Children's Voices: Students' Attitudes about Routine and Nonroutine Mathematics

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CHILDREN’S VOICES: STUDENTS’ ATTITUDES ABOUT ROUTINE AND NONROUTINE MATHEMATICS

by

Deborah E. Seacrest

A DISSERTATION

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This dissertation is a case study focused on the question, “What are students’ attitudes toward school mathematics and toward nonroutine math problems and mathematical games?” It addresses the definitions of some of those terms and then moves on to a literature review that suggests that some change in the curriculum may be needed. In an attempt to begin determining whether nonroutine problems and mathematical games could help, students in two different types of classes were introduced to such problems and games. Their attitudes were assessed using a variety of methods, including observation, interview, and journal writing.

The games and problems used in the classes are explained, along with students’ reactions to them. Overall, students reacted very favorably to the activities. Future work may determine how much students learn from the activities and whether the students are able to apply that knowledge to the mathematics more commonly taught in schools. In the interviews, three of the students discussed the difference between the summer class and “actual math,” so they may see no immediate connection between nonroutine problems, including games, and the mathematics they learned in a more traditional manner.

A connection is made to the Common Core State Standards Initiative, and the Standards for Mathematical Practice are analyzed. The activities done in the summer classes addressed standards at a variety of grade levels, but the focus is on the eight
standards for practice. Finally, some conclusions are drawn about what students appeared to learn in the two classes, and some comments are made concerning future work.
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1 Research Questions and Subquestions

This section introduces the premise of this dissertation, that it is not common practice for all students to be exposed to nonroutine math problems. It includes examples of such problems and discusses the nature of the underlying mathematics. It then discusses children’s attitudes toward mathematics in general and nonroutine problems in particular.

2 Literature Review

This section discusses various aspects of motivation, attitudes, the identification of mathematically gifted students, upper elementary school mathematics, and mathematical self-efficacy.

3 Methods of Exploration

This section describes the plans for summer classes using nonroutine problems. The classes are designed to expose students to nonroutine problems so they can be questioned about their attitudes toward both traditional school mathematics and nonroutine problems. This chapter includes interview questions, journal prompts, and potential problems and how they were minimized.

4 First Class

This section includes data from the first class, which took place for five days in June, 2010.
5 Second Class and Comparisons

This section includes information from the second class, which took place for four days in July, 2010, and makes comparisons between the two classes.

6 Themes

This section lists some themes noted during observation of the two classes.

7 Common Core Standards

This section examines some of the Common Core Standards for Mathematical Practice and illustrates how activities like those used in the summer classes could satisfy those standards.

8 Conclusions

This section summarizes the nature of the findings of the study.
DEDICATION

To everyone who helped and inspired me along the way.
ACKNOWLEDGMENTS

I would like to begin by thanking my family, who introduced me to a wide variety of mathematics at a very early age. I have fond memories of my parents asking interesting and challenging math questions, such as how many steps I would need to take to reach the wall if I initially stood a meter away and moved halfway each time. (I believe I attempted to test this one experimentally, which may not have been the best method.) I also recall long walks with my grandfather while he taught me about various aspects of mathematics. Additionally, I would like to thank my sisters, who allowed me to share some mathematics with them. Not only did my family help prepare me when I was younger, but they also gave me some excellent advice on this dissertation. In particular, I feel very fortunate to be able to say that my mother read my entire dissertation!

I would also like to acknowledge my teachers and professors who went out of their way to encourage my interests in mathematics and mathematics education. I was given many opportunities in mathematics, and I was also allowed chances to teach others from an early age. I am especially grateful to my seventh-grade math teacher, Mrs. Bebe Dennard, who taught me what research mathematics was like by working with me to rediscover important mathematical concepts and procedures, and to Professor Francis Su at Harvey Mudd College, who first gave me the opportunity to do novel research.

My advisor and committee members gave me encouragement and many helpful suggestions along the way, and they answered questions and read countless drafts. I also greatly appreciate the help of my friends, who supported me, gave me ideas, and brainstormed with me. Additionally, I am indebted to the two programs that allowed me to work with their children, the parents who provided consent for their children’s
work to be used and for interviews, and most of all, to the children themselves.

Finally, I would like to thank my husband, who was a great source of support and helped in many ways. Some of the games used were either found by him or adapted from his ideas. He also helped by donating his time to interview children and to help supervise or play games with one class when there were an odd number of students. He always showed great interest in my work, and he let me talk through any questions or problems I had.
Chapter 1

Research Question and Subquestions

Many students dislike mathematics, with some even claiming that it is their least favorite subject (Beck, 1977; Boaler, 2008; Kenschaft, 1997; Stodolsky, 1985). Other students, however, enjoy mathematics very much (Stodolsky, 1985). Part of the difference may be that the latter students encounter very different types of mathematics problems. Students showing initial promise in mathematics, via good grades in math, early interest, or teacher and parent recommendations, are often enrolled in programs where they encounter problems calling for creative, nonroutine problem solving (Hlavaty & Ruderman, 1968). Some of those who do not initially have this interest or achievement, however, are frequently exposed only to more routine problems, often called “plug-and-chug” problems (Duch, 2001) or “skill and drill” problems. This is especially true for students labeled as low achievers, says Boaler (2008), adding that “This [being given plug-and-chug problems] is the last thing these students need” (p. 152). Students then see mathematics as an art of imitation, rather than as something requiring actual thought. This need not remain the case, accord-
ing to Dreyfus and Eisenberg (1996), who believe that students can learn to think mathematically.

The main question this dissertation is designed to address is, “What are students’ attitudes toward school mathematics and toward nonroutine math problems and mathematical games?” Knowing these attitudes may enable teachers to frame mathematics so that students will see it in a more positive light. To answer this question, I attempt to define nonroutine problems, mathematics, and attitudes. I then ask, “Do students see nonroutine math problems, including mathematical games and puzzles, as mathematics or as something else?”

1.1 What are nonroutine problems?

Mayer and Hegarty (1996) wrote, “A nonroutine problem exists when a problem solver has a problem but does not immediately see how to solve it” (p. 32). Some problems are nonroutine simply based on the audience. An example of this follows:

One morning, the farmer looks out in the barnyard and sees a number of pigs and chickens. The farmer says to her daughter, “I count 24 heads and 80 feet. How many pigs and how many chickens are out there?” (Bassarear, 2007, p. 7)

This is a standard, routine problem for eighth-graders in algebra, who may immediately set up a system of equations. However, if the problem is given to fourth-graders, they most likely will not know how to solve systems of equations. Indeed, this may be unlike problems they have solved before, so they will not have a ready solution. Mayer and Hegarty would therefore conclude that this is a non-routine problem for the fourth-graders. With time and effort, however, a very creative and non-algebraic
solution might result. One such solution, which Bassarear observed in his fourth-grade class, was to first draw 24 circles representing the heads of the animals. Then, the student reasoned that every animal had at least two legs, and so she gave each head two legs. She next distributed the remaining 32 legs to the first 16 heads, resulting in 16 four-legged animals (pigs) and 8 two-legged animals (chickens). If students have not seen a similar problem and must find their own solution method, the problem is nonroutine for them. However, if similar problems were given repeatedly, they would cease to become nonroutine. Therefore, one must consider the overall situation when determining whether a problem is routine, rather than simply the problem statement.

According to Novick and Bassok (2005), there are two types of strategies for problem solving: algorithms and heuristics. Algorithms always lead to a solution, such as the standard method of long division, the quadratic formula, or an “exhaustive search,” where one considers every possibility in the set of potential solutions. For instance, to unscramble the letters “eth” into a word, one can simply consider \{eht, eth, het, hte, teh, the\} and note that only the last combination is an English word.

Heuristics, on the other hand, are helpful when the list of possibilities becomes too large to solve by brute force. For instance, if one did not have time to search all six possibilities for “eth,” one could note that “th” is a common consonant blend. Thus, “eth” and “the” are the two most likely possibilities. This does not always work, as with “ith” (hit), but it works enough of the time to be useful. Heuristics are necessary to solve many puzzles and to play most games effectively. For example, the number of legal positions in the game of Go is a number so large that it has not yet been computed. It is estimated to be approximately 171 digits long (Tromp & Farnebäck, 2009). The number of legal moves at any state is not nearly as large, but it is still large enough to preclude an exhaustive search. Therefore, players must
In schools, students are taught algorithms, which they then use to solve routine problems. When faced with nonroutine problems, students need to either modify known algorithms on their own or use heuristics. Heuristics can be developed for mathematics problem solving, such as considering a smaller case or working through examples, but there is no universal algorithm for solving math problems in general.

According to Hlavaty and Ruderman (1968), the right questions can challenge students in ways they have not experienced before and can teach them new mathematical ideas. Those kinds of nonroutine problems can assist in developing problem solving skills and mathematical maturity (London, 1993). The very act of expecting such students to be able to solve those problems is a boost to many students’ self-confidence (Sanguras, 2005).

### 1.2 What is mathematics?

Whether students accept such nonroutine problems as mathematical, or whether they instead feel that nonroutine problems are outside the realm of mathematics, is an important question. Students may not initially believe that the nonroutine problems are mathematical. If they enjoy the nonroutine problems and then later come to agree that those problems are mathematical, that may encourage them to form a more positive view of mathematics.

A simple answer one might expect when someone is asked to define mathematics is that math is anything involving numbers. Gallistel and Gelman (2005), for instance, wrote that, “Mathematics is a system for representing and reasoning about quantities, with arithmetic as its foundation” (p. 559). However, not everything involving numbers is mathematical, and conversely, not everything mathematical involves numbers.
For example, many sports teams assign numbers to their players, but these numbers are simply for reference. Player 5 is not assumed to be as good as Players 2 and 3 combined, for instance, nor is such numbering even necessarily indicative of an ordering on the players at all.

One might consider scheduling as an example of a mathematical problem that need not involve numbers. For instance, suppose a school needed to schedule final exams in such a way that no person has overlapping exams. For the sake of illustration, assume there are five classes and three students. The first student takes classes $A$, $B$, and $C$, the second takes classes $B$, $C$, and $D$, and the last takes class $E$. This can be represented by drawing a vertex for each class and putting edges between classes if there is at least one student taking both classes. Each individual’s picture is shown below in Figure 1.1, and they are then overlaid to form the composite image in Figure 1.2.

![Figure 1.1: Edges are drawn between the students’ classes](image1)

![Figure 1.2: The students’ graphs are overlaid](image2)
Assigning time slots for exams is the same problem as *properly coloring* the graph. In such a coloring, vertices connected by an edge must be different colors. For instance, since vertex $A$ and vertex $B$ have an edge between them, they cannot be the same color. One such proper coloring is shown in Figure 1.3. The three colors required correspond to three time slots needed for the exams. Classes $A$ and $D$ are in the light gray time slot, $B$ is in the dark gray time slot, and $C$ and $E$ are in the black slot. Note that this entire exchange was free of numbers except in the most cursory of ways.

![Figure 1.3: A proper coloring](image)

Because of the previous example, it seems clear that the simplistic answer given above, that mathematics is anything involving numbers, is unsatisfactory. Additionally, there are likely some subjects whose mathematical nature could be debated. For instance, in some universities, computer science is part of the mathematics department, while in others, it is a separate department. There are undoubtedly still some who would claim that computer science is mathematics, while others feel that it is a distinct, albeit related, field. Edwards (2007), for example, notes that,

There are three prevalent views as to the nature of the discipline:

- Computer Science is a branch of engineering
- Computer Science is Mathematics
- Computer Science is a science. (p. 4)
It is possible, and perhaps even likely, that no definition for mathematics exists that would satisfy everyone, or even every mathematician. This does not mean there are no definitions for mathematics, however; it instead means that there are many. For example, Friedrich Ludwig Gottlob Frege (1980), a famous German logician and philosopher, wrote that “Arithmetic thus becomes simply a development of logic, and every proposition of arithmetic a law of logic, albeit a derivative one” (p. 99). More recently, Courant, Robbins, and Stewart (1996) wrote,

Mathematics as an expression of the human mind reflects the active will, the contemplative reason, and the desire for aesthetic perfection. Its basic elements are logic and intuition, analysis and construction, generality and individuality (Courant et al., 1996, “What is Mathematics?” para. 1).

Some physicists, on the other hand, believe that mathematics is “the language of physics” (Schwarz, 2010) or “the foundation of physics” (Feimer, 2010). Cohen (2004), a mathematical biologist, described mathematics as “biology’s next microscope,” asserting that math “can reveal otherwise invisible worlds in all kinds of data” (p. 2017). In Everybody Counts: A Report to the Nation on the Future of Mathematics Education, the authors wrote, “Mathematics is a science of pattern and order” (Mathematical Sciences Education Board & The Board on Mathematical Sciences, National Research Council, 1989, p. 31).

According to these sources, mathematics can be viewed in many different ways. Children form one group whose definitions of mathematics may have been thus far overlooked. Their attitudes towards mathematics have been studied (Beck, 1977; Haladyna et al., 1983; Haladyna & Thomas, 1979; Henry et al., 2007), but their definitions of mathematics were not found in the literature. Those definitions could be quite enlightening to educators, who may be able to use them when determining
how to frame mathematics, and in particular, nonroutine problems.

### 1.3 What are attitudes?

After students engage in nonroutine problems, their mathematical attitudes may reflect some interest in mathematics, whether or not they acknowledge it as such. To begin to determine this, one must better understand what attitudes are.

Ajzen (2005) defined an attitude as “a disposition to respond favorably or unfavorably to an object, person, institution, or event,” (p. 3) and Greenwald and Banaji (1995) had a very similar definition. Oppenheim’s (1992) definition is more general; he wrote that “An attitude is a state of readiness, a tendency to respond in a certain manner when confronted with certain stimuli” (p. 174). In either case, attitudes are rather general, and many of the eight descriptions and definitions given in Greenwald and Banaji’s paper focus on dispositions being positive or negative.

Haladyna et al. (1983) wrote that positive attitudes toward mathematics are important for the following reasons:

1. A positive attitude is an important school outcome in and of itself.
2. Attitude is often positively, although slightly, related to achievement.
3. A positive attitude toward mathematics may increase one’s tendency to elect mathematics courses in high school and college and possibly one’s tendency to elect careers in mathematics or mathematics-related fields (p. 20).

Note that the second reason may be true only for certain populations. Henry et al. (2007) found that first graders’ attitudes toward school, as measured by a survey
Attitudes are based on beliefs and may be either conscious or unconscious (Gerrig & Zimbardo, 2002). Therefore, even with completely open and willing participants, one cannot simply ask for their attitudes about certain subjects. Instead, one must ask questions about beliefs, which, when combined, give some information about those attitudes.

Attitudes affect both attention and behavior (DeLamater & Myers, 2011). For example, if people feel that mathematics is boring, they are more likely to notice instances confirming this (Nickerson, 1998). They are also more likely to attempt to avoid such activities in the future. This confirmation bias, once initiated, may be difficult to alter, since people are less likely to pay attention to positive experiences of mathematics. Positive attitudes toward mathematics, in contrast, would likely cause people to remember what they felt was interesting about mathematics. Furthermore, it may result in them seeking out further mathematical activities.

According to DeLamater and Myers (2011), attitudes consist of three parts: “(1) beliefs or cognitions, (2) an evaluation, and (3) a behavioral disposition” (p. 145). They note that frequently, the beliefs involved cannot be proven either true or false. The authors go on to say that the evaluation consists of “a direction (positive or negative) and an intensity (ranging from very weak to very strong)” (p. 145). This could be simplified to a rating, where attitudes are ranked from 1 to 9, with 1 being a strong negative feeling, 5 being neutral, and 9 being a strong positive feeling. The behavioral disposition includes seeking out or avoiding activities associated with that attitude.

Attitudes can be created by reinforcement, by stimulus-response pairing, or by watching others (DeLamater & Myers, 2011). Reinforcement occurs when people
have direct interaction with the associated activity, person, or institution. Stimulus-response pairing means that various stimuli are associated with each other, and over time, similar reactions occur. Watching others is indirect, with attitudes arising from learning about other people’s experiences.

Attitudes affect behavior, but several factors mitigate this relationship. Attitudes that are accessible and specific are more effective in affecting behavior. One method of testing accessibility is to ask people about certain topics and measure their response time. In one study, respondents were asked which candidate they favored several months before the election. Shorter response times in answering questions about the candidates were correlated with a greater propensity to vote for the initially favored candidate during the election (Gerrig & Zimbardo, 2002).

