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# Effects of middle plane curvature on vibrations of a thickness-shear mode crystal resonator

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## Abstract

We study the effects of a small curvature of the middle plane of a thickness-shear mode crystal plate resonator on its vibration frequencies, modes and acceleration sensitivity. Two-dimensional equations for coupled thickness-shear, flexural and extensional vibrations of a shallow shell are used. The equations are simplified to a single equation for thickness-shear, and two equations for coupled thickness-shear and extension. Equations with different levels of coupling are used to study vibrations of rotated Y-cut quartz and langasite resonators. The influence of the middle plane curvature and coupling to extension is examined. The effect of middle plane curvature on normal acceleration sensitivity is also studied. It is shown that the middle plane curvature causes a frequency shift as large as  $10^{-8} \text{ g}^{-1}$  under a normal acceleration. These results have practical implications for the design of concave-convex and plano-convex resonators.

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*Keyword:* Resonator

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## 1. Introduction

Plano-convex crystal plates (Fig. 1) are widely used as resonators. From a structural point of view, it is obvious that the middle plane (the dotted line in Fig. 1, consisting of points at equal distance to the top and bottom surfaces) of such a resonator is curved. By definition this makes the resonator structure a shell rather than a plate with a flat middle plane. For a plano-convex resonator the curvature of the middle plane is usually small and therefore the resonator is what is called a shallow shell in theories of structures. A plano-convex resonator is in fact a shallow shell with a particular thickness variation such that its bottom surface is flat.

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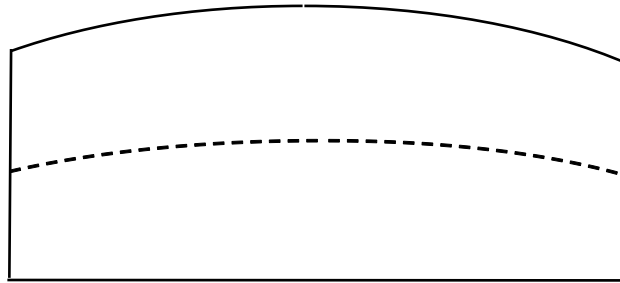


Fig. 1. A plano-convex resonator.

A fundamental difference between a shell and a plate is that in a shell flexural and thickness-shear vibrations are inherently coupled to extensional vibration due to the curvature of the middle plane. This coupling exists irrespective of whether the shell has a uniform or varying thickness. Coupling to extension has consequences in resonator frequency stability, energy trapping and mounting, etc. While there exist extensive results on coupled thickness-shear and flexural vibrations of flat crystal plates, the effects of middle plane curvature and the related coupling to extension are not clear and should be quantified. This is particularly relevant to plano-convex resonators because the curvature of the convex surfaces is being increased in design for stronger energy trapping and smaller resonators.

Another important phenomenon associated with coupling to extension is resonator acceleration sensitivity. Resonators are often mounted on moving objects like missiles and satellites. Accelerations of these objects cause frequency shifts in a resonator through acceleration induced biasing or initial stresses and strains in the resonator. This effect is known to depend strongly on the middle plane extension and therefore is related to the middle plane curvature which causes coupling to extension. It is widely believed (Vig, 2002) that the normal acceleration sensitivity of a perfectly symmetric thickness-shear resonator vanishes. This belief is based on the assumption that under a normal acceleration a plate resonator is in classical flexure without extensional deformation in its middle plane. This is reasonable for a plate resonator with a flat middle plane in deflection. When the middle plane has a curvature, flexure induces an extensional deformation of the middle plane, which can result in a frequency shift. The military requirement to minimize acceleration sensitivity is driving the need to advance from the  $10^{-10} \text{ g}^{-1}$  production technology available today to  $10^{-12} \text{ g}^{-1}$  or better in the near future. At these levels, it is necessary to understand the effect of the middle plane curvature on acceleration sensitivity.

In this paper we study the effects of the middle plane curvature on the vibration of a thickness-shear resonator and its acceleration sensitivity. Our main focus is the coupling to extension due to the curvature. This coupling exists for shallow shells of both varying and uniform thickness. Therefore, for simplicity, we analyze a shallow shell of uniform thickness (see Fig. 2). Such a shell is also used for resonators and is called a concave-convex resonator. Since the resonator in Fig. 2 already has the coupling to extension, it is expected to exhibit the basic effects of this coupling. In Section 2 the structural equations of a shallow shell in coupled thickness-shear, flexural and extensional vibrations are summarized and specialized to the case of plane-strain vibrations of rotated Y-cut quartz or langasite. The equations are simplified by an extended version of the well

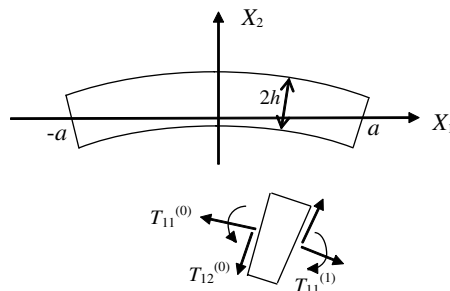


Fig. 2. A concave-convex resonator and resultants.

known thickness-shear approximation (Bleustein and Tiersten, 1968) to obtain a single equation for thickness-shear in Section 3, which will be used to study the basic effects of the curvature on frequency. In Section 4 coupled thickness-shear and extensional vibrations are studied. Normal acceleration sensitivity is analyzed in Section 5. A few discussions are made in Section 6. Finally, some conclusions are drawn in Section 7.

