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STOCHASTIC OPTIMAL CONTROL IN NONLINEAR SYSTEMS

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STOCHASTIC OPTIMAL CONTROL IN NONLINEAR SYSTEMS

by

Celestin Nkundineza

A THESIS

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STOCHASTIC OPTIMAL CONTROL IN NONLINEAR SYSTEMS

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Stochastic control is an important area of research in engineering systems that undergo disturbances. Controlling individual states in such systems is critical. The present investigation is concerned with the application of the stochastic optimal control strategy developed by To (2010) and its implementation as well as providing computed results of linear and nonlinear systems under stationary and nonstationary random excitations. In the strategy the feedback matrix is designed based on the achievement of the objectives for individual states in the system through the application of the Lyapunov equation for the system. Each diagonal element in the gain or associated gain matrix is related to the corresponding states. The strategy is applied to four dynamic engineering systems that are divided into two categories. One category includes two linear systems each of which has two degrees of freedom. The other category encompasses two nonlinear two-degree-of-freedom systems.

Computed results were provided for the optimal control of stationary and nonstationary random displacements. These results were obtained by employing the computer software, MATLAB and were represented graphically. The computed
results include the time-dependent elements of the associated gain matrix. Three-dimensional graphical representations of the controlled mean squares of displacements against the two elements of the feedback gain matrix were included. The latter three-dimensional presentations are important for the design engineer who needs to choose elements of the gain matrix in order to achieve a specific objective in certain states of the system.
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CHAPTER 1  INTRODUCTION

1.1. Motivation of Investigation

Stochastic control has been an important area of research due to its wide applicability in many areas of engineering of systems that undergo wind forces, earthquake excitations, blast shocks and motions on uneven ground profiles and any other random disturbances or forcing uncertainties. Stochastic control has been extensively used in spacecrafts systems [1.1, 1.2], chemical and industrial processes [1.3, 1.4], plate and shell structures [1.5], energy systems, economic and social systems. As pointed out in [1.5-1.10], in existing stochastic optimal control strategies, the linear quadratic Gaussian (LQG) technique minimize the overall system performance index. But there is no guarantee that the individual state performance objective will be achieved. In addition, insufficient information have been obtained about the transients of the individual states. With the LQG technique, the system can have a totally unacceptable transient behavior. The state covariance assignment (SCA) method introduced by Skelton and associates [1.6-1.8] achieves the individual state behavior but is limited to linear systems under stationary random excitations. Recently, To [1.11] has extended the approach in [1.5] to linear and nonlinear multi-degree of freedom (mdof) systems under nonstationary random excitations. However, no computed result was provided in [1.11].

1.2. Primary Objectives of Studies

The primary objectives of the studies reported in the present thesis are two folds. The first primary objective is the application of the stochastic optimal control strategy [1.11] for linear and nonlinear mdof systems under stationary and nonstationary random excitations. The second primary objective is the implementation of the strategy and provision of computed results for four linear and nonlinear mdof systems that are subjected to stationary and nonstationary random
excitations. With this regards various digital computer programs necessary for the computations of related quantities for the optimal stochastic control of mdof linear and nonlinear systems will have to be developed.

1.3. Literature in Control of Systems Under Nonstationary Random Excitations

The present survey of existing literature on optimal control of systems under nonstationary random excitations is focused on computational techniques that can be applied to mdof systems. Sub-section 1.3.1 is concerned with mdof linear systems while mdof nonlinear systems are presented in Sub-section 1.3.2.

1.3.1. Stochastic optimal control of linear systems

The early challenging work in this class of linear systems under nonstationary excitations was on determining the responses of such system with time varying parameters. In [1.12] Wan developed an analytical approach that provides the responses of the equation of the flapping rotor blade advancing in atmospheric turbulence. In this problem a perturbation series solution in stiffness parameter were used to find the stationary random response to a white noise and to an exponentially correlated excitation. In [1.13] Otsuki and Yoshida proposed a synthesis method of nonstationary robust controller for time-varying system. They considered various uncertainties and applied it to controlling a wire of changing length. The robustness lies in the fact that the object parameters undergo the variations.
1.3.2. Stochastic optimal control of nonlinear systems

As mentioned in [1.11] optimal stochastic control strategies for mdof nonlinear systems under nonstationary random excitations are very limited. This may have to do with the fact that general solution of the mdof nonlinear system under nonstationary random is difficult. This may be one of the reasons why the statistical or equivalent linearization (SL) technique [1.14] is frequently employed in such an investigation. In [1.15], for example, Yang, Li and Vongchavalikul have employed this technique together with the LQG for the control of hysteretic structures excited by earthquake excitation treated as nonstationary random excitation. More recently, Pang and Wong [1.16] reported work on a similar problem. Instead of using ground acceleration as input they used ground velocity as input.

1.4. Organization of Thesis

This thesis is organized in the following manner:

Chapter 1 is an introduction on motivation, literature survey, and organization of the thesis.

Chapter 2 gives an outline of the theoretical development of the method on optimal stochastic control of linear and nonlinear multi-degree of freedom systems under nonstationary random excitations and algorithms that are employed for the computations of mean squares of responses.

