7-2006

Taylor Polynomials

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Expository Paper

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In partial fulfillment of the requirements for the Master of Arts Teaching with a Specialization in the Teaching of Middle Level Mathematics in the Department of Mathematics.
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July 2006

Before the age of calculators, studying functions such as $\sin x$, $\cos x$, e^x, and $\ln x$ was quite time consuming. The graphs of these functions are important when studying their characteristics. James Gregory, a Scottish mathematician in the 17th century, made an important discovery about these functions. Using calculus, he wrote a series of terms to approximate very closely the graph of the curve. His main focus was with the function $\ln x$; he was able to calculate any positive value of *x* using a polynomial series. Brook Taylor, an English mathematician, generalized the Maclaurin series, devised by Colin Maclaurin. However, Gregory had actually known about them long before Taylor came into the picture. Taylor invented the method for expanding functions in terms of polynomials about an arbitrary point known as Taylor Series, which he published in 1715. Computing values of polynomials is much easier and less time consuming than evaluating a function like $\sin x$. In this paper, I will look at the background needed before one can truly understand polynomials, the definition of Taylor polynomials, and how to use Taylor polynomials to approximate the functions I mentioned above.

Polynomials

To work with Taylor polynomials, one needs to be able to work comfortably with polynomial functions and their properties. In class, we have discussed linear functions like $f(x) = 2x - 1$ and quadratic functions like $g(x) = x^2 - 3x + 5$. Linear and quadratic functions belong to a family of functions known as polynomial functions.

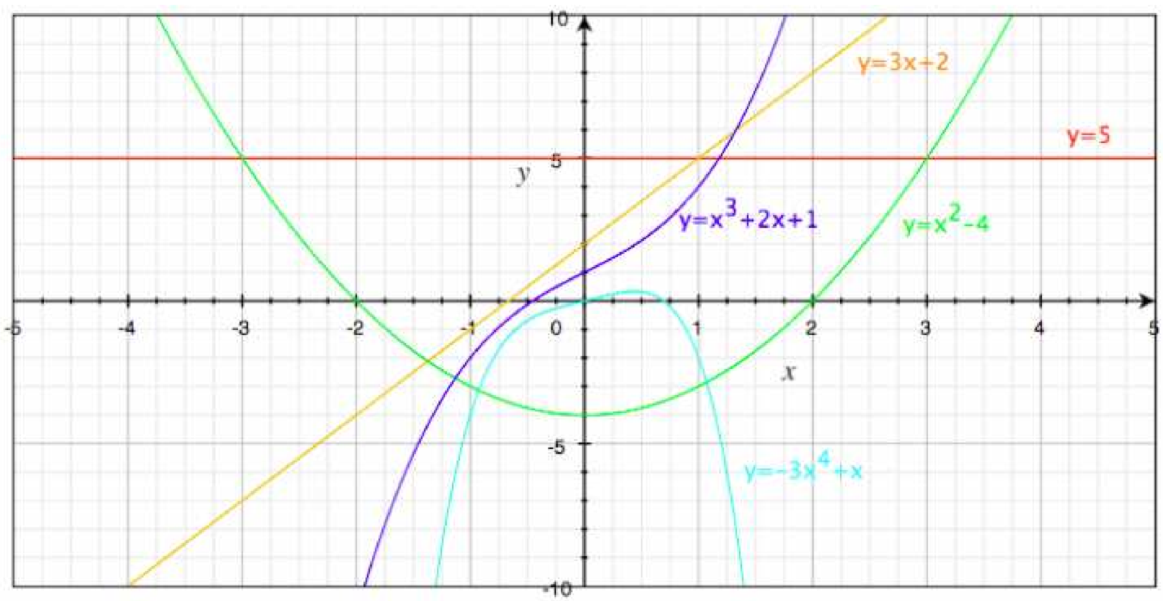
A polynomial is an expression that can be written in the form