Attitudes are often determined via surveys or attitude scales. Scales have been constructed both for mathematics attitudes and for attitudes toward school subjects in general (Aiken, 1996). The $ME$ is a five-question attitude scale developed by Haladyna and Thomas in the 1970s. It is nonverbal, meaning students choose between a happy face, a neutral face, and a sad face as their answers for each question. This has been used for mathematics, as well as for other subjects and for school in general (Haladyna & Thomas, 1979). The Fennema-Sherman Mathematics Attitude Scales are probably the most well-known scales. They were created in 1976 and it takes approximately forty-five minutes to administer all 108 items, which compose nine separate scales (Tapia & Marsh, 2004; Fennema & Sherman, 1976). By 2004, Tapia and Marsh had created the Attitudes Toward Mathematics Inventory, originally a forty-nine item scale that was condensed into a forty-item scale.

To create surveys such as those above, researchers often begin with in-depth interviews. They then create a large pool of attitude statements, which are statements people can agree or disagree with. Next, researchers analyze these statements, of
which there are perhaps sixty to one hundred. Eventually, they usually reduce the number to something more manageable and can give the surveys (Oppenheim, 1992). This dissertation concerns itself with the interviews; determining appropriate attitude statements for a survey is left for future work.

It may be important to determine not only the content of children’s attitudes, but also the intensity of those attitudes, since a child with strong attitudes might be more likely to display behaviors stemming from those attitudes, which could affect other children (Holland et al., 2002). One might eventually also look into the permanence of attitudes and potential causes of attitudes, but that is outside the scope of this dissertation.

1.4 Significance of the study

This study was designed to lead to new knowledge in the field. In particular, children’s attitudes toward classroom mathematics and nonroutine mathematics would be analyzed, and children’s definitions of mathematics would be determined. Thus far, several researchers have considered students’ attitudes towards mathematics. Some researchers, such as Fellows and Koblitz (2000), have given nonroutine problems to students, but no literature could be found about the combination of the two. There were some studies showing that “reflective inquiry” could affect students’ mathematical dispositions positively (Hiebert et al., 1996, p. 17), which is part of how some of the nonroutine problems will be solved, but this does not consider the entire object of study.

One of the goals of the NCTM Standards in 1989 was that students should become “mathematical problem solvers” (National Council of Teachers of Mathematics, 1989). On page 8, the Standards specified that
Students need to work on problems that may take hours, days, and even weeks to solve. Although some may be relatively simple exercises to be accomplished independently, others should involve small groups or an entire class working cooperatively. Some problems also should be open-ended with no right answer, and others need to be formulated.

The 2000 version of the *Standards* added that students need to be given challenging, intriguing tasks (National Council of Teachers of Mathematics, 2000). That advice still applies today, and yet Kenschaft (1997) and Boaler (2008) both note that very few classrooms include such problems.

It is important to know children’s attitudes about nonroutine mathematics before attempting to incorporate, or further incorporate, such mathematics into the curriculum. Because established attitude scales do not address nonroutine problems, they were not appropriate for use in this study. Therefore, observation, interviews, and written responses were used to determine attitudes.

Another important question is whether such nonroutine problems and mathematical games could be used to meet some of the standards set forth by the 2010 Common Core State Standards Initiative. Nonroutine problems and games may meet standards from many different grade levels, making it difficult to analyze.

While results about fourth and fifth grade students may not be immediately generalizable to other age groups, it seems that this research might stimulate others to consider students’ attitudes in other age groups. Note that the fourth and fifth grade age group is between younger elementary school students, who frequently enjoy mathematics, and older students, who often do not (Haladyna & Thomas, 1979).
Chapter 2

Literature Review

External rewards, such as pizza parties or grades, may work to motivate some to work hard in school, but interesting activities may be even better motivators (Deci et al., 2001; Schmakel, 2008). Additionally, external rewards have been shown by some studies to actually decrease intrinsic motivation (Deci et al., 2001). If students consider nonroutine problems to be interesting, then it may be detrimental for them to only be given to students identified as mathematically gifted (Hlavaty & Ruderman, 1968).

There are many reasons why nonroutine problems should be given to average or below-average students. Currently, it is often the case that only those who are identified as especially talented in math have access to these programs (Hlavaty & Ruderman, 1968). However, current methods of identification lead to underrepresentation by non-Asian minorities (Lee et al., 2009; McBee, 2006; Yoon & Gentry, 2009). To reach these minorities, it may be important to target not only those identified as gifted, but also those who are deemed by others to possess only average ability. Not only might there be misidentifications, but Wertheimer wrote that many people believe that mathematical ability is not fixed (Wertheimer, 1999).
Sheffield enumerates what “top students” need. This includes thinking deeply about mathematics, finding beauty in mathematics, and becoming good students for college and for life outside school (Sheffield, 1999). While it makes sense that students who have demonstrated their mathematical prowess need these challenges, it is not clear why these should be restricted to those who have already shown their talent. It seems that all students should think deeply about mathematics, as recommended by the National Council of Teachers of Mathematics (2000), Van de Walle et al. (2010), and many others.

Upper elementary school seems an especially apt time for such a program. For example, Kenschaft discusses the “Fifth Grade Crisis” experienced by many children. She believes this can result in two things: “feeling overwhelmed and infuriating boredom” (Kenschaft, 1997, p. 160), adding that many fifth graders are forced to do simple arithmetic that should have been learned previously. She notes that fifth grade “should focus on solving interesting problems” (Kenschaft, 1997, p. 161).

Feldhaus, reporting on preservice elementary teachers, noted that their dispositions towards mathematics were created in mid to late elementary school. He added that these attitudes were difficult to change after having been formed (Feldhaus, 2010). Such attitudes towards mathematics were listed by the National Research Council as one of five key components of mathematical proficiency (Mathematics Learning Study Committee, 2001, p. 5). By high school, many students have decided that they “hate math” or “can’t do math,” but before then, students’ minds are more open. Similarly, Aiken wrote that “The late elementary and early junior-high grades are viewed as being particularly important to the development of attitude toward mathematics” (Aiken, 1976, p. 296).

Familiarity with nonroutine problems may help enhance students’ mathematical self-efficacy and reduce math and test anxiety. Israeli researchers Birenbaum and
Gutvirtz (1993), for instance, found that test anxiety was correlated with serious mathematical errors, while minor mathematical errors were unrelated. They found two potential explanations for how test anxiety affects performance. The first is that students who are more anxious have trouble retrieving information, and the other is that they have difficulty encoding it. If either of these are true, reducing anxiety may allow students to do better in mathematics. This is in addition to the benefit students will gain from the actual mathematical learning.

The act of participating in a summer math program has the potential to increase students’ respect for and appreciation of mathematics, as well as their confidence in their own abilities to do mathematics. All of these factors are often related to increased achievement (Leung et al., 2006). Indeed, Mighton (2007) noted, “Children’s level of confidence and sense of self will largely determine what they learn” (p. 204).

According to Klein, the National Science Foundation wrote that

> Mathematics and science are learned by doing rather than by passive methods of learning such as watching a teacher work at the chalkboard. Inquiry-based learning and hands-on learning more effectively engage students than lectures (Klein, 2003, p. 194).

Nonroutine problems can be extremely active problems that sometimes call for students to work together to solve them. These problems are all about finding creative methods of problem solving. A class of students may find two or three ways of solving each problem (Burkhardt & Pollak, 2006).

Smith et al. (2004) explain “The mathematics instructional program in the middle grades needs to be more ambitious, setting higher expectations for middle school students and for their teachers” (p. xi). What better way to set higher expectations is there than to use nonroutine problems? Many of these problems were written for
advanced students, but there is no reason they cannot be solved by students who are deemed to have average mathematical ability. It might take them longer, but knowing that they can solve such difficult problems may give them the mathematical self-confidence they need.

Miller and Hom (1997) have shown that eighth-graders are very susceptible to praise and criticism, but not in the fashion that many would expect. When students are praised for doing something easy, others see those students as slow and interpret the praise as surprise that those students were successful. When students are criticized for not doing something well, the teacher evidently expected more, so those students must be good students.

For this reason, simple praise might not be the most effective motivator for many students. Accomplishing something difficult, and feeling competent as a result, might serve as a better motivator (Vallerand et al., 1992). By explaining that the problems they are working on are difficult but doable, one can encourage them to solve problems. They will then get the satisfaction of knowing that they have done something difficult, and done it well.

Roeser and Eccles (1998) found that when students feel their teachers think they are good students, their self-esteem increases. By telling students that they are expected to be able to do these difficult problems, teachers are informing them that they both have the potential to be and are expected to be good students.

Kaplan, Peck, and Kaplan (1997) report that bad grades have detrimental effects on the students who receive them. They wrote that “if they [students] receive low school grades (negative academic experiences), they might engage in a variety of behaviors and adopt attitudes to protect themselves against possible negative feelings”. In other words, they may put less effort into school or try to care less about their grades. In a program where students are given nonroutine problems to work on, they
need not be graded on how many they get correct. They could instead be assessed in a less intrusive or overt manner. The students could be solving math problems for the joy of discovery, rather than for a grade.

Willoughby (1968) wrote that “it is more important to teach people how to learn and use new mathematics than to teach them the mathematics that is presently being used in their chosen profession” (p. 11). He added that

Perhaps the most important aspect of a good mathematics education program is to teach children how to learn and how to be creative. Those who can learn new mathematics when needed and can be creative about the way in which they apply it will always find a use for their talents. Unfortunately, this is also one of the most difficult things to teach (p. 15).

Similarly, it may be more important to inspire children to enjoy mathematics than to teach them by rote memorization. Once they enjoy mathematics, they might be more receptive to learning.

Nonroutine problems may also help prepare students to enter math competitions, should they desire, since those competitions often strive to provide students with nonroutine problems (Doorman et al., 2007). This includes MATHCOUNTS, the American Mathematics Contests (available for grades 8 and under, grades 10 and under, and grades 12 and under), and many smaller regional competitions. As Zirkes and Penna (1984) wrote, “Academic competitions... may stimulate students to discover talents in areas that are new to them; and they can provide a good source of positive news for the school” (p. 94).

James and Smith (1960) wrote that “Mathematics contests have been unusually successful in interesting more people in the subject” (p. 153). If, as is often claimed, the major problem in mathematics learning is the lack of motivation, using nonrou-
tine problems and showing students that they can solve them may lead to students becoming more interested in the subject.

Unfortunately, while much information was available about giving nonroutine problems to students of high mathematical ability, I was unable to find such information on students of average or below-average ability. Some researchers at Tel Aviv University, however, studied a related topic (Tsamir et al., 2009). They looked at one teacher’s evaluation of secondary school students’ proofs in the area of basic number theory. That research helped inform this study, as reasoning and proof plays a major role in nonroutine problems.

The article provides several proofs, both by students and by the teacher. Some of these are incorrect or incomplete, while others are longer and more complicated than they need to be. This is not surprising, as it may have been their first introduction to the concept of proof. Many of the students rose to the challenge and produced at least passable proofs. This was not an advanced class, so this lends credence to the idea that average or below-average students can solve such problems.

Watson (2002) examined the mathematical thinking of below-average students and found that they often showed significantly more promise and mathematical thinking than teachers expected. She says, “In the UK, teaching mathematics to low attaining students in secondary school often involves simplification of the mathematics until it becomes a sequence of small smooth steps which can be easily traversed” (p. 462). She further explains that students should be challenged and allowed to use their mathematical reasoning skills. Watson’s statement could also be applied to the United States, especially with its recent push on achieving basic competence for everyone, as evidenced by the 2002 No Child Left Behind Act (107th Congress, 2002).

In conclusion, giving nonroutine problems to average and below-average students may have the potential to eliminate or at least reduce the gap between those students
and higher-achievers. The problems lead to teamwork and creative thinking, and solving them may yield higher self-confidence in mathematics. Nonroutine problems could result in improved attitudes toward mathematics.
Chapter 3

Methods of Exploration

As stated above, it seems possible that all students, including those deemed average or below-average, should have the opportunity to work with nonroutine math problems. To this end, I worked with varied groups of students in a summer program.

A summer program was ideal for this type of work because the focus could be on nonroutine problems for extended periods of time, rather than during the school year, when schoolwork might take precedence. The first group participated in a week-long summer program where classes meet for three hours a day, five days a week. With the second group, I was able to do a similar program that met for an hour a day for four days.

I served as the teacher for both programs. For the first program, I was assigned two assistants, both of whom were from a local high school. They helped with classroom management and they also presented some of the material, under my guidance. For the second program, the assistance varied. On three of the days, I had several adults sit in on the classes and help with discipline. One adult, a former student of mine, presented one activity to the children, but I presented the rest of the activities.

I collected information via interviews, journals, and observations. Due to its popu-
Table 3.1: Information About Both Groups

<table>
<thead>
<tr>
<th></th>
<th>First Group</th>
<th>Second Group</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Students</strong></td>
<td>11 to 12 students</td>
<td>3 to 12 students</td>
</tr>
<tr>
<td><strong>Entering</strong></td>
<td>5th and 6th</td>
<td>4th and 5th</td>
</tr>
<tr>
<td><strong>Chose the class</strong></td>
<td>Did not choose</td>
<td></td>
</tr>
<tr>
<td><strong>Varying abilities</strong></td>
<td>Yes (some scholarships)</td>
<td>No</td>
</tr>
<tr>
<td><strong>Fees</strong></td>
<td>Classroom</td>
<td>Cafeteria (no board)</td>
</tr>
<tr>
<td><strong>Class Meetings</strong></td>
<td>3 hours/day for 5 days</td>
<td>1 hour/day for 4 days</td>
</tr>
<tr>
<td><strong>Setting</strong></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 3.2: Data Collected

<table>
<thead>
<tr>
<th>Data Type</th>
<th>#: Group 1</th>
<th>#: Group 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Interview Data</td>
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<td>0</td>
</tr>
<tr>
<td>Children’s Journals</td>
<td>6</td>
<td>1</td>
</tr>
<tr>
<td>Lesson Plans</td>
<td>5</td>
<td>4</td>
</tr>
<tr>
<td>My Journal &amp; Comments on Lesson Plans</td>
<td>5</td>
<td>4</td>
</tr>
</tbody>
</table>

of high-risk individuals, including children who have been abused or neglected, the second group does not allow interviews. Therefore, the interviews came solely from the first group of students. The interviews took place on the fourth and fifth days of the class. Both groups wrote in journals every day, which were either photocopied or collected and analyzed with permission. I recorded my observations at the end of each day.

Once I had the data, my analysis procedure was to code the transcripts, and I then looked for common themes. I used the ATLAS.ti software to facilitate this analysis (ATLAS.ti Scientific Software DevelopmentGmbH, 2010). With it, I was able to label phrases with themes I noticed. For instance, when Tia¹ said, “I like that you get to learn new things,” I coded that with the tag “novelty.” I could then look at all quotes tagged with the word “novelty” at once. The screenshot in Figure 3.1

¹All names have been changed.
shows the program with a focus on the theme that math does not need numbers.

![Figure 3.1: Three students agree that “math does not need numbers”](image)

Between the journals, the interviews, and my observations, I was able to discern six themes. Some of the information leading up to the theme of ‘novelty’ is shown in Figure 3.2.

### 3.1 Materials

#### 3.1.1 Interviews

I asked my husband to serve as an outside interviewer, so students could speak more freely about the class I was teaching than if they were speaking with me. The children, of course, were not made aware of any relationship between the two of us. I gave
Evolution of a Theme

Journal
Tia: "I loved doing the magic tricks they were awsome!
(Nobody had seen the magic tricks before.)

Interview
Joe: "I don't like learning new things 'cuz then you have to remember that and then remember everything else you learned."

Observation
Tom began crying when faced with a new way of working with perimeter and area.

Novelty

Supported by:

Literature
Sternberg and Wagner: If an experience is too novel, students may not connect old learning with new learning.

Figure 3.2: The theme of novelty was discovered in several places

him a script and explained that he could ask follow-up questions that he deemed appropriate. The interviewer conducted a mock interview with me playing the role of the child, and we discussed how he could improve his technique for the actual interviews. I had him ask students about their feelings about math, their math classes in school, this summer class, and journaling. Essentially, I was looking at their beliefs about their mathematics experiences and mathematics in general.

The interviewer examined the curriculum for the summer course to familiarize
himself with examples the students might reference during the interviews. He talked with students for roughly twenty minutes each, asking open-ended questions and providing prompts for more information when necessary. Not all questions were asked of each student due to time constraints, and some additional follow-up questions were asked for clarification. When pressured by the time limit, the interviewer tried to focus both on questions that had good responses in previous interviews and on questions that he did not get a chance to ask in previous interviews. The original questions can be found in Appendix A.

The interviewer then asked students to do “think-aloud” problems, where they were asked to say everything that went through their heads during a particular problem (Novick & Bassok, 2005). I chose to include some of these problems because I wanted to see how students solved problems. Two of the problems were similar to problems we had done in class, while the first was a warm-up problem designed to help children feel more comfortable with the process. The interviewer began by modeling a simple think-aloud problem, such as $12 + 15$. He used the same basic model each time. When he was interviewing Sam, he said:

So for this one, when I solve it, I see it’s twelve plus fifteen. Well I know that’s twelve, [and] I know what twelve plus twelve is; that’s a little easier to do because I can just double the twelve to get twenty-four. Of course, that’s not the problem I’m trying to originally solve... I need three more, right, because fifteen is three more than twelve, so I need to increase this by three. And so the sum is twenty-seven.

