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AN ITERATIVE TREATMENT OF DOUBLE-MODE PULSATION

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ABSTRACT

The iterative nonlinear pulsation theory is extended to the bimodal case, and a series of equations derived which are complete to second order. Solutions to these equations are obtained for a number of long-period double-mode Cepheid models in the vicinity of the interaction resonance $\omega_1 + \omega_0 = \omega_3$ and the harmonic resonance $P_2/P_0 = 0.5$. We show that the resonant second-order amplitudes and phases change abruptly as the center of the resonance is approached. In addition, it is demonstrated that, inward of the hydrogen ionization zone, the amplitudes strongly resemble the linear eigenfunctions to which they are resonantly linked. Thus the resonances are shown to have spatial as well as temporal content. Due to nonadiabatic effects the second-order terms are particularly strong in the H I ionization zone, a circumstance which suggests that this region could play an important role in determining limiting-amplitude behavior.

Subject headings: stars: Cepheids — stars: pulsation

I. INTRODUCTION

A dozen years have now passed since the appearance of pioneering nonlinear models of RR Lyrae stars and classical Cepheids (e.g., Christy 1966; Cox, Brownlee, and Ehlers 1966). In the intervening time, despite a very large body of work and considerable advances in sophistication (see the review by Cox 1978), our view of the instability strip remains clouded and problematical. Following the great success of identifying the instability mechanism and showing that it would indeed operate so as to produce oscillations generally resembling the observed ones, the attempt to construct detailed pulsation models in accord with both observations and the “grand theories” of stellar and galactic evolution, has produced results that range from less than completely satisfactory to downright dismaying.

Nowhere are these difficulties more apparent than in the question of the masses of the double-mode Cepheids. A recent list compiled by Stobie (1977) gives 11 such stars, each with two well-determined periods. As is by now well known, these periods cannot be reconciled, simultaneously, with standard assumptions regarding evolution, pulsation, and chemical abundances. Because this conflict depends upon the quantity most reliably determined from both the observational and theoretical sides—namely, the period—many authors consider the double-mode mass anomaly to be the most perplexing of all the problems concerning the instability strip (e.g., Cox 1978).

Numerous authors have employed linear pulsation theory to confirm that the proper periods may be obtained with standard ideas regarding pulsation and composition, provided that very low masses (i.e., $\frac{1}{3}$ to $\frac{1}{2}$ the evolutionary mass) are taken for the models (e.g., Petersen 1978). Such masses, however, appear impossible to reconcile with present ideas concerning stellar evolution (Stobie 1977). On the other hand, it has been shown that the mass discrepancy can be removed if one is willing to accept inhomogeneous, helium-enriched models (Cox, Michaud, and Hodson 1978). This solution has the added attraction of ameliorating the so-called “bump-mass” anomaly involving the middle-period Cepheids. However, while the necessary composition inhomogeneity could be driven by a helium-deficient stellar wind, the problem of continual mixing remains a major one.

Other investigators (e.g., Cogan 1977; Saio, Kobayashi, and Takeuti 1977) have suggested that if the double-mode Cepheids are cool, enhanced convection might be able to favorably alter the model periods, although somewhat unrealistic assumptions seem to be required (e.g., Deupree 1977). Another idea, discussed by Cox (1978), is that a small admixture of nonradial modes in normal radial pulsation models might account for the observed periods. This possibility has not yet been pursued in detail.

Recently, Simon (1978*b*) proposed that the phenomenon of persistent double-mode pulsation be interpreted as involving a resonant interaction between the two linearly unstable modes. If the two linear frequencies are called ω_0 and ω_1 , respectively, their interaction will be resonant whenever $\omega_1 + \omega_0 = \omega_j$, where ω_j is the frequency of the j th normal mode. This interpretation allows the narrow range of period ratio P_1/P_0 observed in the double-mode Cepheids ($0.697 \leq P_1/P_0 \leq 0.711$), as well as in the double-mode AI Velorum stars ($0.768 \leq P_1/P_0 \leq 0.778$) to be explained in a simple manner, provided that the former objects involve the resonance $j = 3$ and the latter objects $j = 4$. The resonance explanation is consistent with undermassive models for the double-mode Cepheids, but probably not with the helium-enriched models, at least in their present form (Cox, Hodson, and King 1979).

With regard to the AI Velorum regime, the resonance hypothesis predicts low masses ($0.2 \lesssim M/M_\odot \lesssim 1$)

which are at odds with the prevailing view that the AI Vel pulsators have masses $\sim 2 M_{\odot}$ and are in a stage of normal post-main-sequence evolution (e.g., Breger 1976). Nonetheless, Simon (1978*b*) has argued that pulsational evidence does not preclude the low masses indicated by the resonance.

The investigation just described was performed entirely with the linear theory, which is capable of only the crudest possible treatment of resonances—namely, in terms of period ratios alone. Further progress cannot be made without a nonlinear scheme. Simon (1977, hereafter S77) employed an iterative nonlinear pulsation theory to study the resonance $P_2/P_0 = 0.5$ in bump Cepheid models. For single-mode pulsations, this theory calculates the second-order harmonic corrections (frequency 2ω) to the linear eigenfunctions (frequency ω). Since the second-order corrections are strongly affected by the resonance, the resonant region may be mapped by a series of iterative models which cross it.

For the case in which two modes are present (say, ω_0 and ω_1), there are four correction terms appearing in second order—the two harmonics ($2\omega_0$ and $2\omega_1$) and the two interaction terms ($\omega_1 + \omega_0$ and $\omega_1 - \omega_0$). The analysis of pulsations in this fashion is particularly germane to the double-mode stars, since the frequencies which appear at any order in the iterative theory are the same as those actually detected in nonlinear Fourier fitting of the observed oscillations (e.g., Simon 1978*a* and references therein).

