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# Fan Cohomology and Equivariant Chow Rings of Toric Varieties

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# FAN COHOMOLOGY AND EQUIVARIANT CHOW RINGS OF TORIC VARIETIES

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FAN COHOMOLOGY AND EQUIVARIANT CHOW RINGS

OF TORIC VARIETIES

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Toric varieties are varieties equipped with a torus action and constructed from cones and fans. In the joint work with Suanne Au and Mark E. Walker, we prove that the equivariant $K$-theory of an affine toric variety constructed from a cone can be identified with a group ring determined by the cone. When a toric variety $X(\Delta)$ is smooth, we interpret equivariant $K$-groups as presheaves on the associated fan space $\Delta$. Relating the sheaf cohomology groups to equivariant $K$-groups via a spectral sequence, we provide another proof of a theorem of Vezzosi and Vistoli: equivariant $K$-theory is formed by patching equivariant $K$-theory of equivariant affine toric subvarieties.

This dissertation studies the sheaf cohomology groups for the equivariant $K$-groups tensored with $\mathbb{Q}$ and completed, and how they relate to the equivariant $K$-groups of non-smooth and non-affine toric varieties. The equivariant $K$-groups tensored with $\mathbb{Q}$ and completed coincide with the equivariant Chow rings for affine toric varieties. For a three-dimensional complete fan , we calculate the dimensions of the sheaf cohomology groups for the symmetric algebra sheaf. When the fan is given by a convex polytope, this information computes the equivariant $K$-groups tensored with $\mathbb{Q}$ and completed as extensions of sheaf cohomology groups.

# ACKNOWLEDGMENTS

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# Contents

# Chapter 1

# Introduction

Equivariant $K$-theory was first developed in the 1980s by Thomason [15] and later by Merkurjev [11] and others. Let $G$ be an algebraic group acting on a variety $X$ over a field $k$. One considers $\mathcal{O}_X$-modules equipped with a $G$-action compatible with the action on $X$. Equivariant $K$-groups are defined using the exact category of locally free coherent $G$-modules. Here we give an overview of the dissertation and details will be given in the later chapters.

A toric variety is a variety over a field $k$ equipped with an action of an algebraic torus $T$ and constructed from a cone or a fan. Toric varieties first appeared in algebraic geometry in the 1970s in connection with compactification problems. (See [12], [5] for references.) The combinatorial structure from the cones and fans allows explicit descriptions of many geometric properties of toric varieties. Toric varieties have provided a rich class of examples in algebraic geometry, linking algebraic geometry with combinatorial and convex geometry, commutative algebra, and many other fields. Let $N$ be a lattice, $M := \mathrm{Hom}(N, \mathbb{Z})$ be the dual lattice, and $\sigma$ be a cone in $N_{\mathbb{R}}$. An affine toric variety $U_\sigma = \mathrm{Spec}[\sigma^\vee \cap M]$ corresponds to a single cone $\sigma$. When cones "glue" together to form a fan $\Delta$, one obtains a toric variety

$$X = X(\Delta) = \varinjlim_{\sigma \in \Delta} U_\sigma.$$

The following results concerning equivariant $K$-theory of affine and smooth toric varieties are due to joint work with Suanne Au and Mark E. Walker. Let $U_\sigma$ be an affine toric variety. The equivariant $K$-groups of $U_\sigma$ are of the form [1, Theorem 4]

$$K_q^T(U_\sigma) \cong K_q(k) \otimes_{\mathbb{Z}} \mathbb{Z}[M_\sigma].$$

where $M_\sigma := \frac{M}{\sigma^\perp \cap M}$. If $X = X(\Delta)$ is a smooth toric variety associated to a fan $\Delta$, a theorem of Vezzosi and Vistoli [17], [18] states that equivariant $K$-groups behave like sheaves and one may patch equivariant $K$-groups of equivariant affine subvarieties to form $K_q^T(X(\Delta))$. More precisely, the following sequence (3.4) is exact.

$$\begin{aligned} 0 \to K_q^T(X) \to \bigoplus_{\sigma \in Max(\Delta)} K_q(k) \otimes_{\mathbb{Z}} \mathbb{Z}[M_\sigma] \to \bigoplus_{\delta < \tau} K_q(k) \otimes_{\mathbb{Z}} \mathbb{Z}[M_{\delta \cap \tau}] \\ \to \bigoplus_{\delta < \tau < \epsilon} K_q(k) \otimes_{\mathbb{Z}} \mathbb{Z}[M_{\delta \cap \tau \cap \epsilon}] \to \cdots . \end{aligned}$$

In particular, one has

$$K_q^T(X) \cong K_q(k) \otimes_{\mathbb{Z}} K_0^T(X).$$

Au, Walker and I give a new proof of the Vezzosi-Vistoli's theorem, interpreting $K_q^T(-)$ as a sheaf on the poset $\Delta$.

Given a fan $\Delta$, a subset $\Lambda \subseteq \Delta$ is open if for all $\sigma \in \Lambda$, if $\tau$ is a face of $\sigma$, then $\tau \in \Lambda$. Let $\langle \sigma \rangle$ denote the smallest open subset containing $\sigma$; it consists of $\sigma$ and all its faces. For an open subset $\Lambda \subseteq \Delta$, define $K_q^T(\Lambda) := K_q^T(X(\Lambda))$. We may view $K_q^T(-)$ as a presheaf on the finite poset $\Delta$ viewed as a topological space. In particular, its stalk at $\sigma$ is

$$K_q^T(\langle \sigma \rangle) := K_q^T(U_\sigma) \cong K_q(k) \otimes_{\mathbb{Z}} \mathbb{Z}[M_\sigma].$$

One can then consider Čech cohomology groups

$$\begin{aligned}
\check{H}^p(\Delta, K_q^T(-)^{\sim}) &= H^p(\bigoplus_{\sigma\in Max(\Delta)} K_q^T(U_\sigma) \to \bigoplus_{\delta<\tau} K_q^T(U_{\delta\cap\tau}) \to \bigoplus_{\delta<\tau<\epsilon} K_q^T(U_{\delta\cap\tau\cap\epsilon}) \to \cdots) \\
&= H^p(\bigoplus_{\sigma\in Max(\Delta)} K_q(k)\otimes_{\mathbb{Z}} \mathbb{Z}[M_\sigma] \to \bigoplus_{\delta<\tau} K_q(k)\otimes_{\mathbb{Z}} \mathbb{Z}[M_{\delta\cap\tau}] \\
&\qquad \to \bigoplus_{\delta<\tau<\epsilon} K_q(k)\otimes_{\mathbb{Z}} \mathbb{Z}[M_{\delta\cap\tau\cap\epsilon}] \to \cdots).
\end{aligned}$$

When $X(\Delta)$ is smooth, there is a spectral sequence

$$\check{H}^p(\Delta, K_q^T(-)^{\sim}) \Longrightarrow K_{q-p}^T(X(\Delta)).$$

using Thomason's work [15] and techniques in [16]. The associated sheaf of equivariant $K$-groups is flasque. It follows that $\check{H}^p(\Delta, K_q^T(-)^{\sim}) = 0$ for all $p > 0$ and the spectral sequence collapses to give the exactness of the sequence (3.4).

When $X = X(\Delta)$ is not smooth but quasi-projecitve, one still has a convergent spectral sequence

$$\check{H}^p(\Delta, K_q^T(-)^{\sim}) \Longrightarrow K_{q-p}^T(X(\Delta))$$

established in [20]. One may therefore understand equivariant $K$-groups on $X(\Delta)$ via computations of Čech cohomology groups.

Let $X = X(\Delta)$ be a toric variety associated to a fan $\Delta$. Consider the map $K_q^T(X(\Delta)) \to K_q(X(\Delta))$ induced by the forgetful functor. Note that the map factors through the completion

$$\begin{tikzcd}
K_q^T(X(\Delta)) \arrow[r] \arrow[d] & K_q(X(\Delta)) \\
\left(K_q^T(X(\Delta))\right)_I^{\wedge} \arrow[ur, dashed] &
\end{tikzcd}$$

where $I$ is the augmented ideal of the representation ring identified with $K_0^T(point)$, and when $X(\Delta)$ is quasi-projective we have the spectral sequence

$$\left(\check{H}^p(\Delta, K_q^T)_{\mathbb{Q}}\right)_I^\wedge \Longrightarrow \left(K_{q-p}^T(X(\Delta))_{\mathbb{Q}}\right)_I^\wedge . \tag{1.1}$$

For a cone $\sigma$, let $S_{\mathbb{Z}}^\bullet(M_\sigma)$ denote the ring of integral polynomial functions on $\sigma$. Over a toric variety $X = X(\Delta)$, Payne proved in [13] that the equivariant Chow ring $A_T^*(X)$ is naturally isomorphic to the ring of piecewise polynomial functions on the fan $\Delta$,

$$PP^*(\Delta) := \{f : |\Delta| \to \mathbb{R} \mid f|_\sigma \in S_{\mathbb{Z}}^\bullet(M_\sigma) \text{ for each } \sigma \in \Delta\}.$$

Let $I$ denote the augmented ideal in $\mathbb{Q}[M]$ and $\mathfrak{m}$ denote the unique homegeneous maximal ideal in $S_{\mathbb{Q}}^\bullet(M_{\mathbb{Q}})$. Over an affine toric variety $U_\sigma$, it follows from Proposition 4.2.1 that we have isomorphisms

$$\left(K_0^T(U_\sigma)_{\mathbb{Q}}\right)_I^\wedge \cong (\mathbb{Q}[M_\sigma])_I^\wedge \cong \left(S_{\mathbb{Q}}^\bullet(M_{\sigma_{\mathbb{Q}}}\right)_{\mathfrak{m}}^\wedge \cong (A_T^*(U_\sigma))_{\mathfrak{m}}^\wedge .$$

Let $\mathcal{F}$ be the sheaf on $\Delta$ associated to the functor $\sigma \mapsto S_{\mathbb{Q}}^\bullet(M_{\sigma_{\mathbb{Q}}})$ together with maps induced by face inclusions. Note $H^0(\Delta, \mathcal{F}) \cong PP^*(\Delta)$. This thesis focuses on computing $\left(\check{H}^p(\Delta, K_q^{T\sim})\right)_I^\wedge \cong H^p\left(\Delta, \mathcal{F}\right)_{\mathfrak{m}}^\wedge \otimes_{\mathbb{Q}} K_q(k)_{\mathbb{Q}}$ for three-dimensional complete fans and connect them to $\left(K_q^T(X(\Delta))\right)_I^\wedge$ when $X = X(\Delta)$ is quasi-projective.

The following is the layout of the thesis.

Chapter 2 gives detailed definitions and outlines properties of toric varieties and equivariant $K$-theory.

Chapter 3 reviews the joint work with Suanne Au and Mark E. Walker mentioned above. The chapter also develops tools for computing sheaf cohomology groups on a poset.

Chapter 4 discusses the the link between equivariant Chow Rings and Equivariant $K$-theory. For $X = X(\Delta)$ where the associated fan $\Delta$ is a simplicial fan, the equivariant

$K$-theory can be described explicitly.

**Theorem 4.4.1** *Let $X = X(\Delta)$ be a toric variety associated to a simlicial fan $\Delta$. Then the symmetric algebra sheaf $\mathcal{F}$ on $\Delta$ is flasque. Moreover,*

$$0 \to \left(K_q^T(X)_{\mathbb{Q}}\right)_I^{\wedge} \to \bigoplus_{\sigma \in Max(\Delta)} \left(K_q^T(U_\sigma)_{\mathbb{Q}}\right)_I^{\wedge} \to \bigoplus_{\delta < \tau} \left(K_q^T(U_{\delta \cap \tau})_{\mathbb{Q}}\right)_I^{\wedge} \to \cdots$$

*and*

$$\begin{aligned} 0 \to \left(K_q^T(X)_{\mathbb{Q}}\right)_I^{\wedge} \to & \bigoplus_{\sigma \in Max(\Delta)} K_q(k)_{\mathbb{Q}} \otimes_{\mathbb{Q}} S^{\bullet}_{\mathbb{Q}}(M_{\sigma_{\mathbb{Q}}})^{\wedge}_{\mathfrak{m}} \\ \to & \bigoplus_{\delta < \tau} K_q(k)_{\mathbb{Q}} \otimes_{\mathbb{Q}} S^{\bullet}_{\mathbb{Q}}(M_{\delta \cap \tau_{\mathbb{Q}}})^{\wedge}_{\mathfrak{m}} \to \bigoplus_{\delta < \tau < \epsilon} K_q(k)_{\mathbb{Q}} \otimes_{\mathbb{Q}} S^{\bullet}_{\mathbb{Q}}(M_{\delta \cap \tau \cap \epsilon_{\mathbb{Q}}})^{\wedge}_{\mathfrak{m}} \to \cdots \end{aligned}$$

*are exact, and there is an isomorphism*

$$\left(K_q^T(X)_{\mathbb{Q}}\right)_I^{\wedge} \cong \left(K_q(k)\right)_{\mathbb{Q}} \otimes_{\mathbb{Q}} \left(A_T^*(X(\Delta))_{\mathbb{Q}}\right)^{\wedge}_{\mathfrak{m}}.$$

For any fan $\Delta$, the two-skeleton is simplicial. It follows from Theorem 4.4.1 that

$$H^p(\Delta, \mathcal{F}) = 0 \text{ for all } p \geq n-1.$$

When $\Delta$ is a three-dimensional complete fan given by a polytope, the spectral sequence (1.1) reduces to

$$0 \to H^1(\Delta, \mathcal{F})^{\wedge}_{\mathfrak{m}} \otimes_{\mathbb{Q}} K_{n+1}(k)_{\mathbb{Q}} \to \left(K_n^T(X(\Delta))_{\mathbb{Q}}\right)_I^{\wedge} \to H^0(\Delta, \mathcal{F})^{\wedge}_{\mathfrak{m}} \otimes_{\mathbb{Q}} K_n(k)_{\mathbb{Q}} \to 0. \quad (1.2)$$

Chapter 5 focuses on the study of $\left(H^p(\Delta, K_q^{T\sim})\right)_I^\wedge \cong H^p\left(\Delta, \mathcal{F}\right)_{\mathfrak{m}}^\wedge \otimes_{\mathbb{Q}} K_q(k)_{\mathbb{Q}}$ for three-dimensional complete fans $\Delta$. We define the degree $r$ symmetric power sheaf $\mathcal{F}^{(r)}$ on $\Delta$ to be the sheaf associated to the functor $\sigma \mapsto S_Q^r(M_{\sigma_{\mathbb{Q}}})$. We construct a new poset $\widetilde{\Delta}$ and associate to it the degree $r$ symmetric power sheaf $\widetilde{\mathcal{F}^{(r)}}$. Using properties of sheaves on projective spaces, we conclude in Theorem 5.4.1 that the degree $r$ symmetric power sheaf $\widetilde{\mathcal{F}^{(r)}}$ on $\widetilde{\Delta}$ is flasque for $r \gg 0$. This technical result allows us to deduce the following.

**Theorem 5.4.2** *If $\Delta$ is a three-dimensional complete fan in general position and every maximal cone has at most $N$ facets, then $H^1(\Delta, \mathcal{F}^{(r)}) = 0$ for $r \geq 2N-5$.*

**Theorem 5.4.3** *For a three-dimensional complete fan $\Delta$, the Hilbert polynomial of $H^1(\Delta, \mathcal{F}^{(r)})$ as a function of $r$ is a constant.*

When $\Delta$ is a complete three-dimensional fan in general position, using the short exact sequence (1.2) we see that only lower degree terms of $H^1\left(\Delta, \mathcal{F}\right)_{\mathfrak{m}}^\wedge$ contribute to $\left(K_n^T(X(\Delta))_{\mathbb{Q}}\right)_I^\wedge$. In particular, we have the following observation.

**Corollary 5.4.5** *If $\Delta$ is a fan given by a three-dimensional polytope in general position, $\left(K_{-1}^T(X(\Delta))_{\mathbb{Q}}\right)_I^\wedge$ is a finite dimensional vector space over $\mathbb{Q}$.*

# Chapter 2

# Background

## 2.1 Toric Varieties

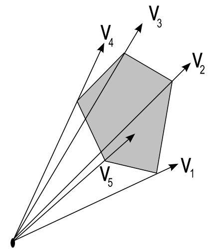
A toric variety is constructed from cones and fans defined on a lattice $N$. The combinatorial structure from cones and fans give concrete descriptions to many properties of the varieties.

### 2.1.1 Cones and Fans

We will follow the notations and the construction in [5]. Let $N$ be a lattice, that is, a free abelian group isomorphic to $\mathbb{Z}^n$ for some $n$. Let $N_{\mathbb{R}}$ be the vector space $N \otimes_{\mathbb{Z}} \mathbb{R}$.

**Definition 2.1.1.** A rational convex polyhedral cone $\sigma$ in $N_{\mathbb{R}}$ generated by $v_1, ..., v_s \in N$ is the set

$$\sigma := \{r_1 v_1 + ... + r_s v_s \in N_{\mathbb{R}} \mid r_i \geq 0\}.$$



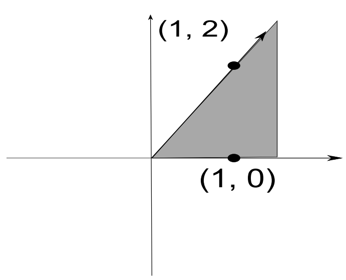

The *dimension* $dim(\sigma)$ of $\sigma$ is the dimension of the vector space $\mathbb{R}\sigma := \sigma + (-\sigma)$ spanned by $\sigma$. A rational convex polyhedral cone is *strongly convex* if $\sigma \cap (-\sigma) = \{0\}$. In the future, a cone refers to a strongly convex rational polyhedral cone unless otherwise specified.

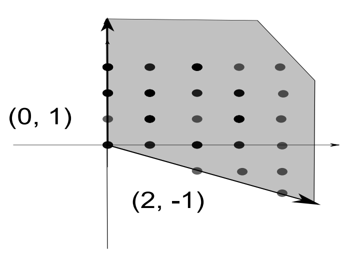
Let $M := \mathrm{Hom}(N, \mathbb{Z})$ denote the dual lattice with dual pairing and $\langle m, n \rangle$ denote the evaluation for all $m \in M$ and $n \in N$.

**Definition 2.1.2.** The dual cone $\sigma^\vee$ of any rational polyhedral cone $\sigma$ is the set of elements in $(N_\mathbb{R})^*$ that are non-negative on $\sigma$:

$$\sigma^\vee := \{u \in (N_\mathbb{R})^* \mid \langle u, v \rangle \geq 0 \text{ for all } v \in \sigma\}.$$

**Example 2.1.3.** Suppose $N = \mathbb{Z}^2$ and identify $M = \mathrm{Hom}(N, \mathbb{Z}) \cong \mathbb{Z}^2$ with $N$. Let $\sigma$ be the two-dimensional cone generated by $(0, 1)$ and $(1, 2)$. Then $\sigma^\vee$ is generated by $(2, -1)$ and $(0, 1)$.




A cone $\sigma$ $\qquad\qquad$ $\sigma^\vee \cap M$

It follows from the theory of convex sets that $(\sigma^\vee)^\vee = \sigma$ for any rational convex polyhedral cone $\sigma$. Note that the dual of a strongly convex rational polyhedral cone is not necessarily strongly convex.

A *face* $\tau$ of $\sigma$ is the intersection of $\sigma$ with any supporting hyperplane,

$$\tau = \sigma \cap u^{\perp} = \{v \in \sigma \mid \langle u, v \rangle = 0\} \text{ for some } u \in \sigma^{\vee}.$$

If $\sigma$ is generated by $v_1, ..., v_s$, the face $\sigma \cap u^{\perp}$ is generated by those vectors $v_i$ such that $\langle u, v_i \rangle = 0$. It follows that a rational convex polyhedral cone has only finitely many faces. If a cone $\sigma$ is strongly convex, the rays generated by a minimal set of generators are exactly the one-dimensional faces of $\sigma$.

A cone is regarded as a face of itself. A *facet* is a face of codimension one.

Here are a few facts about faces of a rational convex polyhedral cone and the associated monoids. See [5] for proofs.

**Proposition 2.1.4.** *Let $\sigma$ be a rational convex polyhedral cone.*

1. *Any face is a rational convex polyhedral cone.*

2. *Any intersection of faces is also a face.*

3. *Any proper face is contained in some facet.*

4. *Any proper face in the intersection of all facets containing it.*

5. *(Farkas' Theorem). The dual of a rational convex polyhedral cone is a rational convex polyhedral cone.*

6. *(Gordan's Lemma). The set $\sigma^{\vee} \cap M$ is a finitely generated monoid.*

7. *If $\tau$ is a face of $\sigma$, then $\sigma^{\vee} \cap \tau^{\perp}$ is a face of $\sigma^{\vee}$, with $dim(\tau) + dim(\sigma^{\vee} \cap \tau^{\perp}) = dim N_{\mathbb{R}}$. This sets up a one-to-one order-reversing correspondence between the faces of $\sigma$ and the faces of $\sigma^{\vee}$.*

8. *Let $u \in \sigma^\vee \cap M$. Then $\tau = \sigma \cap u^\perp$ is a face of $\sigma$. All faces of $\sigma$ have this form, and*

$$\tau^\vee \cap M = (\sigma^\vee \cap M) + \mathbb{Z} \cdot u,$$

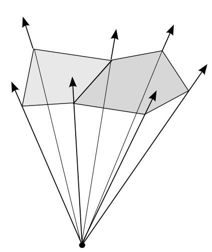
*the submonoid of $M$ generated by $\sigma^\vee \cap M$ and $-u$.*

One may "glue" strongly convex rational polyhedral cones together to form a fan as follows.

