### University of Nebraska - Lincoln Digital Commons@University of Nebraska - Lincoln

Faculty Publications, Department of Statistics

Statistics, Department of

9-1-2006

# On the $\eta$ - $\kappa$ Distribution

Saralees Nadarajah University of Nebraska - Lincoln

Samuel Kotz George Washington University, Washington, DC

Follow this and additional works at: http://digitalcommons.unl.edu/statisticsfacpub



Part of the <u>Statistics and Probability Commons</u>

Nadarajah, Saralees and Kotz, Samuel, "On the  $\eta$  -  $\kappa$  Distribution" (2006). Faculty Publications, Department of Statistics. Paper 6. http://digitalcommons.unl.edu/statisticsfacpub/6

This Article is brought to you for free and open access by the Statistics, Department of at Digital Commons@University of Nebraska - Lincoln. It has been accepted for inclusion in Faculty Publications, Department of Statistics by an authorized administrator of Digital Commons@University of Nebraska - Lincoln.

## **Comments and Corrections**

#### On the $\eta - \kappa$ Distribution

Saralees Nadarajah and Samuel Kotz

Index Terms—Fast fading distribution,  $\eta - \kappa$  distribution.

The recent paper by Yacoub  $et\,al.$  [1] introduces what is referred to as the  $\eta-\kappa$  distribution to describe the statistical variation of the envelope in a fast fading environment. The paper discusses several properties of the distribution. Two of the properties discussed are the nth moment,  $E(P^n)$ , and the cumulative probability function (cpf),  $F_P(\cdot)$ , where P is a random variable representing the normalized envelope. The expression given for  $E(P^n)$  (see equation (10) in Yacoub  $et\,al.$  [1]) is a doubly infinite sum of the Gauss hypergeometric function (which, itself, is an infinite sum). That given for  $F_P(\cdot)$  (see equation (11) in Yacoub  $et\,al.$  [1]) is a triple sum of the incomplete gamma function.

We feel that the expressions in equations (10) and (11) of Yacoub *et al.* [1] are too complicated for practical purposes. In the following, we show how one can derive much simpler forms for  $E(P^n)$  and  $F_P(\cdot)$ . Using equations (5)–(8) in Yacoub *et al.* [1], the probability density function (pdf) of P can be expressed as

$$f_P(p)\frac{\sqrt{h}(1+\kappa)}{\pi \exp(\kappa)} \int_{0}^{2\pi} p \exp(Ap - Bp^2) d\theta, \tag{1}$$

where  $A=2\sqrt{h\kappa(1+\kappa)}\cos\theta$  and  $B=(1+\kappa)h+H(1+\kappa)\cos(2(\theta+\phi))$ . Thus, the nth moment,  $E(P^n)$ , can be expressed as

$$E(P^{n}) = \frac{\sqrt{h}(1+\kappa)}{\pi \exp(\kappa)} \int_{0}^{\infty} \int_{0}^{2\pi} p^{n+1} \exp(Ap - Bp^{2}) d\theta dp$$

$$= \frac{\sqrt{h}(1+\kappa)}{\pi \exp(\kappa)} \int_{0}^{2\pi} \int_{0}^{\infty} p^{n+1} \exp(Ap - Bp^{2}) dp d\theta$$

$$= \frac{\sqrt{h}(1+\kappa)}{\pi \exp(\kappa)} \int_{0}^{2\pi} I(\theta) d\theta.$$
(2)

By equation (2.3.15.3) in Prudnikov *et al.* [2],  $I(\theta)$  can be calculated as

$$I(\theta) = \Gamma(n+2)(2B)^{-(n/2+1)} \exp\left(\frac{A^2}{8B}\right) D_{-n-2} \left(-\frac{A}{\sqrt{2B}}\right), (3)$$

where  $D_p(\cdot)$  denotes the parabolic cylinder function defined by

$$D_p(x) = \frac{\exp(-x^2/4)}{\Gamma(-p)} \int_{0}^{\infty} \frac{\exp\{-(tx + t^2/2)\}}{t^{p+1}} dt.$$

Manuscript received February .

- S. Nadarajah is with the University of Nebraska, Lincoln, NE 68583 USA.
- S. Kotz is with the George Washington University, Washington, DC 20052 USA.