In the present investigation we shall extend the iterative theory to the double-mode case (§ II), and apply it to map the resonance $\omega_1 + \omega_0 = \omega_3$ in a series of double-mode Cepheid models with fundamental periods in the vicinity of the observed period of V367 Sct (§ III). The article closes with a brief discussion (§ IV).

II. THE EXTENDED ITERATIVE THEORY

We wish to consider the case where there are two linearly unstable modes with frequencies ω_0 and ω_1 . Following S77 we write the dependent stellar variables x_i in the form

$$x_i = x_{0i}(1 + \delta x_i/x_{0i}),$$

and expand the structure equations to second order in the perturbations $\delta x_i/x_{0i}$. The index i runs from 1 to 8, with the following correspondences:

$$\begin{aligned} 1 &\leftrightarrow \text{radius } (r); & 2 &\leftrightarrow \text{density } (\rho); & 3 &\leftrightarrow \text{pressure } (P); & 4 &\leftrightarrow \text{temperature } (T); \\ 5 &\leftrightarrow \text{luminosity } (L_r); & 6 &\leftrightarrow \text{opacity } (\kappa); & 7 &\leftrightarrow \text{adiabatic exponent } (\Gamma_1); \\ 8 &\leftrightarrow \text{adiabatic exponent } (\Gamma_3). \end{aligned}$$

A solution is now sought in the form of a superposition of two truncated interacting Fourier series:

$$\begin{aligned} \frac{\delta x_i}{x_{0i}} = & A_{i0} + A_{i1} \cos \omega_0 t - B_{i1} \sin \omega_0 t + Q_{i1} \cos \omega_1 t - R_{i1} \sin \omega_1 t + A_{i2} \cos 2\omega_0 t - B_{i2} \sin 2\omega_0 t \\ & + Q_{i2} \cos 2\omega_1 t - R_{i2} \sin 2\omega_1 t + A_{i+} \cos \omega_+ t - B_{i+} \sin \omega_+ t + A_{i-} \cos \omega_- t - B_{i-} \sin \omega_- t, \end{aligned} \quad (1)$$

where

$$\omega_+ \equiv \omega_1 + \omega_0, \quad \omega_- \equiv \omega_1 - \omega_0.$$

Insertion of equations (1) into the second-order equations yields expressions which are quadratic in the A 's, B 's, Q 's, and R 's. However, consistent with the order of approximation employed, all products are neglected which involve terms other than A_{i1} , B_{i1} , Q_{i1} , and R_{i1} . We then equate coefficients of the given frequencies, each of which yields a separate set of eight equations. The set for ω_0 may be written

$$\mathcal{A}_1 \mathbf{a}_0 = 0, \quad \sigma = \sigma_0, \quad (2)$$

where the vector $\mathbf{a}_0$ is given by

$$\mathbf{a}_0 = (A_{11}, B_{11}, A_{31}, B_{31}, A_{41}, B_{41}, A_{51}, B_{51});$$

σ_0 is the dimensionless angular frequency corresponding to ω_0 ; and $\mathcal{A}_1$ is a linear operator appropriate for reproducing equations (2) in S77.

Continuing this notation, the set for frequency ω_1 , may be written

$$\mathcal{A}_1 \mathbf{a}_1 = 0, \quad \sigma = \sigma_1, \quad (3)$$

where

$$\mathbf{a}_1 = (Q_{11}, R_{11}, Q_{31}, R_{31}, Q_{41}, R_{41}, Q_{51}, R_{51}).$$

Equations (2) and (3) are identical to the ordinary linear nonadiabatic pulsation equations in the limit: (rise time)/period $\rightarrow 0$.

Turning to the second-order terms, the coefficients of frequency $2\omega_0$ yield equations (3)–(10) in S77. These may be written in the form

$$\mathcal{A}_2 \mathbf{b}_0 = \mathbf{d}_0, \quad \sigma = \sigma_0, \quad (4)$$

where

$$\mathbf{b}_0 = (A_{12}, B_{12}, A_{32}, B_{32}, A_{42}, B_{42}, A_{52}, B_{52})$$

and

$$\mathbf{d}_0 = (D_1, E_1, D_3, E_3, D_4, E_4, D_5, E_5).$$

The D 's and E 's are given by equations (11)–(18) (S77).

Similarly, for frequency $2\omega_1$, we obtain

$$\mathcal{A}_2 \mathbf{b}_1 = \mathbf{d}_1, \quad \sigma = \sigma_1, \quad (5)$$

where

$$\mathbf{b}_1 = (Q_{12}, R_{12}, Q_{32}, R_{32}, Q_{42}, R_{42}, Q_{52}, R_{52})$$

and the vector $\mathbf{d}_1$ may be obtained directly from $\mathbf{d}_0$ with the substitutions

$$A_{i2} \rightarrow Q_{i2}, \quad B_{i2} \rightarrow R_{i2}, \quad \sigma_0 \rightarrow \sigma_1. \quad (6)$$