**Definition 2.1.5.** A fan $\Delta$ in $N$ is a set of strongly convex rational polyhedral cones in $N_\mathbb{R}$ such that the following hold:

1. Each face of a cone in $\Delta$ is also a cone in $\Delta$.

2. The intersection of two cones in $\Delta$ is a face of each.



**Proposition 2.1.6.** *[5, Proposition 3, p. 14] If $\sigma$ and $\sigma'$ are rational convex polyhedral cones whose intersection $\tau$ is a face of each, then we have*

$$\tau^\vee \cap M = (\sigma^\vee \cap M) + ((\sigma')^\vee \cap M).$$

### 2.1.2 Building Varieties

A rational convex polyhedral cone $\sigma$ gives a finitely generated monoid $\sigma^\vee \cap M$ where $M = \mathrm{Hom}(N, \mathbb{Z})$. Over a field $k$, one can associate a monoid ring $k[\sigma^\vee \cap M]$ and an *affine toric* *variety* to $\sigma$

$$U_\sigma := \operatorname{Spec} k[\sigma^\vee \cap M].$$

In the monoid ring, operations in the monoid $\sigma^\vee \cap M$ are written multiplicatively using a symbol $\chi$, that is, $\chi^u \cdot \chi^v = \chi^{u+v}$ for all $u, v \in \sigma^\vee \cap M$. For each face $\tau$ of $\sigma$, the inclusion $U_\tau \hookrightarrow U_\sigma$ corresponds to a localization map $\operatorname{Spec} k[\sigma^\vee \cap M] \to \operatorname{Spec} k[\tau^\vee \cap M]$ by Proposition 2.1.4.8. In particular, associated to the zero cone $\{0\}$ we have the monoid ring $k[M]$ and the algebraic torus $T := T_N = \operatorname{Spec} k[M]$. The action of $T$ on itself $T \times T \to T$ is given by the ring map $\chi^u \mapsto \chi^u \otimes \chi^u$ for $u \in M$ from $k[M]$ to $k[M] \otimes k[M]$. The action of $T$ on $U_\sigma$ is given by the map from $k[\sigma^\vee \cap M]$ to $k[M] \otimes k[\sigma^\vee \cap M]$ that sends $\chi^u$ to $\chi^u \otimes \chi^u$ for $u \in M$. The action is compatible with open embeddings of $U_\tau$ for faces $\tau$ of $\sigma$.

**Example 2.1.7.** Recall the two-dimensional cone $\sigma$ generated by $(0, 1)$ and $(1, 2)$ in $N_\mathbb{R}$ where $N = \mathbb{Z}^2$ and $M = \operatorname{Hom}(N, \mathbb{Z}) \cong \mathbb{Z}^2$ is identified with $N$. The monoid $\sigma^\vee \cap M$ is generated by $(2, -1)$, $(1, 0)$, and $(0, 1)$. Hence

$$U_\sigma = \operatorname{Spec} k[x^2 y^{-1}, x, y].$$

The relation among the generators of $\sigma^\vee \cap M$, namely $(0, 1) + (2, -1) - 2(1, 0) = 0$ can be written multiplicatively as $yz - x^2 = 0$, and $k[x^2 y^{-1}, x, y] \cong k[x, y, z]/(yz - x^2)$.

Let $\Delta$ be a fan in $N$. The toric variety associated to $\Delta$ is

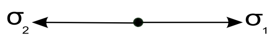
$$X = X(\Delta) := \varinjlim_{\sigma \in \Delta} U_\sigma,$$

the colimit in the category of varieties. Intuitively, $X = X(\Delta)$ is the result of gluing affine toric varieties of maximal cones in $\Delta$ along the affine toric varieties corresponding to their intersections. Moreover, the torus action by $T := T_N$ extends to the whole variety $T \times X \to X$

and $\{U_\sigma \mid \sigma \in \Delta\}$ forms an open cover of $X$ consisting of equivariant affine open subvarieties.

**Example 2.1.8.** $\mathbb{P}^1$

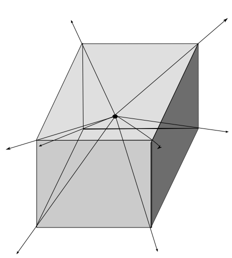
Let $N = \mathbb{Z}$ and identify $M = \mathrm{Hom}(N, \mathbb{Z}) \cong \mathbb{Z}$ with $N$. Consider the following fan $\Delta$ in $\mathbb{R}$.




Note $(\sigma_1)^\vee = \sigma_1$, $(\sigma_2)^\vee = \sigma_2$ and $(\sigma_1 \cap \sigma_2)^\vee = (\{0\})^\vee = \mathbb{R}$. We have $U_{\sigma_1} = \operatorname{Spec} k[x]$, $U_{\sigma_2} = \operatorname{Spec} k[x^{-1}]$, and $U_{\sigma_1 \cap \sigma_2} = \operatorname{Spec} k[x, x^{-1}]$. Hence $X(\Delta) = \varinjlim_{\sigma \in \Delta} U_\sigma = \mathbb{P}^1$.

Suppose there is a homomorphism of lattices $\phi : N' \to N$ and $\Delta$ is a fan in $N$, $\Delta'$ a fan in $N'$, satisfying the condition: for each cone $\sigma'$ in $\Delta'$, there is some cone $\sigma$ in $\Delta$ such that $\phi(\sigma') \subseteq \sigma$. Then there is an induced morphism $\phi_* : X(\Delta') \to X(\Delta)$. If $N = N'$, then the induced morphism commutes with the torus action and we say that the map is $T := T_N$-*equivariant.*

A *convex polytope* $K$ in a finite dimensional vector space $E$ is the convex hull of a finite set of points. A proper face $F$ of $K$ is $F := \{v \in K \mid \langle u, v\rangle = r\}$ where $u \in E^*$ is a function with $\langle u, v\rangle \geq r$ for all $v \in K$. We will assume that a convex polytope always contains the origin in its interior. One can then form a cone over each proper face of the convex polytope $K$ with the origin and the convex polytope gives rise to a fan.



It can be shown that a toric variety is normal [5, Sec. 2.1, p. 29]. On the other hand, a normal variety $X$ containing a torus $T_N$ as a dense open subvariety and having an action of $T_N$ that extends the canonical action of $T_N$ on itself can be realized as a toric variety associated to a fan $\Delta$ in $N_{\mathbb{R}}$ [12, Theorem 1.5].

### 2.1.3 Properties of Toric Varieties Described Combinatorially

Let $X = X(\Delta)$ be a toric variety associated to a fan $\Delta$. Many geometric properties of $X$ can be described on the associated fan $\Delta$.

We say that a cone $\sigma$ is *smooth* if and only if its minimal generators form a part of a basis of the underlying lattice $N$. Let $\mathbb{G}_m^1 := \operatorname{Spec} k[t, t^{-1}]$. It follows that an affine toric variety $U_\sigma$ is smooth if and only if

$$U_\sigma \cong (\mathbb{A}_k)^i \times (\mathbb{G}_m^1)^{n-i} \text{ where } n = dim(N_{\mathbb{R}}) \text{ and } i = dim(\sigma).$$

A fan $\Delta$ is *smooth* if and only if every cone $\sigma \in \Delta$ is smooth. A toric variety $X = X(\Delta)$ is smooth if $\Delta$ is smooth.

A cone $\sigma$ is *simplicial* if $\sigma$ has $i$ generators where $i = dim(\sigma)$. A fan $\Delta$ is *simplicial* if every cone in $\Delta$ is simplicial.

For a fan $\Delta$, its *support* $|\Delta|$ refers to the underlying topological space of the union of the cones in $\Delta$. A fan is *complete* if $|\Delta| = N_{\mathbb{R}}$. A toric variety $X = X(\Delta)$ associated to a fan $\Delta$ is *complete* if and only if $|\Delta| = N_{\mathbb{R}}$ [5, Sec. 2.4, p. 39].

A toric variety $X = X(\Delta)$ is projective if and only if $\Delta$ arises from a convex polytope [12, Cor 2.16, p. 84, Prop. 2.19, p. 87].

## 2.2 Equivariant $K$-theory

In this section, we introduce equivariant $K$-theory for toric varieties. The construction of equivariant $K$-theory in a more general setting can be found in [15].

Let $X = X(\Delta)$ be a toric variety associated to a fan $\Delta$ in $N_{\mathbb{R}}$. The action of the torus $T := T_N$ on $X$ is given by the map $\theta : T \times X \to X$, and it satisfies the usual associative and unital identities for an action. A *quasi-coherent $T$-sheaf* on $X$ is a quasi-coherent $\mathcal{O}_X$-sheaf $\mathcal{F}$ on $X$, together with an isomorphism of $\mathcal{O}_{T\times X}$-modules on $T \times X$

$$\phi : \theta^* \mathcal{F} \xrightarrow{\cong} p_2^* \mathcal{F}$$

where $p_2$ is the projection on the second component.

Let $\mu : T \times T \to T$ be the multiplication, and $e : \operatorname{Spec} k \to T$ be the unit. This isomorphism $\phi$ must be associative in the sense that it satisfies the condition on

$$T \times T \times X : (p_{23}^* \phi) \circ ((1 \times \theta)^* \phi) = (\mu \times 1)^* \phi$$

where $p_{23} : T \times T \times X \to T \times X$ is the projection onto the second and the third components. The unital normalization condition $(e \times 1)^* \phi = id$ follows immediately.

A morphism $\alpha : \mathcal{F} \to \mathcal{G}$ between two quasi-coherent $T$-sheaves is $T$-equivariant if it commutes with the action by $T$ in the obvious sense.

If $\mathcal{F}$ is a quasi-coherent $T$-sheaf that is coherent on $X$, we call it a *coherent $T$-sheaf*. If $\mathcal{F}$ is locally free as a $T$-sheaf, we say $\mathcal{F}$ is a *locally free $T$-sheaf*.

Consider the cateogory of $T$-sheaves on $X$. We have the abelian subcategory of coherent $T$-sheaves and the exact subcategory of locally free coherent $T$-sheaves. Let $\mathcal{K}(T, X)$ denote the algebraic K-theory spectrum of the exact subcategory of locally free coherent $T$-sheaves.

The *equivariant K-groups* are defined as

$$K_q^T(X) := \pi_q \mathcal{K}(T, X) \text{ for } q \geq 0.$$

The tensor product of $T$-sheaves induces a ring structure on $K_0^T(X)$ and a module structure on $K_q^T(X)$ over $K_0^T(X)$ [11]. In fact, $K_*^T(X) := \bigoplus K_q^T(X)$ is a graded ring.

The construction of negative equivariant $K$-groups is analogous to that of classical negative $K$-groups in [16]. See [20, Section 7] for details.

# Chapter 3

# Review of Results of Equivariant $K$-theory

## 3.1 $K_q^T$ on Affine Toric Varieties

### 3.1.1 Action and Grading

Let $X = X(\Delta)$ be a toric variety associated to a fan $\Delta$ in $N_{\mathbb{R}}$ and $\mathcal{F}$ be a quasi-coherent $T$-sheaf on $X$. Note that the character group $\mathrm{Hom}(T, \mathbb{G}_m^1)$ of $T$ is canonically isomorphic to the dual lattice $M = \mathrm{Hom}(N, \mathbb{Z})$. We will show that the group of sections $\Gamma(X, \mathcal{F})$ is graded by $M = \mathrm{Hom}(N, \mathbb{Z})$. In particular, if $X = U_\sigma$ be an affine toric variety associated to a cone $\sigma$, the monoid ring $R := \Gamma(X, \mathcal{O}_X) = k[\sigma^\vee \cap M]$ is an $M$-graded ring, and $\Gamma(X, \mathcal{F})$ is an $M$-graded module over $R$. If $\mathcal{F}$ is a locally free coherent $T$-sheaf over $X = U_\sigma$, then $\Gamma(X, \mathcal{F})$ is an $M$-graded projective module over $R$.

Recall the following definitions and notations from Section 2.2. The action of the torus $T$ on $X$ is given by the map $\theta : T \times X \to X$ satisfying the usual associative and unital identities for an action. A quasi-coherent $T$-sheaf on $X$ is a quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$

on $X$, together with an isomorphism of $\mathcal{O}_{T\times X}$-modules on $T \times X$

$$\phi : \theta^* \mathcal{F} \xrightarrow{\cong} p_2^* \mathcal{F}$$

where $p_2 : T \times X \to X$ is the projection on the second component. This isomorphism $\phi$ must be associative in the sense that it satisfies the equation

$$(p_{23}^* \phi) \circ ((1 \times \theta)^* \phi) = (\mu \times 1)^* \phi$$

where $\mu : T \times T \to T$ is the multiplication and $p_{23} : T \times T \times X \to T \times X$ is the projection onto the second and the third components.

**Lemma 3.1.1.** *Let $\theta_X$ and $\theta_Y$ denote the action of $T$ on $X$ and $Y$ respectively. Let $f : X \to Y$ be a $T$-equivariant morphism. Then the following diagram is a pull-back square.*

$$\begin{tikzcd} T \times X \arrow[r, "\theta_X"] \arrow[d, "1\times f"'] & X \arrow[d, "f"] \\ T \times Y \arrow[r, "\theta_Y"] & Y \end{tikzcd} \tag{3.1}$$

*Proof.* Let $S$ be a testing object such that the outer square commutes.

$$\begin{tikzcd} S \arrow[dr, dotted, "\alpha"] \arrow[drr, bend left, "b"] \arrow[ddr, bend right, "a"'] & & \\ & T \times X \arrow[r, "\theta_X"] \arrow[d, "1\times f"'] & X \arrow[d, "f"] \\ & T \times Y \arrow[r, "\theta_Y"] & Y \end{tikzcd}$$

We need to show that there exists a map $\alpha$ such that $b = \theta_X \circ \alpha$ and $(1 \times f) \circ \alpha = a$.

Let $\gamma$ be the "anti-shearing" map from $T \times X$ to $T \times X$ defined by $(t, x) \mapsto (t, t^{-1}x)$ for all $x \in X$ and $t \in T$. Note $\theta_X \circ \gamma = p_2$ and $\theta_Y \circ \gamma = p_2$.

$$\begin{tikzcd}
 & S \arrow[d, dotted, "\alpha"] \arrow[dd, bend right, "a"'] \arrow[dr, bend left, "b"] & \\
T \times X \arrow[r, "\gamma"] \arrow[d, "1\times f"'] & T \times X \arrow[r, "\theta_X"] \arrow[d, "1\times f"'] & X \arrow[d, "f"] \\
T \times Y \arrow[r, "\gamma"] & T \times Y \arrow[r, "\theta_Y"] & Y
\end{tikzcd}$$

For all $s \in S$, consider $a(s)$ in $T \times Y$. Then its image under $\theta_Y$ in $Y$ is $\theta_Y(a(s)) = f(b(s))$. Because $\theta_Y \circ \gamma = p_2$, there exists $t \in T$ such that $p_2\left((t, \theta_Y(a(s))\right) = \theta_Y(a(s))$ and $\gamma\left((t, \theta_Y(a(s))\right) = (t, t^{-1}\theta_Y(a(s))) = a(s)$ in $T \times Y$. Fix such a $t \in T$ and define $\alpha : S \to T \times X$ as $s \mapsto (t, t^{-1}b(s))$ for all $s \in S$. Then we have $b = \theta_X \circ \alpha$. We also have $(1 \times f) \circ \alpha = a$ since

$$\left((1 \times f) \circ \alpha\right)(s) = (1 \times f)(t, t^{-1}b(s)) = (t, t^{-1}f(b(s))) = (t, t^{-1}\theta_Y(a(s))) = a(s)$$

$\square$

**Lemma 3.1.2.** *Let $f : X \to Y$ be a $T$-equivariant morphism of $k$-varieties and $\mathcal{F}$ be a quasi-coherent $T$-sheaf on $X$. Then $f_*\mathcal{F}$ is a quasi-coherent $T$-sheaf on $Y$.*

*Proof.* Let $p_2$ denote projection on the second component. We have the following pull-back square.

$$\begin{tikzcd}
T \times X \arrow[r, "p_2"] \arrow[d, "1\times f"'] & X \arrow[d, "f"] \\
T \times Y \arrow[r, "p_2"] & Y
\end{tikzcd} \tag{3.2}$$

Since $\mathcal{F}$ is a quasi-coherent $T$-sheaf on $X$, we have an isomorphism of $\mathcal{O}_{T\times X}$-modules on $T \times X$

$$\phi : \theta_X^*\mathcal{F} \xrightarrow{\cong} p_2^*\mathcal{F}.$$

One applies $(1 \times f)_*$ and obtains

$$(1 \times f)_*\phi : (1 \times f)_*\theta_X^*\mathcal{F} \xrightarrow{\cong} (1 \times f)_* p_2^*\mathcal{F}.$$

Note that $T$ is flat over $\operatorname{Spec} k$. It follows that $p_2$, $\theta_X$, and $\theta_Y$ are flat [15]. Then diagram (3.1) gives [7, Proposition III.9.3, p. 255]

$$(1 \times f)_*\theta_X^*\mathcal{F} = \theta_Y^*(f_*\mathcal{F}).$$

Similarly, diagram (3.2) gives

$$(1 \times f)_*\theta_Y^*\mathcal{F} = p_2^*(f_*\mathcal{F}).$$

Together we have an isomorphism of $\mathcal{O}_{T\times Y}$-modules on $T \times Y$

$$(1 \times f)_*\phi : \theta_Y^*(f_*\mathcal{F}) \xrightarrow{\cong} p_2^*(f_*\mathcal{F}).$$

The fact that $p_{23} : T \times T \times Y \to T \times Y$ is flat together with the commutative diagram

$$\begin{tikzcd}
T \times T \times X \arrow[r, "p_{23}"] \arrow[d, "1\times 1\times f"'] & T \times X \arrow[d, "1\times f"] \\
T \times T \times Y \arrow[r, "p_{23}"] & T \times Y
\end{tikzcd}$$

give $(1 \times 1 \times f)_* p_{23}^* = p_{23}^*(1 \times f)_*$. Similarly, the map $1 \times \theta_Y$ is flat and we have $(1 \times 1 \times f)_*(1\times\theta)^* = (1\times\theta)^*(1\times f)_*$; $\mu \times 1$ is flat and we have $(1\times 1\times f)_*(\mu\times 1)^* = (\mu\times 1)^*(1\times f)_*$.

Starting with the equality

$$(p_{23}^*\phi) \circ ((1 \times \theta_X)^*\phi) = (\mu \times 1)^*\phi$$

and composing with $(1 \times f)_*$ gives

$$(p_{23}^*(1 \times f)_*\phi) \circ ((1 \times \theta_Y)^*(1 \times f)_*\phi) = (\mu \times 1)^*(1 \times f)_*\phi.$$

The isomorphism $(1 \times f)_*\phi : \theta_Y^*(f_*\mathcal{F}) \xrightarrow{\cong} p_2^*(f_*\mathcal{F})$ makes $f_*\mathcal{F}$ into a $T$-sheaf on $Y$. Since $X$ is noetherian, $f_*\mathcal{F}$ is quasi-coherent [7, Proposition II.5.8, p.115]. □

**Theorem 3.1.3.** *For any quasi-coherent $T$-sheaf $\mathcal{F}$ on $X$, the global sections $\Gamma(X, \mathcal{F})$ an $M$-graded $\Gamma(X, \mathcal{O}_X)$-module over the $M$-graded ring $\Gamma(X, \mathcal{O}_X)$. The grading is natural with respect to restrictions along equivariant open subvarieties of $X$.*

*Proof.* Consider the map $\pi : X \to \operatorname{Spec} k$. There is the unique $T$-action on $\operatorname{Spec} k$ and the map $\pi$ is $T$-equivariant. Note $\Gamma(X, \mathcal{F}) = \Gamma(\operatorname{Spec} k, \pi_*\mathcal{F})$. We may assume that $X = \operatorname{Spec} k$ and $\mathcal{F} = \widetilde{V}$ where $V$ is a $k$-vector space, not necessarily finite dimensional. Let

$$V_m = \{v \in V \mid \phi(T \times X)(1 \otimes v) = \chi^m \otimes v\}.$$

We will show that $V = \bigoplus_{m \in M} V_m$. Since $\phi$ commutes with the restrictions, the grading commutes with restrictions.

Since $X = \operatorname{Spec} k$, note $\theta = p_2 : T \times X \to X$ and $(\theta^*\mathcal{F})(T \times X) = k[M] \otimes V$. Let

$$\phi(T \times X) : k[M] \otimes V \xrightarrow{=} k[M] \otimes V$$

be given by, for $v \in V$

$$1 \otimes v \mapsto \sum_{m \in M} (\chi^m \otimes v_m).$$

Also, the isomorphism

$$(p_{23}^*\phi)(T \times T \times X) : k[M] \otimes k[M] \otimes V \to k[M] \otimes k[M] \otimes V$$

is given on $v$ by

$$1 \otimes 1 \otimes v \mapsto \sum_{m \in M} (1 \otimes \chi^m \otimes v_m).$$

Because the action by $T$ on $X = \operatorname{Spec} k$ is unique, $(1 \times \theta)^* = p_{13}^*$ and the isomorphism

$$((1 \times \theta)^* \phi)(T \times T \times X) : k[M] \otimes k[M] \otimes V \to k[M] \otimes k[M] \otimes V$$

is given on $v$ by

$$1 \otimes 1 \otimes v \mapsto \sum_{m \in M} (\chi^m \otimes 1 \otimes v_m).$$

The isomorphism

$$((\mu \times 1)^* \phi)(T \times T \times X) : k[M] \otimes k[M] \otimes V \to k[M] \otimes k[M] \otimes V$$

is given on $v$ by

$$1 \otimes 1 \otimes v \mapsto \sum_{m \in M} \chi^m \otimes \chi^m \otimes v_m.$$

On $T \times T \times X$, the equality

$$(p_{23}^* \phi) \circ ((1 \times \theta)^* \phi) = (p_{23}^* \phi) \circ (p_{13}^* \phi) = (\mu \times 1)^* \phi$$

yields

$$\begin{aligned}
&((p_{23}^* \phi) \circ ((1 \times \theta)^* \phi))\,(T \times T \times X)(1 \otimes 1 \otimes v) \\
&= (p_{23}^* \phi)(T \times T \times X) \left( \sum_{m \in M} (\chi^m \otimes 1 \otimes v_m) \right) = \sum_{m \in M} \chi^m \otimes [\phi(T \times X)(1 \otimes v_m)] \\
&= ((\mu \times 1)^* \phi)\,(T \times T \times X)(1 \otimes 1 \otimes v) = \sum_{m \in M} \chi^m \otimes \chi^m \otimes v_m.
\end{aligned}$$

So, $\phi(T \times X)(1 \otimes v_m) = \chi^m \otimes v_m$. Recall $V_m = \{v \in V \mid \phi(T \times X)(1 \otimes v) = \chi^m \otimes v\}$. Given

$v \in V$, we have $v_m \in V_m$ in $\phi(T \times X)(1 \otimes v) = \sum_{m \in M} \chi^m \otimes v_m$.