Chapter 3 deals with determination of mean squares of responses of two two-degree of freedom linear systems under nonstationary excitations. These computed results included those with and without the application of the optimal stochastic control strategy. Here the two systems studied. The first system studied deals with stochastic optimal control of building and secondary equipment under earthquake nonstationary excitation. The second system studied is concerned with the
stochastic optimal control of elastic structure under earthquake nonstationary random excitation.

Chapter 4 deals with the mean squares of responses of two two-dof nonlinear systems obtained by the approach introduced in Chapter 2. The first system is that studied in Chapter 3 except that nonlinearity is included. The second system studied is a vehicle travelling at a constant velocity on a rough road. The random excitations in this case are therefore stationary.

Summary, conclusion and recommendation for future research are included in Chapter 5.
CHAPTER 2  THEORETICAL DEVELOPMENT

2.1 Systems

In the engineering world the word system refers to an entity made of interconnected machine parts which accepts inputs and gives outputs [2.1]. The inputs and/or outputs can be deterministic or random. When the inputs and outputs to the systems have statistical properties such as mean, variance, and covariance that are independent of time they are said to be stationary random. While if the statistical properties are varying with time they are said to be nonstationary random [2.2].

2.2 Eigenvalues and Natural Frequencies of a System

Consider a two degree of freedom (dof) linear system described by the second order differential matrix equations as follows:

\[ \mathbf{M}\ddot{\mathbf{X}} + \mathbf{C}\dot{\mathbf{X}} + \mathbf{K}\mathbf{X} = \mathbf{F} \]  \hspace{1cm} (2.1)

where \( \mathbf{M} = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \) is the mass matrix, \( \mathbf{C} = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} \) is the damping matrix, \( \mathbf{K} = \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} \) is the stiffness matrix, \( \mathbf{X} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \) is the displacement vector, and \( \mathbf{F} = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \) is the externally applied force vector.

The calculation of eigenvalues of this system implies finding out the natural frequencies of the linear system described by these differential equations. To find the natural frequencies of the linear system, the terms that involve the dampings and the excitations can be disregarded. Thus, the equation becomes

\[ \mathbf{M}\ddot{\mathbf{X}} + \mathbf{K}\mathbf{X} = 0. \]  \hspace{1cm} (2.2)
For free vibration, one can assume \( \mathbf{X} = \mathbf{X}_0 \sin(\omega t) \) so that upon substituting this displacement vector and its second derivative with respect to time \( t \) into Equation (2.2) and rearranging we get:

\[
-\omega^2 \mathbf{M} \mathbf{X}_0 + \mathbf{K} \mathbf{X}_0 = \mathbf{0}, \quad \text{or} \quad \begin{bmatrix} k_{11} - \omega^2 m_1 & k_{12} \\ k_{21} & k_{22} - \omega^2 m_2 \end{bmatrix} \begin{bmatrix} x_{o1} \\ x_{o2} \end{bmatrix} = \mathbf{0} \quad (2.3)
\]

Then we solve for the natural frequency \( \omega_n \) by equating to zero the determinant of the coefficient matrix in equation (2.3). That is,

\[
\begin{vmatrix} k_{11} - \omega^2 m_1 & k_{12} \\ k_{21} & k_{22} - \omega^2 m_2 \end{vmatrix} = 0 . \quad (2.4)
\]

The roots of the characteristic Equation (2.4) are the squares of the natural frequencies of the system.

Now consider again Equation (2.1) and assign the state variables to the system as follows:

\[
\mathbf{Z}_1 = \mathbf{X}, \quad \mathbf{Z}_2 = \dot{\mathbf{X}}. \quad (2.5a,b)
\]

Therefore, one can write the state equation as

\[
\begin{align*}
\dot{\mathbf{Z}}_1 &= \mathbf{Z}_2 \\
\dot{\mathbf{Z}}_2 &= -\mathbf{M}^{-1} \mathbf{K} \mathbf{Z}_1 - \mathbf{M}^{-1} \mathbf{C} \mathbf{Z}_2 + \mathbf{M}^{-1} \mathbf{F} 
\end{align*} \quad (2.6a,b)
\]

which can be expressed in a more compact form as:

\[
\dot{\mathbf{Z}} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{M}^{-1} \mathbf{K} & -\mathbf{M}^{-1} \mathbf{C} \end{bmatrix} \mathbf{Z} + \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{M}^{-1} \end{bmatrix} \mathbf{P} \quad (2.7)
\]

or simply as

\[
\dot{\mathbf{Z}} = \mathbf{A} \mathbf{Z} + \mathbf{D} \mathbf{P} \quad (2.8)
\]

where \( \mathbf{A} \) is a 4×4 matrix and is called the amplification or coefficient matrix, \( \mathbf{D} \) is a 4 × 4 matrix and is called the input matrix, \( \mathbf{Z} \) is a 4×1 matrix and \( \mathbf{P} \) is a 4 × 1 matrix.
Taking the Laplace transform of the Equation (2.8) and assuming the system starts from rest, one gets

\[ sZ(s) = AZ(s) + DP(s), \]  
\[ (sI - A)Z(s) = DP(s), \]  
\[ Z(s) = \frac{adj(sI - A)DP(s)}{\det(sI - A)}. \]

Setting the denominator on the right-hand side (RHS) of Equation (2.10) one has the characteristic equation of the system. The roots of the characteristic Equation are the eigenvalues of matrix \( A \). In other words, the eigenvalues of the amplification matrix of a system represented in state variables are the poles of the system.

