modules* N.

Proof. Note that (2) $\Rightarrow$ (1) $\Rightarrow$ (3) is clear, So suppose (3) holds. Let $F_n = \ker d_{n-1}$. Then $0 \to F_n \to F_{n-1} \xrightarrow{d_{n-1}} \cdots \to F_0 \to M \to 0$ is exact. By the dimension shifting lemma, $Tor^R_1(F_n, N) \cong Tor^R_{n+1}(M,N) = 0$ for all $R-$modules N. Hence, F_n is flat. □

Corollary 2.48. *Let* R *be a ring. Then* $fd_R M \leq n$ *for all finitely generated* $R-$*modules* M *if and only if* $fd_R M \leq n$ *for all* $R-$*modules* M.

Proof. We need only prove the forward direction. Let M be an $R-$modules and recall $M = \varinjlim_{M'\ f.g.} M'$. Hence, for all $R-$modules N, $Tor^R_{n+1}(M,N) = \varinjlim Tor^R_{n+1}(M',N) = 0$. □

Proposition 2.49. *Let* (R,m) *be a commutative quasi-local ring and* M *a finitely presented* $R-$*module. Let* $R^m \xrightarrow{\phi} R^n \xrightarrow{\epsilon} M \to 0$ *be a finite presentation. TFAE*

1. $\phi(R^m) \subseteq mR^n$

2. $\phi \otimes 1 : R^m \otimes_R R/m \to R^n \otimes_R R/m$ *is the zero map.*

3. $n = \mu_R(M)$

4. $\ker \epsilon \subseteq mR^n$.

Proof. Note that (1) $\Leftrightarrow$ (4) as the sequence is exact. To show (1) $\Leftrightarrow$ (2), consider the following diagram with exact rows:

$$\begin{array}{ccccccc} R^m \otimes_R R/m & \xrightarrow{\phi\otimes 1} & R^n \otimes_R R/m & \longrightarrow & M \otimes R/m & \longrightarrow & 0 \\ \downarrow \cong & & \downarrow \cong & & \downarrow \cong & & \\ R^m/mR^n & \xrightarrow{\overline{\phi}} & R^n/mR^n & \longrightarrow & M/mM & \longrightarrow & 0 \end{array}$$

Now, $\phi \otimes 1 = 0$ if and only if $\overline{\phi} = 0$ which is if and only if $\phi(R^m) \subseteq mR^n$.

To show (2) $\Leftrightarrow$ (3), note that $\phi \otimes 1 = 0$ if and only if $\overline{\phi} = 0$ which is if and only if $R^n/mR^n \cong M/mM$ which is if and only if $\mu_R(m) = n$ by NAK. □

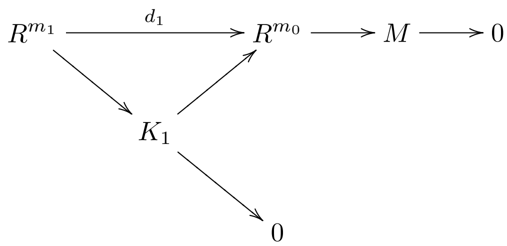
Definition 2.50. *Let* (R,m) *be a commutative Noetherian local ring and* M *a finitely generated* $R-$*module. A* ***minimal free resolution*** *of* M *is a resolution* $F.$ *of* M *such that*

1. F_i *is free of finite rank for all* i.

2. $d_i(F_i) \subseteq mF_{i-1}$ *for all* i.

Lemma 2.51. *Minimal free resolutions exist.*

Proof. Consider the following diagram, where $m_0 = \mu_R(M)$ and $m_1 = \mu_R(K_1)$:



Since $m_0 = \mu_R(M)$, we see $K_1 \subseteq mR^{m_0}$ by the above proposition. Thus $d_1(R^{m_1}) \subseteq mR^{m_0}$. Let $K_2 = \ker d_1$ and let $\widetilde{d_2} = R^{m_2} \to K_2$ be a surjective homomorphism where $m_2 = \mu_R(K_2)$. Let $d_2 : R^{m_2} \to R^{m_1}$ be the composition $R^{m_2} \xrightarrow{d_2} K_2 \hookrightarrow R^{m_1}$. Then $R^{m_2} \xrightarrow{d_2} R^{m_1} \xrightarrow{d_1} R^{m_0} \to M$ is exact and minimal. Now, repeat with $K_3 = \ker d_2$. □

Lemma 2.52. *Let (R, m) be local and $\phi : F \to G$ a map of finitely generated free $R-$modules. Then ϕ is an isomorphism if and only if $\overline{\phi} : F/mF \to G/mG$ is an isomorphism.*

Proof. Let $K = \ker\phi, C = coker\phi$. Since $F/mF \xrightarrow{\overline{\phi}} G/mG \to C/mC \to 0$ is exact and $\overline{\phi}$ is surjective, we have $C = mC$. By NAK, $C = 0$. Now, we have $0 \to K \to F \xrightarrow{\phi} G \to 0$ is exact. As G is free, this sequence splits. Thus $0 \to K/mK \to F/mF \xrightarrow{\overline{\phi}} G/mG \to 0$ is exact. As $\overline{\phi}$ is an isomorphism, $K = mK$ and thus $K = 0$ by NAK. □

Theorem 2.53. *Let (R, m) be a local ring and $f : M \to N$ an isomorphism of finitely generated $R-$modules. Let $F., G.$ be minimal free resolutions of M and N, respectively. Then any chain map $\phi. : F. \to G.$ lifting f is a chain isomorphism. Moreover, $\phi_i|_{B_i(F)} : B_i(F) \to B_i(G)$ is an isomorphism for all $i \geq 0$. (Recall $B_i(F) = imd_{i+1}$).*

Proof. Consider the following diagram:

$$\begin{array}{ccccccccccc}
F. : \cdots & \longrightarrow & F_i & \xrightarrow{d_i} & F_{i-1} & \xrightarrow{d_{i-1}} & \cdots & \xrightarrow{d_1} & F_0 & \longrightarrow & 0 \\
& & \downarrow{\scriptstyle \phi_i} & & \downarrow{\scriptstyle \phi_{i-1}} & & & & \downarrow{\scriptstyle \phi_0} & & \\
G. : \cdots & \longrightarrow & G_i & \xrightarrow{d'_i} & G_{i-1} & \xrightarrow{d'_{i-1}} & \cdots & \longrightarrow & G_0 & \longrightarrow & 0
\end{array}$$

We know $\phi. : F. \to G.$ lifting f exists by the comparison theorem. Now, consider the following diagram with exact rows:

$$\begin{array}{ccccccc}
F_1 & \xrightarrow{d_1} & F_0 & \xrightarrow{\epsilon} & M & \longrightarrow & 0 \\
\downarrow{\scriptstyle \phi_1} & & \downarrow{\scriptstyle \phi_0} & & \downarrow{\scriptstyle f} & & \\
G_1 & \xrightarrow{d'_1} & G_0 & \xrightarrow{\delta} & N & \longrightarrow & 0
\end{array}$$

Now, tensor with R/m to get:

$$\begin{array}{ccccccc}
F_1 \otimes_R R/m & \xrightarrow{d_1 \otimes 1} & F_0 \otimes_R R/m & \xrightarrow{\epsilon \otimes 1} & M/mM & \longrightarrow & 0 \\
\downarrow{\scriptstyle \phi_1 \otimes 1} & & \downarrow{\scriptstyle \phi_0 \otimes 1} & & \downarrow{\scriptstyle \overline{f}} & & \\
G_1 \otimes_R R/m & \xrightarrow{d'_1 \otimes 1} & G_0 \otimes_R R/m & \xrightarrow{\delta \otimes 1} & N/mN & \longrightarrow & 0
\end{array}$$

Note $d_1 \otimes 1 = d'_1 \otimes 1 = 0$ by the above proposition. Thus $\epsilon \otimes 1$ and $\delta \otimes 1$ are isomorphisms. Since $\overline{f}$ is an isomorphism, so is $\phi_0 \otimes 1$.