The interaction terms satisfy very similar relationships. For frequency ω_+ , we have

$$\mathcal{A}_+ \mathbf{b}_+ = \mathbf{d}_+, \quad \sigma = (\sigma_1 + \sigma_0)/2, \quad (7)$$

where $\mathcal{A}_+$ is identical to $\mathcal{A}_2$ except that σ is given by equation (7). Further,

$$\mathbf{b}_+ = (A_{1+}, B_{1+}, A_{3+}, B_{3+}, A_{4+}, B_{4+}, A_{5+}, B_{5+})$$

and

$$\mathbf{d}_+ = (D_{1+}, E_{1+}, D_{3+}, E_{3+}, D_{4+}, E_{4+}, D_{5+}, E_{5+}),$$

where

$$\begin{aligned} D_{1+} = & \frac{1}{2}F_1[-2A_{11}Q_{21} - 2A_{21}Q_{11} - 6A_{11}Q_{11} - 2A_{21}Q_{21} + 2B_{11}R_{21} + 2R_{11}B_{21} + 6B_{11}R_{11} \\ & + 2B_{21}R_{21} + \rho_{PP}(A_{31}Q_{31} - B_{31}R_{31}) + \rho_{TT}(A_{41}Q_{41} - B_{41}R_{41}) \\ & + \rho_{PT}(A_{31}Q_{41} + A_{41}Q_{31} - B_{31}R_{41} - B_{41}R_{31})], \end{aligned} \quad (8)$$

$$\begin{aligned} E_{1+} = & \frac{1}{2}F_1[-2A_{11}R_{21} - 2A_{21}R_{11} - 6A_{11}R_{11} - 2A_{21}R_{21} - 2Q_{11}B_{21} - 2B_{11}Q_{21} - 6B_{11}Q_{11} \\ & - 2Q_{21}B_{21} + \rho_{PP}(A_{31}R_{31} + B_{31}Q_{31}) + \rho_{TT}(A_{41}R_{41} + B_{41}Q_{41}) \\ & + \rho_{PT}(A_{31}R_{41} + A_{41}R_{31} + B_{31}Q_{41} + B_{41}Q_{31})], \end{aligned} \quad (9)$$

$$\begin{aligned} D_{3+} = & B_{11}R'_{31} + R_{11}B'_{31} + B_{11}R_{31} + R_{11}B_{31} - 2B_{11}R_{11} - A_{11}Q'_{31} - Q_{11}A'_{31} \\ & - A_{11}Q_{31} - Q_{11}A_{31} + 2A_{11}Q_{11}, \end{aligned} \quad (10)$$

$$\begin{aligned} E_{3+} = & -A_{11}R'_{31} - R_{11}A'_{31} - A_{11}R_{31} - A_{31}R_{11} + 2A_{11}R_{11} \\ & - Q_{11}B'_{31} - B_{11}Q'_{31} - Q_{11}B_{31} - B_{11}Q_{31} + 2Q_{11}B_{11}, \end{aligned} \quad (11)$$

$$\begin{aligned} D_{4+} = & -\frac{1}{2}\nabla_{\text{rad}}[2A_{61}Q_{61} + 12A_{41}Q_{41} - 4A_{41}Q_{61} - 4Q_{41}A_{61} + 16A_{11}Q_{41} + 16Q_{11}A_{41} \\ & - 4A_{11}Q_{61} - 4Q_{11}A_{61} + 12A_{11}Q_{11} - 2R_{61}B_{61} - 12B_{41}R_{41} \\ & + 4B_{41}R_{61} + 4R_{41}B_{61} - 16B_{11}R_{41} - 16R_{11}B_{41} + 4B_{11}R_{61} \\ & + 4R_{11}B_{61} - 12B_{11}R_{11} - \kappa_{PP}(A_{31}Q_{31} - B_{31}R_{31}) \\ & - \kappa_{TT}(A_{41}Q_{41} - B_{41}R_{41}) \\ & - \kappa_{PT}(A_{31}Q_{41} + A_{41}Q_{31} - B_{31}R_{41} - B_{41}R_{31})] \\ & - \frac{1}{2}[A'_{41}(-Q_{61} + 3Q_{41} + 4Q_{11}) + Q'_{41}(-A_{61} + 3A_{41} + 4A_{11}) \\ & + B'_{41}(R_{61} - 3R_{41} - 4R_{11}) + R'_{41}(B_{61} - 3B_{41} - 4B_{11})], \end{aligned} \quad (12)$$

$$\begin{aligned}
E_{4+} = & \frac{1}{2} \nabla_{\text{rad}} [-2A_{61}R_{61} - 12A_{41}R_{41} + 4A_{41}R_{61} + 4A_{61}R_{41} - 16A_{11}R_{41} - 16A_{41}R_{11} + 4A_{11}R_{61} \\
& + 4A_{61}R_{11} - 12A_{11}R_{11} - 2B_{61}Q_{61} - 12B_{41}Q_{41} + 4B_{41}Q_{61} + 4Q_{41}B_{61} \\
& - 16B_{11}Q_{41} - 16Q_{11}B_{41} + 4B_{11}Q_{61} + 4Q_{11}B_{61} - 12B_{11}Q_{11} \\
& + \kappa_{PP}(A_{31}R_{31} + B_{31}Q_{31}) + \kappa_{TT}(A_{41}R_{41} + B_{41}Q_{41}) \\
& + \kappa_{PT}(A_{31}R_{41} + A_{41}R_{31} + B_{31}Q_{41} + B_{41}Q_{31})] \\
& + \frac{1}{2}[A'_{41}(R_{61} - 3R_{41} - 4R_{11}) - R'_{41}(-A_{61} + 3A_{41} + 4A_{11}) \\
& - B'_{41}(-Q_{61} + 3Q_{41} + 4Q_{11}) + Q'_{41}(B_{61} - 3B_{41} - 4B_{11})], \tag{13}
\end{aligned}$$