Digital Object Identifier 10.1109/TBC.2006.879862

Combining (2) and (3) yields the formula

$$E(P^{n}) = \frac{\Gamma(n+2)\sqrt{h}(1+\kappa)}{2^{n/2+1}\pi \exp(\kappa)} \int_{0}^{2\pi} B^{-(n/2+1)} \times \exp\left(\frac{A^{2}}{8B}\right) D_{-n-2}\left(-\frac{A}{\sqrt{2B}}\right) d\theta. \quad (4)$$

This formula applies for any real number n>-2. If n is a positive integer then, using equation (2.3.15.7) in Prudnikov *et al.* [2],  $I(\theta)$  can be calculated as

$$I(\theta) = \frac{(-1)^{n+1}\sqrt{\pi}}{2\sqrt{B}} \frac{\partial^{n+1}}{\partial q^{n+1}} \times \left[ \exp\left(\frac{q^2}{4B}\right) \operatorname{erfc}\left(\frac{q}{2\sqrt{B}}\right) \right] |_{q=-A},$$
(5)

where  $\mathrm{erfc}(\cdot)$  denotes the complementary error function defined by

$$\operatorname{erfc}(x) = -\frac{2}{\sqrt{\pi}} \int_{0}^{x} \exp(-t^{2}) dt.$$

Combining (2) and (5) yields the simpler formula

(1) 
$$E(P^n) = \frac{(-1)^{n+1}\sqrt{h}(1+\kappa)}{2\sqrt{\pi}\exp(\kappa)} \int_0^{2\pi} B^{-1/2} \frac{\partial^{n+1}}{\partial q^{n+1}} \times \left[ \exp\left(\frac{q^2}{4B}\right) \operatorname{erfc}\left(\frac{q}{2\sqrt{B}}\right) \right]_{q=-4} d\theta.$$
 (6)

Various simple expressions can be obtained from (6) by setting specific values for n. For instance, if n=1, n=3 and n=5 then (6) can be reduced to the simple forms

$$E(P) = \frac{\sqrt{h}(1+\kappa)}{8\pi \exp(\kappa)} \int_{0}^{2\pi} B^{-3} \exp\left(\frac{A^{2}}{4B}\right)$$

$$\times \left[2B^{3/2}\sqrt{\pi} + A^{2}\sqrt{B}\sqrt{\pi} + 2B^{3/2}\sqrt{\pi} \operatorname{erf}\left(\frac{A}{2\sqrt{B}}\right) + A^{2}\sqrt{B}\sqrt{\pi}\operatorname{erf}\left(\frac{A}{2\sqrt{B}}\right)\right]$$

$$+ A^{2}\sqrt{B}\sqrt{\pi}\operatorname{erf}\left(\frac{A}{2\sqrt{B}}\right)$$

$$+2AB \exp\left(-\frac{A^{2}}{4B}\right)d\theta, \qquad (7)$$

$$E(P^{3}) = \frac{\sqrt{h}(1+\kappa)}{32\pi \exp(\kappa)} \int_{0}^{2\pi} B^{-5} \exp\left(\frac{A^{2}}{4B}\right)$$

$$\times \left[12B^{5/2}\sqrt{\pi} + 12A^{2}B^{3/2}\sqrt{\pi} + A^{4}\sqrt{B}\sqrt{\pi} + 12B^{5/2}\sqrt{\pi}\operatorname{erf}\left(\frac{A}{2\sqrt{B}}\right) + 12A^{2}B^{3/2}\right]$$

$$\times \sqrt{\pi}\operatorname{erf}\left(\frac{A}{2\sqrt{B}}\right) + 20AB^{2} \exp\left(-\frac{A^{2}}{4B}\right)$$

$$+ A^{4}\sqrt{B}\sqrt{\pi}\operatorname{erf}\left(\frac{A}{2\sqrt{B}}\right)$$

$$+ 2A^{3}B \exp\left(-\frac{A^{2}}{4B}\right)d\theta \qquad (8)$$

412

and

$$\begin{split} E(P^5) &= \frac{\sqrt{h}(1+\kappa)}{128\pi \exp(\kappa)} \int\limits_0^{2\pi} B^{-7} \exp\left(\frac{A^2}{4B}\right) \\ &\times \left[120B^{7/2}\sqrt{\pi} + 180A^2B^{5/2}\sqrt{\pi} + 30A^4B^{3/2}\sqrt{\pi} \right. \\ &\quad + A^6\sqrt{B}\sqrt{\pi} + 120B^{7/2}\sqrt{\pi}\mathrm{erf}\left(\frac{A}{2\sqrt{B}}\right) \\ &\quad + 180A^2B^{5/2}\sqrt{\pi}\mathrm{erf}\left(\frac{A}{2\sqrt{B}}\right) \\ &\quad + 264AB^3\exp\left(-\frac{A^2}{4B}\right) + 30A^4B^{3/2}\sqrt{\pi}\left(\frac{A}{2\sqrt{B}}\right) \\ &\quad + 56A^3B^2\exp\left(-\frac{A^2}{4B}\right) + A^6\sqrt{B}\sqrt{\pi}\mathrm{erf}\left(\frac{A}{2\sqrt{B}}\right) \\ &\quad + 2A^5B\exp\left(-\frac{A^2}{4B}\right)\right]d\theta\,, \end{split} \tag{9}$$

respectively, where  $\operatorname{erf}(\cdot)$  denotes the error function defined by