By the Lemma, we have ϕ_0 is an isomorphism. By exactness, we have $B_0(F) = imD_1 = \ker\epsilon$ and $B_0(G) = imd'_1 = \ker\delta$. Thus we have the diagram below where $\widetilde{\phi_0} = \phi_0|_{B_0(F)}$:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & B_0(F) & \longrightarrow & F_0 & \xrightarrow{\epsilon} & M & \longrightarrow & 0 \\
& & \downarrow{\scriptstyle \widetilde{\phi_0}} & & \downarrow{\scriptstyle \phi_0} & & \downarrow{\scriptstyle f} & & \\
0 & \longrightarrow & B_0(G) & \longrightarrow & G_0 & \longrightarrow & N & \longrightarrow & 0
\end{array}$$

By the Five Lemma, $\widetilde{\phi_0}$ is an isomorphism.

To complete the proof, apply this argument to the resolutions

$$\begin{array}{ccccccccc}
F'. : \cdots & \longrightarrow & F_2 & \xrightarrow{d_2} & F_1 & \xrightarrow{d_1} & B_0(F) & \longrightarrow & 0 \\
& & \downarrow{\scriptstyle \phi_2} & & \downarrow{\scriptstyle \phi_1} & & \downarrow{\scriptstyle \widetilde{\phi_0}} & & \\
G'. : \cdots & \longrightarrow & G_2 & \xrightarrow{d'_2} & g_1 & \xrightarrow{d'_1} & B_0(G) & \longrightarrow & 0
\end{array}$$

Since $\widetilde{\phi_0}$ is an isomorphism and $F'.$ and $G'.$ are minimal free resolutions of $B_0(F)$ and $B_0(G)$, respectively, we get ϕ_1 is an isomorphism and $\phi_1|_{B_1(F)} : B_1(F) \to B_1(G)$ is an isomorphism. Continue. □

Corollary 2.54. *Let M be a finitely generated $R-$module, where (R, m) is local. Then any two minimal free resolution of M are chain isomorphic.*

Definition 2.55. *Let (R, m) be local and M a finitely generated $R-$module. The i^{th}* ***syzygy*** *of M, denoted $syz_i(M)$, is defined to be $B_i(F)$ where F is any minimal free resolution of M.*

Exercise. Let (R, m) be local, M a finitely generated $R-$module. Let $F., G.$ be free resolutions of M, where $F.$ is minimal. Prove that $G. = F. \oplus L.$ (as chain complexes) for some exact complex of free modules $L.$.

Proof. If $i > 0$, then $0 = H_i(G.) = H_i(F.) \oplus H_i(L.)$, which says $H_i(L.) = 0$ for all $i > 0$. Consider the following diagram, where $\psi.$ and $\phi.$ are liftings of 1_M.

$$\begin{array}{ccccc} F. & \longrightarrow & M & \longrightarrow & 0 \\ \downarrow{\scriptstyle \psi.} & & \downarrow{\scriptstyle 1_M} & & \\ G. & \longrightarrow & M & \longrightarrow & 0 \\ \downarrow{\scriptstyle \phi.} & & \downarrow{\scriptstyle 1_M} & & \\ F. & \longrightarrow & M & \longrightarrow & 0 \end{array}$$

Then $\phi.\psi.$ lifts 1_M. By the theorem, $\phi.\psi.$ is a chain isomorphism. Let $g. : F. \to F.$ be the inverse of $\phi.\psi.$. Then $g.\phi. : G. \to F.$ is the splitting map for the exact sequence $0 \to F. \xrightarrow{\psi.} G\cdot \to G./\psi(F.) \to 0$ (note $\psi.$ is injective as $(g.\phi.)\psi. = 1$). Thus $G. = F. \oplus L.$, where $L. = G./\psi(F.)$. □

Exercise. Let (R, m) be local and $x_1, ..., x_n \in m$ a regular sequence. Then $K.(x_1, ..., x_n)$ is a minimal free resolution of $R/(x_1, ..., x_n)$.

Proof. We will induct on n. For $n = 1$, we have $K.(x)$ is the chain $0 \to R \to R \to 0$. Tensoring with R/m, we have $0 \to R \otimes_R R/m \to R \otimes_R R/m \to 0$ where $a \otimes \overline{b} \mapsto xa \otimes \overline{b} = a \otimes \overline{xb} = 0$ as $x \in m$. Thus $K.(x)$ is minimal by the proposition. So, suppose true for $n > 1$. Recall $K.(x_1, ..., x_{n-1}) = cone(\hat{x_n})$. Suppose we have $\phi_i : K.(x_1, ..., x_{n-1})_{i-1} \oplus K.(x_1, ..., x_{n-1})_i \to K.(x_1, ..., x_{n-1})_{i-2} \oplus K.(x_1, ..., x_{n-1})_{i-1}$ where $(a, b) \mapsto (-\phi'_{i-1}(a), \phi'_i(b) - x_n a)$. Then, applying $- \otimes R/m$ gives us

$$\begin{array}{ccc} (a, b) \otimes \overline{1} & \longrightarrow & (-\phi'_{i-1}(a), \phi'_i(b) - x_n a) \otimes \overline{1} \\ \downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle \cong} \\ (a \otimes \overline{1}, b \otimes \overline{1}) & \longrightarrow & (-\phi'_{i-1}(a) \otimes \overline{1}, (\phi'_i(b) - x_n a) \otimes \overline{1}) \end{array}$$

Now, $-\phi'_{i-1}(a) \otimes \overline{1} = 0 = \phi'_i(b) \otimes \overline{1}$ by induction and $x_n a \otimes \overline{1} = a \otimes \overline{x_n} = 0$ as $x_n \in m$. Thus $\phi_i \otimes 1$ is the zero map, which implies $K.(x_1, ..., x_n)$ is minimal by the proposition. □

Fact. If $x_1, ..., x_n \in R$ and $K.(x_1, ..., x_n)$ is the Koszul complex, then $imd_i \subseteq (x_1, ..., x_n)K_{i-1}$.

Exercise. If $x_1, ...x_n$ for a regular sequence, then $Tor_i^R(R/(x_1, ..., x_n), R/(x_1, ..., x_n)) \cong (R/(x_1, ..., x_n))^{\binom{n}{i}}$.

Definition 2.56. *Let (R, m) be local, M a finitely generated $R-$module. For $i \geq 0$, the i^{th}* ***Betti number*** *of M is defined by $\beta_i(M) = rankF_i$, where $F.$ is a minimal free resolution of M.*

By the exercise, if $x_1, ..., x_n$ form a regular sequence, then $\beta_i(R/(x_1, ..., x_n)) = \binom{n}{i}$.

Open Problem. Let (R, m) be a local ring and M a finitely generated $R-$module such that $pd_R M = n$. Then $\beta_i(M) \geq \binom{n}{i}$ for all $i \geq 0$. Note: This is called the *Buchsbaum-Eisenbud-Horrocks Conjecture.*

Proposition 2.57. *Let (R, m, k) be local, M finitely generated. Then for all $i \geq 0$, $\beta_i(M) = \dim_k Tor_i^R(M, k)$. (By exercise 1 above, as $m = annk$, $m \subseteq annTor_i^R(M, k)$, which says Tor is a $k-$module).*

Proof. Let $F.$ be a minimal free resolution of M. By definition, $F_i \cong R^{\beta_i(M)}$. So $Tor_i^R(M, k) = H_i(F. \otimes_R k)$, which yields the following exact sequence where all of the maps are the zero map (by Proposition 2.49): $\cdots \to R^{\beta_{i+1}(M)} \otimes_R k \to R^{\beta_i(M)} \otimes_R k \to \cdots$. Thus $Tor_i^R(M, k) = R^{\beta_i(M)} \otimes_R k = k^{\beta_i(M)}$. □

Corollary 2.58. *Let (R, m) be local, and M finitely generated. TFAE*

1. *$pd_R M \leq n$.*

2. *$fd_R M \leq n$.*

3. $Tor^R_{n+1}(M,k) = 0$.

