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Area and Perimeter of Polygons

Bryan Engelker

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Expository Paper

Bryan Engelker

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Jim Lewis, Advisor

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A student comes to class excited. She tells you she has figured out a theory you never told the class. She says she has discovered that as the perimeter of a closed figure increases, the area also increases. She shows you two pictures to prove what she is doing. The first picture is of a 4 by 4 square. Of course, its perimeter is 16 and its area is 16. The second picture is of a 4 by 8 rectangle. Here the perimeter is 24 and the area is 32. What do you say to the student?

This is a problem that Jim Lewis and Ruth Heaton, instructors at the University of Nebraska-Lincoln, gave on Day 1 of their geometry class for future elementary teachers in the Fall of 2002. Twenty-four of the thirty-two future elementary teachers believed the child's theory was correct and indicated they would congratulate the child. Eight of the thirty-two questioned the child's theory and only six out of those eight explained that the theory was definitely not true.

Area and perimeter are concepts that seem to cause difficulty for students. They sometimes understand the concepts separately, but the difficulty arises when the concepts are put together. This is understandable since adults in a geometry class for future elementary teachers struggle with the understanding of how area and perimeter are related.

In this paper, I will show how area and perimeter of polygons are related. I will show examples using triangles, quadrilaterals, and pentagons. I will investigate how the area of a polygon can change even if the perimeter is fixed. I will also show what class of specific polygons maximizes the area given a fixed perimeter. This paper will also look at the area and perimeter of polygons with a fixed diameter.

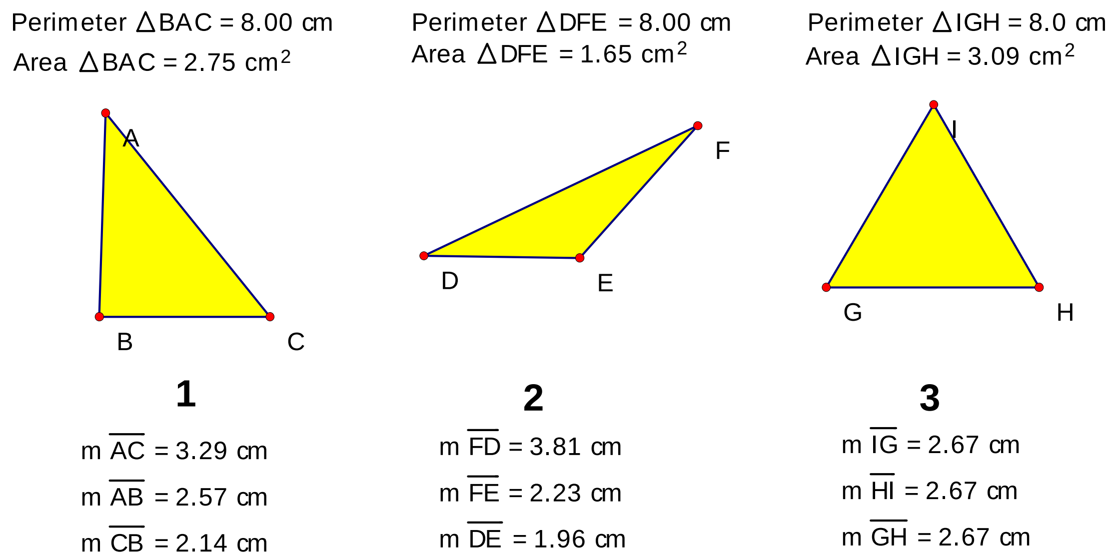
The problem of trying to find the polygon that has the maximum area given a fixed perimeter is called the Isoperimetric Problem. Isoperimetric means having the same perimeter measurement. The isoperimetric problem dates back to the 7^{th} century B.C. in the story of Queen Dido. As legend tells it, Dido had to flee her homeland of Phoenicia after her tyrant brother had killed her wealthy husband and was now after her riches. She sailed away with several boats full of riches, belongings and people. They reached the coast of Africa, in what is now Tunisia.

When they landed the local tribe was not very welcoming. Dido promised the tribal chief a fair amount of riches for as much land as she could mark out with a bull skin. The chief thought this was a great bargain on his end, but he underestimated Dido's knowledge. Dido cut the bull hide into thin strips and sewed the strips together to make one long string. She then used the seashore as one edge of her piece of land and laid the bull skin into a half circle. Dido acquired much more land using this method than what the tribal chief had believed, much to his dismay. Dido and her followers founded on this piece of land what came to be the great and influential city of Carthage.