## 2. Governing equations

Consider the reference configuration of a rotated Y-cut quartz resonator as shown in Fig. 2. Quartz is anisotropic. A particular cut of a quartz plate describes how a plate is taken from a bulk crystal, i.e., the orientation of the plate with respect to the crystal axes (Tiersten, 1969). The equations to be developed will also be applicable to langasite resonators of the same cut because langasite and quartz have the same crystal symmetry. The shell is of length  $2a$  and uniform thickness  $2h$ . It is thin with  $2a \gg 2h$ . Then two-dimensional theories apply. The shell is long in the  $X_3$  direction. Fig. 2 shows a cross-section. We study planar deformations with  $u_3 = 0$  and  $\partial/\partial X_3 = 0$ . For rotated Y-cut quartz such motions are allowed by the three-dimensional exact theory. Let the equation for the middle surface be  $X_2 = f(X_1)$ , and  $Y$  the local coordinate along the shell thickness direction measured from the middle surface. Let  $u_1^{(0)}(X_1, t)$  and  $u_2^{(0)}(X_1, t)$  be the extensional and flexural displacements of the middle surface, and  $u_1^{(1)}(X_1, t)$  the thickness-shear displacement. This notation follows Mindlin (1951). The subscript is the tensor index of the displacement vector. The superscript represents the order in a Taylor expansion in terms of the shell thickness coordinate. Then the three-dimensional displacement fields are approximately

$$u_1 = u_1^{(0)} + Yu_1^{(1)}, \quad u_2 = u_2^{(0)}. \quad (1)$$

From the shell point of view the middle surface ( $Y = 0$ ) is where the shear displacement  $Yu_1^{(1)}$  vanishes, but in general it is not where the total  $u_1$  displacement vanishes because the middle surface may be in extension for a shell. The equations for coupled extensional, flexural and thickness-shear vibration under the shallow shell assumptions are (Xu, 1982)

$$\begin{aligned} T_{11,1}^{(0)} &= 2h\rho_0\ddot{u}_1^{(0)}, \\ -k_1T_{11}^{(0)} + T_{12,1}^{(0)} + F_2^{(0)} &= 2h\rho_0\ddot{u}_2^{(0)}, \\ T_{11,1}^{(1)} - T_{21}^{(0)} &= \frac{2h^3}{3}\rho_0\ddot{u}_1^{(1)}, \end{aligned} \quad (2)$$

where

$$k_1 \cong -\frac{\partial^2 f}{\partial X_1^2} \quad (3)$$

is the curvature of the middle surface, and  $F_2^{(0)}$  is the load in the  $X_2$  direction per unit area of the middle surface. An index following a comma represents partial differentiation with respect to the coordinate associated with the index. We consider a thin and shallow shell with  $k_1h \ll k_1a \ll 1$ . The extensional, shear and bending resultants are given by (Xu, 1982)

$$\begin{aligned} T_{11}^{(0)} &= \int_{-h}^h T_{11} dY = 2h\bar{c}_{11}(u_{1,1}^{(0)} + k_1u_2^{(0)}), \\ T_{12}^{(0)} &= T_{21}^{(0)} = \int_{-h}^h T_{12} dY = 2h\kappa^2c_{66}(u_1^{(1)} + u_{2,1}^{(0)} - k_1u_1^{(0)}), \\ T_{11}^{(1)} &= \int_{-h}^h T_{11} Y dY = \frac{2h^3}{3}\bar{c}_{11}u_{1,1}^{(1)}, \end{aligned} \quad (4)$$

where

$$\bar{c}_{11} = c_{11} - \frac{c_{12}^2}{c_{22}} \quad (5)$$

is the extensional elastic stiffness in the  $X_1$  direction after the stress relaxation in the  $X_2$  direction.  $\kappa$  is the thickness-shear correction factor (Mindlin, 1951). We will use  $\kappa^2 = \pi^2/12$  (Mindlin, 1951) so that in the limit of  $k_1 \rightarrow 0$  the exact frequency for the fundamental pure thickness-shear motion independent of  $X_1$  will be obtained. Due to the curvature of the middle surface, the extensional resultant  $T_{11}^{(0)}$  is present in Eq. (2)<sub>2</sub> for flexure, the flexural displacement  $u_2^{(0)}$  has a contribution to the extensional strain in Eq. (4)<sub>1</sub>, and the extensional displacement  $u_1^{(0)}$  affects the shear strain in Eq. (4)<sub>2</sub>. We consider the case that the middle surface is a parabolic curve. It approximately has a constant curvature when  $f_{,1}^2$  is neglected for a shallow shell. In this case the equations have constant coefficients. With substitutions from Eq. (4), Eq. (2) can be written as three equations for  $u_1^{(0)}$ ,  $u_2^{(0)}$  and  $u_1^{(1)}$