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + ... + a_2 x^2 + a_1 x + a_0$$

where *n* is a nonnegative integer. The expressions $a_n x^n$, $a_{n-1}x^{n-1}$, $a_{n-2}x^{n-2}$, …, $a_2 x^2$, $a_1 x$, and a_0 are called terms of the polynomial, and the numbers a_n, a_{n-1}, a_{n-2}, …, a_2, a_1, and a_0

are called the coefficients of the polynomial. Although the terms of a polynomial can be written in any order, we usually write them in descending powers of *x*. The term containing the highest power of *x* is called the leading term. The coefficient of the leading term is called the leading coefficient, and the power of *x* contained in the leading term is called the degree of the polynomial. The polynomial 0 has no degree. Polynomials of the first few degrees have special names, as indicated in the chart below.

Degree	*Name*	*Example*
0	constant	5
1	linear	$3x+2$
2	quadratic	x^2-4
3	cubic	x^3+2x+1
4	quartic	$-3x^4+x$
5	quintic	$x^5+\pi x^4-3.1x^3+11$



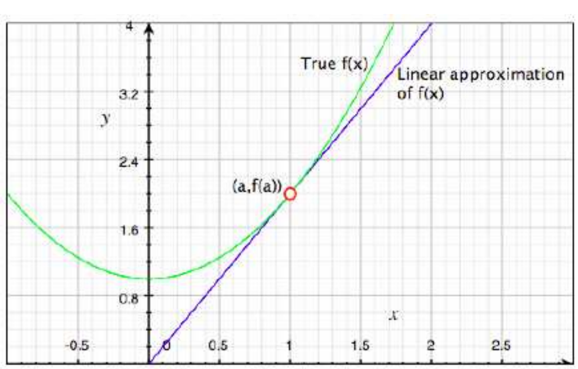
To identify a particular term of a polynomial, we use the name associated with the power of *x* contained in the term. For example, the polynomial x^2-4 has a quadratic term of x^2, no linear term and a constant term of -4. Note that the leading coefficient of x^2-4 is understood to be 1. Every polynomial defines a function, often called *P*. Any value of *x* for which $P(x)=0$ is a root of the equation and a zero of the function.

Linear Approximations

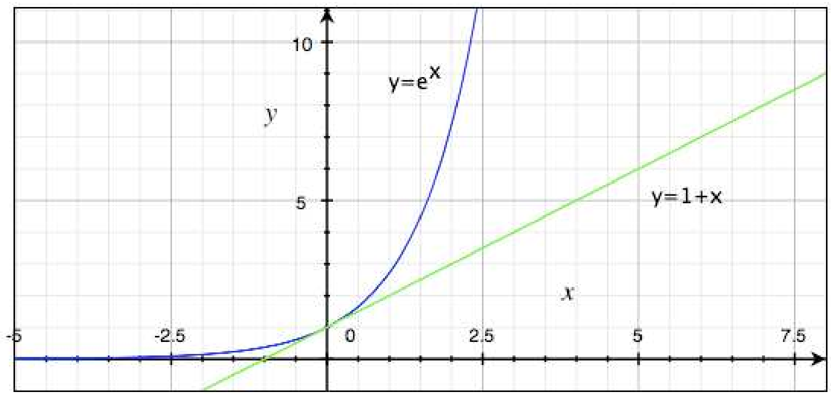
In the class, Concepts of Calculus for Middle Level Teachers, we looked at how to approximate a function using a degree 1 polynomial, namely the tangent line approximation. If any function with a derivative at a point $x = a$ is looked at on a small enough interval close to *a*, the function's graph will resemble a line. We can then use the tangent line to estimate values fairly close to *a*. In class, we used the point-slope form of a line to get the tangent line equation. This equation is $y-y_1=m(x-x_1)$, where *m* is the slope of the tangent line and (x_1,y_1) is a point on the line where the line is tangent to the function. We want to focus on the specific point where $x=a$. This means that our point of tangency is $(a, f(a))$. In calculus, another name for the slope of the tangent line at a point is the derivative at that point. This means that $m = f'(a)$. If we substitute all these values in to our point slope form of our tangent line, we get:

$$y-f(a)=f'(a)(x-a) \qquad \text{or} \qquad P_1(x)=(a)+f'(a)(x-a)$$

The tangent line is the best linear approximation to the function near $x=a$. The tangent line and the curve share the same slope at $x=a$, in other words, they have the same first derivative. This approximation is called a Taylor polynomial of degree 1, noted $P_1(x)$. The diagram below shows that it is a good approximation to the function, but only near the point *a*.