Dido must have been a clever woman to have thought of cutting up the bull skin, sewing it together into a string and using this to mark off the land she bought from the tribal chief. As I will discuss the circle has the greatest area for any shape given a fixed perimeter. This knowledge is what aided Queen Dido in acquiring the largest amount of land possible.

The Greek mathematician Zenodorus (200 B.C.-140 B.C.) studied the area of a figure with fixed diameter. He found that the area of the circle is larger than any polygon having the same perimeter. Jacob Steiner (1796-1863) was a mathematician who extensively studied the isoperimetric problem. He made some of the most essential contributions towards the rigorous proof of the isoperimetric problem. Although Steiner extensively studied the isoperimetric problem, it was Karl Weierstrass (1815-1897) who supplied the first complete proof of the optimality of the circle.

The isoperimetric problem can be applied to any polygon. The isoperimetric problem for triangles asks, for a fixed perimeter, what class of triangles has the greatest area? The triangles below all have a perimeter of 8 cm.



Although all the triangles have a perimeter of 8 cm, their areas are all different. Triangle 1 has an area of 2.75 cm^2, triangle 2 has an area of 1.65 cm^2, and triangle 3 has an area of 3.09 cm^2. Triangle 3 has the largest perimeter out of these three triangles. If we look at the type of triangle that triangle 3 is, we see that it is an equilateral triangle. Triangle 2 is the least equilateral of the three and has the smallest area. Triangle 1 has side lengths that are more similar to each other making it closer to equilateral than triangle 2 and has a larger area than triangle 2. This shows that among all triangles with the same perimeter, the equilateral triangle will have the largest area.

Another relationship between area and perimeter of triangles can be found in Heron's formula. Heron's formula is also known as Hero's formula. Heron was a mathematician in the 1st Century. Although the formula is credited to and named for Heron, it is now believed that Archimedes discovered or at least knew of the formula. Heron's formula relates the area of a triangle to the measure of its three sides. Heron's formula states:

$$A\Delta = \sqrt{s(s-a)(s-b)(s-c)}$$

where s = semiperimeter
a = length of side a
b = length of side b
c = length of side c

The semiperimeter of a triangle is found by adding the three sides of a triangle and dividing the sum by two.

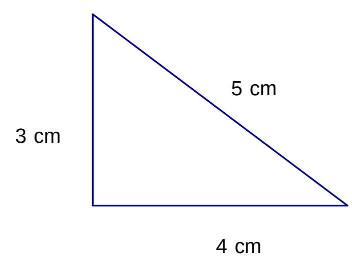
$$s = \frac{(a+b+c)}{2}$$

where a = length of side a

b = length of side b

c = length of side c

Once the semiperimeter is found and we know the lengths of the three sides of the triangle, we can use Heron's formula to find the area of the triangle. We can see an example of Heron's formula for a simple triangle.



First, we need to find the semiperimeter.

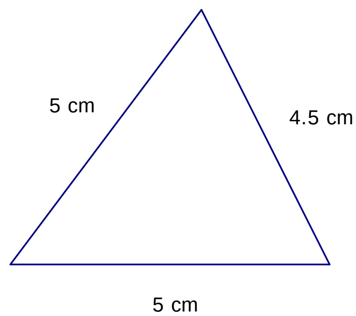
$$s = \frac{(a+b+c)}{2}$$
$$s = \frac{(3+4+5)}{2}$$
$$s = \frac{(12)}{2}$$
$$s = 6cm$$

The semiperimeter is 6 cm. Now we can put the semiperimeter and the side lengths into Heron's formula to find the area.

$$A = \sqrt{s(s-a)(s-b)(s-c)}$$
$$A = \sqrt{6(6-5)(6-4)(6-3)}$$
$$A = \sqrt{6(1)(2)(3)}$$
$$A = \sqrt{6(6)}$$
$$A = \sqrt{36}$$
$$A = 6cm^2$$

$$A = \frac{1}{2}(b \times h)$$
$$A = \frac{1}{2}(4 \times 3)$$
$$A = \frac{1}{2}(12)$$
$$A = 6cm^2$$

Heron's formula shows that for the given triangle the area is $6\,cm^2$. Next to Heron's formula I put in the traditional formula for finding area of a triangle. It also gives the area of the given triangle to be $6\,cm^2$. The cool thing about Heron's formula is that we don't need to know the height of the triangle to find the area as with the formula $A = \frac{1}{2}(b \times h)$. We can look at another example where the height isn't given and we can find the area of the triangle using Heron's formula. In this triangle the height isn't easily known.