From the theory of classical control it is known that for a system to be stable the poles should have their real parts negative. Therefore, in the state variable representation one may conclude that for a system to be stable the eigenvalues of the amplification matrix must have negative real parts.

For the two dof system, it has 4 poles namely \( s_1, s_2, s_3 \) and \( s_4 \). Explicitly, they are given by:

\[ s_1 = -\omega_{n1}\zeta_1 + j\omega_{n1}\sqrt{1 - \zeta_1^2}, \]  
\[ s_2 = -\omega_{n1}\zeta_1 - j\omega_{n1}\sqrt{1 - \zeta_1^2}, \]  
\[ s_3 = -\omega_{n2}\zeta_2 + j\omega_{n2}\sqrt{1 - \zeta_2^2}, \]  
\[ s_4 = -\omega_{n2}\zeta_2 - j\omega_{n2}\sqrt{1 - \zeta_2^2}, \]

where \( \omega_{n1} \) and \( \zeta_1 \) are, respectively, the \( i^{th} \) natural frequencies and damping ratios of the system. For the system without damping, \( \zeta = 0 \) Equation (2.11) gives

\[ s_1 = i\omega_{n1}, \quad s_2 = -i\omega_{n1}, \quad s_3 = i\omega_{n2}, \quad s_4 = -i\omega_{n2}. \]
In conclusion, the imaginary parts of the eigenvalues are related to the natural frequencies of the system.

2.3 Stochastic Optimal Control of Multi-degrees of Freedom Systems

Owing to its many advantages, the state covariance assignment (SCA) technique for control of stationary random responses introduced by Skelton and associates [2.3, 2.4, 2.5] has been applied and extended by To and Chen [2.6, 2.7] to the control of random responses of laminated composite plate and shell structures with piezoelectric components. The approach consists of designing the feedback control gain matrix such that the system achieves the objectives of individual assigned state covariance.

In what follows the approach by To and Chen [2.7] for stochastic optimal control of nonstationary random responses are presented for completeness.

2.3.1 System model equations

Consider a two-degree of freedom (dof) linear or nonlinear system subjected to random excitations. The matrix equation of motion can be expressed as in Equation (2.1) in which \( F \) now is the random excitation vector which can contain stationary or nonstationary or both stationary and nonstationary random excitations.

A nonlinear case of this two dof system subjected to support motion can be applied to the studies of soil-structure interaction, or a primary structure and secondary equipment under earthquake excitation [2.8, 2.9].

For the linear system and with feedback control, a block diagram can be constructed as shown in Figure 2.1.
With reference to the block diagram, the system matrix equation can be expressed as

\[ \dot{Z} = AZ + Bu + D\mu, \]  
\[ u = -GZ, \quad Y = \tilde{C}Z, \]  
\[ \mu = P = \begin{bmatrix} 0 \\ \eta \end{bmatrix} w, \]  

where \( G \) is the gain matrix, the input vector is

\[ \mu = P = \begin{bmatrix} 0 \\ \eta \end{bmatrix} w, \]  

in which \( w \) or \( w(t) \) is the zero mean Gaussian white noise, and \( \eta \) or \( \eta(t) \) is a \( 2 \times 1 \) deterministic modulating function of time vector. The damping matrix in \( A \) of Equation (2.7) for this two dof system is defined by

\[ C = \lambda_m M + \lambda_k K \]  

where \( \lambda_m \) and \( \lambda_k \) are the proportional or Rayleigh mass and stiffness coefficients. It should be noted that the damping matrix \( C \) in Equation (2.7) is not limited to proportional damping.
For discrete or digital control, the continuous white noise process $w$ is replaced by the zero mean Gaussian discrete white noise (GDWN) process $w_D$ so that

$$
\langle w_D \rangle = 0, \; \langle w_D(t_n)w_D(t_n + \Delta t) \rangle = 2\pi S_0 \delta_k(\Delta t), \quad (2.15)
$$

with $S_0$ being the spectral density of the GDWN process and $\delta_k(\Delta t)$ the Kronecker delta function such that $\delta_k(0) = 1$, and $t_n$ is the time step where $n = 0, 1, 2, \ldots N$. The angular brackets denote the mathematical expectation or ensemble average.