$$\begin{aligned} 2h\bar{c}_{11}(u_{1,11}^{(0)} + k_1u_{2,1}^{(0)}) &= 2h\rho_0\ddot{u}_1^{(0)}, \\ -k_12h\bar{c}_{11}(u_{1,1}^{(0)} + k_1u_2^{(0)}) + 2h\kappa^2c_{66}(u_{1,1}^{(1)} + u_{2,11}^{(0)} - k_1u_{1,1}^{(0)}) + F_2^{(0)} &= 2h\rho_0\ddot{u}_2^{(0)}, \\ \frac{2h^3}{3}\bar{c}_{11}u_{1,11}^{(1)} - 2h\kappa^2c_{66}(u_1^{(1)} + u_{2,1}^{(0)} - k_1u_1^{(0)}) &= \frac{2h^3}{3}\rho_0\ddot{u}_1^{(1)}. \end{aligned} \tag{6}$$

### 3. Thickness-shear approximation

Our main interest is the thickness-shear mode  $u_1^{(1)}$  which is the operating mode of the resonator. The flexure  $u_2^{(0)}$  and the extension  $u_1^{(0)}$  are small but they are not zero. During the analysis of thickness-shear resonators, researchers have developed a procedure for eliminating the small flexure in a flat-plate resonator and obtain a simple equation for thickness-shear alone (Bleustein and Tiersten, 1968; Tiersten, 1969; Yang, 1997). This procedure has been called the thickness-shear approximation. In our case, when the middle plane has a curvature, we need to generalize the thickness-shear approximation for flat plates because of the coupling to extension. Consider waves propagating in the  $X_1$  direction with a propagation factor of  $\exp(i(\xi X_1 - \omega t))$ . This transforms the equations into the  $(\omega, \xi)$  plane. The thickness-shear approximation is based on two observations in the  $(\omega, \xi)$  plane. One is that for resonator applications long waves with a small wave number  $\xi$  are of interest. For these waves terms with higher-order derivatives with respect to  $X_1$  can be neglected. This is because a derivative with respect to  $X_1$  is effectively a multiplication by the small wave number  $\xi$ , and a higher-order derivative is a repeated multiplication by the small wave number. The other observation is that the frequency  $\omega$  of long thickness-shear waves is very close to the fundamental plate thickness-shear frequency

$$\omega^2 \cong \omega_0^2 = \frac{\pi^2c_{66}}{4\rho_0h^2} = \frac{3\kappa^2c_{66}}{\rho_0h^2}. \tag{7}$$

With the above two approximations, Eq. (6)<sub>1,2</sub> for  $F_2^{(0)} = 0$  can be approximated as

$$\begin{aligned} 2h\bar{c}_{11}k_1u_{2,1}^{(0)} &= 2h\rho_0(-\omega_0^2)u_1^{(0)}, \\ -k_12h\bar{c}_{11}k_1u_2^{(0)} + 2h\kappa^2c_{66}u_{1,1}^{(1)} &= 2h\rho_0(-\omega_0^2)u_2^{(0)}, \end{aligned} \tag{8}$$

where  $u_{1,11}^{(0)}$  in Eq. (6)<sub>1</sub> is dropped under the long wave approximation. So are  $u_{2,11}^{(0)}$  and  $u_{1,1}^{(0)}$  in Eq. (6)<sub>2</sub>. Solving Eq. (8) approximately under the small curvature assumption, up to  $k_1^2$ , we have

$$\begin{aligned} u_2^{(0)} &\cong -\left(1 + \frac{4k_1^2h^2\bar{c}_{11}}{\pi^2c_{66}}\right)\frac{h^2}{3}u_{1,1}^{(1)}, \\ u_1^{(0)} &\cong \frac{4k_1h^4\bar{c}_{11}}{3\pi^2c_{66}}u_{1,11}^{(1)}. \end{aligned} \tag{9}$$

Substituting Eq. (9) into Eq. (6)<sub>3</sub> gives

$$\frac{2h^3}{3}\left(\bar{c}_{11} + \kappa^2c_{66} + \frac{2}{3}k_1^2h^2\bar{c}_{11}\right)u_{1,11}^{(1)} - 2h\kappa^2c_{66}u_1^{(1)} = \frac{2h^3}{3}\rho_0\ddot{u}_1^{(1)}. \tag{10}$$

Under Eq. (9), the resultants are approximated by

$$\begin{aligned} T_{11}^{(0)} &= -2h\bar{c}_{11}k_1 \left( 1 + \frac{4k_1^2 h^2 \bar{c}_{11}}{\pi^2 c_{66}} \right) \frac{h^2}{3} u_{1,1}^{(1)}, \\ T_{12}^{(0)} &= 2hk^2 c_{66} u_1^{(1)}, \\ T_{11}^{(1)} &= \frac{2h^3}{3} \bar{c}_{11} u_{1,1}^{(1)}. \end{aligned} \quad (11)$$