Proof. (1) $\Rightarrow$ (2) as every projective module is flat. (2) $\Rightarrow$ (3) by the flat resolution lemma. (3) $\Rightarrow$ (1) as then $\beta_i(M) = 0$ for all $i \geq n+1$, which says $pd_R(M) \leq n$ by the proposition. $\square$

Corollary 2.59. *Let (R,m) be local, M finitely generated. Then $pd_R M = fd_R M = \sup\{n | Tor^R_n(M,k) \neq 0\}$.*

Theorem 2.60. *Let (R,m,k) be local. TFAE*

1. *$pd_R M < \infty$ for all finitely generated $R-$modules M.*
2. *$fd_R M < \infty$ for all $R-$modules M.*
3. *$pd_R k < \infty$.*
4. *$Tor^R_n(k,k) = 0$ for some $n \geq 0$.*
5. *For all $R-$modules N, M, there exists ℓ such that $Tor^R_n(M,N) = 0$ for all $n \geq \ell$.*

Proof. Note (2) $\Rightarrow$ (5) follows from the Flat Resolution Lemma, (5) $\Rightarrow$ (4) is clear, (4) $\Rightarrow$ (3) follows from the corollary with $M = k$, (3) $\Rightarrow$ (1) is the corollary (compute $Tor^R_i(M,k)$ using a projective resolution of k), and (1) $\Rightarrow$ (3) is clear. Thus, its enough to show (3) $\Rightarrow$ (2). Let $n = pd_R k$. Then $Tor^R_{n+1}(M,k) = 0$ for all $\ell > n$. If M is finitely generated, then $pd_R M \leq n$, which implies $Tor^R_{n+1}(M,N) = 0$ for all N. For an arbitrary M, $Tor^R_{n+1}(M,N) \cong \varinjlim_{M_i\ f.g.} Tor^R_{n+1}(M_i,N) = 0$ for all N. Thus $fd_R M \leq n$. $\square$

2.3 Regular Local Rings

Generalized Krull's Principal Ideal Theorem. *Let R be a commutative Noetherian ring. Let p be a prime ideal which is minimal over $(x_1, ..., x_n)$. Then $ht(p) \leq n$, where $ht(p) = \sup\{n|$ there exist primes $q_0 \subsetneq q_1 \subsetneq \cdots \subsetneq q_n = p\} = \dim R_p$. In particular, any prime p needs at least $ht(p)$ generators.*

If (R,m) is local, then m needs at least $ht(m) = \dim R$ generators.

Example. Let $R = k[x_1, ..., x_n]$ be a polynomial ring over a field and $m = (x_1, ..., x_n)$. Then $ht(m) \leq n$ by Krull's Principal Ideal Theorem. On the other hand, $ht(m) \geq n$ as $(x_1, ..., x_n) \supsetneq (x_1, ..., x_{n-1}) \supsetneq \cdots \supsetneq (x_1) \supsetneq (0)$ is a chain of primes. Thus $ht(m) = n$.

Definition 2.61. *A local ring (R,m) is called* ***regular*** *if $m = (x_1, ..., x_d)$ where $d = \dim R$ for some $x_1, ..., x_d \in m$.*

Examples. $k[x_1, ..., x_n]_{(x_1,...,x_n)}$, any field, and any local PID (like $\mathbb{Z}_{(p)}$) are all regular local rings.

Note. In the case $R = k[x_1, ..., x_n]_m$ (where $m = (x_1, ..., x_n)_{x_1,...,x_n}$), we have $x_1, ..., x_n$ is a regular sequence in R. So $K.(x_1, ..., x_n)$ is a minimal free resolution of $R/(x_1, ..., x_n) \cong k$. So $pd_R k \leq n$, which implies $pd_R M \leq n$ for all finitely generated $R-$modules M. Its "easy" to see that if R is a regular local ring, then $pd_R R/m < \infty$.

Theorem 2.62 (Auslander, Buchsbaum, Serre, '57)**.** *Let (R,m,k) be a local ring. TFAE*

1. *R is regular*
2. *$pd_R k < \infty$*
3. *$pd_R M < \infty$ for all finitely generated $R-$modules M.*

Corollary 2.63. *Let R be a regular local ring, $p \in SpecR$. Then R_p is a regular local ring.*

Proof. Let $M = R/p$ and $0 \to F_n \to F_{n-1} \to \cdots \to F_0 \to R/p \to 0$ be exact, where F_i are free. Then $0 \to (F_n)_p \to (F_{n-1})_p \to \cdots \to (F_0)_p \to R_p/pR_p \to 0$ is a finite free R_p- resolution of R_p/pR_p. Thus R_p is regular. $\square$

Exercise.

1. Let R be commutative, M, N $R-$modules. Let $p \in SpecR$. Consider the natural $R-$linear map $\phi : Hom_R(M,N)_p \to Hom_{R_p}(M_p, N_p)$, defined by $\frac{f}{s} \mapsto \widetilde{\frac{f}{s}}$ where $\widetilde{\frac{f}{s}} : M_p \to N_p$ is defined by $\frac{m}{t} \mapsto \frac{f(m)}{st}$. Prove that if M is finitely presented, then ϕ is an isomorphism.

Proof. First note the following

$$(Hom_R(R^m, N))_p \cong (\oplus^m Hom_R(R, N))_p \cong (\oplus^m N)_p \cong \oplus N_p \cong \oplus^m Hom_{R_p}(R_p, N_p) \cong Hom_{R_p}(R_p^m, N_p).$$

Note also that this isomorphism is natural as it is a composition of natural isomorphisms. Now, as M is finitely presented, we have $R^n \to R^m \to M \to 0$ is exact. This sequence stays exact if we localize and then Hom, or if we Hom and then localize. Thus we have the following commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & 0 & \longrightarrow & Hom_R(M,N)_p & \longrightarrow & Hom_R(R^m,N)_p & \longrightarrow & Hom_R(R^n,N)_p \\
\downarrow \cong & & \downarrow \cong & & \downarrow \phi & & \downarrow \cong & & \downarrow \cong \\
0 & \longrightarrow & 0 & \longrightarrow & Hom_{R_p}(M_p,N_p) & \longrightarrow & Hom_{R_p}(R_p^m,N_p) & \longrightarrow & Hom_{R_p}(R_p^n,N_p)
\end{array}$$

This is in fact commutative by the naturality of ϕ. By the Five Lemma, ϕ is an isomorphism. □

2. Let R be commutative, M a finitely presented $R-$module. Prove that M is a projective $R-$module if and only if M_p is a free R_p-module for all $p \in SpecR$.