### 2.3.2 Stochastic optimal control of linear systems under stationary random excitations

The Lyapunov equation for the uncontrolled system under nonstationary random excitations can be shown to be [2.7, 2.8]

$$
\frac{dR}{dt} = AR^T + RA^T + S_p \quad (2.16)
$$

where the mean square matrix of the displacement response vector is defined as

$$
R = < ZZ^T >, \text{ and the mean square of the excitation vector}
$$

$$
S_p = 2\pi S_0 D \begin{bmatrix} 0 & 0 \\ \eta & \eta^T \end{bmatrix} D^T = 2\pi S_0 \begin{bmatrix} 0 & 0 \\ 0 & M^{-1} \eta \eta^T (M^{-1})^T \end{bmatrix}.
$$

For the system subjected to stationary random excitations, the mean square matrix $R$ is constant and therefore Equation (2.16) becomes
\[
\mathbf{AR}^T + \mathbf{RA}^T + \mathbf{S}_p = \mathbf{0},
\]
(2.17)

where \( \mathbf{S}_p \) is a constant matrix.

With a feedback control in the system under stationary white noise excitations the Lyapunov equation becomes

\[
(\mathbf{A} - \mathbf{BG})\mathbf{R}^T + \mathbf{R}(\mathbf{A} - \mathbf{BG})^T + \mathbf{S}_p = \mathbf{0}
\]
(2.18)

Hotz and Skelton [2.3] has established a method for solving the above equation for the gain matrix \( \mathbf{G} \) with the given system state covariance objectives.

2.3.3 Lyapunov equation of linear systems under nonstationary random excitations

With reference to Equation (2.18) the controlled system is stable if the term \( \mathbf{A} - \mathbf{BG} \) is stable which means its eigenvalues must have all negative real parts. If the system is subjected to non-stationary random excitation, then the mean square matrix \( \mathbf{R} \) must be positive definite and satisfies the Lyapunov Equation [2.7]

\[
\frac{d\mathbf{R}}{dt} = (\mathbf{A} - \mathbf{BG})\mathbf{R}^T + \mathbf{R}(\mathbf{A} - \mathbf{BG})^T + \mathbf{S}_p.
\]
(2.19)

The objective of the present stochastic optimal control is to determine the gain matrix \( \mathbf{G} \) for the controlled \( \mathbf{R} \) which is different from the uncontrolled \( \mathbf{R} \).

To [2.9] has provided an efficient and straightforward procedure for obtaining the specific elements or all elements of the gain matrix \( \mathbf{G} \). The procedure applies to both linear and nonlinear systems. For completeness, the procedure proposed by To [2.9] is included in what follows.

To begin with, one starts from the uncontrolled system so the Lyapunov equation is

\[
\frac{d\mathbf{R}}{dt} = \mathbf{AR}^T + \mathbf{RA}^T + \mathbf{S}_p.
\]
(2.20)
For the controlled system one defines a modified mean square matrix of generalized displacement vector as

\[ \mathbf{R}_1 = \mathbf{R} - \mathbf{U} \quad (2.21) \]

where the specified mean square matrix \( \mathbf{U} \) is such that the modified mean square matrix of the generalized displacement vector is the desired one.

Replacing \( \mathbf{R} \) by \( \mathbf{R}_1 \) in Equation (2.19) one obtains

\[ \frac{d\mathbf{R}_1}{dt} = (\mathbf{A} - \mathbf{BG})\mathbf{R}_1^T + \mathbf{R}_1(\mathbf{A} - \mathbf{BG})^T + \mathbf{S}_p. \quad (2.22) \]

Substituting Equation (2.20) in Equation (2.22) and after performing some algebraic manipulation and simplifying, one has [2.9]

\[ \frac{d\mathbf{U}}{dt} = (\mathbf{A} - \mathbf{H})\mathbf{U} + \mathbf{U}^T(\mathbf{A} - \mathbf{H})^T + (\mathbf{HR} + \mathbf{R}^T\mathbf{H}^T), \quad (2.23a) \]

or

\[ \frac{d\mathbf{U}}{dt} = \mathbf{HR}_1 + \mathbf{R}_1^T\mathbf{H}^T + \mathbf{A}\mathbf{U} + \mathbf{U}^T\mathbf{A}^T \quad (2.23b) \]

where \( \mathbf{H} = \mathbf{BG} \) and \( \mathbf{U} \) is a matrix of the same order as \( \mathbf{G} \).

Note that one can either control the system displacements or velocities. Clearly, controlling both displacements and velocities is impractical in that velocities depend on displacements and by controlling displacements which, in turn, modify the velocities. Therefore, in this investigation the focus is to determine \( \mathbf{G} \) in order that the desired or controlled mean squares of displacements can be achieved. In other words, by specifying \( \mathbf{U} \) one can determine \( \mathbf{H} \) (and in turn, \( \mathbf{G} \) since \( \mathbf{B} \) is given) from using Equation (2.23) as \( \mathbf{R} \) has already been obtained from the uncontrolled Lyapunov equation. Superficially, Equation (2.23) is difficult to solve for \( \mathbf{H} \). However, as shown by To [2.9] the solution can be achieved efficiently. The next sub-section will deal with the solution of \( \mathbf{H} \) and the mean square of the generalized displacement vector of the controlled system.
2.3.4 Gain matrix and controlled mean square matrix