$$\begin{aligned}
D_{5+} = & \frac{1}{2} \frac{F_2}{\rho_T \nabla_{\text{ad}}} \{ \sigma_0 [B_{21}(Q_{31} + Q_{71} - Q_{21}) + A_{21}(R_{31} + R_{71} - R_{21})] \\
& + \sigma_1 [R_{21}(A_{31} + A_{71} - A_{21}) + Q_{21}(B_{31} + B_{71} - B_{21})] \\
& + (\sigma_1 + \sigma_0) [\rho_{PP}(A_{31}R_{31} + B_{31}Q_{31}) + \rho_{TT}(A_{41}R_{41} + B_{41}Q_{41}) \\
& + \rho_{PT}(A_{31}R_{41} + A_{41}R_{31} + B_{31}Q_{41} + B_{41}Q_{31})] \} \\
& + \frac{1}{2} \left\{ B'_{51} \left[\frac{\Gamma_3}{\Gamma_3 - 1} R_{81} + R_{21} \right] + R'_{51} \left[\frac{\Gamma_3}{\Gamma_3 - 1} B_{81} + B_{21} \right] \right. \\
& \left. - A'_{51} \left[\frac{\Gamma_3}{\Gamma_3 - 1} Q_{81} + Q_{21} \right] - Q'_{51} \left[\frac{\Gamma_3}{\Gamma_3 - 1} A_{81} + A_{21} \right] \right\}, \tag{14}
\end{aligned}$$

$$\begin{aligned}
E_{5+} = & -\frac{1}{2} \frac{F_2}{\rho_T \nabla_{\text{ad}}} \{ \sigma_0 [A_{21}(Q_{31} + Q_{71} - Q_{21}) - B_{21}(R_{31} + R_{71} - R_{21})] \\
& + \sigma_1 [Q_{21}(A_{31} + A_{71} - A_{21}) - R_{21}(B_{31} + B_{71} - B_{21})] \\
& + (\sigma_1 + \sigma_0) [\rho_{PP}(A_{31}Q_{31} - B_{31}R_{31}) + \rho_{TT}(A_{41}Q_{41} - B_{41}R_{41}) \\
& + \rho_{PT}(A_{31}Q_{41} + A_{41}Q_{31} - B_{31}R_{41} - B_{41}R_{31})] \} \\
& - \frac{1}{2} \left\{ A'_{51} \left[\frac{\Gamma_3}{\Gamma_3 - 1} R_{81} + R_{21} \right] + R'_{51} \left[\frac{\Gamma_3}{\Gamma_3 - 1} A_{81} + A_{21} \right] \right. \\
& \left. + B'_{51} \left[\frac{\Gamma_3}{\Gamma_3 - 1} Q_{81} + Q_{21} \right] + Q'_{51} \left[\frac{\Gamma_3}{\Gamma_3 - 1} B_{81} + B_{21} \right] \right\}. \tag{15}
\end{aligned}$$

In the same manner, we obtain for frequency ω_- :

$$\mathcal{A}_- \mathbf{b}_- = \mathbf{d}_-, \quad \sigma = (\sigma_1 - \sigma_0)/2, \tag{16}$$

where $\mathcal{A}_-$ is identical to $\mathcal{A}_2$, except that σ is as given in equation (16). In addition,

$$\mathbf{b}_- = (A_{1-}, B_{1-}, A_{3-}, B_{3-}, A_{4-}, B_{4-}, A_{5-}, B_{5-})$$

and

$$\mathbf{d}_- = (D_{1-}, E_{1-}, D_{3-}, E_{3-}, D_{4-}, E_{4-}, D_{5-}, E_{5-}),$$

where

$$\begin{aligned}
D_{1-} = & \frac{1}{2} F_1 [-2A_{11}Q_{21} - 2A_{21}Q_{11} - 6A_{11}Q_{11} - 2A_{21}Q_{21} - 2B_{11}R_{21} - 2R_{11}B_{21} - 6B_{11}R_{11} \\
& - 2B_{21}R_{21} + \rho_{PP}(A_{31}Q_{31} + B_{31}R_{31}) + \rho_{TT}(A_{41}Q_{41} + B_{41}R_{41}) \\
& + \rho_{PT}(A_{31}Q_{41} + A_{41}Q_{31} + B_{31}R_{41} + B_{41}R_{31})], \tag{17}
\end{aligned}$$

$$\begin{aligned}
E_{1-} = & \frac{1}{2} F_1 [-2A_{11}R_{21} - 2A_{21}R_{11} - 6A_{11}R_{11} - 2A_{21}R_{21} + 2Q_{11}B_{21} + 2B_{11}Q_{21} + 6B_{11}Q_{11} \\
& + 2Q_{21}B_{21} + \rho_{PP}(A_{31}R_{31} - B_{31}Q_{31}) + \rho_{TT}(A_{41}R_{41} - B_{41}Q_{41}) \\
& + \rho_{PT}(A_{31}R_{41} + A_{41}R_{31} - B_{31}Q_{41} - B_{41}Q_{31})], \tag{18}
\end{aligned}$$

$$D_{3-} = -A_{11}Q'_{31} - Q_{11}A'_{31} - A_{11}Q_{31} - Q_{11}A_{31} + 2A_{11}Q_{11} \\ - B_{11}R'_{31} - R_{11}B'_{31} - B_{11}R_{31} - R_{11}B_{31} + 2B_{11}R_{11}, \quad (19)$$

$$E_{3-} = -A_{11}R'_{31} - R_{11}A'_{31} - A_{11}R_{31} - A_{31}R_{11} + 2A_{11}R_{11} \\ + Q_{11}B'_{31} + B_{11}Q'_{31} + Q_{11}B_{31} + B_{11}Q_{31} - 2Q_{11}B_{11}, \quad (20)$$

