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# On the η - κ Distribution

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# Comments and Corrections

## On the $\eta - \kappa$ Distribution

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***Index Terms*—Fast fading distribution, $\eta - \kappa$ distribution.**

The recent paper by Yacoub *et al.* [1] introduces what is referred to as the $\eta - \kappa$ *distribution* to describe the statistical variation of the envelope in a fast fading environment. The paper discusses several properties of the distribution. Two of the properties discussed are the $n$th moment, $E(P^n)$, and the cumulative probability function (cpf), $F_P(\cdot)$, where $P$ is a random variable representing the normalized envelope. The expression given for $E(P^n)$ (see equation (10) in Yacoub *et al.* [1]) is a doubly infinite sum of the Gauss hypergeometric function (which, itself, is an infinite sum). That given for $F_P(\cdot)$ (see equation (11) in Yacoub *et al.* [1]) is a triple sum of the incomplete gamma function.

We feel that the expressions in equations (10) and (11) of Yacoub *et al.* [1] are too complicated for practical purposes. In the following, we show how one can derive much simpler forms for $E(P^n)$ and $F_P(\cdot)$. Using equations (5)–(8) in Yacoub *et al.* [1], the probability density function (pdf) of $P$ can be expressed as

$$f_P(p) \frac{\sqrt{h}(1+\kappa)}{\pi \exp(\kappa)} \int_0^{2\pi} p \exp(Ap - Bp^2) d\theta, \qquad (1)$$

where $A = 2\sqrt{h\kappa(1+\kappa)} \cos\theta$ and $B = (1+\kappa)h + H(1+\kappa)\cos(2(\theta+\phi))$. Thus, the $n$th moment, $E(P^n)$, can be expressed as

$$\begin{aligned} E(P^n) &= \frac{\sqrt{h}(1+\kappa)}{\pi \exp(\kappa)} \int_0^{\infty} \int_0^{2\pi} p^{n+1} \exp(Ap - Bp^2) d\theta dp \\ &= \frac{\sqrt{h}(1+\kappa)}{\pi \exp(\kappa)} \int_0^{2\pi} \int_0^{\infty} p^{n+1} \exp(Ap - Bp^2) dp d\theta \\ &= \frac{\sqrt{h}(1+\kappa)}{\pi \exp(\kappa)} \int_0^{2\pi} I(\theta) d\theta. \end{aligned} \qquad (2)$$

By equation (2.3.15.3) in Prudnikov *et al.* [2], $I(\theta)$ can be calculated as

$$I(\theta) = \Gamma(n+2)(2B)^{-(n/2+1)} \exp\left(\frac{A^2}{8B}\right) D_{-n-2}\left(-\frac{A}{\sqrt{2B}}\right), \qquad (3)$$

where $D_p(\cdot)$ denotes the parabolic cylinder function defined by

$$D_p(x) = \frac{\exp(-x^2/4)}{\Gamma(-p)} \int_0^{\infty} \frac{\exp\left\{-(tx + t^2/2)\right\}}{t^{p+1}} dt.$$

Combining (2) and (3) yields the formula

$$\begin{aligned} E(P^n) = \frac{\Gamma(n+2)\sqrt{h}(1+\kappa)}{2^{n/2+1}\pi \exp(\kappa)} \int_0^{2\pi} B^{-(n/2+1)} \\ \times \exp\left(\frac{A^2}{8B}\right) D_{-n-2}\left(-\frac{A}{\sqrt{2B}}\right) d\theta. \end{aligned} \qquad (4)$$

This formula applies for any real number $n > -2$. If $n$ is a positive integer then, using equation (2.3.15.7) in Prudnikov *et al.* [2], $I(\theta)$ can be calculated as

$$I(\theta) = \frac{(-1)^{n+1}\sqrt{\pi}}{2\sqrt{B}} \frac{\partial^{n+1}}{\partial q^{n+1}} \times \left[\exp\left(\frac{q^2}{4B}\right) \operatorname{erfc}\left(\frac{q}{2\sqrt{B}}\right)\right] |_{q=-A}, \qquad (5)$$

where $\operatorname{erfc}(\cdot)$ denotes the complementary error function defined by

$$\operatorname{erfc}(x) = -\frac{2}{\sqrt{\pi}} \int_0^{x} \exp(-t^2) dt.$$

Combining (2) and (5) yields the simpler formula

$$\begin{aligned} E(P^n) = \frac{(-1)^{n+1}\sqrt{h}(1+\kappa)}{2\sqrt{\pi}\exp(\kappa)} \int_0^{2\pi} B^{-1/2} \frac{\partial^{n+1}}{\partial q^{n+1}} \\ \times \left[\exp\left(\frac{q^2}{4B}\right) \operatorname{erfc}\left(\frac{q}{2\sqrt{B}}\right)\right]\Bigg|_{q=-A} d\theta. \end{aligned} \qquad (6)$$