With displacement control \( \mathbf{U}, \mathbf{H}, \mathbf{A} \) and \( \mathbf{R} \) can be written, respectively as

\[
\mathbf{U} = \begin{bmatrix}
    u_{11} & u_{12} & 0 & 0 \\
    u_{21} & u_{22} & 0 & 0 \\
    0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0
\end{bmatrix}, \tag{2.24}
\]

where \( u_{11} = \gamma_{11}r_{11}, u_{22} = \gamma_{22}r_{22} \) and \( u_{12} = u_{21} = \sqrt{\gamma_{11}}\sqrt{\gamma_{22}}r_{12} \) with \( \gamma_{ij} \) to be called the \textit{reduction ratios} specified for the particular optimal stochastic control,

\[
\mathbf{H} = \begin{bmatrix}
    h_{11} & 0 & 0 & 0 \\
    0 & h_{22} & 0 & 0 \\
    0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0
\end{bmatrix}, \tag{2.25}
\]

\[
\mathbf{A} = \begin{bmatrix}
    a_{11} & a_{12} & a_{13} & a_{14} \\
    a_{21} & a_{22} & a_{23} & a_{24} \\
    a_{31} & a_{32} & a_{33} & a_{34} \\
    a_{41} & a_{42} & a_{43} & a_{44}
\end{bmatrix}, \tag{2.26}
\]

\[
\mathbf{R} = \langle \mathbf{Z}\mathbf{Z}^T \rangle, \quad \mathbf{R} = \langle \begin{bmatrix}
    x_1 \\
    x_2 \\
    \dot{x}_1 \\
    \dot{x}_2
\end{bmatrix} \begin{bmatrix}
    x_1 & x_2 & \dot{x}_1 & \dot{x}_2
\end{bmatrix} \rangle,
\]

\[
\mathbf{R} = \begin{bmatrix}
    r_{11} & r_{12} & r_{13} & r_{14} \\
    r_{21} & r_{22} & r_{23} & r_{24} \\
    r_{31} & r_{32} & r_{33} & r_{34} \\
    r_{41} & r_{42} & r_{43} & r_{44}
\end{bmatrix}, \text{ or } \tag{2.27}
\]

\[
\mathbf{R} = \begin{bmatrix}
    \langle x_1x_1 \rangle & \langle x_1x_2 \rangle & \langle x_1\dot{x}_1 \rangle & \langle x_1\dot{x}_2 \rangle \\
    \langle x_2x_1 \rangle & \langle x_2x_2 \rangle & \langle x_2\dot{x}_1 \rangle & \langle x_2\dot{x}_2 \rangle \\
    \langle \dot{x}_1x_1 \rangle & \langle \dot{x}_1x_2 \rangle & \langle \dot{x}_1\dot{x}_1 \rangle & \langle \dot{x}_1\dot{x}_2 \rangle \\
    \langle \dot{x}_2x_1 \rangle & \langle \dot{x}_2x_2 \rangle & \langle \dot{x}_2\dot{x}_1 \rangle & \langle \dot{x}_2\dot{x}_2 \rangle
\end{bmatrix}.
\]

This matrix can be cast into 4 sub-matrices. They are the displacement-displacement mean square matrix

\[
\mathbf{R}_{11} = \begin{bmatrix}
    \langle x_1x_1 \rangle & \langle x_1x_2 \rangle \\
    \langle x_2x_1 \rangle & \langle x_2x_2 \rangle
\end{bmatrix}.
\]
the velocity-velocity mean square matrix

\[ \mathbf{R}_{22} = \begin{bmatrix} \langle \dot{x}_1 \dot{x}_1 \rangle & \langle \dot{x}_1 \dot{x}_2 \rangle \\ \langle \dot{x}_2 \dot{x}_1 \rangle & \langle \dot{x}_2 \dot{x}_2 \rangle \end{bmatrix}, \]

the displacement-velocity mean square matrix

\[ \mathbf{R}_{12} = \begin{bmatrix} \langle x_1 \dot{x}_1 \rangle & \langle x_1 \dot{x}_2 \rangle \\ \langle x_2 \dot{x}_1 \rangle & \langle x_2 \dot{x}_2 \rangle \end{bmatrix}, \]

and

\[ \mathbf{R}_{21} = \begin{bmatrix} \langle x_1 \dot{x}_1 \rangle & \langle \dot{x}_1 x_2 \rangle \\ \langle x_2 \dot{x}_1 \rangle & \langle x_2 \dot{x}_2 \rangle \end{bmatrix}. \]

It is noted that \( \mathbf{R}_{12} = \mathbf{R}_{21}^T \). Substituting Equations (2.20), (2.22), (2.23), (2.24) in Equation (2.19) and after performing further algebraic manipulation one can show that [2.9]

\[ h_{11} = \frac{(a_{31} + a_{41}) u_{11} + (a_{32} + a_{42}) u_{21}}{r_{14} + r_{13}}, \]  
\[ (2.28a) \]

\[ h_{22} = \frac{(a_{31} + a_{41}) u_{12} + (a_{32} + a_{42}) u_{22}}{r_{24} + r_{23}}. \]  
\[ (2.28b) \]

