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Conjugacy Geodesics in Coxeter Groups

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Conjugacy Geodesic Languages of Coxeter Groups
The University of Nebraska-Lincoln
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Advised by Susan Hermiller and Tim Susse

Reflections of tilings
Given a tiling of a space, we are often interested in its symmetries. By performing one
symmetry after the other, there is a natural multiplication on this set, and the set with
this operation forms an object called a Coxeter group. Different tilings give different
Coxeter groups, and the geometry of a given tiling is encoded by numbers denoted by
$m_1, m_2$. The following Coxeter group has $m_1 = 2, 4$ and $4$.

Reflections as words
We associate each reflection across an edge of the fundamental domain with a letter, called a fundamental
reflection. Any symmetry of the tiling can be realized by a sequence of these reflections, so the fundamental
reflections are said to generate the Coxeter group.

Reflections as paths
By connecting the centers of subsequent reflections of the chosen fundamental domain,
we can also represent sequences of reflections by paths. Therefore we also get a
realization for the space where all paths live.

The word problem

Question (Dehn 1908-1912):
Can we tell if a sequence of reflections gets us back to the original position?

From another point of view, is there a computer algorithm which can decide if a word
defines a loop in the Cayley graph? This is called the word problem.

Paths representing conjugate sequences of reflections.
Dun also posed the conjugacy problem: given two sequences of reflections $w$ and $v$,
are there an algorithm to decide if they are conjugate?
Again, the answer is generally no.

Cyclic and conjugacy geodesics
Just as geodesics tell us about the word problem, there are languages which
tell us about the conjugacy problem. A path is a cyclic geodesic if all of its
cyclic permutations are geodesic, and is a conjugacy geodesic if any other
path which is conjugate to it has equal or greater length.

If the cyclic geodesic and conjugacy geodesic languages of a group are the
same then the conjugacy problem simplifies immensely.

Right-angled Coxeter groups are an important subset of Coxeter groups, and
consist of those Coxeter groups where every $m_1$ is $2$ or $\infty$.

Theorem (Ciobanu, Hermiller, Holt, Rees 2014):
The cyclic geodesic and conjugacy geodesic languages of a right-angled Coxeter
group are the same.

Right-angled Coxeter groups can also be thought of as being “built” as graph
products of one-dimensional Coxeter groups.

A Coxeter group is even if every $m_1$ is even or $\infty$. Note that all right-angled
Coxeter groups are even.

Theorem (Calderon 2016):
The cyclic geodesic and conjugacy geodesic languages of a Coxeter group built of
two-dimensional even Coxeter groups are the same.

Furthermore, this result cannot be significantly strengthened; Coxeter groups of
higher dimension whose higher dimensional cells do not decompose as a
product of lower dimensional cells do not have this property.

Proposition (Calderon 2016):
If $W$ is a Coxeter group with an irreducible special subgroup of dimension 3 or greater,
then the cyclic geodesic and conjugacy geodesic languages of $W$ are not the same.

Open Question:
Are the cyclic geodesic and conjugacy geodesic languages the same for every even
Coxeter group?

Open Question:
Do Coxeter groups have regular conjugacy geodesic languages?

References:
B. Brink and R. Howlett, “A finiteness property and an automatic structure for Coxeter
L. Ciobanu, S. Hermiller, D. Holt, and S. Rees, “Conjugacy languages in groups” Israel J.

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A path which is not a cyclic geodesic.

The Cayley graph of the 2.4.4 Coxeter group.
For a general group of symmetries, the answer is no.
If the geodesic language can be recognized by a simple computer (a property called
regularity), then the word problem is solvable.

The conjugacy problem
Two sequences of reflections $w$ and $v$ are conjugate if there is a sequence of
reflections $u$ such that performing $u$, then $v$, then the inverse of $u$ exhibits the
same symmetry as $w$.

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