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_{0}^{x} \exp(-t^{2}) dt.$$

If n is an even number then one does not need to use (6) since  $E(P^n) = E((X^2 + Y^2)^{n/2})/\{E(X^2 + Y^2)\}^{n/2}$ , where X and Y are independent Gaussian random variables (see equation (15) in Yacoub  $et\ al.\ [1]$ ). The cpf of  $P, F_P(\cdot)$ , can be expressed as

$$F_{P}(p) = 1 - \frac{\sqrt{h}(1+\kappa)}{\pi \exp(\kappa)} \int_{p}^{\infty} \int_{0}^{2\pi} x \exp(Ax - Bx^{2}) d\theta dx$$

$$= 1 - \frac{\sqrt{h}(1+\kappa)}{\pi \exp(\kappa)} \int_{0}^{2\pi} \int_{p}^{\infty} x \exp(Ax - Bx^{2}) dx d\theta$$

$$= 1 - \frac{\sqrt{h}(1+\kappa)}{\pi \exp(\kappa)}$$

$$\times \int_{0}^{2\pi} \left[ \int_{p}^{\infty} (x-p) \exp(Ax - Bx^{2}) dx + p \int_{p}^{\infty} \exp(Ax - Bx^{2}) dx \right] d\theta$$

$$= 1 - \frac{\sqrt{h}(1+\kappa)}{\pi \exp(\kappa)} \int_{0}^{2\pi} \left[ I_{1}(\theta) + p I_{2}(\theta) \right] d\theta. \tag{10}$$

By equation (2.3.15.1) in Prudnikov et~al. [2],  $I_1(\theta)$  and  $I_2(\theta)$  can be calculated as

$$I_1(\theta) = (2B)^{-1} \exp\left\{\frac{A^2}{8B} + \frac{p(A - pB)}{2}\right\} D_{-2} \left(\frac{2pB - A}{\sqrt{2B}}\right)$$
(11)

and

$$I_2(\theta) = (2B)^{-1/2} \exp\left\{\frac{A^2}{8B} + \frac{p(A-pB)}{2}\right\} D_{-1}\left(\frac{2pB-A}{\sqrt{2B}}\right).$$
 (12)

Combining (10), (11) and (12) yields a formula for the cpf  $F_P(\cdot)$ .

Note that all of the formulas in (4), (6), (7), (8), (9) and (10) involve just one integral (with respect to  $\theta$ ) and are much simpler than those in equations (10) and (11) of Yacoub *et al.* [1]. We feel that the formulas given can help the readers and authors of this journal with respect to modeling the statistical variation of the envelope in a fast fading environment.

#### REFERENCES

- [1] M. D. Yacoub, G. Fraidenraich, H. B. Tercius, and F. C. Martins, "The symmetrical  $\eta \kappa$  distribution: A general fading distribution," *IEEE Transactions on Broadcasting*, vol. 51, pp. 504–511, December 2005.
- [2] A. P. Prudnikov, Y. A. Brychkov, and O. I. Marichev, *Integrals and Series*. Amsterdam: Gordon and Breach Science Publishers, 1986, vol. 1.

#### Corrections to "A General SFN Structure With Transmit Diversity for TDS-OFDM System"

Jin-Tao Wang, Jian Song, Jun Wang, Chang-Yong Pan, Zhi-Xing Yang, and Lin Yang

In the above paper [1], the first author's name was misspelled in the byline: "Jian-Tao Wang" should have read: "Jin-Tao Wang".

The corrected byline should read:

Jin-Tao Wang, Jian Song, Jun Wang, Chang-Yopng Pan, Zhi-Xing Yang, and Lin Yang

#### REFERENCES

[1] J.-T. Wang, J. Song, J. Wang, C.-Y. Pan, Z.-X. Yang, and L. Yang, "A general SFN structure with transmit diversity for TDS-OFDM system," *IEEE Trans. Broadcasting*, vol. 52, no. 2, pp. 245–251, Jun. 2006.

Manuscript received June 7, 2006. Digital Object Identifier 10.1109/TBC.2006.883052