First, find the semiperimeter.

$$s = \frac{(5+5+4.5)}{2}$$
$$s = \frac{(14.5)}{2}$$
$$s = 7.25cm$$

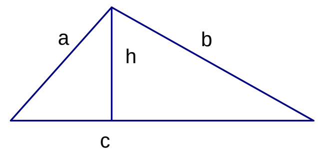
Now, put the semiperimeter and side lengths into Heron's formula.

$$A = \sqrt{7.25(7.25-5)(7.25-5)(7.25-4.5)}$$
$$A = \sqrt{7.25(2.25)(2.25)(2.75)}$$
$$A = \sqrt{7.25(13.92)}$$
$$A = \sqrt{100.92}$$
$$A = 10.05cm^2$$

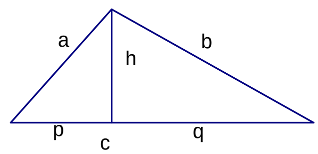
Heron's formula is useful when attempting to find the area of a triangle for which the lengths of the sides are known, but the perpendicular height or altitude is not known. The following is a proof of Heron's formula:

Assume the Pythagorean Theorem and the area formula for a triangle $\frac{1}{2}(b \times h)$ where b is the length of a base and h is the height to that base.

Let a, b, c be the lengths of the sides of our triangle and h be the height to the side of length c.



We have $s = \frac{(a+b+c)}{2}$ so for further reference, 2s = a + b + c; 2(s-a) = -a + b + c; 2(s-b) = a – b + c; 2(s –c) = a + b – c



Let p + q = c then by Pythagorean Theorem $h^2 + p^2 = a^2$ and $h^2 + q^2 = b^2$

Since $q = c - p$, then $q^2 = (c-p)^2$, factor $q^2 = c^2 - 2cp + p^2$

(Add h^2 to both sides) $\quad h^2 + q^2 = h^2 + c^2 - 2cp + p^2$

(Substitute) $\quad b^2 = a^2 - 2cp + c^2$

(Solve for p) $\quad p = \frac{a^2 + c^2 - b^2}{2c}$

Since $h^2 = a^2 - p^2$, substitute for p to get an expression in terms of a, b, and c.

$$h^2 = a^2 - p^2$$

(Factor) $\quad = (a+p)(a-p)$

(Substitute) $\quad = \left[a + \frac{(a^2 + c^2 - b^2)}{2c}\right]\left[a - \frac{(a^2 + c^2 - b^2)}{2c}\right]$