Clearly $V_m$ is a $k$-subspace of $V$, and $V_m \cap V_{m'} = \{0\}$ for all $m \neq m' \in M$. The unital condition $(e \times 1)^* \phi = id$ on $\operatorname{Spec} k \times X$ states that

$$((e \times 1)^* \phi)\,(\operatorname{Spec} k \times X) : V \to V,$$

which is given by $v \mapsto \sum_{m \in M} v_m$, is an equality. We have shown that $V = \bigoplus_{m \in M} V_m$.

□

### 3.1.2 Graded Projective Modules

It follows from the theory of graded rings and modules that an $M$-graded projective module over $R = k[\sigma^\vee \cap M]$ is free [14, Corollary 3.7]. The following is a more precise statement.

**Lemma 3.1.4.** *Let $R = k[\sigma^\vee \cap M]$ and $P$ be a finitely generated $M$-graded $R$-module. If $P$ is $R$-projective, then $P \cong R[m_1] \oplus ... \oplus R[m_r]$ as graded $R$-modules where $m_1, ..., m_r \in M$ and $r$ is the rank of $P$. Each component $R[m_i]$ is $R$ as an $R$-module but the grading is given by $R[m_i]_d := R_{d-m_i}$. In particular, $P$ is a projective object in the category of graded $R$-modules.*

Every object $P$ in the exact category of $M$-graded projective modules over $R$ is described by its rank $r$ and a multiset $m_1, ..., m_r \in M$. A careful analysis of the category yields the description of equivariant $K$-groups over $U_\sigma$ in the following theorem due to Au, Walker and me. Define $M_\sigma := \frac{M}{\sigma^\perp \cap M}$ where $\sigma^\perp := \{u \in \mathbb{R}^n | \langle u, v \rangle = 0 \text{ for all } v \in \sigma\}$ for any cone $\sigma$ in $N_\mathbb{R}$.

**Theorem 3.1.5.** *[1, Theorem 4] For a strongly convex rational cone $\sigma$, there is a natural isomorphism*

$$K_q^T(U_\sigma) \cong K_q(k) \otimes_\mathbb{Z} \mathbb{Z}[M_\sigma].$$

In particular, $K_0^T(U_\sigma) \cong \mathbb{Z}[M_\sigma]$ describes the group completion of the monoid of isomorphism classes of graded projective modules.

Note that $K_0^T(point)$ can be identified with the representation ring $\mathbb{Z}[M]$. The unique map $X(\Delta) \to \{point\}$ is $T$-equivariant for any toric variety $X(\Delta)$ and thus induces a map $K_0^T(point) \to K_0^T(X(\Delta))$, and every equivariant $K$-group $K_q^T(X)$ is hence a $\mathbb{Z}[M]$-module.

## 3.2 Sheaves on a Poset

Given any poset $X$ with order relation $\leq$, one can define the poset topology on $X$ by declaring a subset $Z$ to be open in $X$ if for all $y \in Z$ and $x \leq y$ we have $x \in Z$. For every element $x \in X$, the smallest open subset of $X$ containing $x$ is denoted by $\langle x \rangle = \{y \in X \mid y \leq x\}$. Given a sheaf $\mathcal{F}$ on a poset $X$, we have $\mathcal{F}_x = \mathcal{F}(\langle x \rangle)$ for all $x \in X$.

For this topology, sheaves are uniquely determined by their stalks and the maps between the stalks arising from comparable elements of the poset (see [2, Section 4.1]). We may view the poset $X$ as a category. There is an equivalence between the category of contravariant functors from $X$ to the category of abelian groups and the category of sheaves of abelian groups on $X$. Given a sheaf $\mathcal{F}$ on $X$ with the poset topology, the associated functor $F$ is defined as $F(x) := \mathcal{F}_x = \mathcal{F}(\langle x \rangle)$, and $F$ sends $y \leq x$ to the map $\mathcal{F}_x \to \mathcal{F}_y$ induced by the inclusion $\langle y \rangle \hookrightarrow \langle x \rangle$. Given a contravariant functor $F$, the value of the associated sheaf $\mathcal{F}$ on an open subset $Z$ of $X$ is given by $\mathcal{F}(Z) = \varprojlim_{x \in Z} F(x)$.

A fan $\Delta$ can be thought as a poset with the order relation $\tau \preceq \sigma$ if $\tau$ is a face of $\sigma$. A face inclusion $\tau \preceq \sigma$ gives an inclusion of open sets $\langle \tau \rangle \hookrightarrow \langle \sigma \rangle$. Because two cones intersect at a common face, we have $\langle \tau \rangle \cap \langle \sigma \rangle = \langle \tau \cap \sigma \rangle$.

## 3.3 Computing Sheaf Cohomology on a Poset

### 3.3.1 Godement resolution

Let $X$ be a topological space and $\mathcal{F}$ be a sheaf on $X$. The *Godement resolution of* $\mathcal{F}$ is a resolution of $\mathcal{F}$ by products of skyscraper sheaves. First include $\mathcal{F}$ into the product of skyscraper sheaves $\mathcal{F} \xrightarrow{\phi} \mathcal{G}^0 := \prod_{x \in X} i_{x*}(\mathcal{F}_x)$. Next, include the cokernel sheaf into its product of skyscraper sheaves to obtain $\mathcal{G}^1 := \prod_{x \in X} i_{x*}(coker\phi)_x$. Repeat the process to get a possibly infinite resolution $0 \to \mathcal{F} \to \mathcal{G}^\bullet$ where

$$\mathcal{G}^\bullet : \prod_{x \in X} i_{x_*} \mathcal{F}_x \xrightarrow{d_0} \ldots \xrightarrow{d_{i-1}} \prod_{x \in X} i_{x_*}(coker\ d_{i-2})_x \xrightarrow{d_i} \ldots.$$

Note that a skyscraper sheaf of abelian groups is flasque. For, let $A$ be an abelian group and $i_{x*}A$ be a skyscraper sheaf associated to a point $x$ in $X$. Given open subsets $V \subseteq U$,

$$i_{x*}A(U) \to i_{x*}A(V) = \left\{ \begin{array}{lll} A \xrightarrow{id} A & \text{if} & x \in V \subseteq U \\ A \to 0 & \text{if} & x \in V, x \notin U \\ 0 \to 0 & \text{if} & x \notin V, x \notin U \end{array} \right\} \text{ is always surjective.}$$

The Godement resolution is a flasque resolution and hence the homology groups of its global sections are the sheaf cohomology groups [7, III.2.5.1, p. 208].

If $X$ is a poset with an order relation $\leq$ and $A$ is any abelian group, then, for all $y \in X$,

$$(i_{x*}A)_y = i_{x*}A(\langle y \rangle) = A(\langle y \rangle \cap \{x\}) = \begin{cases} A\ , & \text{if } x \leq y \\ 0\ , & \text{else.} \end{cases}$$

Let $\mathcal{F}$ be a sheaf on a poset $X$, then $(\mathcal{G}^0)_y = \prod_{x \in X} (i_{x*}(\mathcal{F}_x))_y = \prod_{x \leq y} \mathcal{F}_x$. Consider $0 \to \mathcal{F} \xrightarrow{\phi} \mathcal{G}^0$. The sheaf $coker\phi$ is the sheaf associated to the functor $y \mapsto \prod_{x \lneq y} \mathcal{F}_x$ and $\mathcal{G}^1$ is the sheaf associated to the functor $y \mapsto \prod_{x \leq y} \left( \prod_{z \lneq x} \mathcal{F}_z \right) = \prod_{z \lneq x \leq y} \mathcal{F}_z$. Each term in the Godement resolution can be described explicitly as a product of stalks.

### 3.3.2 Čech Cohomology

Let $X$ be a topological space and $\mathcal{U} = \{U_i\}_{i\in I}$ be an open cover of $X$ for a well-ordered index set $I$. For any finite set of indices $i_0 < ... < i_p \in I$, let $U_{i_0<...<i_p}$ denote the intersection $U_{i_0}\cap...\cap U_{i_p}$. Let $\mathcal{F}$ be a sheaf of abelian groups on $X$. We define the *Čech complex* $\check{\mathcal{C}}^\bullet(\mathcal{U},\mathcal{F})$ as follows. For any $p \geq 0$, let $\check{C}^p := \check{C}^p(\mathcal{U},\mathcal{F}) = \prod_{i_0<...<i_p}\mathcal{F}(U_{i_0<...<i_p})$. The boundary map $d : \check{C}^p \to \check{C}^{p+1}$ is defined by

$$(d\alpha)_{i_0<...<i_{p+1}} = \sum_{k=0}^{p+1}(-1)^k\alpha_{i_0<...<\widehat{i_k}<...<\imath_{p+1}}$$

where $\alpha \in \check{C}^p$ and $\alpha_{i_0<...<i_p}$ is in the component indexed by $i_0 < ... < i_p$ and $\widehat{i_k}$ means omitting $i_k$. The cohomology groups of $\check{\mathcal{C}}^\bullet(\mathcal{U},\mathcal{F})$ are *the Čech cohomology groups relative to the cover* $\mathcal{U}$:

$$\check{H}^p_{\mathcal{U}}(X,\mathcal{F}) := H^p(\check{\mathcal{C}}^\bullet(\mathcal{U},\mathcal{F})).$$

**Lemma 3.3.1.** *[7, Exercise III.4.11, p. 225] Let $\mathcal{F}$ be a sheaf on a topological space $X$ and $\mathcal{U}$ an open covering of $X$. If $H^p(U_{i_0<...<i_q},\mathcal{F}|_{U_{i_0<...<i_q}}) = 0$ for all $p > 0$ and all $i_0 < ... < i_q$, then*

$$\check{H}^p_{\mathcal{U}}(X,\mathcal{F}) \xrightarrow{\cong} H^p(X,\mathcal{F}) \text{ for all } p \geq 0.$$

*Proof.* Let $0 \to \mathcal{F} \to \mathcal{I}^\bullet$ be an injective resolution of $\mathcal{F}$. Consider the bi-complex of abelian groups:

$$\begin{tikzcd}
\Gamma(\mathcal{F}) \arrow[r] \arrow[d] & \Gamma(\mathcal{I}^\bullet) \arrow[d] \\
\check{\mathcal{C}}^\bullet(\mathcal{U},\mathcal{F}) \arrow[r] & \check{\mathcal{C}}^\bullet(\mathcal{U},\mathcal{I}^\bullet).
\end{tikzcd}$$

Every injective sheaf $\mathcal{I}$ is flasque [7, III.2.4, p 207]. It follows that $\check{H}^p(\mathcal{U},\mathcal{I}) = 0$ for all $p > 0$ [7, III.4.3, p. 221] and hence all columns but the first one are exact, and $H^p(X,\mathcal{F}) \cong H^p(Tot(\check{\mathcal{C}}^\bullet(\mathcal{U},\mathcal{I}^\bullet)))$ for all $p \geq 0$ where $Tot(-)$ refers to the total complex of a double complex.

The hypothesis $H^p(U_{i_0<...<i_q}, \mathcal{F}|_{U_{i_0<...<i_q}}) = 0$ for all $p > 0$ and all $i_0 < ... < i_q$ gives exactness of all rows but the first one. Then $\check{H}^p_{\mathcal{U}}(X, \mathcal{F}) \cong H^p(Tot(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{I}^\bullet)))$ for all $p \geq 0$, and the result follows. $\square$

**Lemma 3.3.2.** *Let $\Delta$ be a fan and $\mathcal{F}$ a sheaf on $\Delta$. If $\mathcal{U} := \{\langle\sigma\rangle \mid \sigma$ is a maximal cone in $\Delta\}$, then*

$$\check{H}^p_{\mathcal{U}}(\Delta, \mathcal{F}) \cong H^p(\Delta, \mathcal{F}) \text{ for all } p \geq 0.$$

*Proof.* For maximal cones $\sigma_1, ..., \sigma_l$, we have $\langle\sigma_1\rangle \cap \cdots \cap \langle\sigma_l\rangle = \langle\sigma_1 \cap \cdots \cap \sigma_l\rangle$. It suffices to prove $H^p(\langle\tau\rangle, \mathcal{F}) = 0$ for all $p > 0$ for an arbitrary $\tau$ in $\Delta$. This follows from the fact $\mathcal{G}_\tau = \mathcal{G}(\langle\tau\rangle)$ for any $\tau \in \Delta$ and for any sheaf $\mathcal{G}$ on $\Delta$, and an injective resolution remains exact upon taking stalks. $\square$

### 3.3.3 Cellular Complex

The following definitions of orientations of cones and cellular complexes of a complete fan are analogous to the definitions of orientations of cells and homology groups of a CW-complex. See [10, Chapter IX].

Let $N$ be an $n$-dimensional lattice. Recall that a fan $\Delta$ is complete if $|\Delta| = N \otimes_{\mathbb{Z}} \mathbb{R} \cong \mathbb{R}^n$. Fix an isomorphism $N \otimes_{\mathbb{Z}} \mathbb{R} \cong \mathbb{R}^n$ and let $S^{n-1}$ be the unit sphere defined by the Euclidean norm in $\mathbb{R}^n$. Let $\Delta \cap S^{n-1} := \{\tau \cap S^{n-1} \mid \tau \in \Delta\}$ denote the cell decomposition of the sphere $S^{n-1}$ given by $\Delta$. Each cell is mapped homeomorphically onto its image in $S^{n-1}$ and so the cell decomposition is regular. (Recall that a cell decomposition of a topological space is *regular* if each cell is attached the space homeomorphically.) For every non-zero cone $\sigma \in \Delta$ with $dim(\sigma) = i$, let $e^{i-1}_\sigma := \sigma \cap S^{n-1}$ denote the $(i-1)$-dimensional cell on $S^{n-1}$. Recall that for any non-zero cell $e^{i-1}$, the relative homology group $H_{i-1}(e^{i-1}, \partial e^{i-1}) \cong \mathbb{Z}$ where $\partial e^{i-1}$ is the topological boundary of $e^{i-1}$. We define the *orientation of* $\sigma$ to be a choice of a generator of the relative homology group $H_{i-1}(e_\sigma^{i-1}, \partial e_\sigma^{i-1})$.

Let $\mathcal{F}$ be a sheaf on a fan $\Delta$. With an orientation specified on each non-zero cone, the cellular complex $\mathcal{C}^\bullet(\mathcal{F})$ is defined as follows. Let $C^i := \bigoplus_{dim\epsilon=n-i} \mathcal{F}_\epsilon$ for all $0 \leq i \leq n$. The differentials are sums of restrictions with signs determined by whether the orientations of two cones in the restriction agree. The signs in $d_{n-1} : \bigoplus_{dim\epsilon=1} \mathcal{F}_\epsilon \to \mathcal{F}_{\{0\}}$ depend on the orientations of $\epsilon$ with $dim(\epsilon) = 1$.

$$\mathcal{C}^\bullet(\mathcal{F}) : \bigoplus_{dim\sigma=n} \mathcal{F}_\sigma \xrightarrow{d_0} \bigoplus_{dim\tau=n-1} \mathcal{F}_\tau \xrightarrow{d_1} \cdots \xrightarrow{d_{i-1}} \bigoplus_{dim\epsilon=n-i} \mathcal{F}_\epsilon \xrightarrow{d_i} \cdots \xrightarrow{d_{n-1}} \mathcal{F}_{\{0\}}$$

When the fan $\Delta$ is complete, we will prove that cellular cohomology coincides with sheaf cohomology, following the ideas in [3, Proposition 3.3].

**Lemma 3.3.3.** *Let $\Delta$ be a complete fan, and let $\mathcal{F}$ be a sheaf on $\Delta$. Then we have*

$$H^p(\Delta, \mathcal{F}) \cong H^p(\mathcal{C}^\bullet(\mathcal{F})) \textit{ for all } p \geq 0.$$

*Proof.* We will prove that the cellular complex $\mathcal{C}^\bullet(\mathcal{F})$ is indeed a complex. For any $\epsilon \in \Delta$, we need $d^2(\mathcal{F}_\epsilon) = 0$. Up to $C^{n-1}$, the fact that $d^2 = 0$ follows from the fact that $\Delta \cap S^{n-1}$ is a regular cell complex [10, Lemma 7.1, p. 244]. Let $\epsilon$ in $\Delta$ be a two-dimensional cone with $\tau_1$ and $\tau_2$ as facets. The orientation of $\epsilon \cap S^{n-1}$ chooses a direction of the edge and those of $\tau_i \cap S^{n-1}$ assigns $+$ or $-$ to the points. Consider all cases: $-\bullet \longrightarrow \bullet+$ , $+\bullet \longrightarrow \bullet-$ , $+\bullet \longrightarrow \bullet+$ , and $-\bullet \longrightarrow \bullet-$ . The following diagrams tracks the sign of the restriction maps. The maps are labeled 1 if the orientations agree and $-1$ otherwise.

$$\begin{tikzcd}[column sep=small] & \epsilon \arrow[dl, "1"'] \arrow[dr, "1"] & \\ \tau_2 \arrow[dr, "-1"'] & & \tau_1 \arrow[dl, "1"] \\ & \{0\} & \end{tikzcd} \quad \begin{tikzcd}[column sep=small] & \epsilon \arrow[dl, "-1"'] \arrow[dr, "-1"] & \\ \tau_2 \arrow[dr, "-1"'] & & \tau_1 \arrow[dl, "1"] \\ & \{0\} & \end{tikzcd} \quad \begin{tikzcd}[column sep=small] & \epsilon \arrow[dl, "-1"'] \arrow[dr, "1"] & \\ \tau_2 \arrow[dr, "1"'] & & \tau_1 \arrow[dl, "-1"] \\ & \{0\} & \end{tikzcd} \quad \begin{tikzcd}[column sep=small] & \epsilon \arrow[dl, "-1"'] \arrow[dr, "1"] & \\ \tau_2 \arrow[dr, "1"'] & & \tau_{-1} \arrow[dl, "1"] \\ & \{0\} & \end{tikzcd}$$

Then, for all sections $s \in \mathcal{F}_\epsilon$, $d^2(s) = d\left(\begin{bmatrix} s|_{\tau_1} \\ s|_{\tau_2} \end{bmatrix}\right) = s|_\rho - s|_\rho = 0.$

For a sheaf $\mathcal{F}$ on $\Delta$, one forms the Godement resolution $0 \to \mathcal{F} \to \mathcal{G}^\bullet$ and consider the bi-complex of abelian groups:

$$\begin{tikzcd} \Gamma(\mathcal{F}) \arrow[r] \arrow[d] & \mathcal{C}^\bullet(\mathcal{F}) \arrow[d] \\ \Gamma(\mathcal{G}^\bullet) \arrow[r] & \mathcal{C}^\bullet(\mathcal{G}^\bullet). \end{tikzcd}$$

Let $\mathcal{U} := \{\langle\sigma\rangle \mid \sigma \text{ is a maximal cone } \Delta\}$ be the maximal cone cover of $\Delta$. Recall that an element in $H^0(\Delta, \mathcal{F}) = \check{H}^0_{\mathcal{U}}(\Delta, \mathcal{F})$ is a collection of sections $\{s_\sigma \in \mathcal{F}_\sigma\}$ for maximal cones $\sigma \in \Delta$ such that $s_\sigma|_{\sigma\cap\sigma'} = s_{\sigma'}|_{\sigma\cap\sigma'}$, where $dim(\sigma \cap \sigma') = n-1$. Since $\Delta$ is a complete fan, a dimension $n-1$ cone $\tau \in \Delta$ is a facet of exactly two maximal cones $\sigma$ and $\sigma'$. Let $\beta_{\sigma\tau} = 1$ if the orientations of $\sigma$ and $\tau$ agree, and $\beta_{\sigma\tau} = -1$ if the orientations disagree. It follows from the definition of the relative homology groups that $\beta_{\sigma\tau} + \beta_{\sigma'\tau} = 0$. Hence $ker(C^0 \to C^1)$ specifies sections $\{s_\sigma \in \mathcal{F}_\sigma\}$ for maximal cones $\sigma \in \Delta$ that agree on pairwise intersections on dimension $n-1$ cones, and so $ker(C^0 \to C^1) = H^0(\Delta, \mathcal{F})$.

Since $\mathcal{G}_\tau = \mathcal{G}(\langle\tau\rangle)$ for any $\tau \in \Delta$ and for any sheaf $\mathcal{G}$ on $\Delta$, all columns but the first one are exact, and $H^p(\mathcal{C}^\bullet(\mathcal{F})) \cong H^p(Tot(\mathcal{C}^\bullet(\mathcal{G}^\bullet)))$ for all $p \geq 0$ where $Tot(-)$ refers to the total complex of a double complex. If all rows but the first one are exact, then $H^p(\Gamma(\mathcal{G}^\bullet)) \cong H^p(Tot(\mathcal{C}^\bullet(\mathcal{G}^\bullet)))$ for all $p \geq 0$.

It remains to show $H^p(\mathcal{C}^\bullet(\mathcal{G}^j)) = 0$ for all $p > 0$ and for a fixed $j$. Since $\mathcal{G}^j$ is a product of skyscraper sheaves, it suffices to prove exactness for a skyscraper sheaf $i_{\epsilon_*}A$ for some abelian group $A$ and for some cone $\epsilon \in \Delta$.