Consider the thickness-shear modes given by

$$\begin{aligned} u_1^{(1)} &= \cos \frac{n\pi}{2a} X_1 e^{i\omega t}, \quad n = 1, 3, 5, \dots, \\ \text{or } u_1^{(1)} &= \sin \frac{n\pi}{2a} X_1 e^{i\omega t}, \quad n = 2, 4, 6, \dots, \end{aligned} \quad (12)$$

which satisfy the boundary conditions that  $u_1^{(1)}(X_1 = \pm a) = 0$ . Under the thickness-shear approximation these boundary conditions are in fact approximately equivalent to  $T_{12}^{(0)}(X_1 = \pm a) = 0$  (Tiersten, 1969), which can also be seen from Eq. (11)<sub>2</sub>. Substitution of Eq. (12) into Eq. (10) yields

$$\frac{\omega^2}{\omega_0^2} = 1 + n^2 \frac{h^2}{a^2} \left( \frac{\bar{c}_{11}}{c_{66}} + \kappa^2 + \frac{2k_1^2 h^2 \bar{c}_{11}}{3 c_{66}} \right), \quad n = 1, 2, 3, \dots, \quad (13)$$

where the last term in the parentheses is due to the curvature. Eq. (13) shows that the curvature raises the thickness-shear frequency. This agrees qualitatively with the exact results in the thickness-shear vibration analysis of a circular cylindrical crystal shell (Yang and Batra, 1995). It can also be seen that this is a quadratic effect and is expected to be small. Rotated Y-cut quartz plates are a family of plates depending on a parameter  $\theta$ . For some numerical estimates we consider Y-cut and AT-cut quartz which are special cases of rotated Y-cuts with  $\theta = 0^\circ$  and  $35.25^\circ$ , respectively, and Y-cut langasite. We employ the material constants for quartz and langasite as given by Tiersten (1969) and Kosinski and Pastore (2000). For the size and curvature of the resonator, based on the data used by Yang and Tiersten (1995) for a plano-convex resonator, we consider  $k_1 = 1/(20 \text{ cm})$  which is half of the curvature of the convex top surface of the resonator analyzed by Yang and Tiersten (1995),  $a = 1 \text{ cm}$ , and  $h = 0.5 \text{ mm}$ . For an AT-cut resonator these data yield  $\omega_0 = 10.45 \text{ MHz}$ . For Y-cut langasite  $\omega_0 = 8.54 \text{ MHz}$ . From Eq. (13), for  $n = 1$ , we obtain the contribution of curvature to frequency as

$$\frac{1}{2} \frac{h^2}{a^2} \frac{2k_1^2 h^2 \bar{c}_{11}}{3 c_{66}} / \omega_0 \cong \begin{cases} 5.96 \times 10^{-15} & (\text{AT-cut quartz}), \\ 3.69 \times 10^{-15} & (\text{Y-cut quartz}), \\ 7.63 \times 10^{-15} & (\text{Y-cut langasite}), \end{cases} \quad (14)$$

which gives an idea of the order of magnitude of the effect of the middle plane curvature on frequency. This effect seems to be small enough to be neglected in present applications.

#### 4. Coupled thickness-shear and extension

The middle surface curvature does more than the small frequency shift shown in Eq. (14). The related coupling to extension can become strong when the frequency of a higher order extensional mode is the same as the operating frequency of the fundamental thickness-shear mode (i.e., the modes are degenerate). One immediate consequence of degeneracy is that a stress bias can cause the frequencies to split and frequency jumps may result (Kosinski et al., 2003). Furthermore, since thickness-shear can be trapped but extension cannot, this mode coupling is highly undesirable because it affects energy trapping and mounting and should be avoided in design. Therefore it is of practical importance to predict when this coupling occurs. In order to do so we use Eq. (9)<sub>1</sub> to eliminate the flexure in Eq. (6)<sub>1,3</sub> and obtain the following two equations for coupled thickness-shear and extension:

$$\begin{aligned} \bar{c}_{11}u_{1,11}^{(0)} - \frac{1}{3}k_1h^2\bar{c}_{11}u_{1,11}^{(1)} &= \rho_0\ddot{u}_1^{(0)}, \\ \left(\bar{c}_{11} + \kappa^2c_{66} + \frac{1}{3}k_1^2h^2\bar{c}_{11}\right)u_{1,11}^{(1)} - \rho\omega_0^2u_1^{(1)} + k_1\rho_0\omega_0^2u_1^{(0)} &= \rho_0\ddot{u}_1^{(1)}. \end{aligned} \tag{15}$$

The resultants are approximated by

$$\begin{aligned} T_{11}^{(0)} &= 2h\bar{c}_{11}u_{1,1}^{(0)} - 2h\bar{c}_{11}k_1\left(1 + \frac{4k_1^2h^2\bar{c}_{11}}{\pi^2c_{66}}\right)\frac{h^2}{3}u_{1,1}^{(1)}, \\ T_{12}^{(0)} &= 2h\kappa^2c_{66}(u_1^{(1)} - k_1u_1^{(0)}), \\ T_{11}^{(1)} &= \frac{2h^3}{3}\bar{c}_{11}u_{1,1}^{(1)}. \end{aligned} \tag{16}$$