(Simplify) $\quad = \frac{(2ac + a^2 + c^2 - b^2)(2ac - a^2 - c^2 + b^2)}{4c^2}$

(Factor) $$=\frac{\left((a+c)^2-b^2\right)\left(b^2-(a-c)^2\right)}{4c^2}$$

(Factor) $$=\frac{(a+b+c)(a+c-b)(b+a-c)(b-a+c)}{4c^2}$$

(Rearrange) $$=\frac{(a+b+c)(-a+b+c)(a-b+c)(a+b-c)}{4c^2}$$

(Substitute) $$=\frac{2s\cdot 2(s-a)\cdot 2(s-b)\cdot 2(s-c)}{4c^2}$$

Therefore,

$$h^2=\frac{4s(s-a)(s-b)(s-c)}{c^2}$$

$$h=\frac{2\sqrt{s(s-a)(s-b)(s-c)}}{c}$$

Since,

$$A=\frac{1}{2}ch$$

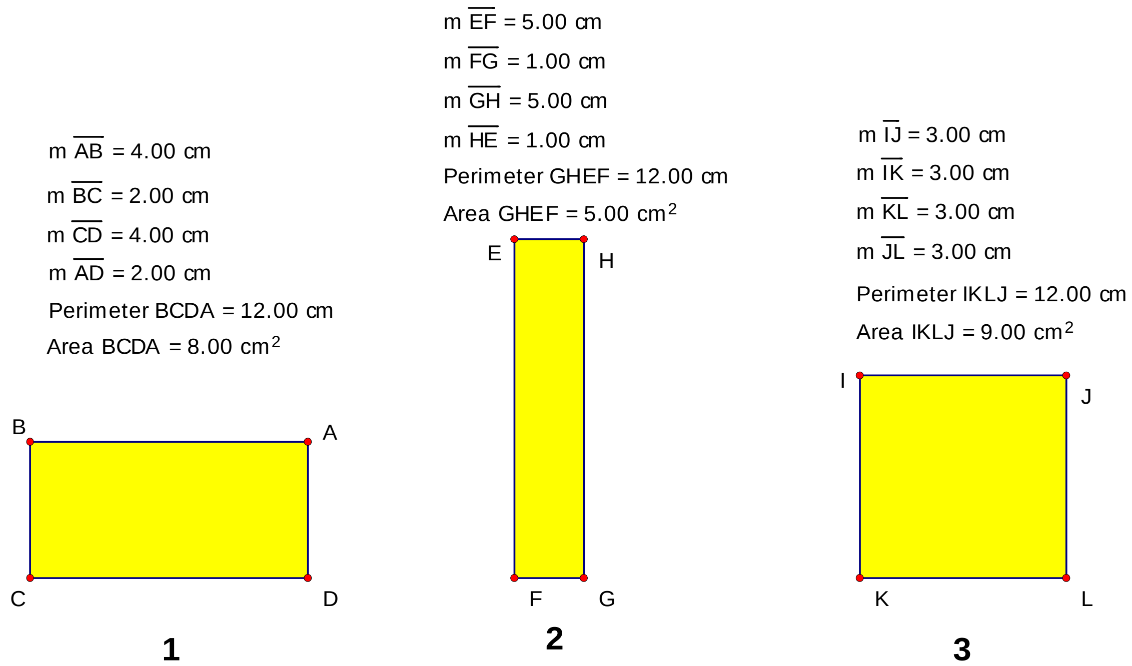
then

$$A=\frac{1}{2}c\left(\frac{2\sqrt{s(s-a)(s-b)(s-c)}}{c}\right)$$

and

$$A=\sqrt{s(s-a)(s-b)(s-c)}$$

Now we will look at the isoperimetric problem for quadrilaterals and try to answer the question, what class of quadrilateral will maximize the area given a fixed diameter? To answer this question we can look at a few examples. The perimeter for the following quadrilaterals is fixed at 12 cm.



Quadrilateral 1 has an area of 8 cm^2, Quadrilateral 2 has an area of 5 cm^2, and Quadrilateral 3 has an area of 9 cm^2. Quadrilateral 3 has the largest area and it is a square. We can put the information into a chart to help see that the square does give the maximum area.

Dimensions	Perimeter	Area
1 cm x 5 cm	12 cm	5 cm^2
2 cm x 4 cm	12 cm	8 cm^2
3 cm x 3 cm	12 cm	9 cm^2
4 cm x 2 cm	12 cm	8 cm^2
5 cm x 1 cm	12 cm	5 cm^2

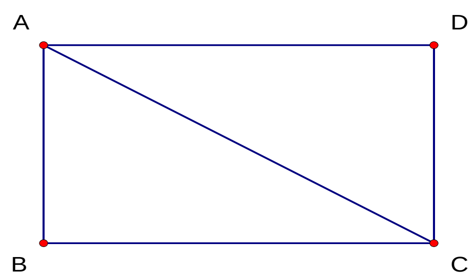
We can use a chart to look at another case. If the perimeter is fixed at one hundred, the chart would show what quadrilateral maximizes the area.

Dimensions	Perimeter	Area
5 x 45	100	225
10 x 40	100	400
15 x 35	100	525
20 x 30	100	600
25 x 25	100	625
30 x 20	100	600
35 x 15	100	525
40 x 10	100	400
45 x 5	100	225

This chart also shows that the area is maximized by the square. So among all quadrilaterals with a fixed perimeter, the square gives the largest area.

I found that when trying to maximize the area of a triangle or quadrilateral with a fixed perimeter, that it was the equilateral triangle and the square that gave the maximum area. These two are both regular polygons, so we can make the conjecture that for any n-gon with a fixed perimeter, the regular polygon alone has the maximum area.