Let's look at a function such as $f(x) = e^x$ near 0. To use the Taylor polynomial of degree 1, we need to know the value of the function and the derivative of the function at 0. The value of the function at 0 is $f(0) = e^0 = 1$. The derivative is $f'(x) = e^x$, so the derivative at 0 is, $f'(0) = e^0 = 1$. We can now plug these values into the formula to get a Taylor polynomial of degree 1 at the point 0. Since the formula when $a = 0$ is $P_1(x) = f(0) + f'(0)x$, after substituting, we get: $P_1(x) = 1 + 1x$. The graph below shows the original function and its linear approximation.



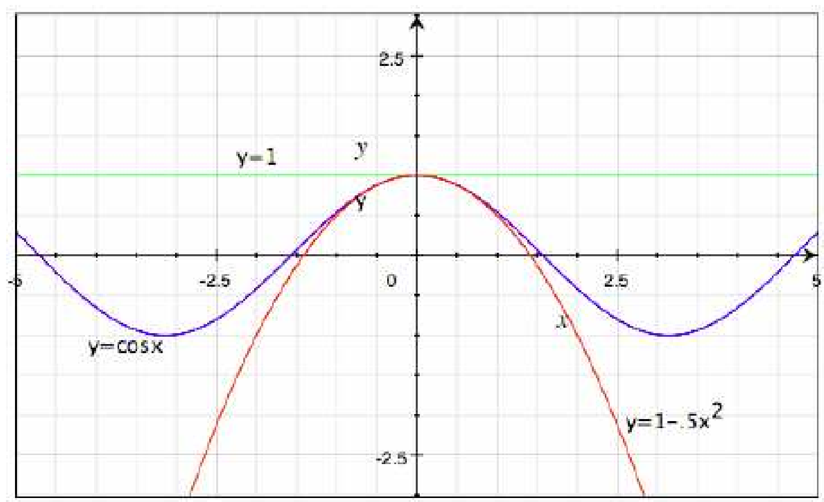
Once again, the tangent line, or Taylor polynomial of degree 1, is a decent approximation close to 0. Once you get too far from 0 though, the error gets worse. Since we used the first derivative to obtain a linear approximation, can we use the second derivative to obtain a better approximation to the original function?

Quadratic Approximations

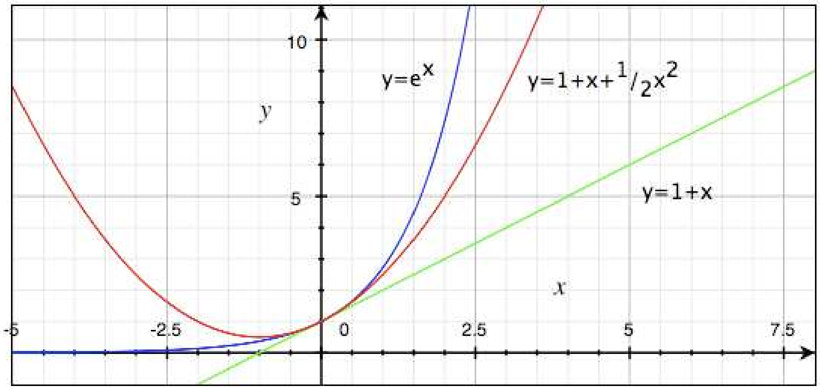
If we want a more accurate approximation to the function, we can use a quadratic function, which not only has the same slope, but also bends in the same way as the original curve. In other words, can we find a function that not only has the same first derivative, but also the same second derivative as the function at and near the value x=a? The quadratic approximation of the function near *a* will have the form:

$$P_2(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 \qquad \text{(Appendix A)}$$