   Proof. For the forward direction, note that if M is projective, then $M \oplus N = R^n$ for some $R-$module N. As localizing commutes with direct sums, this says $M_p \oplus N_p = R_p^n$, which says M_p is a projective R_p module. Of course, as M is finitely presented, M_p is and thus M_p is free as it is a finitely generated projective over a local ring. To prove the backwards direction, we wish to show $Hom_R(M, -)$ is exact. As $Hom_R(M, -)$ is left exact, it is enough to show that for any surjection $\phi : X \to Y$ that $\phi_* : Hom_R(M, X) \to Hom_R(M, Y)$ is also surjective. This is true if it is locally surjective, that is, if $\frac{\phi_*}{1} : (Hom_R(M, X))_p \to (Hom_R(M, Y))_p$ is surjective for all $p \in SpecR$. Of course, by the above exercise, we have the following commutative diagram.

$$\begin{array}{ccc}
(Hom_R(M,X))_p & \xrightarrow{\frac{\phi_*}{1}} & (Hom_R(M,Y))_p \\
\downarrow \cong & & \downarrow \cong \\
Hom_{R_p}(M_p,X_p) & \longrightarrow & Hom_{R_p}(M_p,Y_p)
\end{array}$$

   As M_p is free, it is projective and thus the bottom map is surjective. Thus $\frac{\phi_*}{1}$ is surjective for all primes p, which says ϕ_* is surjective and thus M is projective. □

3. Let $f : R \to S$ be a ring homomorphism, M a flat right $R-$module. Prove $M \otimes_R S$ is a flat right $S-$module. In particular, if R is commutative and M is flat, then M_S is flat as an R_S-module for a mcs S of R.

   Proof. Let A, B be left $S-$modules with $0 \to A \xrightarrow{g} B$ exact. Then $0 \to S \otimes_S A \xrightarrow{1 \otimes g} S \otimes_S B$ is exact. Apply $M \otimes_R -$ to get the following commutative diagram (as the columns are natural isomorphisms):

$$\begin{array}{ccccc}
0 & \longrightarrow & M \otimes_R (S \otimes_S A) & \xrightarrow{1 \otimes (1 \otimes g)} & M \otimes_R (S \otimes_S B) \\
& & \downarrow \cong & & \downarrow \cong \\
& & (M \otimes_R S) \otimes_S A & \xrightarrow{(1 \otimes 1) \otimes g} & (M \otimes_R S) \otimes_S B
\end{array}$$

   Thus $M \otimes_R S$ preserves injections, which implies it is flat. □

   Similarly, if M is flat, then M/IM is a flat $R/I-$module.

4. Let R be commutative, M a finitely presented flat $R-$module. Prove M is projective.

   Proof. First, we shall prove a claim.

   Claim. Let (R, m) be quasi-local, M a finitely presented flat $R-$module. Then M is free.

Proof. As M is finitely presented, it is finitely generated. Thus $R^n \xrightarrow{\phi} M \to 0$ is surjective for $n = \mu_R(M) = \mu_{R/m}(M/mM) = dim_{R/m}(M/mM)$. By a previous exercise, R^n finitely generated and M finitely presented implies $\ker\phi$ is finitely generated. Also, $0 \to \ker\phi \xrightarrow{i} R^n \xrightarrow{\phi} M \to 0$ is exact. This gives us the following long exact sequence on Tor.

$$\cdots \to \underbrace{Tor_1^R(M, R/m)}_{=0} \to \underbrace{Tor_0^R(\ker\phi, R/m)}_{=\ker\phi/m\ker\phi} \to \underbrace{Tor_0^R(R^n, R/m)}_{(R/m)^n} \xrightarrow{f} \underbrace{Tor_0^R(M, R/m)}_{=M/mM} \to 0$$

Clearly, f is surjective. Note it is injective as well as the domain and image are $n-$dimensional vector spaces. So $\ker\phi/m\ker\phi = 0$, which says $\ker\phi = m\ker\phi$. Recall $\ker\phi$ is finitely generated and so $\ker\phi = 0$ by Nakayama's Lemma. Thus M is free.

Now, let M be a finitely presented flat $R-$module. Then M_p is finitely presented by the previous exercise and is thus flat. Further, R_p is local and so M_p is a free R_p-module. By exercise 2 above, M is a projective $R-$module. □

This is also true in the noncommutative case, but the proof is harder.

Exercise. Find a commutative ring R and a finitely generated flat $R-$module M such that M is not projective. (Note: R can not be Noetherian).

3 Cochain Complexes

Definition 3.1. *Let $\mathcal{A}$ be an abelian category. A* ***cochain complex*** *$C^{\cdot}$ in $\mathcal{A}$ is a family of objects $\{C^p\}_{p\in\mathbb{Z}}$ and morphisms $d^p : C^p \to C^{p+1}$ for all p such that $d^{p+1}d^p = 0$ for all p. A cochain complex is written as $C^{\cdot} : \cdots \to C^p \to C^{p+1} \to C^{p+2} \to \cdots$. Let $Z^p(C) = \ker d^p$ be the $p-$**cocycles** of C and $B^p(C) = imd^{p-1}$ be the $p-$**coboundaries** of C. Also, we define $H^p(C) = Z^p(C)/B^p(C)$ to be the p^{th} **cohomology** of C.*

Remarks.

1. Any cochain complex $C^{\cdot}$ in $\mathcal{A}$ can be viewed as a chain complex $C'_{\cdot}$ by letting $C_i = C^{-i}$ and $d_i = d^{-i}$. Then $d^{-i} : C^{-i} \to C^{-i+1}$. Note $H^p(C) = H_{-p}(C')$.

2. Every cochain complex C in $\mathcal{A}$ corresponds uniquely to a chain complex in $\mathcal{A}^{op}$, where $C : \cdots C^i \to C^{i+1} \to \cdots$ maps to $C^{op} : \cdots \leftarrow C^i \leftarrow C^{i+1} \leftarrow \cdots$ (where $C^i = (C^{op})_i$).

Proposition 3.2. *Let $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ be a short exact sequence of cochain complexes. Then there exists a natural long exact sequence on cohomology: $\cdots H^p(A) \xrightarrow{f^*} H^p(B) \xrightarrow{g^*} H^p(C) \xrightarrow{\partial} H^{p+1}(A) \to \cdots$.*

Definition 3.3. *Let $\mathcal{A}$ be an abelian category. An object I of $\mathcal{A}$ is* ***injective*** *if $Hom_{\mathcal{A}}(-, I)$ is an exact functor. $\mathcal{A}$ is said to have* ***enough injectives*** *if every object can be embedded in an injective object.*

Example. $<< R - mod >>$ and $<< mod - R >>$ have enough injectives.

Note.

- $\mathcal{A}$ has enough injectives if and only if $\mathcal{A}^{op}$ has enough projectives.
- An object is injective in $\mathcal{A}$ if and only if it is projective in $\mathcal{A}^{op}$.

Definition 3.4. *Let $\mathcal{A}$ be an abelian category and $M \in Obj\mathcal{A}$. An* ***injective resolution*** *of M is a cochain complex I in $\mathcal{A}$ such that*

1. *$I^i = 0$ for all $i < 0$*
2. *I^i is injective for all i.*
3. *$H^i(I') = \begin{cases} M & \text{if } i = 0, \\ 0 & \text{if } i \neq 0. \end{cases}$*

So $0 \to M \xrightarrow{\epsilon} I^0 \xrightarrow{d^0} I^1 \xrightarrow{d^1} \cdots$ *is exact where* ϵ *is the augmentation map. The* ***injective dimension*** *of* M, *denoted* $\mathrm{id}_R M$, *is the length of the shortest injective resolution of* M.

Note. If $\mathcal{A}$ has enough injectives, then injective resolutions exist:

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & M & \xrightarrow{\epsilon} & I^0 & \overset{d^0}{\dashrightarrow} & & & I^1 & \overset{d^1}{\dashrightarrow} & & & I^2 \cdots \\ & & & & & \searrow & & \nearrow & & \searrow & & \nearrow & \\ & & & & & & coker\epsilon & & & & cokerd^1 & & \\ & & & & & & & \searrow & & & & \searrow & \\ & & & & & & & & 0 & & & & 0 \end{array}$$

Example. $0 \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$ is an injective resolution of $\mathbb{Z}$. This is, in fact, the shortest one. Otherwise, $\mathbb{Z}$ would be injective. However, if that were the case, then we'd have the following

$$\begin{array}{ccccc} & & \mathbb{Z} & & \\ & & \uparrow 1 & \overset{f}{\nwarrow} & \\ 0 & \longrightarrow & \mathbb{Z} & \xrightarrow{2} & \mathbb{Z} \end{array}$$

Say $f(1) = a$. Then $2a = 1$, a contradiction. Thus $\mathbb{Z}$ is not injective (as a $\mathbb{Z}-$module). So $id_{\mathbb{Z}}\mathbb{Z} = 1$.