Another problem mathematicians have worked with is the Isodiametric problem. Isodiametric means having the same diameter measurement. The isodiametric problem looks at polygons with a fixed diameter. The isodiametric problems for polygons was first studied by Karl Reinhardt in 1922. We normally think of diameters in circles, but the diameter of a polygon is the largest possible distance between two vertices on the polygon. In the figure below, $\overline{AC}$ is the diameter of ABCD.

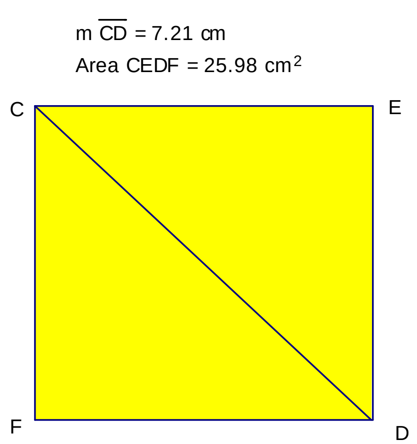


$\overline{AC}$ is the diameter of ABCD

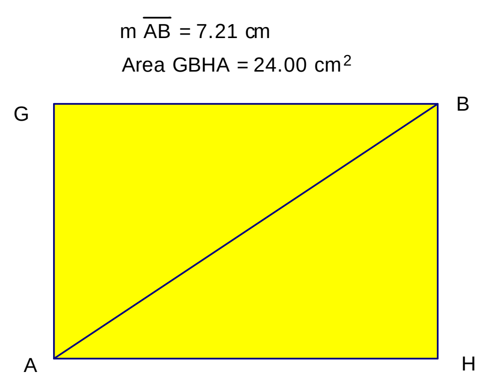
We are going to answer two questions concerning polygons with fixed diameter:

1) What is the maximum area of a polygon with n sides and fixed diameter?
2) What is the maximum perimeter of a polygon with n sides and fixed diameter?

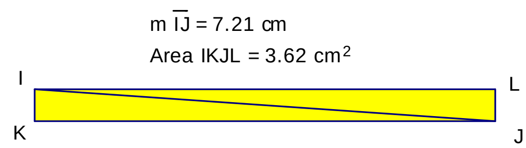
We will look first at the question number 1. We'll start by looking at quadrilaterals that have a fixed diameter. The following quadrilaterals all have a diameter of 7.21 cm.



1



2

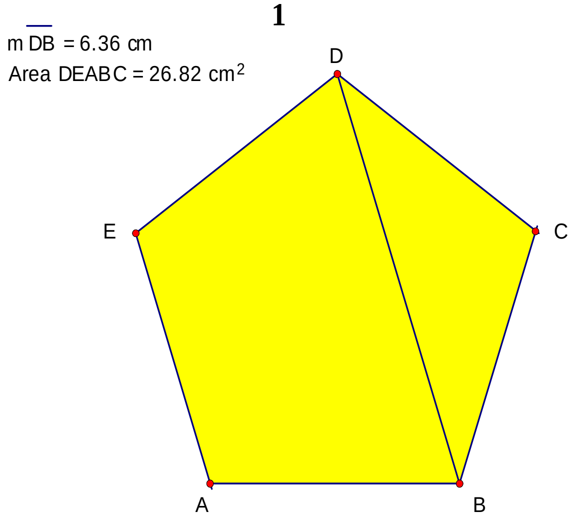


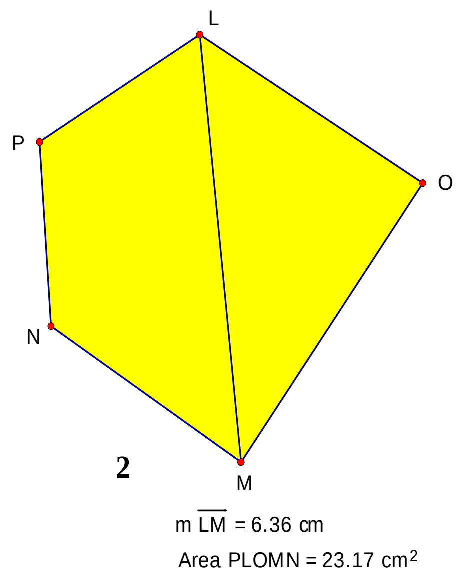
3

We can see that quadrilateral 1 has an area of 25.98 cm ^{2} , quadrilateral 2 has an area of 24 cm ^{2} , and quadrilateral 3 has an area of 3.62 cm ^{2} . These examples show that given a fixed diameter the quadrilateral that maximizes the area is the square or regular quadrilateral.