$$D_{4-} = -\frac{1}{2}\nabla_{\text{rad}}[2A_{61}Q_{61} + 12A_{41}Q_{41} - 4A_{41}Q_{61} - 4Q_{41}A_{61} + 16A_{11}Q_{41} + 16Q_{11}A_{41} \\ - 4A_{11}Q_{61} - 4Q_{11}A_{61} + 12A_{11}Q_{11} + 2R_{61}B_{61} + 12B_{41}R_{41} - 4B_{41}R_{61} - 4R_{41}B_{61} \\ + 16B_{11}R_{41} + 16R_{11}B_{41} - 4B_{11}R_{61} - 4R_{11}B_{61} + 12B_{11}R_{11} \\ - \kappa_{PP}(A_{31}Q_{31} + B_{31}R_{31}) - \kappa_{TT}(A_{41}Q_{41} + B_{41}R_{41}) \\ - \kappa_{PT}(A_{31}Q_{41} + A_{41}Q_{31} + B_{31}R_{41} + B_{41}R_{31})] \\ - \frac{1}{2}[A'_{41}(-Q_{61} + 3Q_{41} + 4Q_{11}) + Q'_{41}(-A_{61} + 3A_{41} + 4A_{11}) \\ - B'_{41}(R_{61} - 3R_{41} - 4R_{11}) - R'_{41}(B_{61} - 3B_{41} - 4B_{11})], \quad (21)$$

$$E_{4-} = \frac{1}{2}\nabla_{\text{rad}}[-2A_{61}R_{61} - 12A_{41}R_{41} + 4A_{41}R_{61} + 4A_{61}R_{41} - 16A_{11}R_{41} - 16A_{41}R_{11} + 4A_{11}R_{61} \\ + 4A_{61}R_{11} - 12A_{11}R_{11} + 2B_{61}Q_{61} + 12B_{41}Q_{41} - 4B_{41}Q_{61} - 4Q_{41}B_{61} + 16B_{11}Q_{41} \\ + 16Q_{11}B_{41} - 4B_{11}Q_{61} - 4Q_{11}B_{61} + 12B_{11}Q_{11} + \kappa_{PP}(A_{31}R_{31} - B_{31}Q_{31}) \\ + \kappa_{TT}(A_{41}R_{41} - B_{41}Q_{41}) + \kappa_{PT}(A_{31}R_{41} + A_{41}R_{31} - B_{31}Q_{41} - B_{41}Q_{31})] \\ + \frac{1}{2}[A'_{41}(R_{61} - 3R_{41} - 4R_{11}) - R'_{41}(-A_{61} + 3A_{41} + 4A_{11}) \\ + B'_{41}(-Q_{61} + 3Q_{41} + 4Q_{11}) - Q'_{41}(B_{61} - 3B_{41} - 4B_{11})], \quad (22)$$

$$D_{5-} = \frac{1}{2} \frac{F_2}{\rho_T \nabla_{\text{ad}}} \{ \sigma_0 [B_{21}(Q_{31} + Q_{71} - Q_{21}) - A_{21}(R_{31} + R_{71} - R_{21})] \\ + \sigma_1 [R_{21}(A_{31} + A_{71} - A_{21}) - Q_{21}(B_{31} + B_{71} - B_{21})] \\ + (\sigma_1 - \sigma_0) [\rho_{PP}(A_{31}R_{31} - B_{31}Q_{31}) + \rho_{TT}(A_{41}R_{41} - B_{41}Q_{41}) \\ + \rho_{PT}(A_{31}R_{41} + A_{41}R_{31} - B_{31}Q_{41} - B_{41}Q_{31})] \} \\ - \frac{1}{2} \left\{ A'_{51} \left[\frac{\Gamma_3}{\Gamma_3 - 1} Q_{81} + Q_{21} \right] + Q'_{51} \left[\frac{\Gamma_3}{\Gamma_3 - 1} A_{81} + A_{21} \right] \right. \\ \left. + B'_{51} \left[\frac{\Gamma_3}{\Gamma_3 - 1} R_{81} + R_{21} \right] + R'_{51} \left[\frac{\Gamma_3}{\Gamma_3 - 1} B_{81} + B_{21} \right] \right\}, \quad (23)$$

$$E_{5-} = -\frac{1}{2} \frac{F_2}{\rho_T \nabla_{\text{ad}}} \{ -\sigma_0 [B_{21}(R_{31} + R_{71} - R_{21}) + A_{21}(Q_{31} + Q_{71} - Q_{21})] \\ + \sigma_1 [Q_{21}(A_{31} + A_{71} - A_{21}) + R_{21}(B_{31} + B_{71} - B_{21})] \\ + (\sigma_1 - \sigma_0) [\rho_{PP}(A_{31}Q_{31} + B_{31}R_{31}) + \rho_{TT}(A_{41}Q_{41} + B_{41}R_{41}) \\ + \rho_{PT}(A_{31}Q_{41} + A_{41}Q_{31} + B_{31}R_{41} + B_{41}R_{31})] \} \\ - \frac{1}{2} \left\{ A'_{51} \left[\frac{\Gamma_3}{\Gamma_3 - 1} R_{81} + R_{21} \right] + R'_{51} \left[\frac{\Gamma_3}{\Gamma_3 - 1} A_{81} + A_{21} \right] \right. \\ \left. - B'_{51} \left[\frac{\Gamma_3}{\Gamma_3 - 1} Q_{81} + Q_{21} \right] - Q'_{51} \left[\frac{\Gamma_3}{\Gamma_3 - 1} B_{81} + B_{21} \right] \right\}. \quad (24)$$