Comparison Theorem for Injective Resolutions. *Let* $\mathcal{A}$ *be an abelian category. Consider the following diagram of cochain complexes in* $\mathcal{A}$:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \longrightarrow & I^0 & \longrightarrow & I^1 & \longrightarrow & \cdots \\ & & \downarrow \phi & & & & & & \\ 0 & \longrightarrow & N & \longrightarrow & E^0 & \longrightarrow & E^1 & \longrightarrow & \cdots \end{array}$$

Suppose the top row is exact and E^i *is injective for all* i. *Then there exists a cochain map* $f^{\cdot} : E^{\cdot} \to I^{\cdot}$ *lifting* ϕ. *Furthermore, any two such liftings are cochain homotopic.*

3.1 Right Derived Functors

Definition 3.5. *Let* $F : A \to B$ *be a covariant, additive, left exact functor on abelian categories. Define the* i^{th} ***right derived functor*** *of* F *by* $R^iF := H^i(F(I^{\cdot}))$, *where* $I^{\cdot}$ *is any injective resolution of* N. *If* $f : N_1 \to N_2$, *let* $I_1^{\cdot}, I_2^{\cdot}$ *be injective resolutions of* N_1, N_2, *respectively. By the comparison theorem, there exists a cochain map* $\phi^{\cdot} : I_1^{\cdot} \to I_2^{\cdot}$ *lifting* f. *Set* $(R^iF)(f) := F(\phi^i)^* : (R^iF)(N_1) \to (R^iF)(N_2)$.

Remark. $R^0F = F$.

Horseshoe Lemma for Injective Resolutions. *Suppose we have the following diagram*

$$\begin{array}{ccccccccc} & & I^{\cdot} & & & & E^{\cdot} & & \\ & & \uparrow & & & & \uparrow & & \\ 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \end{array}$$

where $I^{\cdot}$ *and* $E^{\cdot}$ *are injective resolutions of* A *and* C, *respectively. Then, there exists an injective resolution* $C^{\cdot}$ *of* B *such that* $0 \to I^{\cdot} \to C^{\cdot} \to E^{\cdot} \to 0$ *is exact.*

Theorem 3.6. *Let* $\mathcal{A}$ *be an abelian category with enough injectives. For any short exact sequence* $0 \to A \to B \to C \to 0$ *in* $\mathcal{A}$, *there exists a long exact sequence on right derived functors* $\cdots \to R^iF(A) \to R^iF(B) \to R^iF(C) \to R^{i+1}F(A) \to \cdots$, *which is natural.*

Definition 3.7. *Let* $\mathcal{A} =<< R-mod >>$ *and* $F = Hom_R(M, -)$ *for some* $R-$*modules* M. *Then* $F : \mathcal{A} \to << \mathbb{Z}-mod >>$. *Define* $R^iF(-)$ *to be* $Ext^i(M,-)$.

To compute $Ext^i_R(M,N)$, let $I^\cdot$ be an injective resolution of N. Then $Ext^i_R(M,N) = H^i(Hom_R(M, I^\cdot))$. The Ext functor gets its name from the bijective correspondence of $Ext^1_R(M,N)$ and modules X such that $0 \to N \to X \to M \to 0$ is a short exact sequence. The module X is called a **extension of** M **by** N. This is referred to as the Yoneda description of Ext. In this correspondence, $Ext^1_R(M,N) = 0$ if and only if every extension of M by N splits.

As with Tor, there are two ways to define Ext. The second way is via a right derived functor of a contravariant functor:

Definition 3.8. *Let $F : \mathcal{A} \to \mathcal{B}$ be a contravariant left exact additive functor on abelian categories, where $\mathcal{A}$ has enough projectives. Define the i^{th}* ***right derived functor*** *of F as follows: Let $M \in Obj\mathcal{A}$ and $P.$ a projective resolution of M. Then $F(P.)$ can be viewed as a cochain complex where $F(P_0)^i = F(P_0)_i$ for all i.*

$$\cdots \to \underset{2}{P_2} \to \underset{1}{P_1} \to \underset{0}{P_0} \to 0 \quad \Rightarrow \quad 0 \to \underset{0}{F(P_0)} \to \underset{1}{F(P_1)} \to \underset{2}{F(P_2)} \to \cdots$$

Define $R^iF(M) := H^i(F(P.))$. As before, using the comparison theorem for projective resolutions, one can show R^iF is a well-defined contravariant functor from $\mathcal{A} \to \mathcal{B}$.

Remarks.

1. $R^0F = F$
2. If $0 \to A \to B \to C \to 0$ is a short exact sequence in $\mathcal{A}$, then one obtains the natural long exact sequence $\cdots R^iF(C) \to R^iF(B) \to R^iF(A) \to R^{i+1}F(C) \to \cdots$ for all i.
3. Given $F : \mathcal{A} \to \mathcal{B}$ as above, define $F^{op} : \mathcal{A}^{op} \to \mathcal{B}$ by $F^{op}(A) = F(A)$ for all $A \in Obj\mathcal{A}^{op} = Obj\mathcal{A}$ and $F^{op}(f) = F(f^{op})$ for all morphisms f in $\mathcal{A}^{op}$. Then, we have

$$F: \quad \begin{array}{c} A \xrightarrow{f} B \\ \big\downarrow F \\ F(A) \xrightarrow{F(f)} F(B) \end{array} \qquad\qquad F^{op}: \quad \begin{array}{c} B \xrightarrow{f^{op}} A \\ \big\downarrow F^{op} \\ F^{op}(B) \xrightarrow{F(f)} F^{op}(A) \end{array}$$

So F^{op} is a covariant left exact functor and $R^iF = (R^iF^{op})^{op}$.

Definition 3.9. *If $F = Hom_R(-,N)$ for some left $R-$module N, then we denote $R^iF(-)$ by $\underline{Ext}^i_R(-,N)$.*

Theorem 3.10. *For all $R-$modules M, N, we have $Ext^i_R(M,N) \cong \underline{Ext}^i_R(M,N)$.*

To prove this, we first need to define the following:

Definition 3.11. *A* ***cochain double complex*** *in $\mathcal{A}$ is a family of objects $\{C^{p,q}\}_{p,q\in\mathbb{Z}}$ and morphisms $d^v : C^{p,q} \to C^{p,q+1}$ and $d^h : C^{p,q} \to C^{p+1,q}$ such that $d^vd^h + d^hd^v = (d^v)^2 = (d^h)^2 = 0$. If C is a cochain double complex, then the* ***total complex*** *$Tot^{\oplus}(C)$ is define by $Tot^{\oplus}(C)_n = \bigoplus\limits_{p+q=n} C^{p,q}$ and $d_{tot} = d^v + d^h$.*

Example. Let $P.$ be a chain complex, $I^\cdot$ a cochain complex. Let $Hom_{\mathcal{A}}(P,I)$ denote the cochain double complex $Hom_{\mathcal{A}}(P,I)^{p,q} = Hom_{\mathcal{A}}(P_p, I^q)$ where $d^v : Hom_{\mathcal{A}}(P_p, I^q) \to Hom_{\mathcal{A}}(P^p, I^{q+1})$ is given by $f \mapsto (-1)^{p+q+1}d_If$ and $d^h : Hom_{\mathcal{A}}(P_p, I^q) \to Hom_{\mathcal{A}}(P_{p+1}, I^q)$ is given by $f \mapsto fd_P$ (where d_I is the differential on I and d_P the differential on P.