For example, take $\cos x$ as our function and let $a = 0$. So, we will be using a Taylor polynomial of degree 2 to approximate $\cos x$ near 0. To do this, we need to know the value of the function, first derivative and second derivative at 0. The value of the function at 0 is 1. The first derivative of $\cos x$ is $-\sin x$. So the value of the first derivative at 0 is 0. The second derivative of the function $\cos x$ is $-\cos x$. So the value of the second derivative at 0 is -1. This gives us the quadratic formula to approximate $\cos x$ to be $P_2(x) = 1 + 0(x-0) + \frac{1}{2}(x-0)^2 = 1 - \frac{1}{2}x^2$. As the picture indicates, the quadratic approximation is much more accurate that the linear approximation near 0.



Is this is always the case? Does the Taylor polynomial of degree 2 give a good approximation of the function near *a*? Let's go back to our example where $f(x) = e^x$. We already know our Taylor polynomial of degree 1 is $f(x) = 1 + 1x$ near the point 0, so if we add on the quadratic term, we need to know the function's second derivative. The second derivative of $f(x) = e^x$ is $f''(x) = e^x$, and $f''(0) = e^0 = 1$. So the Taylor polynomial of degree 2 near 0 is $P_2(x) = 1 + 1x + \frac{1}{2}x^2$. The graph below, along with the original function and the first two Taylor polynomials shows that the quadratic approximation is better for points farther from *a* than the linear approximation.



Definition of Taylor Polynomials

We approximate a function $f(x)$ near the point $x = a$ by polynomials. To get more accuracy, we take higher-degree polynomials. The approximation is usually good near the point $x = a$, but often not so good farther from that point. Using Taylor Polynomials, a function can be approximated as closely as desired by polynomial provided the function possesses a sufficient number of derivatives at the point in question. Starting with a generic polynomial and deriving the formula until all coefficients are established will find the formula. (Appendix A)

The definition of Taylor polynomials states: Let the function f and its first n derivatives exist on the closed interval $[x_0, x_1]$. Then, for $a \in (x_0, x_1)$ and $x \in (x_0, x_1)$ the *n*th-degree Taylor Polynomial *f* at *a* is the *n*th-degree polynomial $P(x)$, given by

$$P_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \frac{f^4(a)}{4!}(x-a)^4 + \ldots + \frac{f^n(a)}{n!}(x-a)^n$$

or $P_n(x) = \sum_{i=0}^{n} \frac{f^i(a)}{i!}(x-a)^i$

Maclaurin Polynomials

Often, it is useful to focus on the Taylor polynomials that approximate a function near 0. This is known as a Maclaurin polynomial. The Maclaurin polynomials are named after the Scottish mathematician, Collin Maclaurin. To get the formula for a Maclaurin polynomial, all one needs to do is substitute $a = 0$ into the above Taylor polynomial equation.

$$P_n(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^4(0)}{4!}x^4 + \ldots + \frac{f^n(0)}{n!}x^n$$

For the rest of the paper, I would like to focus on the Maclaurin Polynomials, or Taylor polynomials about 0. To be specific, let's focus on just one function and see the results of taking higher order Taylor polynomials to approximate the function $f(x) = e^x$ about 0.

