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Properties of the Generalized Laplace Transform and Transport Partial Dynamic Equation on Time Scales

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PROPERTIES OF THE GENERALIZED LAPLACE TRANSFORM AND TRANSPORT PARTIAL DYNAMIC EQUATION ON TIME SCALES

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In this dissertation, we first focus on the generalized Laplace transform on time scales. We prove several properties of the generalized exponential function which will allow us to explore some of the fundamental properties of the Laplace transform. We then give a description of the region in the complex plane for which the improper integral in the definition of the Laplace transform converges, and how this region is affected by the time scale in question. Conditions under which the Laplace transform of a power series can be computed term-by-term are given. We develop a formula for the Laplace transform for periodic functions on a periodic time scale. Regressivity and its relationship to the Laplace transform is examined, and the Laplace transform for several functions is explicitly computed. Finally, we explore some inversion formulas for the Laplace transform via contour integration.

In Chapter 4, we develop two recursive representations for the unique solution of the transport partial dynamic equation on an isolated time scale. We then use these representations to explicitly find the solution of the transport equation in several specific cases. Finally, we compare and contrast the behavior with that of the well-known behavior of the solution to the transport partial difference equation in the case where $\mathbb{T} = \mathbb{Z}$.

DEDICATION

This dissertation is dedicated to my wife Carol and my daughter Michelle.

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Chapter 1

Introduction

In this dissertation we will use the tools of the time scales calculus to explore several properties of the generalized exponential function, the generalized Laplace transform on time scales, and the transport partial dynamic equation on time scales.

In 1988 Stefan Hilger introduced the concept of the time scale calculus in his Ph.D. dissertation [16] as a means to unify continuous and discrete analysis. Many results one encounters in the study of both differential and difference equations have analogs in the time scale case. However, the time scale result encompasses both the discrete and continuous results as special cases. In addition, the time scale calculus is a rich source of interesting problems that do not have any natural equivalent in either the continuous or discrete case.

While unification is certainly a goal in studying time scales, another focus is to extend and explain the results of differential and difference equations. For example, if we consider the second-order self-adjoint differential equation $(px')'(t) + q(t)x = 0$, it is well known that nonzero solutions to this equation correspond, via the so-called

Riccati substitution, to solutions of the differential Riccati equation,

$$z' + q(t) + \frac{z^2}{p(t)} = 0. \tag{1.1}$$

In the difference equation case, the well-known second-order self-adjoint equation $\Delta(p\Delta x)(t) + q(t)x(t+1) = 0$ has nonzero solutions that correspond to the solutions of the discrete Riccati equation,

$$\Delta z(t) + q(t) + \frac{z^2(t)}{z(t) + p(t)} = 0. \tag{1.2}$$

It is through analysis using the techniques of time scales that we can not only unify these two concepts, but also explain why the discrete Riccati equation has a different form than the continuous Riccati equation. It turns out that nonzero solutions for the dynamic equation $(px^{\Delta})^{\Delta}(t) + q(t)x^{\sigma}(t) = 0$ on a time scale $\mathbb{T}$ correspond to the solutions of the time scale Riccati equation,

$$z^{\Delta}(t) + q(t) + \frac{z^2(t)}{\mu(t)z(t) + p(t)} = 0. \tag{1.3}$$

Since $z^{\Delta}(t) = z'(t)$ if $\mathbb{T} = \mathbb{R}$, and $z^{\Delta}(t) = \Delta z(t)$ if $\mathbb{T} = \mathbb{Z}$, we find that the $\mu(t)$ term explains the differences between the continuous and discrete cases. If $\mathbb{T} = \mathbb{R}$, then $\mu(t) \equiv 0$ and (1.3) reduces to the well-known continuous case (1.1); and if $\mathbb{T} = \mathbb{Z}$, then $\mu(t) \equiv 1$ and (1.3) reduces to the well-known discrete case (1.2). (Note that in Section 2.1 the notation used here will be introduced and explained.)

After a quick introduction to the time scale calculus, we will look at four new results that deal with the generalized exponential function on time scales (see Section 2.2). These results will be fundamental in many of the proofs we give in Chapter 3. The results pertaining to the generalized exponential function range from asymptotic properties to determining the region of analyticity.

In Chapter 3, we will turn our attention to the generalized Laplace transform on time scales. The definition of the Laplace transform that we will be concerned with here was first given by Bohner and Peterson in the paper "Laplace transform and Z-transform: Unification and extension," (see [5]). Note that this definition differs from that given by Stefan Hilger in [17]. A special case of this transform in the discrete setting is given in Donahue's honors thesis [11] written under the supervision of Paul Eloe. A thorough introduction to the Laplace transform we will define and examine here is also given in [4], and some further properties that we will make use of are given in [3]. A generalization of the Laplace transform to so-called α-derivatives on generalized time scales can be found in [2].

We will define the generalized Laplace transform in Section 3.1 and proceed to examine its properties in the sections that follow. Specifically, we will examine the region in the complex plane for which the Laplace transform converges with various assumptions on the time scale in question. In this section, the conclusions of Theorems 3.2.2 and 3.2.5 are similar to those obtained in [10] by Davis, et al.; however, we have extended and clarified these results by relaxing the assumptions necessary on the time scale. Further, the proofs given here are substantially different than those in [10].

We then work through several results of the Laplace transform when applied to power series and with respect to periodic time scales. We show that in situations involving initial value problems in which the dynamic equation is not regressive on certain time scales, we can still apply the Laplace transform in order to solve these problems. We conclude our examination of some of the properties of the transform by directly calculating several Laplace transforms. Finally, we develop two inversion formulas which are markedly different from the one given by Davis, et al., in [10] as we will use the techniques of contour integration to achieve the results in this dissertation.

In Chapter 4, we change gears and focus on the transport partial dynamic equation. We will look at the specific case when the time scale is isolated. In this case, we show that the solution to the transport partial dynamic equation with a given initial condition is unique. Further, we develop two different recursive representations of this solution. Then, using these representations, we find explicitly the unique solution for several specific time scales. These particular examples prove to be quite interesting since they depart significantly from the behavior observed in the continuous and discrete cases.

Chapter 2

Time Scale Preliminaries

2.1 The Time Scale Calculus

A detailed introduction to the time scale calculus is given in [4] and [15]. In this section we collect the definitions and theorems that will be most useful to us.

Definition 2.1.1. A *time scale*, denoted $\mathbb{T}$, is a nonempty, closed subset of $\mathbb{R}$. For $a, b \in \mathbb{T}$ such that $a < b$, we let $[a, b]_{\mathbb{T}}$ denote the set $[a, b] \cap \mathbb{T}$.

Definition 2.1.2. Let $\mathbb{T}$ be a time scale. For $t \in \mathbb{T}$, we define the *forward jump operator* $\sigma : \mathbb{T} \to \mathbb{T}$ by

$$\sigma(t) := \inf\{s \in \mathbb{T} : s > t\},$$

and the *backward jump operator* $\rho : \mathbb{T} \to \mathbb{T}$ is defined by

$$\rho(t) := \sup\{s \in \mathbb{T} : s < t\}.$$

In these definition we use the convention that $\inf(\emptyset) = \sup(\mathbb{T})$ and $\sup(\emptyset) = \inf(\mathbb{T})$.

For $f : \mathbb{T} \to \mathbb{R}$, we frequently use the notation $f^\sigma(t)$ for the composition $f(\sigma(t))$.

Definition 2.1.3. The *graininess function* $\mu : \mathbb{T} \to [0, \infty)$ is defined by

$$\mu(t) := \sigma(t) - t.$$

The above definitions for the forward and backward jump operators lend to a natural classification of the points in a time scale:

Definition 2.1.4. Let $t \in \mathbb{T}$. If $\sigma(t) = t$ and $t \neq \sup(\mathbb{T})$, then t is *right-dense*. If $\sigma(t) > t$, then t is *right-scattered*. Similarly, if $\rho(t) = t$ and $t \neq \inf(\mathbb{T})$, then t is *left-dense*, and if $\rho(t) < t$, then t is *left-scattered*. If a point $t \in \mathbb{T}$ is both right-scattered and left-scattered, we say that t is an *isolated point*.

Definition 2.1.5. If a time scale $\mathbb{T}$ is composed completely of isolated points, we say that $\mathbb{T}$ is an *isolated time scale*.

Definition 2.1.6. If $\sup(\mathbb{T}) = m$ such that m is left-scattered, then define $\mathbb{T}^\kappa := \mathbb{T} \backslash \{m\}$; otherwise, define $\mathbb{T}^\kappa := \mathbb{T}$.

Definition 2.1.7. A function $f : \mathbb{T} \to \mathbb{R}$ is *rd-continuous* provided it is continuous at right-dense points in $\mathbb{T}$ and its left-sided limits exist and are finite at all left-dense points in $\mathbb{T}$.

A function $p : \mathbb{T} \to \mathbb{R}$ is *regressive* provided $1 + \mu(t)p(t) \neq 0$ for all $t \in \mathbb{T}^\kappa$. The set of all regressive and rd-continuous functions on a time scale $\mathbb{T}$ is denoted by $\mathcal{R} = \mathcal{R}(\mathbb{T})$. A function $p : \mathbb{T} \to \mathbb{R}$ is *positively regressive* provided $1 + \mu(t)p(t) > 0$ for all $t \in \mathbb{T}^\kappa$. The set of all positively regressive and rd-continuous functions is denoted by $\mathcal{R}^+ = \mathcal{R}^+(\mathbb{T})$.

Throughout, we will use the following abuse of notation: we will write $z \in \mathbb{C} \cap \mathcal{R}$ to mean $z \in \mathbb{C}$ and $1 + z\mu(t) \neq 0$ for all $t \in \mathbb{T}$. In other words, when z is viewed as a constant function from $\mathbb{T}$ to $\mathbb{C}$, it is indeed an element of $\mathcal{R}$. However, we will always take $\mathbb{C} \cap \mathcal{R} \subseteq \mathbb{C}$.

We are now in a position to define the generalization of the classical derivative, the so-called Δ-derivative, as well as the generalization of the classical integral, the Δ-integral, for an arbitrary time scale.

Definition 2.1.8. Let $f : \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}^\kappa$. If t is a right-scattered point, the Δ-*derivative* of f is defined to be

$$f^\Delta(t) := \frac{f(\sigma(t)) - f(t)}{\sigma(t) - t} = \frac{f^\sigma(t) - f(t)}{\mu(t)}.$$

Otherwise, we define it to be

$$f^\Delta(t) := \lim_{s \to t} \frac{f(t) - f(s)}{t - s}$$

provided this limit exists.

Definition 2.1.9. The *Cauchy* Δ-*integral* of a function $f : \mathbb{T} \to \mathbb{R}$ is defined as

$$\int_a^b f(t)\Delta t := F(b) - F(a),$$

where $F : \mathbb{T} \to \mathbb{R}$ is an antiderivative of f.

Of course, defining the Δ-integral to be antidifferentiation as is done in Definition 2.1.9 is extremely restrictive. It turns out that the Δ-integral can be developed as a Riemann integral (see, for example, [14]). In fact, this development is almost identical to what one would encounter in an introductory analysis course for the classical Riemann integral. Further, the Δ-integral can be developed as a Lebesgue integral. Since this will prove useful to us in this dissertation, some of this development is given below starting with Theorem 2.1.18. A thorough treatment of building the Δ-integral as a Lebesgue integral can be found in [8] and [13].

Definition 2.1.10. The *generalized Taylor monomials*, $h_k : \mathbb{T} \times \mathbb{T} \to \mathbb{R}$ for $k \in \mathbb{N}_0$, are defined recursively as follows:

$$h_0(t,s) \equiv 1, \text{ and}$$
$$h_{k+1}(t,s) = \int_s^t h_k(\tau,s)\Delta\tau,$$

for all $s,t \in \mathbb{T}$, and $k \in \mathbb{N}_0$.

In the next definition, we define an important function on a time scale: the generalized exponential function, $e_p(t,t_0)$. Properties of this function will play a pivotal role in obtaining many of the results found in Chapter 3.

Definition 2.1.11. For $p \in \mathcal{R}$, the *generalized exponential function* $e_p : \mathbb{T} \times \mathbb{T} \to \mathbb{R}$ is defined by

$$e_p(t,s) = \exp\left(\int_s^t \xi_{\mu(\tau)}(p(\tau))\Delta\tau\right),$$

for $s,t \in \mathbb{T}$, where the *cylinder transformation*, $\xi_h : \mathbb{C}\backslash\left\{-\frac{1}{h}\right\} \to \mathbb{C}$ for $h > 0$, is given by

$$\xi_h(z) = \frac{1}{h}\operatorname{Log}(1+zh),$$

and $\xi_0(z) = z$.

The proofs of the following theorems can be found in [4].

Theorem 2.1.12. Let $\mathbb{T}$ be a time scale and $t_0 \in \mathbb{T}$. If $p \in \mathcal{R}^+$, then

$$1 + \int_{t_0}^t p(\tau)\Delta\tau \leq e_p(t,t_0) \quad \text{for all} \quad t \in [t_0,\infty)_{\mathbb{T}}.$$

Definition 2.1.13. For $p, q \in \mathcal{R}$, we define *circle plus addition*, denoted $\oplus$, and *circle minus*, denoted $\ominus$, as follows:

$$(p \oplus q)(t) := p(t) + q(t) + \mu(t)p(t)q(t) \quad \text{and} \quad (\ominus p)(t) := \frac{-p(t)}{1 + p(t)\mu(t)},$$

for all $t \in \mathbb{T}^\kappa$.

It should be noted that $(\mathcal{R}, \oplus)$ is an abelian group. With $\oplus$ and $\ominus$ in hand, we are able to obtain several basic properties of the generalized exponential function. The proofs of the next two theorems are given in [4], Theorem 2.36 and Theorem 2.44, respectively.

Theorem 2.1.14. If $p, q \in \mathcal{R}$ and $t, s, r \in \mathbb{T}$, then

(a) $e_0(t, s) \equiv 1$ and $e_p(t, t) \equiv 1$,

(b) $e_p(\sigma(t), s) = (1 + \mu(t)p(t))e_p(t, s)$,

(c) $e_p(s, t) = \frac{1}{e_p(t,s)} = e_{\ominus p}(t, s)$,

(d) $e_p(t, s)e_p(s, r) = e_p(t, r)$, and

(e) $e_p(t, s)e_q(t, s) = e_{p \oplus q}(t, s)$.

Note that (d) is often referred to as the semigroup property.

Theorem 2.1.15. Assume $p \in \mathcal{R}^+$ and $t_0 \in \mathbb{T}$. Then $e_p(t, t_0) > 0$ for all $t \in \mathbb{T}$.

Definition 2.1.16. For $p : \mathbb{T} \to \mathbb{R}$ such that $\mu p^2 \in \mathcal{R}$, we define the *trigonometric functions* by

$$\cos_p(t, t_0) = \frac{e_{ip}(t, t_0) + e_{-ip}(t, t_0)}{2} \quad \text{and} \quad \sin_p(t, t_0) = \frac{e_{ip}(t, t_0) - e_{-ip}(t, t_0)}{2i}.$$

Definition 2.1.17. For each $t \in [t_0, \infty)_{\mathbb{T}}$, we define

$$T_{rd}(t) := \{s \in [t_0, t)_{\mathbb{T}} : \sigma(s) = s\}, \quad \text{and} \quad T_{rs}(t) := \{s \in [t_0, t)_{\mathbb{T}} : \sigma(s) > s\}.$$

Note that if t_0 is right-scattered, we take $T_{rs}(t_0) = \{t_0\}$, and similarly, if t_0 is right-dense, we take $T_{rd}(t_0) = \{t_0\}$.

So, $T_{rd}(t)$ is the set of all right-dense points in $[t_0, \infty)_{\mathbb{T}}$ strictly less than t. For consistency in notation, we will denote the set of *all* right-dense points in $[t_0, \infty)_{\mathbb{T}}$ by $T_{rd}(\infty)$, and similarly, the set of all right-scattered points in $[t_0, \infty)_{\mathbb{T}}$ will be denoted by $T_{rs}(\infty)$.

As noted above, Δ-integration can be realized via the Lebesgue Δ-measure, μ_Δ, on $\mathbb{T}$. This measure is briefly introduced in [6, Section 5.7] and fleshed out in detail by Cabada and Vivero in [8]. Hence, we can apply general measure theory results (such as the Dominated Convergence Theorem) to the Δ-integral.

Throughout this work, we would like to evaluate the Δ-integral over $[t_0, t)_{\mathbb{T}}$ by integrating over the sets $T_{rd}(t)$ and $T_{rs}(t)$, and then adding. This is only legal if $T_{rd}(t)$ and $T_{rs}(t)$ are Δ-measurable.

Theorem 2.1.18. The set of all right-scattered points, $T_{rs}(\infty)$, is countable.

Proof. For each $k \in \mathbb{Z}$ and $n \in \mathbb{N}$, define

$$A_{n,k} := \left\{ t \in [t_0, \infty)_{\mathbb{T}} \cap [k, k+1) : \mu(t) \geq \frac{1}{n} \right\}.$$

Note that $A_{n,k} \subset [k, k+1)$ is bounded, and for any distinct $s, t \in A_{n,k}$, we have $|t - s| \geq \frac{1}{n}$. Therefore, $A_{n,k}$ has only finitely many elements. Hence, for any fixed $k \in \mathbb{Z}$, $A_k := \bigcup_{n=1}^{\infty} A_{n,k}$ is countable. Further, by construction, A_k is precisely the set of all right-scattered points in $[t_0, \infty)_{\mathbb{T}} \cap [k, k+1)$. Therefore, $T_{rs}(\infty) = \bigcup_{k=-\infty}^{\infty} A_k$, which is the countable union of countable sets, and hence is countable. $\square$

Note that in the proceeding theorem, we proved the set of all right-scattered points in the unbounded set $[t_0, \infty)_{\mathbb{T}}$ is countable. Cabada and Vivero prove an identical result for a bounded time scale via a different method. They define a monotone increasing function on a real interval that has jump discontinuities precisely at the right-scattered points of $\mathbb{T}$, and hence the set of right-scattered points is countable (see [8, Lemma 3.1]).

Corollary 2.1.19. For any $t \in [t_0, \infty)_{\mathbb{T}}$, the sets $T_{rs}(t)$ and $T_{rd}(t)$ are Δ-measurable. Further, the sets $T_{rs}(\infty)$ and $T_{rd}(\infty)$ are Δ-measurable.

Proof. Since for any $t \in [t_0, \infty)_{\mathbb{T}}$, the singleton set $\{t\}$ is Δ-measurable (see [6, Section 5.7]), it follows that $T_{rs}(\infty)$, a countable union of singletons, is Δ-measurable. Since $T_{rd}(\infty) = [t_0, \infty)_{\mathbb{T}} \cap [T_{rs}(\infty)]^c$, we have that $T_{rd}(\infty)$ is Δ-measurable. Finally, fix $t \in [t_0, \infty)_{\mathbb{T}}$. Since all intervals of the form $[a, b)_{\mathbb{T}}$ for $a, b \in \mathbb{T}$ are Δ-measurable, $T_{rd}(t) = T_{rd}(\infty) \cap [t_0, t)_{\mathbb{T}}$ and $T_{rs}(t) = T_{rs}(\infty) \cap [t_0, t)_{\mathbb{T}}$ are Δ-measurable. $\square$

Theorem 2.1.20. Let $E \subseteq [t_0, \infty)_{\mathbb{T}}$ be nonempty such that E is Δ-measurable and contains no right-scattered points. Let $f : E \to \mathbb{R}$ be a Δ-measurable function. Then,

$$\int_E f(t) d\mu_\Delta(t) = \int_E f(t) dm(t),$$

where m is the usual Lebesgue measure on $\mathbb{R}$.

Proof. Let $A \subseteq E$ be an arbitrary, Δ-measurable set. Since A is Δ-measurable, for each $k \in \mathbb{N}_0$, we have that each $A_k := A \cap [t_0 + k, t_0 + k + 1)$ is also Δ-measurable. Since each A_k is bounded, $t_0 + k + 1 \notin A_k$, and A_k contains no right-scattered points, by Proposition 3.1 in [8], it follows that $\mu_\Delta(A_k) = m(A_k)$ for all $k \in \mathbb{N}_0$. Noting that $A_i \cap A_j = \emptyset$ for $i \neq j$, we have

$$\mu_\Delta(A) = \mu_\Delta\left(\bigcup_{k=0}^{\infty} A_k\right) = \sum_{k=0}^{\infty} \mu_\Delta(A_k) = \sum_{k=0}^{\infty} m(A_k) = m\left(\bigcup_{k=0}^{\infty} A_k\right) = m(A).$$

Therefore, since $\mu_\Delta(A) = m(A)$ for all Δ-measurable $A \subseteq E$, it follows from standard measure theory arguments that

$$\int_E f(t) d\mu_\Delta(t) = \int_E f(t) dm(t).$$

□

We conclude this section by stating a very useful result from Cabada and Vivero (see [8, Theorem 5.2]). We have rephrased the statement slightly to match our notation.

Theorem 2.1.21. Let $E \subseteq [t_0, \infty)_{\mathbb{T}}$ be any bounded, Δ-measurable set, and $f : \mathbb{T} \to \mathbb{R}$ be Δ-measurable. If $\sup(E) \in E$ and is a right-scattered point, then take $\tilde{E} = E \backslash \{\sup(E)\}$. Otherwise, take $\tilde{E} = E$. Then,

$$\int_E f(t) d\mu_\Delta(t) = \int_E f(t) dm(t) + \sum_{t \in \tilde{E} \cap T_{rs}(\infty)} f(t)\mu(t).$$

From this point forward, when writing integrals, we will stop explicitly mentioning the measures in question. i.e., $\int_E f(t)\Delta t$ denotes integration with respect to the μ_Δ measure, and $\int_E f(t)dt$ denotes integration with respect to the Lebesgue measure.

2.2 Properties of the Generalized Exponential Function

In addition to the basic properties of the generalized exponential function outlined in the previous section and in Chapter 2 of [4], we will utilize several other properties throughout. The proofs of these four results first appeared in my paper [1].

In the first lemma, we compute the limit as t goes to infinity of $e_{\alpha\ominus x}(t,t_0)$ where $\alpha, x \in \mathbb{R}$ such that $x > \alpha$ and $1+\alpha\mu(t) > 0$ for all $t \in \mathbb{T}$. Of course, this result is obvious when the time scale in question is $\mathbb{T} = \mathbb{R}$ since, in this case, $e_{\alpha\ominus x}(t,t_0) = e^{(\alpha-x)(t-t_0)}$. However, since the constant $\alpha - x$ generalizes to a function dependent on t, namely $(\alpha\ominus x)(t)$, some care is required in obtaining the desired result on an arbitrary time scale. This lemma and Lemma 2.2.2 (which relates a generalized exponential involving a complex number z to a generalized exponential involving $\text{Re}(z)$, the real part of z) will be used in tandem to generalize several well-known results about the classical Laplace transform on $\mathbb{R}$. In Lemma 2.2.3, we will give a detailed proof that the generalized exponential function is analytic on the domain $\Omega := \mathbb{C}\backslash\{x \in \mathbb{R} : x \leq 0\}$. Finally, we show that complex conjugation behaves as expected with the generalized exponential function; if $p : \mathbb{T} \to \mathbb{R}$, then $\overline{e_{ip}(t,t_0)} = e_{-ip}(t,t_0)$.

Lemma 2.2.1. Let $\mathbb{T}$ be unbounded above. Fix $t_0 \in \mathbb{T}$ and let $x > \alpha$ such that $\alpha \in \mathcal{R}^+$ be given. Then,

$$\lim_{t\to\infty} e_{x\ominus\alpha}(t,t_0) = +\infty, \quad \text{and} \quad \lim_{t\to\infty} e_{\alpha\ominus x}(t,t_0) = 0.$$

Proof. We first consider the limit of $e_{x\ominus\alpha}(t,t_0)$ as $t \to \infty$.

Since $\mathbb{T}$ is unbounded above, at least one of the following must hold:

(a) $m(T_{rd}(\infty)) = +\infty$, or

(b) $\sum_{t \in T_{rs}(\infty)} \mu(t) = +\infty$.

Here m is the classical Lebesgue measure on $\mathbb{R}$.

Note that $(x \ominus \alpha)(t) = \frac{x-\alpha}{1+\alpha\mu(t)} > 0$, and hence $1 + (x \ominus \alpha)(t)\mu(t) > 0$ for all $t \in \mathbb{T}$. Thus, $x \ominus \alpha \in \mathcal{R}^+$. By Theorem 2.1.12, we have

$$\begin{aligned} e_{x\ominus\alpha}(t, t_0) &\geq 1 + \int_{t_0}^{t} (x \ominus \alpha)(\tau)\Delta\tau \\ &= 1 + \int_{T_{rd}(t)} \frac{x-\alpha}{1+\alpha\mu(\tau)}\Delta\tau + \int_{T_{rs}(t)} \frac{x-\alpha}{1+\alpha\mu(\tau)}\Delta\tau \\ &= 1 + (x-\alpha)\int_{T_{rd}(t)} d\tau + (x-\alpha)\sum_{\tau\in T_{rs}(t)} \frac{\mu(\tau)}{1+\alpha\mu(\tau)} \quad \text{(by Theorem 2.1.21)} \\ &= 1 + (x-\alpha)m(T_{rd}(t)) + (x-\alpha)\sum_{\tau\in T_{rs}(t)} \frac{\mu(\tau)}{1+\alpha\mu(\tau)}. \end{aligned} \tag{2.1}$$

In the above calculation, when applying Theorem 2.1.21 to the integral over $T_{rs}(t)$, we note that $\sup\{T_{rs}(t)\} \leq t$ and $t \notin T_{rs}(t)$ by the definition of the set $T_{rs}(t)$, and hence the integral reduces to a sum over the entire set $T_{rs}(t)$.

Case 1: Assume $m(T_{rd}(\infty)) = +\infty$ holds.

From (2.1), since every term is nonnegative, we have $e_{x\ominus\alpha}(t, t_0) \geq (x-\alpha)m(T_{rd}(t))$. Since $m(T_{rd}(\infty)) = \infty$, it follows that $T_{rd}(t) \to \infty$ as $t \to \infty$. Therefore, $e_{x\ominus\alpha}(t, t_0) \to \infty$ as $t \to \infty$.

Case 2: Assume $\sum_{t\in T_{rs}(\infty)} \mu(t) = +\infty$ holds.

Again from (2.1), since every term is nonnegative, we have that

$$e_{x\ominus\alpha}(t, t_0) \geq (x-\alpha)\sum_{\tau\in T_{rs}(t)} \frac{\mu(\tau)}{1+\alpha\mu(\tau)}. \tag{2.2}$$

Because $\sum_{t\in T_{rs}(\infty)} \mu(t) = \infty$, we can choose $\{t_1, t_2, \cdots\} \subseteq T_{rs}(\infty)$ such that

(a) $t_1 < t_2 < \cdots$,

(b) $\lim_{n\to\infty} t_n = \infty$, and

(c) $\sum_{n=1}^{\infty} \mu(t_n) = \infty$.

Note that for each $t \in \mathbb{T}$, $T_{rs}(t) \cap \{t_1, t_2, \cdots\}$ is a finite set as $\sup\{T_{rs}(t)\} \le t$ and $\lim_{n\to\infty} t_n = \infty$. Hence, for each $t \in \mathbb{T}$ where $t \ge t_1$, there exists $n_0 = n_0(t) \in \mathbb{N}$ such that $T_{rs}(t) \cap \{t_1, t_2, \cdots\} = \{t_1, t_2, \cdots, t_{n_0}\}$. Further, by the definition of $T_{rs}(t)$, the construction of $\{t_n\}_{n=1}^{\infty}$, and our choice of n_0, we have that as $t \to \infty$, $n_0 \to \infty$. Therefore, from (2.2), for $t \ge t_1$,

$$\begin{aligned} e_{x\ominus\alpha}(t, t_0) &\ge (x-\alpha) \sum_{\tau\in T_{rs}(t)} \frac{\mu(\tau)}{1+\alpha\mu(\tau)} \\ &\ge (x-\alpha) \sum_{k=1}^{n_0} \frac{\mu(t_k)}{1+\alpha\mu(t_k)}. \end{aligned} \tag{2.3}$$

We aim to show that the series $\sum_{k=1}^{\infty} \frac{\mu(t_k)}{1+\alpha\mu(t_k)}$ diverges. We will consider two subcases.

<u>Subcase 1:</u> Assume $\lim_{n\to\infty} \mu(t_n) = 0$.

Note that in this case

$$\lim_{n\to\infty} \frac{\mu(t_n)}{1+\alpha\mu(t_n)} \left(\frac{1}{\mu(t_n)}\right) = \lim_{n\to\infty} \frac{1}{1+\alpha\mu(t_n)} = 1 > 0,$$

so by the Limit Comparison Test, since $\sum_{n=1}^{\infty} \mu(t_n)$ diverges, $\sum_{n=1}^{\infty} \frac{\mu(t_n)}{1+\alpha\mu(t_n)}$ also diverges.

<u>Subcase 2:</u> Assume $\lim_{n\to\infty} \mu(t_n) \ne 0$.

In this case, there exists a subsequence $\{t_{n_k}\}_{k=1}^{\infty}$ such that $\mu(t_{n_k}) \ge \epsilon_0 > 0$, for some ϵ_0. Note that if we define $f(x) := \frac{x}{1+\alpha x}$, then $f'(x) = \frac{1}{(1+\alpha x)^2}$. So, in particular, if $\alpha \geq 0$, then $f'(x) > 0$ for all $x \in (0, \infty)$, and if $\alpha < 0$, then $f'(x) > 0$ for all $x \in \left(0, -\frac{1}{\alpha}\right)$.