Note $(i_{\epsilon_*}A)_\sigma = \begin{cases} A \quad , & \text{if } \epsilon \preceq \sigma \\ 0 \quad , & \text{else.} \end{cases}$

Say that the cone $\epsilon$ is of dimension $n-i$. The cellular complex is

$$\mathcal{C}^\bullet(i_{\epsilon_*}A): \bigoplus_{\substack{dim\sigma=n\\ \epsilon\preceq\sigma}} A \xrightarrow{d_0} \bigoplus_{\substack{dim\tau=n-1\\ \epsilon\preceq\tau}} A \xrightarrow{d_1} \dots \xrightarrow{d_{i-2}} \bigoplus_{\substack{dim\rho=n-i+1\\ \epsilon\preceq\rho}} A \xrightarrow{d_{i-1}} A \xrightarrow{d_i} 0$$

Let $Star(\epsilon) := \{\sigma \in \Delta \mid \epsilon \preceq \sigma\}$, a closed subset of $\Delta$. Let $\langle Star(\epsilon)\rangle$ denote the subfan of $\Delta$ generated by $Star(\epsilon)$. Let $E$ denote the sub CW-complex $\{\tau \cap S^{n-1} \mid \tau \in \langle Star(\epsilon)\rangle\}$ of $\Delta \cap S^{n-1}$ given by $\langle Star(\epsilon)\rangle$. If $\epsilon = \{0\}$, then $Star(\epsilon) = \langle Star(\epsilon)\rangle = \Delta$ and $E = S^{n-1}$. If $\epsilon \neq \{0\}$, then there exists a maximal cone $\eta$ in $\Delta$ not containing $\epsilon$. Indeed, let $v$ be a ray contained in $\epsilon$. A maximal cone containing $-v$ does not contain $v$ because of strong convexity. We may stereographically project $E$ to $\mathbb{R}^{n-1}$ using a point in the interior of $\eta$. Note that the image of $E$ is star convex. (Recall that a non-empty subset of an Euclidean space is *star convex* if it contains a point such that the line segment joining that point to every other point in the subset lies inside the subset.) It follows that $E$ is contractible and $\partial E \cong S^{n-2}$.

If $\epsilon = \{0\}$, the cellular complex $\mathcal{C}^\bullet(i_{\epsilon_*}A)$ is the augmented chain complex computing the reduced homology of $S^{n-1}$ with coefficients in $A$ [19, Theorem 2.21, p.58]. In particular, we have [19, Theorem 1.15, p.24]

$$H^p(\mathcal{C}^\bullet(i_{\epsilon_*})A) = \widetilde{H}_{n-p-1}(S^{n-1}) = 0 \text{ for all } p \geq 1.$$

If $dim(\epsilon) = n - i > 0$, the cellular complex $\mathcal{C}^\bullet(i_{\epsilon_*}A)$ coincides with the augmented chain complex computing the reduced relative cellular homology of $(E, \partial E)$ with coefficient in $A$. Because $E \cong D^{n-1}$ is contractible, $\widetilde{H}_{i-p-1}(E, \partial E) \cong \widetilde{H}_{i-p-1}(\partial E)$. It follows that $H^p(\mathcal{C}^\bullet(i_{\epsilon_*}A)) = \widetilde{H}_{i-p-1}(E, \partial E) \cong \widetilde{H}_{i-p-1}(\partial E) \cong \widetilde{H}_{i-p-1}(S^{n-2})$. For all $p \geq 1$, we have $i - p - 1 < n - 2$ and $H^p(\mathcal{C}^\bullet(i_{\epsilon_*}A)) \cong \widetilde{H}_{i-p-1}(S^{n-2}) = 0$. $\square$

## 3.4 $K_q^T$ on Smooth Toric Varieties

Let $X = X(\Delta)$ be a smooth toric variety associated to a fan $\Delta$ in $N_{\mathbb{R}}$. Au, Walker and I use Theorem 3.1.5, the theory of sheaves on poset associated to $\Delta$, and the foundational results of Thomason [15] concerning equivariant $K$-theory to recover a result due to Vezzosi and Vistoli [17, 18]: the sequence

$$0 \to K_q^T(X) \to \bigoplus_{\sigma \in Max(\Delta)} K_q^T(U_\sigma) \to \bigoplus_{\delta < \tau} K_q^T(U_{\delta \cap \tau}) \to \cdots \tag{3.3}$$

is exact. Here $Max(\Delta)$ denote the set of maximal cones with an arbitrary ordering.

Consider a fan $\Delta$ with the poset topology. For any subfan $\Lambda \subseteq \Delta$, define $K_q^T(\Lambda) := K_q^T(X(\Lambda))$. Note that $\left(K_q^T(-)\right)_\sigma = K_q^T(\langle\sigma\rangle) = K_q^T(U_\sigma)$. Whenever $\Lambda' \subseteq \Lambda$ as subfans of $\Delta$, we have an induced map $K_q^T(\Lambda) := K_q^T(X(\Lambda)) \to K_q^T(\Lambda') := K_q^T(X(\Lambda'))$. This defines $K_q^T(-)$ as a presheaf on $\Delta$.

Let $\mathcal{U}$ denote the open cover $\{\langle\sigma\rangle \mid \sigma \in Max(\Delta)\}$ of $\Delta$. The sequence (3.3) coincides with the Čech complex of the presheaf $K_q^T$ relative to $\mathcal{U}$ on $\Delta$. By Theorem 3.1.5, the exactness of (3.3) follows from the exactness of

$$\begin{aligned} 0 \to K_q^T(X) \to \bigoplus_{\sigma \in Max(\Delta)} K_q(k) \otimes_{\mathbb{Z}} \mathbb{Z}[M_\sigma] &\to \bigoplus_{\delta < \tau} K_q(k) \otimes_{\mathbb{Z}} \mathbb{Z}[M_{\delta \cap \tau}] \\ &\to \bigoplus_{\delta < \tau < \epsilon} K_q(k) \otimes_{\mathbb{Z}} \mathbb{Z}[M_{\delta \cap \tau \cap \epsilon}] \to \cdots . \end{aligned} \tag{3.4}$$

**Theorem 3.4.1.** *[1, Theorem 6] Let $X = X(\Delta)$ be a smooth toric variety. Then the sheaf $\Lambda \mapsto K_q^T(X(\Lambda))$ on $\Delta$ is a flasque sheaf. Moreover, sequences (3.3) and (3.4) are exact, and there is an isomorphism*

$$K_q^T(X) \cong K_q(k) \otimes_{\mathbb{Z}} K_0^T(X).$$

*Proof.* (See [1].) Let $\mathcal{A}_q$ be the sheaf on $\Delta$ associated to the functor sending a cone $\sigma$ to $K_q(k) \otimes \mathbb{Z}[M_\sigma]$ and a face inclusion $\tau \prec \sigma$ to the map induced by the canonical quotient $M_\sigma \twoheadrightarrow M_\tau$.

The sheaf $\mathcal{A}_0$ is flasque by [2]. Since $\mathcal{A}_0$ is a flasque sheaf of torsion free abelian groups, the presheaf $K_q(k) \otimes_{\mathbb{Z}} \mathcal{A}_0$ is actually a sheaf. Indeed, for any open subset $U$ and open covering $U = \cup_i V_i$ of it, the map from $\mathcal{A}_0(U)$ to the associated Čech complex is a quasi-isomorphism by [7, III.4.3], and since $\mathcal{A}_0$ is torsion free, this map remains a quasi-isomorphism upon tensoring with any abelian group. It now follows from the correspondence between functors and sheaves that $\mathcal{A}_q \cong K_q(k) \otimes \mathcal{A}_0$. In particular, $\mathcal{A}_q$ is also flasque.

For a subfan $\Lambda$ of $\Delta$, let $\mathcal{V}$ be the Zariski open covering $\{U_\sigma \mid \sigma \text{ is a maximal cone in } \Lambda\}$ of $X(\Lambda)$ and let $\mathcal{U}$ be the open covering $\{\langle\sigma\rangle \mid \sigma \in Max(\Delta)\}$ of $\Lambda$. By Theorem 3.1.5, the Čech cohomology complex of the presheaf $\mathcal{K}_q^T(-)$ on $X(\Lambda)$ for the open covering $\mathcal{V}$ coincides with the Čech cohomology complex of the sheaf $\mathcal{A}_q$ for the open covering $\mathcal{U}$. Since the higher Čech cohomology of flasque sheaves vanishes [7, III.4.3], we have

$$\check{H}^p_{\mathcal{V}}(X(\Lambda), K_q^T) = \check{H}^p_{\mathcal{U}}(\Delta, \mathcal{A}_q) = 0, \text{ for all } p > 0. \tag{3.5}$$

Thomason [15] has proven that $\mathcal{K}^T$ coincides with equivariant $G$-theory (defined from equivariant coherent sheaves) and that the latter satisfies the usual localization property relating $X$, an equivariant closed subscheme, and its open complement. From this one deduces that if $X(\Lambda) = U \cup V$ is a covering by equivariant open subschemes, then

$$\begin{tikzcd}
\mathcal{K}^T(X(\Lambda)) \arrow[r] \arrow[d] & \mathcal{K}^T(U) \arrow[d] \\
\mathcal{K}^T(V) \arrow[r] & \mathcal{K}^T(U \cap V)
\end{tikzcd}$$

is a homotopy cartesian square. Arguing just as in [16, Section 8], one obtains a convergent spectral sequences

$$\check{H}^p_{\mathcal{V}}(X(\Lambda), K_q^T) \Longrightarrow K_{q-p}^T(X(\Lambda)).$$

By (3.5), this spectral sequence collapses to give

$$\check{H}^0_{\mathcal{V}}(X(\Lambda), K_q^T) \cong K_q^T(X(\Lambda)), \text{ for all } q. \tag{3.6}$$

Combining (3.6) and (3.5) gives that the complexes

$$0 \to K_q^T(X(\Lambda)) \to \bigoplus_{\sigma} K_q^T(U_\sigma) \to \bigoplus_{\delta<\tau} K_q^T(U_{\delta\cap\tau}) \to \cdots$$

and

$$0 \to \mathcal{A}_q(\Lambda) \to \bigoplus_{\sigma} K_q(k) \otimes_{\mathbb{Z}} \mathbb{Z}[M_\sigma] \to \bigoplus_{\delta<\tau} K_q(k) \otimes_{\mathbb{Z}} \mathbb{Z}[M_{\delta\cap\tau}] \to \cdots$$

are exact and isomorphic to each other. In particular, $\Lambda \mapsto \mathcal{K}_q^T(X(\Lambda))$ is isomorphic to the flasque sheaf $\mathcal{A}_q$. $\square$

# Chapter 4

# Equivariant Chow Rings and Equivariant $K$-theory

## 4.1 Equivariant Chow Rings

The equivariant Chow cohomology refers to the "operational" theory developed by Fulton and MacPherson [6, Chapter 17] and discussed in the context of toric varieties in [13]. For a group $G$ and a $G$-space $X$, Edidin and Graham developed equivariant intersection theory [4], defining $A^i_G(X)$ to be the operational Chow cohomology group $A^i(X \times_G U)$, where $U$ is an equivariant open subset of a representation of $G$ of an affine space such that $G$ acts freely on $U$ and the codimension of the complement of $U$ is greater than $i$. The definition is independent of the choice the representation and the open subset $U$.

There is a multiplication map in equivariant Chow cohomology $A^i_G(X) \otimes A^j_G(X) \to A^{i+j}_G(X)$ making $A^*_G(X) := \bigoplus A^i_G(X)$ into a ring. In our context, $X = X(\Delta)$ is a toric variety associated to a fan $\Delta$ with an action of the algebraic torus $T$. It turns out that an equivariant Chow ring $A^*_T(X)$ has a very explicit description.

For any cone $\sigma$, let $S^\bullet_{\mathbb{Z}}(M_\sigma)$ denote the ring of polynomial functions with integer coefficients on $\sigma$, where $M_\sigma = \frac{M}{\sigma^\perp \cap M}$. Let $X = X(\Delta)$ be a toric variety defined by the fan $\Delta$. The ring of piecewise polynomial functions with integer coefficients on $\Delta$ is defined by

$$PP^*(\Delta) = \{f : |\Delta| \to \mathbb{R} \mid f|_\sigma \in S^\bullet_{\mathbb{Z}}(M_\sigma) \text{ for each } \sigma \in \Delta\}.$$

Payne has proved the following correspondence between $A^*_T(X)$ and $PP^*(\Delta)$.

**Theorem 4.1.1.** *[13, Theorem 1] There is a natural isomophism from $A^*_T(X)$ to $PP^*(\Delta)$.*

Let $\mathcal{F}$ be a locally free coherent $T$-sheaf over $X(\Delta)$. By Lemma 3.1.4, for each $\sigma \in \Delta$, $\Gamma(U_\sigma, \mathcal{F})$ is an $M$-graded projective module over $k[\sigma^\vee \cap M]$ of the form $\bigoplus_{i=1}^r k[\sigma^\vee \cap M][m_i]$ for $m_1, ..., m_r \in M$ and $r$ the rank of $\Gamma(U_\sigma, \mathcal{F})$. Let $\mathbf{u}_\sigma = \{m_1, ..., m_r\}$ denote the multiset.

**Theorem 4.1.2.** *[13, Theorem 3] The equivariant Chern class $c_i^T(\mathcal{F})$ in $A^i_T(X(\Delta))$ is naturally identified with the piecewise polynomial function whose restriction to $\sigma$ is the $i$-th elementary symmetric function $e_i(\mathbf{u}_\sigma)$.*

## 4.2 Tensoring with $\mathbb{Q}$ and Completing

Over an affine toric variety $U_\sigma$ associated to a cone $\sigma$, we have $K_0^T(U_\sigma) \cong \mathbb{Z}[M_\sigma]$ from Theorem 3.1.5. We will prove that equivariant $K$-theory and the equivariant Chow cohomology ring are isomorphic upon tensoring with $\mathbb{Q}$ and completing.

**Proposition 4.2.1.** *Let $M$ be a free abelian group. Let $I := \langle \chi^m - 1 \mid m \in M\rangle$ denote the augmentation ideal in the group ring $\mathbb{Q}[M]$ and let $\mathfrak{m}$ denote the maximal homogeneous ideal in the symmetric algebra $S^\bullet_{\mathbb{Q}}(M_{\mathbb{Q}})$. Then $\mathbb{Q}[M]^\wedge_I$ and $S^\bullet_{\mathbb{Q}}(M_{\mathbb{Q}})^\wedge_{\mathfrak{m}}$ are isomorphic as rings. Moreover, the isomorphism is natural in $M$.*

*Proof.* For all $m \in M$, define

$$e^m := \sum_{i=0}^{\infty} \frac{m^i}{i!} = 1 + m + \frac{m^2}{2!} + \frac{m^3}{3!} + \frac{m^4}{4!} + \cdots \text{ in } S^\bullet_{\mathbb{Q}}(M_{\mathbb{Q}})^\wedge_{\mathfrak{m}}.$$

The mapping is additive, that is, $e^{m+m'} = e^m \cdot e^{m'}$. Indeed, using $\mathbb{Z}^2 \to M$ where $(1,0) \mapsto m$ and $(0,1) \mapsto m'$, we may assume $M = \mathbb{Z}^2$. It follows that $S^\bullet_{\mathbb{Q}}(M_{\mathbb{Q}}) \cong \mathbb{Q}[x,y]$ and $m = x, m' = y$. The equality $e^{m+m'} = e^m \cdot e^{m'}$ holds because the power series expansions of $e^{x+y}$ coincides with that of $e^x \cdot e^y$ in $\mathbb{Q}[x,y]^\wedge_{(x,y)}$. Thus, the assignment $m \mapsto e^m$ for $m \in M$ is a group homomorphism from $M$ to units in $S^\bullet_{\mathbb{Q}}(M_{\mathbb{Q}})^\wedge_{\mathfrak{m}}$ and thus induces a ring homomorphism $\mathbb{Q}[M] \to S^\bullet_{\mathbb{Q}}(M_{\mathbb{Q}})^\wedge_{\mathfrak{m}}$.

For each generator of the augmentation ideal $I$, $\chi^m - 1$ maps to

$$e^m - 1 = m + \frac{m^2}{2!} + \frac{m^3}{3!} + \frac{m^4}{4!} + \cdots = m\left(1 + \frac{m}{2!} + \frac{m^2}{3!} + \frac{m^3}{4!} + \cdots\right)$$

where $1 + \frac{m}{2!} + \frac{m^2}{3!} + \frac{m^3}{4!} + \cdots$ is a unit in $S^\bullet_{\mathbb{Q}}(M_{\mathbb{Q}})^\wedge_{\mathfrak{m}}$. So, $IS^\bullet_{\mathbb{Q}}(M_{\mathbb{Q}})^\wedge_{\mathfrak{m}} = \mathfrak{m}S^\bullet_{\mathbb{Q}}(M_{\mathbb{Q}})^\wedge_{\mathfrak{m}}$. It follows $S^\bullet_{\mathbb{Q}}(M_{\mathbb{Q}})^\wedge_{\mathfrak{m}} \cong (S^\bullet_{\mathbb{Q}}(M_{\mathbb{Q}})^\wedge_{\mathfrak{m}})^\wedge_I$ in the commutative diagram

$$\begin{tikzcd}
\mathbb{Q}[M] \arrow[r] \arrow[d] & S^\bullet_{\mathbb{Q}}(M_{\mathbb{Q}})^\wedge_{\mathfrak{m}} \arrow[d] \\
\mathbb{Q}[M]^\wedge_I \arrow[r] & (S^\bullet_{\mathbb{Q}}(M_{\mathbb{Q}})^\wedge_{\mathfrak{m}})^\wedge_I
\end{tikzcd}$$

and we obtain a ring homomorphism $\mathbb{Q}[M_{\mathbb{Q}}]^\wedge_I \xrightarrow{f} S^\bullet_{\mathbb{Q}}(M_{\mathbb{Q}})^\wedge_{\mathfrak{m}}$.

On the other hand, consider the mapping $M \to \mathbb{Q}[M]^\wedge_I$ given by sending $m \in M$ to

$$\begin{aligned}
\ln(\chi^m) :=& \sum_{i=1}^{\infty}(-1)^{i-1}\frac{(\chi^m-1)^i}{i} = (\chi^m - 1) - \frac{(\chi^m-1)^2}{2} + \frac{(\chi^m-1)^3}{3} - \frac{(\chi^m-1)^4}{4} + \cdots \\
=& (\chi^m - 1)\left(1 - \frac{\chi^m - 1}{2} + \frac{(\chi^m-1)^2}{3} - \frac{(\chi^m-1)^3}{4} + \cdots\right).
\end{aligned}$$

For any $m$ and $m' \in M$, we need to show $\ln(\chi^{m+m'}) = \ln(\chi^m) + \ln(\chi^{m'})$ in $\mathbb{Q}[M]^\wedge_I$. The equality follows from that in the power series ring $\mathbb{Q}[x,y]^\wedge_{(x-1,y-1)}$ where the equation $\ln(xy) = \ln(x) + \ln(y)$ holds for the power series expansions.

Extending the map $\mathbb{Q}$-linearly defines a $\mathbb{Q}$-vector space map $M_\mathbb{Q} \to \mathbb{Q}[M]^\wedge_I$ and induces a ring map $S^\bullet_\mathbb{Q}(M_\mathbb{Q}) \to \mathbb{Q}[M]^\wedge_I$. Since $1 - \frac{\chi^m - 1}{2} + \frac{(\chi^m-1)^2}{3} - \frac{(\chi^m-1)^3}{4} + \cdots$ is a unit in $\mathbb{Q}[M]^\wedge_I$, $\mathfrak{m}\mathbb{Q}[M]^\wedge_I = I\mathbb{Q}[M]^\wedge_I$ and we obtain a ring homomorphism $S^\bullet_\mathbb{Q}(M_\mathbb{Q})^\wedge_\mathfrak{m} \xrightarrow{g} \mathbb{Q}[M_\mathbb{Q}]^\wedge_I$.

Note the composition $g \circ f$ is the identity because the equation $e^{\ln x} = x$ holds in $\mathbb{Q}[x]^\wedge_{(x-1)}$ for the power series expansion. Similarly, $(f \circ g)$ is the identity on $S^\bullet_\mathbb{Q}(M_\mathbb{Q})^\wedge_\mathfrak{m}$. We have $\mathbb{Q}[M]^\wedge_I \xrightarrow{\cong} S^\bullet_\mathbb{Q}(M_\mathbb{Q})^\wedge_\mathfrak{m}$ as rings. From the definition, the isomorphism is clearly natural on $M$. □

**Remark 4.2.2.** Let $U_\sigma$ be an affine toric variety. With the notations as above, the proposition shows

$$\left(K_0^T(U_\sigma)_\mathbb{Q}\right)^\wedge_I \cong \left(\mathbb{Q}[M_\sigma]\right)^\wedge_I \cong \left(S^\bullet_\mathbb{Q}(M_{\sigma_\mathbb{Q}})\right)^\wedge_\mathfrak{m} \cong \left(A^*_T(U_\sigma)_\mathbb{Q}\right)^\wedge_\mathfrak{m}.$$

and the isomorphisms are natural with respect to face inclusions.

## 4.3 Relating Sheaf Cohomology and Equivariant Chow Rings

We recall some notations and definitions from Chapter 3 here. Let $X = X(\Delta)$ be a toric variety associated to the fan $\Delta$. There is an equivariant open cover

$$\mathcal{V} := \{U_\sigma \mid \sigma \text{ is a maximal cone in } \Delta\} \text{ of } X(\Delta).$$

On the other hand, the fan $\Delta$ is equipped with the poset topology and

$$\mathcal{U} := \{\langle\sigma\rangle \mid \sigma \text{ is a maximal cone in } \Delta\}$$

is an open cover of $\Delta$. Let $K_q^T(-)$ be the presheaf on $\Delta$ defined by $\Lambda \mapsto K_q^T(X(\Lambda))$ for all subfans $\Lambda$ in $\Delta$. Then

$$\left(K_q^T\right)_\sigma = K_q^T(\langle\sigma\rangle) = K_q^T(U_\sigma)$$

and

$$\check{H}^p_{\mathcal{V}}(X(\Delta), K_q^T) = \check{H}^p_{\mathcal{U}}(\Delta, K_q^T).$$

When $X = X(\Delta)$ is a smooth toric variety, we have Thomason's spectral sequence [15], [16, Section 8]

$$\check{H}^p_{\mathcal{U}}(\Delta, K_q^T) \Longrightarrow K^T_{q-p}(X(\Delta))$$

allowing us to use Čech cohomology to study equivariant $K$-theory in Theorem 3.4.1.