He explained that there were many correct answers, and he then posed the problems found in Appendix A.
3.1.2 Journals

Ball and Bass (2003) note that “Some initiatives and moves that teachers might make to support the development of students’ mathematical reasoning... is to create lasting records of established as well as currently negotiated mathematical knowledge” (p. 43). To do this, I had students write in journals. They did much of their mathematics in the journals, and I also gave them some time twice a day to write down their thoughts and questions about our activities. I then collected them, read them, and commented on them. I asked permission to make photocopies of them, as well.

Clarke et al. (1993) hoped that journals would help students begin “searching for patterns, making and testing conjectures, generalizing, asking Why?, trying to be systematic, classifying, transforming, searching for methods, deciding on rules, defining, agreeing on equivalences, reasoning, demonstrating, expressing doubt, and proving” (p. 239). Therefore, while the journals were partly for data collection, they had value for the children, as well.

According to Clarke et al. (1993), it is important for students to understand what they are expected to write in a mathematics journal. To that end, I considered giving them a sample journal entry. However, I did not wish to stifle their creativity, so I gave some general guidelines instead. I told students that journal entries may be thought of in several ways. They can use their journals as something to look back upon in a few months or years to remember what they’ve learned, as a way to share what they’ve done with parents, siblings, or friends, as a place to write about mathematics, or as a place to write about their feelings about mathematics. Additionally, they can use them to communicate with me if they feel uncomfortable raising a question or saying something in class.
For the second group of students, the journals also included short writing assignments where they responded to prompts. These prompts came from the interview questions that some of the first group of students received. These writing assignments stemmed from the fact that interviewing is not allowed at the second program, but the organizers of the program do allow copies to be made of student work. The journal prompts that were used are given in Appendix B.

In addition to conducting interviews and reading students’ journals, I kept my own journal about what I did with the students each day. With this journal, I was able to easily review what concepts I covered during each class.

### 3.2 Limitations

There were several limitations in this study, some of which were anticipated and some of which were not. Because these are elementary school students, I needed the permission of the parents and the two programs, as well as that of the children. While this took some work, it was eventually obtained, albeit not from as many parents as I had hoped. With the first group, I was given permission to interview five children and copy the journals of six. For the second group, which did not allow interviews at all, I was only able to copy one of the journals.

Second, the time span was limited. A week is enough to expose students to nonroutine problems, but not enough to determine any changes that those problems might have on the students. For this reason, it would be helpful to be able to do a more in-depth study, where I could spend more time with the students and observe how they react to nonroutine problems and games, both at the beginning and after having had some exposure to them.

Another problem was that because I was teaching the classes, I could not focus on
just one group to watch as their understanding of the activities progressed. Instead, I had to move around the classroom and observe all students, helping when needed. Therefore, while I was able to observe many strategies, I did not see how students’ strategies changed over time. It would be beneficial to conduct further research with individual pairs or small groups of students. I might use the strategy employed by Bright and Harvey (1986), which was to create a matrix with one column for each turn and one row for each strategy. I could then record a 1 in cells if the strategy was used on a particular turn, and a 0 otherwise.

As the teacher, I also had an obligation to spend time solely on activities for the students’ benefit, with the exception of the twenty-minute interviews, for which I had to obtain permission. In a program where I had the consent of all students and their parents, I could spend more time on data collection activities that may not directly benefit the students.

One advantage to being the teacher is that I thoroughly understood the material I was presenting, and I was able to explain the rules and adapt them if necessary. If other people were using the activities I set up and did not fully understand the subtleties, the activities might not run so smoothly. For instance, for time purposes, I simplified a domino game, explained in Section 5.1 on page 67, from ten squares to six squares for the second group. If people did not fully understand the strategy behind the game, they might have reduced it to five, which yields a completely different and much more difficult strategy than the strategy for even numbers.

Additionally, it was unfortunate that the second program did not allow interviews, so all of my interviews came from the first program. The journals allowed some insight into the children’s thoughts, but while most of the first group of students read my comments and questions, they rarely followed up and actually answered the questions posed in their journals. Because of time constraints and attendance
variation, the journals were not passed back to the second group, so there was no
chance for continued interaction. A larger study with a group that allowed interviews
might shed some light on what the children from the second group might have had
to say.

Some of the students seemed very concerned about writing the “right thing” in
their journals, which may have made it more difficult to collect information about
what they were really thinking. I attempted to assure them that there was no single
right answer, and that I wanted to read whatever they put. This seemed to satisfy
some of them, but it was not clear whether it satisfied everyone.

Of the fourteen children enrolled in the first class, two did not attend either the
first or the second days (out of five total days), and thus did not have the opportunity
to take a permission slip home. Of the remaining twelve, only six gave consent for
me to copy journals, and only five of those allowed the interview.

The second class had an even lower return rate; out of a comparable number of
students, only one returned her consent form. Many students said they had forgotten
their forms at home. This may be because the second program was an all-day affair
consisting of many different activities, while children in the first program were coming
for just one or two classes. Therefore, the class, and thus the consent form, may have
been a higher priority for the first group of children. It may also have been the case
that students who were not living with their parents wanted their parents to sign,
rather than their foster parents or temporary guardians. I had one girl ask if she
could have her mother sign it, only to find out later that she only saw her mother
once a week, and would not see her again until after the class had ended.
Chapter 4

First Class

The first class took place during June 14-18, 2010. While I was teaching this class, I had an independent observer perform the interviews. Before he did the interviews, he and I went over what was being done in class in detail, and we also had a practice interview. All students who were interviewed had parents or guardians who paid full price for the program. The three children in the class who were on need-based scholarships had extremely poor attendance, and thus did not participate in the interviews.

<table>
<thead>
<tr>
<th>Pseudonym</th>
<th>Age</th>
<th>Grade Entering</th>
<th>Gender</th>
</tr>
</thead>
<tbody>
<tr>
<td>Jen</td>
<td>10</td>
<td>5</td>
<td>Female</td>
</tr>
<tr>
<td>Joe</td>
<td>11</td>
<td>6</td>
<td>Male</td>
</tr>
<tr>
<td>Sam</td>
<td>10</td>
<td>5</td>
<td>Female</td>
</tr>
<tr>
<td>Tia</td>
<td>10</td>
<td>6</td>
<td>Female</td>
</tr>
<tr>
<td>Tom</td>
<td>9</td>
<td>5</td>
<td>Male</td>
</tr>
</tbody>
</table>

Table 4.1: Student Information
4.1 School Math

When asked about their math classes from school last year, most children spoke of doing practice problems. Joe\(^1\) noted that they did practice problems every day. According to the children, the classes did not focus only on written work. Many also discussed visualizations, such as drawing pictures to represent fractions. Joe’s teacher even brought an apple to class and divided it into pieces, and then explained how fourths went together to form halves, and halves to form wholes. Sam, on the other hand, could not recall visualizing division problems, for instance. She claimed that she understood the procedure, but her class did not delve any deeper than the surface.

Two of the students mentioned that their teachers tried to ensure that everyone in the class understood. Jen noted that with her teacher, “We would keep going over it until everybody got it.” In contrast, Tia’s teacher would apparently “go to the back and help those kids [who did not yet understand] while the ones that did get it keep working.” The method Tia claimed her teacher used allowed students who did understand to move on instead of having to sit through another explanation. However, it requires a space in the classroom for the teacher to work with students who do not understand, and it also singles out those students more than Jen’s teacher’s supposed practice.

A third possibility, which was not mentioned by any of the children, is to group students by ability. This method was discussed by Thomas Hoffer, who found that, “Ability grouping thus appears to benefit advanced students, to harm slower students, and to have a negligible overall effect as the benefits and liabilities cancel each other out” (Hoffer, 1992). However, even if ability grouping were only beneficial, such

\(^1\)All names have been changed.
grouping does not eliminate the differences between students. In fact, Tia mentioned that she is in a “diff” class, and there were still some students who did not understand immediately. The diff stands for differentiated, and the Handbook for Parents of Gifted Students in the Lincoln Public Schools explains that,

Diff math is also accelerated and compacted; part of the mathematics curriculum should be offered in a gifted group, at a challenging level, and the rest of the teaching should be “compacted” to avoid repetitive worksheets that reinforce already-mastered skills. The regular math curriculum includes pre-tests that demonstrate content the students already know, so that compacting can be ensured (Gifted Community Review Committee, 2005, p. 25)

Even with ability grouping, some method must be employed to help struggling students. There may be no ideal method that works for every classroom; it seems entirely possible that both teachers are right. This is especially true because Tia mentioned she was in a differentiated class while Jen did not, so it may be that different techniques are needed for different types of students.

4.2 Math in the Summer Class

Table 4.2 lists the activities done in the first group. Note that this section will only discuss activities that were only done in the first group. Activities done in both groups will be discussed in Chapter 5.
<table>
<thead>
<tr>
<th>Activity</th>
<th>Date</th>
<th>Purpose</th>
</tr>
</thead>
<tbody>
<tr>
<td>Towers of Hanoi</td>
<td>June 14</td>
<td>Powers of Two</td>
</tr>
<tr>
<td>Handshakes</td>
<td>June 14</td>
<td>Additive &amp; Multiplicative Reasoning</td>
</tr>
<tr>
<td>Stars</td>
<td>June 14</td>
<td>Definitions</td>
</tr>
<tr>
<td>Factorials</td>
<td>June 14</td>
<td>Factorials</td>
</tr>
<tr>
<td>Penny Game</td>
<td>June 14</td>
<td>Modular Arithmetic</td>
</tr>
<tr>
<td>Fractals</td>
<td>June 15</td>
<td>Fractals</td>
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<tr>
<td>Graph Coloring</td>
<td>June 15</td>
<td>Graph Coloring</td>
</tr>
<tr>
<td>Paper Folding</td>
<td>June 15</td>
<td>Definitions</td>
</tr>
<tr>
<td>+/- and +/- Games</td>
<td>June 15</td>
<td>Order of Operations</td>
</tr>
<tr>
<td>River Crossing</td>
<td>June 15</td>
<td>Logical Reasoning</td>
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<tr>
<td>Rock, Paper, Scissors, Dynamite</td>
<td>June 15</td>
<td>Game Theory</td>
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<td>Area and Perimeter</td>
<td>June 15</td>
<td>Area and Perimeter</td>
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<td>Pig</td>
<td>June 16</td>
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<tr>
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<td>June 16</td>
<td>Base 5</td>
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<td>Even or Odd</td>
<td>June 16</td>
<td>Modular Arithmetic</td>
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<td>Train Cars</td>
<td>June 16</td>
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<tr>
<td>Addition Trick</td>
<td>June 16</td>
<td>“Complementary” Numbers</td>
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<td>Five Card Trick</td>
<td>June 16</td>
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<td>June 17</td>
<td>Strategy</td>
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<td>Parity</td>
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<td>June 17</td>
<td>Definitions, Square Numbers</td>
</tr>
<tr>
<td>Move Forward Game</td>
<td>June 17</td>
<td>Invariants</td>
</tr>
<tr>
<td>Cartesian Coordinates</td>
<td>June 17</td>
<td>Cartesian Coordinates</td>
</tr>
<tr>
<td>Sixteen Card Trick</td>
<td>June 17</td>
<td>Coordinates</td>
</tr>
<tr>
<td>Hog</td>
<td>June 18</td>
<td>Probability</td>
</tr>
<tr>
<td>Guess My Rule</td>
<td>June 18</td>
<td>Categorization</td>
</tr>
<tr>
<td>Flashlights on a Bridge</td>
<td>June 18</td>
<td>Logical Reasoning</td>
</tr>
</tbody>
</table>

Table 4.2: Activities done in the first classes

4.2.1 Shape Configurations

When discussing the first summer class, students tended to bring up three things: visualizations, hands-on activities, and games. For example, in one activity that was mentioned by multiple students, the students were shown various configurations of overlapping shapes: a triangle, a square, and a circle. Two such configurations are shown in Figure 4.1. They were asked to place dots in the shapes such that, for
instance, the triangle and square had an odd number of dots, but the circle had an even number. Once they had accomplished that, they were asked to minimize the number of dots needed to do such a task. In all examples we did in class, either one or two dots sufficed. Based on journal reports, the students seemed to enjoy this. They made several discoveries, such as that two dots in the same region could be eliminated. Some had to be reminded by other students that zero was an even number.

Figure 4.1: Two configurations of shapes with minimal numbers of dots

4.2.2 Base Five

Another activity many students mentioned in their journals was learning about base five. For instance, Tom spoke proudly about “learning stuff that most adults don’t know.” The students learned base five by dealing with something familiar: United States currency. Because a nickel is worth five pennies and a quarter is worth five nickels, those three coins can form a basis for learning up to three-digit numbers in base five. The number 21 in base five, for instance, represented two nickels and one penny, which the students knew was worth eleven cents. The children noticed that the digits 5 through 9 never appear in base five, and they were able to construct a base five number line. Additionally, students quickly grasped addition and subtraction of up to three digit numbers in base five, eagerly doing problems on the board. They were
not satisfied with leaving the answer in base five, however, feeling that the problem was not finished until the answer was in base ten. While students were relatively comfortable working in base five, they seemed to believe that it was not as valid a system as base ten.

Jen made an interesting point, noting that the hands-on activities were easier to do in the summer classroom due to its small size. There were never more than twelve students at a time, and there were two assistants to help with activities when needed. It would take further study to determine whether such hands-on activities are feasible in a traditional classroom. The activities were clearly popular with the children, based on journals and interviews.

### 4.2.3 Modular Arithmetic

The children also looked at modular arithmetic, also known as clock arithmetic. They began by analyzing a train whose cars alternated in color between red and blue. The first train car was red, the second blue, and so on. Students immediately noted the even/odd pattern and were able to tell what color any given car would be. When there were three colors, they found it more difficult. One tried to adapt the even/odd rule, but found that he could not. Another student stepped in and talked about multiples of three.

With six colors, when the fifth car was blue, one student suggested that every multiple of five would be blue. The mistake is understandable, since when every second car was blue, every multiple of two was blue. Other students pointed to car 10 as a counterexample, and everyone understood that even a single counterexample meant that the original conjecture was flawed. Shortly afterward, the class correctly noted that blue cars were all one less than multiples of six.
4.2.4 Domino and Token Games

During the course of the week, the children were introduced to many games, almost all of which they reported as being completely new to them. These games were referenced by three of the five interviewees, and a fourth mentioned them indirectly, calling them puzzles and “mind-bendery.” Most games were two-player games, but a few could be played with more.

The Move-forward game was taken from the paper, “Loop invariants, exploration of regularities, and mathematical games” (Ginat, 2001). It is a simpler variation of Northcott’s game, discussed by Richard Guy in his survey of impartial mathematical games (Guy, 1991). The Move-forward game is played on board consisting of two rows of squares. There are two light gray tokens on the leftmost squares and two dark gray tokens on the rightmost squares, as shown in Figure 4.2. The light gray player moves first, moving either of her tokens to the right as many squares as she likes. The dark gray retaliates, moving left, with the caveat that he cannot jump over her token. Play continues in this manner, and the last player who can move wins.

Figure 4.2: The starting configuration for the Move-forward game

As with most games, some students understood this somewhat quickly, and others took longer. One boy kept making the same mistake of moving one of his counters all the way to the other side on his first move, at which point his partner could immediately win by countering with the same move. It took some time to convince him that this would never work for him. The optimal strategy for this consists of
mirroring one’s opponent. In this case, the second player always wins, because he can simply mimic the first player’s moves on the opposite row. For instance, if the first player moves her top piece three spaces to the right, the second player should move his bottom piece three spaces to the left. Playing the domino game, discussed in Section 5.1 on page 67, may have helped prepare them for this game.

4.2.5 Factorials, Fractals, and the Pigeonhole Principle

The children seemed fascinated by large numbers, and they enjoyed learning about factorials. Some of them asked about large factorials in their journals. Fractals were also popular, being mentioned by two of the students interviewed. The children were introduced to the Sierpinski Triangle and the Koch Snowflake, partially shown below in Figures 4.3 and 4.4, and they were given opportunities to draw both as best they could. A few had trouble, especially with the snowflake, but most produced relatively good approximations. Bob did six iterations of the Sierpinski Triangle, and Joe did seven. Tia did the most iterations on the Koch snowflake, with four.