For boundary conditions we consider the case that

$$u_1^{(0)}(X_1 = \pm a) = 0, \quad u_1^{(1)}(X_1 = \pm a) = 0. \tag{17}$$

Consider the following modes:

$$u_1^{(0)} = A \cos \zeta X_1 e^{i\omega t}, \quad u_1^{(1)} = B \cos \zeta X_1 e^{i\omega t}, \tag{18}$$

where  $A$  and  $B$  are constants, and  $\zeta$  is the wave number in the  $X_1$  direction. For Eq. (18) to satisfy Eq. (17) we must have

$$\cos \zeta a = 0, \quad \zeta = \frac{n\pi}{2a}, \quad n = 1, 3, 5, \dots \tag{19}$$

Substituting Eq. (18) into Eq. (15), for nontrivial solutions of  $A$  and  $B$ , we obtain the following frequency equation

$$\begin{vmatrix} \rho_0\omega^2 - \bar{c}_{11}\zeta^2 & \frac{1}{3}k_1h^2\bar{c}_{11}\zeta^2 \\ k_1\rho\omega_0^2 & \rho_0\omega^2 - (\bar{c}_{11} + \kappa^2c_{66} + \frac{1}{3}k_1^2h^2\bar{c}_{11})\zeta^2 - \rho_0\omega_0^2 \end{vmatrix} = 0. \tag{20}$$

Similarly, there exist another set of modes

$$u_1^{(0)} = C \sin \zeta X_1 e^{i\omega t}, \quad u_1^{(1)} = D \sin \zeta X_1 e^{i\omega t}, \tag{21}$$

where  $C$  and  $D$  are constants, and

$$\sin \zeta a = 0, \quad \zeta = \frac{n\pi}{2a}, \quad n = 2, 4, 6, \dots \tag{22}$$

The frequency equation is still given by Eq. (20).

For the special case of  $k_1 = 0$ , Eq. (20) reduces to two sets of frequencies for uncoupled extension and thickness-shear. When  $k_1 \neq 0$  but is small, Eq. (20) yields two sets of frequencies with one being essentially extensional and the other essentially thickness-shear. We plot the two sets of frequencies versus the aspect ration  $a/h$  in Fig. 3 for AT-cut quartz. This is called a spectral plot and is very useful in resonator design (Mindlin, 1951). The figure shows that for certain values of  $a/h$  the two sets of curves intersect. These are the aspect ratios at which a thickness-shear mode and an extensional mode have the same resonant frequency and should be avoided in resonator design. The extensional and thickness-shear frequency spectra for Y-cut langasite are shown in Fig. 4. Figs. 3 and 4 are qualitatively the same but the quantitative differences are significant for resonator design.

### 5. Normal acceleration sensitivity

Finally, we consider the influence of middle plane curvature on normal acceleration sensitivity. It is worthwhile to note here that the effect we are considering is entirely different from that considered by others with regard to the in-plane acceleration sensitivity of contoured resonators (e.g., Zhou and Tiersten, 1992). Consider a resonator with a slightly curved middle plane under a normal acceleration  $a_2$  as shown (Fig. 5).



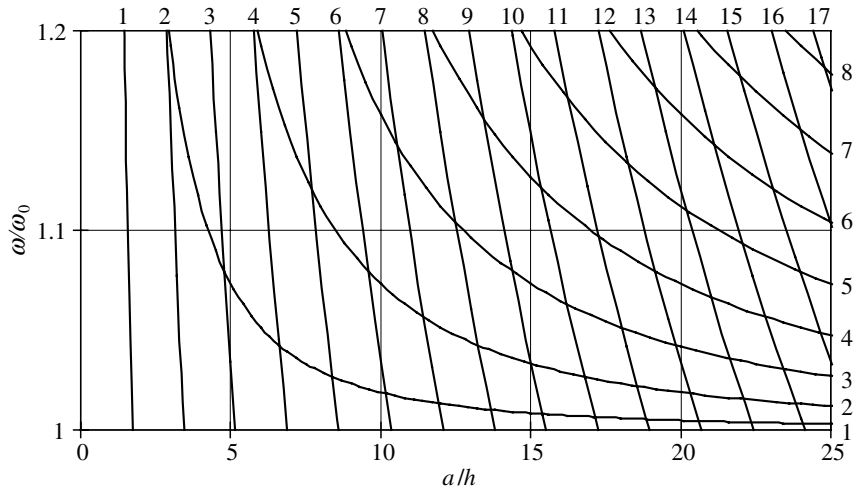


Fig. 3. Frequency spectra for coupled thickness-shear and extension in AT-cut quartz. (The essentially thickness-shear modes are numbered on the right. The essentially extensional modes are numbered on the top.)

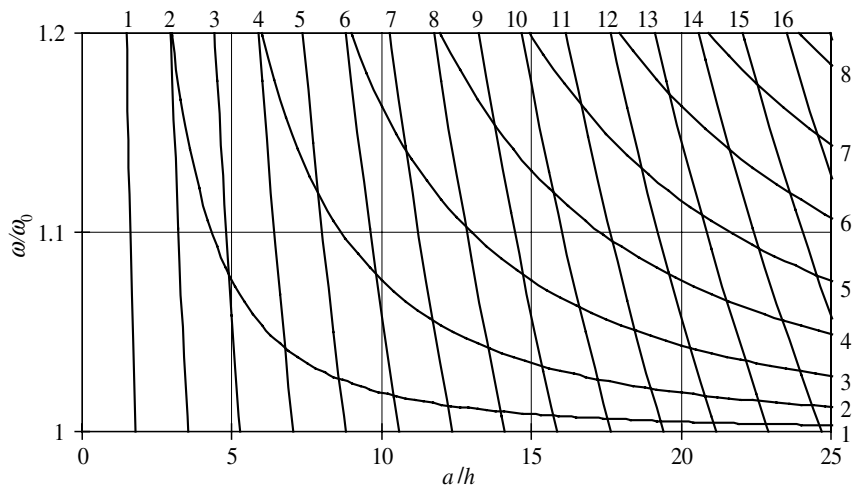


Fig. 4. Frequency spectra for coupled thickness-shear and extension in Y-cut langsite.