When $X = X(\Delta)$ is a quasi-projective toric variety, not necessarily smooth, Walker proves in [20, Theorem 8.2] that there is a convergent spectral sequence of $\mathbb{Z}[M]$-modules

$$\check{H}^p_{\mathcal{U}}(\Delta, K_q^T) \Longrightarrow K^T_{q-p}(X(\Delta)).$$

Note that the spectral sequence involves negative equivariant $K$-groups when $q - p < 0$.

Let $(-)_\mathbb{Q}$ denote tensoring with $\mathbb{Q}$ over $\mathbb{Z}$. Let $I$ be the augmentation ideal in $\mathbb{Q}[M]$ generated by elements of the form $1 - \chi^m$ where $m \in M$. Let $(-)^\wedge_I := - \otimes_{\mathbb{Q}[M]} \mathbb{Q}[M]^\wedge_I$ denote the completion with respect to the augmentation ideal $I$. It follows from Lemma 3.3.2 that $\check{H}^p_{\mathcal{U}}(\Delta, K_q^T) = H^p(\Delta, K_q^T(-)^\sim)$ for all $p \geq 0$ where $H^p(\Delta, K_q^T(-)^\sim)$ refers to the sheaf cohomology groups of the sheaf $K_q^T(-)^\sim$. Upon tensoring with $\mathbb{Q}$ and completing, we have that

$$\left(H^p(\Delta, K_q^T(-)^{\sim})_{\mathbb{Q}}\right)_I^{\wedge} \Longrightarrow \left(K_{q-p}^T(X(\Delta))_{\mathbb{Q}}\right)_I^{\wedge}.$$

We have isomorphisms

$$\begin{aligned}
&\left[H^p\left(\Delta, K_q^T(-)^{\sim}\right)_{\mathbb{Q}}\right]_I^{\wedge} \cong \left[\check{H}^p\left(\Delta, K_q^T(-)^{\sim}\right)_{\mathbb{Q}}\right]_I^{\wedge} = \left[\check{H}^p\left(\Delta, K_q^T\right)_{\mathbb{Q}}\right]_I^{\wedge} \\
&\cong \left[H^p\left(\bigoplus_{\sigma} K_q^T(\langle\sigma\rangle) \to \bigoplus_{\delta<\tau} K_q^T(\langle\delta\cap\tau\rangle) \to \bigoplus_{\delta<\tau<\epsilon} K_q^T(\langle\delta\cap\tau\cap\epsilon\rangle) \to \cdots\right)_{\mathbb{Q}}\right]_I^{\wedge} \\
&\cong H^p\left(\bigoplus_{\sigma}\left(K_q^T(\langle\sigma\rangle)_{\mathbb{Q}}\right)_I^{\wedge} \to \bigoplus_{\delta<\tau}\left(K_q^T(\langle\delta\cap\tau\rangle)_{\mathbb{Q}}\right)_I^{\wedge} \to \bigoplus_{\delta<\tau<\epsilon}\left(K_q^T(\langle\delta\cap\tau\cap\epsilon\rangle)_{\mathbb{Q}}\right)_I^{\wedge} \to \cdots\right).
\end{aligned}$$

We can take tensoring with $\mathbb{Q}$ and completion of the homology into the complex because tensoring with $\mathbb{Q}$ over $\mathbb{Z}$ is exact and the map $\mathbb{Q}[M] \to (\mathbb{Q}[M])_I^{\wedge}$ is flat.

For all $\epsilon \in \Delta$ we have

$$\begin{aligned}
\left(K_q^T(\langle\epsilon\rangle)_{\mathbb{Q}}\right)_I^{\wedge} &\cong \left((\mathbb{Z}[M_\epsilon] \otimes_{\mathbb{Z}} K_q(k))_{\mathbb{Q}}\right)_I^{\wedge} \cong \left(\left((\mathbb{Z}[M_\epsilon] \otimes_{\mathbb{Z}} K_q(k))_{\mathbb{Q}}\right)_I^{\wedge}\right)_I^{\wedge} \\
&\cong \left((\mathbb{Q}[M_\epsilon] \otimes_{\mathbb{Q}} K_q(k)_{\mathbb{Q}})_I^{\wedge}\right)_I^{\wedge} \cong (\mathbb{Q}[M_\epsilon] \otimes_{\mathbb{Q}} K_q(k)_{\mathbb{Q}}) \otimes_{\mathbb{Q}[M]} \mathbb{Q}[M]_I^{\wedge} \otimes_{\mathbb{Q}[M]} \mathbb{Q}[M]_I^{\wedge} \\
&\cong (\mathbb{Q}[M_\epsilon])_I^{\wedge} \otimes_{\mathbb{Q}} (K_q(k)_{\mathbb{Q}})_I^{\wedge} \cong (\mathbb{Q}[M_\epsilon])_I^{\wedge} \otimes_{\mathbb{Q}} K_q(k)_{\mathbb{Q}}
\end{aligned}$$

where $(K_q(k)_{\mathbb{Q}})_I^{\wedge} \cong K_q(k)_{\mathbb{Q}}$ since $IK_q(k)_{\mathbb{Q}} = 0$.

It follows that

$$
\begin{aligned}
&H^p\left(\bigoplus_{\sigma}\left(K_q^T(\langle\sigma\rangle)_{\mathbb{Q}}\right)_I^{\wedge} \to \bigoplus_{\delta<\tau}\left(K_q^T(\langle\delta\cap\tau\rangle)_{\mathbb{Q}}\right)_I^{\wedge} \to \bigoplus_{\delta<\tau<\epsilon}\left(K_q^T(\langle\delta\cap\tau\cap\epsilon\rangle)_{\mathbb{Q}}\right)_I^{\wedge} \to \cdots\right)\\
&\cong H^p\left(\bigoplus_{\sigma}(\mathbb{Q}[M_\sigma])_I^{\wedge}\otimes_{\mathbb{Q}} K_q(k)_{\mathbb{Q}} \to \bigoplus_{\delta<\tau}(\mathbb{Q}[M_{\delta\cap\tau}])_I^{\wedge}\otimes_{\mathbb{Q}} K_q(k)_{\mathbb{Q}}\right.\\
&\qquad\qquad\left.\to \bigoplus_{\delta<\tau<\epsilon}(\mathbb{Q}[M_{\delta\cap\tau\cap\epsilon}])_I^{\wedge}\otimes_{\mathbb{Q}} K_q(k)_{\mathbb{Q}} \to \cdots\right)\\
&\cong H^p\left(\bigoplus_{\sigma}(\mathbb{Q}[M_\sigma])_I^{\wedge} \to \bigoplus_{\delta<\tau}(\mathbb{Q}[M_{\delta\cap\tau}])_I^{\wedge} \to \bigoplus_{\delta<\tau<\epsilon}(\mathbb{Q}[M_{\delta\cap\tau\cap\epsilon}])_I^{\wedge} \to \cdots\right)\otimes_{\mathbb{Q}} K_q(k)_{\mathbb{Q}}
\end{aligned}
$$

The last isomorphism holds because $-\otimes K_q(k)_{\mathbb{Q}}$ is exact.

We also have $(\mathbb{Q}[M_\epsilon])_I^{\wedge} \cong \left(S^\bullet_{\mathbb{Q}}(M_{\epsilon\mathbb{Q}})\right)_{\mathfrak{m}}^{\wedge}$ as $(\mathbb{Q}[M])_I^{\wedge} \cong \left(S^\bullet_{\mathbb{Q}}(M_{\mathbb{Q}})\right)_{\mathfrak{m}}^{\wedge}$-modules for all $\epsilon\in\Delta$ where $\mathfrak{m}$ is the unique homogeneous maximal ideal in $S^\bullet_{\mathbb{Q}}(M_{\mathbb{Q}})$. The $E^2$-terms of the spectral sequence upon tensoring with $\mathbb{Q}$ and completing becomes

$$
\begin{aligned}
\left[H^p\left(\Delta, K_q^T(-)^{\sim}\right)_{\mathbb{Q}}\right]_I^{\wedge} &\cong H^p\left(\Delta, \left(\sigma\mapsto \left(S^\bullet_{\mathbb{Q}}(M_{\sigma\mathbb{Q}})\right)_{\mathfrak{m}}^{\wedge}\right)^{\sim}\right)\otimes_{\mathbb{Q}} K_q(k)_{\mathbb{Q}}\\
&\cong H^p\left(\Delta, \left(\sigma\mapsto S^\bullet_{\mathbb{Q}}(M_{\sigma\mathbb{Q}})\otimes_{\mathbb{Q}} S^\bullet_{\mathbb{Q}}(M_{\mathbb{Q}})_{\mathfrak{m}}^{\wedge}\right)^{\sim}\right)\otimes_{\mathbb{Q}} K_q(k)_{\mathbb{Q}}\\
&\cong \left[H^p\left(\Delta, \left(\sigma\mapsto S^\bullet_{\mathbb{Q}}(M_{\sigma\mathbb{Q}})\right)^{\sim}\otimes_{\mathbb{Q}} S^\bullet_{\mathbb{Q}}(M_{\mathbb{Q}})_{\mathfrak{m}}^{\wedge}\right)\right]\otimes_{\mathbb{Q}} K_q(k)_{\mathbb{Q}}\\
&\cong \left[H^p\left(\Delta, \left(\sigma\mapsto S^\bullet_{\mathbb{Q}}(M_{\sigma\mathbb{Q}})\right)^{\sim}\right)\right]\otimes_{\mathbb{Q}} S^\bullet_{\mathbb{Q}}(M_{\mathbb{Q}})_{\mathfrak{m}}^{\wedge}\otimes_{\mathbb{Q}} K_q(k)_{\mathbb{Q}}\\
&\cong \left[H^p\left(\Delta, \left(\sigma\mapsto S^\bullet_{\mathbb{Q}}(M_{\sigma\mathbb{Q}})\right)^{\sim}\right)\right]_{\mathfrak{m}}^{\wedge}\otimes_{\mathbb{Q}} K_q(k)_{\mathbb{Q}}
\end{aligned}
$$

Let $\mathcal{F}$ be the sheaf associated to the functor $\sigma\mapsto S^\bullet_{\mathbb{Q}}(M_{\sigma\mathbb{Q}})$ for all $\sigma\in\Delta$ along the maps induced by face inclusions. In more detail, given $\tau\preceq\sigma$, we have $\mathcal{F}_\sigma := S^\bullet_{\mathbb{Q}}(M_{\sigma_{\mathbb{Q}}}) \to \mathcal{F}_\tau := S^\bullet_{\mathbb{Q}}(M_{\tau_{\mathbb{Q}}})$ induced by the natural surjection $M_\sigma = \frac{M}{\sigma^\perp\cap M}\to M_\tau = \frac{M}{\tau^\perp\cap M}$. We have

$$
\left[H^p\left(\Delta, K_q^T(-)^{\sim}\right)_{\mathbb{Q}}\right]_I^{\wedge} \cong H^p\left(\Delta,\mathcal{F}\right)_{\mathfrak{m}}^{\wedge}\otimes_{\mathbb{Q}} K_q(k)_{\mathbb{Q}} \tag{4.1}
$$

and in particular

$$H^0(\Delta,\mathcal{F})^\wedge_{\mathfrak{m}} \cong (PP^*(\Delta)_{\mathbb{Q}})^\wedge_{\mathfrak{m}} \cong (A_T^*(X(\Delta))_{\mathbb{Q}})^\wedge_{\mathfrak{m}}. \tag{4.2}$$

The spectral sequence is of the form

$$H^p(\Delta,\mathcal{F})^\wedge_{\mathfrak{m}} \otimes_{\mathbb{Q}} K_q(k)_{\mathbb{Q}} \Longrightarrow \left(K^T_{q-p}(X(\Delta))_{\mathbb{Q}}\right)^\wedge_I. \tag{4.3}$$

## 4.4 Immediate Consequences

Recall that a cone $\sigma$ is simplicial if $\sigma$ has $i$ generators where $i = dim(\sigma)$ and that a fan $\Delta$ is simplicial if every cone in $\Delta$ is simplicial.

**Theorem 4.4.1.** *Let $X = X(\Delta)$ be a toric variety associated to a simplicial fan $\Delta$. Then the symmetric algebra sheaf $\mathcal{F}$ on $\Delta$ is flasque. Moreover,*

$$0 \to \left(K_q^T(X)_{\mathbb{Q}}\right)^\wedge_I \to \bigoplus_{\sigma\in Max(\Delta)} \left(K_q^T(U_\sigma)_{\mathbb{Q}}\right)^\wedge_I \to \bigoplus_{\delta<\tau}\left(K_q^T(U_{\delta\cap\tau})_{\mathbb{Q}}\right)^\wedge_I \to \cdots$$

*and*

$$\begin{aligned} 0 \to \left(K_q^T(X)_{\mathbb{Q}}\right)^\wedge_I \to & \bigoplus_{\sigma\in Max(\Delta)} K_q(k)_{\mathbb{Q}} \otimes_{\mathbb{Q}} S^\bullet_{\mathbb{Q}}(M_{\sigma_{\mathbb{Q}}})^\wedge_{\mathfrak{m}} \\ \to & \bigoplus_{\delta<\tau} K_q(k)_{\mathbb{Q}} \otimes_{\mathbb{Q}} S^\bullet_{\mathbb{Q}}(M_{\delta\cap\tau_{\mathbb{Q}}})^\wedge_{\mathfrak{m}} \to \bigoplus_{\delta<\tau<\epsilon} K_q(k)_{\mathbb{Q}} \otimes_{\mathbb{Q}} S^\bullet_{\mathbb{Q}}(M_{\delta\cap\tau\cap\epsilon_{\mathbb{Q}}})^\wedge_{\mathfrak{m}} \to \cdots \end{aligned}$$

*are exact, and there is an isomorphism*

$$\left(K_q^T(X)_{\mathbb{Q}}\right)^\wedge_I \cong K_q(k)_{\mathbb{Q}} \otimes_{\mathbb{Q}} (A_T^*(X(\Delta))_{\mathbb{Q}})^\wedge_{\mathfrak{m}}.$$

*Proof.* To show that the sheaf $\mathcal{F}$ is flasque, it suffices to prove $\mathcal{F}(\langle\sigma\rangle) \twoheadrightarrow \mathcal{F}(\partial\langle\sigma\rangle)$ for all $\sigma \in \Delta$ where $\partial\langle\sigma\rangle := \langle\sigma\rangle \setminus \{\sigma\}$ [2, Lemma 4.1]. In other words, for a fixed cone $\sigma \in \Delta$, we have to show the surjectivity of the map

$$S^\bullet_{\mathbb{Q}}(M_{\sigma\mathbb{Q}}) \to ker\left(\bigoplus_{\substack{\tau_i \prec \sigma \\ \tau_i \text{ a facet of } \sigma}} S^\bullet_{\mathbb{Q}}\left(M_{\tau_i\mathbb{Q}}\right) \to \bigoplus_{i<j} S^\bullet_{\mathbb{Q}}\left(M_{\tau_i\cap\tau_j\mathbb{Q}}\right)\right). \tag{4.4}$$

We claim that we may assume that $\sigma$ is generated by a part of the standard basis $e_1, ..., e_n$ in $N$ where $n = rank(N)$. Indeed, let $\alpha : N_{\mathbb{Q}} \to N_{\mathbb{Q}}$ denote a $\mathbb{Q}$-change of basis such that $\alpha(\sigma)$ is generated by a part of the standard basis $e_1, ..., e_n$ in $N$. Let $\alpha^* : M_{\mathbb{Q}} \to M_{\mathbb{Q}}$ denote the composition with $\alpha$. The map $\alpha^*$ induces isomorphisms

$$M_{\alpha(\tau)_{\mathbb{Q}}} = \frac{M_{\mathbb{Q}}}{\alpha(\tau)^\perp \cap M_{\mathbb{Q}}} \xrightarrow[\cong]{\alpha^*} M_{\tau\mathbb{Q}} = \frac{M_{\mathbb{Q}}}{\tau^\perp \cap M_{\mathbb{Q}}} \text{ for all } \tau \preceq \sigma,$$

and the isomorphisms commute with face inclusions. When $\sigma$ is generated by a part of the standard basis $e_1, ..., e_n$ in $N$, the surjection of (4.4) is obvious.

It follows that $H^p(\Delta, \mathcal{F}) = 0$ for all $p > 0$. The exactness of the two complexes follows from (4.1). At the same time, the spectral sequence (4.3) collapses to give

$$H^0(\Delta, \mathcal{F})^\wedge_{\mathfrak{m}} \otimes K_q(k)_{\mathbb{Q}} \cong \left(K^T_q(X(\Delta))_{\mathbb{Q}}\right)^\wedge_I \text{ for all } q \geq 0$$

and by (4.2)

$$(A^*_T(X(\Delta))_{\mathbb{Q}})^\wedge_{\mathfrak{m}} \otimes K_q(k)_{\mathbb{Q}} \cong \left(K^T_q(X(\Delta))_{\mathbb{Q}}\right)^\wedge_I \text{ for all } q \geq 0$$

□

**Remark 4.4.2.** In the study of $H^p(\Delta, \mathcal{F})$ for the symmetric algebra sheaf $\mathcal{F}$ on a fan $\Delta$, we may assume that $\Delta$ is a complete fan. If $\Delta$ is not complete, one can complete it using simplicial cones. Let $\Delta_s$ denote the subfan generated by the added simplicial cones and $\Delta_c$ denote the complete fan. Clearly, we have $\Delta_c = \Delta \cup \Delta_s$. Consider the long exact sequence of sheaf cohomology groups

$$\cdots \to H^p(\Delta_c, \mathcal{F}) \to H^p(\Delta, \mathcal{F}) \bigoplus H^p(\Delta_s, \mathcal{F}) \to H^p(\Delta_s \cap \Delta, \mathcal{F}) \to H^{p+1}(\Delta, \mathcal{F}) \to \cdots .$$

Since $H^p(\Delta_s, \mathcal{F}) = 0$ and $H^p(\Delta_s \cap \Delta, \mathcal{F}) = 0$ for all $p > 0$ by Theorem 4.4.1, we have $H^p(\Delta_c, \mathcal{F}) \cong H^p(\Delta, \mathcal{F})$ for all $p \geq 1$.

Given a fan $\Delta$, let $\Delta^{\leq 2}$ denote the 2-skeleton of the fan, consisting of all cones in $\Delta$ with dimension less than or equal to two. Note that $\Delta^{\leq 2}$ is a simplicial subfan of $\Delta$ and $\Delta^{\leq 2} \hookrightarrow \Delta$ is an open map.

**Proposition 4.4.3.** *Let $\Delta$ be a complete fan in $N_{\mathbb{R}}$ where $dim(N_{\mathbb{R}}) = n$ and let $\mathcal{G}$ be a sheaf on $\Delta$. Consider $\mathcal{G}' := \mathcal{G}|_{\Delta^{\leq 2}}$. Suppose $H^p(\langle\sigma\rangle^{\leq 2}, \mathcal{G}) = 0$ for any cone $\sigma \in \Delta$ and for all $p > 0$. Then*

$$H^p(\Delta, \mathcal{G}) = H^p(\Delta^{\leq 2}, \mathcal{G}')) \text{ for all } p \geq n - 1.$$

*Proof.* Let $j : \Delta^{\leq 2} \hookrightarrow \Delta$ denote the inclusion map. There exists a spectral sequence [21, 5.8.5, p. 152]

$$H^p(\Delta, \mathcal{R}^q j_* \mathcal{G}') \Longrightarrow H^{p+q}(\Delta^{\leq 2}, \mathcal{G}')$$

where the derived right functor of the direct image sheaf $\mathcal{R}^q j_* \mathcal{G}'$ is the associated sheaf of the presheaf $V \mapsto H^q(V \cap \Delta^{\leq 2}, \mathcal{G}') = H^q(V \cap \Delta^{\leq 2}, \mathcal{G})$.

For all $\sigma \in \Delta$ and for all $q > 0$,

$$(\mathcal{R}^q j_* \mathcal{G}')_\sigma = H^q(\langle\sigma\rangle^{\leq 2}, \mathcal{G}) = 0$$

by the hypothesis. So, $H^p(\Delta, j_* \mathcal{G}') = H^p(\Delta^{\leq 2}, \mathcal{G}')$.

On the other hand, $H^p(\Delta, j_* \mathcal{G}') = H^p(\mathcal{C}^\bullet(j_* \mathcal{G}'))$ where $\mathcal{C}^\bullet$ is the cellular complex and when $p \geq n - 1$, $H^p(\mathcal{C}^\bullet(j_* \mathcal{G}')) = H^p(\mathcal{C}^\bullet(\mathcal{G})) = H^p(\Delta, \mathcal{G})$. $\square$

**Corollary 4.4.4.** *Let $\mathcal{F}$ be the symmetric algebra sheaf associated to $\sigma \mapsto S^{\bullet}_{\mathbb{Q}}(M_{\sigma_{\mathbb{Q}}})$ on a fan $\Delta$ in $N_{\mathbb{R}}$ where $dim(N_{\mathbb{R}}) = n$. Then*

$$H^p(\Delta, \mathcal{F}) = 0 \textit{ for all } p \geq n-1.$$

*Proof.* The 2-skeleton $\Delta^{\leq 2}$ is a simplicial subfan of $\Delta$. By Theorem 4.4.1, we have $H^p(\Delta^{\leq 2}, \mathcal{F}|_{\Delta^{\leq 2}}) = 0$ for all $p > 0$, and the result follows from Proposition 4.4.3. $\square$

# Chapter 5

# Three-dimensional Complete Fans

In this chapter, we study the equivariant $K$-theory of three-dimensional projective toric varieties. More precisely, we study the rational equivariant $K$-groups completed at the augmentation ideal of $\mathbb{Q}[M]$. We start by outlining the constructions and results in this chapter.

Let $\Delta$ be a three-dimensional complete fan. Let $\mathcal{F}$ be the sheaf on $\Delta$ associated to the functor $\sigma \mapsto S^{\bullet}_{\mathbb{Q}}(M_{\sigma_Q})$. Viewing it degree-wise, we have sheaves $\mathcal{F}^{(r)}$ on $\Delta$ defined by $\sigma \mapsto S^{r}_{\mathbb{Q}}(M_{\sigma_Q})$ with maps induced by face inclusions.