Students were also introduced to the Pigeonhole Principle, also known as Dirichlet’s Principle. During introductions, I had everyone state their name and birthday. I asked students ahead of time whether they thought we would have two people with their birthdays in the same month. With twelve students, two assistants, and one instructor, several of them spoke up to say that there must be some duplicate. This principle is actually a very important one to mathematics, and the students already understood it quite well. They also understood that it was not necessarily true that all months would be represented, and indeed, there were several months with no birthdays in them.
4.2.6 Definitions

Definitions are a crucial part of mathematics, and it is important to ensure that people mean the same thing when they use certain terms. To see this, we discussed the definition of a star and drew some objects which were stars and some which were not. I was surprised at how inclusive the children were; while some did not count all of them, the majority of the class called the objects in Figure 4.5 stars. I expected
students to decide that a point was not a star because it did not have an interior, for instance, but they did not. The students appeared surprised that I did not have a set definition in mind, and that they were allowed to make up their own definitions. This is exactly what mathematicians frequently do, so it makes sense to also allow children to explore the formulation and role of definitions.

Figure 4.5: Stars, according to the first class

4.2.7 Squares in a Checkerboard

Another case where definitions were important was when I asked students to count the number of squares in a six by six checkerboard. Initially, many of them said thirty-six, until I told them there were more than that. Gradually, they realized that there were larger squares, as well, such as the square composed of the entire board. Many felt it was too hard to count them well and began guessing. For students who had no idea how to proceed, I suggested they classify all of the squares and then count them. For instance, they could classify squares by the lengths of their sides and then count each size. Even then, it was not necessary to count each square. One girl, Tia, noticed that there were $6 \times 6$ squares of side length one, $5 \times 5$ squares of side length two, and so on, down to $1 \times 1$ square of side length six. For students who did not see this pattern, I suggested that they look at the upper left corner of the square of the
size in question. For example, the upper left corner of a three by three square could be in any of the shaded squares shown in Figure 4.6. This explains why the pattern Tia noticed works.

![Figure 4.6: Possible locations for the upper left corner of a 3 by 3 square](image)

### 4.2.8 Maximization

I also developed a very simple game where I gave students three numbers, a plus sign, and a minus sign, and I asked them to make the largest number possible using each number and symbol exactly once. For instance, if I gave them 4, 7, 9, they should either write $7 + 9 - 4$ or $9 + 7 - 4$. Students noticed several things about problems like these. Some students reasoned through the process, while others tried all combinations. Both groups noticed that the smallest number is the number that should be subtracted, while the two largest should be added together. I wrote out that rule on the board:

$$\text{big number} + \text{middle number} - \text{small number}$$

or

$$\text{middle number} + \text{big number} - \text{small number}.$$ 

I then explained that people did not like to write everything out, so I asked if it was okay if I used just a $b$ for the biggest number, an $m$ for the middle number, and an $s$
for the small number. They agreed that it would still be clear, so I wrote:

\[ b + m - s \text{ or } m + b - s. \]

That simple step was very important, as it introduced the children to variables. Variables are part of the Common Core Standards for third-graders, who are asked to “represent these problems using equations with a letter standing for the unknown quantity.” (Common Core State Standards Initiative, 2010, p. 23). The Common Core Standards are discussed more thoroughly in Chapter 7. Students were also asked what would happen if two of the numbers were the same, or even if all of the numbers were the same. They correctly answered that it was okay for \( b \) to equal \( m \), for instance.

I followed this game with a similar game using a multiplication sign and a plus sign. Again, the students quickly found the rule, which is that one should multiply the two larger numbers and then add the smaller number. This time, they were able to write the rule using \( b, m, \) and \( s \) on their own. For instance, given \( 4, 7, \) and \( 9 \), students would write \( 9 \times 7 + 4 = 67 \). Some students cleverly noted that if parentheses were allowed, then they could make the answer larger, by adding the two smallest numbers and then multiplying by the largest number. The same example as before, \( 4, 7, \) and \( 9 \), yields \( (4 + 7) \times 9 = 99 \). This understanding of the order of operations demonstrates that students have learned another third-grade Common Core Standard, order of operations on addition, subtraction, multiplication, and division. Their use of parentheses shows their competence with yet another Standard, this time from the fifth grade, which states that students should be able to “use parentheses, brackets, or braces in numerical expressions, and evaluate expressions with these symbols” (Common Core State Standards Initiative, 2010, p. 35).
4.2.9 An Extension of Rock, Paper, Scissors

Another game that we played was the traditional “Rock, Paper, Scissors.” In it, two players count to three and then hold out their hands in the shape of a rock, paper, or scissors. If one player made a rock and the other made paper, the latter wins, because “paper wraps around rock.” With paper and scissors, scissors wins, since “scissors cut paper.” Finally, with rock and scissors, the rock wins, since “rock smashes scissors.” If the players choose the same object, they try again. This is frequently used to decide which person receives a certain advantage.

A pure strategy, such as always choosing rock, will fail against an intelligent player. The other player can simply respond by always choosing paper. An unequal mixed strategy, such as choosing rock 40% of the time, paper 30% of the time, and scissors 30% of the time, can be beaten 40% of the time and tied 30% of the time by always choosing paper. The best strategy is to choose each one with equal probability. Some children use strategies such as trying to beat what the player picked most recently, assuming the opponent might make the same choice as before, but once they are discovered, the opponent can take advantage of that strategy to always win.

After the children had played a few games of Rock, Paper, Scissors, I introduced a revised version, entitled “Rock, Paper, Scissors, Dynamite,” where dynamite beats paper and rock, but scissors beats dynamite (since the scissors can snip the fuse). At first, the children were very excited about the new option, so many of them chose it. However, once they caught on, they started choosing scissors, since it beats dynamite. From there, they moved on to choosing rock, since it beats scissors, and eventually they realized that a mixed strategy would again be best.

Finding that mixed strategy was not easy for the students. Initially, one could make a table like Table 4.3. Paper and rock each win twice in the table, while scissors...
Table 4.3: The Rock, Paper, Scissors, Dynamite game

<table>
<thead>
<tr>
<th></th>
<th>Rock</th>
<th>Paper</th>
<th>Scissors</th>
<th>Dynamite</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rock</td>
<td>Tie</td>
<td>Paper</td>
<td>Rock</td>
<td>Dynamite</td>
</tr>
<tr>
<td>Paper</td>
<td>Tie</td>
<td>Scissors</td>
<td>Tie</td>
<td>Dynamite</td>
</tr>
<tr>
<td>Scissors</td>
<td>Dynamite</td>
<td>Dynamite</td>
<td>Scissors</td>
<td>Tie</td>
</tr>
</tbody>
</table>

and dynamite appear four times each. One could conjecture that each object should be chosen in proportion to how often it appears in the table, but that is not the correct answer. Actually, dynamite is strictly better than paper, since both beat rock and are beaten by scissors, but dynamite beats paper. Therefore, one should never choose paper, so the game is reduced to Rock, Scissors, Dynamite. This is analogous to Rock, Paper, Scissors, and the same mixed strategy is ideal. The children did not reach this conclusion on their own, due in part to time constraints, but with some help, they came to understand the optimal strategy.

4.2.10 Area and Perimeter

The Common Core Standards for third grade state that students should “measure the area of a shape by finding the total number of same-size units of area required to cover the shape without gaps or overlaps, a square with sides of unit length being the standard unit for measuring area” (Common Core State Standards Initiative, 2010, p. 21). I gave pairs of students sixteen toothpicks and asked them to form different rectangles and measure the area in “toothpick-squares.” This caused some confusion, as shown in Figure 4.7, because some students were using overlapping toothpicks and considering the smallest square they could form to be the unit square. Thus, the leftmost square was said to have area one unit, and the rightmost large square was deemed to have area four units.
Figure 4.7: The two large squares were judged by some to have different area

Once students understood that they were to make rectangles with no interior edges, we created a chart on the board, similar to the one in Table 4.4. Thus, not only did the children learn about geometry, but they also worked on table-making skills, discussed in the Common Core Standards for grade six (Common Core State Standards Initiative, 2010). The children agreed that there was no need to count both a $3 \times 5$ rectangle and a $5 \times 3$ rectangle, so the chart was shorter than it might have otherwise been. Some students even imagined non-integer extensions, shown in grey. This led them to practice multiplication of fractions, a fifth-grade skill (Common Core State Standards Initiative, 2010).

<table>
<thead>
<tr>
<th>Height</th>
<th>Width</th>
<th>Area</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>4</td>
<td>16</td>
</tr>
<tr>
<td>3</td>
<td>5</td>
<td>15</td>
</tr>
<tr>
<td>2</td>
<td>6</td>
<td>12</td>
</tr>
<tr>
<td>1/2</td>
<td>7</td>
<td>$3\frac{3}{4}$</td>
</tr>
</tbody>
</table>

Table 4.4: Rectangles with perimeter 16

After this activity, I gave them 16 Starbursts, which are candies in the shape of square prisms, and asked them to make rectangles and measure the perimeters. Some students initially tried to form rectangles with no Starbursts in the center, but I explained that they had to be filled in. Overall, they found this task much simpler than working with toothpicks, and the children quickly formed charts like Table 4.5.
Again, some even chose to imagine fractional Starbursts.

<table>
<thead>
<tr>
<th>Height</th>
<th>Width</th>
<th>Perimeter</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>4</td>
<td>16</td>
</tr>
<tr>
<td>2</td>
<td>8</td>
<td>20</td>
</tr>
<tr>
<td>1</td>
<td>16</td>
<td>34</td>
</tr>
<tr>
<td>1/2</td>
<td>32</td>
<td>65</td>
</tr>
</tbody>
</table>

Table 4.5: Rectangles with area 16

4.2.11 “Magic” Tricks

I also taught the children some “magic tricks.” One is a very simple arithmetic trick that teaches flexibility in computation, while the other uses some more sophisticated mathematics, including modular arithmetic and permutations.

For the first trick, a volunteer gives a four digit number, such as 2847, and the mathematician performing the trick says that she and her volunteer will together come up with four more numbers such that the sum of all five numbers is 22,845. They will take turns giving numbers, all of four digits or less, with the volunteer going first. If the volunteer says 4973, the performer says 5026. The volunteer might then say 9742, and the performer would respond with 257. Note that each pair of numbers adds to 9999, and in fact, the corresponding digits of the numbers add to nine. For example, the thousands place of 4973 is a four, so the thousands place of the next number must be five, since that is nine minus four.

The children really liked this trick, because it seems amazing that someone could do complex calculations so quickly, but the calculations are actually quite simple. Several of them performed the trick for their parents at home or at the open house. Not only does this trick teach addition of multiple multi-digit numbers, but it also
encourages flexibility in computing. In particular, rather than adding 19,998 to the original number, it is much easier to add 20,000 and then subtract 2.

The second trick is Fitch Cheney’s Five Card Trick (Mulcahy, 2003), which involves a performer, a trained assistant, and a volunteer from the audience. In fact, the assistant does most of the work. He shuffles a standard deck of 52 cards (four suits of Ace, 2, 3, 4, 5, 6, 7, 8, 9, 10, Jack, Queen, King) and then asks the volunteer to choose any five. Everyone except the performer looks at these cards, and the assistant explains that he will show four of the cards to the performer, who will then ascertain the identity of the fifth card.

Determining the suit of the missing card is quite simple. Because there are five cards and only four suits, the Pigeonhole principle, discussed on page 36, states that some suit must be duplicated. Therefore, the assistant chooses one of those cards as the secret card, and places the other at the front of the remaining four. Finding the value of the card is more difficult, and it relies upon the fact that there are $3! = 6$ possible permutations of the three remaining cards. Before the trick, the performer and assistant must agree upon a ranking of the cards. Traditionally, aces are the lowest and kings the highest. In the case of a tie, such as the two of spades and the two of diamonds, cards are ranked alphabetically by suit, with clubs being the lowest and spades the highest. For example, with a seven of hearts, two of spades, and two of clubs, the two of clubs would count as the lowest card, and the seven of hearts the highest.

Each permutation of those three cards corresponds to a different number, one through six, as shown in Table 4.6. Since there are only six permutations, if the secret card is a nine of hearts and the other heart is a two, one must simply switch the two cards and work modulo thirteen. Nine plus six is fifteen, which is two modulo thirteen. Therefore, one would arrange the cards high, middle, low to indicate that
the performer needs to add six to the nine she is given.

<table>
<thead>
<tr>
<th>First card</th>
<th>Second card</th>
<th>Third card</th>
<th>Amount to add</th>
</tr>
</thead>
<tbody>
<tr>
<td>Low</td>
<td>Middle</td>
<td>High</td>
<td>1</td>
</tr>
<tr>
<td>Low</td>
<td>High</td>
<td>Middle</td>
<td>2</td>
</tr>
<tr>
<td>Middle</td>
<td>Low</td>
<td>High</td>
<td>3</td>
</tr>
<tr>
<td>Middle</td>
<td>High</td>
<td>Low</td>
<td>4</td>
</tr>
<tr>
<td>High</td>
<td>Low</td>
<td>Middle</td>
<td>5</td>
</tr>
<tr>
<td>High</td>
<td>Middle</td>
<td>Low</td>
<td>6</td>
</tr>
</tbody>
</table>

Table 4.6: Amount to add to the value of the first card

This trick involves a lot of higher level mathematics, including the concepts of permutations, factorials, and modular arithmetic. The Common Core Standards do not introduce permutations until high school, and they do not introduce factorials or modular arithmetic at all. That does not mean that they should not be included in the curriculum, however; merely that they were not considered to be crucial.

Another card trick was introduced by one of the students, Tia. In it, she laid out sixteen cards in a four by four grid, by rows, as in the left side of Figure 4.8. She then asked a volunteer to pick a card and tell her what row it was in. For instance, if they had chosen the seven of hearts, they would tell her it was in the second row. Tia then carefully gathered them up and laid them down in another four by four grid, this time putting them down by columns, so the picture was transposed, or reflected across the main diagonal. The second layout is shown on the right side of Figure 4.8. She would again ask what row the card was in, and would this time be told it was the third row. Since she knew it was in the second column of this new arrangement, she could determine that the card chosen must be the seven of hearts.

This relatively simple card trick uses the fact that a row and a column are sufficient to determine an entry in a matrix. While this may seem obvious, it is actually quite important. One could extend it to coordinates in the plane, and note that each
combination of integers yields exactly one lattice point, and that different combinations imply different lattice points. Additionally, this trick is reminiscent of linear algebra, where one often needs to transpose the matrix. Linear algebra is usually reserved for college classes, but it makes sense to set the foundation for it earlier, when possible.

### 4.2.12 Cartesian Coordinates

Another topic relating to the plane is that of the Cartesian coordinate system. I asked students to draw pictures using lattice points and straight lines, and to write down those points. For instance, I drew the picture shown in Figure 4.9, which would be represented by

\[
(1, 4) \text{ to } (1, 3) \\
(3, 4) \text{ to } (3, 3) \\
(2, 2) \\
(0, 2) \text{ to } (1, 1) \text{ to } (3, 1) \text{ to } (4, 2).
\]
I then had students write down the coordinates for their pictures and trade those with their partners. They then tried to reconstruct the pictures from the coordinates.

![Figure 4.9: A face drawn with lattice points and line segments](image)

Some of the kids were familiar with the Cartesian coordinate system and some were not, so some students needed individual help. Additionally, not all students drew their pictures at the same scale, so some of them had to redraw their axes. Eventually, however, students were successful. Graphing points with Cartesian coordinates is listed in the Common Core Standards for fifth grade (Common Core State Standards Initiative, 2010).

### 4.2.13 The Four Color Theorem

Students were also introduced to a famous mathematics problem, the Four Color Theorem. This theorem, proved in 1976 by Kenneth Appel and Wolfgang Haken, states that any planar map is four-colorable, meaning that any collection of regions can be colored using four or fewer colors such that if two regions share a border, they are different colors. The students were first shown several two-colorable pictures and asked to color them. Then, they were told to design a map that required three colors. Someone came up with a design similar to the one shown in Figure 4.10. I then asked for one requiring four colors, and eventually one member of the class came up with
the image in Figure 4.11. I then asked for a map requiring five colors. After showing that all of their proposals were actually four-colorable, I told them that there was no map that needed five colors. I mentioned also that while five colors are not needed, they are often used by cartographers, because it can be difficult to find a proper four-coloring.

![Figure 4.10: A map requiring three colors](image)

![Figure 4.11: A map requiring four colors](image)

This theorem introduces children to some relatively recent mathematics. For many children, this is the first piece of mathematics that they have seen that was shown within the past century. It shows students that mathematics is not yet complete, and there are still discoveries to be made.

### 4.3 Definitions of Math

During the interview, the children were asked to define numbers for a student who had never been introduced to math before. This was intended to have students think about basic principles. Sam indicated that she would relate numbers to letters, saying, “They serve the same general purpose.” Joe tried to define numbers based on their qualities, saying that a number “is either positive or negative. With the exception
of zero, I think.” When pressed, he gave examples of numbers, going from one to sixteen before being guided by the interviewer into explaining the number three. He then explained that numbers had an order to them. Jen said, “Numbers are symbols that represent counting,” which may be a sufficient definition for elementary school, but fails in higher grades. Tia was told that the student was good in subjects other than math, so she related numbers to something familiar to this new student. She said, “Numbers are like chapters in a book. You can add numbers to make a whole thing and chapters to make a whole book.”