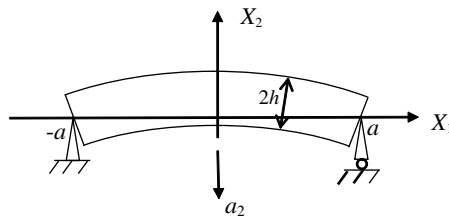


Fig. 5. A simply supported crystal resonator under a normal acceleration  $a_2$ .

*5.1. Perturbation integral for frequency shifts in resonators*

The first-order description of frequency shifts in resonators due to biasing mechanical fields can be calculated by the following integral from a first-order perturbation analysis (Tiersten, 1978):

$$\Delta\omega = \frac{1}{2\omega} \frac{\int_V \hat{c}_{L\gamma M\alpha} u_{\gamma,L} u_{\alpha,M} dV}{\int_V \rho_0 u_{\alpha} u_{\alpha} dV}, \tag{23}$$

where  $\omega$  and  $u_{\alpha}$  are the unperturbed resonant frequency and the corresponding mode when there are no biasing fields.  $\Delta\omega$  is the frequency perturbation due to the biasing fields.  $V$  is the volume of the crystal resonator in the reference configuration when there are no biasing fields.  $\hat{c}_{L\gamma M\alpha}$ , the change of the effective elastic constants of the crystal under biasing fields, are given by

$$\hat{c}_{K\alpha L\gamma} = c_{K\alpha LN} w_{\gamma,N} + c_{KML\gamma} w_{\alpha,M} + c_{K\alpha L\gamma AB} w_{A,B} + c_{KLAB} w_{A,B} \delta_{\alpha\gamma}, \tag{24}$$

where  $\mathbf{w}$  is the biasing displacement vector and  $\delta_{\alpha\gamma}$  the Kronecker delta.  $c_{ABCD}$  and  $c_{ABCDEF}$  are the second- and third-order fundamental elastic constants.

*5.2. Biasing deformations due to a normal acceleration*

We consider here the case of cylindrical flexure ( $w_3 = 0, \partial/\partial X_3 = 0$ ). For the biasing deformation due to a normal acceleration, the classical theory for coupled extension and flexure without shear deformation is sufficient. This result can be obtained from the above equations by eliminating thickness-shear as follows. First, we neglect the effect of the rotatory inertia in Eq. (2)<sub>3</sub> and obtain

$$T_{21}^{(0)} = T_{11,1}^{(1)}. \tag{25}$$

Substitution of Eq. (25) into Eq. (2)<sub>2</sub> yields

$$-k_1 T_{11}^{(0)} + T_{11,11}^{(1)} + F_2^{(0)} = 2h\rho_0 \ddot{w}_2^{(0)}, \tag{26}$$

where, to be consistent with the notation in Eq. (24), we have used  $\mathbf{w}$  for the biasing displacement. We also set the shell shear strain in Eq. (4)<sub>2</sub> to zero and obtain

$$w_1^{(1)} = -w_{2,1}^{(0)} + k_1 w_1^{(0)}. \tag{27}$$

Under Eqs. (25) and (27), Eq. (4) becomes

$$\begin{aligned} T_{11}^{(0)} &= 2h\bar{c}_{11}(w_{1,1}^{(0)} + k_1 w_2^{(0)}), \\ T_{12}^{(0)} &= -\frac{2h^3}{3}\bar{c}_{11}(w_{2,111}^{(0)} - k_1 w_{1,11}^{(0)}), \\ T_{11}^{(1)} &= -\frac{2h^3}{3}\bar{c}_{11}(w_{2,11}^{(0)} - k_1 w_{1,1}^{(0)}). \end{aligned} \tag{28}$$

Substituting Eq. (28) into Eq. (2)<sub>1</sub> and Eq. (26), we have the following two equations for  $w_1^{(0)}$  and  $w_2^{(0)}$ :

$$\begin{aligned} 2h\bar{c}_{11}(w_{1,11}^{(0)} + k_1 w_{2,1}^{(0)}) &= 0, \\ -k_1 2h\bar{c}_{11}(w_{1,1}^{(0)} + k_1 w_2^{(0)}) - \frac{2h^3}{3}\bar{c}_{11}(w_{2,1111}^{(0)} - k_1 w_{1,1111}^{(0)}) + \rho_0 2ha_2 &= 0, \end{aligned} \tag{29}$$

where, for a shallow shell under a constant normal acceleration  $a_2$  (Fig. 5), we have used