The auxiliary relations which follow equation (18) in S77 also hold for frequencies ω_1 and $2\omega_1$ with the substitutions $A_{i1} \rightarrow Q_{i1}$, $B_{i1} \rightarrow R_{i1}$, $A_{i2} \rightarrow Q_{i2}$, $B_{i2} \rightarrow R_{i2}$. For frequency ω_+ , the second, third, seventh, and eighth of these relationships become, respectively,

$$\begin{aligned} A_{2+} &= \rho_P A_{3+} + \rho_T A_{4+} + \frac{1}{2}[\rho_{PP}(A_{31}Q_{31} - B_{31}R_{31}) + \rho_{TT}(A_{41}Q_{41} - B_{41}R_{41}) \\ &\quad + \rho_{PT}(A_{31}Q_{41} + A_{41}Q_{31} - B_{31}R_{41} - B_{41}R_{31})], \\ B_{2+} &= \rho_P B_{3+} + \rho_T B_{4+} + \frac{1}{2}[\rho_{PP}(A_{31}R_{31} + B_{31}Q_{31}) + \rho_{TT}(A_{41}R_{41} + B_{41}Q_{41}) \\ &\quad + \rho_{PT}(A_{31}R_{41} + A_{41}R_{31} + B_{31}Q_{41} + B_{41}Q_{31})], \\ A_{6+} &= \kappa_P A_{3+} + \kappa_T A_{4+} + \frac{1}{2}[\kappa_{PP}(A_{31}Q_{31} - B_{31}R_{31}) + \kappa_{TT}(A_{41}Q_{41} - B_{41}R_{41}) \\ &\quad + \kappa_{PT}(A_{31}Q_{41} + A_{41}Q_{31} - B_{31}R_{41} - B_{41}R_{31})], \\ B_{6+} &= \kappa_P B_{3+} + \kappa_T B_{4+} + \frac{1}{2}[\kappa_{PP}(A_{31}R_{31} + B_{31}Q_{31}) + \kappa_{TT}(A_{41}R_{41} + B_{41}Q_{41}) \\ &\quad + \kappa_{PT}(A_{31}R_{41} + A_{41}R_{31} + B_{31}Q_{41} + B_{41}Q_{31})]. \end{aligned}$$

For frequency ω_- , the same relationships are, respectively,

$$\begin{aligned} A_{2-} &= \rho_P A_{3-} + \rho_T A_{4-} + \frac{1}{2}[\rho_{PP}(A_{31}Q_{31} + B_{31}R_{31}) + \rho_{TT}(A_{41}Q_{41} + B_{41}R_{41}) \\ &\quad + \rho_{PT}(A_{31}Q_{41} + A_{41}Q_{31} + B_{31}R_{41} + B_{41}R_{31})], \\ B_{2-} &= \rho_P B_{3-} + \rho_T B_{4-} + \frac{1}{2}[\rho_{PP}(A_{31}R_{31} - B_{31}Q_{31}) + \rho_{TT}(A_{41}R_{41} - B_{41}Q_{41}) \\ &\quad + \rho_{PT}(A_{31}R_{41} + A_{41}R_{31} - B_{31}Q_{41} - B_{41}Q_{31})], \\ A_{6-} &= \kappa_P A_{3-} + \kappa_T A_{4-} + \frac{1}{2}[\kappa_{PP}(A_{31}Q_{31} + B_{31}R_{31}) + \kappa_{TT}(A_{41}Q_{41} + B_{41}R_{41}) \\ &\quad + \kappa_{PT}(A_{31}Q_{41} + A_{41}Q_{31} + B_{31}R_{41} + B_{41}R_{31})], \\ B_{6-} &= \kappa_P B_{3-} + \kappa_T B_{4-} + \frac{1}{2}[\kappa_{PP}(A_{31}R_{31} - B_{31}Q_{31}) + \kappa_{TT}(A_{41}R_{41} - B_{41}Q_{41}) \\ &\quad + \kappa_{PT}(A_{31}R_{41} + A_{41}R_{31} - B_{31}Q_{41} - B_{41}Q_{31})]. \end{aligned}$$

Boundary conditions for equations (2) and (3) are the "standard" linear boundary conditions (Iben 1971). For equation (4), the boundary conditions are given in equations (19)–(22) in S77. Equation (5) satisfies the same boundary conditions as equation (4), but with the substitutions $A_{i1} \rightarrow Q_{i1}$, $B_{i1} \rightarrow R_{i1}$, $A_{i2} \rightarrow Q_{i2}$, $B_{i2} \rightarrow R_{i2}$.

For the interaction equation (7), the boundary conditions in the interior are

$$A_{1+} = B_{1+} = A_{5+} = B_{5+} = 0. \quad (25)$$

At the surface, we have

$$A'_{3+} = B'_{3+} = 0 \quad (26)$$

and

$$A_{5+} = 2A_{1+} + 4A_{4+} + F_{A+}, \quad (27)$$

$$B_{5+} = 2B_{1+} + 4B_{4+} + F_{B+}, \quad (28)$$

where

$$F_{A+} = A_{11}Q_{11} - B_{11}R_{11} + 6(A_{41}Q_{41} - B_{41}R_{41}) + 4(A_{11}Q_{41} + Q_{11}A_{41} - B_{11}R_{41} - R_{11}B_{41}) \quad (29)$$

$$F_{B+} = A_{11}R_{11} + B_{11}Q_{11} + 6(A_{41}R_{41} + B_{41}Q_{41}) + 4(A_{11}R_{41} + R_{11}A_{41} + B_{11}Q_{41} + Q_{11}B_{41}). \quad (30)$$