$f(2) = e^2 = 7.389$

Degree 1: (linear) $P_1(x) = 1 + x$ — $P_1(2) = 1 + 2 = 3$

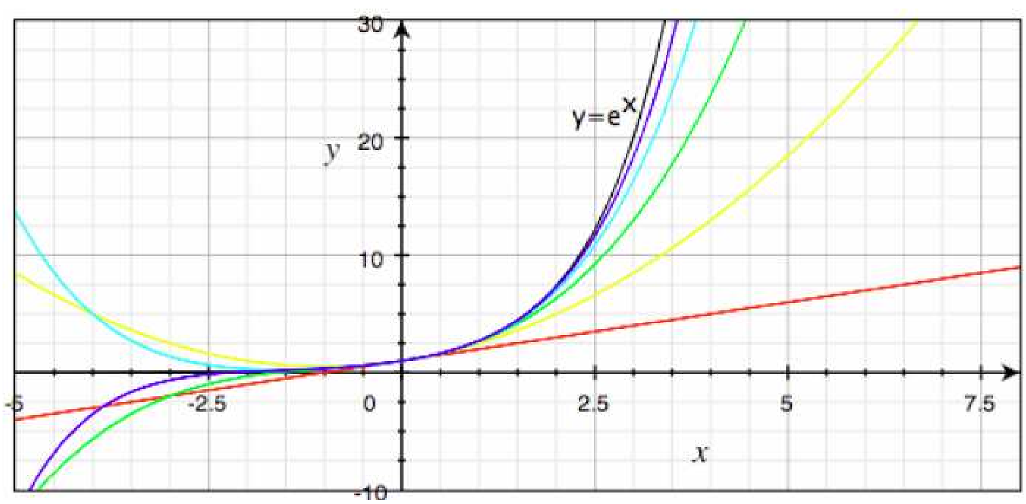
Degree 2: (Quadratic) $P_2(x) = 1 + x + \frac{1}{2}x^2$ — $P_2(2) = 1 + 2 + \frac{1}{2}2^2 = 5$

Degree 3: (Cubic) $P_3(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3$ — $P_3(2) = 1 + 2 + \frac{1}{2}2^2 + \frac{1}{6}2^3 = 6\frac{1}{3}$

Degree 4: (Quartic) $P_4(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4$ — $P_4(2) = 1 + 2 + \frac{1}{2}2^2 + \frac{1}{6}2^3 + \frac{1}{24}2^4 = 7$

Degree 5: (Quintic) $P_5(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{120}x^5$

$P_5(2) = 1 + 2 + \frac{1}{2}2^2 + \frac{1}{6}2^3 + \frac{1}{24}2^4 + \frac{1}{120}2^5 = 7.2\overline{6}$



Error Analysis

It is important to discuss the error involved when using a Taylor polynomial. The polynomial is just a good approximation for the function near *a*, but how accurate is it? If we define the error to be $E_m(x)$, then we could write that $f(x) = P_n(x) + E_m(x)$, this means that $E_m(x) = f(x) - P_n(x)$. Beyond this point, the error function gets quite complicated. The formula for the error is:

$$E_m(x) = f(x) - P_n(x) = \frac{1}{(n+1)!}(x-a)^{n+1} f^{n+1}(c_x) = \frac{1}{n!}\int_a^x (x-t)^n f^{(n+1)}(t)dt$$

The point c_x is restricted to the interval bounded by *x* and *a*, and otherwise c_x is unknown. It is important that the function can be derived *n* + 1 times. When discussing Taylor polynomials, it is important to understand that there will always be a small error. The error gets larger as the value gets farther from *a*. By taking higher order polynomials, our approximation gets closer to the actual function on the interval.

Taylor Series

A Taylor series is an infinite series created just like the Taylor Polynomials. The series has infinitely many terms rather than stopping at the n^{th} term like the polynomials. By definition, the Taylor polynomial is just a truncated series. When the series approaches infinity, it is no longer considered an approximation, but instead equal to the actual function if $\lim_{n\to\infty} E_m(x) = 0$. The series is written:

$$P(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \ldots + \frac{f^n(a)}{n!}(x-a)^n + \ldots$$

or

$$P(x) = \sum_{i=0}^{\infty} \frac{f^i(a)}{i!}(x-a)^i$$

For the Taylor series to exist at a point *x*, the error, $E_m(x)$, must approach zero as the number of terms approaches infinity. So, when discussing Taylor series, it is important to establish that the error term approaches zero as *m* approaches infinity. In other words, it becomes negligible for large *m*.

The Taylor Series for $\sin x$, $\cos x$, and e^x:

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \dots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \dots$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$$

Without the use of technology, finding $\sin(2)$ can be quite difficult and time consuming. However, having the knowledge of Taylor polynomials and Taylor series is like having a new technology, one that can be used without electricity or batteries. With a knowledge polynomial functions and some basic calculus, one can approximate functions very accurately. What was once nearly impossible is made easy by Taylor polynomials.