Various simple expressions can be obtained from (6) by setting specific values for $n$. For instance, if $n = 1$, $n = 3$ and $n = 5$ then (6) can be reduced to the simple forms

$$\begin{aligned} E(P) = \frac{\sqrt{h}(1+\kappa)}{8\pi \exp(\kappa)} \int_0^{2\pi} B^{-3} \exp\left(\frac{A^2}{4B}\right) \\ \times \Bigg[ 2B^{3/2}\sqrt{\pi} + A^2\sqrt{B}\sqrt{\pi} + 2B^{3/2}\sqrt{\pi}\operatorname{erf}\left(\frac{A}{2\sqrt{B}}\right) \\ + A^2\sqrt{B}\sqrt{\pi}\operatorname{erf}\left(\frac{A}{2\sqrt{B}}\right) \\ + 2AB \exp\left(-\frac{A^2}{4B}\right)\Bigg] d\theta, \end{aligned} \qquad (7)$$

$$\begin{aligned} E(P^3) = \frac{\sqrt{h}(1+\kappa)}{32\pi \exp(\kappa)} \int_0^{2\pi} B^{-5} \exp\left(\frac{A^2}{4B}\right) \\ \times \Bigg[ 12B^{5/2}\sqrt{\pi} + 12A^2B^{3/2}\sqrt{\pi} + A^4\sqrt{B}\sqrt{\pi} \\ + 12B^{5/2}\sqrt{\pi}\operatorname{erf}\left(\frac{A}{2\sqrt{B}}\right) + 12A^2B^{3/2} \\ \times \sqrt{\pi}\operatorname{erf}\left(\frac{A}{2\sqrt{B}}\right) + 20AB^2 \exp\left(-\frac{A^2}{4B}\right) \\ + A^4\sqrt{B}\sqrt{\pi}\operatorname{erf}\left(\frac{A}{2\sqrt{B}}\right) \\ + 2A^3B \exp\left(-\frac{A^2}{4B}\right)\Bigg] d\theta \end{aligned} \qquad (8)$$

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and

$$
\begin{aligned}
E(P^5) = {} & \frac{\sqrt{h}(1+\kappa)}{128\pi \exp(\kappa)} \int_0^{2\pi} B^{-7} \exp\left(\frac{A^2}{4B}\right) \\
& \times \Big[ 120B^{7/2}\sqrt{\pi} + 180A^2B^{5/2}\sqrt{\pi} + 30A^4B^{3/2}\sqrt{\pi} \\
& + A^6\sqrt{B}\sqrt{\pi} + 120B^{7/2}\sqrt{\pi}\mathrm{erf}\left(\frac{A}{2\sqrt{B}}\right) \\
& + 180A^2B^{5/2}\sqrt{\pi}\mathrm{erf}\left(\frac{A}{2\sqrt{B}}\right) \\
& + 264AB^3\exp\left(-\frac{A^2}{4B}\right) + 30A^4B^{3/2}\sqrt{\pi}\left(\frac{A}{2\sqrt{B}}\right) \\
& + 56A^3B^2\exp\left(-\frac{A^2}{4B}\right) + A^6\sqrt{B}\sqrt{\pi}\mathrm{erf}\left(\frac{A}{2\sqrt{B}}\right) \\
& + 2A^5B\exp\left(-\frac{A^2}{4B}\right)\Big] \, d\theta, \qquad (9)
\end{aligned}
$$

respectively, where $\mathrm{erf}(\cdot)$ denotes the error function defined by

$$
\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x \exp(-t^2)dt.
$$

If $n$ is an even number then one does not need to use (6) since $E(P^n) = E((X^2+Y^2)^{n/2})/\{E(X^2+Y^2)\}^{n/2}$, where $X$ and $Y$ are independent Gaussian random variables (see equation (15) in Yacoub *et al.* [1]).

The cpf of $P$, $F_P(\cdot)$, can be expressed as

$$
\begin{aligned}
F_P(p) &= 1 - \frac{\sqrt{h}(1+\kappa)}{\pi \exp(\kappa)} \int_p^{\infty}\int_0^{2\pi} x\exp(Ax - Bx^2)d\theta dx \\
&= 1 - \frac{\sqrt{h}(1+\kappa)}{\pi \exp(\kappa)} \int_0^{2\pi}\int_p^{\infty} x\exp(Ax - Bx^2)dx d\theta \\
&= 1 - \frac{\sqrt{h}(1+\kappa)}{\pi \exp(\kappa)} \\
&\quad \times \int_0^{2\pi}\left[\int_p^{\infty}(x-p)\exp(Ax - Bx^2)dx \right. \\
&\quad \left. + p\int_p^{\infty}\exp(Ax - Bx^2)dx\right] d\theta \\
&= 1 - \frac{\sqrt{h}(1+\kappa)}{\pi \exp(\kappa)} \int_0^{2\pi}[I_1(\theta) + pI_2(\theta)]\, d\theta. \qquad (10)
\end{aligned}
$$

By equation (2.3.15.1) in Prudnikov *et al.* [2], $I_1(\theta)$ and $I_2(\theta)$ can be calculated as

$$
I_1(\theta) = (2B)^{-1}\exp\left\{\frac{A^2}{8B} + \frac{p(A-pB)}{2}\right\} D_{-2}\left(\frac{2pB - A}{\sqrt{2B}}\right) \qquad (11)
$$

and

$$
I_2(\theta) = (2B)^{-1/2}\exp\left\{\frac{A^2}{8B} + \frac{p(A-pB)}{2}\right\} D_{-1}\left(\frac{2pB - A}{\sqrt{2B}}\right). \qquad (12)
$$

Combining (10), (11) and (12) yields a formula for the cpf $F_P(\cdot)$.

Note that all of the formulas in (4), (6), (7), (8), (9) and (10) involve just one integral (with respect to $\theta$) and are much simpler than those in equations (10) and (11) of Yacoub *et al.* [1]. We feel that the formulas given can help the readers and authors of this journal with respect to modeling the statistical variation of the envelope in a fast fading environment.

## References


[1] M. D. Yacoub, G. Fraidenraich, H. B. Tercius, and F. C. Martins, "The symmetrical $\eta-\kappa$ distribution: A general fading distribution," *IEEE Transactions on Broadcasting*, vol. 51, pp. 504–511, December 2005.

[2] A. P. Prudnikov, Y. A. Brychkov, and O. I. Marichev, *Integrals and Series*. Amsterdam: Gordon and Breach Science Publishers, 1986, vol. 1.