Boundary conditions for equation (16) are also given by equations (25)–(28) in which the subscript plus becomes a subscript minus, and where we have

$$F_{A-} = A_{11}Q_{11} + B_{11}R_{11} + 6(A_{41}Q_{41} + B_{41}R_{41}) + 4(A_{11}Q_{41} + Q_{11}A_{41} + B_{11}R_{41} + R_{11}B_{41}), \quad (31)$$

$$F_{B-} = A_{11}R_{11} - B_{11}Q_{11} + 6(A_{41}R_{41} - B_{41}Q_{41}) + 4(A_{11}R_{41} + R_{11}A_{41} - B_{11}Q_{41} - Q_{11}B_{41}). \quad (32)$$

The six equation sets (2), (3), (4), (5), (7), and (16) may now be solved to provide an iterative solution for bimodal pulsation complete to second order. The linear sets are solved first. As in S77, we choose, for convenience, to treat the familiar linear nonadiabatic pulsation equations instead of equations (2) and (3). For all models considered here the error ($\sim \sigma_i/\sigma_r$) incurred by doing so is 1% or less. Once the linear solutions are known, the right-hand

sides of equations (4), (5), (7), and (16) may be constructed, and these equations solved for the second-order quantities. Solution techniques are as described in S77. A typical model has some 380 zones, and complete second-order solutions require about $2\frac{1}{2}$ minutes per model on the CDC 7600.

III. RESULTS OF THE CALCULATIONS

Static models were constructed as in S77, except that the analytic formula of Stellingwerf (1975) was used for the opacity. Three series of models were treated, all having $M = 3.3 M_{\odot}$. The luminosity of the principal series was $L = 1500 L_{\odot}$ ($L_{\odot} = 3.826 \times 10^{33}$ ergs s⁻¹). Models in the other two series had $L = 1200 L_{\odot}$ and $L = 1850 L_{\odot}$, respectively. In all cases, the composition was $X = 0.70$, $Z = 0.02$. We note that all these models have masses about 60% of the evolutionary mass.

For convenience, let us rewrite the time-dependent part of Equation (1) in the form

$$\frac{\delta x_j}{x_{0j}} = \alpha_j \cos(\omega_0 t + \phi_{j1}) + \beta_j \cos(2\omega_0 t + \phi_{j2}) + \zeta_j \cos(\omega_1 t + \psi_{j1})$$
$$+ \eta_j \cos(2\omega_1 t + \psi_{j2}) + \mu_{j+} \cos(\omega_+ t + \chi_{j+}) + \mu_{j-} \cos(\omega_- t + \chi_{j-}).$$

(33)

For subsequent discussion, the subscript j will stand for r , p , t , and l , so that equation (33) expresses the bimodal oscillations of radius, pressure, temperature, and luminosity, respectively.

Table 1 gives linear results for some representative models. The linear calculations were scaled to the values $\alpha_r = \zeta_r = 1.0$ at the surface of each model. In the principal series, 16 models were calculated over the range: $6100 \leq T_e \leq 5300$.

The second-order results for the principal series are presented in Figures 1–4, in which the second-order amplitudes β_j , η_j , μ_{j+} , and μ_{j-} ($j = r, p, t, l$) and phases ϕ_{j2} , ψ_{j2} , χ_{j+} , and χ_{j-} are plotted against effective temperature.

According to the iterative theory, a resonant interaction ought to make itself felt in the second-order interaction term (μ_{j+} , χ_{j+}). Let us define the resonance more precisely in terms of the quantity (Simon 1978*b*)

$$d_j \equiv (P_{1+0} - P_j)/P_{1+0},$$

which measures the normalized distance of any model from the resonant condition $\omega_1 + \omega_0 = \omega_j$. Here $P_{1+0} \equiv 2\pi/(\omega_1 + \omega_0)$ is the interaction period.

Table 1 gives values of d_3 for selected models of the principal series. At the top of Figures 1–4, the value of d_3 is indicated by arrows for models at 5800 K and 5500 K, respectively. The change in sign of d_3 over this range indicates that the resonance center ($d_3 = 0$) lies between these models. The resonance effects are seen to appear as rapid changes in μ_{r+} and χ_{r+} (Fig. 1) and μ_{p+} , χ_{p+} (Fig. 2). Changes in luminosity and temperature (Figs. 3 and 4) are much less striking, as discussed in S77.

Figure 5 shows the spatial runs of the interaction amplitudes μ_{r+} and μ_{r-} for model 3.3.550. The abscissa is zone number, which increases inward, so that zone 1 marks the model surface. The H I and He II ionization regions are indicated by vertical lines. Plotted for comparison are the linear nonadiabatic eigenfunctions for the fundamental (α_r), the first overtone (ζ_r), and the third overtone (ξ_3). Inward of the hydrogen ionization zone, it can be

TABLE 1
LINEAR NONADIABATIC PERIODS AND GROWTH RATES FOR SELECTED MODELS

Model No.	T_{eff}	P_0 (days)	$-(100\sigma_l/\sigma_r)_0$	P_1 (days)	$-(100\sigma_l/\sigma_r)_1$	d_3
$L = 1500 L_{\odot}$						
3.3.610.....	6100	4.50	0.19	3.24	0.68	-0.049
3.3.580.....	5800	5.40	0.57	3.85	1.13	-0.012
3.3.555.....	5550	6.35	0.88	4.48	1.25	0.020
3.3.550.....	5500	6.57	0.92	4.62	1.23	0.027
3.3.530.....	5300	7.57	1.00	5.24	1.08	0.055
$L = 1200 L_{\odot}$						
B3.3.580....	5800	4.45	0.43	3.20	1.02	
B3.3.550....	5500	5.40	0.64	3.84	1.07	
B3.3.520....	5200	6.67	0.67	4.65	0.82	
$L = 1850 L_{\odot}$						
A3.3.620....	6200	5.09	0.03	3.64	0.32	
A3.3.580....	5800	6.47	0.73	4.57	1.15	
A3.3.550....	5500	7.91	1.25	5.49	1.36

FIG. 1.—Amplitudes and phases of the second-order corrections to the linear radius perturbations. Notation follows eq. (33). The plotted quantities have been evaluated near optical depth $\tau = 0.2$. Arrows at the top indicate values of d_3 at 5800 and 5500 K.