When $\Delta$ is given by a convex polytope, the associated toric variety $X(\Delta)$ is projective. We have a convergent spectral sequence [20]

$$H^p\left(\Delta, \mathcal{F}\right)^{\wedge}_{\mathfrak{m}} \otimes_{\mathbb{Q}} K_q(k)_{\mathbb{Q}} \cong \left(\prod_r H^p\left(\Delta, \mathcal{F}^{(r)}\right)\right) \otimes_{\mathbb{Q}} K_q(k)_{\mathbb{Q}} \Longrightarrow \left(K^T_{q-p}(X(\Delta))_{\mathbb{Q}}\right)^{\wedge}_{I}.$$

The isomorphism follows from the sequence of isomorphisms

$$\begin{aligned}
&H^p\left(\Delta, \mathcal{F}\right)^{\wedge}_{\mathfrak{m}} := \left[H^p\left(\Delta, \left(\sigma \mapsto S^{\bullet}_{\mathbb{Q}}(M_{\sigma\mathbb{Q}})\right)^{\sim}\right)\right]^{\wedge}_{\mathfrak{m}} \cong H^p\left(\Delta, \left(\sigma \mapsto \left(S^{\bullet}_{\mathbb{Q}}(M_{\sigma\mathbb{Q}})\right)^{\wedge}_{\mathfrak{m}}\right)^{\sim}\right)\\
&\cong H^p\left(\Delta, \left(\sigma \mapsto \prod_r S^r_{\mathbb{Q}}(M_{\sigma\mathbb{Q}})\right)^{\sim}\right) \cong H^p\left(\Delta, \prod_r \left(\sigma \mapsto S^r_{\mathbb{Q}}(M_{\sigma\mathbb{Q}})\right)^{\sim}\right)\\
&\cong \prod_r H^p\left(\Delta, \left(\sigma \mapsto S^r_{\mathbb{Q}}(M_{\sigma\mathbb{Q}})\right)^{\sim}\right)\\
&\cong \prod_r H^p\left(\Delta, \mathcal{F}^{(r)}\right).
\end{aligned}$$

If $\Delta$ is a fan associated to a three-dimensional polytope, then this spectral sequence yields a short exact sequence

$$0 \to H^1\left(\Delta, \mathcal{F}\right)^{\wedge}_{\mathfrak{m}} \otimes_{\mathbb{Q}} K_{n+1}(k)_{\mathbb{Q}} \to \left(K_n^T(X(\Delta))_{\mathbb{Q}}\right)^{\wedge}_I \to H^0(\Delta, \mathcal{F})^{\wedge}_{\mathfrak{m}} \otimes_{\mathbb{Q}} K_n(k)_{\mathbb{Q}} \to 0.$$

Thus, we may understand $\left(K_q^T(X(\Delta))_{\mathbb{Q}}\right)^{\wedge}_I$ by computing $H^1\left(\Delta, \mathcal{F}\right)^{\wedge}_{\mathfrak{m}}$ and $H^0(\Delta, \mathcal{F})^{\wedge}_{\mathfrak{m}}$. It follows from (4.2) that we have isomorphisms

$$H^0(\Delta, \mathcal{F})^{\wedge}_{\mathfrak{m}} \cong \prod_r H^0\left(\Delta, \mathcal{F}^{(r)}\right) \cong (PP^*(\Delta)_{\mathbb{Q}})^{\wedge}_{\mathfrak{m}} \cong (A^*_T(X(\Delta))_{\mathbb{Q}})^{\wedge}_{\mathfrak{m}}.$$

Given a fan $\Delta$, we have $H^0(\Delta, \mathcal{F}^{(r)}) = \{f_\sigma \in S^r_{\mathbb{Q}}(M_{\sigma_{\mathbb{Q}}}) \mid f_\sigma = f_{\sigma'} \in S^r_{\mathbb{Q}}(M_{\sigma\cap\sigma'})\}$. If we choose representatives $f_\sigma \in S^r_{\mathbb{Q}}(M_{\mathbb{Q}})$ in $H^0(\Delta, \mathcal{F}^{(r)})$, then

$$H^0(\Delta, \mathcal{F}^{(r)}) \subseteq \{\text{Continuous Functions on } |\Delta|\}$$

and an element in $H^0(\Delta, \mathcal{F}^{(r)})$ is a piecewise homogeneous polynomial function of degree $r$ with rational coefficients. Each $f_\sigma \in S^r_{\mathbb{Q}}(M_{\mathbb{Q}})$ is a function defined on $\mathbb{Q}\sigma$, the vector space spanned by $\sigma$.

Let $\Delta$ be a three-dimensional complete fan. Let $\sigma$ be one of its maximal cones and $\tau_1$ and

$\tau_2$ be any two of its facets. For any $f \in H^0(\Delta, \mathcal{F}^{(r)})$, $f|_{\mathbb{Q}\tau_1}$ and $f|_{\mathbb{Q}\tau_2}$ agree on $\mathbb{Q}\tau_1 \cap \mathbb{Q}\tau_2$ even if $\tau_1$ and $\tau_2$ intersect only at the origin. In Section 5.1, we modify the poset $\Delta$ and obtain a new poset $\widetilde{\Delta}$ to take into account the added intersection property. After defining symmetric power sheaves on the new poset $\widetilde{\Delta}$ in Section 5.2, we argue that the symmetric power sheaves on $\widetilde{\Delta}$ are flasque for sufficiently high powers (Theorem 5.4.1). We then conclude the following.

**Theorem 5.4.2** *If $\Delta$ is a three-dimensional complete fan in general position and every maximal cone has at most $N$ facets, then $H^1(\Delta, \mathcal{F}^{(r)}) = 0$ for $r \geq 2N - 5$.*

**Theorem 5.4.3** *For a three-dimensional complete fan $\Delta$, the Hilbert polynomial of $H^1(\Delta, \mathcal{F}^{(r)})$ as a function of $r$ is a constant.*

We define the *$r$-th Euler characteristic* of $\Delta$ to be

$$\chi^r(\Delta) = \sum (-1)^i dim_{\mathbb{Q}} H^i(\Delta, \mathcal{F}^{(r)}).$$

If $\Delta$ is a complete fan, the cellular complex $\mathcal{C}^\bullet(\mathcal{F}^{(r)})$ computes sheaf cohomology and

$$\chi^r(\Delta) = \sum (-1)^i dim_{\mathbb{Q}} C^i(\mathcal{F}^{(r)}).$$

Let $\Delta$ be a three-dimensional complete fan. It follows from Proposition 4.4.1 that $H^p(\Delta, \mathcal{F}^{(r)}) = 0$ for all $p \geq 2$. Let $f$, $e$, and $v$ denote the numbers of three-dimensional, two-dimensional, and one-dimensional cones in $\Delta$. Since one has $\begin{pmatrix} s+r-1 \\ r \end{pmatrix}$ degrees of freedom in specifying polynomials of degree $r$ in $s$ variables,

$$\begin{aligned} \chi^r(\Delta) &= dim_{\mathbb{Q}} H^0(\Delta, \mathcal{F}^{(r)}) - dim_{\mathbb{Q}} H^1(\Delta, \mathcal{F}^{(r)}) \\ &= f \left( \begin{array}{c} 3+r-1 \\ r \end{array} \right) - e \left( \begin{array}{c} 2+r-1 \\ r \end{array} \right) + v \left( \begin{array}{c} 1+r-1 \\ r \end{array} \right) \end{aligned} \tag{5.1}$$

for all $r \geq 1$. When $r = 0$, the cellular cohomology coincides with the reduced homology of the sphere. We have $\chi^0(\Delta) = 1 = dim_{\mathbb{Q}} H^0(\Delta, \mathcal{F}^{(0)})$ and $dim_{\mathbb{Q}} H^1(\Delta, \mathcal{F}^{(0)}) = 0$. Using the results of Theorem 5.4.3 and linear algebra, we calculate $dim_{\mathbb{Q}} H^0(\Delta, \mathcal{F}^{(r)})$ and $dim_{\mathbb{Q}} H^1(\Delta, \mathcal{F}^{(r)})$ for all $r \geq 0$ and use the information to understand the equivariant $K$-groups.

## 5.1 Posets with Ghost Elements

Recall that $\langle \sigma \rangle$ is the smallest fan generated by a cone $\sigma$, and it consists of $\sigma$ and all its faces. If $\tau$ is a face of $\sigma$, we have an open inclusion $\langle \tau \rangle \hookrightarrow \langle \sigma \rangle$. The poset associated to the fan $\Delta$ is the colimit of the posets $\langle \sigma \rangle$ for all $\sigma \in \Delta$ along the maps induced by face inclusions. Let $\mathbb{Q}\tau := \mathbb{Q}(\tau \cap N_{\mathbb{Q}})$ denote the set of points of $\tau$ with rational coordinates. Now, for each cone $\sigma$, we define a collection of vector spaces associated to $\sigma$,

$$\widetilde{\langle \sigma \rangle} := \{ \mathbb{Q}\tau_1 \cap ... \cap \mathbb{Q}\tau_l \subseteq N_{\mathbb{Q}} \mid \tau_1, ..., \tau_l \preceq \sigma \}.$$

The set $\widetilde{\langle \sigma \rangle}$ is a poset with respect to vector space containment.

**Lemma 5.1.1.** *For a convex polyhedral cone $\sigma$ and a proper face $\epsilon$,*

$$\mathbb{Q}\epsilon = \bigcap_{\substack{\epsilon \preceq \tau \\ \tau \text{ is a facet of } \sigma}} \mathbb{Q}\tau.$$

*Proof.* The proof is similar to that of the fact in [5, p. 10]: every proper face of a convex polyhedral cone is the intersection of all facets containing it.

The proof is by induction on the codimension of $\epsilon$. If the codimension is one, $\epsilon$ is a facet. Suppose that the codimension of $\epsilon$ is larger than two. Clearly $\mathbb{Q}\epsilon \subseteq \mathbb{Q}\tau$ for some facet $\tau$ of $\sigma$. By induction $\mathbb{Q}\epsilon = \cap_{\substack{\epsilon \preceq \eta \\ \eta \text{ is a facet of } \tau}} \mathbb{Q}\eta$. The image $\overline{\sigma}$ of $\sigma$ in $\mathbb{Q}\sigma/\mathbb{Q}\eta$ is a two-dimensional cone. Hence each $\mathbb{Q}\eta$ is the intersection of two vector spaces of the form $\mathbb{Q}\tau$ for two facets $\tau$ of $\sigma$, and $\mathbb{Q}\epsilon = \cap_{\substack{\epsilon \preceq \tau \\ \tau \text{ is a facet of } \sigma}} \mathbb{Q}\tau$. $\square$

By sending each face of $\sigma$ to the vector space it generates, we have an injective order-preserving map $\langle\sigma\rangle \hookrightarrow \widetilde{\langle\sigma\rangle}$.

Recall that the *dimension* of a cone $\sigma$ is defined as $dim(\sigma) := dim_{\mathbb{R}}(\mathbb{R}\sigma)$. A set of codimension one vector spaces $\{H_1, ..., H_l\}$ in a $d$-dimensional vector space is *in general position* if for $1 \leq i_1 < \cdots < i_n \leq l$ we have

$$dim(H_{i_1} \cap ... \cap H_{i_n}) = \max\{d-n, 0\}$$

A three-dimensional complete fan is *in general position* if the collection of two-dimensional vector spaces generated by its two-dimensional cones is in general position. A three-dimensional convex polytope is *in general position* if and only if the three-dimensional fan it defines is in general position. Since the two-dimensional cones correspond to edges on a convex polytope, a convex polytope is in general position if and only if there are no parallel edges.

**Lemma 5.1.2.** *Given a three-dimensional cone $\sigma$, the fan $\langle\sigma\rangle$ is in general position.*

*Proof.* Recall that there is a one-to-one order reversing correspondence between faces of a cone and the faces of the dual cone. More precisely, given $\tau$ a face of $\sigma$, $\tau^{\perp} \cap \sigma^{\vee}$ is a face of $\sigma^{\vee}$. Our assertion is equivalent to the following: in the arrangement of one dimensional vector spaces corresponding to the rays of a strongly convex rational polyhedral cone, no three of them lie on the same plane.

Suppose that there are three rays of $\sigma$ lying on the same plane and let $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ be the minimal lattice points on the rays. Then, there are real numbers $r$ and $s$ such that $\mathbf{w} = r\mathbf{u} + s\mathbf{v}$.

Let $\mathbf{a}$ be a vector in the relative interior of the facet of the dual cone $\sigma^\vee$ defined by $\mathbf{u}$. That is, $\langle \mathbf{a}, \mathbf{u} \rangle = 0$ but $\langle \mathbf{a}, \mathbf{v} \rangle > 0$ and $\langle \mathbf{a}, \mathbf{w} \rangle > 0$. Evaluating $\mathbf{a}$ at $\mathbf{w} = r\mathbf{u} + s\mathbf{v}$ yields $s > 0$. Similarly, $r > 0$. Let $\mathbf{c}$ be a point on the facet of the dual cone defined by $\mathbf{w}$. We have $\langle \mathbf{c}, \mathbf{w} \rangle = 0 = \langle c, (r\mathbf{u} + s\mathbf{v}) \rangle > 0$, a contradiction. $\square$

The following fact follows immediately.

**Corollary 5.1.3.** *Let $\sigma$ be a three-dimensional cone and $\epsilon$ and $\tau$ be two of its faces. If $\mathbb{Q}\epsilon \subseteq \mathbb{Q}\tau$, then $\epsilon \preceq \tau$.*

**Remark 5.1.4.** If $\sigma$ is a four dimensional cone, then the fan $\langle \sigma \rangle$ is not necessarily in general position. Let $e_1, ..., e_4$ be the standard basis of $\mathbb{R}^4 = N \otimes_{\mathbb{Z}} \mathbb{R}$ and let $\sigma$ be generated by $e_1, e_2, e_3, e_4$, and $e_1 + e_2 + e_3 - e_4$. The cone $\sigma$ is strongly convex. The codimension one supporting hyperplanes of $\sigma$ correspond to the rays of $\sigma^\vee$, and $\sigma^\vee$ is generated by $e_1, e_2, e_3, e_1 + e_4, e_2 + e_4$, and $e_3 + e_4$. Note that four of the generators $e_1, e_2$, $e_1 + e_4$, and $e_2 + e_4$ lie on the same codimension one subspace $z = 0$. The corresponding four hyperplanes intersect in a one dimensional subspace. The fan $\langle \sigma \rangle$ is not in general position.

Consider the set $\widetilde{\langle \sigma \rangle} = \{\mathbb{Q}\tau_1 \cap ... \cap \mathbb{Q}\tau_l \mid \tau_1, ..., \tau_l \preceq \sigma\}$ for $\sigma$ a three-dimensional cone. By the preceding lemma, any distinct three-way intersection among members of $\widetilde{\langle \sigma \rangle} \backslash \{\mathbb{Q}\sigma\}$ is zero. We will call an element a "ghost" element in $\widetilde{\langle \sigma \rangle}$ if it is not in the image of $\langle \sigma \rangle$ in the inclusion $\langle \sigma \rangle \hookrightarrow \widetilde{\langle \sigma \rangle}$. The only ghost elements in $\widetilde{\langle \sigma \rangle}$ are the two-way intersections of vector spaces corresponding to two facets that intersect only at the origin. For example, let $\sigma$ be a pyramidal cone generated by $(\pm 1, \pm 1, 1)$ opening around the positive $z$-axis. The ghost elements in $\widetilde{\langle \sigma \rangle}$ are the $x$-axis and the $y$-axis, the result of intersecting the vector spaces spanned by two opposite facets. If $dim(\sigma) \leq 2$, $\widetilde{\langle \sigma \rangle}$ and $\langle \sigma \rangle$ are isomorphic as posets. Whenever $\tau \preceq \sigma$, there is a poset inclusion $\widetilde{\langle \tau \rangle} \hookrightarrow \widetilde{\langle \sigma \rangle}$.

Recall that a set equipped with a reflexive and transitive order relation is a *preordered set*. The category of preordered sets consists of preordered sets and order-preserving maps. The colimit of a diagram of preordered sets exists. The underlying set of the colimit is the colimit of the diagram in the category of sets. An order relation between two elements in a preordered set gives an order relation between their images in the colimit. The order relation in the colimit is generated transitively by the order relations in the preordered sets in the diagram.

**Lemma 5.1.5.** *The colimit of a diagram in the category of posets can be obtained by first taking the colimit of the diagram in the category of preordered sets and then modding out the equivalence relation $a \sim b \Leftrightarrow a \leq b$ and $b \leq a$. In particular, colimits exist in the category of posets.*

*Proof.* The category of posets $Pos$ is a full subcategory of the category of preordered sets $PreO$. Let $F : Pos \to PreO$ be the forgetful functor. Let $G : PreO \to Pos$ be the functor sending $Q \mapsto Q/\sim$ and taking an order-preserving map between two preordered sets $Q \to Q'$ to the induced map from $Q/\sim$ to $Q'/\sim$. Clearly, $G \circ F \cong id_{Pos}$. The two functors $F$ and $G$ form an adjoint pair, that is, given a preordered set $A$ and a poset $B$, we have

$$\mathrm{Hom}_{Pos}(G(A), B) \cong \mathrm{Hom}_{PreO}(A, F(B)).$$

In detail, an order-preserving map $G(A) \to B$ in $Pos$ defines a map $A \to F(B)$ in $PreO$ by composition with $A \to G(A)$. An order-preserving map $A \to F(B)$ in $PreO$ corresponds to the induced map $G(A) \to B$ in *Pos*.

Let $\{P_i\}_{i\in I}$ be a diagram of posets. As preordered sets, the colimit $\varinjlim_{i\in I}{}^{PreO} F(P_i)$ exists. Since the functor G has a right adjoint and it preserves colimits [9, V.5, p.115], we have

$$G\left(\varinjlim_{i\in I}{}^{PreO} F(P_i)\right) = \varinjlim_{i\in I}{}^{Pos} G(F(P_i)) = \varinjlim_{i\in I}{}^{Pos} P_i.$$

□

As we define a topology on a poset in Chapter 2, given a preordered set $A$, one can define a topology on $A$ by declaring that a subset is open if it is closed under going down with respect to the order relation. Equivalently, a subset of $A$ is closed if it is closed under going up with respect to the order relation.

**Lemma 5.1.6.** *Let $A$ and $B$ be two preordered sets. A map $f : A \to B$ is continuous if and only if it is order-preserving.*

*Proof.* Let $V \subseteq B$ be a closed subset. Take $a' \in A$ such that $a' \geq a$ for some $a \in f^{-1}(V) = \{a \in A \mid f(a) \in V\}$. If $f$ is order-preserving, $f(a') \geq f(a)$ and $f(a') \in V$. So, $a' \in f^{-1}(V)$ and $f^{-1}(V)$ is closed in $A$. On the other hand, suppose $f$ is continuous but there exist $a$ and $a'$ in $A$ such that $a' \geq a$ but $f(a') \ngeq f(a)$ in $B$. Note that $a \in f^{-1}(\overline{\{f(a)\}})$ where $\overline{\{f(a)\}}$ denotes the closure of $\{f(a)\}$ in $A$. The subset $f^{-1}(\overline{\{f(a)\}})$ is closed by continuity of $f$ and $a' \in f^{-1}(\overline{\{f(a)\}})$. Then $f(a') \in \overline{\{f(a)\}}$ meaning $f(a') \geq f(a)$, a contradiction. □

**Definition 5.1.7.** Given a fan $\Delta$, define $\widehat{\Delta} := \varinjlim_{\sigma\in\Delta} \widetilde{\langle\sigma\rangle}$, the colimit of the functor $\Delta \to$ *Posets* given by $\sigma \mapsto \widetilde{\langle\sigma\rangle}$.

Whenever $\tau \prec \sigma$, the diagram

$$\begin{tikzcd}
\langle\sigma\rangle \arrow[r] & \widetilde{\langle\sigma\rangle} \arrow[r] & \widehat{\Delta} \\
\langle\tau\rangle \arrow[u] \arrow[r] & \widetilde{\langle\tau\rangle} \arrow[u] \arrow[ur] &
\end{tikzcd}$$

commutes and hence the universal property of the colimit induces a map of posets $\Delta \to \widehat{\Delta}$.

The posets in the diagrams $\{\langle\sigma\rangle\}_{\sigma\in\Delta}$ and $\{\widetilde{\langle\sigma\rangle}\}_{\sigma\in\Delta}$ are isomorphic if $dim(\sigma) \leq 2$. Over the three-dimensional fan $\Delta$, maximal cones meet along common faces of dimensions less than or equal to 2. The elements identified in the two colimits $\Delta = \varinjlim^{Pos}_{\sigma\in\Delta}\langle\sigma\rangle$ and $\widehat{\Delta} = \varinjlim^{Pos}_{\sigma\in\Delta}\widetilde{\langle\sigma\rangle}$ are the same. It follows that $\widetilde{\langle\sigma\rangle} \to \widehat{\Delta}$ is an injection for all $\sigma \in \Delta$. The poset $\widehat{\Delta}$ contains an isomorphic copy of $\Delta$ with additional ghost elements, and we have $\widehat{\Delta} = \varinjlim^{Pos}_{\sigma\in\Delta}\widetilde{\langle\sigma\rangle} \cong \varinjlim^{PreO}_{\sigma\in\Delta}\widetilde{\langle\sigma\rangle}$.