The students were also told that this hypothetical student asked, “What is math?” Tom responded that math was, “Using numbers to figure out the answer to a number problem, and also figuring out how things work, like if you’re strategizing.” He, Sam, and Tia felt that math did not need to include numbers, especially certain kinds of math, such as finding game strategies or labeling parts of geometric figures. Jen disagreed, noting that it was not possible to add or subtract without numbers. She seemed to feel that mathematics was based on addition and subtraction. Despite three of the students agreeing that mathematics was possible without numbers, many students felt that games were not mathematical. Joe noted that his fifth grade class has more of a “focus on math” than the summer class, and Tia and Tom both spoke about “actual math,” as opposed to games, with Tom noting that this class did not “teach curriculum stuff.”

### 4.4 Attitudes Toward Math

Some of the students were also asked if they were good at math. Tom responded that he was, because he was “pretty much right with all the answers to my problems every time.” Other qualities of people who were good at math included studying it,
<table>
<thead>
<tr>
<th>School “Actual Math”</th>
<th>Summer Program “Puzzle Math”</th>
</tr>
</thead>
<tbody>
<tr>
<td>Contents: Fractions, Multiplication, Long Division, Pre-Algebra, Algebra, Geometry, Order of Operations</td>
<td>Patterns, Shapes, Base 5, Towers of Hanoi, Fractals, Puzzles, Dice Games</td>
</tr>
<tr>
<td>Methods: Teacher Explanation, Logic Puzzles, Demonstrations, Timed Tests, Free Practice, Games, Long Lectures</td>
<td>Examples, Partner Work, Hands-On Activities, Magic Tricks, Games</td>
</tr>
</tbody>
</table>

Table 4.7: Contents and methods used in school and in the first program, according to the children

Three students said that they liked to be challenged in math or that they enjoyed learning new things. For instance, in her journal, Jen wrote, “I liked how hard you made the handshake problem. I am always up for challenges!” Joe, however, specifically said that while he liked a challenge, he did not like learning new things, and preferred instead to do mathematics he found more comfortable. This is a very interesting distinction. Many students find learning new things difficult. For instance, they might rather converse in their native language than in a second language. However, learning new things is necessary for growth and progress.

It would be interesting to attempt to determine the differences between Joe’s attitude and the attitudes of Tia and Tom. Tia stated that she enjoyed learning new things, and Tom said that he liked math but disliked doing math problems, because “it gets boring after a while.” Joe and Tom seem to have opposing views. One possible explanation is age. Tom is nine, Tia is ten, and Joe is eleven. It seems feasible that younger students would be more likely to enjoy learning because their
brains contain more synapses and are set up for learning (Bruer, 1997). Such thinking is only speculation, but it could lead to an interesting study.

When asked what they disliked about mathematics, some students mentioned specific areas. Tia, for instance, said, “Algebra was not my favorite thing.” She noted that her teacher “didn’t get algebra either, so it was kind of hard for her to teach it.” This serves to emphasize the importance of teacher education and professional development. If a fifth grade student knows that her teacher is having trouble with math, she may have trouble believing that she herself can learn it. Jen spoke about her difficulties with fractions, noting simply that, “They’re too hard.” Joe said he did not like multiplying and dividing fractions. Other students, however, could not think of anything they disliked about math.

Students were also asked about the math journals they kept that week. They had a designated time to write in their journals twice each day, and they also made notes during class. Jen noted that she liked the math journals, because she could ask questions that were somewhat off-topic or that there was no time for in class and “get like a real answer.” Only Joe said he had kept a math journal before, and his was only for taking notes. Students also said that the journals could be used to make the rest of the class and next year’s class better.

4.5 Think-Aloud Problems

The children were asked three think-aloud problems, designed to show how students solved various types of problems. For these problems, the students say everything they think during a particular problem. The students were all provided with pencils and paper. The first problem, twelve times fifteen, was largely to introduce them to the idea of a think-aloud problem. I speculated that students might try non-standard
algorithms, since a non-standard algorithm was modeled for them for twelve plus fifteen, but only minor changes were made to the algorithm, and those only by two students. Rather than multiplying one times two, moving to the tens place, and then multiplying one times one, both students performed one times twelve in a single step. One student, Tia, began by doing twelve times twelve instead of twelve times fifteen, mimicking the model, which could have worked. However, she became confused and decided to use the standard algorithm instead.

The next think-aloud problem was a hand-shaking problem. In the hand-shaking problem done in class, everyone shook hands with everyone else. In this problem, there were five members on each team, and each player only shook the hands of players on the opposite team. Therefore, there should be twenty-five handshakes total. Several students drew pictures like Figure 4.12, with some drawing all twenty-five connections and others only drawing five and extrapolating. All reached the number twenty-five at some point in the process, some using multiplication and others using repeated addition. However, two students asserted that there were twenty-five handshakes for the red team and twenty-five for the blue, resulting in fifty total handshakes. This was despite the similar problem from class, where students realized they needed to account for the fact that Alice shaking hands with Bob was equivalent to Bob shaking hands with Alice.

The final think-aloud problem asked students to find the optimal strategy for a game. The game consists of two players taking turns removing pennies from a pot. The pot begins with twelve pennies, and players may take either one or two at a time. The optimal strategy for the second player is to reverse everything the first player does, so the sum of the pennies taken in each round is three. This guarantees that the second player will be able to take the last penny.

Students played the same game in class, except that they were allowed to take
one, two, or three pennies. This game has a similar strategy; player two takes pennies such that the amount taken per round equals four. This strategy was discussed in class, and students were given an opportunity to practice it.

One student, Joe, immediately saw the correct strategy of complementing the other’s choice, which is shown in Figure 4.13. Tom and Sam each constructed a game, choosing the amounts each player would take arbitrarily. Once the number of pennies was reduced to four, however, Tom and Sam each saw that the player who takes the number of pennies down to three wins. Jen, on the other hand, simply noted that if every player took two, the second player would take the last penny, so she felt that was player two’s optimal strategy. She neglected to consider the fact that player one might choose to take only one penny, thus preventing her strategy from being successful.
4.6 Conclusions

The interviews, journals, and observations provided a lot of information about how the students viewed their math classes in school and the class over the summer. It is important to note that there are many differences between the two, not all of which are directly relevant to mathematics learning. For instance, some of the children in the first program were only there for a half-day, three hours, instead of being in school for a full school day. However, the comments from the students indicated that they saw differences in the content and presentation of mathematics, and they were able to articulate some of these differences to the interviewer.

The think-aloud process allowed students to formulate and share their thoughts as they solved problems, which gave some insight into their solution methods. Students were able to use the process relatively quickly, and it detailed how they obtained their answers. Such a process is labor-intensive, but as Furner, Yahya, and Duffy (2005) observe, “Encouraging children to think aloud when solving problems helps teachers pinpoint students’ difficulties in solving math problems” (p. 18). They also note that this process technique allows self-corrections and peer-corrections to be made more
easily.

One alternative to the think-aloud process is using written explanations. Pu-
galee (2004) found that written explanations were superior to verbal explanations in helping children perform problem-solving tasks, but he studied ninth-graders, who are more comfortable writing than elementary school students are. It is unclear whether his findings would extend to younger children. Additionally, verbal explanations would likely be analyzed and discussed immediately, while written explanations might not be studied and returned until after students have lost interest in the prob-
lem.
Chapter 5

Second Class and Comparisons

The second class took place at a Nebraskan non-profit organization serving children and teenagers who have been abused or neglected, as well as their families. Many of the participants are in foster care, but others remain with their families, while the families receive help and training. This group has two Community Learning Centers at nearby elementary schools. Students from both centers met at one school during July 26-29, 2010 for science classes and for the class I offered.

Because this program serves a sensitive population, they do not allow interviews. Therefore, the children were asked to write about certain topics. There were several other differences between this population and the first class’s population. The first group of children were exiting fourth and fifth grades, while the other children were entering fourth and fifth. Since the classes took place one month apart, the second group of children were approximately eleven months younger than the first group of children.

Additionally, the first class was part of a paid program serving hundreds of children in dozens of classes. Each class took place in a classroom. With the second organization, there were fewer children and fewer classes, and we met in the cafete-
ria. The children were seated at a long, rectangular table instead of at desks. There were also science and art projects scattered throughout the room, which some of the students found distracting, as evidenced by some of the students getting out of their seats and interacting with them when they were supposed to be working. Finally, there was no board, so information had to be given verbally or passed out on paper.

The second program was also much shorter, lasting only an hour, and the first five or ten minutes of each day consisted of separating out the children who should be there from the younger children who wanted to watch but needed to be elsewhere. The first class lasted from 1 to 4 in the afternoon, with a short break for a snack and recess. Almost every activity done in the second class was done in the first, but the reverse was not true. Finally, the first group of children remained relatively constant, while the second group ranged in size from three to twelve, depending on the day. Therefore, I could not plan on building on previous days’ work.

Rather than try to work with two groups of students from extremely similar backgrounds and circumstances, I decided to try to find students from different circumstances, hoping that their differences might provide valuable insights. The minor setbacks I experienced, such as the lack of a good classroom environment, might be something a teacher in a school district with fewer resources might have to overcome.

### 5.1 Math and Comparisons

Table 5.1 shows activities done in the second class. Because all but one of these were also performed in the first class, they will be discussed together in this chapter.

The activity that was only done in the second class was multiplication by nines, which arose spontaneously. I showed the students that if they multiplied a nonzero whole number by nine and then repeatedly added the digits until arriving at a single
digit, their answer would be nine. I then had them check the result with numbers of their own. Unfortunately, I failed to specify a maximum size, and one girl chose to multiply an extremely large number by nine. When I checked her paper and realized this would take much longer than planned, I asked her to cut her number short and add the digits. However, even adding the digits she had found served to be rather time-consuming.

5.1.1 Towers of Hanoi

The Towers of Hanoi game proved to be extremely popular with both groups. In the most common version of this game, students begin with three discs on the leftmost peg of a board consisting of three pegs. They are allowed to move one disc at a time, and at no time can a larger disc be on top of a smaller disc. The goal is to move all discs to the rightmost peg. In class, we played this game, as well as variations with zero, one, two, and four discs. We wrote the minimum number of moves on a piece of paper, forming a table like the one in Table 5.2.

<table>
<thead>
<tr>
<th>Activity</th>
<th>Date</th>
<th>Purpose</th>
</tr>
</thead>
<tbody>
<tr>
<td>Towers of Hanoi</td>
<td>July 26</td>
<td>Powers of Two</td>
</tr>
<tr>
<td>Handshakes</td>
<td>July 26</td>
<td>Additive &amp; Multiplicative Reasoning</td>
</tr>
<tr>
<td>Multiplication by Nines</td>
<td>July 26</td>
<td>Pattern Recognition</td>
</tr>
<tr>
<td>Fractals</td>
<td>July 27</td>
<td>Fractals</td>
</tr>
<tr>
<td>Pig</td>
<td>July 27</td>
<td>Probability</td>
</tr>
<tr>
<td>Guess My Rule</td>
<td>July 27</td>
<td>Categorization</td>
</tr>
<tr>
<td>Penny Game</td>
<td>July 28</td>
<td>Modular Arithmetic</td>
</tr>
<tr>
<td>Hog</td>
<td>July 28</td>
<td>Probability</td>
</tr>
<tr>
<td>River Crossing</td>
<td>July 29</td>
<td>Logical Reasoning</td>
</tr>
<tr>
<td>Dominoes</td>
<td>July 29</td>
<td>Strategy</td>
</tr>
<tr>
<td>Dominoes on a Checkerboard</td>
<td>July 29</td>
<td>Parity</td>
</tr>
<tr>
<td>Flashlights on a Bridge</td>
<td>July 29</td>
<td>Logical Reasoning</td>
</tr>
</tbody>
</table>

Table 5.1: Activities done in the second class
At the second program, the three students present that day then looked for patterns. They noticed that all of the minimum numbers were odd except the first, but were unable to determine how to get from one minimum to the next. I therefore added a third column, consisting of twice the minimum number of moves, as shown in Table 5.3. The children still could not find a pattern, so I asked how one would get from the 0 in the upper right of the table to the 1 in the center of the second row. They responded that I could do so by adding one. I repeated the question for getting from the 2 to the 3, and they then saw the pattern. I then asked how many moves they thought it would take for five discs. One student responded thirty, but the other two saw that it would be thirty-one.

Table 5.2: Minimum numbers of moves for $n$ discs

<table>
<thead>
<tr>
<th>$n$</th>
<th># Moves</th>
<th>Twice # Moves</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
<td>6</td>
</tr>
<tr>
<td>3</td>
<td>7</td>
<td>14</td>
</tr>
<tr>
<td>4</td>
<td>15</td>
<td>30</td>
</tr>
</tbody>
</table>

Table 5.3: Minimum numbers of moves for $n$ discs and twice the minimum

When I did this activity with the first group of students, they found the pattern on their own. Possible reasons for this difference include age, education, interest, and the number of students present. As mentioned previously, the second group consisted of children who were on average eleven months younger than the first group of children,
and one grade below them. Additionally, the second group of children are often in foster care, and therefore may have had less consistent schooling. Moving from one school to another may cause gaps in their education. The first group of children all signed up for a class in the “Math, Science, and Technology” category, while other children were there for games and education in general. Finally, there were only three students present at the second group this day, while in the first class, there were twelve.

In each case, students were able to calculate the minimum number of moves for $n$ discs based on the minimum for $n - 1$ discs. They did not see that the minimum was $2^n - 1$, likely because they were mostly or completely unfamiliar with the concept of exponents. This problem taught pattern recognition and the effects of repeatedly applying the same function.

5.1.2 Handshakes

In both classes, we calculated the number of handshakes that would occur if everyone shook everyone else’s hand. For both groups, someone calculated the answer for $n$ people as $n(n - 1)$. The second class was small enough that we were able to actually test this. With four people, we saw that there were only six handshakes, which did not match the hypothesis of twelve. We then discussed why there was a discrepancy, and realized there was overcounting. One girl likened it to the number of girls in her family. She had three sisters, and each sister had three sisters, but there were not twelve girls total. To find the number of handshakes for five people, the students added four, since the four people from before each needed to shake the hand of the new person. Thus, the students never had a formula for the number of handshakes, as they would have if they had recalculated multiplicatively. The students instead
used recursion.

In the first class, we had twelve students, many of whom initially calculated the answer to be 132. We did not actually count handshakes in this class, but I told those students that their answers were a bit too high, and that they should think about why that would be. Most were able to then reduce their answers by a factor of two, though some had help from a classmate seated near them. Because these students were working with higher numbers to begin with, many of them generated a formula for the number of handshakes. Some students justified their answers additively, while others explained them multiplicatively.

This activity helped solidify the connection between addition and multiplication, because students saw that both methods of reasoning could be used to find an answer. It also showed that in this case, one method was advantageous. The multiplicative rule yielded a direct answer, while the additive rule relied upon recursion. Depending on the number of students, this topic may also allow students to practice their multi-digit multiplication. Finally, a table can be very helpful in this case, so students can practice data collection and analysis skills.

5.1.3 Dice Games

The students also played two dice games, Pig and Hog, which many of the children liked. Neller and Presser discussed optimal strategies in their 2005 paper (Neller & Presser, 2005). These games have been used with middle school, high school, and college students for teaching probability and computer programming, as discussed by Neller et al. (2006). In Pig, a player’s turn consists of rolling a six-sided die as many times as he or she likes, provided a one does not appear. The score gained each round is zero, if a 1 is ever rolled, and the sum of the numbers appearing, if no 1 is rolled.
For example, if Sam rolled a two, a three, a five, and then a one, she would earn zero points that round. If she had stopped at the five, however, she would have earned ten points. The optimal choice to maximize the score is to keep rolling until at or above twenty points in a round. To calculate a strategy for maximizing their scores, the students were introduced to the concepts of expected value and multiplying fractions by integers.

Before learning the optimal strategy\(^2\), we discussed the children’s strategies. These included only rolling once, rolling twice, rolling until you beat the other player’s score, and to keep rolling until you get a one!

Many of the children in the second class used this last strategy, even though it prevents them from accumulating any points. They were all curious to see how many points they could acquire in a given round before losing them. This strategy was adopted by one or two children in the first class, as well. It is unclear whether the students did not completely understand the rules of the game or simply wanted to play the game their own way. Many students did not have fixed strategies; they simply rolled until they decided they had acquired enough points, with that limit varying depending on their mood.

Hog is similar to Pig, except that players choose the number of dice to roll and roll them all at once. For this game, I distributed ten dice to each pair. Many of the children in the second class split them up into two groups of five, and then each child would roll his or her own set of five dice. This eliminated the strategy-finding that is a major component of the game, so I encouraged students to try rolling different numbers of dice.