Proof of Theorem (Sketch). Let $P.$ be a projective resolution of M and $I^\cdot$ an injective resolution of N. Form the cochain double complex $C = Hom_{\mathcal{A}}(P., I^\cdot)$. As $I^\cdot$ is injective, the rows are exact except in the 0^{th} spot and as $P.$ is projective, the columns are exact except in the 0^{th} spot. From this, we create a new cochain double complex T by adding $Hom_{\mathcal{A}}(M, I^\cdot)$ in the $p = -1$ column and $Hom_{\mathcal{A}}(P., N)$ in the $q = -1$ row. Then T has exact rows and exact columns. As with Tor, one can use the acycle assembly lemma (for cochain double complexes in the third quadrant) to conclude $Tot(T)$ is exact. There are morphisms of cochain double complexes $f : C \to Hom_{\mathcal{A}}(M, I^\cdot)$ and $g : C \to Hom_{\mathcal{A}}(P., N)$. These induce chain maps $\widetilde{f} : Tot(C) \to Hom_{\mathcal{A}}(M, I^\cdot)$ and $\widetilde{g} : Tot(C) \to Hom(P., N)$. As with Tor, $cone(\widetilde{f}) = Tot(T)[-1]$ and $cone(\widetilde{g}) = Tot(T)[-1]$, both of which are exact. Thus $\widetilde{f}, \widetilde{g}$ are quasi-isomorphisms and thus induce maps on homology. Hence, $Ext^i_R(M,N) = H^i(Hom_{\mathcal{A}}(M, I^\cdot)) \cong H^i(Hom_{\mathcal{A}}(P., N)) = \underline{Ext}^i_R(M,N)$. □

Dimension Shifting Lemma. *Let $0 \to N \to I^0 \to I^1 \to \cdots \to I^{n-1} \to C \to 0$ be an exact sequence in $\mathcal{A}$ and suppose I^i is injective for all i. Then, for all objects M in $\mathcal{A}$, $Ext^i_{\mathcal{A}}(M, C) \cong Ext^{i+n}_{\mathcal{A}}(M, N)$ for all $i \geq 1$.*

Proof. Let $n = 1$. Then $0 \to N \to I^0 \to C \to 0$ is exact. Apply $Hom_{\mathcal{A}}(M, -)$ to get the long exact sequence

$$\cdots \to \underbrace{Ext^i_{\mathcal{A}}(M, I^0)}_{=0\ for\ i\geq 1} \to Ext^i_{\mathcal{A}}(M, C) \to Ext^{i+1}_{\mathcal{A}}(M, N) \to \underbrace{Ext^{i+1}_{\mathcal{A}}(M, I^0)}_{=0\ for\ i\geq 1} \to \cdots$$

where the first and last modules are zero as I^0 is injective. Thus $Ext^i_{\mathcal{A}}(M, C) \cong Ext^{i+1}_{\mathcal{A}}(M, N)$ for all $i \geq 1$. For $n > 1$, we have $0 \to M \to I^0 \to \cdots \to I^{n-2} \to K \to 0$ and $0 \to K \to I^{n-1} \to C \to 0$ are exact. By the $n-1$ and $n = 1$ cases, done. ☐

Lemma 3.12. *Suppose $0 \to K \to P_{n-1} \to \cdots \to P_0 \to M \to 0$ is exact, where the P_i are projective for all i. Then for all objects N of $\mathcal{A}$, we have $Ext^i_{\mathcal{A}}(K, N) \cong Ext^{i+n}_{\mathcal{A}}(M, N)$ for all $i \geq 1$.*

Proposition 3.13. *Let $\mathcal{A}$ be an abelian category with enough projectives. Let $M \in Obj\mathcal{A}$ and $n \in \mathbb{Z}$. Then TFAE*

1. *$pdM \leq n$.*
2. *Given any exact sequence $0 \to K \to P_{n-1} \to \cdots \to P_0 \to M \to 0$, where P_i are projective for all i, then K is projective.*
3. *$Ext^{n+1}_{A}(M, N) = 0$ for all objects N of $\mathcal{A}$.*

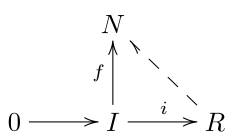
Proposition 3.14. *Let R be a ring, N a left $R-$module, and $n \in \mathbb{Z}$. TFAE*

1. *$id_R N \leq n$.*
2. *For all short exact sequences $0 \to N \to I^0 \to \cdots \to I^{n-1} \to C \to 0$ where I^i is injective for all i, we have C is injective.*
3. *$Ext^{n+1}_R(M, N) = 0$ for all M.*
4. *$Ext^{n+1}_R(R/I, N) = 0$ for all left ideals I.*

Proof. As $<< R - mod >>$ has enough injectives, (2) $\Rightarrow$ (1) $\Rightarrow$ (3) $\Rightarrow$ (4) are clear. Thus, we need only show (4) $\Rightarrow$ (2). To do so, we will first prove the following claim:

Claim. N is injective if and only if $Ext^1_R(R/I, N) = 0$ for all left ideals I of R.

Proof. By Baer's Criterion, N is injective if whenever we have the following diagram



where the bottom row is exact, there exists $g : R \to N$ making the diagram commute. Now, consider $0 \to I \to R \to R/I \to 0$ and apply $Hom_R(-, N)$. Then we have

$$0 \to Hom_R(R/I, N) \to Hom_R(R, N) \xrightarrow{i^*} Hom_R(I, N) \to \underbrace{Ext^1_R(R/I, N)}_{=0\ by\ hypothesis} \to \cdots$$

where $i^*(g) = gi$. By exactness, i^* is surjective, which implies there exists $g \in Hom_R(R, N)$ such that $gi = i^*(g) = f$. Thus N is injective.

Now, we will induct on n. For $n = 1$, we have $0 \to I^0 \hookrightarrow C \to 0$ and $Ext^2_R(R/J, N) = 0$ for all left ideals J. By dimension shifting, this says $Ext^1_R(R/J, C) = 0$. Thus C is injective by the lemma. ☐

Exercise. Let R be a ring, M a left $R-$module. Prove TFAE

1. M is flat.
2. For all right ideals I of R, the map $I \otimes_R M \to R \otimes_R M$ where $i \otimes m \mapsto i \otimes m$ is injective.

3. $Tor_1^R(R/I, M) = 0$ for all right ideals I of R.

Proof. First, we prove (1) $\Rightarrow$ (2). Suppose M is flat and I is a right ideal. Then $0 \to I \xrightarrow{i} R \to R/I \to 0$ is exact. As M is flat, $0 \to I \otimes M \xrightarrow{i\otimes 1} R \otimes M \to R/I \otimes M \to 0$ is exact and thus $i \otimes 1$ is injective.

To prove (2) $\Rightarrow$ (3), suppose $i \otimes 1$ is injective and I is a right ideal. Then $0 \to I \to R \to R/I \to 0$ is exact. Thus we have the following long exact sequence on Tor:

$$\cdots \to \underbrace{Tor_1^R(R, M)}_{=0 \ as \ R \ is \ flat} \to Tor_1^R(R/I, M) \to \underbrace{Tor_0^R(I, M)}_{\cong I\otimes M} \to \underbrace{Tor_0^R(R, M)}_{\cong R\otimes M} \to \cdots$$

Let f be the map from $Tor_1^R(R/I, M) \to I \otimes M$. Then f is injective, which implies $Tor_1^R(R/I, M) \cong imf = \ker(i \otimes 1_M) = 0$.