If $\alpha \geq 0$, then $1 + \alpha x \neq 0$ for all $x \in [0, \infty)$ and f is continuous (and increasing) on the entire interval $(0, \infty)$. Therefore, since $\mu(t_{n_k}) \geq \epsilon_0$, it follows that $f(\mu(t_{n_k})) \geq f(\epsilon_0)$.

If $\alpha < 0$, then f is continuous on $\left(0, -\frac{1}{\alpha}\right)$ and is increasing on this interval. Now, $\alpha \in \mathcal{R}^+$ implies $1 + \alpha\mu(t) > 0$ which implies $\mu(t) < \frac{-1}{\alpha}$ for all $t \in \mathbb{T}$ since $\alpha < 0$. Therefore, $\mu(t) \in \left[0, \frac{-1}{\alpha}\right)$ for all $t \in \mathbb{T}$. So again, we have that $\mu(t_{n_k}) \geq \epsilon_0$ for all $k \in \mathbb{N}$ which implies $f(\mu(t_{n_k})) \geq f(\epsilon_0)$. So, regardless of the value of α, we have that $f(\mu(t_{n_k})) \geq f(\epsilon_0)$.

Thus,

$$f(\mu(t_{n_k})) = \frac{\mu(t_{n_k})}{1 + \alpha\mu(t_{n_k})} \geq \frac{\epsilon_0}{1 + \alpha\epsilon_0} = f(\epsilon_0) > 0,$$

for all $k \in \mathbb{N}$. Therefore, $\lim_{n\to\infty} \frac{\mu(t_n)}{1+\alpha\mu(t_n)} \neq 0$ and so $\sum_{n=1}^{\infty} \frac{\mu(t_n)}{1+\alpha\mu(t_n)}$ diverges.

As $t \to \infty$, we noted above that $n_0 = n_0(t) \to \infty$. Therefore, $\sum_{k=1}^{n_0} \frac{\mu(t_k)}{1+\alpha\mu(t_k)} \to \infty$ as $t \to \infty$. It follows from (2.3) that $e_{x\ominus\alpha}(t, t_0) \to \infty$ as $t \to \infty$.

As for the $\lim_{t\to\infty} e_{\alpha\ominus x}(t, t_0)$, note that using Theorem 2.1.14(c),

$$\lim_{t\to\infty} e_{\alpha\ominus x}(t, t_0) = \lim_{t\to\infty} e_{\ominus(x\ominus\alpha)}(t, t_0) = \lim_{t\to\infty} \frac{1}{e_{x\ominus\alpha}(t, t_0)} = 0.$$

□

When working with the standard exponential function defined on the complex numbers, a commonly used identity is $|e^z| = e^x$ where $z = x + iy$. Unfortunately, this does not necessarily hold for the generalized exponential function on an arbitrary time scale. However, we are guaranteed the following inequality in the case where $\mathrm{Re}(z) \in \mathcal{R}^+ \cap \mathbb{R}$.

Lemma 2.2.2. Let $p : \mathbb{T} \to \mathbb{C}$ such that $p(t) = u(t) + iv(t)$ where $u, v : \mathbb{T} \to \mathbb{R}$ and $u \in \mathcal{R}^+$. Then,

(a) $|e_p(t, t_0)| \geq e_u(t, t_0)$, and

(b) $|e_{\ominus p}(t, t_0)| \leq e_{\ominus u}(t, t_0)$,

for all $t \in [t_0, \infty)_{\mathbb{T}}$.

Proof. First note that $p \in \mathcal{R}$. To see this, consider for any $t \in \mathbb{T}$,

$$1 + p(t)\mu(t) = 1 + [u(t) + iv(t)]\mu(t) = [1 + u(t)\mu(t)] + iv(t)\mu(t) \neq 0,$$

since $u \in \mathcal{R}^+$.

Now for any $t \in \mathbb{T}$,

$$|1 + p(t)\mu(t)|^2 = \Big(1 + u(t)\mu(t)\Big)^2 + \Big(v(t)\mu(t)\Big)^2 \geq \Big(1 + u(t)\mu(t)\Big)^2,$$

and hence $|1 + p(t)\mu(t)| \geq 1 + u(t)\mu(t)$. Therefore, using the cylinder transform given in Definition 2.1.11,

$$\begin{aligned}
\mathrm{Re}(\xi_{\mu(t)}(p(t))) &= \begin{cases} \mathrm{Re}\left[\frac{1}{\mu(t)} \mathrm{Log}(1 + p(t)\mu(t))\right], & \mu(t) > 0, \\ \mathrm{Re}(p(t)), & \mu(t) = 0 \end{cases} \\
&= \begin{cases} \frac{1}{\mu(t)} \ln|1 + p(t)\mu(t)|, & \mu(t) > 0, \\ u(t), & \mu(t) = 0 \end{cases} \\
&\geq \begin{cases} \frac{1}{\mu(t)} \ln(1 + u(t)\mu(t)), & \mu(t) > 0, \\ u(t), & \mu(t) = 0 \end{cases}
\end{aligned}$$

$$= \begin{cases} \frac{1}{\mu(t)} \operatorname{Log}(1 + u(t)\mu(t)), & \mu(t) > 0, \\ u(t), & \mu(t) = 0 \end{cases}$$

$$= \xi_{\mu(t)}(u(t)),$$

where the second to last equality follows since $u \in \mathcal{R}^+$; i.e., $1 + u(t)\mu(t) > 0$ for all $t \in \mathbb{T}$. Finally, note that by the definition of the generalized exponential function (Definition 2.1.11), we have

$$\begin{aligned} |e_p(t, t_0)| &= \left| \exp\left(\int_{t_0}^t \xi_{\mu(\tau)}(p(\tau))\Delta\tau \right) \right| \\ &= \left| \exp\left(\int_{t_0}^t \operatorname{Re}(\xi_{\mu(\tau)}(p(\tau)))\Delta\tau + i \int_{t_0}^t \operatorname{Im}(\xi_{\mu(\tau)}(p(\tau)))\Delta\tau \right) \right| \\ &= \exp\left(\int_{t_0}^t \operatorname{Re}(\xi_{\mu(\tau)}(p(\tau)))\Delta\tau \right) \geq \exp\left(\int_{t_0}^t \xi_{\mu(\tau)}(u(\tau))\Delta\tau \right) = e_u(t, t_0). \end{aligned}$$

The second statement of the lemma follows immediately as $u \in \mathcal{R}^+$ and $e_{\ominus u}(t, t_0) = \frac{1}{e_u(t,t_0)}$. □

In practice, at least in this dissertation, we will apply Lemma 2.2.2 by taking $z \in \mathbb{C} \cap \mathcal{R}$ with $z = x + iy$ and concluding that $|e_z(t, t_0)| \geq e_x(t, t_0)$.

Lemma 2.2.3. Let $\mathbb{T}$ be a time scale. Define $\Omega := \mathbb{C} \backslash \{x \in \mathbb{R} : x \leq 0\}$ and fix any $s, t \in \mathbb{T}$. Then, $e_z(t, s)$ is analytic on the domain Ω. Further,

$$\frac{d}{dz} [e_z(t, s)] = e_z(t, s) \int_s^t \frac{\Delta\tau}{1 + z\mu(\tau)},$$

for all $z \in \Omega$.

Proof. Let $s, t \in \mathbb{T}$ be arbitrary. First consider $f(z) := \int_s^t \xi_{\mu(\tau)}(z)\Delta\tau$. We claim that f is analytic on Ω.

Fix $z_0 \in \Omega$. Let $z_0 = x_0 + iy_0$. If $x_0 > 0$, then take $R := \frac{1}{2}|z_0|$. If $x_0 \le 0$, then take $R := \frac{1}{2}|y_0|$. Note that since $z_0 \in \Omega$, if $x_0 \le 0$, then $y_0 \ne 0$. So, in either case, $R > 0$. Since Ω is open, there exists $0 < r < R$ such that $B_r(z_0) \subseteq \Omega$. Note that

$$\begin{aligned}\lim_{z\to z_0} \frac{f(z) - f(z_0)}{z - z_0} &= \lim_{z\to z_0} \frac{1}{z - z_0}\left[\int_s^t \xi_{\mu(\tau)}(z)\Delta\tau - \int_s^t \xi_{\mu(\tau)}(z_0)\Delta\tau\right] \\ &= \lim_{z\to z_0} \int_s^t \frac{\xi_{\mu(\tau)}(z) - \xi_{\mu(\tau)}(z_0)}{z - z_0}\Delta\tau.\end{aligned}$$

We will apply the Dominated Convergence Theorem in order to interchange this integral and limit.

Fix $\tau \in \mathbb{T}$. Since the principle logarithm is analytic on Ω and $z \in \Omega$ implies $1 + z\mu(\tau) \in \Omega$, by direct calculation,

$$\begin{aligned}\frac{d}{dz}\xi_{\mu(\tau)}(z) &= \begin{cases} \frac{d}{dz}\left[\frac{1}{\mu(\tau)}\operatorname{Log}(1 + z\mu(\tau))\right], & \mu(\tau) > 0, \\ \frac{d}{dz}[z], & \mu(\tau) = 0 \end{cases} \\ &= \begin{cases} \frac{1}{1+z\mu(\tau)}, & \mu(\tau) > 0, \\ 1, & \mu(\tau) = 0 \end{cases} \\ &= \frac{1}{1 + z\mu(\tau)},\end{aligned}$$

for any $z \in \Omega$. Hence, for any fixed $\tau \in \mathbb{T}$,

$$\lim_{z\to z_0} \frac{\xi_{\mu(\tau)}(z) - \xi_{\mu(\tau)}(z_0)}{z - z_0} = \frac{d}{dz}\xi_{\mu(\tau)}(z)\Big|_{z=z_0} = \frac{1}{1 + z_0\mu(\tau)}.$$

We will now show that $\frac{\xi_{\mu(\tau)}(z)-\xi_{\mu(\tau)}(z_0)}{z-z_0}$ is dominated by a Δ-integrable function on

$[s,t]_{\mathbb{T}}$. If $\mu(\tau)=0$, we have that

$$\left|\frac{\xi_{\mu(\tau)}(z)-\xi_{\mu(\tau)}(z_0)}{z-z_0}\right| = \left|\frac{z-z_0}{z-z_0}\right| = 1. \tag{2.4}$$

Assume $\mu(\tau)>0$. In this case, note that

$$\begin{aligned}\left|\frac{\xi_{\mu(\tau)}(z)-\xi_{\mu(\tau)}(z_0)}{z-z_0}\right| &= \left|\frac{\operatorname{Log}(1+z\mu(\tau))-\operatorname{Log}(1+z_0\mu(\tau))}{\mu(\tau)(z-z_0)}\right| \\ &= \left|\frac{1}{\mu(\tau)(z-z_0)}\operatorname{Log}\left(\frac{1+z\mu(\tau)}{1+z_0\mu(\tau)}\right)\right| \\ &= \left|\frac{1}{\mu(\tau)(z-z_0)}\operatorname{Log}\left(1+\frac{(z-z_0)\mu(\tau)}{1+z_0\mu(\tau)}\right)\right|. \end{aligned} \tag{2.5}$$

We claim that $\left|\frac{(z-z_0)\mu(\tau)}{1+z_0\mu(\tau)}\right| < 1$ for $z\in B_r(z_0)$ and any $\tau\in\mathbb{T}$. To show this, we define $g:[0,\infty)\to\mathbb{R}$ by

$$g(x) := \frac{x}{|1+z_0x|} = \frac{x}{\sqrt{(1+xx_0)^2+(xy_0)^2}},$$

where $z_0=x_0+iy_0$. Note that g is continuous on $[0,\infty)$ because $z_0\in\Omega$ which implies $1+z_0x\neq 0$ for all $x\in[0,\infty)$. Also, $g'(x)=\frac{1+xx_0}{((1+xx_0)^2+(xy_0)^2)^{\frac{3}{2}}}$, and so, provided $x_0\neq 0$, $x=\frac{-1}{x_0}$ is a possible critical point. We will now consider two cases.

Case: Assume $x_0\geq 0$.

It follows that g has no critical points in $(0,\infty)$. Further, $x_0\geq 0$ implies $g'(x)>0$ for $x\in[0,\infty)$; hence, g is increasing on $[0,\infty)$. Also, note that $\lim_{x\to\infty} g(x)=\frac{1}{|z_0|}$ and g is continuous on $[0,\infty)$. Therefore, in this case, $g(x)\leq\frac{1}{|z_0|}$ for all $x\in[0,\infty)$. Thus, for $z\in B_r(z_0)$,

$$\left|\frac{(z-z_0)\mu(\tau)}{1+z_0\mu(\tau)}\right| = |z-z_0|\, g(\mu(\tau)) \leq \frac{|z-z_0|}{|z_0|} < \frac{r}{|z_0|} < \frac{1}{2},$$

by the choice of r.

<u>Case:</u> Assume $x_0 < 0$.

Since $z_0 \in \Omega$, $y_0 \neq 0$. Then $\frac{-1}{x_0}$ is a critical point in $(0, \infty)$ which corresponds to a local maximum. Note that

$$g\left(\frac{-1}{x_0}\right) = \frac{-1}{x_0}\left|\frac{x_0}{y_0}\right| = \frac{1}{|y_0|}.$$

Therefore, in this case, $g(x) \leq \frac{1}{|y_0|}$ for all $x \in [0, \infty)$. Thus, for $z \in B_r(z_0)$,

$$\left|\frac{(z - z_0)\mu(\tau)}{1 + z_0\mu(\tau)}\right| = |z - z_0|\, g(\mu(\tau)) \leq \frac{|z - z_0|}{|y_0|} < \frac{r}{|y_0|} < \frac{1}{2},$$

again by the choice of r.

Thus, we have shown for any $\tau \in \mathbb{T}$ and $z \in B_r(z_0)$,

$$\left|\frac{(z - z_0)\mu(\tau)}{1 + z_0\mu(\tau)}\right| < \frac{1}{2}.$$

Thus, for $z \in B_r(z_0)$, $\frac{(z-z_0)\mu(\tau)}{1+z_0\mu(\tau)}$ is in the radius of convergence for the Taylor series expansion of $\operatorname{Log}\left(1 + \frac{(z-z_0)\mu(\tau)}{1+z_0\mu(\tau)}\right)$.

So, from (2.5), we have for $z \in B_r(z_0)$,

$$\begin{aligned}
\left|\frac{\xi_{\mu(\tau)}(z) - \xi_{\mu(\tau)}(z_0)}{z - z_0}\right| &= \left|\frac{1}{\mu(\tau)(z - z_0)} \operatorname{Log}\left(1 + \frac{(z - z_0)\mu(\tau)}{1 + z_0\mu(\tau)}\right)\right| \\
&= \frac{1}{|z - z_0|\, \mu(\tau)} \left|\sum_{k=0}^{\infty} \frac{(-1)^k}{k+1} \left(\frac{(z - z_0)\mu(\tau)}{1 + z_0\mu(\tau)}\right)^{k+1}\right| \\
&\leq \frac{1}{|z - z_0|\, \mu(\tau)} \sum_{k=0}^{\infty} \frac{1}{k+1} \left(\frac{|z - z_0|\, \mu(\tau)}{|1 + z_0\mu(\tau)|}\right)^{k+1}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{|1+z_0\mu(\tau)|}\sum_{k=0}^{\infty}\frac{1}{k+1}\left(\frac{|z-z_0|\,\mu(\tau)}{|1+z_0\mu(\tau)|}\right)^k \\
&\le \frac{1}{|1+z_0\mu(\tau)|}\sum_{k=0}^{\infty}\frac{1}{k+1}\left(\frac{1}{2}\right)^k \\
&\le \frac{1}{|1+z_0\mu(\tau)|}\sum_{k=0}^{\infty}\left(\frac{1}{2}\right)^k = \frac{2}{|1+z_0\mu(\tau)|}.
\end{aligned} \tag{2.6}
$$

From (2.4) and (2.6), we have shown that for $z \in B_r(z_0)$ and any $\tau \in \mathbb{T}$,

$$\left|\frac{\xi_{\mu(\tau)}(z) - \xi_{\mu(\tau)}(z_0)}{z - z_0}\right| \le \frac{2}{|1+z_0\mu(\tau)|}.$$

We will now show that $\frac{2}{|1+z_0\mu(\tau)|}$ is Δ-integrable on $[s,t]_{\mathbb{T}}$. Consider $h : [0,\infty) \to \mathbb{R}$ given by $h(x) = \frac{2}{|1+z_0x|}$. Since $z_0 \in \Omega$, $1 + z_0x \in \Omega$ for all $x \in [0,\infty)$, which implies $1 + z_0x \neq 0$ for all $x \in [0,\infty)$. Therefore, h is continuous on $[0,\infty)$. So, h satisfies the following properties: (a) h is continuous on $[0,\infty)$, (b) $h(0) = 2$, and (c) $\lim_{x\to\infty} h(x) = 0$.

Therefore, it must be the case that $h(x)$ is bounded on $[0,\infty)$. Thus, $h(\mu(\tau)) = \frac{2}{|1+z_0\mu(\tau)|}$ is bounded for $\tau \in \mathbb{T}$. In particular, $\frac{2}{|1+z_0\mu(\tau)|}$ is bounded on the interval $[s,t]_{\mathbb{T}}$. Since bounded functions on a finite interval are Δ-integrable (see [6, Theorem 5.20]), $\frac{2}{|1+z_0\mu(\tau)|}$ is Δ-integrable on $[s,t]_{\mathbb{T}}$.

Applying the Dominated Convergence Theorem, we have

$$
\begin{aligned}
\lim_{z\to z_0}\frac{f(z)-f(z_0)}{z-z_0} &= \lim_{z\to z_0}\int_s^t \frac{\xi_{\mu(\tau)}(z) - \xi_{\mu(\tau)}(z_0)}{z - z_0}\Delta\tau \\
&= \int_s^t \lim_{z\to z_0}\frac{\xi_{\mu(\tau)}(z) - \xi_{\mu(\tau)}(z_0)}{z - z_0}\Delta\tau = \int_s^t \frac{\Delta\tau}{1+z_0\mu(\tau)}.
\end{aligned}
$$

Since $z_0 \in \Omega$ was arbitrary, we have that for any $z \in \Omega$, the derivative exists and

$$\frac{d}{dz}\left[\int_s^t \xi_{\mu(\tau)}(z)\Delta\tau\right] = \int_s^t \frac{\Delta\tau}{1+z\mu(\tau)}.$$

Finally, since e^z is entire, by the chain rule,

$$\begin{aligned}\frac{d}{dz}\left[e_z(t,s)\right] &= \frac{d}{dz}\left[\exp\left(\int_s^t \xi_{\mu(\tau)}(z)\Delta\tau\right)\right] \\ &= \exp\left(\int_s^t \xi_{\mu(\tau)}(z)\Delta\tau\right)\frac{d}{dz}\left[\int_s^t \xi_{\mu(\tau)}(z)\Delta\tau\right] = e_z(t,s)\int_s^t \frac{\Delta\tau}{1+z\mu(\tau)},\end{aligned}$$

for all $z \in \Omega$. □

Lemma 2.2.4. Let $\mathbb{T}$ be any time scale. If $p : \mathbb{T} \to \mathbb{R}$, then $\overline{e_{ip}(t,t_0)} = e_{-ip}(t,t_0)$.

Proof. Since $p(t)$ is real-valued, $1 \pm i\mu(t)p(t) \neq 0$ for $t \in \mathbb{T}$, and hence $\pm ip \in \mathcal{R}$. Note that for any $t \in \mathbb{T}$,

$$\begin{aligned}\operatorname{Log}(1-ip(t)\mu(t)) &= \ln|1-ip(t)\mu(t)| + i\operatorname{Arg}(1-ip(t)\mu(t)) \\ &= \ln|1+ip(t)\mu(t)| + i\arctan(-p(t)\mu(t)) \\ &= \ln|1+ip(t)\mu(t)| - i\arctan(p(t)\mu(t)) \\ &= \ln|1+ip(t)\mu(t)| - i\operatorname{Arg}(1+ip(t)\mu(t)) \\ &= \overline{\operatorname{Log}(1+ip(t)\mu(t))}.\end{aligned}$$

We have used here the fact that arctan is an odd function. So,

$$\begin{aligned}
\xi_{\mu(t)}(-ip(t)) &= \begin{cases} \frac{1}{\mu(t)}\operatorname{Log}(1-ip(t)\mu(t)), & \mu(t)>0,\\ -ip(t), & \mu(t)=0 \end{cases}\\
&= \begin{cases} \overline{\frac{1}{\mu(t)}\operatorname{Log}(1+ip(t)\mu(t))}, & \mu(t)>0,\\ \overline{ip(t)}, & \mu(t)=0. \end{cases}\\
&= \overline{\xi_{\mu(t)}(ip(t))},
\end{aligned}$$

which implies

$$\int_{t_0}^{t}\xi_{\mu(\tau)}(-ip(\tau))\Delta\tau = \overline{\int_{t_0}^{t}\xi_{\mu(\tau)}(ip(\tau))\Delta\tau}.$$

Since $\overline{e^{z}} = e^{\overline{z}}$, by using the definition of the generalized exponential function, we have

$$\begin{aligned}
\overline{e_{ip}(t,t_0)} = \overline{\exp\left(\int_{t_0}^{t}\xi_{\mu(\tau)}(ip(\tau))\Delta\tau\right)} &= \exp\left(\overline{\int_{t_0}^{t}\xi_{\mu(\tau)}(ip(\tau))\Delta\tau}\right)\\
&= \exp\left(\int_{t_0}^{t}\xi_{\mu(\tau)}(-ip(\tau))\Delta\tau\right) = e_{-ip}(t,t_0).
\end{aligned}$$

□

Chapter 3

The Generalized Laplace Transform

3.1 Introduction

Throughout, we let $\mathbb{T}$ be a time scale that is unbounded above with $t_0 \in \mathbb{T}$ fixed.

Definition 3.1.1. Assume that $f : \mathbb{T} \to \mathbb{R}$ is a locally Δ-integrable function. Then the *generalized Laplace transform* of f is defined by

$$\mathcal{L}\{f\}(z) := \int_{t_0}^{\infty} f(t) e^{\sigma}_{\ominus z}(t, t_0) \Delta t \quad \text{for } z \in \mathcal{D}\{f\},$$

where $\mathcal{D}\{f\} \subseteq \mathbb{C}$ consists of all $z \in \mathbb{C} \cap \mathcal{R}$ for which the improper integral exists.

Recall that we are using the notation $z \in \mathcal{R} \cap \mathbb{C}$ to mean $z \in \mathbb{C}$ and $1 + z\mu(t) \neq 0$ for all $t \in \mathbb{T}$. Thus, $\mathcal{R} \cap \mathbb{C} \subseteq \mathbb{C}$ (see the explanation after Definition 2.1.7).

In Section 3.2 we will discuss the set $\mathcal{D}\{f\} \subseteq \mathbb{C}$. We will show that for reasonable assumptions on the function f, $\mathcal{D}\{f\}$ is nonempty regardless of the time scale in question. Further, if we put some restrictions on the time scale, we can say even more about $\mathcal{D}\{f\}$.

Definition 3.1.2. A function $f : \mathbb{T} \to \mathbb{R}$ is said to be of *exponential order* α provided there exist constants $M > 0$, $\alpha \in \mathcal{R}^+ \cap \mathbb{R}$ such that $|f(t)| \leq M e_\alpha(t, t_0)$ for all $t \in [t_0, \infty)_\mathbb{T}$.

We will always assume that $f : \mathbb{T} \to \mathbb{R}$ is a locally Δ-integrable function, sometimes without explicit mention. By making this assumption, we are assured that the only problem we must worry about in the definition of $\mathcal{L}\{f\}(z)$ is whether or not the improper integral converges for a particular $z \in \mathbb{C}$.

3.2 Region of Convergence of the Laplace Transform

The following theorem appears in [4, Example 3.91]. Here, Bohner and Peterson via direct computation obtain the following result:

Theorem 3.2.1. Let $\mathbb{T}$ be any time scale that is unbounded above. Fix $t_0 \in \mathbb{T}$. Then for $z \in \mathbb{C} \cap \mathcal{R}$ and $\alpha \in \mathcal{R}$,

$$\mathcal{L}\{e_\alpha(\cdot, t_0)\}(z) = \frac{1}{z - \alpha}$$

provided $\lim_{t\to\infty} e_{\alpha\ominus z}(t, t_0) = 0$.

This limit defines precisely the set of $z \in \mathbb{C} \cap \mathcal{R}$ for which the improper integral in the definition of the Laplace transform converges to $\frac{1}{z-\alpha}$. In other words, given any time scale $\mathbb{T}$ that is unbounded above and a fixed $t_0 \in \mathbb{T}$, we have that

$$\mathcal{D}\{e_\alpha(\cdot, t_0)\} = \left\{ z \in \mathbb{C} \cap \mathcal{R} : \lim_{t\to\infty} e_{\alpha\ominus z}(t, t_0) = 0 \right\}.$$

Depending on the complexity of the time scale in question, computing the set in the complex plane that this limit describes can be a difficult task. In fact, for a given $f : \mathbb{T} \to \mathbb{R}$ and choice of α, it may not be immediately clear that $\mathcal{D}\{f\}$ is nonempty.

In this section, we will show that with the standard assumption that $f : \mathbb{T} \to \mathbb{R}$ is of exponential order α, the set $\mathcal{D}\{f\}$ is nonempty for any time scale; and further, $\mathcal{D}\{f\}$ contains the right-half plane $z \in \mathbb{C}$ such that $\mathrm{Re}(z) > \alpha$. If we restrict the time scale in question such that the graininess is bounded below (away from 0), we can say even more about $\mathcal{D}\{f\}$. Finally, if we have a specific time scale in hand, we will show via an example, that we can determine $\mathcal{D}\{f\}$ even more explicitly.

Theorem 3.2.2. Let $\mathbb{T}$ be any time scale that is unbounded above. Fix $t_0 \in \mathbb{T}$. If $f : [t_0, \infty)_{\mathbb{T}} \to \mathbb{R}$ is of exponential order α where $\alpha \in \mathcal{R}^+ \cap \mathbb{R}$, then

$$\{z \in \mathbb{C} : \mathrm{Re}(z) > \alpha\} \subseteq \mathcal{D}\{f\}.$$

Furthermore, the improper integral in the definition of the Laplace transform converges absolutely for z in this region.

Proof. Fix $z \in \mathbb{C}$ such that $z = x + iy$ and $x > \alpha$. Here $x, y \in \mathbb{R}$. Since $\alpha \in \mathcal{R}^+$, $1 + x\mu(t) > 1 + \alpha\mu(t) > 0$ for any $t \in \mathbb{T}$. Thus, $1 + z\mu(t) = 1 + x\mu(t) + iy\mu(t) \neq 0$ and so $z \in \mathcal{R}$.

We aim to show that $\int_{t_0}^{\infty} \left| f(t) e^{\sigma}_{\ominus z}(t, t_0) \right| \Delta t$ converges. Since $x > \alpha$ and $\alpha \in \mathcal{R}^+$, by Lemma 2.2.1, $\lim_{t\to\infty} e_{\alpha \ominus x}(t, t_0) = 0$, which implies, by Theorem 3.2.1, that $\mathcal{L}\{e_\alpha(\cdot, t_0)\}(x)$ exists and equals $\frac{1}{x-\alpha}$.

For any $L \in [t_0, \infty)_{\mathbb{T}}$, we have

$$\begin{aligned}\int_{t_0}^{L} \left|f(t)e^{\sigma}_{\ominus z}(t,t_0)\right| \Delta t &\leq M \int_{t_0}^{L} e_\alpha(t,t_0) \left|e^{\sigma}_{\ominus z}(t,t_0)\right| \Delta t \\ &\leq M \int_{t_0}^{L} e_\alpha(t,t_0) e^{\sigma}_{\ominus x}(t,t_0) \Delta t \quad \text{(by Lemma 2.2.2)} \\ &\leq M \int_{t_0}^{\infty} e_\alpha(t,t_0) e^{\sigma}_{\ominus x}(t,t_0) \Delta t \\ &= M\mathcal{L}\{e_\alpha(\cdot,t_0)\}(x) \\ &= \frac{M}{x-\alpha}.\end{aligned}$$

As $L \to \infty$, the real-valued function of L, $\int_{t_0}^{L} \left|f(t)e^{\sigma}_{\ominus z}(t,t_0)\right| \Delta t$, is monotone increasing and bounded above, and so it follows that $\lim_{L\to\infty} \int_{t_0}^{L} \left|f(t)e^{\sigma}_{\ominus z}(t,t_0)\right| \Delta t$ converges. i.e., the desired improper integral converges, and we have that $\mathcal{L}\{f\}(z)$ converges absolutely. Hence, $z \in \mathcal{D}\{f\}$. □

Example 3.2.3. Consider the time scale $\mathbb{T} = \mathbb{R}$ and $f(t) = e_\alpha(t,0) = e^{\alpha t}$ for some $\alpha \in \mathbb{R}$. In this case, the Laplace transform is simply the classical Laplace transform. It is well-known that in the classical case, $\mathcal{L}\{e^{\alpha t}\}(z)$ exists if and only if $\text{Re}(z) > \alpha$. i.e., we have found a function of exponential order α and a time scale such that $\{z \in \mathbb{C} : \text{Re}(z) > \alpha\} = \mathcal{D}\{f\}$.