$$F_2^{(0)} = \rho_0 2ha_2, \tag{30}$$

and set the dynamic terms to zero. The boundary conditions corresponding to Fig. 5 are

$$\begin{aligned} T_{11}^{(0)}(X_1 = \pm a) &= 0, \\ w_2^{(0)}(X_1 = \pm a) &= 0, \\ T_{11}^{(1)}(X_1 = \pm a) &= 0, \end{aligned} \tag{31}$$

which can be satisfied by

$$\begin{aligned} w_2^{(0)} &= \sum_{m \text{ odd}} A_m \cos \alpha_m X_1, & w_1^{(0)} &= \sum_{m \text{ odd}} B_m \sin \alpha_m X_1, \\ \alpha_m &= \frac{m\pi}{2a}, & m &= 1, 3, 5, \dots \end{aligned} \quad (32)$$

Substituting Eq. (32) into Eq. (29), to the first order of the small curvature  $k_1$ , we have

$$A_m = \frac{12\rho_0 a_2}{h^2 m \pi \bar{c}_{11} \alpha_m^4} \sin \frac{m\pi}{2}, \quad B_m = -\frac{k_1}{\alpha_m} A_m. \quad (33)$$

As indicated by Eq. (24), we will need  $w_{K,L}$  to calculate a frequency shift. We begin with

$$\begin{aligned} w_{K,L} &= E_{LK} + \Omega_{LK}, \\ E_{LK} &= (w_{K,L} + w_{L,K})/2 = E_{KL}, \\ \Omega_{LK} &= (w_{K,L} - w_{L,K})/2 = -\Omega_{KL}. \end{aligned} \quad (34)$$

Since  $w_3 = 0$  and  $\partial/\partial X_3 = 0$ , we immediately have

$$E_{31} = E_{32} = E_{33} = 0, \quad \Omega_{31} = \Omega_{32} = 0. \quad (35)$$

Once  $w_1^{(0)}$  and  $w_2^{(0)}$  are found, from Eqs. (1) and (27) the biasing displacement field is given by

$$w_1 = w_1^{(0)} + Y(-w_{2,1}^{(0)} + k_1 w_1^{(0)}), \quad w_2 = w_2^{(0)}, \quad (36)$$

from which we have

$$\begin{aligned} E_{11} &= w_{1,1}^{(0)} + Y(-w_{2,11}^{(0)} + k_1 w_{1,1}^{(0)}), \\ E_{21} &= k_1 w_1^{(0)}, \\ \Omega_{21} &= -2w_{2,1}^{(0)} + k_1 w_1^{(0)}. \end{aligned} \quad (37)$$

Finally, from the following stress relaxation condition of a thin shell:

$$c_{21}E_{11} + c_{22}E_{22} = 0, \quad (38)$$

we obtain the thickness strain as

$$E_{22} = -\frac{c_{12}}{c_{22}} E_{11} = -\frac{c_{12}}{c_{22}} \left[ w_{1,1}^{(0)} + Y(-w_{2,11}^{(0)} + k_1 w_{1,1}^{(0)}) \right]. \quad (39)$$

### 5.3. Unperturbed modes

For the unperturbed modes we are interested in the mode with  $n = 1$  in Eq. (13)

$$u_1 = Y u_1^{(1)} = Y \cos \frac{\pi}{2a} X_1 e^{i\omega t}, \quad (40)$$

with frequency

$$\frac{\omega^2}{\omega_0^2} = 1 + \frac{h^2}{a^2} \left( \frac{\bar{c}_{11}}{c_{66}} + \kappa^2 + \frac{2k_1^2 h^2}{3} \frac{\bar{c}_{11}}{c_{66}} \right). \quad (41)$$

### 5.4. Acceleration sensitivity

It is known that a flexure in the form of Eq. (32)<sub>1</sub> by itself produces zero first-order frequency shifts under a normal acceleration in a symmetric resonator (Vig, 2002). Therefore we need to consider only the biasing

extensional deformation. To the first order of  $k_1$ , we need to consider only the following components of the biasing deformation gradient:

$$w_{1,1} = w_{1,1}^{(0)}, \quad w_{2,2} = -\frac{c_{12}}{c_{22}} w_{1,1}^{(0)}. \tag{42}$$

From Eqs. (24), (32) and (37) we obtain the nonzero components of  $\hat{c}_{K\alpha L\gamma}$  as

$$\begin{aligned} \hat{c}_{1111} &= \left[ 3c_{11} + c_{111} - \frac{c_{12}}{c_{22}}(c_{12} + c_{112}) \right] w_{1,1}, \\ \hat{c}_{2121} &= \left[ 2c_{66} + c_{12} + c_{166} - \frac{c_{12}}{c_{22}}(c_{22} + c_{266}) \right] w_{1,1}. \end{aligned} \tag{43}$$

Then from Eqs. (23) and (37) we have

$$\begin{aligned} \frac{\Delta\omega}{\omega} &= -k_1 \sum_{m \text{ odd}} \frac{A_m \sin \frac{m\pi}{2}}{m\pi(m^2 - 4)} \\ &\times \frac{\left[ 3c_{11} + c_{111} - \frac{c_{12}}{c_{22}}(c_{12} + c_{112}) \right] \pi^2 h^2 (m^2 - 2) - \left[ 2c_{66} + c_{12} + c_{166} - \frac{c_{12}}{c_{22}}(c_{22} + c_{266}) \right] 24a^2}{2\omega_0^2 \rho_0 a^2 h^2}. \end{aligned} \tag{44}$$