To prove (3) $\Rightarrow$ (1), we will first prove the following claim:

Claim. Suppose $Tor_1^R(R/I, M) = 0$ for all right ideals I. Then $Tor_1^R(N, M) = 0$ for all finitely generated right $R-$modules N.

Proof. Induct on the number of generators of N. First, suppose $N = n_1 R$ for some $n_1 \in R$. Then $N \cong R/I$ where $I = Ann_R n_1$. Then we have the short exact sequence $0 \to 0 \to N \to R/I \to 0$, giving us the following long exact sequence on Tor:

$$\cdots \to 0 \to Tor_1^R(N, M) \to \underbrace{Tor_1^R(R/I, M)}_{=0} \to 0 \to \cdots.$$

So $Tor_1^R(N, M) = 0$. So suppose $N = n_1 R + ... + n_k R$ and that the claim holds for modules with $k-1$ generators. Let $N' = n_1 R + ... + n_{k-1} R$. So $N = N' + n_k R$. Then, $0 \to N' \to N \to N/N' \to 0$ is a short exact sequence, giving us the following long exact sequence on Tor:

$$\cdots \to \underbrace{Tor_1^R(N', M)}_{=0 \ by \ induction} \to Tor_1^R(N, M) \to \underbrace{Tor_1^R(N/N', M)}_{=0 \ by \ n=1 \ case} \to \cdots$$

Thus $Tor_1^R(N, M) = 0$.

Recall that M is flat if and only if $Tor_1^R(N, M) = 0$ for all finitely generated $R-$modules N. □

Corollary 3.15. *Let R be a ring, M a left $R-$module. Then TFAE*

1. $fd_R M \leq n$.
2. $Tor_{n+1}^R(R/I, M) = 0$ *for all right ideals of R.*

Proof. Follows from the exercise and dimension shifting. □

Corollary 3.16. *Let R be a ring, $n \in \mathbb{Z}$. TFAE*

1. $fd_R M \leq n$ *for all left $R-$modules M.*
2. $fd_R R/I \leq n$ *for all left ideals I of R.*
3. $fd_R N \leq n$ *for all right $R-$modules N.*
4. $fd_R R/I \leq n$ *for all right ideals I of R.*
5. $Tor_{n+1}^R(M, N) = 0$ *for all right $R-$modules M and left $R-$modules N.*

If there exists $n \in \mathbb{Z}$ which satisfies the conditions above, the least such n is called the ***weak dimension*** *or* ***Tor dimension*** *of R.*

Proof. We've already shown (1) $\Leftrightarrow$ (5) $\Leftrightarrow$ (3). To prove (2) $\Rightarrow$ (3), if $fd_R R/I \leq n$ for all left ideals I, then $Tor_{n+1}^R(M, R/I) = 0$ for all M and I. Thus $fd_R M \leq n$ for all right $R-$modules by the corollary. To prove (5) $\Rightarrow$ (4), we have $Tor_{n+1}^R(R/I, N) = 0$ for all N, which implies $fd_R R/I \leq n$. Lastly, to prove (4) $\Rightarrow$ (2), let I be a left ideal. Then for any right ideal J, $Tor_{n+1}^R(R/J, R/I) = 0$, which says $fd_R R/I \leq n$ by the corollary. □

Theorem 3.17 (Auslander '55)**.** *Let R be a ring and $n \in \mathbb{Z}$. TFAE*

1. $pd_R M \leq n$ *for all left* $R-$*modules* M.
2. $pd_R M \leq n$ *for all finitely generated left* $R-$*modules* M.
3. $pd_R R/I \leq n$ *for all left ideals* I *of* R.
4. $id_R N \leq n$ *for all left* $R-$*modules* N.
5. $Ext_R^{n+1}(M, N) = 0$ *for all left* M, N.

If such an n exists, the least such n is called the ***left global dimension*** *of R, denoted $l.gl.dim R$.*

Theorem 3.18. *Note that* $(1) \Rightarrow (2) \Rightarrow (3)$ *is clear. To prove* $(3) \Rightarrow (4)$, *note that if* $Ext_R^{n+1}(R/I, N) = 0$ *for all left ideals I of R, then $id_R N \leq n$ by Proposition 3.14. Now,* $(4) \Rightarrow (5)$ *is clear and* $(5) \Rightarrow (1)$ *follows from Proposition 3.13.*

Similarly, we can define the **right global dimension** with an analogous theorem on right $R-$modules.

Example. Let R be a PID, not a field. Then $pd_R M \leq 1$ for all finitely generated $R-$modules M (by the Structure Theorem for finitely generated modules over a PID). By Auslander's Theorem, $pd_R M \leq 1$ for all $R-$modules M. In particular, if we have $F \xrightarrow{\phi} Q \to 0$ where Q is the field of fractions and F is free, then $\ker \phi$ is free.

Fact. If R is a ring and M a finitely presented left or right flat $R-$module, then M is projective. (We proved in the commutative case.)

Exercise. Let R be left Noetherian. Then

1. $fd_R M = pd_R M$ for all finitely generated left $R-$modules M.
2. $l.gl.dim R = weak\ dim R$.

Proof.

1. We know $fd_R M \leq pd_R M$ as a projective resolution is a flat resolution. So we need only show $fd_R M \geq pd_R M$. Let M be a finitely generated flat left $R-$module. Then M is finitely presented (as R is Noetherian). Recall a finitely presented flat module is projective, and so M is projective. Thus every flat resolution of finitely generated modules is a projective resolution. Now, suppose $fd_R M = n$ (if ∞, we are done). Let $F_{n-1} \xrightarrow{\phi} \cdots \to F_0 \to M \to 0$ be a finitely generated flat chain (the "start" of a projective resolution). Recall $\ker \phi$ is a finitely generated projective. Thus $0 \to K \to F_{n-1} \to \cdots \to F_0 \to 0$ is a projective resolution. Thus $pd_R M \leq n$.
2. Recall that $l.gl.dim R$ is the least such n such that $pd_R R/I \leq n$ for all I and the $weak\ dim R$ is the least such n such that $fd_R R/I \leq n$ for all I. By part 1, these are the same. □

An analogous result holds when R is right Noetherian.

Theorem 3.19. *Let R be a ring. TFAE*

1. $l.gl.dim R = 0$
2. $r.gl.dim R = 0$
3. *R is left Noetherian and* $weak\ dim R = 0$
4. *R is right Noetherian and* $weak\ dim R = 0$
5. *R is semisimple.*

Proof. Recall that R is semisimple if and only if every left $R-$module is projective which is if and only if every right $R-$module is projective. Then $(1) \Leftrightarrow (5)$ follows as every left $R-$module is projective and $(5) \Leftrightarrow (2)$ follows as every right $R-$module is projective. Now $(3) \Leftrightarrow (4) \Leftrightarrow (5)$ follows from the above exercise. □

Exercise. Let R be a ring. TFAE

1. $weak\ dim R = 0$

2. R is von Neumann regular

Proof. First, we will prove the forward direction. Let I be a finitely generated ideal. Now, $0 \to I \to R \to R/I \to 0$ is exact and as R, I are finitely generated, we see R/I is finitely presented. As $weak\ dim R = 0$, R/I is flat, and thus is projective. Thus the sequence splits and $R = I \oplus R/I$.

To prove the backward direction, suppose I is a finitely generated ideal of R. Then $R = I \oplus R/I$. Then R/I is projective (it is a direct summand of a free module) and hence flat. Thus $fd_R R/I = 0$. Now, suppose I is not finitely generated. Let $I = \varinjlim_{I_\alpha \in J} I_\alpha$ where J is the set of all finitely generated ideals.

Claim. $R/I = \varinjlim_{I_\alpha \in J} R/I_\alpha$.