Remark 3.2.4. In light of Example 3.2.3 and Theorem 3.2.2, for an arbitrary time scale and an arbitrary function of exponential order, Theorem 3.2.2 is the "best" we can do. The set $\{z \in \mathbb{C} : \text{Re}(z) > \alpha\}$ is the largest set (not depending on the time scale $\mathbb{T}$) for which we can say

$$\{z \in \mathbb{C} : \text{Re}(z) > \alpha\} \subseteq \mathcal{D}\{f\}.$$

So, for the generalized Laplace transform, if we restrict our attention to functions

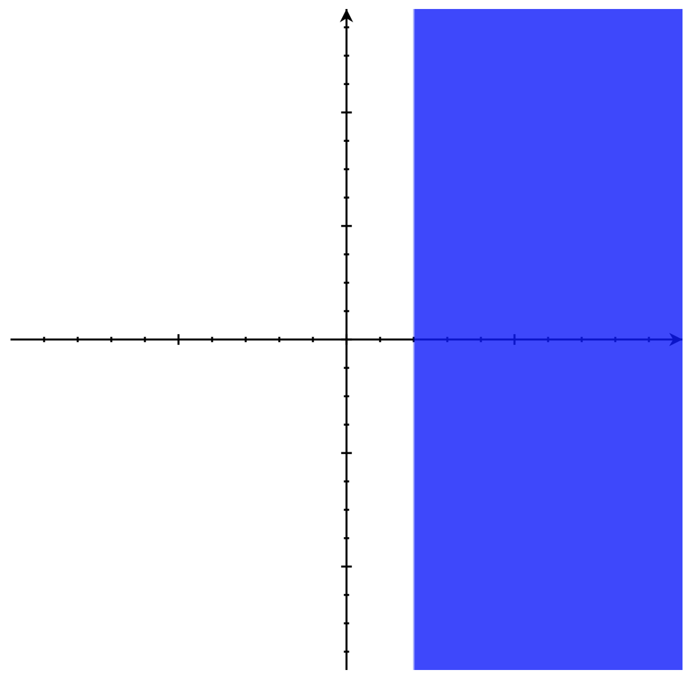


Figure 3.1: The region of convergence guaranteed by Theorem 3.2.2 for an arbitrary time scale $\mathbb{T}$.

of exponential order, the region of convergence is extremely similar to the classical Laplace transform on $\mathbb{R}$. However, we have shown that we have a half-plane of convergence for *any* time scale $\mathbb{T}$. This half-plane is shown in Figure 3.1.

We will now show that if we put some restrictions on the time scale, we can obtain an even larger region in the complex plane (independent of $\mathbb{T}$) for which the Laplace transform of a function of exponential order converges.

Theorem 3.2.5. Let $\mathbb{T}$ be unbounded above such that $0 < \mu_{min} \leq \mu(t)$ for all $t \in \mathbb{T}$. If $f : [t_0, \infty)_{\mathbb{T}} \to \mathbb{R}$ is of exponential order $\alpha > 0$, then

$$\{z \in \mathbb{C} : |1 + z\mu_{min}| > 1 + \alpha\mu_{min}\} \subseteq \mathcal{D}\{f\}.$$

Furthermore, the improper integral in the definition of the Laplace transform converges absolutely for z in this region.

To prove this theorem, we use the following lemma.

Lemma 3.2.6. Let $\mathbb{T}$ be unbounded above such that $0 < \mu_{min} \leq \mu(t)$ for all $t \in \mathbb{T}$. Let $\alpha > 0$ be given. Then, for $z \in \mathbb{C}$ such that $|1 + z\mu_{min}| > 1 + \alpha\mu_{min}$, there exists a constant $\delta = \delta(\alpha, z, \mu_{min}) > 0$ such that

$$\frac{1 + \alpha\mu(t)}{|1 + z\mu(t)|} \leq \delta < 1 \quad \text{for all } t \in \mathbb{T}.$$

Proof. Fix $z \in \mathbb{C}$ such that

$$|1 + z\mu_{min}| > 1 + \alpha\mu_{min}. \tag{3.1}$$

This implies $\frac{1+\alpha\mu_{min}}{|1+z\mu_{min}|} < 1$. Let $z = u + iv$ and define $f : [0, \infty) \to \mathbb{R}$ by

$$f(x) = \frac{1 + \alpha x}{|1 + zx|} = \frac{1 + \alpha x}{\sqrt{(1 + ux)^2 + (vx)^2}}.$$

Thus,

$$f'(x) = \frac{\alpha - u + x(\alpha u - (u^2 + v^2))}{[(1 + ux)^2 + (vx)^2]^{\frac{3}{2}}} = \frac{\alpha - \operatorname{Re}(z) + x(\alpha \operatorname{Re}(z) - |z|^2)}{|1 + zx|^3}.$$

Note that it follows from assumption (3.1) that

$$1 + \alpha\mu_{min} < |1 + z\mu_{min}| \leq 1 + |z| \mu_{min} \implies \alpha < |z| . \tag{3.2}$$

If $\alpha\,\mathrm{Re}(z) - |z|^2 = 0$, then using (3.2) we have

$$0 = \alpha\,\mathrm{Re}(z) - |z|^2 < \alpha\,\mathrm{Re}(z) - \alpha^2 = \alpha(\mathrm{Re}(z) - \alpha) \implies \mathrm{Re}(z) - \alpha > 0,$$

where the final implication is obtained by dividing by $\alpha > 0$. Thus, in this case,

$$f'(x) = \frac{\alpha - \mathrm{Re}(z)}{|1 + zx|^3} < 0,$$

and hence f is strictly decreasing on $(0, \infty)$ and $f(0) = 1$.

We now assume $\alpha\,\mathrm{Re}(z) - |z|^2 \neq 0$. Thus, the only possible critical point, x_0, of f is given by

$$x_0 = \frac{\mathrm{Re}(z) - \alpha}{\alpha\,\mathrm{Re}(z) - |z|^2}. \tag{3.3}$$

We will now consider two cases.

<u>Case:</u> Assume $\mathrm{Re}(z) \geq \alpha$.

Then

$$\alpha\,\mathrm{Re}(z) \leq [\mathrm{Re}(z)]^2 \leq [\mathrm{Re}(z)]^2 + [\mathrm{Im}(z)]^2 = |z|^2$$

and so $\alpha\,\mathrm{Re}(z) - |z|^2 < 0$. (We have strict inequality here since $z \neq \alpha$ which follows from assumption (3.1).) Therefore, $x_0 \leq 0$ and so f does not have a critical point on the interval $(0, \infty)$. Further, it is straightforward to see that f is continuous on $(0, \infty)$, f is strictly decreasing on $(0, \infty)$, and $f(0) = 1$.

Thus, $0 < \mu_{\min}$ implies $f(\mu_{min}) < 1$. Further, $\mu_{min} \leq \mu(t)$ for all $t \in \mathbb{T}$ implies $f(\mu_{min}) \geq f(\mu(t))$ for all $t \in \mathbb{T}$. Define $\delta := f(\mu_{min})$, and so, in this case we have

$$f(\mu(t)) = \frac{1 + \alpha\mu(t)}{|1 + z\mu(t)|} \leq \frac{1 + \alpha\mu_{min}}{|1 + z\mu_{min}|} = f(\mu_{min}) = \delta < 1 \quad \text{for all } t \in \mathbb{T}.$$

<u>Case:</u> Assume $\mathrm{Re}(z) < \alpha$.

If $\mathrm{Re}(z) = 0$, then $\mathrm{Im}(z) \neq 0$ by assumption (3.1). Thus, $\alpha\,\mathrm{Re}(z) - |z|^2 = -[\mathrm{Im}(z)]^2 < 0$. If $\mathrm{Re}(z) < 0$, then $\alpha\,\mathrm{Re}(z) < 0 \implies \alpha\,\mathrm{Re}(z) - |z|^2 < 0$. If $\mathrm{Re}(z) > 0$, then from (3.2) we have $\alpha\,\mathrm{Re}(z) < \mathrm{Re}(z)\,|z| \leq |z|^2 \implies \alpha\,\mathrm{Re}(z) - |z|^2 < 0$.

So, for $\mathrm{Re}(z) < \alpha$, $\alpha\,\mathrm{Re}(z) - |z|^2 < 0$, and hence $x_0 > 0$ given by (3.3) is a critical point for f in the interval $(0, \infty)$. Again, it is straightforward to see that f is continuous on $(0, \infty)$, f is strictly increasing on $(0, x_0)$, and f is strictly decreasing on (x_0, ∞). Since $f(0) = 1$, it follows that for $x \in [0, x_0)$, $f(x) \geq 1$.

Further, $\lim_{x\to\infty} f(x) = \frac{\alpha}{|z|} < 1$ by (3.2). By assumption (3.1), $f(\mu_{min}) < 1$, and so $\mu_{min} \in (x_0, \infty)$. So, in particular, f is decreasing on the interval (μ_{min}, ∞). Thus, $\mu_{min} \leq \mu(t)$ for all $t \in \mathbb{T}$ implies $f(\mu_{min}) \geq f(\mu(t))$ for all $t \in \mathbb{T}$. Define $\delta := f(\mu_{min})$, and so again we have

$$f(\mu(t)) = \frac{1 + \alpha\mu(t)}{|1 + z\mu(t)|} \leq \frac{1 + \alpha\mu_{min}}{|1 + z\mu_{min}|} = f(\mu_{min}) = \delta < 1 \quad \text{for all } t \in \mathbb{T}.$$

□

Proof of Theorem 3.2.5. Since $0 < \mu_{min} \leq \mu(t)$ for all $t \in \mathbb{T}$, the time scale $\mathbb{T}$ is an isolated time scale. Let $\mathbb{T} = \{t_0, t_1, \cdots\}$ where $t_0 < t_1 < \cdots$. Fix $z \in \mathbb{C}$ such that $|1 + z\mu_{min}| > 1 + \alpha\mu_{min}$.

Note that by the definition of the generalized exponential function, for any $n \in \mathbb{N}_0$,

$$\begin{aligned} |e_{\alpha\ominus z}(t_n, t_0)| &= \left|\exp\left(\int_{t_0}^{t_n} \xi_{\mu(\tau)}((\alpha \ominus z)(\tau)\Delta\tau\right)\right| \\ &= \exp\left(\int_{t_0}^{t_n} \mathrm{Re}[\xi_{\mu(\tau)}((\alpha \ominus z)(\tau)]\Delta\tau\right) \\ &= \exp\left(\int_{t_0}^{t_n} \frac{1}{\mu(\tau)} \ln\left|\frac{1 + \alpha\mu(\tau)}{1 + z\mu(\tau)}\right| \Delta\tau\right) \end{aligned}$$

$$= \exp\left(\sum_{i=0}^{n-1} \ln\left|\frac{1+\alpha\mu(t_i)}{1+z\mu(t_i)}\right|\right) \quad \text{(by Theorem 2.1.21)}$$
$$= \prod_{i=0}^{n-1}\left|\frac{1+\alpha\mu(t_i)}{1+z\mu(t_i)}\right|$$
$$\leq \prod_{i=0}^{n-1}\delta \quad \text{(by Lemma 3.2.6)}$$
$$= \delta^n,$$

where $\delta < 1$.

In an argument that is very similar to that given in Lemma 3.2.6 (and also given in the proof of Lemma 2.2.3), it follows that there exists a constant $C = C(z) > 0$ such that $\left|\frac{\mu(t)}{1+z\mu(t)}\right| \leq C$ for all $t \in [t_0, \infty)_{\mathbb{T}}$. Thus, for any $k \in \mathbb{N}_0$, we have

$$\begin{aligned}
\int_{t_0}^{t_k} \left|f(t)e^{\sigma}_{\ominus z}(t,t_0)\right| \Delta t &\leq M \int_{t_0}^{t_k} e_\alpha(t,t_0)\left|e^{\sigma}_{\ominus z}(t,t_0)\right| \Delta t \\
&\leq M \int_{t_0}^{t_k} \left|\frac{1}{1+z\mu(t)}\right| |e_{\alpha\ominus z}(t,t_0)| \Delta t \\
&= M \sum_{i=0}^{k-1} \left|\frac{\mu(t_i)}{1+z\mu(t_i)}\right| |e_{\alpha\ominus z}(t_i,t_0)| \quad \text{(by Theorem 2.1.21)} \\
&\leq MC\sum_{i=0}^{k-1}\delta^i \leq MC\sum_{i=0}^{\infty}\delta^i = \frac{MC}{1-\delta}.
\end{aligned}$$

As $k \to \infty$, the real-valued sequence $\int_{t_0}^{t_k} \left|f(t)e^{\sigma}_{\ominus z}(t,t_0)\right| \Delta t$ is monotone increasing and bounded above, thus $\lim_{k\to\infty} \int_{t_0}^{t_k} \left|f(t)e^{\sigma}_{\ominus z}(t,t_0)\right| \Delta t$ converges. i.e., the desired improper integral converges, and we have that $\mathcal{L}\{f\}(z)$ converges absolutely. Hence, $z \in \mathcal{D}\{f\}$. $\square$

Example 3.2.7. Let $\mathbb{T} = h\mathbb{Z}$, $h > 0$, and $f(t) = e_\alpha(t,0)$ for some $\alpha > 0$. On this time scale, it is straightforward to see that $e_{\alpha\ominus z}(t,0) = \left(\frac{1+\alpha h}{1+zh}\right)^{\frac{t}{h}}$. Using Theorem

2.1.21, we have

$$\begin{aligned}\mathcal{L}\{e_\alpha(\cdot,0)\}(z) &= \int_0^\infty e_\alpha(t,0)e^\sigma_{\ominus z}(t,0)\Delta t\\ &= \sum_{t=0}^\infty \frac{h}{1+zh}e_{\alpha\ominus z}(t,0) = \frac{h}{1+zh}\sum_{t=0}^\infty\left(\frac{1+\alpha h}{1+zh}\right)^{\frac{t}{h}}.\end{aligned}$$

Thus, the Laplace transform converges if and only if $\left|\frac{1+\alpha h}{1+zh}\right| < 1$ if and only if $1+\alpha h < |1+zh|$. Therefore, in conjunction with Theorem 3.2.5, we can conclude $\{z \in \mathbb{C} : |1+zh| > 1+\alpha h\} = \mathcal{D}\{f\}$.

Remark 3.2.8. Similar to what we saw before, because of Example 3.2.7 and Theorem 3.2.5, for a time scale such that $0 < \mu_{min} \leq \mu(t)$ for all $t \in \mathbb{T}$ and an arbitrary function of exponential order α, Theorem 3.2.5 is the "best" we can do. The set $\{z \in \mathbb{C} : |1+z\mu_{min}| > 1+\alpha\mu_{min}\}$ is the largest set (not depending on the time scale $\mathbb{T}$) for which we can say

$$\{z \in \mathbb{C} : |1+z\mu_{min}| > 1+\alpha\mu_{min}\} \subseteq \mathcal{D}\{f\}.$$

So, for the generalized Laplace transform, if we restrict the graininess of our time scale as indicated and only focus on functions of exponential order α, the region of convergence is extremely similar to the classical Z-transform from difference equations (see, for example, [12, Section 6.1]). This region of convergence for the generalized Laplace transform is shown in Figure 3.2.

Note that this region is slightly different from what we expect for the Z-transform from difference equations. For the Z-transform, the transform converges for all points outside a ball centered at the origin. However here, we have convergence for all points outside a ball centered at $\frac{-1}{\mu_{min}}$. The reason for this is that the generalized Laplace

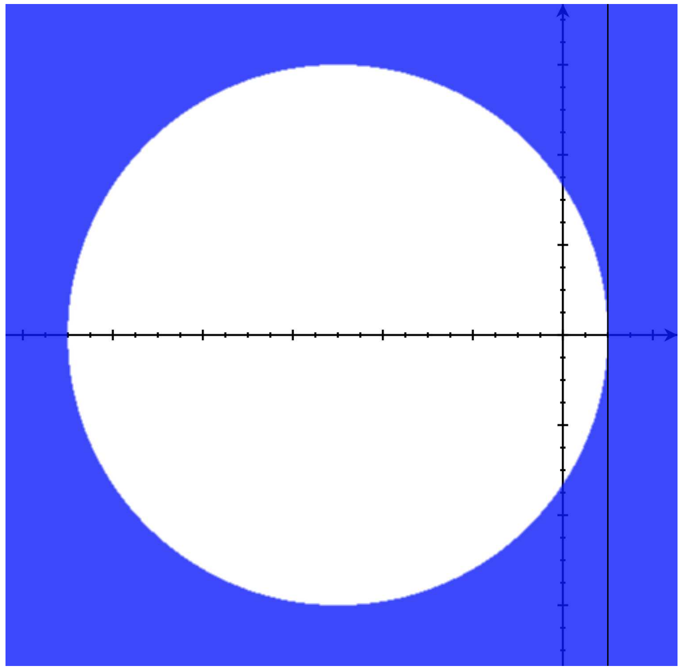


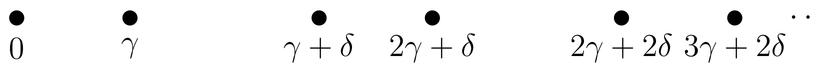
Figure 3.2: The region of convergence guaranteed by Theorem 3.2.5 for a time scale such that $0 < \mu_{min} \leq \mu(t)$ for all $t \in \mathbb{T}$.

transform on $\mathbb{Z}$ is a shift of the classical Z-transform. In other words,

$$(z+1)\mathcal{L}\{f\}(z) = \mathcal{Z}\{f\}(z+1).$$

So far we have shown that for any time scale, the region of convergence of the generalized Laplace transform behaves very similar to that of the classical Laplace transform on $\mathbb{R}$. We have also shown that by bounding the graininess below away from zero, the region of convergence of the generalized Laplace transform behaves very similar to that of the classical Z-transform on $\mathbb{Z}$. We conclude this section with an interesting example that shows that when we take a time scale that is not either of the classical cases $\mathbb{R}$ or $\mathbb{Z}$, then we find that the region of convergence of the generalized Laplace transform behaves in a substantially different way than the two classical cases.

Example 3.2.9. In this example we will consider the function $f(t) = e_\alpha(t, t_0)$, for some $\alpha > 0$, on the time scale defined by $\mathbb{T} := \{0, \gamma, \gamma+\delta, 2\gamma+\delta, 2\gamma+2\delta, \cdots\}$, where $\gamma > 0$ and $\delta > 0$. We will enumerate $\mathbb{T} = \{t_0, t_1, t_2, \cdots\}$.



It is straightforward to see that on this time scale $\mu(t_n) = \gamma$ if n is even, $\mu(t_n) = \delta$ if n is odd, and

$$e_{\alpha\ominus z}(t_n, t_0) = \begin{cases} \left(\frac{1+\alpha\delta}{1+z\delta}\right)^{\frac{n}{2}} \left(\frac{1+\alpha\gamma}{1+z\gamma}\right)^{\frac{n}{2}}, & n \text{ even}, \\ \left(\frac{1+\alpha\delta}{1+z\delta}\right)^{\frac{n+1}{2}} \left(\frac{1+\alpha\gamma}{1+z\gamma}\right)^{\frac{n-1}{2}}, & n \text{ odd}. \end{cases}$$

Thus,

$$\begin{aligned}
\mathcal{L}\{e_\alpha(\cdot, t_0)\}(z) &= \int_{t_0}^{\infty} e_\alpha(t, t_0) e^\sigma_{\ominus z}(t, t_0) \Delta t \\
&= \sum_{n=0}^{\infty} \frac{\mu(t_n)}{1 + z\mu(t_n)} e_{\alpha\ominus z}(t_n, t_0) \quad \text{(by Theorem 2.1.21)} \\
&= \sum_{\substack{n=0 \\ n \text{ even}}}^{\infty} \frac{\gamma}{1+z\gamma} \left(\frac{1+\alpha\delta}{1+z\delta}\right)^{\frac{n}{2}} \left(\frac{1+\alpha\gamma}{1+z\gamma}\right)^{\frac{n}{2}} \\
&\qquad + \sum_{\substack{n=0 \\ n \text{ odd}}}^{\infty} \frac{\delta}{1+z\delta} \left(\frac{1+\alpha\delta}{1+z\delta}\right)^{\frac{n+1}{2}} \left(\frac{1+\alpha\gamma}{1+z\gamma}\right)^{\frac{n-1}{2}} \\
&= \left(\frac{\gamma}{1+z\gamma}\right) \sum_{k=0}^{\infty} \left(\frac{1+\alpha\delta}{1+z\delta}\right)^{k} \left(\frac{1+\alpha\gamma}{1+z\gamma}\right)^{k} \\
&\qquad + \left(\frac{\delta}{1+z\delta}\right) \sum_{k=0}^{\infty} \left(\frac{1+\alpha\delta}{1+z\delta}\right)^{k+1} \left(\frac{1+\alpha\gamma}{1+z\gamma}\right)^{k}
\end{aligned}$$

$$= \left[\frac{\gamma}{1+z\gamma} + \frac{\delta(1+\alpha\delta)}{(1+z\delta)^2}\right] \sum_{k=0}^{\infty} \left[\left(\frac{1+\alpha\delta}{1+z\delta}\right)\left(\frac{1+\alpha\gamma}{1+z\gamma}\right)\right]^k.$$

Noting that we have ended up with a geometric series, it follows that the Laplace transform of $e_\alpha(t, t_0)$ converges on this time scale if and only if $\left|\left(\frac{1+\alpha\delta}{1+z\delta}\right)\left(\frac{1+\alpha\gamma}{1+z\gamma}\right)\right| < 1$. This inequality defines the set of $z \in \mathbb{C}$ for which the generalized Laplace transform converges, $\mathcal{D}\{e_\alpha(t, t_0)\}$, on this particular time scale. If we fix $\alpha = \frac{1}{4}$ and $\gamma = \frac{1}{2}$, Figures 3.3, 3.4, 3.5, and 3.6 show how the region of convergence varies as we modify the time scale by adjusting the value of δ.

3.3 Power Series on Time Scales

From the study of the classical Laplace transform on $\mathbb{R}$, it is known that we are not always able to expand a function via its power series representation, take the Laplace transform term-by-term, and obtain a representation of the Laplace transform of the original function (see, for example, [19, Example 1.17]).

The main objective of this section is to give conditions on when we can take the generalized Laplace transform of a power series representation of a function $f : \mathbb{T} \to \mathbb{R}$ term-by-term. As in Bohner and Guseinov's paper [3], we will define $\mathcal{F}$ to be the set of all functions $f : [t_0, \infty)_\mathbb{T} \to \mathbb{C}$ of the form

$$f(t) = \sum_{n=0}^{\infty} c_n h_n(t, t_0), \quad t \in [t_0, \infty)_\mathbb{T}, \tag{3.4}$$

where $|c_n| \leq K\alpha^n$ for all $n \in \mathbb{N}_0$. Here $K > 0, \alpha > 0$ are constants independent of n. Also note that $h_n(t, t_0)$ is the generalized monomial found in Definition 2.1.10. Bohner and Guseinov prove in [3], that for $f \in \mathcal{F}$, the series in (3.4) converges uniformly on any compact interval $[a, b]_\mathbb{T}$ where $b > a$. They also show for $z \in \mathbb{C} \cap \mathcal{R}$,

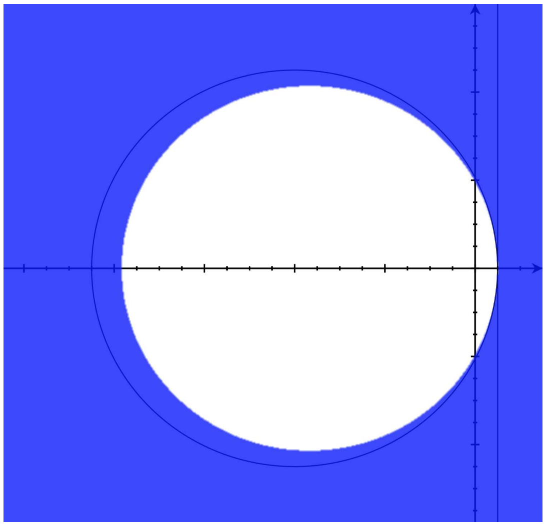


Figure 3.3: The region of convergence guaranteed by Example 3.2.9 with $\gamma = \frac{1}{2}$ and $\delta = \frac{3}{5}$.

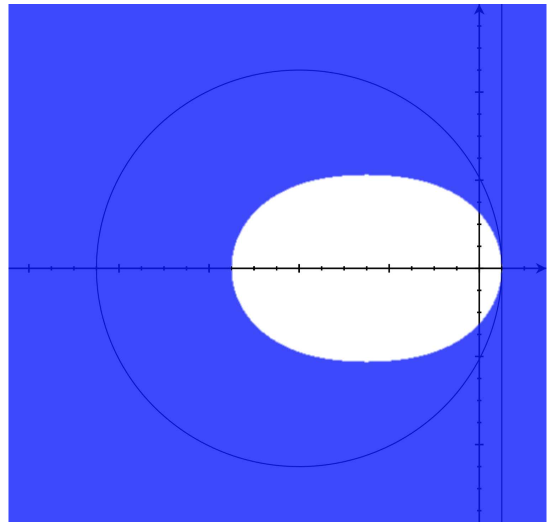


Figure 3.5: The region of convergence guaranteed by Example 3.2.9 with $\gamma = \frac{1}{2}$ and $\delta = 2$.

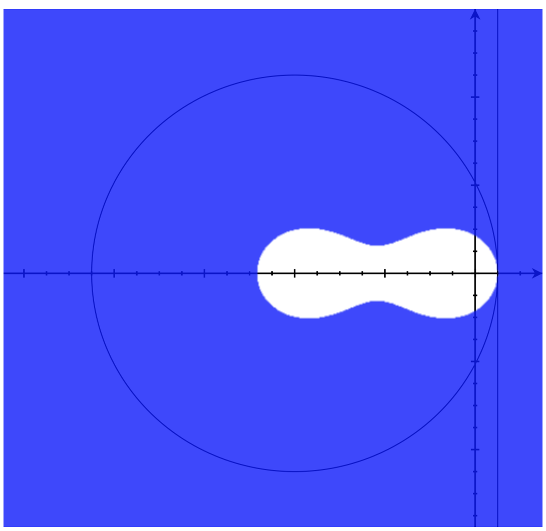


Figure 3.4: The region of convergence guaranteed by Example 3.2.9 with $\gamma = \frac{1}{2}$ and $\delta = 6$.

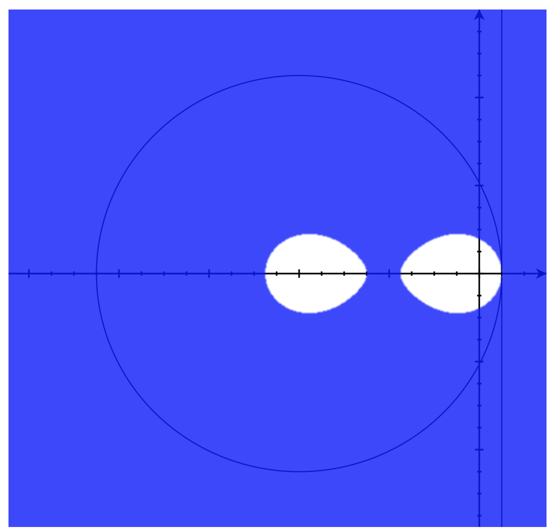


Figure 3.6: The region of convergence guaranteed by Example 3.2.9 with $\gamma = \frac{1}{2}$ and $\delta = 8$.

Note that Theorem 3.2.2 guarantees the half-plane of convergence to the right of the vertical line in these figures. Similarly, since in all four cases, the time scale satisfies $0 < \frac{1}{2} \leq \mu(t)$, Theorem 3.2.5 guarantees convergence outside of a ball of radius $\frac{9}{4}$ centered at -2 indicated by the circle in these figures.

the generalized exponential function is given by the expansion

$$e_z(t,t_0) = \sum_{n=0}^{\infty} z^n h_n(t,t_0).$$

Further, Bohner and Guseinov define

$$\mathcal{D}_n := \left\{ z \in \mathbb{C} \cap \mathcal{R} : \lim_{t\to\infty} h_n(t,t_0) e_{\ominus z}(t,t_0) = 0 \right\},$$

and $\mathcal{D} := \bigcap_{n=0}^{\infty} \mathcal{D}_n$. This definition relies on the fact that, via direct calculation, $\mathcal{L}\{h_k(t,t_0)\}(z) = \frac{1}{z^{k+1}}$ provided $\lim_{t\to\infty} h_k(t,t_0)e_{\ominus z}(t,t_0) = 0$ (see [4, Theorem 3.90]). Hence, $\mathcal{D}$ describes the set in the complex plane for which the Laplace transform converges when applied to *each* term of the expansion of the function f.

In [3], it is assumed that $\mathcal{D}$ is nonempty. However, after the work done in Section 3.2, we merely need to show that for each $n \in \mathbb{N}_0$, $h_n(t,t_0)$ is of exponential order α for some $\alpha > 0$ in order to find some sets that are contained in $\mathcal{D}$ regardless of the time scale.

Theorem 3.3.1. Let $\mathbb{T}$ be unbounded above, and fix $t_0 \in \mathbb{T}$. For any $n \in \mathbb{N}_0$ and any $\alpha > 0$, the generalized monomial $h_n(t,t_0)$ is of exponential order α.