With reference to Equation (2.28) one can see that the determination of the elements of matrix \( \mathbf{H} \) requires the elements of the mean square matrix \( \mathbf{R} \) of the uncontrolled system. The latter mean square matrix can, in theory, be obtained in closed-form since the system is linear. However, for convenience and for the solution to nonlinear systems the Runge-Kutta fourth order (RK4) numerical integration scheme is adopted in the present investigation. For the present system, the RK4 algorithm for the evaluation of \( \mathbf{R} \) becomes [2.10]

\[ t = n \Delta t, \]  
\[ (2.29a) \]

\[ f(t, \mathbf{R}) = \frac{d \mathbf{R}}{dt} = A \mathbf{R}^T + \mathbf{R}^T A + \mathbf{S}_p, \]  
\[ (2.29b) \]
\[ \varphi_1 = \Delta t f(t, R), \quad \varphi_2 = \Delta t f(t + \frac{\Delta t}{2}, R + \frac{\varphi_1}{2}), \quad \text{(2.29c,d)} \]

\[ \varphi_3 = \Delta t f(t + \frac{\Delta t}{2}, R + \frac{\varphi_2}{2}), \quad \varphi_4 = \Delta t f(t + \Delta t, R + \varphi_3), \quad \text{(2.29e,f)} \]

\[ R(t + \Delta t) = R(t) + \frac{1}{6} (\varphi_1 + 2\varphi_2 + 2\varphi_3 + \varphi_4). \quad \text{(2.29g)} \]

Once \( R \) is computed, Equation (2.28) is applied to evaluate the elements of \( H \) and, in turn, the gain matrix \( G \) can be obtained accordingly since \( B \) is a given matrix. In other words, the determined gain matrix can be applied to verify the controlled mean square matrix of the feedback control system, if one so desires. That is, one can subtract \( H \) from the amplification matrix \( A \) and the resulting matrix, \( A - H \) can be employed to replace \( A \) in Equation (2.20) which can then be evaluated for the controlled \( R \) by the RK4 scheme.

Before leaving this sub-section it may be appropriate to note that \( H \) is time dependent and in general the time histories of its elements are very irregular or have discontinuities. Therefore, in practice, the corresponding \( G \) matrix can be difficult to design. A practical approach to circumvent this difficulty is the following. Once the time-dependent elements of \( H \) are evaluated constant values with reference to the elements are selected so that desired controlled \( R \) can be obtained. The implementation of this procedure and computed results will be performed and presented in Chapters 3 and 4.

### 2.4 Stochastic Optimal Control of Nonlinear Systems

The optimal stochastic control strategy in the last section can only be applied to linear systems. For nonlinear systems more algebraic manipulations are necessary before the optimal stochastic control procedure can be implemented. In the present investigation the statistical or equivalent linearization (SL) technique [2.11-2.13] is applied to the nonlinear system at every time step. The SL technique in [2.13] and applied by To and Liu in [2.11] at every time step for a multi-degrees of freedom
(mdof) nonlinear system under nonstationary random excitations is introduced in the following.

In general, the matrix equation of motion for a nonlinear mdof system under nonstationary random excitations can be expressed as

\[ G(\ddot{X}, \dot{X}, X) = F(t) \]

(2.30)

where \( F(t) \) is the uniformly modulated zero mean Gaussian white noise excitation vector and is expressed as \( F(t) = \eta(t)w(t) \) in which \( \eta(t) \) is the vector of deterministic modulating functions and \( w(t) \) is the zero mean Gaussian white noise defined in Equation (2.13). The equivalent linear equation to (2.30) is

\[ M_e\ddot{X} + C_e\dot{X} + K_e X = F(t) \]

(2.31)

where the matrices \( M_e, C_e \) and \( K_e \) are the equivalent assembled mass, damping and stiffness matrices of the mdof nonlinear system, respectively. The error vector can be written as

\[ E = G(\ddot{X}, \dot{X}, X) - M_e\ddot{X} - C_e\dot{X} - K_e X. \]

(2.32)

To obtain the equivalent assembled mass, damping and stiffness matrices that minimize the error \( E \), the mean square value of \( E \) has to satisfy the following necessary conditions [2.13]

\[
\frac{\partial (E^T E)}{\partial m_{eij}} = 0, \quad \frac{\partial (E^T E)}{\partial c_{eij}} = 0, \quad \frac{\partial (E^T E)}{\partial k_{eij}} = 0, \quad i, j = 1,2,...,n_s \quad (2.33a, b, c)
\]

where \( n_s \) is the number of dof of the system, and \( m_{eij}, c_{eij} \) and \( k_{eij} \) are respectively the elements of the equivalent linear assembled mass, damping, and stiffness matrices.