We let $(V,\sigma)$ denote a typical element in $\widehat{\Delta}$ where the vector space $V = \mathbb{Q}\sigma$ or $V = \mathbb{Q}\tau_1 \cap \ldots \cap \mathbb{Q}\tau_l \in \widetilde{\langle\sigma\rangle}$ for some facets $\tau_1, \ldots, \tau_l$ of a maximal cone $\sigma$. Note $(0,\sigma) = (0,\sigma')$ for all $\sigma, \sigma' \in \Delta$ and $(0,\sigma)$ is the unique minimal element in $\widehat{\Delta}$. Two elements $(W,\sigma)$ and $(W',\sigma')$ are equal if and only if $W = W' = \mathbb{Q}\tau$ for some $\tau$ with $\tau \preceq \sigma$ and $\tau \preceq \sigma'$. Let $\leq$ denote the order relation on $\widehat{\Delta}$ and $\leq_\sigma$ denote the order relation on $\widetilde{\langle\sigma\rangle}$. If $(V,\sigma) \leq (V',\sigma')$ in $\widehat{\Delta}$, then there exist $(W_i,\sigma_j) \in \widehat{\Delta}$ such that

$$\begin{aligned}
(V,\sigma=\sigma_0) \leq_{\sigma_0} (W_1,\sigma_0) &= (W_1,\sigma_1) \leq_{\sigma_1} (W_2,\sigma_1) = (W_2,\sigma_2) \leq_{\sigma_2} \cdots \\
\cdots &\leq_{\sigma_{n-1}} (W_n,\sigma_{n-1}) = (W_n,\sigma_n=\sigma') \leq_{\sigma'} (V',\sigma').
\end{aligned}$$

We define $dim(V,\sigma) = dimV$ in $\widehat{\Delta}$. The dimension function is order-preserving. Moreover, it preserves strict inequalities. Indeed, on the image of $\widetilde{\langle\sigma\rangle}$ in $\widehat{\Delta}$, if $(V,\sigma) <_\sigma (V',\sigma)$, clearly $dim(V,\sigma) < dim(V',\sigma)$. If $(V,\sigma) < (V',\sigma')$, then $dim(V,\sigma) < dim(V',\sigma')$ because at least one inequality $\leq_{\sigma_i}$ in the chain in strict.

Now, we define a new relation $\sim$ on $\widehat{\Delta}$ as follows:

$$(V, \sigma) \sim (W, \sigma') \Leftrightarrow V = W \subseteq \mathbb{Q}(\sigma \cap \sigma')$$

. If two elements are equivalent in this relation, both of them have to be ghost elements. Suppose otherwise. Say $(\mathbb{Q}\epsilon, \sigma) \sim (\mathbb{Q}\epsilon', \sigma')$ for faces $\epsilon$ and $\epsilon'$ of $\sigma$ and $\sigma'$ respectively, that is, $\mathbb{Q}\epsilon = \mathbb{Q}\epsilon' \subseteq \mathbb{Q}(\sigma \cap \sigma')$. By Corollary 5.1.3, both $\epsilon$ and $\epsilon'$ are faces of $\sigma \cap \sigma'$. Since $dim(\mathbb{Q}(\sigma \cap \sigma')) \leq 2$ and $\sigma \cap \sigma'$ is strongly convex, the faces defining the same vector spaces must be the same. Suppose $(\mathbb{Q}\epsilon, \sigma) \sim (W, \sigma')$ for a face $\epsilon$ of $\sigma$ and a ghost element $(W, \sigma')$ in $\widetilde{\langle \sigma' \rangle}$ such that $\mathbb{Q}\epsilon = W \subseteq \mathbb{Q}(\sigma \cap \sigma')$. By Corollary 5.1.3, $\epsilon$ is a face of $\sigma \cap \sigma'$ which is in turn a face of $\sigma'$. From the construction of $\widetilde{\langle \sigma' \rangle}$, $\mathbb{Q}\epsilon \neq W$.

**Definition 5.1.8.** Define $\widetilde{\Delta} := \widehat{\Delta}/\langle \sim \rangle$, the quotient of $\widehat{\Delta}$ obtained by modding out the equivalence relation generated by $\sim$. For any $[(V, \sigma)]$ and $[(W, \tau)]$ in $\widetilde{\Delta}$, we define $[(V, \sigma)] \preceq [(W, \tau)]$ if $(V, \sigma) \leq (W, \tau)$ in $\widehat{\Delta}$.

**Proposition 5.1.9.** *The quotient $\widetilde{\Delta} := \widehat{\Delta}/\langle \sim \rangle$ of $\widehat{\Delta}$ is a poset with respect to the order relation $\preceq$.*

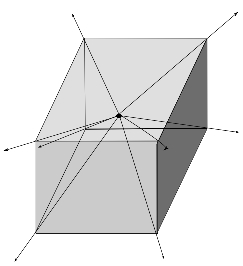
*Proof.* The set $\widetilde{\Delta}$ together with $\preceq$ is a preordered set. We will show that $\preceq$ is anti-symmetric by using the dimension function. Note that if $(V, \sigma) \sim (W, \tau)$ in $\widehat{\Delta}$, then $dimV = dimW$. For every element $[(V, \sigma)] \in \widetilde{\Delta}$, we define $dim[(V, \sigma)] = dimV$. The dimension function from $\widetilde{\Delta}$ to $\mathbb{N} \cup \{0\}$ is order-preserving. Also, a strict inequality in $\widehat{\Delta}$ gives a strict inequality in $\widetilde{\Delta}$. The dimension function preserves strict inequalities in $\widetilde{\Delta}$. In other words, if $[(V, \sigma)] \preceq [(W, \tau)]$ and $dim[(V, \sigma)] = dim[(W, \tau)]$, then $[(V, \sigma)] = [(W, \tau)]$. It follows that if $[(V, \sigma)] \preceq [(W, \tau)]$ and $[(W, \tau)] \preceq [(V, \sigma)]$, then $[(V, \sigma)] = [(W, \tau)]$. $\square$

All maps in the construction of $\widehat{\Delta}$ and $\widetilde{\Delta}$ are order-preserving and hence are continuous with respect to the poset topology, and we have a commutative triangle of posets.

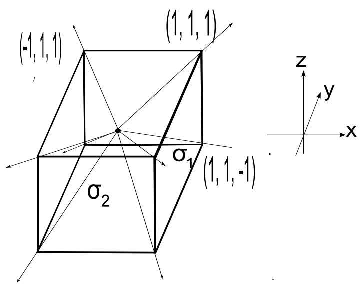
$$\begin{tikzcd} \Delta \arrow[r, hook, "i"] \arrow[d, hook, "j"'] & \widetilde{\Delta} \\ \widehat{\Delta} \arrow[ur, two heads, "f"'] & \end{tikzcd}$$

From now on we write $(W, \sigma)$ for a typical element in $\widetilde{\Delta}$ instead of $[(W, \sigma)]$. Recall that $\sigma$ is a maximal cone and the vector space $W$ is in $\widetilde{\langle \sigma \rangle}$. The smallest open set generated by $(W, \sigma)$ is denoted by $\langle (W, \sigma) \rangle$.

**Example 5.1.10.** Let $\Delta$ be the complete fan over the faces of the perfect cube with vertices $(\pm 1, \pm 1, \pm 1)$. The fan $\Delta$ has six maximal cones opening around positive and negative $x$, $y$, and $z$-axis respectively.



Let $\Delta'$ be the subfan of $\Delta$ consisting of four pyramidal cones opening around the positive and negative $x$-axis and $y$-axis respectively. For the maximal cone $\sigma_1$ opening around the positive $x$-axis, the set $\widetilde{\langle \sigma_1 \rangle}$ has two ghost elements, namely the $y$-axis and the $z$-axis. Between any two adjacent pyramidal cones $\sigma_1$ and $\sigma_2$, the $z$-axis, which is a ghost element in both $\widetilde{\langle \sigma_1 \rangle}$ and $\widetilde{\langle \sigma_2 \rangle}$, is equivalent under the relation $\sim$ since it is contained in $\mathbb{Q}(\sigma_1 \cap \sigma_2)$.




**Example 5.1.11.** Let $\Delta''$ be the subfan of $\Delta$ with only two pyramidal cones opening around positive and negative $x$-axis. Although each pyramidal cone still has two ghost elements (namely the $y$-axis and the $z$-axis), they are not equivalent since the intersection between the two pyramidal cones is the origin.

## 5.2 Symmetric Power Sheaves on $\widetilde{\Delta}$

On a three-dimensional fan $\Delta$, recall that the sheaf $\mathcal{F}^{(r)}$ is associated to the functor $\sigma \mapsto \mathcal{F}^{(r)}_{\sigma} := S^r_{\mathbb{Q}}(M_{\sigma_{\mathbb{Q}}})$. On $\widetilde{\Delta}$, we define the sheaf $\widetilde{\mathcal{F}}^{(r)}$ to be the one associated to the functor $(W,\sigma) \mapsto S^r_{\mathbb{Q}}(W^*)$ with maps induced by the order relations as follows: given $(W,\sigma) \preceq (V,\tau)$ in $\widetilde{\Delta}$, one has $W \subseteq V$ and then $V^* \twoheadrightarrow W^*$ and $S^r_{\mathbb{Q}}(V^*) \twoheadrightarrow S^r_{\mathbb{Q}}(W^*)$.

Consider the map $\Delta \xhookrightarrow{i} \widetilde{\Delta}$. For $(W,\sigma) \in \widetilde{\Delta}$, if $(W,\sigma) = (\mathbb{Q}\tau,\sigma)$ for some $\tau$ in $\Delta$, then the inverse image of $\langle(W,\sigma)\rangle$ in $\Delta$ is $\langle\tau\rangle$. If $(W,\sigma)$ is a ghost element, that is, a one dimensional space in $\widetilde{\langle\sigma\rangle}$ obtained from intersecting two vector spaces generated by two facets of $\sigma$ only meeting at the origin, then $i^{-1}\langle(W,\sigma)\rangle$ consists only of the zero cone in $\Delta$. It follows that

$$\begin{aligned}\left(i_*\mathcal{F}^{(r)}\right)_{(W,\sigma)} &= i_*\mathcal{F}^{(r)}\left(\langle(W,\sigma)\rangle\right) = \mathcal{F}^{(r)}\left(i^{-1}\langle(W,\sigma)\rangle\right)\\ &= \begin{cases} \mathcal{F}^{(r)}(\langle\tau\rangle) = \left(\mathcal{F}^{(r)}\right)_{\tau} = \left(\widetilde{\mathcal{F}}^{(r)}\right)_{(W,\sigma)}, & \text{if } (W,\sigma) = (\mathbb{Q}\tau,\sigma) \text{ for } \tau \preceq \sigma \in \Delta \\ 0 & , \text{ else.}\end{cases}\end{aligned}$$

Define a morphism of sheaves $\widetilde{\mathcal{F}}^{(r)} \to i_*\mathcal{F}^{(r)}$ associated to the natural transformation between the two functors given by $\left(\widetilde{\mathcal{F}}^{(r)}\right)_{(W,\sigma)} \to \left(i_*\mathcal{F}^{(r)}\right)_{(W,\sigma)}$ as either the identity or the surjection to zero. Let $\mathcal{G}^{(r)}$ be the kernel of $\widetilde{\mathcal{F}}^{(r)} \to i_*\mathcal{F}^{(r)}$. We have

$$\mathcal{G}^{(r)}_{(W,\sigma)} = \begin{cases} 0 & , \ \text{if } (W,\sigma) = (\mathbb{Q}\tau,\sigma) \text{ for } \tau \preceq \sigma \in \Delta \\ \left(\widetilde{\mathcal{F}}^{(r)}\right)_{(W,\sigma)}, & \text{else,} \end{cases}$$

and a short exact sequence of sheaves on $\widetilde{\Delta}$

$$0 \to \mathcal{G}^{(r)} \to \widetilde{\mathcal{F}}^{(r)} \to i_*\mathcal{F}^{(r)} \to 0. \tag{5.2}$$

Let $\langle \widetilde{\Delta} \setminus i(\Delta) \rangle$ denote the smallest open subset of $\widetilde{\Delta}$ containing the ghost elements. The set $\langle \widetilde{\Delta} \setminus i(\Delta) \rangle$ consists of ghost elements and zero. Let $j : \langle \widetilde{\Delta} \setminus i(\Delta) \rangle \to \widetilde{\Delta}$ be the inclusion map.

**Lemma 5.2.1.** *$\mathcal{G}^{(r)} \cong \mathcal{G}^{(r+1)}$ for all $r \geq 1$.*

*Proof.* We will show that $\mathcal{G}^{(r)}$ and $\mathcal{G}^{(r+1)}$ are naturally isomorphic as functors for any $r \geq 1$. Fix $r$. The functor $\mathcal{G}^{(r)}$ is non-zero only on ghost elements. For any two distinct ghost elements, there is no order relation between them. All ghost elements are of dimension one and we can fix an isomorphism $S^r(W^*) \cong S^{r+1}(W^*)$ for each $r$ and for each ghost element $(W,\sigma)$ in $\widetilde{\Delta}$. Given any $(V,\sigma) \prec (W,\tau)$, the diagram

$$\begin{array}{ccc} \left(\mathcal{G}^{(r)}\right)_{(V,\sigma)} & \longrightarrow & \left(\mathcal{G}^{(r+1)}\right)_{(V,\sigma)} \\ \downarrow & & \downarrow \\ \left(\mathcal{G}^{(r)}\right)_{(W,\tau)} & \longrightarrow & \left(\mathcal{G}^{(r+1)}\right)_{(W,\tau)} \end{array}$$

has at least one row equal to zero and obviously commutes. $\square$

Let $\mathcal{U} := \{\langle(\mathbb{Q}\sigma,\sigma)\rangle | \ \sigma \text{ is a maximal cone in } \Delta\}$.

**Lemma 5.2.2.** *Let $\mathcal{F}$ be a sheaf on $\widetilde{\Delta}$ with $\mathcal{F}_{(0,\sigma)} = 0$. Then its Čech cohomology with respect to the maximal cone cover $\mathcal{U}$ coincides with the sheaf cohomology.*

*Proof.* Let $\sigma_1, ..., \sigma_l$ be the maximal cones in $\Delta$. Let $U_i := \langle(\mathbb{Q}\sigma_i,\sigma_i)\rangle$ and $V = U_{i_1} \cap ... \cap U_{i_n}$ be any intersection of $U_i$'s. By Lemma 3.3.1, it suffices to show $H^p(V, \mathcal{F}|_V) = 0$ for all $p > 0$.

Let $\mathcal{G}^\bullet$ be the Godement resolution of $\mathcal{F}$. If $V = \langle(W,\sigma_i)\rangle$ for some $(W,\sigma_i) \in \widetilde{\Delta}$, then $H^p(V,\mathcal{F}|_V) = H^p(\mathcal{G}^\bullet(V)) = 0$ for all $p > 0$.

If $V$ is not a principal open subset in $\widetilde{\Delta}$, then $V$ consists of ghost elements and the zero cone. The cokernel sheaf of $\mathcal{F}|_V \hookrightarrow \mathcal{G}^0|_V$ is the zero sheaf and the Godement resolution is

$$0 \to \mathcal{F}|_V \to \prod_{(W,\sigma)\in V} i_*\mathcal{F}_{(W,\sigma)} \to 0.$$

It follows $H^p(V,\mathcal{F}|_V) = H^p(\mathcal{G}^\bullet(V)) = 0$ for all $p > 0$ and the sheaf $\mathcal{F}|_V$ is flasque. □

In the map of posets $\Delta \xhookrightarrow{i} \widetilde{\Delta}$, note $i^{-1}(\langle(\mathbb{Q}\sigma_1,\sigma_1)\rangle \cap ... \cap \langle(\mathbb{Q}\sigma_l,\sigma_l)\rangle) = \langle\sigma_1 \cap ... \cap \sigma_l\rangle$ for maximal cones $\sigma_i$'s in $\Delta$. Let $i^{-1}\mathcal{U} := \{\langle\sigma_i\rangle \mid \sigma_i \text{ is a maximal cone in } \Delta\}$ denote the maximal cone cover on $\Delta$.

**Lemma 5.2.3.** *The Čech cohomology of $i_*\mathcal{F}^{(r)}$ on $\widetilde{\Delta}$ with respect to $\mathcal{U}$ coincides with the Čech cohomology of $\mathcal{F}^{(r)}$ on $\Delta$ with respect to $i^{-1}\mathcal{U}$ and $H^p(\widetilde{\Delta}, i_*\mathcal{F}^{(r)}) \cong H^p(\Delta,\mathcal{F}^{(r)})$.*

*Proof.* It follows from the values of the two sheaves at their stalks. □

The aim now is to use the sheaf cohomology groups associated to the short exact sequence (5.2) to study $H^p(\Delta,\mathcal{F}^{(r)})$. We will show that $\widetilde{\mathcal{F}}^{(r)}$ is flasque for $r \gg 0$. In order to do so, we need some facts about sheaves on projective spaces, which is the topic of the next section.

## 5.3 Sheaves on Projective Spaces

Recall that set of $\{H_1, ..., H_l\}$ of codimension one planes in $\mathbb{P}^d$ is *in general position* if for $1 \leq i_1 < ... < i_n \leq l$

$$dim(H_{i_1} \cap ... \cap H_{i_n}) = \begin{cases} d-n & , \quad \text{if } d \geq n \\ -1 & , \quad \text{if } d < n. \end{cases}$$

We use the convention that $dim(\varnothing) = -1$. A set of homogeneous linear forms in $k[X_0, X_1, ..., X_d]$ is *in general position* if the associated hyperplanes they define are in general position.

**Lemma 5.3.1.** *Let $F_1, ..., F_N$ be homogeneous linear forms in general position in $k[X, Y, Z]$ and $L_1, ..., L_N$ be the associated reduced schemes in $\mathbb{P}^2_k := \text{Proj } \ k[X, Y, Z]$ for some field $k$. Let the map $\mathcal{O}_{\mathbb{P}^2} \to \bigoplus_{i=1}^N i_*\mathcal{O}_{L_i}$ be coordinate-wise projections and let $\bigoplus_{i=1}^N i_*\mathcal{O}_{L_i} \to \bigoplus_{i<j} i_*\mathcal{O}_{L_i \cap L_j}$ be defined by*

$$(f_i)_{1\leq i \leq l} \in \bigoplus_{i=1}^N i_*\mathcal{O}_{L_i}(U) \mapsto (f_j - f_i)_{1 \leq i < j \leq l} \in \bigoplus_{i<j} i_*\mathcal{O}_{L_i \cap L_j}(U) \text{ for any } U \text{ open in } \mathbb{P}^2.$$

*Then the sequence of coherent sheaves on $\mathbb{P}^2_k$*

$$\mathcal{A}'^{\bullet} : 0 \to \mathcal{O}_{\mathbb{P}^2}(-N) \xrightarrow{L_1 ... L_N} \mathcal{O}_{\mathbb{P}^2} \to \bigoplus_{i=1}^N i_*\mathcal{O}_{L_i} \to \bigoplus_{i<j} i_*\mathcal{O}_{L_i \cap L_j} \to 0$$

*is exact.*

*Proof.* We may assume that the field $k$ is algebraically closed, and it is enough to prove the statement locally at a point. Without loss of generality, we may assume that the point is the origin in a copy of $\mathbb{A}^2 \subseteq \mathbb{P}^2$. Let $R := k[x, y]_{(x,y)}$ denote the local coordinate ring and let

$l_i$ be the linear polynomial in $R$ corresponding to $L_i$. We will show

$$0 \to R \xrightarrow{l_1 \ldots l_N} R \to \bigoplus_{i=1}^{N} R/l_i \to \bigoplus_{i<j} R/(l_i, l_j) \to 0 \tag{5.3}$$

is exact.

Since there are at most two linear polynomials intersecting at a point in $\mathbb{P}^2$, at most two $l_i$'s are non-units in $R$. If $l_1$ is the only non-unit linear polynomial in $R$, then the sequence (5.3) becomes $0 \to R \xrightarrow{l_1} R \to R/l_1 \to 0$ and its exactness is clear. Say $l_1$ and $l_2$ are the only non-unit linear polynomials in $R$. They are not associates. The exactness of the sequence $0 \to R \xrightarrow{l_1 l_2} R \to R/l_1 \oplus R/l_2 \to R/(l_1, l_2) \to 0$ in fact holds for any UFD. Indeed, the exactness at $R/l_1 \oplus R/l_2$ is shown as follows. Let $(a, b) \in R/l_1 \oplus R/l_2$ such that $\overline{a} = \overline{b} \in R/(l_1, l_2)$. Then $a - b = rl_1 + sl_2$ for some $r, s \in R$. It follows that for arbitrary lifting $\tilde{a}$ and $\tilde{b}$ for $a$ and $b$ respectively, $\tilde{a} - rl_1 = \tilde{b} - sl_2$ in $R$ maps to $a \in R/l_1$ and $b \in R/l_2$. □

Given a coherent sheaf $\mathcal{A}$ on $\mathbb{P}^d$, there is an integer $r_{\mathcal{A}}$ such that $H^p(X, \mathcal{A}(r)) = 0$ for all $p > 0$ and for all $r \geq r_{\mathcal{A}}$ [7, Theorem 5.2].

**Lemma 5.3.2.** *Let $X = \mathbb{P}^d$. If $\mathcal{G}^\bullet : 0 \to \mathcal{G}_1 \to \mathcal{G}_2 \to ... \to \mathcal{G}_l \to 0$ is an exact sequence of coherent sheaves over $\mathbb{P}^d$, then*

$$\Gamma(X, \mathcal{G}^\bullet(r)) : 0 \to \Gamma(X, \mathcal{G}_1(r)) \to \Gamma(X, \mathcal{G}_2(r)) \to ... \to \Gamma(X, \mathcal{G}_l(r)) \to 0$$

*is exact for $r \geq max(r_{\mathcal{G}_1}, r_{\mathcal{G}_2}, ..., r_{\mathcal{G}_{l-2}})$.*

*Proof.* Every short exact sequence of sheaves $0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{C} \to 0$ induces a long exact sequence of cohomology groups. Over $\mathbb{P}^d$, the sequence $0 \to \mathcal{A}(r) \to \mathcal{B}(r) \to \mathcal{C}(r) \to 0$ remains exact. It follows that the sequence $0 \to H^0(X, \mathcal{A}(r)) \to H^0(X, \mathcal{B}(r)) \to H^0(X, \mathcal{C}(r)) \to 0$ is exact for all $r \geq r_{\mathcal{A}}$, and $H^p(X, \mathcal{C}(r)) = 0$ for all $p > 0$ for all $r \geq max(r_{\mathcal{A}}, r_{\mathcal{B}})$.