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Derivation of the Formula for Taylor Polynomials

Let's derive this formula to understand where it comes from. I will begin with a generic formula of an *n*th-degree polynomial.

$$f(x) = a_0 + a_1(x-a) + a_2(x-a)^2 + a_3(x-a)^3 + ... + a_n(x-a)^n$$

Find the value of the function at the point *a*:

$$f(a) = a_0 + a_1(a-a) + a_2(a-a)^2 + a_3(a-a)^3 + ... + a_n(a-a)^n$$

Since we know that $(a-a)=0$, this leads us to:

$$f(a) = a_0 + a_1(0) + a_2(0)^2 + a_3(0)^3 + ... + a_n(0)^n$$

This tells me that $f(a) = a_0$. I can substitute this value into my equation, giving me:

$$f(x) = f(a) + a_1(x-a) + a_2(x-a)^2 + a_3(x-a)^3 + ... + a_n(x-a)^n$$

Now, take the derivative of the function:

$$f'(x) = 0 + a_1 + 2a_2(x-a)^1 + 3a_3(x-a)^2 + ... + na_n(x-a)^{n-1}$$

Find the value of the derivative at point *a*.

$$f'(a) = 0 + a_1 + 2a_2(a-a)^1 + 3a_3(a-a)^2 + ... + na_n(a-a)^{n-1}$$

Once again, $(a-a)=0$ so,

$$f'(x) = 0 + a_1 + 2a_2(0)^1 + 3a_3(0)^2 + ... + na_n(0)^{n-1}$$

This tells me that $f'(x) = a_1$. I can substitute this value into my equation, giving me:

$$f(x) = f(a) + f'(a)(x-a) + a_2(x-a)^2 + a_3(x-a)^3 + ... + a_n(x-a)^n$$

This gives the linear approximation: $P_1(x) = f(a) + f'(a)(x-a)$

Now, take the second derivative of the function:

$$f''(x) = 0 + 2a_2 + 3\cdot 2a_3(x-a)^1 + ... + n\cdot(n-1)a_n(x-a)^{n-2}$$

Find the value of the second derivative at the point *a*.

$$f''(a) = 0 + 2a_2 + 3\cdot 2a_3(a-a)^1 + ... + n\cdot(n-1)a_n(a-a)^{n-2}$$

Once again, $(a-a)=0$, this gives:

$$f''(a) = 0 + 2a_2 + 3\cdot 2a_3(0)^1 + ... + n\cdot(n-1)a_n(0)^{n-2}$$

This tells me that $f''(a) = 2a_2$, or $\frac{f''(a)}{2} = a_2$. Substitute back into the original function to get:

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + a_3(x-a)^3 + ... + a_n(x-a)^n$$

This gives the quadratic approximation: $P_2(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2$

Take the third derivative of the function:

$$f'''(x) = 0 + 3\cdot 2a_3 + 4\cdot 3\cdot 2a_4(x-a)^1 + ... + n\cdot(n-1)\cdot(n-2)a_n(x-a)^{n-3}$$

Find the value of the third derivative at the point *a*.

$$f'''(a) = 0 + 3\cdot 2a_3 + 4\cdot 3\cdot 2a_4(a-a)^1 + ... + n\cdot(n-1)\cdot(n-2)a_n(a-a)^{n-3}$$

Once again, $(a-a) = 0$, this gives:

$$f'''(a) = +0 + 3\cdot 2a_3 + ... + n\cdot(n-1)\cdot(n-2)a_n(0)^{n-3}$$

This tells me that $f'''(a) = 3\cdot 2a_3$, or $\frac{f'''(a)}{3\cdot 2} = a_3$. Substitute back into the original function:

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \frac{f'''(a)}{3\cdot 2}(x-a)^3 + ... + a_n(x-a)^n$$

This pattern can be followed until the n^th^ derivative to give the final formula:

$$P_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \frac{f^4(a)}{4!}(x-a)^4 + ... + \frac{f^n(a)}{n!}(x-a)^n$$