It can be shown, after some algebraic manipulation [2.13] that the linearized mass, damping, and stiffness matrices are

\[
m_{eij} = \left\langle \frac{\partial G_i(\ddot{X}, \dot{X}, X)}{\partial \ddot{x}_j} \right\rangle, \quad c_{eij} = \left\langle \frac{\partial G_i(\ddot{X}, \dot{X}, X)}{\partial \dot{x}_j} \right\rangle, \quad (2.34a, b)
\]
\[ k_{eij} = \left\langle \frac{\partial G_i(\dot{X}, \ddot{X}, X)}{\partial x_j} \right\rangle, \quad i, j = 1, 2, \ldots, n_s. \] (2.34c)

Equation (2.30) can be re-cast in a more familiar form as

\[ M\ddot{X} + C\dot{X} + KX + g(X) = F(t) \]

whose equivalent linearized equation becomes

\[ M\ddot{X} + C\dot{X} + K_eX = F(t) \] (2.35)

where

\[ K_e = K + K_n \] (2.36)

where \( K \) is the linear stiffness and \( K_n \) is the associated nonlinear part of the stiffness matrix which is obtained by minimizing the mean square \( \langle E^T E \rangle \) where now \( E = g(X) - K_nX \).
Following the results in [2.11, 2.13], the linearized stiffness matrix becomes

\[
(K_n)_{ij} = \langle \frac{\partial g_i}{\partial X_j} \rangle
\]  

(2.37)

The approach to implement the computation of mean squares of generalized displacement vector in Equation (3.35) requires the ensemble average or mathematical expectation of the generalized displacement vector. In the present investigation the approach adopted is that provided by To and Liu [2.11, 2.14] which is based on the SCD method. It gives time dependant recursive mean vector \( \Lambda \) at the next time step as

\[
\Lambda(n + 1) = N_2\Lambda(n) + N_3\Lambda(n - 1) - \Delta t^2 N_1 g(X(n))
\]  

(2.38)

where \( n \), as applied in Equation (2.15), is the time step such that \( t_n - t_{n-1} = \Delta t \),

\[
\Lambda(n) = \{ \langle x_1(n) \rangle \}, \quad N_1 = (M + 0.5 \Delta t C_e)^{-1},
\]

(2.39a,b)

\[
N_2 = N_1 [2M - \Delta t^2 K_e], \quad N_3 = N_1 (0.5 \Delta t C_e - M),
\]

(2.39c,d)

where \( C_e = \lambda_m M + \lambda_k K_e \), by making use of Equation (2.14).

After some algebraic manipulation, Equation (2.35) with feedback control can be represented in the state variable form as Equation (2.12),

\[
\dot{Z} = A_e Z + Bu + D\mu, \quad u = -GZ, \quad Y = \hat{C}Z,
\]

(2.40a,b,c)

where \( A_e \) is the linearized amplification matrix resulting from the equivalent linearized system such that

\[
A_e = \begin{bmatrix} 0 & 1 \\ -M^{-1}K_e & -M^{-1}C_e \end{bmatrix},
\]

and the remaining symbols have their usual meaning.

Similar to the procedure for the linear system presented in Section 2.3, the Lyapunov equations for the above linearized nonlinear system without feedback control and with feedback control become, respectively
\[
\frac{d\mathbf{R}}{dt} = \mathbf{A}_e \mathbf{R}^T + \mathbf{R}(\mathbf{A}_e)^T + \mathbf{S}_p, \quad \text{and} \quad (2.41)
\]
\[
\frac{d\mathbf{R}}{dt} = (\mathbf{A}_e - \mathbf{B}\mathbf{G})\mathbf{R}^T + \mathbf{R}(\mathbf{A}_e - \mathbf{B}\mathbf{G})^T + \mathbf{S}_p. \quad (2.42)
\]

Note that the linearized or equivalent amplification matrix \( \mathbf{A}_e \) changes at every time step since it is dependent on the mean and mean squares of displacements of the system. Therefore, the maximum time step size that gives stable solution varies at every time step during the computation process.

It was shown that \([2.14, 2.8]\) the relation between \( \Delta t \) and \( \omega_n \) is

\[
\Delta t = 0.83 - 0.72 \log_{10} \omega_n, \quad 1.0 \leq \omega_n < 5.0 \quad (2.43a),
\]
\[
\Delta t = 1.0 - 0.053 \omega_n - 0.12 \omega_n^2, \quad \omega_n \leq 1.0 \quad (2.43b)
\]

where the relation can be used for cases where the time steps and natural frequencies are dimensionless. With reference to Equation (2.43), it is obvious that the higher the natural frequency the smaller the time step size required for stable computation. Therefore, one needs to select the highest natural frequency among the natural frequencies obtained for a mdof system so that the time step size can be determined from Equation (2.43) for stable solution.

### 2.5 Closing Remarks

Before leaving this chapter four remarks are in order. First, a MatLab program has been developed for the implementation of the RK4 algorithm in Equation (2.29) for computation of the mean square matrix \( \mathbf{R} \) without feedback and with feedback control. Second, another MatLab program has also been developed for the computation of \( \mathbf{H} \) as presented in Equation (2.28). Third, Application of the developed computer programs has been made to the optimal stochastic control of linear and nonlinear two dof systems. Their computed results will be presented and discussed in the following two chapters, Chapters 3 and 4. Fourth, explicit expressions for \( \mathbf{K}_e \) in equation (2.36) are required before the RK4 algorithm can be
applied for specific nonlinear systems. Such explicit expressions will be included in Chapter 4.
CHAPTER 3 STOCHASTIC OPTIMAL CONTROL OF LINEAR SYSTEMS

3.1 Introduction

The approach for stochastic optimal control of mdof linear systems under nonstationary random excitations described in Section 2.3 in Chapter 2 is applied in this chapter to two linear systems. For tractability and availability of results for direct comparison the first two dof system is the one studied by Masri [3.1] and To and Orisamolu [3.2]. The second two dof linear system studied in the present investigation is the linear part of the hysteretic structure considered by Yang, Li and Vongchavalitkul [3.3]. The results of these two systems are presented in the following sections, Sections 3.2 and 3.3. Section 3.4 includes closing remarks.