In this game, the best strategy is to roll either five or six dice. Before being

\(^2\)The strategy is optimal for maximizing one’s score, but a more complicated strategy is needed to maximize the chance of winning. Such a strategy would need to take into account each player’s score.
told the strategy, children developed their own strategies. Most were simply, “Roll \( n \) dice” for various values of \( n \). One student from the first class, Tom, rolled only dice that showed a one on the previous roll, explaining that it was unlikely for any given die to show a one twice in a row. Thinking that past rolls affect future rolls is a relatively common belief, one that is erroneous as long as the dice are assumed to be fair. Surprisingly, Tom was the first student in that class whose score surpassed one hundred!

The first group found adding the numbers shown on the dice to be a fairly trivial task, but it was not so for the second group. For them, adding several numbers at a time was still a chore, and it was interesting to see how they dealt with this. Some students used the “make ten” strategy, where they moved dice around and tried to form groups of ten whenever possible. For example, when rolling a three, a four, two fives, and a six, one might place the four and the six in one group, the fives in another, and the three by itself to get the sum 23. Another popular strategy used multiplication; one could group all of the like dice together and then add the resulting products. For instance, two fours and three fives would be \( 8 + 15 = 23 \). A third strategy did not take advantage of the fact that dice are movable objects; it was to simply add the numbers left to right, as one might do on paper. Several students who used this strategy counted on their fingers. This strategy was likely used by children who were not as comfortable with mathematical properties like the commutative property or as flexible in their thinking.

Probability was a major factor in both Pig and Hog. In Pig, each roll yielded a \( \frac{1}{6} \) chance of immediately losing all points accumulated that round. In Hog, the probability was at least \( \frac{1}{6} \) of scoring zero points that round, and the probability increased with each added die. The optimal strategy for Hog was beyond the scope of the class, but many good strategies were developed and discussed.
5.1.4 “Guess My Rule”

Another activity was a game I called “Guess My Rule.” In this game, one person chooses a rule such that some numbers follow the rule and others do not. Other students then guess numbers, usually integers, and are told whether each number follows the rule. In the first class, my first example was prime numbers and my next was square numbers. The first example, students found fairly quickly, though not all of them were familiar with the word “prime.” Fewer students were familiar with square numbers, but they eventually guessed that the rule was that all of my numbers were products of a number with itself. With the second group of children, who were not as mathematically sophisticated, I felt that these rules would be too difficult. Therefore, my rules were even numbers, which was guessed fairly quickly, and numbers containing the digit 3, which took a little more time.

The students then created their own rules. Some of the first class’s rules included factors of 100, positive numbers, must contain the digit 1, multiples of 3, and must consist of an even digit and an odd digit. Many of these rules were very creative. The rules the second group of children developed, however, focused entirely on multiples. They had multiples of 3, of 5, and of 10, and of 2. One girl pointed out that we had already done multiples of two when we did even numbers, and the class eventually agreed that those two rules were the same. I believe the students may have been focused on multiples because many of them had just come from third grade, which places a large emphasis on the multiplication table.

The “Guess My Rule” game teaches students about categorization and the many ways to divide numbers into two groups. It reinforces types of numbers, such as primes, squares, factors, and multiples, and it encourages children to think of new and unique rules. This type of thinking may be what Willoughby (1968) had in mind.
when he wrote that children should be taught to be creative. In fact, the game can also be adapted to objects other than numbers, such as shapes or equations. It could even be used in classes other than math class.

5.1.5 Penny Game

Both groups also played the penny game, where students take 1, 2, or 3 pennies from a pot each turn and try to take the last penny. I chose to modify the presentation of the game slightly for the second group. The first group started with twelve pennies and took some time to discover that it was desirable to reduce the pot to four pennies. I then suggested that they split the twelve penny game into the eight penny game and the four penny game, and they saw that they could further divide the eight penny game into four and four. Essentially, they played three different games, with the second player doing the opposite of the first player each time, keeping the total number of pennies taken in each round at four. We then looked at games where we did not start with multiples of four, and we saw that the first player could win.

In the second group, I decided to start by showing that the second player could always win the four penny game. To do so, I acted it out with an assistant. We played several games before the children decided that the first player could never win. I then had them play the five penny game, and after some experimentation, some of the students discovered that the first player could always win. We went to the six penny game next, although some students jumped straight to the twelve penny game. Once the number of pennies grew large enough, most students abandoned strategies and either chose randomly or always chose three, until the pot was reduced to a previously solved game.

Two groups in the second class of children decided to combine their pennies and
play a four-player game. In this game, players can work together to ensure that others have no chance to win. For instance, when five pennies are left, the first player to go cannot win. However, he or she can decide who can win. If the first player takes one penny, the third player can win, but if the first player takes two or three, the second player can win. Therefore, it became a strategy game where some children were able to decide the winners.

Although it was not explicitly stated in class, the penny game taught modular arithmetic. In the original, two-player game, the second player can always win games where the number of pennies is divisible by four, while the first player can win all other games. In other words, the first player wins when there are $1, 2, \text{ or } 3 \pmod{4}$ pennies, and the second player wins when there are $0 \pmod{4}$ pennies. In general, if players can take $1, 2, 3, \ldots, \text{ or } n$ pennies, the second player wins when there are $0 \pmod{n+1}$ pennies and the first player wins otherwise, assuming optimal play.

### 5.1.6 Domino Game and Tiling

The domino game is played on a grid consisting of one row of squares. The children in the first class started with the ten-square game. Two players take turns placing dominoes, which take two squares each, on the grid in such a way that they fit into the squares. Dominoes are not allowed to hang off of the grid, and they cannot be stacked. The last person who can place a domino wins.

![Figure 5.1: A sample completed domino game, won by the second player](image)

In this case, two girls in the first class discovered the optimal strategy within just
a few minutes of learning the game. They found that if the first player placed her domino in the center of the board, she could simply mirror the second player’s moves. Not all of the students looked for a strategy, so they were told that there was an optimal move for the first player, and that they should try to find it. With that hint, several other pairs of students were able to find the strategy, though none articulated it quite as well as the first two.

Due to time constraints and differences in age and mathematical maturity, I chose to modify the dominoes game slightly for the second class. Rather than beginning with a grid of ten squares, the second group of children began with six. I acted out several games with an assistant to make sure they understood the rules, and I then let the children try the game on their own. Some were able to master the strategy for six squares relatively quickly, while others did not until their partners explained it to them. I then had them expand it to the ten square game. One girl immediately generalized her strategy to the ten square game, and then to the sixteen square game. Most, however, moved arbitrarily until near the end.

This game teaches the importance of symmetry. In a game with an even number of squares, the first player can always win by moving in the middle first and then mirroring the second player’s moves. However, this parity is broken when the board has an odd number of squares. One method of adjusting the game so the first player always wins involves not requiring dominoes to fit in the squares. For instance, the first player could put his domino in the exact middle of a seven-square board, leaving a space on either side in which other dominoes can be placed, as shown in Figure 5.2. This modification introduces children to the differences between continuous and discrete distributions.

I then had students look at a six by six checkerboard pattern with two light squares in opposite corners removed, as shown in Figure 5.3. They attempted to tile
the board with dominoes and found that each time, they were left with at least two
dark squares that they could not tile. Once every group had discovered that, I asked
them what each domino covered. They told me it covered two squares, and with some
more prompting, they added that each domino always covered one light square and
one dark square. I then had them count the numbers of each type of square, and they
found sixteen light squares and eighteen dark squares. One student then explained
to the rest that that meant there would always be at least two dark squares without
“partners.” This took slightly more prompting in the second class than it did in the
first class.

This activity also reinforced the importance of parity and the concept of a one-
to-one matching. Additionally, it indirectly taught children about the importance of
proof. A single example would be sufficient to prove that it was possible to tile the
modified board, but an explanation was required to prove its impossibility.

5.1.7 Puzzles

I also gave both groups of students some puzzles, such as the following:

You have some carrots, a rabbit, and a wolf, and you want to get them all across a river. The boat will only hold you and one other item or animal. If left alone, the rabbit will eat the carrots and the wolf will eat the rabbit. How can you get them all across?

We acted this out, and the students in the first class quickly realized that the rabbit must go over first. The second group of children tried all three combinations, some several times, before I pointed out that whenever they sent over anything except the rabbit, something was eaten. Once the person acting out the part of the protagonist returned to the first side of the river, the students were split as to which to take over next. In fact, due to symmetry, the choice does not matter. However, that is a rather sophisticated notion, and the students did not realize it.

If the children chose to send the carrots over next, they saw that the protagonist could not leave both the carrots and the rabbit on the far side of the river. Some of them decided that meant they had made the wrong decision, but others persisted, noting that the rabbit could be taken back over to the first side of the river, thus preserving the carrots. From there, the problem is relatively straightforward. The rabbit could be left and the wolf taken over, where it could be left on the far side with the carrots. Finally, the protagonist could return, take the rabbit, and make a final trip to the far side of the river.

This problem requires students to think creatively. It would be most efficient for the protagonist to make five trips total, with three of them being to the far
side while carrying an item or animal, and two of them being empty-handed return trips. However, it is impossible to do this without something being eaten. Therefore, students must think of the idea of bringing an item or animal back to the near side of the river. This sort of thinking is not listed as a standard, but it does fit with four of the eight mathematical practices listed for each grade level. The practices are

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning (Common Core State Standards Initiative, 2010, p. 6),

and they will be discussed in detail in Chapter 7. This problem addresses practices 1, 2, 3, and 5, where the appropriate tools in this case would be manipulatives or pictures.

In addition, both sets of students were introduced to the flashlight problem, which addresses the same practices:

Four children want to cross a bridge, but the bridge can only hold two people at a time. It is dark, and they have only one flashlight, which they need in order to cross the bridge. Alice takes one minute to cross, Bob takes two, Charles takes five, and Diane takes ten minutes. The faster child will slow down to the speed of the slower child while making the trip. How quickly can they all cross to the other side?
Several students thought of original and unorthodox solutions, such as throwing or rolling the flashlight back, but I explained that if the fastest runner still took one minute to cross, it would be impossible to get the flashlight across the bridge without someone actively moving it. Then groups came up with solutions like the following:

Charles and Diane cross, taking ten minutes. Charles goes back, which takes another five minutes. Then Alice and Bob run over, in two minutes, so it’s been seventeen. Then Alice runs back, for eighteen, and Alice and Charles cross, adding another five minutes. It takes twenty-three minutes.

Students were able to improve upon it by having Alice do all the running:

Alice and Diane cross, which takes ten minutes. Alice runs back, so that’s eleven. Alice and Charles cross the bridge, adding another five minutes. It’s now been sixteen minutes, and Alice runs back to make seventeen. Then Alice and Bob run over, and it’s been nineteen minutes.

Most agreed that since Alice had done all of the running and she was the fastest, this could not be improved. However, the optimal solution takes only seventeen. It is:

Alice and Bob take two minutes to cross, and then Alice runs back, for a total of three minutes. Charles and Diane cross, adding ten minutes to the trip, making it thirteen minutes long. Bob runs back, so it’s been fifteen minutes, and then he and Alice cross, taking another two minutes. They have all crossed in seventeen minutes.

In this, students were introduced to the idea of optimizing, something not introduced in the Common Core Standards until high school.
5.2 Writing Time

We spent eight to fifteen minutes each day writing, depending upon the length of the journal prompt. Each day, students were very concerned about writing the “right” thing, as if there were only one correct answer to these questions. Many students wanted to tell me their answers before they wrote them down. Only one student returned her consent form that allowed me to use her written answers, so it was fortunate that many students wanted to share verbally as well as write.

On the first day, I asked students to describe someone who was good at math and explain how they knew he or she was good. I also asked if it was possible to become better at math, and if so, how. Alejandra\(^1\) gave the name of a boy in her class, explaining that he was good “because when we do math probloms [sic] he gets them right.” Many of these students expressed the opinion that mathematics consists solely of arithmetic problems. This was reinforced when Alejandra explained that the only way to get better at math was to do math problems during the summer or after school. She noted verbally that she probably would not practice outside of school.

The next day, I asked whether the children liked math, and I requested that they explain why or why not. I then asked those who liked math to try to think of something they disliked about math, and those who disliked math to try to think of something mathematical that they enjoyed. Alejandra said she did not like math because it was boring, and was unable to think of anything about math that she liked. Many of the students said that they did enjoy math, but did not verbally explain their reasons for this.

The third day, I asked students to explain mathematics to a new student who had never heard of mathematics before. They found this very confusing, because they

\(^{1}\text{Not her real name.}\)
did not know anyone like this. It took some explaining before they understood that they were supposed to pretend. Some students spoke about using counting or age to explain math, while others said they would start with simple addition and subtraction problems. Alejandra was not present this day. In fact, attendance was so poor that no student was present for all four days.

The last day’s prompt was rather long. I asked what students did in math class last year, and Alejandra responded that she did “+ − × ÷” and, after some prompting, she added “shapes time.” It’s unclear what she meant by that last comment; she may have been learning the names of different shapes or playing with pattern blocks. She was unable to explain how she learned these things. Other students explained simply that the teacher taught them. I also asked students what we did in this summer class, and Alejandra said we “played games.” When asked whether the class this summer was math or something else, she responded that it was a “little of both.” It was similar to math class in that in both cases, she did some math problems and some games. This seems to contradict her previous statement, that math was boring, unless she felt that the games played in her traditional math class were boring games. She did not explain how it was different from her math class.

5.3 Results

While the first group of children was older and perhaps more mathematically sophisticated, both groups claimed to enjoy the classes, and both worked on a variety of mathematical skills. The second group of children did not always work on the same skills as the first group, but they were still learning mathematics. In fact, these activities may be especially appropriate for mixed-ability classrooms due to the fact that there are many different things that can be learned from them. Table 5.5 shows some
<table>
<thead>
<tr>
<th>Day</th>
<th>Prompt</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Describe someone who is good at math. Explain how you know he or she is good. Is it possible to become better at math, and if so, how?</td>
</tr>
<tr>
<td>2</td>
<td>Do you like math? If so, why? Is there any part you dislike? If not, why not? Is there any part you like?</td>
</tr>
<tr>
<td>3</td>
<td>Imagine there’s a new kid at school who has never had a math class before, or even heard of math. How might you explain math to him or her? Remember, this student might not know words like “addition” or “subtraction,” so you’ll have to use non-math words.</td>
</tr>
<tr>
<td>4</td>
<td>What did you do in math class last year? Try to give topics, like “long division”, and methods, like “we did lots of practice problems”. What have you done in here this summer? Again, try to give both topics and methods. Is what we’re doing math or something else? How is it like math class and how is it different?</td>
</tr>
</tbody>
</table>

Table 5.4: Writing Prompts

<table>
<thead>
<tr>
<th>Activity</th>
<th>First Class</th>
<th>Second Class</th>
</tr>
</thead>
<tbody>
<tr>
<td>Towers of Hanoi</td>
<td>patterns and iteration</td>
<td>patterns and iteration</td>
</tr>
<tr>
<td>Handshakes</td>
<td>primarily multiplication, some addition</td>
<td>primarily addition, some multiplication</td>
</tr>
<tr>
<td>Hog</td>
<td>probability and strategy</td>
<td>addition strategies, some probability</td>
</tr>
<tr>
<td>Guess My Rule</td>
<td>primes, factors, squares, multiples</td>
<td>multiples</td>
</tr>
<tr>
<td>Penny Game</td>
<td>large games broken into smaller games</td>
<td>small games</td>
</tr>
<tr>
<td>Domino Game</td>
<td>parity and strategy-stealing</td>
<td>some parity and strategy-stealing</td>
</tr>
</tbody>
</table>

Table 5.5: Skills Learned by Children

of what the students seemed to work on most.
Chapter 6

Themes

As expected, different children appreciated different things. I now discuss two major themes and four minor themes observed during the classes. These themes were chosen by analyzing the interview transcripts, the journals, and my own observations, which were written in my journal.

6.1 Novelty

Many of the children appreciated the novelty of the games and activities. This is not surprising, as Ames (1992) noted that “students are more likely to approach and engage in learning in a manner consistent with a mastery goal when they perceive meaningful reasons for engaging in an activity; that is, when they are focused on... gaining new skills” (p. 263). Since there were no exams in this class, it makes sense that students would be trying for mastery, rather than just performance.

To keep students interested, one could introduce new situations and contexts (Krapp et al., 1992). For example, many teachers use pie or pizza to teach fractions. It may be helpful to use other items, such as distance, age, or volume. Similarly, after ex-
plaining different ways to find irrational numbers, one might ask students to each try to come up with an irrational number so unusual that nobody else in the class has the same number. Justifying the irrationality of this number could be an interesting exercise. This process could encourage students to think in new ways.