Proof. Note that we have the following commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & I_\alpha & \longrightarrow & R & \longrightarrow & R/I_\alpha & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle incl} & & \downarrow{\scriptstyle =} & & \downarrow{\scriptstyle f} & & \\
0 & \longrightarrow & I_\beta & \longrightarrow & R & \longrightarrow & R/I_\beta & \longrightarrow & 0
\end{array}$$

where $f(r + I_\alpha) = r + I_\beta$. As $\varinjlim$ is exact, we get $0 \to \varinjlim I_\alpha \to R \to \varinjlim R/I_\alpha \to 0$ is exact. As $I = \varinjlim I_\alpha$, we have $R/I \cong \varinjlim R/I_\alpha$ by exactness.

Let M be a left $R-$module. Then, $Tor_0^R(R/I, M) = Tor_0^R(\varinjlim R/I_\alpha, M) = \varinjlim Tor_0^R(R/I_\alpha, M) = 0$. □

Note. Let (R, m, k) be a regular local ring. Then $m = (x_1, ..., x_d)$ where $d = \dim R$. Then $pd_R k = d$ as the Koszul Complex is a minimal free resolution of k. Then $Tor_{d+1}^R(M, k) = 0$ for all $R-$modules M. This implies $\beta_{d+1} = 0$ if M is finitely generated. Thus $pd_R M \leq d$ for all finitely generated $R-$modules M.

Theorem 3.20. *In a Noetherian local ring* (R, m, k), *TFAE*

1. $gl.dim R = n$

2. *R is a regular local ring of dimension n.*

Proof. (2) ⇒ (1) follows from the above note. To prove (1) ⇒ (2), note that $pd_R k \leq n$ implies R is a regular local ring by Theorem 2.62. Then, by the note, $\dim R = pd_R k = gl.dim R = n$. □

Examples.

1. $R = k[x]_{(x)}$ is a regular local ring of dimension 1 (where k is a field and x a variable)

2. $R = k[x_1, ..., x_t]_{(x_1,...,x_t)}$ is a regular local ring of dimension t.

Exercise. Let (R, m) be a local, Noetherian, commutative ring. Prove R is a regular local ring if and only if $id_R k < \infty$.

Proof. For the forward direction, if R is a regular local ring of dimension n, then $gl.dim R = n$. By Auslander's Theorem, this says $id_R N \leq n$ for all modules N. In particular, $id_R k < \infty$.

For the backward direction, suppose $id_R k < \infty$. Then $Ext_R^i(M, k) = 0$ for all $i \geq n+1$ and all $R-$modules M. Let F be a minimal free resolution of k. Recall $Ext_R^i(k, k) = H^i(Hom(F., k))$. Thus $0 = H^i(Hom(F., k))$ for all $i \geq n+1$. Now, recall that $k = Hom_R(k, k)$ and thus we have the following naturally commutative diagram:

$$\begin{array}{ccccccc}
\cdots & \longrightarrow & Hom(F_i, Hom_R(k,k)) & \xrightarrow{\psi} & Hom(F_{i-1}, Hom_R(k,k)) & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle \cong} & & \\
 & & Hom(k \otimes F_i, k) & \xrightarrow{\psi'} & Hom(k \otimes F_{i-1}, k) & &
\end{array}$$

Now, recall that if we have an exact sequence $R^m \xrightarrow{\phi} R^n \to k \to 0$ and we apply $- \otimes_R k$, we get that $\phi \otimes 1 : R^m \otimes k \to R^n \otimes k$ is the 0 map. So $k \otimes F_i \to k \otimes F_{i-1}$ is the zero map, which implies $\psi' = 0$ and thus $\psi = 0$. Thus $0 = H_i(Hom(F., k)) = Hom(F_i, k) \cong Hom(\oplus^{\beta_i} R, k) \cong \oplus^{\beta_i} Hom(R, k) \cong k^{\beta_i(k)}$ where $\beta_i(k) = rank F_i$. Thus $\beta_i(k) = 0$, which says $F.$ is a finite projective resolution of k. Thus $pd_R k < \infty$ and so R is a regular local ring by Theorem 2.62. □

Exercise. Let $\phi : R \to S$ be a ring homomorphism of commutative rings such that S is flat as an $R-$module. Prove

1. If M is a finitely presented left $R-$module, then $Hom_R(M,N) \otimes_R S \cong Hom_S(M \otimes_R S, N \otimes_R S)$ for all left $R-$modules.

   Proof. First note that $Hom_R(R^n, N) \otimes_R S \cong (\oplus_{i=1}^n N) \otimes_R S \cong \oplus_{i=1}^n (N \otimes_R S) \cong Hom_S(S^n, N \otimes_R S)$, where the isomorphisms are natural. Now, let $R^m \to R^n \to M \to 0$ be exact. Then $0 \to Hom_R(M,N) \to Hom_R(R^n,N) \to Hom_R(R^m,N)$ is exact. Now, if we apply $- \otimes_R S$, we stay exact as S is flat. Thus, we have the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & 0 & \longrightarrow & Hom_R(M,N) \otimes S & \longrightarrow & Hom_R(R^n,N) \otimes S & \longrightarrow & Hom_R(R^m,N) \otimes S \\ \downarrow \cong & & \downarrow \cong & & & & \downarrow \cong & & \downarrow \cong \\ 0 & \longrightarrow & 0 & \longrightarrow & Hom_S(M \otimes_R S, N \otimes_R S) & \longrightarrow & Hom_S(S^n, N \otimes S) & \longrightarrow & Hom_S(S^n, N \otimes S) \end{array}$$

   By the exactness and the Five Lemma, done. □

2. If R is Noetherian, then $(Ext_R^i(M,N)) \otimes_R S \cong Ext_S^i(M \otimes_R S, N \otimes_R S)$ if M is finitely generated.

   Proof. Let $P.$ be a finitely generated projective resolution for M. Then

$$\begin{aligned} Ext_R^i(M,N) \otimes_R S &= H^i(Hom_R(P.,N)) \otimes_R S \\ &= H^i(Hom_R(P.,N) \otimes_R S) \\ &= H^i(Hom_S(P.S, N \otimes S) \\ &= Ext_S^i(M \otimes_R S, N \otimes_R S) \text{ as } P. \otimes S \text{ is a projective resolution as } S \text{ is flat} \end{aligned}$$
   □

Corollary 3.21. *Let R be Noetherian, W a multiplicatively closed subset of R, M a finitely generated $R-$module. Then $Ext_R^i(M,N)_W \cong Ext_{R_W}^i(M_W, N_W)$.*

Corollary 3.22. *Let R be Noetherian, E an injective $R-$module. Then E_S is an injective R_S-module for all multiplicatively closed subsets S of R.*

Proof. Recall $id_R N = 0$ if and only if $Ext_R^1(R/I,N) = 0$ for all I. Let I_S be an ideal of R_S for an ideal I of R. Then $Ext_R^1(R_S/I_S, E_S) \cong (Ext_R^1(R/I,E))_S = 0$ as E is injective. Thus E_S is injective. □

Definition 3.23. *Let (R,m) be a commutative, local, Noetherian ring. R is called **Gorenstein** if $id_R R < \infty$.*

Corollary 3.24. *If (R,m) is Gorenstein, so is R_p for any $p \in SpecR$.*

Proof. Say $0 \to R \to I^0 \to \cdots \to I^n \to 0$ is exact. Then $0 \to R_p \to I_p^0 \to \cdots \to I_p^n \to 0$ is exact and I_p^i are injective. Thus $id_{R_p} R_p < \infty$. □

Corollary 3.25. *Regular local rings are Gorenstein.*

Proof. In a regular local ring, $id_R M < \infty$ for all modules M. □