One decomposes the sequence $\mathcal{G}^\bullet : 0 \to \mathcal{G}_1 \to \mathcal{G}_2 \to \ldots \to \mathcal{G}_l \to 0$ into a series of short exact sequences. Upon applying sufficiently large degree twists, each short exact sequence gives a short exact sequence of global sections. Pasting together, we have $0 \to \Gamma(X, \mathcal{G}_1(r)) \to \Gamma(X, \mathcal{G}_2(r)) \to \ldots \to \Gamma(X, \mathcal{G}_l(r)) \to 0$ exact for $r \geq max(r_{\mathcal{G}_1}, r_{\mathcal{G}_2}, \ldots, r_{\mathcal{G}_{l-2}})$. □

## 5.4 Flasquenss of $\widetilde{\mathcal{F}}^{(r)}$

**Theorem 5.4.1.** *Let $\Delta$ be a three-dimensional complete fan and $N$ be the largest number of facets among maximal cones in $\Delta$. Then the sheaf $\widetilde{\mathcal{F}}^{(r)}$ on $\widetilde{\Delta}$ is flasque for $r \geq 2N-5$.*

*Proof.* Being flasque is equivalent to $\widetilde{\mathcal{F}}^{(r)}(\langle(W,\sigma)\rangle) \twoheadrightarrow \widetilde{\mathcal{F}}^{(r)}(\partial\langle(W,\sigma)\rangle)$ for all $(W,\sigma) \in \widetilde{\Delta}$ where $\partial\langle(W,\sigma)\rangle := \langle(W,\sigma)\rangle \backslash \{(W,\sigma)\}$ [2, Lemma 4.1]. We will prove this by cases depending on the dimension $dim(W,\sigma)$.

If $dim(W,\sigma) = 1$, the statement is trivial since $\widetilde{\mathcal{F}}^{(r)}\langle(\partial(W,\sigma))\rangle = 0$.

If $dim(W,\sigma) = 3$ where $W = \mathbb{Q}\sigma$ and $\sigma$ is a maximal cone, one can "projectivize", embedding $\langle(W,\sigma)\rangle$ in $\mathbb{P}^2$. The element $(W,\sigma)$ corresponds to $\mathbb{P}^2$, and its facets are defined by linear forms $F_i$ where $1 \leq i \leq l$. The linear forms $F_i$'s are in general position by Lemma 5.1.2. Let $L_i$ denote the associated reduced schemes in $\mathbb{P}^2$.

It follows that $\widetilde{\mathcal{F}}^{(r)}(\langle(W,\sigma)\rangle)$ coincides with $\Gamma(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(r))$ and $\widetilde{\mathcal{F}}^{(r)}(\partial\langle(W,\sigma)\rangle)$ is the kernel of $\Gamma\left(\mathbb{P}^2, \bigoplus_{i=1}^{l} i_*\mathcal{O}_{L_i}(r)\right) \to \Gamma\left(\mathbb{P}^2, \bigoplus_{i<j} i_*\mathcal{O}_{L_i \cap L_j}(r)\right)$. Recall the exact sequence of sheaves on $\mathbb{P}^2$ from Lemma 5.3.1

$$\mathcal{A}'^\bullet : 0 \to \mathcal{O}_{\mathbb{P}^2}(-l) \to \mathcal{O}_{\mathbb{P}^2} \to \bigoplus_{i=1}^{l} i_*\mathcal{O}_{L_i} \to \bigoplus_{i<j} i_*\mathcal{O}_{L_i \cap L_j} \to 0.$$

The surjectivity of $\widetilde{\mathcal{F}}^{(r)}(\langle(W,\sigma)\rangle) \to \widetilde{\mathcal{F}}^{(r)}(\partial\langle(W,\sigma)\rangle)$ holds if $\Gamma(\mathbb{P}^2, \mathcal{A}'^\bullet(r))$ is exact.

To figure out what degree twist to apply so the sequence $\Gamma(\mathbb{P}^2, \mathcal{A}'^\bullet(r))$ is exact, we use the following facts about each term [7, Theorem 5.1]:

- $H^p(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(r)) = 0$ for all $p > 0$ if $r \geq -2$.

- $H^p(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(-l + r)) = 0$ for all $p > 0$ if $r - l \geq -2 \Leftrightarrow r \geq l - 2$.

- $H^p(\mathbb{P}^2, i_* \mathcal{O}_{L_i}(r)) = 0$ for all $p > 0$ if $r \geq -1$ since $\mathcal{O}_{L_i} \cong \mathcal{O}_{\mathbb{P}^1}$.

- $H^p(\mathbb{P}^2, i_* \mathcal{O}_{L_i \cap L_j}(r)) = 0$ for all $p > 0$ if $r \geq 0$ since $i_* \mathcal{O}_{L_i \cap L_j} \cong \mathcal{O}_{\{point\}}$.

By Lemma 5.3.2, $\Gamma(\mathbb{P}^2, \mathcal{A}'^{\bullet}(r))$ is exact if $r \geq l - 2$ where $l$ is the number of facets of the maximal cone $\sigma$. Since $N$ is the largest number of facets among maximal cones in $\Delta$, note $2N - 5 \geq l - 2$.

If $dim(W, \sigma) = 2$, then $W = \mathbb{Q}\tau$ for some two-dimensional cone $\tau$ in $\Delta$ and $\tau$ is a facet of exactly two maximal cones $\sigma_1 := \sigma$ and $\sigma_2$. Note $(W, \sigma_1) = (W, \sigma_2)$ in $\widetilde{\Delta}$. Let $(W_0, \sigma_1) := (W, \sigma_1), (W_1, \sigma_1), ..., (W_{l_1}, \sigma_1)$ be the elements in $\widetilde{\Delta}$ corresponding to the $l_1 + 1$ facets of $\sigma_1$, and $(W_0, \sigma_2) := (W, \sigma_2), (W_{l_1+1}, \sigma_2), ..., (W_{l_1+l_2}, \sigma_2)$ be those elements corresponding to $l_2 + 1$ facets.of $\sigma_2$.

In $\widetilde{\Delta}$, the set $\partial\langle(W, \sigma_1)\rangle = \langle(W, \sigma_1)\rangle \setminus \{(W, \sigma_1)\}$ is covered by open subsets of the form $\langle(W_0 \cap W_i, \sigma_1)\rangle$ or $\langle(W_0 \cap W_j, \sigma_2)\rangle$ where $dim(W_0 \cap W_i, \sigma_1) = dim(W_0 \cap W_j, \sigma_2) = 1$. Since the two-dimensional cone $\tau$ has two one-dimensional faces, each of which can be written as an intersection of two facets in a maximal cone, there are $l := l_1 + l_2 - 2$ distinct open subsets covering $\partial\langle(W, \sigma_1)\rangle$.

Note $\langle(W_0 \cap W_i, \sigma_m)\rangle \cap \langle(W_0 \cap W_j, \sigma_n)\rangle = (\{0\}, \sigma_m)$ for all $1 \leq i \leq l_1$, $1 \leq j \leq l_2$, and $m, n = 1$ or $2$. Starting here, we write $W_0 \cap W_i$ for $(W_0 \cap W_i, \sigma_m)$, dropping the reference to maximal cones.

$$\begin{aligned}
\widetilde{\mathcal{F}}^{(r)}(\partial\langle(W,\sigma_1)\rangle &= ker\left(\bigoplus_{j=1}^{l}\widetilde{\mathcal{F}}^{(r)}\langle W_0\cap W_j\rangle \to \bigoplus_{1\le i<j\le l}\widetilde{\mathcal{F}}^{(r)}\left(\langle W_0\cap W_i\rangle\cap\langle W_0\cap W_j\rangle\right)\right)\\
&= ker\left(\bigoplus_{j=1}^{l} S^r\left((W_0\cap W_j)^*\right) \to \bigoplus_{1\le i<j\le l} S^r\left(((W_0\cap W_i)\cap(W_0\cap W_j))^*\right)\right)\\
&= ker\left(\bigoplus_{j=1}^{l} S^r\left((W_0\cap W_j)^*\right) \to \bigoplus_{1\le i<j\le l} S^r(\{0\}^*) = 0\right)\\
&= \bigoplus_{j=1}^{l} S^r\left((W_0\cap W_j)^*\right)
\end{aligned}$$

Elements in $\widetilde{\Delta}$ can be viewed as subvarieties in $\mathbb{P}^2$. Let $P_j \in \mathbb{P}^2$ be the point corresponding to $W_0 \cap W_j \in \widetilde{\Delta}$. The map $\widetilde{\mathcal{F}}^{(r)}(\langle W\rangle) \to \widetilde{\mathcal{F}}^{(r)}(\partial\langle W\rangle)$ becomes $\Gamma(\mathbb{P}^2, i_*\mathcal{O}_L(r)) \to \Gamma(\mathbb{P}^2, \bigoplus_{j=1}^{l} i_*\mathcal{O}_{P_j}(r)) \cong \bigoplus_{j=1}^{l}\mathbb{Q}$. The surjectivity holds when $r \geq l-1$. Since $N$ is the largest number of facets of maximal cones, $l-1 = (l_1+l_2-2)-1 \leq (N-1)+(N-1)-3 = 2N-5$. The sheaf $\widetilde{\mathcal{F}}^{(r)}$ is flasque for $r \geq 2N-5$. □

**Theorem 5.4.2.** *If $\Delta$ is a three-dimensional complete fan in general position and every maximal cone has at most $N$ facets, then $H^1(\Delta, \mathcal{F}^{(r)}) = 0$ for $r \geq 2N-5$.*

*Proof.* Let $i : \Delta \to \widetilde{\Delta}$ be the inclusion map. We have $H^p(\Delta, \mathcal{F}^{(r)}) = H^p(\widetilde{\Delta}, i_*\mathcal{F}^{(r)})$ by Lemma 5.2.3. Recall the short exact sequence (5.2) of sheaves $0 \to \mathcal{G}^{(r)} \to \widetilde{\mathcal{F}}^{(r)} \to i_*\mathcal{F}^{(r)} \to 0$ on $\widetilde{\Delta}$ and the associated long exact sequence of sheaf cohomology. Since $\widetilde{\mathcal{F}}^{(r)}$ is flasque for $r \geq 2N-5$ by Theorem 5.4.1, we have $H^p(\Delta, \widetilde{\mathcal{F}}^{(r)}) = 0$ for all $p \geq 1$ and for all $r \geq 2N-5$. It follows that $H^p(\widetilde{\Delta}, i_*\mathcal{F}^{(r)}) \cong H^{p+1}(\widetilde{\Delta}, \mathcal{G}^{(r)})$ for $p \geq 1$ and for all $f \geq 2N-5$. In particular, we are interested in $H^1(\widetilde{\Delta}, i_*\mathcal{F}^{(r)}) \cong H^2(\widetilde{\Delta}, \mathcal{G}^{(r)})$.

Let $\{\sigma_i\}$ be the maximal cones of $\Delta$. By definition $\mathcal{G}^{(r)}\left(\langle(\mathbb{Q}\sigma_i, \sigma_i)\rangle\right) = 0$. Since $\Delta$ is in general position, all the ghost elements are distinct. The construction of $\widetilde{\Delta}$ requires no identification of ghost elements and $\bigcap\langle(\mathbb{Q}\sigma_{i_j}, \sigma_{i_j})\rangle = \langle(\mathbb{Q}(\bigcap\sigma_{i_j}), \sigma_{i_j})\rangle$. The Čech complex with respect to the cover of maximal cones of $\mathcal{G}^{(r)}$ is the zero complex. Hence

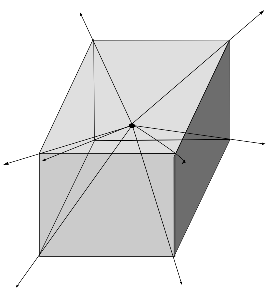
$$H^1(\Delta, \mathcal{F}^{(r)}) \cong H^1(\widetilde{\Delta}, i_*\mathcal{F}^{(r)}) \cong H^1(\widetilde{\Delta}, \widetilde{\mathcal{F}}^{(r)}) \cong H^2(\widetilde{\Delta}, \mathcal{G}^{(r)}) = 0$$

for $r \geq 2N - 5$. $\square$

**Theorem 5.4.3.** *For a three-dimensional complete fan $\Delta$, the Hilbert polynomial of $H^1(\Delta, \mathcal{F}^{(r)})$ as a function of $r$ is a constant.*

*Proof.* Theorem 5.4.1 asserts that $\widetilde{\mathcal{F}}^{(r)}$ is flasque for $r \gg 0$ and we have $H^1(\Delta, \mathcal{F}^{(r)}) \cong H^1(\widetilde{\Delta}, i_*\mathcal{F}^{(r)}) \cong H^2(\widetilde{\Delta}, \mathcal{G}^{(r)})$. The result follows from the fact $dim H^i(\Delta, \mathcal{G}^{(r)}) = dim H^i(\Delta, \mathcal{G}^{(1)})$ for all $r \geq 1$ and for all $i$ from Lemma 5.2.1. $\square$

**Example 5.4.4.** Perfect Cube



Let $\Delta$ be the complete fan over the faces of the perfect cube with vertices $(\pm 1, \pm 1, \pm 1)$. Theorem 5.4.1 asserts that $\widetilde{\mathcal{F}}^{(r)}$ is flasque for $r \geq 3$ and we have $H^1(\Delta, \mathcal{F}^{(r)}) \cong H^1(\widetilde{\Delta}, i_*\mathcal{F}^{(r)}) \cong H^2(\widetilde{\Delta}, \mathcal{G}^{(r)})$.

Let $\{\sigma_i\}_{i=1}^6$ be the set of maximal cones, and we write $\langle \mathbb{Q}\sigma_i \rangle$ for $\langle (\mathbb{Q}\sigma_i, \sigma_i) \rangle$. Consider the Čech complex of $\mathcal{G}^{(r)}$ on $\widetilde{\Delta}$:

$$\bigoplus_{i=1}^{6}\mathcal{G}^{(r)}\left(\langle\mathbb{Q}\sigma_i\rangle\right)\to\bigoplus_{1\le i<j\le 6}\mathcal{G}^{(r)}\left(\langle\mathbb{Q}\sigma_i\rangle\cap\langle\mathbb{Q}\sigma_j\rangle\right)\to\bigoplus_{1\le i<j<k\le 6}\mathcal{G}^{(r)}\left(\langle\mathbb{Q}\sigma_i\rangle\cap\langle\mathbb{Q}\sigma_j\rangle\cap\langle\mathbb{Q}\sigma_k\rangle\right)\to$$
$$\bigoplus_{1\le i<j<k<l\le 6}\mathcal{G}^{(r)}\left(\langle\mathbb{Q}\sigma_i\rangle\cap\langle\mathbb{Q}\sigma_j\rangle\cap\langle\mathbb{Q}\sigma_k\rangle\cap\langle\mathbb{Q}\sigma_l\rangle\right)\to$$
$$\bigoplus_{1\le i<j<k<l<m\le 6}\mathcal{G}^{(r)}\left(\langle\mathbb{Q}\sigma_i\rangle\cap\langle\mathbb{Q}\sigma_j\rangle\cap\langle\mathbb{Q}\sigma_k\rangle\cap\langle\mathbb{Q}\sigma_l\rangle\cap\langle\mathbb{Q}\sigma_m\rangle\right)\to$$
$$\bigoplus\mathcal{G}^{(r)}\left(\langle\mathbb{Q}\sigma_1\rangle\cap\langle\mathbb{Q}\sigma_2\rangle\cap\langle\mathbb{Q}\sigma_3\rangle\cap\langle\mathbb{Q}\sigma_4\rangle\cap\langle\mathbb{Q}\sigma_5\rangle\cap\langle\mathbb{Q}\sigma_6\rangle\right).$$

For a pyramidal cone $\sigma_i$, a pair of parallel edges on the square base gives rise to a ghost element, namely the line as the intersection between two planes each of which is generated by an edge and the origin. The construction process of $\widetilde{\Delta}$ identifies the the common ghost elements between any two adjacent faces of the cube. In $\widetilde{\Delta}$, between any two opposite faces $\sigma_i$ and $\sigma_j$ of the cube, $\langle\mathbb{Q}\sigma_i\rangle$ and $\langle\mathbb{Q}\sigma_i\rangle$ have two ghost elements in common. In other words, $dim\ \mathcal{G}^{(r)}\left(\langle\mathbb{Q}\sigma_i\rangle\cap\langle\mathbb{Q}\sigma_j\rangle\right)=2$. Any three faces $\sigma_i$, $\sigma_j$, and $\sigma_k$ of the cube three pairs of parallel edges have a ghost element in common in $\langle\mathbb{Q}\sigma_i\rangle\cap\langle\mathbb{Q}\sigma_j\rangle\cap\langle\mathbb{Q}\sigma_k\rangle$. Similarly, any four faces $\sigma_i$, $\sigma_j$, $\sigma_k$, and $\sigma_l$ with four pairs of parallel edges have a ghost element in common in $\langle\mathbb{Q}\sigma_i\rangle\cap\langle\mathbb{Q}\sigma_j\rangle\cap\langle\mathbb{Q}\sigma_k\rangle\cap\langle\mathbb{Q}\sigma_l\rangle$. Any five-way intersection $\langle\mathbb{Q}\sigma_i\rangle\cap\langle\mathbb{Q}\sigma_j\rangle\cap\langle\mathbb{Q}\sigma_k\rangle\cap\langle\mathbb{Q}\sigma_l\rangle\cap\langle\mathbb{Q}\sigma_m\rangle$ with $1\le i<j<k<l\le 6$ is empty.

The Čech complex becomes

$$0\to\bigoplus_{1\le i<j\le 6}\mathcal{G}^{(r)}\left(\langle\mathbb{Q}\sigma_i\rangle\cap\langle\mathbb{Q}\sigma_j\rangle\right)\to\bigoplus_{1\le i<j<k\le 6}\mathcal{G}^{(r)}\left(\langle\mathbb{Q}\sigma_i\rangle\cap\langle\mathbb{Q}\sigma_j\rangle\cap\langle\mathbb{Q}\sigma_k\rangle\right)\to$$
$$\bigoplus_{1\le i<j<k<l\le 6}\mathcal{G}^{(r)}\left(\langle\mathbb{Q}\sigma_i\rangle\cap\langle\mathbb{Q}\sigma_j\rangle\cap\langle\mathbb{Q}\sigma_k\rangle\cap\langle\mathbb{Q}\sigma_l\rangle\right)\to 0$$

and the dimensions of the terms are 6, 12, and 3 respectively. It follows $dimH^2(\widetilde{\Delta},\mathcal{G}^{(r)})=dimH^1(\Delta,\mathcal{F}^{(r)})=3$ for all $r\ge 3$.

The perfect cube has $f = 6$, $e = 12$, and $v = 8$ and Equation (5.1) becomes

$$3(r+2)(r+1) - 12(r+1) + 8 = dim_{\mathbb{Q}} H^0(\Delta, \mathcal{F}^{(r)}) - dim_{\mathbb{Q}} H^1(\Delta, \mathcal{F}^{(r)})$$

for all $r \geq 1$. For all $r \geq 3$, having $dim_{\mathbb{Q}} H^1(\Delta, \mathcal{F}^{(r)}) = 3$ completely determines $dim_{\mathbb{Q}} H^0(\Delta, \mathcal{F}^{(r)})$. In low degrees, Payne computed $dim_{\mathbb{Q}} H^0(\Delta, \mathcal{F}^{(r)})$ for $0 \leq r \leq 3$ in [8, Example 4.2] and then $dim_{\mathbb{Q}} H^1(\Delta, \mathcal{F}^{(r)})$ is determined.

| $r$ | $\chi^r(\Delta)$ | $dim_{\mathbb{Q}} H^0(\Delta, \mathcal{F}^{(r)})$ | $dim_{\mathbb{Q}} H^1(\Delta, \mathcal{F}^{(r)})$ |
|---|---|---|---|
| 0 | 1 | 1 | 0 |
| 1 | 2 | 4 | 2 |
| 2 | 8 | 11 | 3 |
| 3 | 20 | 23 | 3 |
| 4 | 38 | 41 | 3 |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |
| $r$ | $3r^2 - 3r + 2$ | $3r^2 - 3r + 5$ | 3 |

When $\Delta$ is a three-dimensional complete fan given by a convex polytope, the spectral sequence

$$\left[H^p\left(\Delta, K_q^T\right)_{\mathbb{Q}}\right]_I^{\wedge} \Longrightarrow \left(K_{q-p}^T(X(\Delta))_{\mathbb{Q}}\right)_I^{\wedge}$$

reduces to a short exact sequence

$$0 \to H^1\left(\Delta, \mathcal{F}\right)_{\mathfrak{m}}^{\wedge} \otimes_{\mathbb{Q}} K_{n+1}(k)_{\mathbb{Q}} \to \left(K_n^T(X(\Delta))_{\mathbb{Q}}\right)_I^{\wedge} \to H^0(\Delta, \mathcal{F})_{\mathfrak{m}}^{\wedge} \otimes_{\mathbb{Q}} K_n(k)_{\mathbb{Q}} \to 0.$$

The computation of $H^1(\Delta, \mathcal{F}^{(r)})$ for $r \gg 0$ shows that $H^1\left(\Delta, \mathcal{F}\right)_{\mathfrak{m}}^{\wedge} \otimes_{\mathbb{Q}} K_{n+1}(k)_{\mathbb{Q}}$ contributes to $\left(K_n^T(X(\Delta))_{\mathbb{Q}}\right)_I^{\wedge}$ mainly through lower degree terms.

**Corollary 5.4.5.** *If $\Delta$ is a fan given by a three-dimensional convex polytope in general position, $\left(K_{-1}^T(X(\Delta))_{\mathbb{Q}}\right)_I^{\wedge}$ is a finite dimensional vector space over $\mathbb{Q}$.*

*Proof.* By Theorem 5.4.2, $H^1(\Delta, \mathcal{F}^{(r)}) = 0$ for $r \gg 0$ and $H^1\left(\Delta, \mathcal{F}\right)^\wedge_{\mathfrak{m}}$ is a finite dimensional vector space. When $n = -1$, we have $K_{-1}(k) = 0$ and $\left(K^T_{-1}(X(\Delta))_{\mathbb{Q}}\right)^\wedge_I \cong H^1\left(\Delta, \mathcal{F}\right)^\wedge_{\mathfrak{m}}$. $\square$

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