Sternberg and Gardner (1994) wrote that if an experience was too novel, students may be unable to make connections between previous work and current work. For instance, if students who were just learning “skip counting” were suddenly introduced to exponents, most could not understand the process, since they had not yet mastered multiplication. Conversely, if an experience has too little novelty, it will not be helpful in evaluating students’ intelligence.

Tia stated that she liked that “you get to learn new things” in math, but not everyone enjoys novelty. Joe told the interviewer that while he liked a challenge, he did not enjoy learning new things. He explained, “I don’t like learning new things because then you have to remember that and then remember everything else you learned.” Tom, on the other hand, disliked a lack of novelty. He said, “I hate doing math problems sometimes, because it gets boring after a while. It’s like, I know how to do this.”

However, during a discussion of perimeter and area, Tom began crying because he said his teacher taught it differently. While he liked math, perhaps he was unhappy with both a lack of novelty and too much novelty. It may be that some students may need to be eased into new approaches. For instance, rather than simply saying, “Instead of ‘borrowing’, we will use the word ‘regrouping’,” one might explain why the term “borrowing” is misleading and how “regrouping” is more accurate.
6.2 Cooperative, Competitive, and Solitary Play

Many children liked working with others, but a few seemed to prefer working alone. This may be in part because when playing a game, an uncooperative opponent may do something unexpected, forcing a child to change his or her strategy. While this helped students learn optimal strategies, it upset a few children who wanted the other player to behave a certain way.

Students were also asked to work together on things besides games, such as the base five activity from page 33 of Section 4.2.2 or when they were trying to tile a checkerboard missing two opposite corner squares with dominoes. They seemed to enjoy this, perhaps because they were working cooperatively instead of competitively. Others, like Tom, were more competitive, preferring games like Pig (page 62) and the domino game (page 67). Additionally, in the four player penny game, some of the students realized that, while they could not win, they could determine who would win.

It would be interesting to examine these behaviors in the future and attempt to determine which students tend to prefer cooperation and which prefer competition. Inkpen et al. (1995) found that when two children shared a computer to play a puzzle-solving computer game, they did significantly better than when the children played alone, and that this tendency was especially strong for girls. They also found that motivation was mediated by whether the children had partners, with children playing alone being more likely to end the game before the thirty-minute time limit. Again, the difference was particularly noticeable with girls, who left early 21.3% of the time when playing alone and 4.3% of the time when playing together on a single computer.
6.3 Structure

Some students preferred playing games with set rules, such as Pig and Hog, described in Section 5.1.3 on page 62, or the domino game, discussed in Section 4.2.4 on page 67. For instance, Tom’s favorite activity of the first four days of the week was Pig. With Pig and with some of the other games, the children were allowed certain moves or options, and they just had to choose between those possibilities. Other students enjoyed more unstructured work, sometimes even creating their own rules or modifications to a game. For instance, on July 28, two groups playing the penny game, described in Section 4.5 on page 53, decided to work together and create a four player game\(^1\).

Hofferth and Sandberg (2001) noted that unstructured free play could be quite important for children, including “playing cards, board games, and puzzles” (p. 296). However, they wrote that many children might be involved in so many structured activities that they had little time for unstructured play. Additionally, they found that “free play time at home was lower when mothers were employed than when they were not” (p. 303). This may indicate that it is particularly important to include some time for unstructured games in the school setting, so everyone has a chance to participate in unstructured play.

6.4 Games Are/Are Not Math

Two students explained in their interviews that games were not “actual math.” Joe noted that there was “more like a focus on math in fifth grade,” as opposed to in the summer class. Tom, on the other hand, said that his favorite part of math was playing strategy games, indicating that he believed that games could be mathematical.

\(^1\)Institutional rules prohibit further description of these four students.
Determining whether games are part of mathematics is one part of finding out what students think about the definition of mathematics in general. Unfortunately, most of the children interviewed were unable to define mathematics very fully, and indeed, many adults may have difficulty constructing coherent definitions.

### 6.5 Math Is Only/Is More Than Just Numbers

In their interviews, Tia, Sam, and Tom said that math did not have to include numbers, though Sam had previously said that math was “pretty much based on numbers” and could not provide an example of math that did not use numbers. Tia suggested that with “the radius and the chords and stuff, you kind of have to use numbers, but not completely use numbers.” In other words, she noted that numbers were often associated with geometric figures, but were not necessarily needed.

Jen felt that math required numbers, explaining that “I think it has to, because you do, like, adding and subtracting and stuff, and so, you have to have numbers to do that, otherwise it would be pretty much impossible to do it.” She may not have had as much experience with geometry as some of the other students, or she might not consider that “real math”.

### 6.6 When Are Answers Right?

Another interesting subject of study is when students feel their work is correct and complete. For instance, when doing arithmetic in base five, the students did not feel their answers were complete until they had converted them back to base ten. Some students may feel that their answers are right as soon as they are done, while others wait until they have checked their work. Still others are even more cautious, not
feeling that they are done until the back of the book or their teachers verify their work. It is important that students find an appropriate balance. One would not want students to not bother to check their work at all, but at the same time, one does not want them to become overly dependent on others. When appropriate, they should have confidence in their own answers.
Chapter 7

Common Core Standards

The Common Core State Standards Initiative is a recent movement to develop standards for mathematics and English that can be used by various states. Indeed, these standards have already been adopted by a large number of states, as shown at http://www.corestandards.org/in-the-states in map form. States adopting these standards earn points toward their applications for Race to the Top, a recent federally funded program that is “a competitive grant program to encourage and reward States that are implementing significant reforms in the four education areas described in the ARRA [American Recovery and Reinvestment Act of 2009]: enhancing standards and assessments, improving the collection and use of data, increasing teacher effectiveness and achieving equity in teacher distribution, and turning around struggling schools” (U.S. Department of Education, 2010, p. 3).

The Common Core Standards are intended to ensure that every student in the United States is educated well and prepared for college or for a career. These standards may also help college admissions personnel, who could be assured that students from various states would have some similarities in their educational backgrounds. They would also be advantageous to people who move frequently and whose children
must change schools often.

7.1 Standards for Mathematical Practice

In the Common Core document, the authors list eight standards for mathematical practice:

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning (Common Core State Standards Initiative, 2010).

Despite the classes lasting for only one week, many of these standards were addressed. The following pages will discuss how these standards were met in the summer courses.

7.1.1 Make sense of problems and persevere in solving them

The Common Core Standards state that students should “consider analogous problems, and try special cases and simpler forms of the original problem in order to gain insight into its solution” (Common Core State Standards Initiative, 2010, p. 6). This is exactly what students did when trying to solve many of the problems and games.
For instance, the penny game, described on page 53, initially started with either 20 or 12 pennies, depending upon the class. Many groups began by working with smaller cases to better understand the problem. For instance, it is easy to show that the 4 penny game can always be won by the second player, and that the first player can always win the 3 penny game. This technique also worked well with the domino game from page 67.

7.1.2 Reason abstractly and quantitatively

According to the Common Core Standards, “quantitative reasoning entails... attending to the meaning of quantities, not just how to compute them” (Common Core State Standards Initiative, 2010, p. 6). Students did this in one of the very first activities in each class, when they calculated how many handshakes would take place if all students shook hands with every other classmate. They had to understand that Alice shaking hands with Bob and Bob shaking hands with Alice counted as only one handshake, meaning that many naïve calculations needed to be halved. This led to the discussion of the difference between handshakes and bows. While both were signs of respect, there would be half as many handshakes as bows, because two people form a single handshake, while two people give two bows.

7.1.3 Construct viable arguments and critique the reasoning of others

In Section 4.2.3, the children learned about modular arithmetic by considering a train whose cars alternated in color. The first car was red, the second was blue, the third red, and so on. The students were quickly able to determine the color of any train car, including very large numbers, by noting whether the number was even or odd.
Several adaptations were considered, including a train where there were six colors: red, orange, yellow, green, blue, and purple. When I asked students to describe all blue cars, one student noted that the fifth car would be blue, and he generalized it to say that every multiple of five would be blue. A classmate argued with his reasoning, pointing out the specific example that car number ten was green, not blue. In other words, they understood that the discovery of a single counterexample was enough to discredit his hypothesis. Working together, the children discovered the true rule, which was that blue cars were cars with numbers that were one less than a multiple of six.

7.1.4 Model with mathematics

In the first class, students played what I called the $+/−$ game, described in Section 4.2.8. Students were given three numbers, an addition sign, and a subtraction sign, and they were asked to maximize the result. The students quickly deduced the correct answers to given problems, and then we turned to trying to generalize the problem. They noted that it is always best to add the two largest numbers, in either order, and subtract the smallest number. We first wrote this as:

\[
\text{big number} + \text{middle number} - \text{small number}
\]

or

\[
\text{middle number} + \text{big number} - \text{small number}.
\]

We then decided that it would be clear to represent the big number with a $b$, the middle number with an $m$, and the small number with an $s$, resulting in

\[
b + m - s \quad \text{or} \quad m + b - s.
\]
While this is a simple model, it may have been the first time some of these children had seen variables used. The students were later able to adapt this to the $+\times$ game, where they were given three numbers, an addition sign, and a multiplication sign.

### 7.1.5 Use appropriate tools strategically

Because of the constraints of the classroom and cafeteria where the programs were held, calculators and computers were not used. However, I did build some concrete models for the Towers of Hanoi, described in Section 5.1.1. The models each consisted of three pegs attached to a platform and four differently-sized discs with holes in the middle, so they fit on the pegs. As discussed earlier, the rules were that larger discs could not go on top of smaller discs, and the discs must all be moved, one disc at a time, from the leftmost peg to the rightmost peg. This is trivial with one peg and simple with two, and both of these solutions can be found fairly easily without using the model. Three and four pegs prove to be more difficult, however, and students readily took advantage of the models available.

### 7.1.6 Attend to precision

Because “mathematically proficient students try to communicate precisely to others,” (Common Core State Standards Initiative, 2010, p. 7) we discussed the importance of making sure that definitions are shared, or at least explained thoroughly. To illustrate this, in the first class, we examined the mathematical meaning of the word “star.” All students agreed that the traditional five-pointed star was a star, and that the six-pointed star was, as well. Precise definitions of stars ranged widely, however, with some students feeling that anything with a “sharp point” was a star (including the examples in Figure 4.5) and others feeling that stars were more limited. The
students seemed to understand that since different people had different ideas about what a star was, it was important to make clear their definitions when discussing different concepts.

7.1.7 Look for and make use of structure

According to the Common Core Standards, “Mathematically proficient students look closely to discern a pattern or structure” (Common Core State Standards Initiative, 2010, p. 8). When learning about base five, the students noticed several patterns. They quickly realized that the digits 5 through 9 never occurred in base five numbers. By making the comparison to United States currency, as shown in Section 4.2.2, the children rapidly learned how to add and subtract base five numbers.

7.1.8 Look for and express regularity in repeated reasoning

Repeated reasoning aided children working with the Towers of Hanoi, discussed in Section 5.1.1. Once students had discovered the solution to the problem with \( n - 1 \) discs, they could find the solution for \( n \) discs simply by moving the top \( n - 1 \) discs to the second peg, moving the \( n \)th disc to the last peg, and then transferring the top \( n - 1 \) discs to the last peg. The penny game from Section 4.5 also used repeated reasoning, as the solution to the \( n \) penny game could be obtained from the solution to the \( n - 4 \) penny game.
Chapter 8

Conclusions

For this dissertation, I set out to study students’ attitudes towards mathematics, both math taught in schools and recreational math. Previously, some research had been done about attitudes toward school math, and there was some research on nonroutine mathematics, but nothing could be found about the intersection of those two areas. I found that even students who claimed they did not like math enjoyed the mathematical games and puzzles. This could lead to an avenue for reaching students who do not enjoy, or who are even fearful of, the traditional mathematics taught in schools.

I also examined students’ definitions of mathematics, focusing in particular on whether the nonroutine activities and games used in the two classes were considered mathematics. I found that, while some students did consider the games to be mathematics, most did not. This could be beneficial, because it means that students’ preconceptions about mathematics would not hold for these games and activities. This may allow students who report not liking mathematics to be more open-minded towards the mathematical games and problems. If, after appreciating the games and activities, students are convinced that they are a part of mathematics, it is possible
that the students will then be more open to future mathematics in schools.

Upper elementary school children were chosen for a variety of reasons. They were between lower elementary school children, who generally report enjoying mathematics, and older children, who often do not (Haladyna & Thomas, 1979). They also are approximately the age that Kenschaft (1997) said needed to work on “interesting problems” (p. 161). Finally, Feldhaus (2010) found that preservice elementary teachers’ mathematical dispositions were formed in mid to late elementary school; it seems likely that that would be the case for others, as well. In both classes, there were some who said they liked mathematics and others who said they did not, so a variety of children were exposed to the nonroutine problems and games. The suitability of most of the games and activities to a variety of age groups means that future work can be done with a wider age range.

Students in both programs appeared to greatly enjoy the games and activities in the classes, as evidenced by observations and their journals and interviews. In fact, all five interviewees were asked what their least favorite part of the class was, and only one was able to think of an answer. He said he disliked coloring, which they did when learning about the Four Color Theorem. The fact that four of the students had no negative comments may indicate that the students found the material intrinsically motivating; they were working for their own reasons, rather than for a grade.

The themes encountered in the interviews, journals, and observations showed that students had different viewpoints. For instance, while one student explained that she liked math because she enjoys exploring new material, another noted that he did not enjoy learning new things, because it was hard to remember everything. A third indicated that he did not like doing activities lacking in novelty, because they were boring. Clearly, children are individuals, and they express these differences early in their lives.
Because one student said he disliked a lack of novelty but then cried when faced with an activity he felt was too new, it seems that there may be a spectrum of novelty, or perhaps that novelty can be tempered by connections to older material. It is also possible that there were other factors at work, and that this same child might have enjoyed the activity had he been in a more positive frame of mind from the start.

When asked about their favorite activities of the week, some students listed individual activities, while others preferred multi-player games. This may be because some students prefer working together, while others would rather work alone. Additionally, there were differences between competitive activities, such as Pig and Hog from Section 5.1.3, and cooperative activities, such as attempting to tile a checkerboard missing two opposite corners with dominoes.

Some children preferred structured games, such as Pig and Hog, while others preferred more open-ended activities, or even to create their own rules. Hofferth and Sandberg (2001) have shown that it is important for children to spend time working with activities with less structure, though that is certainly not to say that structured activities are not important.

Three students reported that the games and other activities done in the summer classes were not “actual math,” while another said that strategy games were his favorite part of math. It is unclear why some students viewed games as mathematics and others did not. On a related topic, three students said that mathematics did not necessarily explicitly require numbers, with some citing geometry as an area where numbers need not always be used. One girl felt that mathematics did require numbers, noting that without numbers, one could neither add nor subtract. She may have seen mathematics as consisting of little more than more advanced versions of addition and subtraction, or it is possible that she interpreted the question differently than some of the others.
Finally, students showed interesting beliefs about when mathematics problems were finished. For example, students quickly became proficient at working with base five addition and subtraction problems, including problems with three or more addends. However, none of them were content to leave the answer in base five; they all converted their answers to base ten, even when the problems were given in base five. It appeared that the students believed answers were only correct when given in base ten.

When the five students from the first program were interviewed, they were asked about their experiences and given three “think-aloud” problems, time permitting. For these think-aloud problems, students were asked to say everything they thought while they worked, in an attempt to determine the students’ thought processes. While there was not enough information to analyze whether these thought processes were affected by the class, I was able to determine that the students were able to present their thoughts relatively clearly. Therefore, I believe further study with think-aloud problems would be beneficial.

Two of these three think-aloud problems resembled problems done in class, but one of these resemblances was only superficial. In class, we calculated the number of handshakes if everyone shook hands with everyone else, and for the think-aloud problem, students were asked to calculate the number of handshakes if five people on one team shook hands with five people on another team. The latter problem is significantly simpler. None of the children mistakenly attempted to apply the calculation from the in-class problem to the think-aloud problem, despite the superficial similarity.

The Common Core State Standards Initiative lists many specific content standards for kindergarten through high school in both mathematics and English. It has been adopted by over forty states and the District of Columbia. There are also eight
standards for mathematical practice, explained in Chapter 7. All eight of these standards were addressed in the classes, indicating a large degree of concordance between these activities and the Common Core Standards. The Common Core, in turn, has opinions in line with the recommendations of the National Council of Teachers of Mathematics and the National Research Council.

For this dissertation, I worked with two groups of children, and those groups differed in many areas. The first group was older and showed evidence of higher socioeconomic status. The class was held in a much nicer facility, with recess and a snack built into the three hour class. The second class was shorter and took place in a cafeteria. In both cases, however, the students enjoyed the games and activities. Indeed, several students who were too young to qualify for the second group tried to join anyway, because the students who were participating were demonstrating such enthusiasm.