The series expansion for the flexural deformation given in Eq. (32) converges rapidly. Using only the leading term rather than keeping the entire series introduces an error of less than 1% for the center deflection (Kosinski et al., 2002). If we consider only the dominant first term of the series then the frequency shift of Eq. (44) is approximately

$$\frac{\Delta\omega}{\omega} = -128\rho_0 k_1 a_2 a^2 \times \frac{\pi^2 \left[ 3c_{11} + c_{111} - \frac{c_{12}}{c_{22}}(c_{12} + c_{112}) \right] + 24(a/h)^2 \left[ 2c_{66} + c_{12} + c_{166} - \frac{c_{12}}{c_{22}}(c_{22} + c_{266}) \right]}{\pi^8 \bar{c}_{11} c_{66} \left[ 1 + \left( \frac{\bar{c}_{11}}{c_{66}} + \kappa^2 + \frac{2k_1^2 h^2}{3} \frac{\bar{c}_{11}}{c_{66}} \right) / (a/h)^2 \right]}. \tag{45}$$

We note that Eq. (45) shows a first order effect linear in  $k_1$  and  $a_2$ .

### 5.5. Numerical results

We consider numerical results for quartz and langasite resonators. The fourteen independent third-order elastic constants of quartz are taken from Thurston et al. (1966), and the other 17 nonzero third-order constants are determined using the relations given by Nelson (1979). Using the tensor transformation rules the third-order elastic constants of rotated Y-cut quartz can be obtained systematically for any  $\theta$ . Consider an AT-cut quartz resonator with the same geometric parameters as the one in Section 3 before Eq. (14). Under  $a_2 = g$ , from Eq. (45) we found

$$\frac{\Delta\omega}{\omega} = 9.56 \times 10^{-8}. \tag{46}$$

For langasite, the 14 independent third-order elastic constants with respect to the crystallographic axes are given by Aleksandrov et al. (1995) and Sorokin et al. (1996). For a Y-cut langasite resonator with the same geometry, we obtain, for  $a_2 = g$ ,

$$\frac{\Delta\omega}{\omega} = 1.26 \times 10^{-8}. \tag{47}$$

## 6. Discussion

The results of this analysis are important for the  $10^{-10} \text{ g}^{-1}$  production technology. They suggest that, due to the curvature of the middle plane, the normal acceleration sensitivity of symmetric plano-convex or

concave–convex resonators is not zero and is bounded from below by this effect. It is also interesting to note that the Y-cut langasite resonator is expected to be substantially better than an AT-cut quartz resonator.

The extensional deformation of the middle plane, the accompanying thickness contraction, and the related frequency shift can be due to different effects. Kosinski et al. (2002) and the present paper consider two of these different effects. In Kosinski et al. (2002) the frequency shift was due to the second-order effect of a relatively large deflection under a relatively large normal acceleration. It exists for both flat plate and contoured resonators but disappears when the acceleration is small. In the present paper the frequency shift is due to the middle plane curvature. This effect exists when the acceleration is small but disappears when the resonator is a flat plate. For a plano-convex resonator under  $a_2 = g$ , the frequency shift for the cylindrical flexure considered in this paper is as large as  $10^{-8} \text{ g}^{-1}$  which is much larger than the effect of  $10^{-11} \text{ g}^{-1}$  considered in Kosinski et al. (2002). We plan to conduct a systematic study of the present effect for the case of non-cylindrical flexure, and to analyze such phenomena as aspect-ratio compensation and the functional dependence on the radius of curvature (contour).

In all works before Kosinski et al. (2002) and before the present paper, the above two effects were neglected. For example, for a perfectly symmetric AT-cut plano-convex resonator the normal acceleration sensitivity was calculated to be zero (Tiersten and Shick, 1990; Zhou and Tiersten, 1991) unless some other effects like the rotation of the support system (Tiersten and Shick, 1990) or the offset of the mode center (Zhou and Tiersten, 1991) destroy the symmetry. The effect of support rotation is on the orders of  $10^{-13} \text{ g}^{-1}$  (Tiersten and Shick, 1990), and the effect of mode center offset is on the order of  $10^{-10} \text{ g}^{-1}$  per millimeter of offset (Zhou and Tiersten, 1991). These are smaller than the effect of middle plane curvature.

## 7. Conclusion

A plano-convex resonator or a concave–convex resonator has a curved middle plane and is a shallow shell. Middle plane curvature causes coupling to extension. This coupling raises the thickness-shear resonant frequencies through a small, second order effect. To avoid strong coupling of the fundamental thickness-shear mode to a higher order extensional mode due to middle plane curvature, certain values of the aspect ratio should be avoided. Middle plane curvature causes coupling to extension when the resonator is under a normal acceleration, which can induce a frequency shift in AT-cut quartz and Y-cut langasite resonators of the order of  $10^{-8} \text{ g}^{-1}$ .

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