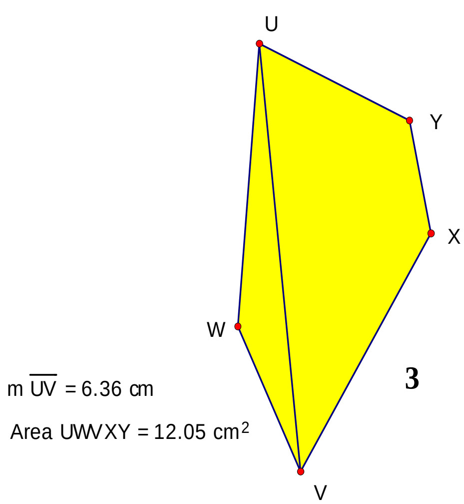
Another example using pentagons also demonstrates that given a fixed diameter the regular pentagon maximizes the area. The following pentagons all have a diameter of 6.36 cm.

1





2

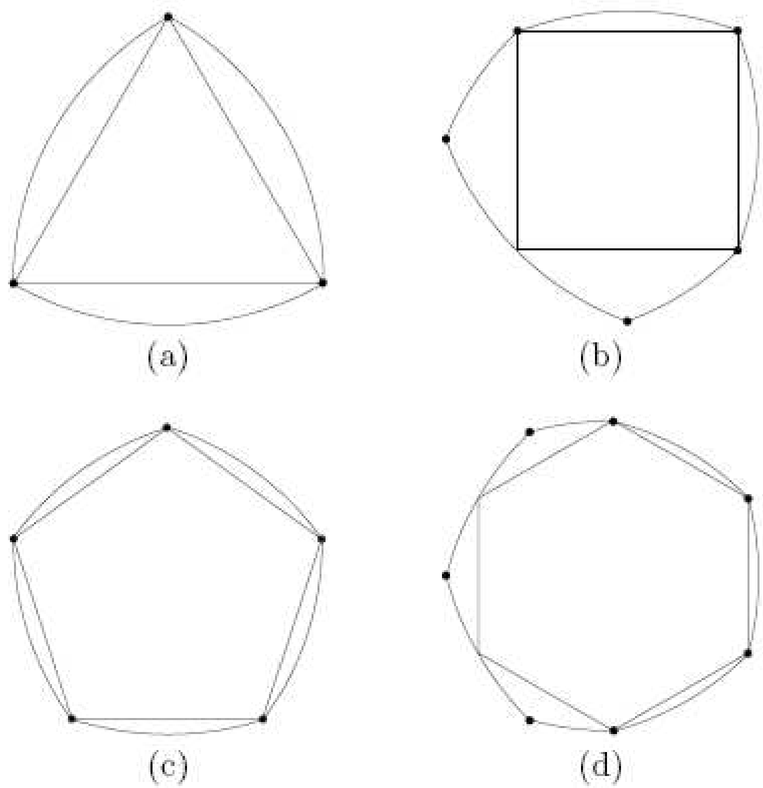


3

The diagram shows that pentagon 1 has the greatest area of the three at 26.82 cm^2. Pentagon 1 is also a regular pentagon. Again, the regular pentagon maximizes the area when given a fixed diameter.

It seems that we could make the conjecture that the regular n-gon maximizes the area given a fixed diameter. Although this seems like a good conjecture, Reinhardt and other mathematicians have shown that the regular n-gon gives maximum area only when n (number of sides) is odd or $n < 6$. So among all n-gons of odd n, the regular n-gon has the maximum area. This is not the case for n-gons of even n. The two examples earlier in this paper do meet the criteria of being less than six so the regular square and regular pentagon do in deed maximize the area.

When Reinhardt was studying the isodiametric problem for polygons, he used Reuleaux polygons in his proofs. A Reuleux polygon is defined as a set of constant width whose boundary consists of a finite number of circular arcs of the same radius. A Reuleaux polygon is not a polygon in the traditional sense, since its edges are not line segments. The following are examples of Reuleaux polygons made around regular polygons.