This work has led to a possible partial solution to the problem of finding ways to interest students in mathematics. Because such games can teach a wide variety of mathematical skills and concepts, they may be used to help motivate students who previously would never have learned to enjoy mathematics.

These findings have several implications for research. It seems that further research is needed to discover what students can learn from these games and whether children’s interest in these games and the associated mathematical concepts is retained over time. Additionally, research should be done that looks at many different age groups. Because so many of the games used this summer are appropriate for a wide variety of ages, it should be relatively straightforward to extend this research to students of other ages, from the very young to adults.

The research is not yet conclusive enough to give definite instructions for teaching, but it may be that teachers should incorporate mathematical games into the
classroom. If so, teachers may need to be properly trained on the use of these games. Having students simply play the games may teach some of them some concepts, but skillful introductions and guidance could greatly increase the effectiveness of the games. This is yet another area where research is needed; it seems likely that both too little and too much guidance could be problematic. To test this, students could be given games with varying amount of guidance, and their responses could be observed. This could help determine whether there is an optimal level of guidance, and if so, how important that guidance is. If proper guidance makes a significant difference, teacher education and professional development may be advantageous.
Appendix A

Interview Script

Below is the script given to the interviewer. He was able to interview five students from the first class. Notes in italics were for the interviewer and were not read to the children.

1. What did you do in math class last year? *Try to get topics, like ‘long division’, and methods, like ‘we did lots of practice problems’.*

2. What are you doing in the class this summer? *Again, try to get both topics and methods.*

   a) Is the class math or something else?

      i. *If something else*: What would you call it?

3. How does this class compare to your math class last year? *Try to get similarities and differences.*

4. Imagine there’s a new kid at school who has never had a math class before, or even heard of math. How might you explain math to him or her?
a) *If student says “addition” or some other operation:* What if he/she has never heard of addition before? Could you explain it using non-math words?

5. Do you know someone who is good at math?
   
a) What makes him or her good at math?

6. Are you good at math?
   
a) How do you know?
   
b) What does it mean to be good at math?

7. Do you like math?
   
a) Why (or why not)? Is there any part you dislike (or like)?

8. Why do you think you wrote in your math journals this week?
   
a) Did you like writing in a journal?
   
b) Have you used a journal before in any class?

There were three “think-aloud” problems that students were asked to answer. The interviewer first modeled twelve plus fifteen aloud, so the students would be more comfortable with the concept of a think-aloud problem. Again, notes in italics were for the interviewer and were not read aloud.

1. What is twelve times fifteen?

2. There are five people on the red team and five people on the blue team. If each person shakes hands with everyone on the other team (and nobody on the same team), how many handshakes take place?
a) Note that they have already done a handshake problem, but the methods for solving this problem are very different.

3. There are twelve pennies in the pot, and on each turn, a player may remove one or two pennies from the pot. The last player to remove a penny wins. What is the optimal strategy?

a) This is also similar to a problem posed in class, but players could remove one, two, or three pennies in the game in class.
Appendix B

Journal Prompts

1. Describe someone who is good at math and explain how you know that he or she is good.

2. Is it possible to become better at math? If so, how?

3. Do you like math?
   a) Why (or why not)? Is there any part you dislike (or like)?

4. Imagine there’s a new kid at school who has never had a math class before, or even heard of math. How might you explain math to him or her? Remember, this student might not know words like “addition” or “subtraction,” so you’ll have to use non-math words.

5. What did you do in math class last year? Try to give topics, like ‘long division,’ and methods, like ‘we did lots of practice problems.’

6. What have you done in here this summer? Again, try to give both topics and methods.
   a) Is what we’re doing math or something else?
7. How is it like math class and how is it different?
Appendix C

Informed Consent Forms and Letters to Parents

Included are the letters home to the parents and guardians, as well as informed consent forms that adults and children had to sign. The first letter and forms are labeled with my maiden name, Berg, because those classes took place before the date of my wedding, while the other program took place afterward.
Dear Parents and Guardians,

I am a graduate student in mathematics education at the University of Nebraska-Lincoln. I am currently working on my dissertation, tentatively titled "Students' Attitudes about Routine and Nonsensical Mathematics Problems." This is a research project investigating post-4th and post-5th graders' views about different aspects of mathematics, and I would appreciate your help.

There are two ways for your child to participate in this study. The first takes no additional time. Students will be doing some journal assignments as part of the class, and I would like to photocopy and analyze their work. I would also like to select some students for fifteen to twenty minute interviews. This would be done by a trained interviewer during class. You may choose to have your child participate in just the journal aspect or both the journal and the interview, or not at all. Participating or not participating will in no way affect your child's status in the class or with the program.

Attached are the official University of Nebraska permission slips that give me permission to copy your child's work and to interview your child. If you would like your child to participate, either with the journal and interview or just the journal, please sign the Adult Consent form and send it in with your child. If you return a signed Adult Consent form, your child will then be invited to read and sign the Child Assent form. Both forms must be signed and returned in order to participate. You will receive copies of the forms after they have been returned.

If you have any questions, please contact me at [Redacted], or you can talk to me before or after class.

Thank you very much!

Debbie Berg

Figure C.1: Letter Home for the First Class
Parental Informed Consent Form

You are invited to permit your child to participate in this research study, tentatively titled “Children’s Voices: Students’ Attitudes about Routine and Nonroutine Mathematics Problems.” The following information is provided in order to help you make an informed decision whether or not to allow your child to participate. If you have any questions please do not hesitate to ask.

The purpose of this study is to collect and analyze students’ work on grades 4-5 on nonroutine math problems and determine the students’ attitudes towards both traditional and nontraditional mathematics. Participation in the study will allow your child’s mathematics journals to be photocopied and analyzed. This study will take place in the classroom. If you give additional permission by checking the boxes below, your child may also be selected for a 15-20 minute interview taking place during class. The conversation will be audio recorded for note-taking purposes; the recordings will not be made public.

There are no known risks associated with this research. Your child may find the learning experience enjoyable and educational. The information gained from this study may help us to better understand students’ mathematical attitudes.

Any information obtained during this study which could identify your child will be kept strictly confidential. The data will be stored indefinitely in the principal investigator’s office or on a password-protected laptop and will only be seen by the investigators. The information obtained in this study may be published in a doctoral dissertation or educational journals or may be presented at educational meetings but your child’s identity will be kept strictly confidential.

Your child’s rights as a research subject have been explained to you. If you have any additional questions about the study, please contact Debbie Berg at [redacted] or David Fowler at [redacted]. If you have any questions about your child’s rights as a research participant that have not been answered by the investigator or to report any concerns about the study, you may contact the University of Nebraska-Lincoln Institutional Review Board (UNL IRB), telephone [redacted].

Your child’s participation in this project is voluntary and your child may decide not to participate in the study or to withdraw at any time without adversely affecting their relationship with the investigator, the [redacted] program, or the University of Nebraska-Lincoln. Your decision on whether to allow your child to participate will not affect his or her standing in the course in any way.

DOCUMENTATION OF INFORMED CONSENT
You are voluntarily making a decision whether or not to allow your child to participate in the research study. Your signature certifies that you have decided to allow your child to participate and his or her work to be photocopied and analyzed, having read and understood the information presented. Checking the boxes below means that you have also decided to allow your child to be interviewed and audio recorded. You will be given a copy of this consent form to keep.

______________________________
Child’s Name

I agree to allow my child to be interviewed.
I agree to allow my child’s interview to be audio recorded.

______________________________
Signature of Parent

______________________________
Date

Deborah E. Berg, M.S., Principal Investigator
David Fowler, Ph.D., Secondary Investigator

E-mail: [redacted]
E-mail: [redacted]

Phone: [redacted]
Phone: [redacted]

231 Mabel Lee Hall / Lincoln, NE 68588
(402) 472-2913
Child Assent Form

I would like to invite you to participate in a study. You will be doing some journal assignments as part of the first class, and I would like to photocopy and look at what you write. I would also like to talk about math and your thoughts about math for 15-20 minutes in an interview. This would be done by a trained interviewer during class. You may choose to do one or both of these, or neither.

If you are interviewed, the interviewer will ask some questions about what you think about math, and he will ask you to solve some math problems aloud. The interview will be audio-recorded to help take notes, but only the researchers will hear the recording. Your work and what you say during the interview may be reported, but I won’t use your real name.

Your parents will also be asked to give their permission for you to take part in this study. Please talk with your parents before you decide whether or not to participate. You do not have to be in this study if you do not want to. If you decide to participate in the study, you can stop at any time. Your decision to participate in the research or not will not affect your standing in the class.

You may ask any questions concerning this research and have those questions answered before, during, or after the study by contacting the investigators using the contact information below or by talking to Debbie before, during, or after class.

If you sign this form, it means that you have decided to participate and have read everything that is on this form. You and your parents will be given a copy of this form to keep.

I agree to be interviewed.
I agree that my interview can be audio-recorded.

__________________________  __________________________
Child’s Printed Name                  Date

Deborah E. Berg, MS, Principal Investigator  E-mail: __________________________
David Fowler, Ph.D., Secondary Investigator  E-mail: __________________________

231 Mabel Lee Hall / Lincoln, NE 68588
(402) 472-2913

Figure C.3: Child Informed Consent Form for the First Class
Dear Parents and Guardians,

I am a graduate student in mathematics education at the University of Nebraska-Lincoln. I am currently working on my dissertation, tentatively titled "Students’ Attitudes about Routine and Non-routine Mathematics Problems." This is a research project investigating post-4th and post-6th graders’ views about different aspects of mathematics, and I would appreciate your help.

Participating in this study takes no additional time. Students will be doing some journal assignments as part of the class, and I would like to photocopy and analyze their work. Participating or not participating will in no way affect your child’s status in the class or with the program.

Attached are the official University of Nebraska permission slips that give me permission to copy your child’s work. If you would like your child to participate, please sign the Adult Consent form and send it in with your child. If you return a signed Adult Consent form, your child will then be invited to read and sign the Child Assent form. Both forms must be signed and returned in order to participate. You will receive copies of the forms after they have been returned.

If you have any questions, please contact me or you can talk to me before or after class.

Thank you very much!

Debbie Seacrest

Figure C.4: Letter Home for the Second Class
Parental Informed Consent Form

You are invited to permit your child to participate in this research study, tentatively titled "Children's Voices: Students' Attitudes about Routine and Nonroutine Mathematics Problems". The following information is provided in order to help you make an informed decision whether or not to allow your child to participate. If you have any questions please do not hesitate to ask.

The purpose of this study is to collect and analyze students’ work in grades 4-5 on nonroutine math problems and determine the students’ attitudes towards both traditional and nontraditional mathematics. Participation in the study will allow your child's mathematics journals to be photocopied and analyzed. Your child will not be asked to do anything beyond the activities normally part of the program. This study will take place with the program.

There are no known risks associated with this research. Your child may find the learning experience enjoyable and educational. The information gained from this study may help us to better understand students’ mathematical attitudes.

Any information obtained during this study which could identify your child will be kept strictly confidential. The data will be stored indefinitely in the principal investigator’s office or on a password-protected laptop and will only be seen by the investigators. The information obtained in this study may be published in a doctoral dissertation or educational journals or may be presented at educational meetings but your child's identity will be kept strictly confidential.

Your child’s rights as a research subject have been explained to you. If you have any additional questions about the study, please contact Debbie Searesat or David Fowler at . If you have any questions about your child’s rights as a research participant that have not been answered by the investigator or to report any concerns about the study, you may contact the University of Nebraska-Lincoln Institutional Review Board (UNL IRB), telephone .

Your child’s participation in this project is voluntary and your child may decide not to participate in this study or to withdraw at any time without adversely affecting their relationship with the investigator, the program, or the University of Nebraska-Lincoln. Your decision about participation will not affect your child's standing in the program in any way.

DOCUMENTATION OF INFORMED CONSENT

You are voluntarily making a decision whether or not to allow your child to participate in the research study. Your signature certifies that you have decided to allow your child to participate and his or her work to be photocopied and analyzed, having read and understood the information presented. You will be given a copy of this consent form to keep.

______________________________
Child’s Name

______________________________  ______________________________
Signature of Parent  Date

Deborah E. Seares, M.S., Principal Investigator  E-mail:  Phone:  
David Fowler, Ph.D., Secondary Investigator  E-mail:  Phone:  

231 Mabel Lee Hall / Lincoln, NE 68588  (402) 472-2913
Child Assent Form

I would like to invite you to participate in a study. You will be doing some journal assignments this summer and I would like to photocopy and look at what you write. Your work may be reported, but I won’t use your real name.

Your parents or guardians will also be asked to give their permission for you to take part in this study. Please talk with them before you decide whether or not to participate. You do not have to be in this study if you do not want to. If you decide to participate in the study, you can stop at any time. Your decision to participate in the research or not will not affect your standing in the class.

You may ask any questions concerning this research and have those questions answered before, during, or after the study by contacting the investigators using the contact information below or by talking to Debbie before, during, or after class.

If you sign this form, it means that you have decided to participate and have read everything that is on this form. You and your parents will be given a copy of this form to keep.

_________________________  ______________________
Child’s Printed Name                                             Date

_________________________  ______________________
Signature of Child                                                 Date

Deborah E. Seacrest, MS, Principal Investigator
David Fowler, Ph.D., Secondary Investigator

E-mail: [REDACTED]

231 Mabel Lee Hall / Lincoln, NE 68588
(402) 472-2913

Figure C.6: Child Informed Consent Form for the Second Class
Appendix D

List of Activities

The following chart consists of the activities done in the two classes. Note that there was one activity from the second class that was not done in the first class, but many activities from the first class were not used in the second class. This is mostly because there was much more instructional time in the first class.
<table>
<thead>
<tr>
<th>Activity</th>
<th>First Class</th>
<th>Second Class</th>
</tr>
</thead>
<tbody>
<tr>
<td>Towers of Hanoi</td>
<td>June 14</td>
<td>July 26</td>
</tr>
<tr>
<td>Handshakes</td>
<td>June 14</td>
<td>July 26</td>
</tr>
<tr>
<td>Stars</td>
<td>June 14</td>
<td>-</td>
</tr>
<tr>
<td>Factorials</td>
<td>June 14</td>
<td>-</td>
</tr>
<tr>
<td>Penny Game</td>
<td>June 14</td>
<td>July 28</td>
</tr>
<tr>
<td>Fractals</td>
<td>June 15</td>
<td>July 27</td>
</tr>
<tr>
<td>Graph Coloring</td>
<td>June 15</td>
<td>-</td>
</tr>
<tr>
<td>Paper Folding</td>
<td>June 15</td>
<td>-</td>
</tr>
<tr>
<td>$+/-$ and $+/\times$ Games</td>
<td>June 15</td>
<td>-</td>
</tr>
<tr>
<td>River Crossing</td>
<td>June 15</td>
<td>July 29</td>
</tr>
<tr>
<td>Rock, Paper, Scissors, Dynamite</td>
<td>June 15</td>
<td>-</td>
</tr>
<tr>
<td>Area and Perimeter</td>
<td>June 15</td>
<td>-</td>
</tr>
<tr>
<td>Pig</td>
<td>June 16</td>
<td>July 27</td>
</tr>
<tr>
<td>Base 5</td>
<td>June 16</td>
<td>-</td>
</tr>
<tr>
<td>Even or Odd</td>
<td>June 16</td>
<td>-</td>
</tr>
<tr>
<td>Train Cars</td>
<td>June 16</td>
<td>-</td>
</tr>
<tr>
<td>Addition Trick</td>
<td>June 16</td>
<td>-</td>
</tr>
<tr>
<td>Five Card Trick</td>
<td>June 16</td>
<td>-</td>
</tr>
<tr>
<td>Dominoes</td>
<td>June 17</td>
<td>July 29</td>
</tr>
<tr>
<td>Dominoes on a Checkerboard</td>
<td>June 17</td>
<td>July 29</td>
</tr>
<tr>
<td>Squares on a Checkerboard</td>
<td>June 17</td>
<td>-</td>
</tr>
<tr>
<td>Move Forward Game</td>
<td>June 17</td>
<td>-</td>
</tr>
<tr>
<td>Cartesian Coordinates</td>
<td>June 17</td>
<td>-</td>
</tr>
<tr>
<td>Sixteen Card Trick</td>
<td>June 17</td>
<td>-</td>
</tr>
<tr>
<td>Hog</td>
<td>June 18</td>
<td>July 28</td>
</tr>
<tr>
<td>Guess My Rule</td>
<td>June 18</td>
<td>July 27</td>
</tr>
<tr>
<td>Flashlights on a Bridge</td>
<td>June 18</td>
<td>July 29</td>
</tr>
<tr>
<td>Multiplication by Nines</td>
<td>-</td>
<td>July 26</td>
</tr>
</tbody>
</table>

Table D.1: Activities done in the two classes
Bibliography