Proof. Fix $n \in \mathbb{N}_0$ and $\alpha > 0$. We first claim $\lim_{t\to\infty} h_n(t,t_0)e_{\ominus\alpha}(t,t_0) = 0$. To see this, we will apply a version of L'Hôpital's Rule for time scales (see [4, Theorem 1.120]):

Assume f and g are Δ-differentiable on $\mathbb{T}$ with $\lim_{t\to\infty} g(t) = \infty$. Suppose $g(t) > 0$, $g^\Delta(t) > 0$ for all $t \in [t_0,\infty)_\mathbb{T}$. Then $\lim_{t\to\infty} \frac{f^\Delta(t)}{g^\Delta(t)} = r \in \overline{\mathbb{R}}$ implies $\lim_{t\to\infty} \frac{f(t)}{g(t)} = r$.

Using the definition of $h_n(t,t_0)$, note that $h_n(t,t_0) \geq 0$ for all $t \in [t_0,\infty)_\mathbb{T}$. Also, since $\alpha > 0$, it follows that $1 + \alpha\mu(t) > 0$ for all $t \in [t_0,\infty)_\mathbb{T}$, and so $e_\alpha(t,t_0) > 0$ for all $t \in [t_0, \infty)_{\mathbb{T}}$ by Theorem 2.1.15. Thus, for $t \in [t_0, \infty)_{\mathbb{T}}$,

$$0 \leq h_n(t, t_0) e_{\ominus\alpha}(t, t_0) = \frac{h_n(t, t_0)}{e_\alpha(t, t_0)}. \tag{3.5}$$

Since $\alpha > 0$, $\lim_{t\to\infty} e_\alpha(t, t_0) = \infty$ by Lemma 2.2.1. Further, since $e_\alpha(t, t_0) > 0$, we have $e_\alpha^\Delta(t, t_0) = \alpha e_\alpha(t, t_0) > 0$. Continuing to take Δ-derivatives shows that for every $0 \leq k \leq n$, $e_\alpha^{\Delta^k}(t, t_0) = \alpha e_\alpha^{\Delta^{k-1}}(t, t_0) = \cdots = \alpha^k e_\alpha(t, t_0) > 0$. Therefore,

$$\frac{h_n^{\Delta^n}(t, t_0)}{e_\alpha^{\Delta^n}(t, t_0)} = \frac{h_{n-1}^{\Delta^{n-1}}(t, t_0)}{\alpha e_\alpha^{\Delta^{n-1}}(t, t_0)} = \cdots = \frac{1}{\alpha^n e_\alpha(t, t_0)}.$$

Hence, by n applications of L'Hôpital's Rule, we have

$$\lim_{t\to\infty} \frac{h_n(t, t_0)}{e_\alpha(t, t_0)} = \lim_{t\to\infty} \frac{h_n^{\Delta^n}(t, t_0)}{e_\alpha^{\Delta^n}(t, t_0)} = \frac{1}{\alpha^n} \lim_{t\to\infty} \frac{1}{e_\alpha(t, t_0)} = 0.$$

So, by (3.5), it follows that

$$\lim_{t\to\infty} h_n(t, t_0) e_{\ominus\alpha}(t, t_0) = 0.$$

In other words, there exists a $T \in [t_0, \infty)_{\mathbb{T}}$ such that for all $t \in [T, \infty)_{\mathbb{T}}$, we have $h_n(t, t_0) e_{\ominus\alpha}(t, t_0) < 1$. Define

$$M := \max\left\{ \sup_{t\in[t_0,T]_{\mathbb{T}}} h_n(t, t_0) e_{\ominus\alpha}(t, t_0),\ 1 \right\}.$$

Then, $h_n(t, t_0) e_{\ominus\alpha}(t, t_0) \leq M$ for all $t \in [t_0, \infty)_{\mathbb{T}}$. Therefore, $h_n(t, t_0) \leq M e_\alpha(t, t_0)$ for all $t \in [t_0, \infty)_{\mathbb{T}}$. $\square$

Using this theorem, we can now show that for any time scale, $\{z \in \mathbb{C} : \mathrm{Re}(z) > 0\} \subseteq \mathcal{D}$.

Corollary 3.3.2. For any time scale $\mathbb{T}$, if $z \in \mathbb{C}$ such that $\text{Re}(z) > 0$, then $z \in \mathcal{D}$.

Proof. Let $z \in \mathbb{C}$ be such that $\text{Re}(z) > 0$. Since $\text{Re}(z) > 0$, there exists a constant α such that $0 < \alpha < \text{Re}(z)$. So for any $n \in \mathbb{N}_0$, $h_n(t, t_0)$ is of exponential order α, and by Theorem 3.2.2, $z \in \mathcal{D}_n$. This holds for any $n \in \mathbb{N}_0$; therefore, $z \in \bigcap_{n=0}^{\infty} \mathcal{D}_n = \mathcal{D}$. $\square$

We are now set to prove the main result of this section:

Theorem 3.3.3. Assume $f : \mathbb{T} \to \mathbb{R}$ is such that there exist constants $K > 0$, $\alpha > 0$ with $f(t) = \sum_{n=0}^{\infty} c_n h_n(t, t_0)$ for all $t \in [t_0, \infty)_{\mathbb{T}}$ where $|c_n| \leq K\alpha^n$ for all $n \in \mathbb{N}_0$. i.e., $f \in \mathcal{F}$. Then,

$$\mathcal{L}\{f\}(z) = \sum_{n=0}^{\infty} c_n \mathcal{L}\{h_n(t, t_0)\}(z) = \sum_{n=0}^{\infty} \frac{c_n}{z^{n+1}},$$

for all $z \in \mathbb{C}$ such that $\text{Re}(z) > \alpha$.

Proof. By definition of the generalized monomial, for each $n \in \mathbb{N}_0$, $h_n(t, t_0) \geq 0$ for all $t \in [t_0, \infty)_{\mathbb{T}}$. Let $z \in \mathbb{C}$ such that $z = x + iy$ and $x > \alpha$. Let $\epsilon > 0$ be given. Since $x > \alpha$, we have that the geometric series $\frac{1}{x} \sum_{n=0}^{\infty} \left(\frac{\alpha}{x}\right)^n = \frac{1}{x}\left(\frac{1}{1-\frac{\alpha}{x}}\right) = \frac{1}{x-\alpha}$. Hence, there exists $M \in \mathbb{N}$ such that for all $N \geq M$,

$$\frac{1}{x-\alpha} - \frac{1}{x} \sum_{n=0}^{N} \left(\frac{\alpha}{x}\right)^n < \frac{\epsilon}{K}.$$

Now, for $N \geq M$, we have

$$\begin{aligned} \left| \mathcal{L}\{f\}(z) - \sum_{n=0}^{N} c_n \mathcal{L}\{h_n(t, t_0)\}(z) \right| &= \left| \mathcal{L}\left\{ f(t) - \sum_{n=0}^{N} c_n h_n(t, t_0) \right\}(z) \right| \\ &= \left| \mathcal{L}\left\{ \sum_{n=N+1}^{\infty} c_n h_n(t, t_0) \right\}(z) \right| \end{aligned}$$

$$
\begin{aligned}
&\leq \int_{t_0}^{\infty} \left(\sum_{n=N+1}^{\infty} |c_n|\, h_n(t,t_0) \right) \left| e^{\sigma}_{\ominus z}(t,t_0) \right| \Delta t \\
&\leq K \int_{t_0}^{\infty} \left(\sum_{n=N+1}^{\infty} \alpha^n h_n(t,t_0) \right) e^{\sigma}_{\ominus x}(t,t_0) \Delta t \\
&= K \int_{t_0}^{\infty} \left(\sum_{n=0}^{\infty} \alpha^n h_n(t,t_0) - \sum_{n=0}^{N} \alpha^n h_n(t,t_0) \right) e^{\sigma}_{\ominus x}(t,t_0) \Delta t \\
&= K \int_{t_0}^{\infty} \left(e_\alpha(t,t_0) - \sum_{n=0}^{N} \alpha^n h_n(t,t_0) \right) e^{\sigma}_{\ominus x}(t,t_0) \Delta t \\
&= K \int_{t_0}^{\infty} e_\alpha(t,t_0) e^{\sigma}_{\ominus x}(t,t_0) \Delta t \\
&\qquad - K \sum_{n=0}^{N} \alpha^n \int_{t_0}^{\infty} h_n(t,t_0) e^{\sigma}_{\ominus x}(t,t_0) \Delta t.
\end{aligned}
$$

We have used Lemma 2.2.2 to obtain the second inequality above. Since $x > \alpha > 0$, and $e_\alpha(t,t_0)$ is of exponential order α, it follows that $x \in \mathcal{D}\{e_\alpha(t,t_0)\}$ by Theorem 3.2.2. Thus, $\mathcal{L}\{e_\alpha(\cdot,t_0)\}(x)$ exists and equals $\frac{1}{x-\alpha}$.

Similarly, since $x > 0$, and $h_n(t,t_0)$ is of exponential order α for any $n \in \mathbb{N}_0$, it follows that $x \in \mathcal{D}$ by Corollary 3.3.2. Thus, $\mathcal{L}\{h_n(\cdot,t_0)\}(x)$ exists and equals $\frac{1}{x^{n+1}}$ for each $n \in \mathbb{N}_0$ by [4, Theorem 3.90]. So, continuing our calculation, we have

$$
\begin{aligned}
&= K\mathcal{L}\{e_\alpha(\cdot,t_0)\}(x) - K \sum_{n=0}^{N} \alpha^n \mathcal{L}\{h_n(\cdot,t_0)\}(x) \\
&= K \left[\frac{1}{x-\alpha} - \sum_{n=0}^{N} \frac{\alpha^n}{x^{n+1}} \right] \\
&= K \left[\frac{1}{x-\alpha} - \frac{1}{x} \sum_{n=0}^{N} \left(\frac{\alpha}{x} \right)^n \right] \\
&< \epsilon.
\end{aligned}
$$

Therefore, since $\epsilon > 0$ was arbitrary, $\mathcal{L}\{f\}(z) = \sum_{n=0}^{\infty} c_n \mathcal{L}\{h_n(t,t_0)\}(z)$. Finally, by [4, Theorem 3.90], we have

$$\mathcal{L}\{f\}(z) = \sum_{n=0}^{\infty} c_n \mathcal{L}\{h_n(t,t_0)\}(z) = \sum_{n=0}^{\infty} \frac{c_n}{z^{n+1}}.$$

□

3.4 Basic Properties of the Laplace Transform

In this section, we collect some basic properties of the generalized Laplace transform on time scales. From Theorem 3.2.2, if $f : \mathbb{T} \to \mathbb{R}$ is of exponential order α, the Laplace transform $\mathcal{L}\{f\}(z)$ converges absolutely for $z \in \mathbb{C}$ such that $\mathrm{Re}(z) > \alpha$ where $\alpha \in \mathcal{R}^+ \cap \mathbb{R}$. In fact, the proof gives the bound

$$|\mathcal{L}\{f\}(z)| \le \int_{t_0}^{\infty} \left| f(t) e^{\sigma}_{\ominus z}(t,t_0) \right| \Delta t \le \frac{M}{\mathrm{Re}(z) - \alpha}.$$

Thus, two immediate corollaries are as follows:

Corollary 3.4.1. Let $f : \mathbb{T} \to \mathbb{R}$ be of exponential order α, and take $x_0 > \alpha$ where $\alpha \in \mathcal{R}^+ \cap \mathbb{R}$. Then, the Laplace transform $\mathcal{L}\{f\}(z)$ is uniformly bounded on $\{z \in \mathbb{C} : \mathrm{Re}(z) \ge x_0\}$.

Proof. For any $z = x + iy$ such that $x \ge x_0 > \alpha$, we have that $|\mathcal{L}\{f\}(z)| \le \frac{M}{x-\alpha} \le \frac{M}{x_0-\alpha}$. Here, the bound $\frac{M}{x_0-\alpha}$ is independent of z. □

Corollary 3.4.2. Let $f : \mathbb{T} \to \mathbb{R}$ be of exponential order α where $\alpha \in \mathcal{R}^+ \cap \mathbb{R}$. Then,

$$\lim_{\mathrm{Re}(z) \to \infty} \mathcal{L}\{f\}(z) = 0.$$

It turns out that the Laplace transform satisfies an even stronger form of convergence than absolute convergence. In the next theorem, we show that the Laplace transform converges uniformly on a closed half-plane.

Theorem 3.4.3. Let $f : \mathbb{T} \to \mathbb{R}$ be of exponential order α, and take $x_0 > \alpha > 0$. Then, the Laplace transform $\mathcal{L}\{f\}(z)$ converges uniformly on $\{z \in \mathbb{C} : \text{Re}(z) \geq x_0\}$.

Proof. First note that for $x \in (0, \infty)$, $e_{\alpha \ominus x}(t, t_0)$ is decreasing with respect to x. To see this, let $x > 0$ be given. Then $x \in \Omega$ as defined by Lemma 2.2.3. Thus, using this lemma, we have

$$\begin{aligned}\frac{d}{dx}\left[e_{\alpha \ominus x}(t, t_0)\right] &= \frac{d}{dx}\left[\frac{e_\alpha(t, t_0)}{e_x(t, t_0)}\right] \\ &= -e_\alpha(t, t_0) e_{\ominus x}(t, t_0) \int_{t_0}^{t} \frac{\Delta\tau}{1 + x\mu(\tau)} \\ &= -e_{\alpha \ominus x}(t, t_0) \int_{t_0}^{t} \frac{\Delta\tau}{1 + x\mu(\tau)}.\end{aligned}$$

Since

$$1 + (\alpha \ominus x)(t)\mu(t) = 1 + \frac{\alpha - x}{1 + x\mu(t)}\mu(t) = \frac{1 + \alpha\mu(t)}{1 + x\mu(t)} > 0$$

on $[t_0, \infty)_{\mathbb{T}}$, by Theorem 2.1.15, $e_{\alpha \ominus x}(t, t_0) > 0$ for all $t \in [t_0, \infty)_{\mathbb{T}}$. Further, since $x > 0$, $\int_{t_0}^{t} \frac{\Delta\tau}{1+x\mu(\tau)} > 0$ for all $t \in (t_0, \infty)_{\mathbb{T}}$. Therefore, $\frac{d}{dx}\left[e_{\alpha \ominus x}(t, t_0)\right] < 0$ for all $x > 0$. It follows that for any $x \geq x_0$, $e_{\alpha \ominus x}(t, t_0) \leq e_{\alpha \ominus x_0}(t, t_0)$.

Let $\epsilon > 0$ be given. Let $z = x + iy$ such that $x \geq x_0$ be given. By Lemma 2.2.1, $\lim_{t \to \infty} e_{\alpha \ominus x_0}(t, t_0) = 0$. Hence, there exists $T \in \mathbb{T}$, sufficiently large, such that for all $t \in [T, \infty)_{\mathbb{T}}$,

$$e_{\alpha \ominus x_0}(t, t_0) < \frac{(x_0 - \alpha)\epsilon}{M}.$$

So, for any $T_1 \in [T,\infty)_{\mathbb{T}}$, we have

$$\begin{aligned}
\left|\int_{T_1}^{\infty} f(t)e^{\sigma}_{\ominus z}(t,t_0)\Delta t\right| &\le M\int_{T_1}^{\infty} e_\alpha(t,t_0)\left|e^{\sigma}_{\ominus z}(t,t_0)\right|\Delta t\\
&\le M\int_{T_1}^{\infty} e_\alpha(t,t_0)e^{\sigma}_{\ominus x}(t,t_0)\Delta t \quad \text{(by Lemma 2.2.2)}\\
&= M\int_{T_1}^{\infty} \frac{1}{1+x\mu(t)}e_{\alpha\ominus x}(t,t_0)\Delta t\\
&= \frac{M}{\alpha-x}\int_{T_1}^{\infty} \frac{\alpha-x}{1+x\mu(t)}e_{\alpha\ominus x}(t,t_0)\Delta t\\
&= \frac{M}{\alpha-x}\int_{T_1}^{\infty} (\alpha\ominus x)(t)e_{\alpha\ominus x}(t,t_0)\Delta t\\
&= \frac{M}{\alpha-x}e_{\alpha\ominus x}(t,t_0)\Big|_{t=T_1}^{t\to\infty}\\
&= \frac{Me_{\alpha\ominus x}(T_1,t_0)}{x-\alpha} \le \frac{Me_{\alpha\ominus x_0}(T_1,t_0)}{x_0-\alpha} < \epsilon.
\end{aligned}$$

Since $\epsilon > 0$ was arbitrary, the Laplace transform converges uniformly on $\{z \in \mathbb{C} : \mathrm{Re}(z) \ge x_0 > \alpha\}$. □

3.5 Periodic Time Scales

Definition 3.5.1. A time scale $\mathbb{T}$ is said to be *periodic with period T* provided $t \in \mathbb{T}$ implies $t + T \in \mathbb{T}$.

Let $\mathbb{T}$ be a periodic time scale with period T. Note that necessarily, periodic time scales are unbounded above. If we consider a function $f : \mathbb{T} \to \mathbb{R}$ that has period T, we can write the Laplace transform of f in terms of $\tilde{f} : \mathbb{T} \to \mathbb{R}$ defined by $\tilde{f}(t) = f(t)$ for $t \in [t_0, t_0 + T]_{\mathbb{T}}$ and $\tilde{f}(t) = 0$ elsewhere. In other words, in this case we can find the Laplace transform merely by performing an appropriate integration over the first period of the function f.

Theorem 3.5.2. Let $\mathbb{T}$ be a periodic time scale with period T. Further, assume

$f : \mathbb{T} \to \mathbb{R}$ is periodic with period T. Then,

$$\mathcal{L}\{f\}(z) = \frac{1}{1 - e_{\ominus z}(t_0 + T, t_0)} \mathcal{L}\{\tilde{f}\}(z),$$

where $\mathcal{L}\{\tilde{f}\}(z) = \int_{t_0}^{t_0+T} f(t) e^{\sigma}_{\ominus z}(t, t_0)\Delta t$.

Proof. Since $\mathbb{T}$ is a periodic time scale with period T, $\mu(t+T) = \mu(t)$ for all $t \in \mathbb{T}$. By Theorem 2.1.21, for any Δ-integrable $g : \mathbb{T} \to \mathbb{R}$, we have

$$\int_a^b g(\tau)\Delta\tau = \int_{[a,b]_{\mathbb{T}}} g(\tau) d\tau + \sum_{\tau \in [a,b)_{\mathbb{T}} \cap T_{rs}(\infty)} g(\tau)\mu(\tau),$$

where the first integral is the classical Lebesgue integral and $[a, b)_{\mathbb{T}} \cap T_{rs}(\infty)$ is the set of right-scattered points in the interval $[a, b)_{\mathbb{T}}$. By the translation invariance of the Lebesgue integral, and reindexing the sum we obtain

$$\begin{aligned}
\int_a^b g(\tau)\Delta\tau &= \int_{[a,b]_{\mathbb{T}}} g(\tau) d\tau + \sum_{\tau \in [a,b)_{\mathbb{T}} \cap T_{rs}(\infty)} g(\tau)\mu(\tau) \\
&= \int_{[a-T,b-T]_{\mathbb{T}}} g(\tau + T) d\tau + \sum_{\tau \in [a-T,b-T)_{\mathbb{T}} \cap T_{rs}(\infty)} g(\tau + T)\mu(\tau + T) \\
&= \int_{[a-T,b-T]_{\mathbb{T}}} g(\tau + T) d\tau + \sum_{\tau \in [a-T,b-T)_{\mathbb{T}} \cap T_{rs}(\infty)} g(\tau + T)\mu(\tau) \\
&= \int_{a-T}^{b-T} g(\tau + T)\Delta\tau,
\end{aligned}$$

where we have applied Theorem 2.1.21 again to obtain the final equality.

Further, for any $z \in \mathbb{C} \cap \mathcal{R}$, we have that the cylinder transformation satisfies $\xi_{\mu(t)}(z) = \xi_{\mu(t+T)}(z)$ for all $t \in \mathbb{T}$ since $\mu(t) = \mu(t+T)$. Using this, it is straightforward to see that $e_z(t + T, t_0 + T) = e_z(t, t_0)$.

Now by the periodicity of f and μ, as well as the semigroup property of the generalized exponential function, we have

$$\begin{aligned}
\mathcal{L}\{f\}(z) &= \int_{t_0}^{\infty} f(t)e^{\sigma}_{\ominus z}(t,t_0)\Delta t \\
&= \int_{t_0}^{t_0+T} f(t)e^{\sigma}_{\ominus z}(t,t_0)\Delta t + \int_{t_0+T}^{\infty} f(t)e^{\sigma}_{\ominus z}(t,t_0)\Delta t \\
&= \int_{t_0}^{t_0+T} f(t)e^{\sigma}_{\ominus z}(t,t_0)\Delta t + \int_{t_0}^{\infty} f(\tau+T)e^{\sigma}_{\ominus z}(\tau+T,t_0)\Delta \tau \\
&= \int_{t_0}^{t_0+T} f(t)e^{\sigma}_{\ominus z}(t,t_0)\Delta t \\
&\qquad + \int_{t_0}^{\infty} f(\tau)\left(\frac{1}{1+z\mu(\tau+T)}\right) e_{\ominus z}(\tau+T,t_0)\Delta \tau \\
&= \int_{t_0}^{t_0+T} f(t)e^{\sigma}_{\ominus z}(t,t_0)\Delta t \\
&\qquad + \int_{t_0}^{\infty} f(\tau)\left(\frac{1}{1+z\mu(\tau)}\right) e_{\ominus z}(\tau+T,t_0+T)e_{\ominus z}(t_0+T,t_0)\Delta \tau \\
&= \int_{t_0}^{t_0+T} f(t)e^{\sigma}_{\ominus z}(t,t_0)\Delta t \\
&\qquad + e_{\ominus z}(t_0+T,t_0)\int_{t_0}^{\infty} f(\tau)\left(\frac{1}{1+z\mu(\tau)}\right) e_{\ominus z}(\tau,t_0)\Delta \tau \\
&= \int_{t_0}^{t_0+T} f(t)e^{\sigma}_{\ominus z}(t,t_0)\Delta t + e_{\ominus z}(t_0+T,t_0)\int_{t_0}^{\infty} f(\tau)e^{\sigma}_{\ominus z}(\tau,t_0)\Delta \tau \\
&= \int_{t_0}^{t_0+T} f(t)e^{\sigma}_{\ominus z}(t,t_0)\Delta t + e_{\ominus z}(t_0+T,t_0)\mathcal{L}\{f\}(z).
\end{aligned}$$

Solving for $\mathcal{L}\{f\}(z)$, we obtain

$$\mathcal{L}\{f\}(z) = \frac{1}{1-e_{\ominus z}(t_0+T,t_0)}\int_{t_0}^{t_0+T} f(t)e^{\sigma}_{\ominus z}(t,t_0)\Delta t.$$

□

3.6 Regressivity

In Theorem 3.2.1, direct calculation from the definition of the Laplace transform gives us that if $\alpha \in \mathbb{C} \cap \mathcal{R}$, $\mathcal{L}\{e_\alpha(\cdot, t_0)\}(z) = \frac{1}{z-\alpha}$ for $z \in \mathbb{C} \cap \mathcal{R}$ such that $\lim_{t\to\infty} e_{\alpha \ominus z}(t, t_0) = 0$. When applying this formula, it is essential that $\alpha \in \mathcal{R}$ to ensure that the generalized exponential function $e_\alpha(t, t_0)$ is well defined. So, this formula holds only on specific time scales; namely, time scales such that $\alpha \in \mathcal{R}$. The question we concern ourselves with in this section is on a time scale $\mathbb{T}$ for which $\alpha \notin \mathcal{R}$, does there exist $f : \mathbb{T} \to \mathbb{R}$ such that $\mathcal{L}\{f\}(z) = \frac{1}{z-\alpha}$? The answer is affirmative, and we will show by example that we can use this to solve initial value problems.

Let $\mathbb{T}$ be a time scale that is unbounded above and fix $t_0 \in \mathbb{T}$. Let $\alpha \notin \mathcal{R}([t_0, \infty)_\mathbb{T})$ be given. Thus, there exists $t_1 \in [t_0, \infty)_\mathbb{T}$ such that $1 + \alpha\mu(t_1) = 0$ and for all $t_0 \le t < t_1$, $1 + \alpha\mu(t) \neq 0$. In other words, t_1 is the first point in the time scale for which $1 + \alpha\mu(t_1) = 0$.

Consider $\tilde{\mathbb{T}} := [t_0, t_1]_\mathbb{T}$. Note that $\tilde{\mathbb{T}}$ is a bounded time scale such that $\alpha \in \mathcal{R}(\tilde{\mathbb{T}})$. Hence, on $\tilde{\mathbb{T}}$, the generalized exponential, $e_\alpha(t, t_0)$, is well defined. Note that there is a subtle difference between the graininess functions for the time scales $\mathbb{T}$ and $\tilde{\mathbb{T}}$. For any $t \in [t_0, t_1)_\mathbb{T}$, $\mu(t) = \tilde{\mu}(t)$. However, $\mu(t_1) = \frac{-1}{\alpha}$ but $\tilde{\mu}(t_1) = 0$, and then for any $t \in \mathbb{T}$ such that $t > t_1$, $\tilde{\mu}(t_1)$ is not even defined. Define $f : \mathbb{T} \to \mathbb{R}$ by

$$f(t) := \begin{cases} e_\alpha(t, t_0), & t \in \tilde{\mathbb{T}}, \\ 0, & t \notin \tilde{\mathbb{T}}. \end{cases}$$

By construction, f is well defined on all of $\mathbb{T}$.

Theorem 3.6.1. Let $\mathbb{T}$, $\tilde{\mathbb{T}}$, α, $t_1 \in \mathbb{T}$, and $f : \mathbb{T} \to \mathbb{R}$ be as above. Then,

$$\mathcal{L}\{f\}(z) = \frac{1}{z - \alpha},$$

for all $z \in \mathbb{C} \cap \mathcal{R}$.

Proof. Using the definition of the Laplace transform, we have

$$\begin{aligned}
\mathcal{L}\{f\}(z) &= \int_{t_0}^{\infty} f(t) e^{\sigma}_{\ominus z}(t, t_0)\Delta t \\
&= \int_{t_0}^{\sigma(t_1)} e_{\alpha}(t, t_0) e^{\sigma}_{\ominus z}(t, t_0)\Delta t \\
&= \int_{t_0}^{t_1} \frac{e_{\alpha}(t, t_0)}{1 + z\mu(t)} e_{\ominus z}(t, t_0)\Delta t + \int_{t_1}^{\sigma(t_1)} \frac{e_{\alpha}(t, t_0)}{1 + z\mu(t)} e_{\ominus z}(t, t_0)\Delta t \\
&= \int_{t_0}^{t_1} \frac{1}{1 + z\mu(t)} e_{\alpha \ominus z}(t, t_0)\Delta t + \frac{\mu(t_1)}{1 + z\mu(t_1)} e_{\alpha \ominus z}(t_1, t_0) \\
&= \frac{1}{\alpha - z} \int_{t_0}^{t_1} \frac{\alpha - z}{1 + z\mu(t)} e_{\alpha \ominus z}(t, t_0)\Delta t + \frac{-\frac{1}{\alpha}}{1 - \frac{z}{\alpha}} e_{\alpha \ominus z}(t_1, t_0) \\
&= \frac{1}{\alpha - z} \int_{t_0}^{t_1} (\alpha \ominus z)(t) e_{\alpha \ominus z}(t, t_0)\Delta t - \frac{1}{\alpha - z} e_{\alpha \ominus z}(t_1, t_0) \\
&= \frac{1}{\alpha - z} e_{\alpha \ominus z}(t, t_0)\Big|_{t=t_0}^{t=t_1} - \frac{1}{\alpha - z} e_{\alpha \ominus z}(t_1, t_0) \\
&= \frac{1}{\alpha - z} e_{\alpha \ominus z}(t_1, t_0) - \frac{1}{\alpha - z} e_{\alpha \ominus z}(t_0, t_0) - \frac{1}{\alpha - z} e_{\alpha \ominus z}(t_1, t_0) \\
&= \frac{1}{z - \alpha}.
\end{aligned}$$

The fourth equality is obtained by applying Theorem 2.1.21 to the second integral. □

Example 3.6.2. Let $\mathbb{T}$ be the time scale $\{0, 3, 6, 7\} \cup \mathbb{T}_a$ where $\mathbb{T}_a$ is any time scale such that $\mathbb{T}_a \subseteq [7, \infty)$. Note that, in particular, $\mu(0) = \mu(3) = 3$ and $\mu(6) = 1$. Thus, $-1 \notin \mathcal{R}(\mathbb{T})$.