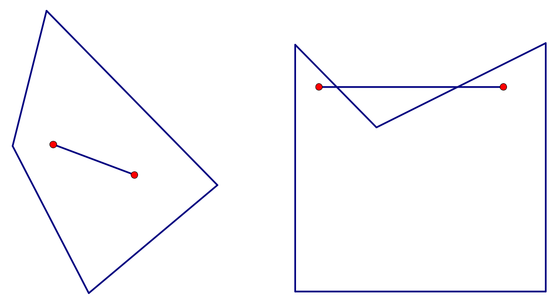


(a) (b) (c) (d)

The Reuleaux polygons are named for the 19th Century German mechanical engineer Franz Reuleaux. Reuleaux was a brilliant engineer. He created over 300 models of simple machines and is remembered for the Reuleaux triangle. The Reuleaux triangle is a curve with constant width that has been used in Mazda Car Company's rotary engines. The Mazda RX-7 is equipped with a rotary engine. The rotary engine reduces vibration and noise because there is no opportunity for metal surfaces to strike.

As can be seen by the pictures above, the Reuleaux polygons obtained when the number of sides is even look dramatically different than those made when the number of sides is odd. Reinhardt found that this discrepancy is ultimately the source of the difference in the even and odd cases for the isodiametric problem for area.

The second question asked, what is the maximum perimeter of a polygon with n sides and fixed diameter? One restriction needs to be added to this question. The restriction is that we are only looking at cases of convex polygons. A polygon is convex if the line segment joining two points inside the polygon also lies inside the polygon.



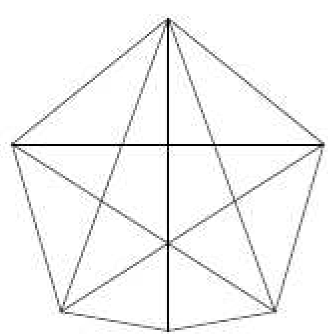
Convex Concave

To answer this question, I could use the same quadrilaterals and pentagons that were used to show the maximum area for a polygon with fixed diameter, but the same conditions apply to perimeter. Reinhardt found that the convex regular polygon only achieves maximum perimeter when n is odd and $n < 6$.

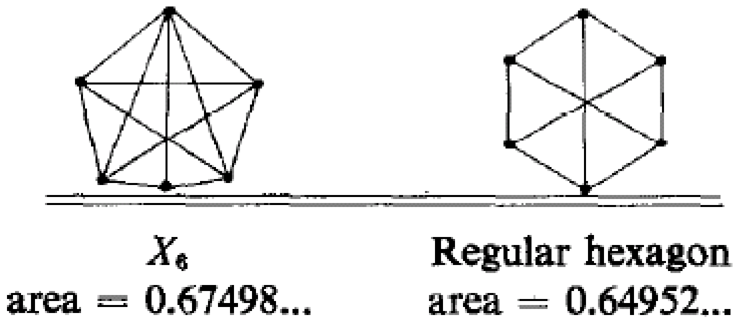
The answer to both questions can be answered in the statement; Among all convex polygons with n sides and fixed diameter where $n > 5$, the regular n-gon has neither maximal area nor maximal perimeter.

The actual proofs for the isoperimetric and isodiametric problems are very complex and lengthy. Reinhardt and others have made great strides in proving maximal area and maximal perimeter, but there are more questions about maximal area and perimeter for cases where n is even and $n > 5$. Many mathematicians still today are looking for those answers.

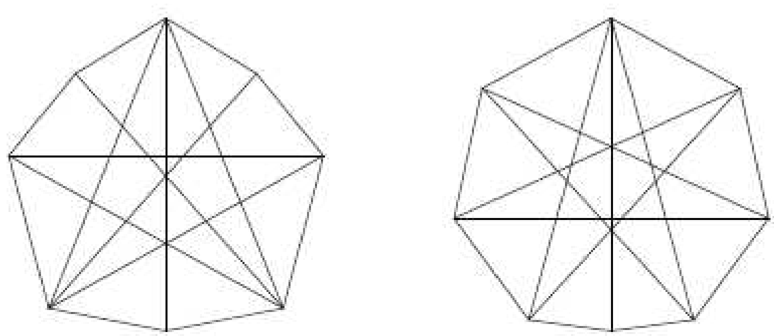
Dr. Robert Bieri, a German mathematician, proved in 1961 that the hexagon shown below has maximum area among all hexagons with fixed diameter that possess axes of symmetry.