FIG. 2.—Same as Fig. 1, except that pressure corrections are plotted

FIG. 3.—Same as Fig. 1, except that temperature corrections are plotted. These corrections were evaluated at the photosphere.

FIG. 4.—Same as Fig. 3, except that luminosity corrections are plotted

FIG. 5.—Spatial runs of a number of radius pulsation amplitudes in the model 3.3.550 (5500 K): the linear amplitudes α_r (fundamental), ζ_r (first overtone), and ξ_3 (third overtone); and the second-order interactive corrections μ_{r+} and μ_{r-} . Zone number increases inward. The H I and He II ionization zones are indicated. Note the similarity of μ_{r+} and ξ_3 inward of zone 260.

seen that the behavior of μ_{r+} is strikingly like that of ξ_3 . On the other hand, the nonresonant interaction amplitude μ_{r-} bears no resemblance to any of the linear eigenfunctions.

Further examination of Figure 5 shows that in the surface regions of the model (H I region and outward), nonadiabatic effects are very large for both μ_{r+} and μ_{r-} . In the hydrogen ionization zone itself, the amplitudes are seen to peak very sharply; this behavior is reminiscent of that of the linear nonadiabatic temperature perturbations. Because the interactive second-order corrections μ_{r+} and μ_{r-} show by far their greatest strength in the H I region, this zone, in spite of its limited extent in mass, deserves serious consideration as the possible seat of modal selection. This is especially true for those models in which hydrogen provides a substantial fraction of the linear driving.

As one moves away from the resonant region, the spatial run of μ_{r+} begins to depart from a trinodal form. Figure 6 illustrates this for models 3.3.580 and 3.3.530. In the former case one notices the appearance of a very weak fourth node, a circumstance anticipating the approach to the resonance $\omega_1 + \omega_0 = \omega_4$, which would appear in a model of still higher temperature. Conversely, for model 3.3.530, we see that the outer node is beginning to disappear. Presumably, models at yet cooler temperatures would begin to show a binodal μ_{r+} under the growing influence of the resonance $\omega_1 + \omega_0 = \omega_2$. It is important to note here that no such changes occur in the nonresonant amplitude μ_{r-} , the nodal structure of which remains remarkably steady over a wide range of model temperatures.

Returning to Figure 1, we now remark the extremely sharp rise of the fundamental harmonic amplitude, β_r , in the vicinity of 5300 K. This behavior signals the presence of the harmonic resonance $P_2/P_0 = 0.5$ (S77). In fact, the nonadiabatic period ratio P_2/P_0 for the model 3.3.530 is 0.498. The sharp change of β_r in this region is accompanied by similar behavior in the associated phase, ϕ_{r2} (Fig. 1).

Figure 7 shows the spatial run of β_r as well as that of the first overtone harmonic amplitude η_r , and the linear nonadiabatic eigenfunction of the second overtone ξ_2 . Inward of the H I region β_r shows a strong structural similarity to ξ_2 , the eigenfunction of the linear mode to which it is resonantly linked. This is the analog of the resonant behavior of μ_{r+} shown in Figure 5. One notices by contrast from Figure 7 that the nonresonant η_r is much weaker in amplitude and poorly defined in nodal structure. Strongly nonadiabatic behavior for both β_r and η_r may be marked once again in the hydrogen ionization zone. Thus it is possible that this region may be the

FIG. 6.—Second-order interaction amplitude μ_{r+} versus zone number for models at 5800 and 5300 K. Note that in both cases the trinodal structure is beginning to erode.

seat of important effects involving the onset of single-mode limiting amplitude, in addition to its potential role, noted above, in modal selection.

IV. DISCUSSION

In the present investigation, we have extended the second-order iterative theory to the bimodal case and employed it to treat a number of models in the vicinity of the interaction resonance $\omega_1 + \omega_0 = \omega_3$, and the harmonic

FIG. 7.—Spatial runs of the second-order harmonic correction amplitudes β_r (fundamental) and η_r (first overtone) and the linear eigenfunction ξ_2 (second overtone) in the model 33.530 (5300 K). Note the similarity of β_r and ξ_2 inward of zone 260.

resonance $P_2/P_0 = 0.5$. In each case, we have shown that the resonant second-order amplitude and phase display abrupt changes in behavior as the resonance center is approached. Furthermore, it has been demonstrated that, inward of the H I ionization zone, the resonant amplitude strongly resembles the linear eigenfunction to which it is linked. Thus the resonances are shown to have spatial content over and above their temporal definition in terms of commensurate periods.

Simon (1978b) has suggested that the second-order amplitudes and phases may hold important clues to the mechanics of modal selection. If this is the case, then three of the regions encompassed by the principal series of models above (see Fig. 1) seem especially worthy of extended treatment. In terms of effective temperature, these are, approximately: (1) 5800 K, where μ_{r+} and β_r are both at or near relative minimum; (2) 5600 K, where β_r is still at minimum, but μ_{r+} has reached resonant maximum; and (3) 5300 K, where μ_{r+} is again minimal, but β_r has now risen to maximum. These regions will be the focus of further investigation employing a fully nonlinear code. We shall report the results of this treatment elsewhere.

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