Consider the initial value problem

$$x^{\Delta} + x = 1, \quad x(0) = 0.$$

Applying the Laplace transform, we have

$$\begin{aligned} z\mathcal{L}\{x\}(z) + \mathcal{L}\{x\}(z) = \frac{1}{z} &\implies \mathcal{L}\{x\}(z) = \frac{1}{z(z+1)} = \frac{1}{z} - \frac{1}{z+1} \\ &\implies x(t) = \mathcal{L}^{-1}\left\{\frac{1}{z}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{z+1}\right\} \\ &\implies x(t) = 1 - \mathcal{L}^{-1}\left\{\frac{1}{z+1}\right\}. \end{aligned}$$

Since $-1 \notin \mathcal{R}(\mathbb{T})$, we must apply the previous theorem in order to find a function f such that $\mathcal{L}\{f\}(z) = \frac{1}{z+1}$ on this particular time scale. In this case, $\tilde{\mathbb{T}} = \{0, 3, 6\}$ and on this time scale, it is straightforward to compute that $e_{-1}(t, 0) = (-2)^{\frac{t}{3}}$. Hence, defining $f(t) := (-2)^{\frac{t}{3}}$ for $t \in \{0, 3, 6\}$ and 0 otherwise, we have $\mathcal{L}\{f\}(z) = \frac{1}{z+1}$ by Theorem 3.6.1. Therefore,

$$\begin{aligned} x(t) &= 1 - \begin{cases} (-2)^{\frac{t}{3}}, & t \in \{0, 3, 6\}, \\ 0, & \text{otherwise} \end{cases} \\ &= \begin{cases} 1 - (-2)^{\frac{t}{3}}, & t \in \{0, 3, 6\}, \\ 1, & \text{otherwise.} \end{cases} \end{aligned}$$

Direct calculation verifies that this is indeed a solution of the initial value problem on $\mathbb{T}$.

3.7 Inversion Integrals

In this section we present two inversion theorems for the Laplace transform that make use of contour integration. The result given in Theorem 3.7.1 is specific to time scales that are unbounded above and such that $0 < \mu_{min} \leq \mu(t)$ for all $t \in \mathbb{T}$. In this particular case, the inversion integral appears very much like the classic inversion integral in the case of the Z-transform on the integers.

In the second theorem given in this section (Theorem 3.7.3), we turn our attention to an arbitrary time scale (i.e., we require no restriction on the graininess as needed in Theorem 3.7.1).

Theorem 3.7.1. Assume $\mathbb{T}$ is unbounded above such that $0 < \mu_{min} \leq \mu(t)$ for all $t \in \mathbb{T}$. Let $f : \mathbb{T} \to \mathbb{R}$ such that $F(z) := \mathcal{L}\{f\}(z)$ is analytic on $\{z \in \mathbb{C} : |z| > R\}$ for some $R > 0$, $F(z)$ has only isolated singularities, and the Laplace transform of f converges uniformly on $\Omega := \{z \in \mathbb{C} : |z| \geq r\}$ where $r > R$. Then,

$$f(t) = \frac{1}{2\pi i} \int_C F(z) e_z(t, t_0) dz,$$

where C is a positively oriented, simple, closed path in Ω such that $\mathrm{Ind}_C\left(\frac{-1}{\mu(t)}\right) = 1$ for all $t \in [t_0, \infty)_{\mathbb{T}}$, and $\mathrm{Ind}_C(z) = 1$ at each singularity of F. Here, $\mathrm{Ind}_C(z)$ is the *index* of z with respect to the path C, and is defined by $\mathrm{Ind}_C(z) := \frac{1}{2\pi i} \int_C \frac{d\xi}{\xi - z}$.

Proof. Note that since F is analytic on $\{z \in \mathbb{C} : |z| > R\}$ and has only isolated singularities, it follows that F has only a finite number of singularities. Further, since $0 < \mu_{min} \leq \mu(t)$ we have that $\frac{-1}{\mu_{min}} \leq \frac{-1}{\mu(t)} < 0$ for all $t \in [t_0, \infty)_{\mathbb{T}}$. Hence, there exists a positively oriented, simple, closed path in Ω that encircles the singularities of F and the points $-\frac{1}{\mu(t)}$, $t \in [t_0, \infty)_{\mathbb{T}}$.

Since $0 < \mu_{min} \leq \mu(t)$ for all $t \in \mathbb{T}$, it follows that $\mathbb{T}$ is an isolated time scale.

Thus, let $\mathbb{T} = \{t_0, t_1, \cdots\}$ where $t_0 < t_1 < \cdots$. Then, for any $t_n \in \mathbb{T}$, by Theorem 2.1.21,

$$\begin{aligned} e_z(t_n, t_0) &= \exp\left(\int_{t_0}^{t_n} \xi_{\mu(\tau)}(z)\Delta\tau\right) \\ &= \exp\left(\sum_{k=0}^{n-1} \operatorname{Log}(1 + z\mu(t_k))\right) = \prod_{k=0}^{n-1} (1 + z\mu(t_k)). \end{aligned}$$

This implies $e_{\ominus z}(t_n, t_0) = \prod_{k=0}^{n-1} \frac{1}{1+z\mu(t_k)}$.

Again using Theorem 2.1.21, we have

$$F(z) = \mathcal{L}\{f\}(z) = \int_{t_0}^{\infty} f(t)e^{\sigma}_{\ominus z}(t, t_0)\Delta t = \sum_{n=0}^{\infty} f(t_n)\mu(t_n) \prod_{k=0}^{n} \frac{1}{1 + z\mu(t_k)}.$$

Fix $t_m \in \mathbb{T}$. Then, using uniform convergence to interchange the sum and integral, we have

$$\begin{aligned} &\frac{1}{2\pi i} \int_C F(z)e_z(t_m, t_0)dz \\ &\quad= \frac{1}{2\pi i} \int_C \left(\sum_{n=0}^{\infty} f(t_n)\mu(t_n) \prod_{k=0}^{n} \frac{1}{1 + z\mu(t_k)}\right) \prod_{k=0}^{m-1} (1 + z\mu(t_k))dz \\ &\quad= \sum_{n=0}^{\infty} f(t_n)\mu(t_n) \left[\frac{1}{2\pi i} \int_C \prod_{k=0}^{n} \frac{1}{1 + z\mu(t_k)} \prod_{k=0}^{m-1} (1 + z\mu(t_k))dz\right] \\ &\quad= \sum_{n=0}^{m-1} f(t_n)\mu(t_n) \frac{1}{2\pi i} \int_C \prod_{k=n+1}^{m} (1 + z\mu(t_k))dz + \frac{1}{2\pi i} \int_C \frac{f(t_m)\mu(t_m)}{1 + z\mu(t_m)} dz \\ &\qquad\quad + \sum_{n=m+1}^{\infty} f(t_n)\mu(t_n) \frac{1}{2\pi i} \int_C \prod_{k=m}^{n} \frac{1}{1 + z\mu(t_k)} dz. \end{aligned} \tag{3.6}$$

Note that for any $0 \le n \le m-1$, $\prod_{k=n+1}^{m}(1+z\mu(t_k))$ is entire and hence the contour integral $\int_C \prod_{k=n+1}^{m}(1 + z\mu(t_k))dz = 0$ for such n. Therefore, the first summation in (3.6) is 0.

As for the second term of (3.6), since $\frac{f(t_m)\mu(t_m)}{1+z\mu(t_m)}$ has a pole at $\frac{-1}{\mu(t_m)}$ of order one which is encircled by C, by the Cauchy Residue Theorem we have

$$\begin{aligned}\frac{1}{2\pi i}\int_C \frac{f(t_m)\mu(t_m)}{1+z\mu(t_m)}dz &= \frac{f(t_m)\mu(t_m)}{2\pi i}\int_C \frac{1}{1+z\mu(t_m)}dz\\ &= f(t_m)\mu(t_m)\operatorname*{Res}_{z=-\frac{1}{\mu(t_m)}}\frac{1}{1+z\mu(t_m)}\\ &= f(t_m)\mu(t_m)\frac{1}{\mu(t_m)} = f(t_m).\end{aligned}$$

We now aim to handle the third term of (3.6) by showing that

$$\frac{1}{2\pi i}\int_C \prod_{k=m}^{n}\frac{1}{1+z\mu(t_k)}dz = 0$$

for every $n \geq m+1$. To do so, fix $n \geq m+1$. Define $f_n(z) := \prod_{k=m}^{n}\frac{1}{1+z\mu(t_k)}$. Note that C encircles all the poles of f_n and f_n is analytic on $\{z \in \mathbb{C} : |z| > R\}$. Thus, we can apply the so-called residue theorem at infinity (see, for example, [7, page 185]):

$$\begin{aligned}\frac{1}{2\pi i}\int_C f_n(z)dz &= -\operatorname*{Res}_{z=0}\frac{1}{z^2}f_n\left(\frac{1}{z}\right) = -\operatorname*{Res}_{z=0}\frac{1}{z^2}\prod_{k=m}^{n}\frac{1}{1+\frac{1}{z}\mu(t_k)}\\ &= -\operatorname*{Res}_{z=0}\frac{1}{z^2}\left(\frac{z}{z+\mu(t_m)}\right)\left(\frac{z}{z+\mu(t_{m+1})}\right)\prod_{k=m+2}^{n}\frac{z}{z+\mu(t_k)}\\ &= -\operatorname*{Res}_{z=0}\left(\frac{1}{z+\mu(t_m)}\right)\left(\frac{1}{z+\mu(t_{m+1})}\right)\prod_{k=m+2}^{n}\frac{z}{z+\mu(t_k)} = 0,\end{aligned}$$

since $\left(\frac{1}{z+\mu(t_m)}\right)\left(\frac{1}{z+\mu(t_{m+1})}\right)\prod_{k=m+2}^{n}\frac{z}{z+\mu(t_k)}$ does not have a singularity at $z = 0$. Since $n \geq m+1$ was arbitrary, it follows that the third term of (3.6) is 0. Therefore, we have shown that

$$\frac{1}{2\pi i}\int_C F(z)e_z(t_m,t_0)dz = f(t_m),$$

for all $m \in \mathbb{N}_0$. □

Turning our attention to the case of an arbitrary time scale, we first consider a result from complex analysis. An extension of Cauchy's integral formula given in [9, Theorem 62.1] is as follows:

Theorem 3.7.2. Let f be analytic in the half-plane $\mathrm{Re}(z) \geq \gamma > 0$ such that there exist constants $M > 0$, $k > 0$ such that $|f(z)| \leq \frac{M}{|z|^k}$ for $|z| > r_0$ in the half-plane for some $r_0 > 0$. If $z_0 \in \mathbb{C}$ with $\mathrm{Re}(z_0) > \gamma$, then

$$f(z_0) = \frac{-1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} \frac{f(z)}{z - z_0} dz = \frac{-1}{2\pi} \int_{-\infty}^{\infty} \frac{f(z)}{z - z_0} dy,$$

where the integration is along the line $x = \gamma$; so $z = \gamma + iy$.

To get an idea of what the inversion integral should be, consider the following. Suppose $F(z) = \mathcal{L}\{f\}(z)$. Assuming we can interchange the inverse Laplace transform operator $\mathcal{L}^{-1}$ and a certain integral, we would have

$$\begin{aligned} f(t) &= \mathcal{L}^{-1}\{\mathcal{L}\{f\}(z)\}(t) \\ &= \mathcal{L}^{-1}\{F(z)\}(t) \\ &= \mathcal{L}^{-1}\left\{\frac{-1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} \frac{F(s)}{s - z} ds\right\}(t) \\ &= \frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} F(s)\mathcal{L}^{-1}\left\{\frac{1}{z - s}\right\}(t) ds \\ &= \frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} F(s) e_s(t, t_0) ds. \end{aligned}$$

Here we have used the fact that $\mathcal{L}\{e_s(\cdot, t_0)\}(z) = \frac{1}{z-s}$ on any time scale for an appropriate choice of $s \in \mathbb{C}$. Of course these are merely formal manipulations, but it does give us an indication as to what the time scale equivalent of the Laplace inversion formula should be.

Theorem 3.7.3. Let $F(z)$ be analytic in the half-plane $\text{Re}(z) > \alpha > 0$ such that for each fixed $t \in [t_0, \infty)_{\mathbb{T}}$, there exist $M > 0$, $k > 1$, and $\gamma > \alpha$ such that

$$|F(\gamma \oplus iy)|\,|e_{iy}(t,t_0)| \leq \frac{M}{y^k} \quad \text{for all } y \geq y_0 > 0,$$

for some $y_0 > 0$. Further, assume $F(z)$ is real-valued whenever $z \in \mathbb{R}$ and $\text{Re}(z) > \alpha$. Then, the following integral along the line $x = \gamma$ converges to some function, say $f : \mathbb{T} \to \mathbb{R}$. i.e.,

$$f(t) = \frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} e_z(t,t_0)F(z)dz.$$

Proof. Note $1 + z\mu(t) = 0$ if and only if $z = -\frac{1}{\mu(t)}$. Hence, for any $z \notin \mathcal{R}$, it must be the case that z is on the negative real axis. Since we are working in the half-plane $\text{Re}(z) > \alpha > 0$, non-regressivity is not an issue.

Fix $t \in [t_0, \infty)_{\mathbb{T}}$. Now, for any $\beta > 0$, we consider $\frac{1}{2\pi i}\int_{\gamma - i\beta}^{\gamma + i\beta} F(z)e_z(t,t_0)dz$. Consider the substitution:

$$\begin{aligned} y &= \frac{z-\gamma}{i(1+\gamma\mu(t))} \implies z = \gamma + iy(1+\gamma\mu(t)) = \gamma \oplus iy, \\ dy &= \frac{1}{i(1+\gamma\mu(t))}dz \implies dz = i(1+\gamma\mu(t))dy. \end{aligned}$$

Also, the limits of integration with this substitution become

$$z = \gamma + i\beta \implies y = \frac{\beta}{1+\gamma\mu(t)} \text{ and } z = \gamma - i\beta \implies y = \frac{-\beta}{1+\gamma\mu(t)}.$$

Thus,

$$
\begin{aligned}
\frac{1}{2\pi i}&\int_{\gamma-i\beta}^{\gamma+i\beta} F(z)e_z(t,t_0)dz \\
&= \frac{i(1+\gamma\mu(t))}{2\pi i}\int_{\frac{-\beta}{1+\gamma\mu(t)}}^{\frac{\beta}{1+\gamma\mu(t)}} e_{\gamma\oplus iy}(t,t_0)F(\gamma\oplus iy)dy \\
&= \frac{(1+\gamma\mu(t))e_\gamma(t,t_0)}{2\pi}\int_{\frac{-\beta}{1+\gamma\mu(t)}}^{\frac{\beta}{1+\gamma\mu(t)}} e_{iy}(t,t_0)F(\gamma\oplus iy)dy \\
&= \frac{e_\gamma^\sigma(t,t_0)}{2\pi}\left[\int_{\frac{-\beta}{1+\gamma\mu(t)}}^{0} e_{iy}(t,t_0)F\Big(\gamma+iy(1+\gamma\mu(t))\Big)dy \right.\\
&\qquad\left. + \int_0^{\frac{\beta}{1+\gamma\mu(t)}} e_{iy}(t,t_0)F\Big(\gamma+iy(1+\gamma\mu(t))\Big)dy\right] \\
&= \frac{e_\gamma^\sigma(t,t_0)}{2\pi}\left[\int_0^{\frac{\beta}{1+\gamma\mu(t)}} e_{-iy}(t,t_0)F\Big(\gamma-iy(1+\gamma\mu(t))\Big)dy \right.\\
&\qquad\left. + \int_0^{\frac{\beta}{1+\gamma\mu(t)}} e_{iy}(t,t_0)F\Big(\gamma+iy(1+\gamma\mu(t))\Big)dy\right].
\end{aligned}
$$

Since $F(z)$ is real-valued whenever $z \in \mathbb{R}$ and $\mathrm{Re}(z) > \alpha$, by the Schwarz Reflection Principle, it follows that $F(\overline{z}) = \overline{F(z)}$. Further, by Lemma 2.2.4, $\overline{e_{iy}(t,t_0)} = e_{-iy}(t,t_0)$. So, continuing the calculation we have

$$
\begin{aligned}
&= \frac{e_\gamma^\sigma(t,t_0)}{\pi}\int_0^{\frac{\beta}{1+\gamma\mu(t)}} \mathrm{Re}\left[e_{iy}(t,t_0)F\Big(\gamma+iy(1+\gamma\mu(t))\Big)\right]dy \\
&= \frac{e_\gamma^\sigma(t,t_0)}{\pi}\int_0^{\frac{\beta}{1+\gamma\mu(t)}} \Big[U(\gamma,y(1+\gamma\mu(t)))\cos_y(t,t_0) \\
&\qquad\qquad - V(\gamma,y(1+\gamma\mu(t)))\sin_y(t,t_0)\Big]dy,
\end{aligned}
$$

where we have used Euler's formula (see [4, Exercise 3.27]) and taken $F(u+iv) = U(u,v) + iV(u,v)$.

So, if $\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} F(z)e_z(t,t_0)dz$ converges, it is real-valued. Finally, to see that it does in fact converge, using the same substitution as above, note that for any sufficiently large β,

$$\begin{aligned}
&\left|\frac{1}{2\pi i}\int_{\gamma-i\beta}^{\gamma+i\beta} e_z(t,t_0)F(z)dz\right| \\
&\qquad = \frac{e_\gamma^\sigma(t,t_0)}{\pi}\left|\int_0^{\frac{\beta}{1+\gamma\mu(t)}} \operatorname{Re}\left[e_{iy}(t,t_0)F(\gamma\oplus iy)\right]dy\right| \\
&\qquad \le \frac{e_\gamma^\sigma(t,t_0)}{\pi}\int_0^{\frac{\beta}{1+\gamma\mu(t)}} \left|F(\gamma\oplus iy)e_{iy}(t,t_0)\right|dy \\
&\qquad \le \frac{e_\gamma^\sigma(t,t_0)}{\pi}\int_0^{\infty} \left|F(\gamma\oplus iy)e_{iy}(t,t_0)\right|dy \\
&\qquad \le \frac{e_\gamma^\sigma(t,t_0)}{\pi}\left[\int_0^{y_0} \left|F(\gamma\oplus iy)e_{iy}(t,t_0)\right|dy + \int_{y_0}^{\infty}\frac{M}{y^k}dy\right] \\
&\qquad \le \frac{e_\gamma^\sigma(t,t_0)}{\pi}\left[C_{t,\gamma} + \frac{My_0^{1-k}}{k-1}\right],
\end{aligned}$$

since $k > 1$. Here the bound is independent of β. Thus, as a real-valued function of β,

$$\left|\frac{1}{2\pi i}\int_{\gamma-i\beta}^{\gamma+i\beta} e_z(t,t_0)F(z)dz\right|,$$

is monotonically increasing, and bounded above, and hence converges as $\beta \to \infty$. Thus, the contour integral $\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e_z(t,t_0)F(z)dz$ converges. □

3.8 Some Calculations

Recall that in the case of the classical Laplace transform on $\mathbb{R}$, for $n \in \mathbb{N}_0$ we have

$$\frac{1}{n!}\mathcal{L}\{t^n e^{\alpha t}\}(z) = \frac{1}{(z-\alpha)^{n+1}}.$$

The question naturally arises: what is the time scale analog of $\frac{t^n}{n!}$ in the context of the Laplace transform?

Definition 3.8.1. Fix $\alpha \in \mathbb{C} \cap \mathcal{R}$. For each $k \in \mathbb{N}_0$, we define the functions $j_k : \mathbb{T} \times \mathbb{T} \to \mathbb{C}$ recursively by taking $j_0(t, t_0) := 1$, and

$$j_{k+1}(t, t_0) := \int_{t_0}^{t} \frac{1}{1 + \alpha\mu(\tau)} j_k(\tau, t_0)\Delta\tau.$$

Theorem 3.8.2. Let $\alpha \in \mathbb{C} \cap \mathcal{R}$, and $n \in \mathbb{N}_0$ be given. Then

$$\mathcal{L}\left\{j_n(\cdot, t_0)e_\alpha(\cdot, t_0)\right\}(z) = \frac{1}{(z - \alpha)^{n+1}},$$

provided

$$\lim_{t\to\infty} j_k(t, t_0)e_{\alpha\ominus z}(t, t_0) = 0 \text{ for each } k = 0, 1, \cdots, n.$$

Proof. For $n = 0$, from Theorem 3.2.1, we have

$$\mathcal{L}\{j_0(\cdot, t_0)e_\alpha(\cdot, t_0)\}(z) = \mathcal{L}\{e_\alpha(\cdot, t_0)\}(z) = \frac{1}{z - \alpha}.$$

Note that for any $n \in \mathbb{N}$,

$$j_n^\Delta(t, t_0) = \left(\int_{t_0}^{t} \frac{1}{1 + \alpha\mu(\tau)} j_{n-1}(\tau, t_0)\Delta\tau\right)^\Delta = \frac{1}{1 + \alpha\mu(t)} j_{n-1}(t, t_0).$$

Proceeding by induction, assume $\mathcal{L}\left\{j_{n-1}(\cdot, t_0)e_\alpha(\cdot, t_0)\right\}(z) = \frac{1}{(z-\alpha)^n}$ for some $n \geq 1$. Now consider

$$\begin{aligned}
&\mathcal{L}\{j_n(\cdot, t_0)e_\alpha(\cdot, t_0)\}(z)\\
&\qquad = \int_{t_0}^{\infty} j_n(t, t_0)e_\alpha(t, t_0)e^\sigma_{\ominus z}(t, t_0)\Delta t\\
&\qquad = \int_{t_0}^{\infty} j_n(t, t_0)(1 + \mu(t)(\ominus z)(t))e_{\alpha\ominus z}(t, t_0)\Delta t
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{\alpha - z} \int_{t_0}^{\infty} j_n(t,t_0) \left(\frac{\alpha - z}{1 + z\mu(t)} \right) e_{\alpha \ominus z}(t,t_0) \Delta t \\
&= \frac{1}{\alpha - z} \int_{t_0}^{\infty} j_n(t,t_0) (\alpha \ominus z)(t) e_{\alpha \ominus z}(t,t_0) \Delta t \\
&= \frac{1}{\alpha - z} \left[j_n(t,t_0) e_{\alpha \ominus z}(t,t_0) \Big|_{t=t_0}^{t \to \infty} - \int_{t_0}^{\infty} j_n^{\Delta}(t,t_0) e_{\alpha \ominus z}^{\sigma}(t,t_0) \Delta t \right] \\
&\qquad \text{(by integration by parts)} \\
&= \frac{1}{\alpha - z} \left[0 - \int_{t_0}^{\infty} j_n^{\Delta}(t,t_0) e_{\alpha \ominus z}^{\sigma}(t,t_0) \Delta t \right] \\
&= \frac{1}{z - \alpha} \int_{t_0}^{\infty} j_n^{\Delta}(t,t_0) \left(1 + \mu(t)(\alpha \ominus z)(t)\right) e_{\alpha \ominus z}(t,t_0) \Delta t \\
&= \frac{1}{z - \alpha} \int_{t_0}^{\infty} \left(\frac{1}{1 + \alpha\mu(t)} \right) j_{n-1}(t,t_0) \left(\frac{1 + \alpha\mu(t)}{1 + z\mu(t)} \right) e_{\alpha \ominus z}(t,t_0) \Delta t \\
&= \frac{1}{z - \alpha} \int_{t_0}^{\infty} j_{n-1}(t,t_0) e_{\alpha}(t,t_0) \left(\frac{1}{1 + z\mu(t)} \right) e_{\ominus z}(t,t_0) \Delta t \\
&= \frac{1}{z - \alpha} \int_{t_0}^{\infty} j_{n-1}(t,t_0) e_{\alpha}(t,t_0) \left(1 + \mu(t)(\ominus z)(t)\right) e_{\ominus z}(t,t_0) \Delta t \\
&= \frac{1}{z - \alpha} \int_{t_0}^{\infty} j_{n-1}(t,t_0) e_{\alpha}(t,t_0) e_{\ominus z}^{\sigma}(t,t_0) \Delta t \\
&= \frac{1}{z - \alpha} \mathcal{L}\left\{ j_{n-1}(\cdot,t_0) e_{\alpha}(\cdot,t_0) \right\}(z) = \frac{1}{(z - \alpha)^{n+1}}.
\end{aligned}
$$

□

Theorem 3.8.3. Let $\alpha, \beta \in \mathbb{C}$ be regressive. Then

$$
\mathcal{L}\left\{ e_{\frac{\alpha}{1+\beta\mu}}(\cdot,t_0) e_{\beta}(\cdot,t_0) \right\}(z) = \mathcal{L}\{e_{\beta}(\cdot,t_0)\}(z - \alpha) = \frac{1}{z - (\alpha + \beta)},
$$

for those $z \in \mathcal{R} \cap \mathbb{C}$ such that $\lim_{t \to \infty} e_{(\alpha+\beta) \ominus z}(t,t_0) = 0$.

Proof. Note that for any $t \in \mathbb{T}$,

$$\begin{aligned}\frac{\alpha}{1+\beta\mu(t)} \oplus \beta &= \frac{\alpha}{1+\beta\mu(t)} + \beta + \frac{\alpha\beta\mu(t)}{1+\beta\mu(t)} \\ &= \frac{\alpha+\beta+\beta^2\mu(t)+\alpha\beta\mu(t)}{1+\alpha\beta\mu(t)} \\ &= \frac{(\alpha+\beta)(1+\beta\mu(t))}{1+\beta\mu(t)} \\ &= \alpha+\beta.\end{aligned}$$

Hence

$$\mathcal{L}\left\{e_{\frac{\alpha}{1+\beta\mu}}(\cdot,t_0)e_\beta(\cdot,t_0)\right\}(z) = \mathcal{L}\left\{e_{\alpha+\beta}(\cdot,t_0)\right\}(z) = \frac{1}{z-(\alpha+\beta)}.$$

On the other hand, $\mathcal{L}\{e_\beta(\cdot,t_0)\}(z-\alpha) = \frac{1}{z-\beta}\Big|_{z=z-\alpha} = \frac{1}{z-(\alpha+\beta)}$. □

Theorem 3.8.4. For $\alpha \neq 0$,

$$\mathcal{L}^{-1}\left\{\frac{1}{(z^2+\alpha^2)^2}\right\}(t) = \frac{\sin_\alpha(t,t_0)}{2\alpha^3} - \frac{\cos_\alpha(t,t_0)}{2\alpha^2}\int_{t_0}^{t}\frac{\Delta\tau}{1+\alpha^2\mu^2(\tau)} - \frac{\sin_\alpha(t,t_0)}{2\alpha}\int_{t_0}^{t}\frac{\mu(\tau)\Delta\tau}{1+\alpha^2\mu^2(\tau)},$$

for those $z \in \mathcal{R} \cap \mathbb{C}$ such that

$$\lim_{t\to\infty} j_k(t,t_0)e_{i\alpha\ominus z}(t,t_0) = 0 \text{ and } \lim_{t\to\infty} j_k(t,t_0)e_{-i\alpha\ominus z}(t,t_0) = 0, \quad k = 0,1.$$

Proof. Let $\alpha \neq 0$ be given. Note that by partial fraction decomposition, we have

$$\frac{1}{(z^2+\alpha^2)^2} = \frac{-1}{4\alpha^3 i(z+i\alpha)} - \frac{1}{4\alpha^2 i(z+i\alpha)^2} + \frac{1}{4\alpha^3 i(z-i\alpha)} - \frac{1}{4\alpha^2 i(z-i\alpha)^2}.$$

We now take the inverse Laplace transform and apply Theorem 3.2.1 and Theorem

3.8.2, to obtain

$$\begin{aligned}
&\mathcal{L}^{-1}\left\{\frac{1}{(z^2+\alpha^2)^2}\right\}(t)\\
&\quad=\frac{-1}{4\alpha^3 i}\mathcal{L}^{-1}\left\{\frac{1}{z+i\alpha}\right\}(t)-\frac{1}{4\alpha^2 i}\mathcal{L}^{-1}\left\{\frac{1}{(z+i\alpha)^2}\right\}(t)\\
&\qquad\quad+\frac{1}{4\alpha^3 i}\mathcal{L}^{-1}\left\{\frac{1}{z-i\alpha}\right\}(t)-\frac{1}{4\alpha^2 i}\mathcal{L}^{-1}\left\{\frac{1}{(z-i\alpha)^2}\right\}(t)\\
&\quad=\frac{-1}{4\alpha^3 i}e_{-i\alpha}(t,t_0)-\frac{1}{4\alpha^2 i}\left(e_{-i\alpha}(t,t_0)\int_{t_0}^{t}\frac{\Delta\tau}{1-i\alpha\mu(\tau)}\right)\\
&\qquad\quad+\frac{1}{4\alpha^3 i}e_{i\alpha}(t,t_0)-\frac{1}{4\alpha^2 i}\left(e_{i\alpha}(t,t_0)\int_{t_0}^{t}\frac{\Delta\tau}{1+i\alpha\mu(\tau)}\right)\\
&\quad=\frac{1}{2\alpha^3}\left(\frac{e_{i\alpha}(t,t_0)-e_{-i\alpha}(t,t_0)}{2i}\right)-\frac{1}{4\alpha^2}\left(e_{-i\alpha}(t,t_0)\int_{t_0}^{t}\frac{1+i\alpha\mu(\tau)}{1+\alpha^2\mu^2(\tau)}\Delta\tau\right.\\
&\qquad\qquad\qquad\qquad\left.+e_{i\alpha}(t,t_0)\int_{t_0}^{t}\frac{1-i\alpha\mu(\tau)}{1+\alpha^2\mu^2(\tau)}\Delta\tau\right)\\
&\quad=\frac{\sin_\alpha(t,t_0)}{2\alpha^3}-\frac{1}{2\alpha^2}\left(\frac{e_{-i\alpha}(t,t_0)+e_{i\alpha}(t,t_0)}{2}\right)\int_{t_0}^{t}\frac{\Delta\tau}{1+\alpha^2\mu^2(\tau)}\\
&\qquad\quad-\frac{1}{2\alpha}\left(\frac{e_{i\alpha}(t,t_0)-e_{-i\alpha}(t,t_0)}{2i}\right)\int_{t_0}^{t}\frac{\mu(\tau)\Delta\tau}{1+\alpha^2\mu^2(\tau)}\\
&\quad=\frac{\sin_\alpha(t,t_0)}{2\alpha^3}-\frac{\cos_\alpha(t,t_0)}{2\alpha^2}\int_{t_0}^{t}\frac{\Delta\tau}{1+\alpha^2\mu^2(\tau)}-\frac{\sin_\alpha(t,t_0)}{2\alpha}\int_{t_0}^{t}\frac{\mu(\tau)\Delta\tau}{1+\alpha^2\mu^2(\tau)}.
\end{aligned}$$

□

Similarly we can show for $\alpha \neq 0$,

$$\mathcal{L}^{-1}\left\{\frac{z}{(z^2+\alpha^2)^2}\right\}(t)=\frac{\sin_\alpha(t,t_0)}{2\alpha}\int_{t_0}^{t}\frac{\Delta\tau}{1+\alpha^2\mu^2(\tau)}-\frac{\cos_\alpha(t,t_0)}{2}\int_{t_0}^{t}\frac{\mu(\tau)\Delta\tau}{1+\alpha^2\mu^2(\tau)}.$$

Now, since

$$\frac{z^2}{(z^2+\alpha^2)^2}=\frac{1}{z^2+\alpha^2}-\frac{\alpha^2}{(z^2+\alpha^2)^2}\text{ and }\frac{z^3}{(z^2+\alpha^2)^2}=\frac{z}{z^2+\alpha^2}-\frac{\alpha^2 z}{(z^2+\alpha^2)^2},$$

we can apply the above to obtain:

$$\mathcal{L}^{-1}\left\{\frac{z^2}{(z^2+\alpha^2)^2}\right\}(t) = \frac{\sin_\alpha(t,t_0)}{2\alpha} + \frac{\cos_\alpha(t,t_0)}{2}\int_{t_0}^{t}\frac{\Delta\tau}{1+\alpha^2\mu^2(\tau)} - \frac{\alpha\sin_\alpha(t,t_0)}{2}\int_{t_0}^{t}\frac{\mu(\tau)\Delta\tau}{1+\alpha^2\mu^2(\tau)},$$

and

$$\mathcal{L}^{-1}\left\{\frac{z^3}{(z^2+\alpha^2)^2}\right\}(t) = \cos_\alpha(t,t_0) - \frac{\alpha\sin_\alpha(t,t_0)}{2}\int_{t_0}^{t}\frac{\Delta\tau}{1+\alpha^2\mu^2(\tau)} - \frac{\alpha^2\cos_\alpha(t,t_0)}{2}\int_{t_0}^{t}\frac{\mu(\tau)\Delta\tau}{1+\alpha^2\mu^2(\tau)}.$$

These, and a few other similar calculations are collected in Tables 3.1 and 3.2. Note that in these tables we have that $\mathcal{L}\{f\}(z) = F(z)$.