In 1975, R.L. Graham(1935-) proved that this hexagon is in fact optimal for all hexagons with fixed diameter. This hexagon's area is about 3.92% larger than that of the regular hexagon.

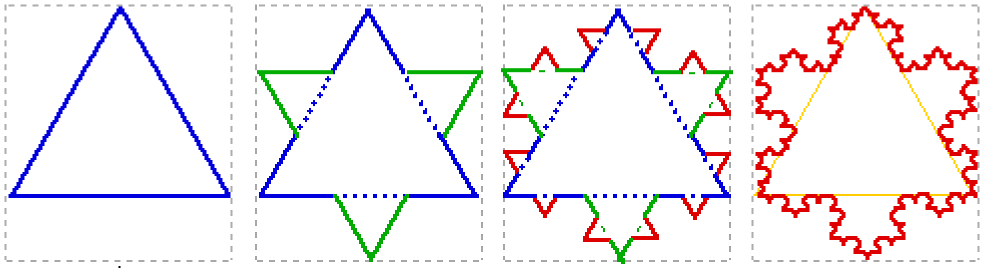


In 2002, Audet, Hansen, Messene and Xiong found two octagons with areas larger than the regular octagon. Those two octagons are shown below. Octagon A's area is 0.52% larger than the regular octagon and Octagon B's area is about 2.79% larger than the regular octagon.



Octagon A Octagon B

A fun and curious relationship between area and perimeter is the Koch Snowflake Curve. The Koch Snowflake Curve is named after Swedish mathematician Niels Fabian Helge von Koch (1870-1924). The Koch Snowflake is derived by a process that starts with an equilateral triangle with side lengths of 1. Next, equilateral triangles with side lengths of $\frac{1}{3}$ are built onto the middle third of each side of the original. The next step takes equilateral triangles with side lengths of $\frac{1}{9}$ and adds these onto the middle third of the existing twelve sides. This process in continued an infinite number of times using smaller and smaller equilateral triangles.



Interestingly, as this process is carried out the area of the snowflake encloses a finite amount of area, but the perimeter goes to infinity. So an infinite perimeter encloses a finite area.

The Koch Snowflake adds an interesting twist to the first case discussed in this paper. We looked at how an equilateral triangle will maximize the area for a given perimeter. But in the Koch Snowflake, although built with equilateral triangles, the perimeter is more or less being maximized as it approaches infinity and the area reaches a finite amount.

SUMMARY

This topic was very interesting to me because I have taught area and perimeter to my fifth grade students. Students do struggle with understanding how area and perimeter are interrelated. I was not familiar with the isoperimetric and isodiametric problem before receiving this topic. I had known that to maximize the area of a quadrilateral with a fixed perimeter that you wanted the most square-like polygon. I never realized that all regular polygons maximize the area for a fixed perimeter.

I also learned about Queen Dido and how she achieved acquiring the largest piece of land possible. I read different accounts of Dido’s life and it was interesting to see how certain aspects were common, but others were not. Almost every account I read mentioned in some form the story of using the bull skin to acquire land.

Reuleaux polygons were another concept that was new to me. I had never heard about Reuleaux polygons or the Reuleaux triangle. I found some interesting websites that I think I can use with my students. It also interested me in the fact that Mazda uses Reuleaux triangles in their rotary engines.

The most important mathematical learning I received from researching this topic and writing this paper was Heron’s formula for finding the area of triangles. This formula intrigued me. When I teach finding the area of triangles, students struggle with trying to find the altitude. When using Heron’s formula if you know the lengths of the three sides, you can find the area. I find it amazing that Heron’s formula is not more widely known and more widely used. I think upper elementary students would benefit from learning both methods for finding the area of a triangle. Once those students were familiar with the square root they could find the area of any triangle.

Teaching Heron’s formula is a way to get students to think mathematically, to teach concepts more deeply, and to show multiple representations on how to solve a problem. Exposing students to multiple ways to solve problems will help strengthen their mathematical knowledge and foundation for future learning.

The biggest challenge I faced while researching this topic was finding information that I could read and understand about the isodiametric problem. The proofs are very complex and hard to understand. Most of the information I found was to technical for me to sift through. I did enjoy reading Michael Mossinghoff's paper, A $1 Problem. It was at times confusing, but for the most part, if I sat and carefully read through it and thought about it I could understand it. It is also interesting that he suggests a new coin shouldn't be round.

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