$F(z)$	$f(t)$
$\frac{1}{(z-\alpha)^{n+1}}$, $n \in \mathbb{N}_0$	$j_n(t,t_0)e_\alpha(t,t_0)$
$\frac{\alpha\beta}{(z^2+\alpha^2)(z^2+\beta^2)}$	$\frac{\alpha \sin_\beta(t,t_0) - \beta \sin_\alpha(t,t_0)}{\alpha^2-\beta^2}$
$\frac{z}{(z^2+\alpha^2)(z^2+\beta^2)}$	$\frac{\cos_\beta(t,t_0) - \cos_\alpha(t,t_0)}{\alpha^2-\beta^2}$
$\frac{z^2}{(z^2+\alpha^2)(z^2+\beta^2)}$	$\frac{\alpha \sin_\alpha(t,t_0) - \beta \sin_\beta(t,t_0)}{\alpha^2-\beta^2}$
$\frac{z^3}{(z^2+\alpha^2)(z^2+\beta^2)}$	$\frac{\alpha^2 \cos_\alpha(t,t_0) - \beta^2 \cos_\beta(t,t_0)}{\alpha^2-\beta^2}$

Table 3.1: Laplace Transform formulas for $\alpha \neq \beta$.

$F(z)$	$f(t)$
$\frac{1}{(z^2+\alpha^2)^2}$	$\frac{\sin_\alpha(t,t_0)}{2\alpha^3} - \frac{\cos_\alpha(t,t_0)}{2\alpha^2}\int_{t_0}^t \frac{\Delta\tau}{1+\alpha^2\mu^2(\tau)} - \frac{\sin_\alpha(t,t_0)}{2\alpha}\int_{t_0}^t \frac{\mu(\tau)\Delta\tau}{1+\alpha^2\mu^2(\tau)}$
$\frac{z}{(z^2+\alpha^2)^2}$	$\frac{\sin_\alpha(t,t_0)}{2\alpha}\int_{t_0}^t \frac{\Delta\tau}{1+\alpha^2\mu^2(\tau)} - \frac{\cos_\alpha(t,t_0)}{2}\int_{t_0}^t \frac{\mu(\tau)\Delta\tau}{1+\alpha^2\mu^2(\tau)}$
$\frac{z^2}{(z^2+\alpha^2)^2}$	$\frac{\sin_\alpha(t,t_0)}{2\alpha} + \frac{\cos_\alpha(t,t_0)}{2}\int_{t_0}^t \frac{\Delta\tau}{1+\alpha^2\mu^2(\tau)} - \frac{\alpha\sin_\alpha(t,t_0)}{2}\int_{t_0}^t \frac{\mu(\tau)\Delta\tau}{1+\alpha^2\mu^2(\tau)}$
$\frac{z^3}{(z^2+\alpha^2)^2}$	$\cos_\alpha(t,t_0) - \frac{\alpha\sin_\alpha(t,t_0)}{2}\int_{t_0}^t \frac{\Delta\tau}{1+\alpha^2\mu^2(\tau)} - \frac{\alpha^2\cos_\alpha(t,t_0)}{2}\int_{t_0}^t \frac{\mu(\tau)\Delta\tau}{1+\alpha^2\mu^2(\tau)}$

Table 3.2: Laplace Transform formulas for $\alpha \neq 0$.

Chapter 4

The Transport Equation

4.1 Introduction

In partial differential equations, the transport initial value problem

$$u_t(t,s) + u_s(t,s) = 0, \quad t \geq s \geq 0,$$
$$u(t,0) = f(t), \quad t \in [0,\infty),$$

has a unique solution given by $u(t,s) = f(t-s)$. If we would like to consider the analog of the transport equation on time scales, we run into an immediate problem: for an arbitrary time scale, we are not guaranteed that $t - s \in \mathbb{T}$ and hence $f(t-s)$ could be a nonsense statement on that particular time scale, and so $f(t-s)$ has no chance of being the solution of the transport equation in this case. In this chapter we will investigate this problem in the special case that $\mathbb{T}$ is an isolated time scale. In particular, we will develop two recursive representations for the unique solution of the transport dynamic initial value problem, as well as give several examples.

In the examples considered here we will find closed-form solutions to the transport partial dynamic initial value problem on particular time scales; however, it should be noted that the recursive representations developed in Section 4.2 are sufficient for numerically finding the solution of the transport dynamic equation on *any* isolated time scale.

Note that in the case of the transport equation on $\mathbb{R}$ or $\mathbb{Z}$, we expect the initial function to simply "shift" in time. In other words, we have a right traveling wave solution and the wave is not distorted in any fashion, just shifted. However, as we will see in Examples 4.3.2, 4.3.4, and 4.3.6, once we find a closed-form solution to the transport dynamic equation and graph the solution, we will find that some distortion occurs. Why? It turns out that when the graininess of the time scale is not constant, as the solution progresses in time, it must compensate for this nonconstant graininess. It does so by distorting the initial function in some fashion.

One final note in way of introduction: while studying this particular partial dynamic equation is interesting in its own right, it should be pointed out that it is crucial in the development of the convolution theorem on time scales (see [3]).

Throughout, we will let $\mathbb{T}$ be an isolated time scale that is unbounded above with $t_0 \in \mathbb{T}$ fixed. Since $\mathbb{T}$ is isolated, we can take $\mathbb{T} = \{t_0, t_1, \cdots\}$ where $t_0 < t_1 < t_2 < \cdots$. The transport partial dynamic initial value problem that we will be considering is given by

$$u^{\Delta_t}(t, \sigma(s)) + u^{\Delta_s}(t, s) = 0, \quad t, s \in \mathbb{T}, \quad t \geq s \geq t_0, \tag{4.1}$$

$$u(t, t_0) = f(t), \quad t \in [t_0, \infty)_{\mathbb{T}}, \tag{4.2}$$

where $f : [t_0, \infty)_{\mathbb{T}} \to \mathbb{C}$.

There is a useful result from Bohner and Guseinov's paper on the convolution on time scales (see [3, Lemma 2.4]) that we will make use of in this chapter:

Lemma 4.1.1. Let $\mathbb{T}$ be any time scale. If u is a solution of the partial dynamic initial value problem (4.1)–(4.2), then $u(t,t) = f(t_0)$ for any $t \in [t_0, \infty)_{\mathbb{T}}$.

Throughout, we will use some notation that is in the spirit of difference equations. For $k \in \mathbb{N}$, $n \in \mathbb{N}_0$ such that $1 \leq k \leq n+1$, define

$$[\mu(t_n)]^{\underline{k}} := \prod_{i=n-k+1}^{n} \mu(t_i) = \mu(t_n)\mu(t_{n-1}) \cdots \mu(t_{n-k+2})\mu(t_{n-k+1}).$$

4.2 The Transport Equation on an Isolated Time Scale

We will now examine the solution of the transport equation (4.1)–(4.2) in the case that the time scale in question is isolated. First, in Theorem 4.2.1 we show that on an isolated time scale, the solution of (4.1)–(4.2) is unique. In Theorems 4.2.2 and 4.2.3 we develop two representations of this unique solution, both of which use recursion. This section is concluded by proving a quick result that has to do with the form of the solution of the transport equation on an isolated time scale (see Theorem 4.2.6).

Theorem 4.2.1. Let $\mathbb{T}$ be an isolated time scale. Then the partial dynamic initial value problem (4.1)–(4.2) has a unique solution.

Proof. Assume u is a solution to (4.1)–(4.2). First note that since u solves the partial dynamic equation (4.1) and $\mathbb{T}$ is an isolated time scale, for $t \geq s \geq t_0$ we have

$$0 = u^{\Delta_t}(t, \sigma(s)) + u^{\Delta_s}(t,s) = \frac{u(\sigma(t), \sigma(s)) - u(t, \sigma(s))}{\mu(t)} + \frac{u(t, \sigma(s)) - u(t,s)}{\mu(s)}.$$

Solving for $u(\sigma(t), \sigma(s))$, it follows that

$$u(\sigma(t), \sigma(s)) = u(t, \sigma(s)) \left(1 - \frac{\mu(t)}{\mu(s)}\right) + \frac{\mu(t)}{\mu(s)} u(t, s). \tag{4.3}$$

We need to show that $u(t, s)$ is uniquely determined for any $t \geq s \geq t_0$. Let $t, s \in \mathbb{T}$ such that $t \geq s \geq t_0$ be arbitrary but fixed. Since $\mathbb{T}$ is isolated, there exist $n, m \in \mathbb{N}_0$ such that $t = t_n$ and $s = t_m$. So, for any $n, m \in \mathbb{N}_0$ with $n \geq m$, we must show that $u(t_n, t_m)$ is uniquely determined. We will do this in several cases. First, consider the case when $m = 0$. Note that $u(t_n, t_0) = f(t_n)$ for any $n \geq 0$ by the initial condition (4.2). i.e., $u(t_n, t_0)$ is uniquely determined from the initial data.

Case: Assume $m = 1$.

We claim that $u(t_n, t_1)$ is uniquely determined for $n \geq 1$. We will prove this using induction on n. For the base case note that by Lemma 4.1.1, $u(t_1, t_1) = f(t_0)$ (and hence is uniquely determined), and from equation (4.3), it follows that

$$u(t_2, t_1) = u(\sigma(t_1), \sigma(t_0)) = u(t_1, t_1) \left(1 - \frac{\mu(t_1)}{\mu(t_0)}\right) + \frac{\mu(t_1)}{\mu(t_0)} u(t_1, t_0).$$

Since $u(t_1, t_1) = f(t_0)$ and $u(t_1, t_0) = f(t_1)$, it follows that $u(t_2, t_1)$ is uniquely determined by the initial data.

Proceeding by induction, assume $u(t_n, t_1)$ for some $n \geq m = 1$ is uniquely determined from the initial data. Then, by equation (4.3),

$$u(t_{n+1}, t_1) = u(\sigma(t_n), \sigma(t_0)) = u(t_n, t_1) \left(1 - \frac{\mu(t_n)}{\mu(t_0)}\right) + \frac{\mu(t_n)}{\mu(t_0)} u(t_n, t_0).$$

Here $u(t_n, t_1)$ is uniquely determined by the induction hypothesis, and $u(t_n, t_0) = f(t_n)$ is given by the initial condition. Therefore, $u(t_{n+1}, t_1)$ is uniquely determined from the initial data. Hence, it follows by induction that $u(t_n, t_1)$ is uniquely determined for any $n \geq m = 1$.

Case: Assume $m = 2$.

We now claim that $u(t_n, t_2)$ is uniquely determined for $n \geq m = 2$. As above, we will prove this using induction on n. For the base case note that by Lemma 4.1.1, $u(t_2, t_2) = f(t_0)$ (and hence is uniquely determined).

Proceeding by induction, assume that for some $n \geq m = 2$, $u(t_n, t_2)$ is uniquely determined from the initial data. Then, by equation (4.3),

$$u(t_{n+1}, t_2) = u(\sigma(t_n), \sigma(t_1)) = u(t_n, t_2) \left(1 - \frac{\mu(t_n)}{\mu(t_1)}\right) + \frac{\mu(t_n)}{\mu(t_1)} u(t_n, t_1).$$

Here $u(t_n, t_2)$ is uniquely determined by the induction hypothesis, and $u(t_n, t_1)$ is uniquely determined by the previous case. Therefore, $u(t_{n+1}, t_2)$ is uniquely determined. Hence, it follows by induction that $u(t_n, t_2)$ is uniquely determined for any $n \geq m = 2$.

Continuing this process inductively, we find that for any $n, m \in \mathbb{N}_0$ such that $n \geq m$, $u(t_n, t_m)$ is uniquely determined which implies that for any $t \geq s \geq t_0$, $u(t, s)$ is uniquely determined from the initial data. Therefore, u is the unique solution to (4.1). $\square$

Theorem 4.2.2. For an isolated time scale $\mathbb{T} = \{t_0, t_1, \cdots\}$ where $t_0 < t_1 < \cdots$, the unique solution of (4.1)–(4.2) is given by

$$u(t_{n+j}, t_n) = \sum_{i=0}^{j} A_{j,i}(n) f(t_i), \quad n, j \in \mathbb{N}_0,$$

where

$$A_{j,i}(n) := \begin{cases} \displaystyle\sum_{k=0}^{n-1} \frac{[\mu(t_{n+j-1})]^{\underline{j}}}{[\mu(t_{n+j-1-k})]^{\underline{j}}} \left(1 - \frac{\mu(t_{n+j-1-k})}{\mu(t_{n-1-k})}\right) A_{j-1,i}(n-k), & \text{for } i < j, \\ \dfrac{[\mu(t_{n+j-1})]^{\underline{j}}}{[\mu(t_{j-1})]^{\underline{j}}}, & \text{for } i = j. \end{cases}$$

Before we prove Theorem 4.2.2, it should be noted that the solution given here is a linear combination of the initial data and that each $A_{j,i}(n)$ is simply the coefficient of the $f(t_i)$ term in this linear combination. So, when presented with a specific isolated time scale, in order to find the solution to (4.1)–(4.2), we simply need to resolve the recursion in the definition of the $A_{j,i}(n)$ coefficients for each $0 \leq i \leq j$.

Proof. Let $t \geq s \geq t_0$ be given. Then there exist $n \in \mathbb{N}$ and $j \in \mathbb{N}_0$ such that $t = t_{n+j}$ and $s = t_n$. Define $u(t_{n+j}, t_n)$ as in the statement of the theorem.

Note that

$$\begin{aligned} u^{\Delta_t}(t, \sigma(s)) &= u^{\Delta_t}(t_{n+j}, t_{n+1}) \\ &= \frac{1}{\mu(t_{n+j})} \left[u(t_{n+j+1}, t_{n+1}) - u(t_{n+j}, t_{n+1})\right] \\ &= \frac{1}{\mu(t_{n+j})} \left[\sum_{i=0}^{j} A_{j,i}(n+1) f(t_i) - \sum_{i=0}^{j-1} A_{j-1,i}(n+1) f(t_i)\right] \\ &= \frac{1}{\mu(t_{n+j})} \left[A_{j,j}(n+1) f(t_j) + \sum_{i=0}^{j-1} \Big(A_{j,i}(n+1) - A_{j-1,i}(n+1)\Big) f(t_i)\right]. \end{aligned} \tag{4.4}$$

Similarly,

$$\begin{aligned}
-u^{\Delta_s}(t,s) &= -u^{\Delta_s}(t_{n+j},t_n) \\
&= \frac{1}{\mu(t_n)}\left[u(t_{n+j},t_n) - u(t_{n+j},t_{n+1})\right] \\
&= \frac{1}{\mu(t_n)}\left[\sum_{i=0}^{j} A_{j,i}(n)f(t_i) - \sum_{i=0}^{j-1} A_{j-1,i}(n+1)f(t_i)\right] \\
&= \frac{1}{\mu(t_n)}\left[A_{j,j}(n)f(t_j) + \sum_{i=0}^{j-1}\Big(A_{j,i}(n) - A_{j-1,i}(n+1)\Big)f(t_i)\right]. \qquad (4.5)
\end{aligned}$$

Hence, $u^{\Delta_t}(t,\sigma(s)) = -u^{\Delta_s}(t,s)$ (and so (4.1) is satisfied) if and only if (4.4) equals (4.5). To see that this is in fact the case, we will show that for each $0 \le i \le j$, the coefficients of $f(t_i)$ in (4.4) and (4.5) are equal.

First consider the coefficient of $f(t_j)$ in (4.4). Note that

$$\begin{aligned}
\frac{1}{\mu(t_{n+j})}A_{j,j}(n+1) &= \frac{1}{\mu(t_{n+j})}\frac{[\mu(t_{n+j})]^{\underline{j}}}{[\mu(t_{j-1})]^{\underline{j}}} \\
&= \frac{\mu(t_{n+j-1})\cdots\mu(t_{n+1})}{[\mu(t_{j-1})]^{\underline{j}}} \\
&= \frac{1}{\mu(t_n)}\frac{\mu(t_{n+j-1})\cdots\mu(t_{n+1})\mu(t_n)}{[\mu(t_{j-1})]^{\underline{j}}} \\
&= \frac{1}{\mu(t_n)}\frac{[\mu(t_{n+j-1})]^{\underline{j}}}{[\mu(t_{j-1})]^{\underline{j}}} \\
&= \frac{1}{\mu(t_n)}A_{j,j}(n).
\end{aligned}$$

Now fix i such that $0 \le i \le j-1$. We aim to show that the coefficient of $f(t_i)$ in

(4.4) matches the coefficient of $f(t_i)$ in (4.5). To see this, first consider

$$\begin{aligned}
\frac{1}{\mu(t_{n+j})}A_{j,i}(n+1) &= \frac{1}{\mu(t_{n+j})}\sum_{k=0}^{n}\frac{[\mu(t_{n+j})]^{\underline{j}}}{[\mu(t_{n+j-k})]^{\underline{j}}}\left(1-\frac{\mu(t_{n+j-k})}{\mu(t_{n-k})}\right)A_{j-1,i}(n+1-k) \\
&= \frac{1}{\mu(t_{n+j})}\sum_{k=0}^{n}\frac{\mu(t_{n+j})\mu(t_{n+j-1})\cdots\mu(t_{n+1})}{\mu(t_{n+j-k})\mu(t_{n+j-1-k})\cdots\mu(t_{n+1-k})} \\
&\qquad\qquad \cdot\left(1-\frac{\mu(t_{n+j-k})}{\mu(t_{n-k})}\right)A_{j-1,i}(n+1-k) \\
&= \frac{1}{\mu(t_{n})}\sum_{k=0}^{n}\frac{\mu(t_{n+j-1})\cdots\mu(t_{n+1})\mu(t_n)}{\mu(t_{n+j-k})\mu(t_{n+j-1-k})\cdots\mu(t_{n+1-k})} \\
&\qquad\qquad \cdot\left(1-\frac{\mu(t_{n+j-k})}{\mu(t_{n-k})}\right)A_{j-1,i}(n+1-k) \\
&= \frac{1}{\mu(t_{n})}\sum_{k=0}^{n}\frac{[\mu(t_{n+j-1})]^{\underline{j}}}{[\mu(t_{n+j-k})]^{\underline{j}}}\left(1-\frac{\mu(t_{n+j-k})}{\mu(t_{n-k})}\right)A_{j-1,i}(n+1-k) \\
\text{(reindexing)} &= \frac{1}{\mu(t_{n})}\sum_{k=-1}^{n-1}\frac{[\mu(t_{n+j-1})]^{\underline{j}}}{[\mu(t_{n+j-1-k})]^{\underline{j}}}\left(1-\frac{\mu(t_{n+j-1-k})}{\mu(t_{n-k-1})}\right)A_{j-1,i}(n-k) \\
&= \frac{1}{\mu(t_{n})}\left[\frac{[\mu(t_{n+j-1})]^{\underline{j}}}{[\mu(t_{n+j})]^{\underline{j}}}\left(1-\frac{\mu(t_{n+j})}{\mu(t_{n})}\right)A_{j-1,i}(n+1)\right] \\
&\qquad + \frac{1}{\mu(t_{n})}\sum_{k=0}^{n-1}\frac{[\mu(t_{n+j-1})]^{\underline{j}}}{[\mu(t_{n+j-1-k})]^{\underline{j}}}\left(1-\frac{\mu(t_{n+j-1-k})}{\mu(t_{n-k-1})}\right)A_{j-1,i}(n-k) \\
&= \left[\left(\frac{1}{\mu(t_{n+j})}-\frac{1}{\mu(t_{n})}\right)A_{j-1,i}(n+1)\right]+\frac{1}{\mu(t_{n})}A_{j,i}(n).
\end{aligned}$$

Since i was arbitrary, for any $0 \le i \le j-1$, it follows that

$$\frac{1}{\mu(t_{n+j})}\Big(A_{j,i}(n+1)-A_{j-1,i}(n+1)\Big) = \frac{1}{\mu(t_n)}\Big(A_{j,i}(n)-A_{j-1,i}(n+1)\Big).$$

Therefore, for each $i = 0, \cdots, j$, the coefficients of $f(t_i)$ are equal in (4.4) and (4.5), and so we can conclude that $u^{\Delta_t}(t,\sigma(s)) = -u^{\Delta_s}(t,s)$.

To see that the initial condition is satisfied, note that $A_{j,j}(0) = 1$ and for $i \neq j$, $A_{j,i}(0) = 0$ by the convention that a sum from 0 to -1 is taken to be 0. Therefore,

$u(t_j,t_0)=\sum_{i=0}^{j}A_{j,i}(0)f(t_i)=f(t_j)$. So, we have shown that $u(t,s)$ solves the partial dynamic equation as well as satisfies the initial condition. Finally, by Theorem 4.2.1, u is the unique solution of (4.1)–(4.2). □

In practice, when explicitly calculating the $A_{j,i}(n)$ terms for a particular time scale, the recursive definition given in Theorem 4.2.2 lends to some cancellation. For example, noting that

$$A_{j,j}(n)=\frac{[\mu(t_{n+j-1})]^{\underline{j}}}{[\mu(t_{j-1})]^{\underline{j}}},$$

it follows that

$$\begin{aligned}
A_{j,j-1}(n) &= \sum_{k=0}^{n-1}\frac{[\mu(t_{n+j-1})]^{\underline{j}}}{[\mu(t_{n+j-1-k})]^{\underline{j}}}\left(1-\frac{\mu(t_{n+j-1-k})}{\mu(t_{n-1-k})}\right)A_{j-1,j-1}(n-k)\\
&= \sum_{k=0}^{n-1}\frac{[\mu(t_{n+j-1})]^{\underline{j}}}{[\mu(t_{n+j-1-k})]^{\underline{j}}}\left(1-\frac{\mu(t_{n+j-1-k})}{\mu(t_{n-1-k})}\right)\left(\frac{[\mu(t_{n+j-2-k})]^{\underline{j-1}}}{[\mu(t_{j-2})]^{\underline{j-1}}}\right)\\
&= \frac{[\mu(t_{n+j-1})]^{\underline{j}}}{[\mu(t_{j-2})]^{\underline{j-1}}}\sum_{k=0}^{n-1}\frac{[\mu(t_{n+j-2-k})]^{\underline{j-1}}}{[\mu(t_{n+j-1-k})]^{\underline{j}}}\left(1-\frac{\mu(t_{n+j-1-k})}{\mu(t_{n-1-k})}\right)\\
&= \frac{[\mu(t_{n+j-1})]^{\underline{j}}}{[\mu(t_{j-2})]^{\underline{j-1}}}\sum_{k=0}^{n-1}\frac{1}{\mu(t_{n+j-1-k})}\left(1-\frac{\mu(t_{n+j-1-k})}{\mu(t_{n-1-k})}\right)\\
&= \frac{[\mu(t_{n+j-1})]^{\underline{j}}}{[\mu(t_{j-2})]^{\underline{j-1}}}\sum_{k=0}^{n-1}\left(\frac{1}{\mu(t_{n+j-1-k})}-\frac{1}{\mu(t_{n-1-k})}\right).
\end{aligned}\tag{4.6}$$

Similarly, it can be shown that

$$\begin{aligned}
A_{j,j-2}(n) = \frac{[\mu(t_{n+j-1})]^{\underline{j}}}{[\mu(t_{j-3})]^{\underline{j-2}}}&\sum_{k_1=0}^{n-1}\left(\frac{1}{\mu(t_{n+j-1-k_1})}-\frac{1}{\mu(t_{n-1-k_1})}\right)\\
&\cdot\sum_{k_2=0}^{n-1-k_1}\left(\frac{1}{\mu(t_{n+j-2-k_1-k_2})}-\frac{1}{\mu(t_{n-1-k_1-k_2})}\right).
\end{aligned}\tag{4.7}$$

The calculations (4.6) and (4.7) motivate the following representation for the solution of (4.1)–(4.2).

Theorem 4.2.3. The coefficients $A_{j,i}(n)$ in Theorem 4.2.2 can be written as follows:

$$A_{j,i}(n) = \frac{[\mu(t_{n+j-1})]^{\underline{j}}}{[\mu(t_{i-1})]^{\underline{i}}} B_{j,i,0}(n),$$

where

$$B_{j,i,\Delta}(n) = \begin{cases} \displaystyle\sum_{k=0}^{n-1} \left(\frac{1}{\mu(t_{n+j-\Delta-1-k})} - \frac{1}{\mu(t_{n-1-k})} \right) B_{j,i,\Delta+1}(n-k), & \text{for } j-i < \Delta, \\ 1, & \text{for } j-i = \Delta, \end{cases}$$

and $0 \le \Delta \le j-i$.

Of course, this representation still uses recursion for the $B_{j,i,\Delta}(n)$ terms. Note that in each step of the recursion, we add one to Δ and hence we will eventually arrive at $\Delta = j-i$ and thus the recursion will terminate after a finite number of steps. Therefore, when completely expanded out, $B_{j,i,\Delta}(n)$ will consist of $j-i$ nested summations. Despite the use of recursion again here, this representation for the solution is an improvement as the cancellation that occurs when expanding the $A_{j,i}(n)$ terms from Theorem 4.2.2 is already accounted for. Before proving Theorem 4.2.3, we need a technical lemma.

Lemma 4.2.4. For any $i, m, n \in \mathbb{N}_0$, we have

$$B_{i+m,i,0}(n) = B_{i+m+1,i,1}(n),$$

where $B_{j,i,\Delta}(n)$ is as defined in Theorem 4.2.3.

Proof. The overall idea of this proof is to expand out $B_{i+m,i,0}(n)$ using the recursive definition to obtain m nested summations. We will then rewrite the indices of the points in the time scale in an appropriate way in order to collapse the expansion back down and ultimately end up with $B_{i+m+1,i,1}(n)$. While this idea is straightforward, the details do become a little bit cumbersome.

Let $i, n \in \mathbb{N}_0$ be arbitrary but fixed. If $m = 0$, then by definition of $B_{j,i,\Delta}(n)$,

$$B_{i+m,i,0}(n) = B_{i,i,0}(n) = 1, \text{ and}$$
$$B_{i+m+1,i,1}(n) = B_{i+1,i,1}(n) = 1.$$

So, in the base case where $m = 0$, we have $B_{i+m,i,0}(n) = B_{i+m+1,i,1}(n)$ as desired. Assume $m \geq 1$. Then,

$$\begin{aligned}
B_{i+m,i,0}(n) &= \sum_{k_1=0}^{n-1} \left(\frac{1}{\mu(t_{n+(i+m)-0-1-k_1})} - \frac{1}{\mu(t_{n-1-k_1})} \right) B_{i+m,i,1}(n-k_1) \\
&= \sum_{k_1=0}^{n-1} \left(\frac{1}{\mu(t_{n+i+m-1-k_1})} - \frac{1}{\mu(t_{n-1-k_1})} \right) B_{i+m,i,1}(n-k_1) \\
&= \sum_{k_1=0}^{n-1} \left(\frac{1}{\mu(t_{n+i+m-1-k_1})} - \frac{1}{\mu(t_{n-1-k_1})} \right) \\
&\quad \cdot \sum_{k_2=0}^{n-k_1-1} \left(\frac{1}{\mu(t_{n+i+m-2-k_1-k_2})} - \frac{1}{\mu(t_{n-1-k_1-k_2})} \right) B_{i+m,i,2}(n-k_1-k_2) \\
&= \sum_{k_1=0}^{n-1} \left(\frac{1}{\mu(t_{n+i+m-1-k_1})} - \frac{1}{\mu(t_{n-1-k_1})} \right) \\
&\quad \cdot \sum_{k_2=0}^{n-k_1-1} \left(\frac{1}{\mu(t_{n+i+m-2-k_1-k_2})} - \frac{1}{\mu(t_{n-1-k_1-k_2})} \right) \\
&\quad \cdot \sum_{k_3=0}^{n-k_1-k_2-1} \left(\frac{1}{\mu(t_{n+i+m-3-k_1-k_2-k_3})} - \frac{1}{\mu(t_{n-1-k_1-k_2-k_3})} \right) \\
&\quad \cdot B_{i+m,i,3}(n-k_1-k_2-k_3).
\end{aligned}$$

Continuing this process, after $m-1$ steps we have

$$
\begin{aligned}
&= \sum_{k_1=0}^{n-1} \left(\frac{1}{\mu(t_{n+i+m-1-k_1})} - \frac{1}{\mu(t_{n-1-k_1})} \right) \\
&\quad \cdot \sum_{k_2=0}^{n-k_1-1} \left(\frac{1}{\mu(t_{n+i+m-2-k_1-k_2})} - \frac{1}{\mu(t_{n-1-k_1-k_2})} \right) \\
&\quad \cdot \ldots \cdot \\
&\quad \cdot \sum_{k_{m-1}=0}^{\substack{n-k_1-\cdots \\ -k_{m-2}-1}} \left(\frac{1}{\mu(t_{n+i+m-(m-1)-k_1-\cdots-k_{m-1}})} - \frac{1}{\mu(t_{n-1-k_1-\cdots-k_{m-1}})} \right) \\
&\quad \cdot B_{i+m,i,m-1}(n-k_1-\cdots-k_{m-1}) \\
&= \sum_{k_1=0}^{n-1} \left(\frac{1}{\mu(t_{n+i+m-1-k_1})} - \frac{1}{\mu(t_{n-1-k_1})} \right) \\
&\quad \cdot \sum_{k_2=0}^{n-k_1-1} \left(\frac{1}{\mu(t_{n+i+m-2-k_1-k_2})} - \frac{1}{\mu(t_{n-1-k_1-k_2})} \right) \\
&\quad \cdot \ldots \cdot \\
&\quad \cdot \sum_{k_{m-1}=0}^{\substack{n-k_1-\cdots \\ -k_{m-2}-1}} \left(\frac{1}{\mu(t_{n+i+m-(m-1)-k_1-\cdots-k_{m-1}})} - \frac{1}{\mu(t_{n-1-k_1-\cdots-k_{m-1}})} \right) \\
&\quad \cdot \sum_{k_m=0}^{\substack{n-k_1-\cdots \\ -k_{m-1}-1}} \left(\frac{1}{\mu(t_{n+i+m-m-k_1-\cdots-k_m})} - \frac{1}{\mu(t_{n-1-k_1-\cdots-k_m})} \right) \\
&\quad \cdot B_{i+m,i,m}(n-k_1-\cdots-k_m). \qquad (4.8)
\end{aligned}
$$

By definition of $B_{j,i,\Delta}(n)$, it follows that both $B_{i+m,i,m}(n-k_1-\cdots-k_m)=1$ and $B_{i+m+1,i,m+1}(n-k_1-\cdots-k_m)=1$. Hence, we can replace $B_{i+m,i,m}(n-k_1-\cdots-k_m)$ with $B_{i+m+1,i,m+1}(n-k_1-\cdots-k_m)$ in (4.8). Further, in the innermost summation of (4.8), we will rewrite the index of t in an appropriate way. So continuing from (4.8), we have

$$= \sum_{k_1=0}^{n-1} \left(\frac{1}{\mu(t_{n+i+m-1-k_1})} - \frac{1}{\mu(t_{n-1-k_1})} \right)$$
$$\cdot \sum_{k_2=0}^{n-k_1-1} \left(\frac{1}{\mu(t_{n+i+m-2-k_1-k_2})} - \frac{1}{\mu(t_{n-1-k_1-k_2})} \right)$$
$$\cdot \ldots \cdot$$
$$\cdot \sum_{k_{m-1}=0}^{\substack{n-k_1-\cdots \\ -k_{m-2}-1}} \left(\frac{1}{\mu(t_{n+i+m-(m-1)-k_1-\cdots-k_{m-1}})} - \frac{1}{\mu(t_{n-1-k_1-\cdots-k_{m-1}})} \right)$$
$$\cdot \sum_{k_m=0}^{\substack{n-k_1-\cdots \\ -k_{m-1}-1}} \left(\frac{1}{\mu(t_{(n-k_1-\cdots-k_{m-1})+(i+m+1)-m-1-k_m})} - \frac{1}{\mu(t_{(n-k_1-\cdots-k_{m-1})-1-k_m})} \right)$$
$$\cdot B_{i+m+1,i,m+1}(n-k_1-\cdots-k_m). \tag{4.9}$$

Looking at the innermost summation, we see that when compared to the definition of $B_{j,i,\Delta}(n)$ given in the statement of Theorem 4.2.3, n in the definition is $n-k_1-\cdots-k_{m-1}$ here, j in the definition is $i+m+1$ here, and Δ in the definition is m here. Therefore, the innermost sum in (4.9) reduces to

$$\sum_{k_m=0}^{\substack{n-k_1-\cdots \\ -k_{m-1}-1}} \left(\frac{1}{\mu(t_{(n-k_1-\cdots-k_{m-1})+(i+m+1)-m-1-k_m})} - \frac{1}{\mu(t_{(n-k_1-\cdots-k_{m-1})-1-k_m})} \right)$$
$$\cdot B_{i+m+1,i,m+1}(n-k_1-\cdots-k_m)$$
$$= B_{i+m+1,i,m}(n-k_1-\cdots-k_{m-1}).$$

Therefore, substituting this in and continuing from (4.9), we obtain:

$$
\begin{aligned}
&= \sum_{k_1=0}^{n-1}\left(\frac{1}{\mu(t_{n+i+m-1-k_1})} - \frac{1}{\mu(t_{n-1-k_1})}\right) \\
&\quad \cdot \sum_{k_2=0}^{n-k_1-1}\left(\frac{1}{\mu(t_{n+i+m-2-k_1-k_2})} - \frac{1}{\mu(t_{n-1-k_1-k_2})}\right) \\
&\quad \cdot \ldots \cdot \\
&\quad \cdot \sum_{k_{m-1}=0}^{\substack{n-k_1-\cdots \\ -k_{m-2}-1}}\left(\frac{1}{\mu(t_{n+i+m-(m-1)-k_1-\cdots-k_{m-1}})} - \frac{1}{\mu(t_{n-1-k_1-\cdots-k_{m-1}})}\right) \\
&\quad \cdot B_{i+m+1,i,m}(n-k_1-\cdots-k_{m-1}) \\
&= \sum_{k_1=0}^{n-1}\left(\frac{1}{\mu(t_{n+i+m-1-k_1})} - \frac{1}{\mu(t_{n-1-k_1})}\right) \\
&\quad \cdot \sum_{k_2=0}^{n-k_1-1}\left(\frac{1}{\mu(t_{n+i+m-2-k_1-k_2})} - \frac{1}{\mu(t_{n-1-k_1-k_2})}\right) \\
&\quad \cdot \ldots \cdot \\
&\quad \cdot \sum_{k_{m-1}=0}^{\substack{n-k_1-\cdots \\ -k_{m-2}-1}}\left(\frac{1}{\mu(t_{(n-k_1-\cdots-k_{m-2})+(i+m+1)-(m-1)-1-k_{m-1}})} - \frac{1}{\mu(t_{(n-k_1-\cdots-k_{m-2})-1-k_{m-1}})}\right) \\
&\quad \cdot B_{i+m+1,i,m}(n-k_1-\cdots-k_{m-1}). \qquad (4.10)
\end{aligned}
$$

Again looking at the innermost summation, we see that when compared to the definition of $B_{j,i,\Delta}(n)$ given in the statement of Theorem 4.2.3, n in the definition is $n-k_1-\cdots-k_{m-2}$ here, j in the definition is $i+m+1$ here, and Δ in the definition is $m-1$ here. Therefore, the innermost sum in (4.10) reduces to

$$\sum_{\substack{k_{m-1}=0}}^{\substack{n-k_1-\cdots\\-k_{m-2}-1}} \left(\frac{1}{\mu(t_{(n-k_1-\cdots-k_{m-2})+(i+m+1)-(m-1)-1-k_{m-1}})} - \frac{1}{\mu(t_{(n-k_1-\cdots-k_{m-2})-1-k_{m-1}})} \right)$$

$$\cdot B_{i+m+1,i,m}(n-k_1-\cdots-k_{m-1})$$

$$= B_{i+m+1,i,m-1}(n-k_1-\cdots-k_{m-2}).$$

Substituting and continuing this iterative process from (4.10), it follows that

$$= \sum_{k_1=0}^{n-1} \left(\frac{1}{\mu(t_{n+i+m-1-k_1})} - \frac{1}{\mu(t_{n-1-k_1})} \right)$$

$$\cdot \sum_{k_2=0}^{n-k_1-1} \left(\frac{1}{\mu(t_{n+i+m-2-k_1-k_2})} - \frac{1}{\mu(t_{n-1-k_1-k_2})} \right)$$

$$\cdot \ldots \cdot$$

$$\cdot B_{i+m+1,i,m-1}(n-k_1-\cdots-k_{m-2})$$

$$\vdots$$

$$= \sum_{k_1=0}^{n-1} \left(\frac{1}{\mu(t_{n+i+m-1-k_1})} - \frac{1}{\mu(t_{n-1-k_1})} \right)$$

$$\cdot \sum_{k_2=0}^{n-k_1-1} \left(\frac{1}{\mu(t_{n+i+m-2-k_1-k_2})} - \frac{1}{\mu(t_{n-1-k_1-k_2})} \right)$$

$$\cdot B_{i+m+1,i,3}(n-k_1-k_2)$$

$$
\begin{aligned}
&= \sum_{k_1=0}^{n-1} \left(\frac{1}{\mu(t_{n+i+m-1-k_1})} - \frac{1}{\mu(t_{n-1-k_1})} \right) \\
&\qquad \cdot \sum_{k_2=0}^{n-k_1-1} \left(\frac{1}{\mu(t_{(n-k_1)+(i+m+1)-2-1-k_2})} - \frac{1}{\mu(t_{(n-k_1)-1-k_2})} \right) \\
&\qquad \cdot B_{i+m+1,i,3}(n-k_1-k_2) \\
&= \sum_{k_1=0}^{n-1} \left(\frac{1}{\mu(t_{n+i+m-1-k_1})} - \frac{1}{\mu(t_{n-1-k_1})} \right) B_{i+m+1,i,2}(n-k_1) \\
&= \sum_{k_1=0}^{n-1} \left(\frac{1}{\mu(t_{n+(i+m+1)-1-1-k_1})} - \frac{1}{\mu(t_{n-1-k_1})} \right) B_{i+m+1,i,2}(n-k_1) \\
&= B_{i+m+1,i,1}(n).
\end{aligned}
$$

Recall that this lengthy calculation began with $B_{i+m,i,0}(n)$, and hence we have shown for any $i, m, n \in \mathbb{N}_0$,

$$
B_{i+m,i,0}(n) = B_{i+m+1,i,1}(n).
$$

□

With this lemma in hand, we are now in a position to prove Theorem 4.2.3.

Proof of Theorem 4.2.3. Let $i, j \in \mathbb{N}_0$ such that $j \geq i$ be arbitrary but fixed. We will use induction on $m := j - i$. Hence, $j = i + m$, and we aim to show that for any $i, n, m \in \mathbb{N}_0$, we have

$$
A_{j,i}(n) = A_{i+m,i}(n) = \frac{[\mu(t_{n+i+m-1})]^{\underline{i+m}}}{[\mu(t_{i-1})]^{\underline{i}}} B_{i+m,i,0}(n) = \frac{[\mu(t_{n+j-1})]^{\underline{j}}}{[\mu(t_{i-1})]^{\underline{i}}} B_{j,i,0}(n). \tag{4.11}
$$

The base case is $m = 0$ which implies $i = j$. In this case, starting with the right-hand side of (4.11), we have

$$\frac{[\mu(t_{n+j-1})]^{\underline{j}}}{[\mu(t_{i-1})]^{\underline{i}}} B_{j,i,0}(n) = \frac{[\mu(t_{n+j-1})]^{\underline{j}}}{[\mu(t_{j-1})]^{\underline{j}}} B_{j,j,0}(n) = \frac{[\mu(t_{n+j-1})]^{\underline{j}}}{[\mu(t_{j-1})]^{\underline{j}}} = A_{j,j}(n),$$

where we have used the definition of $A_{j,j}(n)$ given in the statement of Theorem 4.2.2.

Proceeding by induction, assume that for some $m \geq 0$, and any $i, n \in \mathbb{N}_0$, that statement (4.11) holds. Now, from the definition of $A_{j,i}(n)$, it follows that

$$\begin{aligned} A_{i+m+1,i}(n) &= \sum_{k=0}^{n-1} \frac{[\mu(t_{n+(i+m+1)-1})]^{\underline{i+m+1}}}{[\mu(t_{n+(i+m+1)-1-k})]^{\underline{i+m+1}}} \\ &\qquad \cdot \left(1 - \frac{\mu(t_{n+(i+m+1)-1-k})}{\mu(t_{n-1-k})}\right) A_{i+m,i}(n-k) \\ \text{(by induction hypothesis)} &= \sum_{k=0}^{n-1} \frac{[\mu(t_{n+i+m})]^{\underline{i+m+1}}}{[\mu(t_{n+i+m-k})]^{\underline{i+m+1}}} \left(1 - \frac{\mu(t_{n+i+m-k})}{\mu(t_{n-1-k})}\right) \\ &\qquad \cdot \left(\frac{[\mu(t_{(n-k)+i+m-1})]^{\underline{i+m}}}{[\mu(t_{i-1})]^{\underline{i}}} B_{i+m,i,0}(n-k)\right) \\ &= \frac{[\mu(t_{n+i+m})]^{\underline{i+m+1}}}{[\mu(t_{i-1})]^{\underline{i}}} \sum_{k=0}^{n-1} \frac{[\mu(t_{(n-k)+i+m-1})]^{\underline{i+m}}}{[\mu(t_{n+i+m-k})]^{\underline{i+m+1}}} \\ &\qquad \cdot \left(1 - \frac{\mu(t_{n+i+m-k})}{\mu(t_{n-1-k})}\right) B_{i+m,i,0}(n-k). \end{aligned} \tag{4.12}$$

Focusing our attention on the summation in (4.12), note that

$$\begin{aligned} &\sum_{k=0}^{n-1} \frac{[\mu(t_{(n-k)+i+m-1})]^{\underline{i+m}}}{[\mu(t_{n+i+m-k})]^{\underline{i+m+1}}} \left(1 - \frac{\mu(t_{n+i+m-k})}{\mu(t_{n-1-k})}\right) B_{i+m,i,0}(n-k) \\ &\quad = \sum_{k=0}^{n-1} \frac{\mu(t_{n-k+i+m-1})\mu(t_{n-k+i+m-2}) \cdots \mu(t_{n-k+i+m-1-(i+m)+1})}{\mu(t_{n-k+i+m})\mu(t_{n-k+i+m-1}) \cdots \mu(t_{n-k+i+m-(i+m+1)+1})} \\ &\qquad \cdot \left(1 - \frac{\mu(t_{n+i+m-k})}{\mu(t_{n-1-k})}\right) B_{i+m,i,0}(n-k) \end{aligned}$$

$$
\begin{aligned}
&= \sum_{k=0}^{n-1} \frac{1}{\mu(t_{n-k+i+m})} \left(1 - \frac{\mu(t_{n+i+m-k})}{\mu(t_{n-1-k})}\right) B_{i+m,i,0}(n-k) \\
&= \sum_{k=0}^{n-1} \left(\frac{1}{\mu(t_{n-k+i+m})} - \frac{1}{\mu(t_{n-1-k})}\right) B_{i+m,i,0}(n-k) \\
&= \sum_{k=0}^{n-1} \left(\frac{1}{\mu(t_{n-k+i+m})} - \frac{1}{\mu(t_{n-1-k})}\right) B_{i+m+1,i,1}(n-k) \qquad \text{(by Lemma 4.2.4)} \\
&= \sum_{k=0}^{n-1} \left(\frac{1}{\mu(t_{n+(i+m+1)-0-1-k})} - \frac{1}{\mu(t_{n-1-k})}\right) B_{i+m+1,i,1}(n-k) \\
&= B_{i+m+1,i,0}(n).
\end{aligned}
$$

For the final equality, we have used the definition of $B_{j,i,\Delta}(n)$ given in the statement of Theorem 4.2.3: j in the definition is $i+m+1$ here, and Δ in the definition is 0 here. Therefore, from (4.12) we have

$$
\begin{aligned}
A_{i+m+1,i}(n) &= \frac{[\mu(t_{n+i+m})]^{\underline{i+m+1}}}{[\mu(t_{i-1})]^{\underline{i}}} \sum_{k=0}^{n-1} \frac{[\mu(t_{(n-k)+i+m-1})]^{\underline{i+m}}}{[\mu(t_{n+i+m-k})]^{\underline{i+m+1}}} \\
&\qquad \cdot \left(1 - \frac{\mu(t_{n+i+m-k})}{\mu(t_{n-1-k})}\right) B_{i+m,i,0}(n-k) \\
&= \frac{[\mu(t_{n+i+m})]^{\underline{i+m+1}}}{[\mu(t_{i-1})]^{\underline{i}}} B_{i+m+1,i,0}(n) \\
&= \frac{[\mu(t_{n+i+(m+1)-1})]^{\underline{i+(m+1)}}}{[\mu(t_{i-1})]^{\underline{i}}} B_{i+(m+1),i,0}(n)
\end{aligned}
$$

as desired. Hence, by induction, for any $n, j, i \in \mathbb{N}_0$ such that $j \geq i$, we have

$$
A_{j,i}(n) = \frac{[\mu(t_{n+j-1})]^{\underline{j}}}{[\mu(t_{i-1})]^{\underline{i}}} B_{j,i,0}(n).
$$

□

We conclude this section with one quick result about the coefficients of the $f(t_i)$ terms, $0 \leq i \leq j$, in the solution of (4.1)–(4.2) given in Theorem 4.2.2. It turns out, regardless of the initial function $f(t)$, for any $j, n \in \mathbb{N}_0$, the coefficients $A_{j,i}(n)$ sum to one when summing with respect to i. To prove this, we first need a straightforward lemma.

Lemma 4.2.5. For any time scale $\mathbb{T}$, the solution of the transport dynamic equation (4.1) with a constant initial condition—i.e., in (4.2), $f(t) \equiv c$ for some constant c—is $u(t,s) \equiv c$.

Proof. Assume $u(t,s) = c$ for all $t, s \in \mathbb{T}$ such that $t \geq s$. Then $u^{\Delta_s}(t,s) = 0$ and since, $u(t,\sigma(s)) = c$ as well, we further have that $u^{\Delta_t}(t,\sigma(s)) = 0$. Therefore u satisfies (4.1)–(4.2) when $f(t) \equiv c$. □

Theorem 4.2.6. Let $\mathbb{T}$ be an isolated time scale such that $\mathbb{T} = \{t_0, t_1, \cdots\}$ where $t_0 < t_1 < \cdots$. Then the coefficients of the $f(t_i)$ terms of the solution given in the statement of Theorem 4.2.2 sum to one. In other words, for each $j, n \in \mathbb{N}_0$,

$$\sum_{i=0}^{j} A_{j,i}(n) = 1.$$

Proof. Let $f(t) \equiv 1$. By Lemma 4.2.5, $u(t_{n+j}, t_n) = 1$ for any $j, n \in \mathbb{N}_0$. Thus, by the uniqueness of the solution to (4.1)–(4.2) on an isolated time scale (see Theorem 4.2.1), it follows that

$$1 = u(t_{n+j}, t_n) = \sum_{i=0}^{j} A_{j,i}(n) f(t_i) = \sum_{i=0}^{j} A_{j,i}(n).$$

Since each $A_{j,i}(n)$ is independent of f, we have proven the desired result. □

4.3 Examples

In this section we collect several specific examples of solutions to the transport dynamic equation (4.1)–(4.2) that are obtained using Theorems 4.2.2 and 4.2.3.

Example 4.3.1. Let $\mathbb{T} = \mathbb{N}_0$. It is well known from the study of difference equations that if $f : \mathbb{N}_0 \to \mathbb{R}$, then the unique solution of

$$\Delta_t u(t, s+1) + \Delta_s u(t,s) = 0, \quad t, s \in \mathbb{N}_0, \quad t \geq s \geq 0,$$
$$u(t, 0) = f(t), \quad t \in \mathbb{N}_0,$$

is given by $u(t, s) = f(t - s)$. (Note that this partial difference equation is simply (4.1)–(4.2) in which we have taken $\mathbb{T} = \mathbb{N}_0$ and $t_0 = 0$.) In this example, we will show that by using Theorem 4.2.3, we obtain this same solution as expected.

By the definition of $B_{j,i,\Delta}(n)$ given in Theorem 4.2.3, and since $\mu(t) \equiv 1$ on this time scale, for any $j > i$, it follows that

$$\begin{aligned} B_{j,i,0}(n) &= \sum_{k=0}^{n-1} \left(\frac{1}{\mu(t_{n+j-1-k})} - \frac{1}{\mu(t_{n-1-k})} \right) B_{j,i,1}(n-k) \\ &= \sum_{k=0}^{n-1} (1-1)\, B_{j,i,1}(n-k) \\ &= 0, \end{aligned}$$

and for $j = i$, we have

$$B_{j,i,0}(n) = 1.$$

Hence,

$$A_{j,i}(n) = \frac{[\mu(t_{n+j-1})]^{\underline{j}}}{[\mu(t_{i-1})]^{\underline{i}}} B_{j,i,0}(n) = B_{j,i,0}(n) = \begin{cases} 0, & j > i, \\ 1, & j = i. \end{cases}$$

Let $t, s \in \mathbb{T} = \mathbb{N}_0$ such that $t \geq s$. Then there exist $n, j \in \mathbb{N}_0$ such that $t = t_{n+j}$ and $s = t_n$. So, by Theorem 4.2.2,

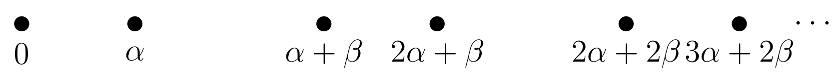
$$u(t_{n+j}, t_n) = \sum_{i=0}^{j} A_{j,i}(n) f(t_i) = f(t_j). \tag{4.13}$$

Because $\mathbb{T} = \{0, 1, 2, \cdots\} = \{t_0, t_1, t_2, \cdots\}$, we have that $t_{n+j} = n + j$ and $t_n = n$. Thus,

$$t - s = t_{n+j} - t_n = n + j - n = j = t_j$$

and so equation (4.13) reduces to $u(t, s) = f(t - s)$.

Example 4.3.2. Let $\alpha > 0$ and $\beta > 0$ be given. Consider the isolated time scale $\mathbb{T} = \{t_0, t_1, t_2, \cdots\} = \{0, \alpha, \alpha + \beta, 2\alpha + \beta, 2\alpha + 2\beta, \cdots\}$.



Note that for this particular time scale, for any $n \in \mathbb{N}_0$,

$$\mu(t_n) = \begin{cases} \alpha, & n \text{ even}, \\ \beta, & n \text{ odd}. \end{cases} \tag{4.14}$$

We will use Theorem 4.2.2 to show that the unique solution of (4.1)–(4.2) on this particular time scale is given by

$$u(t_{n+j}, t_n) = \begin{cases} \left(1 - \frac{\beta}{\alpha}\right) f(t_{j-1}) + \frac{\beta}{\alpha} f(t_j), & \text{both } n, j \text{ odd}, \\ f(t_j), & \text{otherwise}, \end{cases}$$

where $n, j \in \mathbb{N}_0$.

Proof. Let $n, j \in \mathbb{N}_0$ be arbitrary but fixed. By Theorem 4.2.2, the unique solution of (4.1)–(4.2) is given by

$$u(t_{n+j}, t_n) = \sum_{i=0}^{j} A_{j,i}(n) f(t_i).$$

Thus, we need to determine $A_{j,i}(n)$ for each $0 \leq i \leq j$. First we will consider the case when $i = j$. Note that

$$A_{j,j}(n) = \frac{[\mu(t_{n+j-1})]^{\underline{j}}}{[\mu(t_{j-1})]^{\underline{j}}} = \frac{\mu(t_{n+j-1})\mu(t_{n+j-2})\cdots\mu(t_n)}{\mu(t_{j-1})\mu(t_{j-2})\cdots\mu(t_0)}.$$

If n is even, then using (4.14) we have

$$\begin{aligned}
n + j - 1 \equiv j - 1 \mod 2 &\implies \mu(t_{n+j-1}) = \mu(t_{j-1}) \\
n + j - 2 \equiv j - 2 \mod 2 &\implies \mu(t_{n+j-2}) = \mu(t_{j-2}) \\
&\vdots \\
n + 1 \equiv 1 \mod 2 &\implies \mu(t_{n+1}) = \mu(t_1) \\
n \equiv 0 \mod 2 &\implies \mu(t_n) = \mu(t_0).
\end{aligned}$$

Hence, if n is even,

$$A_{j,j}(n) = \frac{\mu(t_{n+j-1})\mu(t_{n+j-2})\cdots\mu(t_n)}{\mu(t_{j-1})\mu(t_{j-2})\cdots\mu(t_0)} = \frac{\mu(t_{j-1})\mu(t_{j-2})\cdots\mu(t_0)}{\mu(t_{j-1})\mu(t_{j-2})\cdots\mu(t_0)} = 1. \tag{4.15}$$

Similarly, if n is odd, then

$$\begin{aligned} n+j-1 \equiv j \mod 2 &\implies \mu(t_{n+j-1}) = \mu(t_j) \\ n+j-2 \equiv j-1 \mod 2 &\implies \mu(t_{n+j-2}) = \mu(t_{j-1}) \\ &\vdots \\ n+1 \equiv 2 \mod 2 &\implies \mu(t_{n+1}) = \mu(t_2) \\ n \equiv 1 \mod 2 &\implies \mu(t_n) = \mu(t_1). \end{aligned}$$

So, in this case

$$A_{j,j}(n) = \frac{\mu(t_{n+j-1})\mu(t_{n+j-2})\cdots\mu(t_n)}{\mu(t_{j-1})\mu(t_{j-2})\cdots\mu(t_0)} = \frac{\mu(t_j)\mu(t_{j-1})\cdots\mu(t_1)}{\mu(t_{j-1})\mu(t_{j-2})\cdots\mu(t_0)} = \frac{\mu(t_j)}{\mu(t_0)}.$$

It follows that if n is odd, then $A_{j,j}(n) = 1$ when j is even, and $A_{j,j}(n) = \frac{\beta}{\alpha}$ when j is odd. So in the case $i = j$, we have shown that

$$A_{j,j}(n) = \begin{cases} \frac{\beta}{\alpha}, & j \text{ odd}, n \text{ odd}, \\ 1, & \text{otherwise}. \end{cases} \tag{4.16}$$

Now let $0 \le i \le j-1$ be given. By definition,

$$A_{j,i}(n) = \sum_{k=0}^{n-1} \frac{[\mu(t_{n+j-1})]^{\underline{j}}}{[\mu(t_{n+j-1-k})]^{\underline{j}}} \left(1 - \frac{\mu(t_{n+j-1-k})}{\mu(t_{n-1-k})}\right) A_{j-1,i}(n-k).$$

Note that

$$\left(1 - \frac{\mu(t_{n+j-1-k})}{\mu(t_{n-1-k})}\right) = 0 \iff \mu(t_{n+j-1-k}) = \mu(t_{n-1-k})$$
$$\iff n+j-1-k \equiv n-1-k \mod 2$$
$$\iff j \equiv 0 \mod 2.$$

Thus, if j is even, $\left(1 - \frac{\mu(t_{n+j-1-k})}{\mu(t_{n-1-k})}\right) = 0$ for every $0 \leq k \leq n-1$. Hence, regardless of the parity of n, if j is even, then $A_{j,i}(n) = 0$ for any $0 \leq i \leq j-1$.

Now assume j is odd. If n is even, then

$$\begin{aligned}
A_{j,i}(n) &= \sum_{k=0}^{n-1} \frac{[\mu(t_{n+j-1})]^{\underline{j}}}{[\mu(t_{n+j-1-k})]^{\underline{j}}} \left(1 - \frac{\mu(t_{n+j-1-k})}{\mu(t_{n-1-k})}\right) A_{j-1,i}(n-k) \\
&= \sum_{\substack{k=0 \\ k \text{ even}}}^{n-1} \frac{[\mu(t_{n+j-1})]^{\underline{j}}}{[\mu(t_{n+j-1-k})]^{\underline{j}}} \left(1 - \frac{\mu(t_{n+j-1-k})}{\mu(t_{n-1-k})}\right) A_{j-1,i}(n-k) \\
&\quad + \sum_{\substack{k=0 \\ k \text{ odd}}}^{n-1} \frac{[\mu(t_{n+j-1})]^{\underline{j}}}{[\mu(t_{n+j-1-k})]^{\underline{j}}} \left(1 - \frac{\mu(t_{n+j-1-k})}{\mu(t_{n-1-k})}\right) A_{j-1,i}(n-k) \\
&= \sum_{\substack{k=0 \\ k \text{ even}}}^{n-1} 1 \left(1 - \frac{\alpha}{\beta}\right) A_{j-1,i}(n-k) + \sum_{\substack{k=0 \\ k \text{ odd}}}^{n-1} \frac{\alpha}{\beta} \left(1 - \frac{\beta}{\alpha}\right) A_{j-1,i}(n-k) \\
&= \sum_{\substack{k=0 \\ k \text{ even}}}^{n-1} \left(1 - \frac{\alpha}{\beta}\right) A_{j-1,i}(n-k) + \sum_{\substack{k=0 \\ k \text{ odd}}}^{n-1} (-1) \left(1 - \frac{\alpha}{\beta}\right) A_{j-1,i}(n-k) \\
&= \sum_{k=0}^{n-1} (-1)^k \left(1 - \frac{\alpha}{\beta}\right) A_{j-1,i}(n-k).
\end{aligned} \tag{4.17}$$

In the case where n is odd, an almost identical calculation yields

$$A_{j,i}(n) = \sum_{k=0}^{n-1} (-1)^k \left(1 - \frac{\beta}{\alpha}\right) A_{j-1,i}(n-k). \tag{4.18}$$

If $i \neq j-1$, then, since j is odd, $j-1$ is even and so $A_{j-1,i}(n) = 0$ for any n

from work done above. Hence, if $i \neq j-1$, then using either equation (4.17) or (4.18) depending on the parity of n, we have $A_{j,i}(n) = 0$ regardless of n.

So finally, the last case we must consider is when $i = j-1$ and j is odd. If n is even, from (4.17), it follows that

$$\begin{aligned} A_{j,i}(n) = A_{j,j-1}(n) &= \sum_{k=0}^{n-1} (-1)^k \left(1 - \frac{\alpha}{\beta}\right) A_{j-1,j-1}(n-k) \\ &= \sum_{k=0}^{n-1} (-1)^k \left(1 - \frac{\alpha}{\beta}\right) \qquad \text{(by (4.16) since } j-1 \text{ is even)} \\ &= 0, \end{aligned}$$

since n is even and hence this summation has an even number of terms. If n is odd, from (4.18) it follows that

$$\begin{aligned} A_{j,i}(n) = A_{j,j-1}(n) &= \sum_{k=0}^{n-1} (-1)^k \left(1 - \frac{\beta}{\alpha}\right) A_{j-1,j-1}(n-k) \\ &= \sum_{k=0}^{n-1} (-1)^k \left(1 - \frac{\beta}{\alpha}\right) \qquad \text{(by (4.16) since } j-1 \text{ is even)} \\ &= 1 - \frac{\beta}{\alpha}, \end{aligned}$$

since n is odd and hence this summation has an odd number of terms.

So, in the case that $0 \leq i \leq j-1$, putting everything together, we have

$$A_{j,i}(n) = \begin{cases} 1 - \frac{\beta}{\alpha}, & j \text{ odd, } n \text{ odd, and } i = j-1, \\ 0, & \text{otherwise.} \end{cases} \tag{4.19}$$

Therefore, using (4.16) and (4.19), we can write the solution of (4.1)–(4.2) as

$$u(t_{n+j}, t_n) = \sum_{i=0}^{j} A_{j,i}(n) f(t_i) = \begin{cases} \left(1 - \frac{\beta}{\alpha}\right) f(t_{j-1}) + \frac{\beta}{\alpha} f(t_j), & \text{both } n, j \text{ odd,} \\ f(t_j), & \text{otherwise.} \end{cases}$$

□

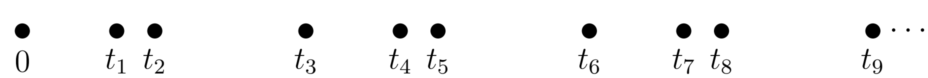
We will now use this result to graph the solution of (4.1)–(4.2) for a specific time scale. Let $\mathbb{T} = \{0, 1, 3, 4, 6, \cdots\}$. Hence $\mathbb{T}$ is of the form given in Example 4.3.2 with $\alpha = 1$ and $\beta = 2$. Therefore, the solution is

$$u(t_{n+j}, t_n) = \begin{cases} 2f(t_j) - f(t_{j-1}), & \text{both } n, j \text{ odd,} \\ f(t_j), & \text{otherwise.} \end{cases}$$

If we take the initial function to be the identity function, $f(t) = t$, we can graph the solution of (4.1)–(4.2), $u(t, s)$, on this particular time scale which we do in Figure 4.2.

Remark 4.3.3. As mentioned in the introduction, since the graininess on this time scale is not constant, we see that we do not have a traveling wave solution as we would expect in the case where $\mathbb{T} = \mathbb{Z}$. Instead, we have distortion of the initial function as s increases. Since the graininess on this time scale only attains two values in a periodic fashion, the distortion is not too significant and is quite predictable. In fact, using the enumeration of the time scale $\mathbb{T} = \{t_0, t_1, t_2, \cdots\}$, if we fix $s = t_k$ where $k \equiv 0 \mod 2$, then we have the initial function only shifted.

Example 4.3.4. Let $\alpha > 0$, $\beta > 0$, and $\gamma > 0$ be given. Consider the isolated time scale $\mathbb{T} = \{t_0, t_1, t_2, \cdots\} = \{0, \alpha, \alpha + \beta, \alpha + \beta + \gamma, 2\alpha + \beta + \gamma, 2\alpha + 2\beta + \gamma \cdots\}$.



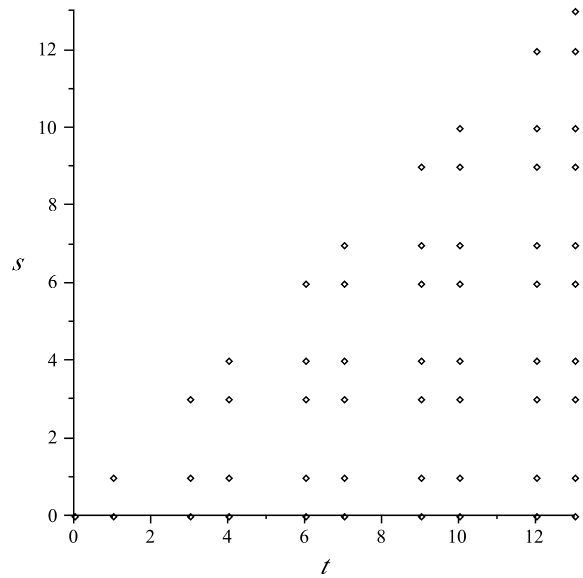


Figure 4.1: The domain of the solution of (4.1)–(4.2) found in Example 4.3.2.

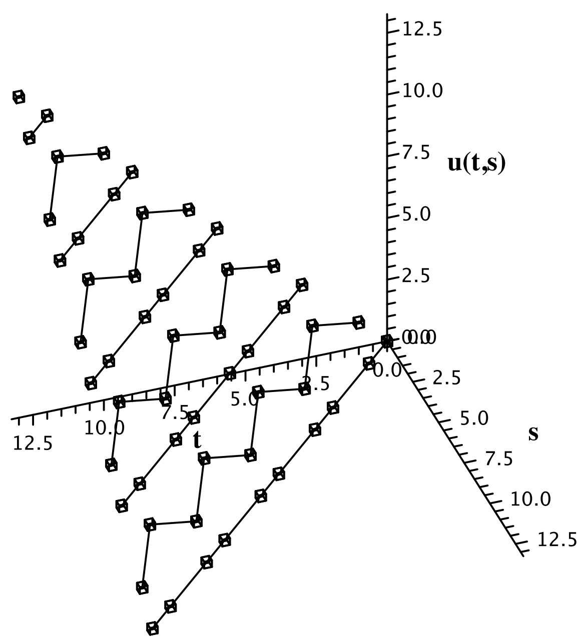


Figure 4.2: The solution of (4.1)–(4.2) found in Example 4.3.2 on $\mathbb{T} = \{0, 1, 3, 4, 6, \cdots\}$ with the initial function given by $f(t) = t$.

Note that for this particular time scale, for any $n \in \mathbb{N}_0$,

$$\mu(t_n) = \begin{cases} \alpha, & n \equiv 0 \mod 3, \\ \beta, & n \equiv 1 \mod 3, \\ \gamma, & n \equiv 2 \mod 3. \end{cases} \tag{4.20}$$

In an argument very similar to that given in Example 4.3.2, we can show that:
If $j \equiv 0 \mod 3$,

$$u(t_{n+j}, t_n) = f(t_j) \quad \text{for all } n \in \mathbb{N}_0.$$

If $j \equiv 1 \mod 3$,

$$u(t_{n+j}, t_n) = \begin{cases} f(t_j), & n \equiv 0 \mod 3, \\ \frac{\beta}{\alpha} f(t_j) + \left(1 - \frac{\beta}{\alpha}\right) f(t_{j-1}), & n \equiv 1 \mod 3, \\ \frac{\gamma}{\alpha} f(t_j) + \left(1 - \frac{\gamma}{\alpha}\right) f(t_{j-1}), & n \equiv 2 \mod 3. \end{cases}$$

If $j \equiv 2 \mod 3$,

$$u(t_{n+j}, t_n) = \begin{cases} f(t_j), & n \equiv 0 \mod 3, \\ \begin{array}{l} \dfrac{\gamma}{\alpha} f(t_j) + \dfrac{\beta}{\alpha}\left(1 - \dfrac{\gamma}{\alpha}\right) f(t_{j-1}) \\ \qquad + \left(1 - \dfrac{\gamma}{\alpha} - \dfrac{\beta}{\alpha}\left(1 - \dfrac{\gamma}{\alpha}\right)\right) f(t_{j-2}) \end{array}, & n \equiv 1 \mod 3, \\ \dfrac{\gamma}{\beta} f(t_j) + \left(1 - \dfrac{\gamma}{\beta}\right) f(t_{j-1}), & n \equiv 2 \mod 3. \end{cases}$$

Let $\mathbb{T} = \{0, 1, 3, 6, 7, 9, 12, \cdots\}$. Note that $\mathbb{T}$ is of the form given in Example 4.3.4 with $\alpha = 1$, $\beta = 2$, and $\gamma = 3$. Hence, we can use the solution that we explicitly calculated above to graph the solution of (4.1)–(4.2) with the initial function being the identity function, $f(t) = t$. We do this in Figure 4.4.

Remark 4.3.5. In this figure (Figure 4.4), we notice that the behavior is similar to that which we observed in Example 4.3.2. However here, since the time scale is periodic with 3 points in its period, we find that if $s = t_k$ where $k \equiv 0 \mod 3$, then we have the initial function only shifted.

Example 4.3.6. Let $\mathbb{T}$ be the so-called harmonic time scale: let $t_0 = 1$, $t_n = \sum_{i=1}^{n} \frac{1}{i}$ for $n \in \mathbb{N}$, and define $\mathbb{T} = \{t_n : n \in \mathbb{N}_0\}$. Note that $\mathbb{T}$ is unbounded above and $\mu(t_n) = t_{n+1} - t_n = \frac{1}{n+1}$.

We will use Theorem 4.2.3 to show that the unique solution of (4.1)–(4.2) on this particular time scale is given by

$$u(t_{n+j}, t_n) = \sum_{i=0}^{j} \left[\frac{i!(n+j-i-1)^{\underline{j-i}} \cdot j^{\underline{j-i}}}{(n+j)^{\underline{j}}(j-i)!} \right] f(t_i),$$

where $n, j \in \mathbb{N}_0$.

Proof. Let $n, j \in \mathbb{N}_0$ be given. If $j = 0$, then $u(t_n, t_n) = f(t_0)$ by Lemma 4.1.1. So, assume $j \geq 1$. Recall from Theorem 4.2.2 that

$$u(t_{n+j}, t_n) = \sum_{i=0}^{j} A_{j,i}(n) f(t_i),$$

and by Theorem 4.2.3 we know that

$$A_{j,i}(n) = \frac{[\mu(t_{n+j-1})]^{\underline{j}}}{[\mu(t_{i-1})]^{\underline{i}}} B_{j,i,0}(n),$$

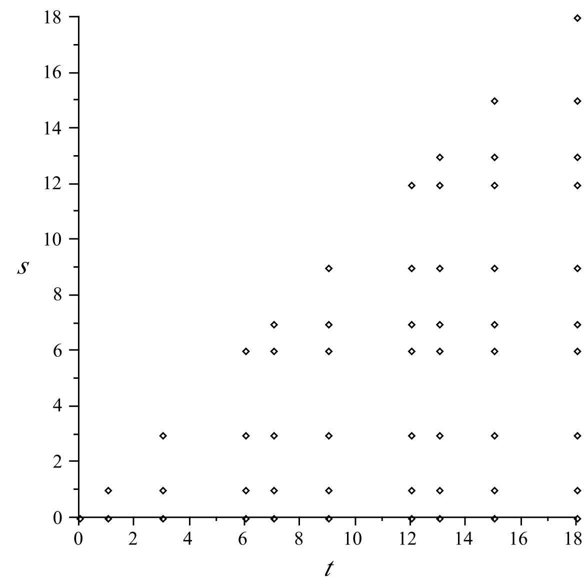


Figure 4.3: The domain of the solution of (4.1)–(4.2) found in Example 4.3.4.

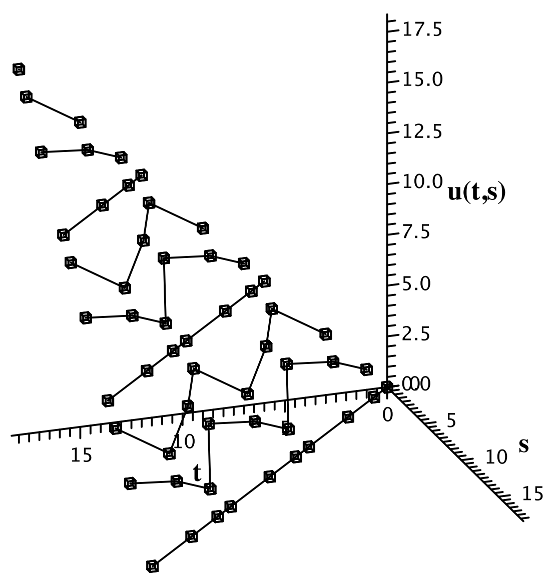


Figure 4.4: The solution of (4.1)–(4.2) found in Example 4.3.4 on $\mathbb{T} = \{0, 1, 3, 6, 7, 9, 12, \cdots\}$ with the initial function given by $f(t) = t$.

where

$$B_{j,i,\Delta}(n) = \begin{cases} \displaystyle\sum_{k=0}^{n-1} \left(\frac{1}{\mu(t_{n+j-\Delta-1-k})} - \frac{1}{\mu(t_{n-1-k})} \right) B_{j,i,\Delta+1}(n-k), & \text{for } j-i<\Delta, \\ 1, & \text{for } j-i=\Delta. \end{cases}$$

We now aim to compute $B_{j,i,0}(n)$ on this time scale. For $j \geq 1$, note that

$$\begin{aligned} B_{j,i,0}(n) &= \sum_{k_1=0}^{n-1} \left(\frac{1}{\mu(t_{n+j-1-k_1})} - \frac{1}{\mu(t_{n-1-k_1})} \right) B_{j,i,1}(n-k_1) \\ &= \sum_{k_1=0}^{n-1} \Big((n+j-k_1) - (n-k_1) \Big) B_{j,i,1}(n-k_1) \\ &= j \sum_{k_1=0}^{n-1} B_{j,i,1}(n-k_1). \end{aligned} \tag{4.21}$$

If $i+1<j$, then

$$\begin{aligned} B_{j,i,1}(n-k_1) &= \sum_{k_2=0}^{n-k_1-1} \left(\frac{1}{\mu(t_{n-k_1+j-2-k_2})} - \frac{1}{\mu(t_{n-1-k_1-k_2})} \right) B_{j,i,2}(n-k_1-k_2) \\ &= \sum_{k_2=0}^{n-k_1-1} \Big((n-k_1+j-1-k_2) - (n-k_1-k_2) \Big) B_{j,i,2}(n-k_1-k_2) \\ &= (j-1) \sum_{k_2=0}^{n-k_1-1} B_{j,i,2}(n-k_1-k_2). \end{aligned}$$

Hence, combining this with (4.21), it follows that

$$\begin{aligned} B_{j,i,0}(n) &= j(j-1) \sum_{k_1=0}^{n-1} \sum_{k_2=0}^{n-k_1-1} B_{j,i,2}(n-k_1-k_2) \\ &= j^{\underline{2}} \sum_{k_1=0}^{n-1} \sum_{k_2=0}^{n-k_1-1} B_{j,i,2}(n-k_1-k_2). \end{aligned}$$

Continuing this process inductively, we obtain

$$\begin{aligned}
B_{j,i,0}(n) &= j^{\underline{2}} \sum_{k_1=0}^{n-1} \sum_{k_2=0}^{n-k_1-1} B_{j,i,2}(n-k_1-k_2) \\
&= j^{\underline{3}} \sum_{k_1=0}^{n-1} \sum_{k_2=0}^{n-k_1-1} \sum_{k_3=0}^{\substack{n-k_1\\-k_2-1}} B_{j,i,3}(n-k_1-k_2-k_3) \\
(j-i \text{ iterations}) \quad &\vdots \\
&= j^{\underline{j-i}} \sum_{k_1=0}^{n-1} \sum_{k_2=0}^{n-k_1-1} \cdots \sum_{k_{j-i}=0}^{\substack{n-k_1-\cdots\\-k_{j-i-1}-1}} B_{j,i,j-i}(n-k_1-\cdots-k_{j-i}) \\
&= j^{\underline{j-i}} \underbrace{\sum_{k_1=0}^{n-1} \sum_{k_2=0}^{n-k_1-1} \cdots \sum_{k_{j-i}=0}^{\substack{n-k_1-\cdots\\-k_{j-i-1}-1}}}_{j-i \text{ sums}} 1 \\
&= j^{\underline{j-i}} \left[\frac{(n+j-i-1)^{\underline{j-i}}}{(j-i)!} \right].
\end{aligned}$$

The proof for the final equality is given below in Lemma 4.3.7.

Next note that

$$\begin{aligned}
\frac{[\mu(t_{n+j-1})]^{\underline{j}}}{[\mu(t_{i-1})]^{\underline{i}}} &= \frac{\mu(t_{n+j-1})\mu(t_{n+j-2})\cdots\mu(t_n)}{\mu(t_{i-1})\mu(t_{i-2})\cdots\mu(t_0)} \\
&= \frac{\frac{1}{(n+j)(n+j-1)\cdots(n+1)}}{\frac{1}{(i)(i-1)\cdots 1}} \\
&= \frac{i!}{(n+j)^{\underline{j}}}.
\end{aligned}$$

Therefore, by Theorem 4.2.3,

$$A_{j,i}(n) = \frac{[\mu(t_{n+j-1})]^{\underline{j}}}{[\mu(t_{i-1})]^{\underline{i}}} B_{j,i,0}(n) = \frac{i!}{(n+j)^{\underline{j}}} \left(j^{\underline{j-i}} \left[\frac{(n+j-i-1)^{\underline{j-i}}}{(j-i)!} \right] \right),$$

and so by Theorem 4.2.2,

$$u(t_{n+j}, t_n) = \sum_{i=0}^{j} \left[\frac{i!(n+j-i-1)^{\underline{j-i}} \cdot j^{\underline{j-i}}}{(n+j)^{\underline{j}}(j-i)!} \right] f(t_i),$$

where $n, j \in \mathbb{N}_0$. We graph this solution of (4.1)–(4.2) for the harmonic time scale in Figure 4.6. □

Lemma 4.3.7. For any $n, M \in \mathbb{N}$,

$$\underbrace{\sum_{k_1=0}^{n-1} \sum_{k_2=0}^{n-k_1-1} \cdots \sum_{k_{M-1}=0}^{\substack{n-k_1-\cdots \\ -k_{M-2}-1}} \sum_{k_M=0}^{\substack{n-k_1-\cdots \\ -k_{M-1}-1}}}_{M \text{ sums}} 1 = \frac{(n+M-1)^{\underline{M}}}{M!}.$$

Proof. Fix $n \in \mathbb{N}$. If $M = 1$, the result is obvious, so assume that $M > 1$.

Note that

$$\sum_{k_1=0}^{n-1} \sum_{k_2=0}^{n-k_1-1} \cdots \sum_{k_{M-1}=0}^{\substack{n-k_1-\cdots \\ -k_{M-2}-1}} \sum_{k_M=0}^{\substack{n-k_1-\cdots \\ -k_{M-1}-1}} 1 = \sum_{k_1=0}^{n-1} \sum_{k_2=0}^{n-k_1-1} \cdots \sum_{k_{M-1}=0}^{\substack{n-k_1-\cdots \\ -k_{M-2}-1}} (n-k_1-\cdots-k_{M-1}). \quad (4.22)$$

Focusing on the innermost summation of (4.22), note that when $k_{M-1} = 0$, the summand $n - k_1 - \cdots - k_{M-1} = n - k_1 - \cdots - k_{M-2}$. Similarly, when $k_{M-1} = 1$, the summand is $n - k_1 - \cdots - k_{M-2} - 1$. We continue this process in the following table:

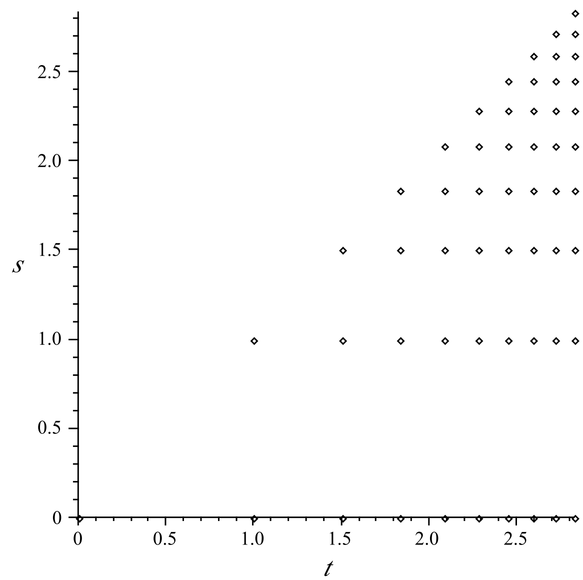


Figure 4.5: The domain of the solution of (4.1)–(4.2) found in Example 4.3.6.

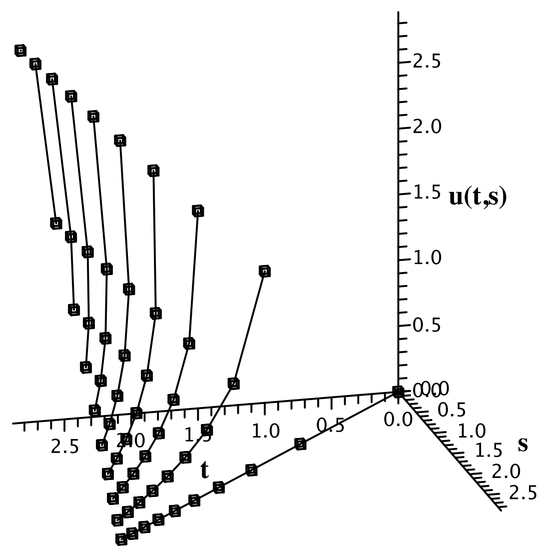


Figure 4.6: The solution of (4.1)–(4.2) found in Example 4.3.6 on the so-called harmonic time scale with the initial function given by $f(t) = t$.

k_{M-1}	$n-k_1-\cdots-k_{M-1}$
0	$n-k_1-\cdots-k_{M-2}$
1	$n-k_1-\cdots-k_{M-2}-1$
2	$n-k_1-\cdots-k_{M-2}-2$
$\vdots$	$\vdots$
$n-k_1-\cdots-k_{M-1}-2$	2
$n-k_1-\cdots-k_{M-1}-1$	1

Hence, we can reindex the innermost summation of (4.22) as follows:

$$\begin{aligned}\sum_{k_{M-1}=0}^{\substack{n-k_1-\cdots\\-k_{M-2}-1}}(n-k_1-\cdots-k_{M-1}) &= \sum_{j=1}^{\substack{n-k_1-\cdots\\-k_{M-2}}} j \\ &= \sum_{j=1}^{\substack{n-k_1-\cdots\\-k_{M-2}}} j^{\underline{1}} \\ &= \frac{j^{\underline{2}}}{2}\Big|_{j=1}^{j=n-k_1-\cdots-k_{M-2}+1} \\ &= \frac{(n-k_1-\cdots-k_{M-2}+1)^{\underline{2}}}{2}.\end{aligned}$$

Here we have used standard results from the theory of summation from difference equations (see, for example, [18, Section 2.2]). Note that when we evaluate the lower limit in the second to last expression, we get $\frac{1^{\underline{2}}}{2}=0$. This will occur at every stage of this calculation. So, continuing from (4.22) we have

$$\begin{aligned}&= \sum_{k_1=0}^{n-1}\sum_{k_2=0}^{n-k_1-1}\cdots\sum_{k_{M-2}=0}^{\substack{n-k_1-\cdots\\-k_{M-3}-1}}\sum_{k_{M-1}=0}^{\substack{n-k_1-\cdots\\-k_{M-2}-1}}(n-k_1-\cdots-k_{M-1}) \\ &= \frac{1}{2}\sum_{k_1=0}^{n-1}\sum_{k_2=0}^{n-k_1-1}\cdots\sum_{k_{M-2}=0}^{\substack{n-k_1-\cdots\\-k_{M-3}-1}}(n-k_1-\cdots-k_{M-2}+1)^{\underline{2}}. \qquad (4.23)\end{aligned}$$

Again we can reindex this innermost summation:

k_{M-2}	$n-k_1-\cdots-k_{M-2}+1$
0	$n-k_1-\cdots-k_{M-3}+1$
1	$n-k_1-\cdots-k_{M-3}$
2	$n-k_1-\cdots-k_{M-3}-1$
$\vdots$	$\vdots$
$n-k_1-\cdots-k_{M-2}-2$	3
$n-k_1-\cdots-k_{M-2}-1$	2

Thus, continuing from (4.23), we have

$$= \frac{1}{2}\sum_{k_1=0}^{n-1}\sum_{k_2=0}^{n-k_1-1}\cdots\sum_{k_{M-3}=0}^{\substack{n-k_1-\cdots\\-k_{M-4}-1}}\sum_{k_{M-2}=0}^{\substack{n-k_1-\cdots\\-k_{M-3}-1}}(n-k_1-\cdots-k_{M-2}+1)^{\underline{2}}$$

$$= \frac{1}{2}\sum_{k_1=0}^{n-1}\sum_{k_2=0}^{n-k_1-1}\cdots\sum_{k_{M-3}=0}^{\substack{n-k_1-\cdots\\-k_{M-4}-1}}\sum_{j=2}^{\substack{n-k_1-\cdots\\-k_{M-3}+1}} j^{\underline{2}}$$

$$= \frac{1}{2}\sum_{k_1=0}^{n-1}\sum_{k_2=0}^{n-k_1-1}\cdots\sum_{k_{M-3}=0}^{\substack{n-k_1-\cdots\\-k_{M-4}-1}}\left.\frac{j^{\underline{3}}}{3}\right|_{j=2}^{j=n-k_1-\cdots-k_{M-3}+2}$$

$$= \frac{1}{3!}\sum_{k_1=0}^{n-1}\sum_{k_2=0}^{n-k_1-1}\cdots\sum_{k_{M-3}=0}^{\substack{n-k_1-\cdots\\-k_{M-4}-1}}(n-k_1-\cdots-k_{M-3}+2)^{\underline{3}}$$

$$\vdots$$

$$= \frac{1}{(M-2)!}\sum_{k_1=0}^{n-1}\sum_{k_2=0}^{n-k_1-1}(n-k_1-k_2+M-3)^{\underline{M-2}}$$

$$= \frac{1}{(M-2)!}\sum_{k_1=0}^{n-1}\sum_{j=M-2}^{n-k_1+M-3} j^{\underline{M-2}}$$

$$
\begin{aligned}
&= \frac{1}{(M-1)!} \sum_{k_1=0}^{n-1} j^{\underline{M-1}} \Big|_{j=M-2}^{j=n-k_1+M-2} \\
&= \frac{1}{(M-1)!} \sum_{k_1=0}^{n-1} (n-k_1+M-2)^{\underline{M-1}} \\
&= \frac{1}{(M-1)!} \sum_{j=M-1}^{n+M-2} j^{\underline{M-1}} \\
&= \frac{1}{M!} j^{\underline{M}} \Big|_{j=M-1}^{j=n+M-1} \\
&= \frac{(n+M-1)^{\underline{M}}}{M!}.
\end{aligned}
$$

□

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