It should be noted that the main objectives of this chapter are: (a) the determination of the matrix $H$ (and therefore, in turn, the gain matrix $G$) of the controlled system, and (b) verification of the correctness of the approach by way of evaluating the mean square matrices $R$ of displacements after applying $H$ to the uncontrolled systems.

3.2 Linear Two Degree-of-freedom System

The problem studied here has the following system parameters [3.1, 3.2]. The mass matrix is $M = \begin{bmatrix} 1.0 & 0 \\ 0 & 0.1 \end{bmatrix}$, the stiffness matrix $K = \begin{bmatrix} 1.1 & -0.1 \\ -0.1 & 0.1 \end{bmatrix}$, the Rayleigh damping and stiffness coefficients are, respectively $\lambda_m = 0.009875$ and $\lambda_k = 0.009875$; the deterministic modulating functions are $\eta_1 = e^{-0.1t} - e^{-1.5t}$ and $\eta_2 = 0.1\eta_1$; and the spectral density of the zero mean Gaussian noise excitation is $S_0 = \frac{1}{2\pi} \times 10^{-6}$. The two dimensionless natural frequencies are $\omega_1 = 0.854$, and $\omega_2 = 1.170$ while the initial condition for the implementation of the RK4 algorithm outlined in Equation (2.29) is
3.2.1 Uncontrolled system responses

The first step is to find the mean squares or covariances of responses of the system without control. Various time steps were used for the numerical computations applying the RK4 scheme and it was found that the largest time step size is 0.83 which is consistent with that predicted by Equation (2.43). It is interesting to note that the latter equation was derived for the stochastic central difference (SCD) method [3.4] while the present computer program makes use of the RK4 algorithm. The results with time step size of 0.83 are identical to those applying smaller time step sizes. The modulating functions and covariances of responses are presented in Figures 3.1 through 3.5. Note that the results in Figures 3.4 and 3.5 are identical, indicating that $\mathbf{R}_{12}$ is the transpose of $\mathbf{R}_{21}$. Note also that the magnitudes of the peaks of the plots in Figures 3.2 through 3.5 ($\mathbf{R}_{11}$: the covariances of displacements, $\mathbf{R}_{12}$: the covariances of displacement versus velocities, $\mathbf{R}_{21}$: covariances of velocities versus displacements, $\mathbf{R}_{22}$: covariances of velocities) for time step size $\Delta t = 0.83$ agree very well with those for $\Delta t = 0.083$ where high frequency components are clearly present.
Figure 3.1  Modulating functions of random excitations.
Figure 3.2 Top: Covariances of displacements ($\Delta t = 0.83$);
Bottom: Covariances of displacements ($\Delta t = 0.083$).
Figure 3.3  Top: Covariances of velocities ($\Delta t = 0.83$) ;
Bottom: Covariances of velocities  ($\Delta t = 0.083$).
Figure 3.4 Top: Covariances of displacements and velocities ($\Delta t = 0.83$); Bottom: Covariances of displacements and velocities ($\Delta t = 0.083$).
Figure 3.5  Top: Covariances of velocity and displacements ($\Delta t = 0.83$);  
Bottom: Covariances of velocity and displacements ($\Delta t = 0.083$).
By comparing the Figures 3.2 and 3.3, it can be noted that the velocity mean square profiles follows the displacement mean square profiles. However the magnitudes of the maxima and minima differ by a certain constant factor.

### 3.2.2 Gain matrix of controlled system

Once the covariance or mean squares of responses are obtained the associated gain matrices $\mathbf{H}$ can be evaluated by Equation (2.28), and, in turn, the gain matrix $\mathbf{G}$ can be deduced. Without loss of generality and for simplicity one can assume $\mathbf{B}$ being a unity matrix so that $\mathbf{G} = \mathbf{H}$. Henceforth, computed results for the associated matrices will be studied.

Applying Equation (2.28) for various reduction ratios, the associated gain matrices $\mathbf{H}$ are presented in Figures 3.6 through 3.12 where all results were obtained with time step size $\Delta t = 0.083$. With reference to the presented results it is observed that the value of the largest positive peak of each element of the associated gain matrix is proportional to the corresponding reduction factor. This makes sense in that for higher reduction ratios it requires more energy to control or reduce the system responses and therefore higher gain is required.
Figure 3.6  Associated gain matrix: $U_{11} = 0.5r_{11}$ and $U_{22} = 0.75r_{22}$.

Figure 3.7  Associated gain matrix: $U_{11} = 0.5r_{11}$ and $U_{22} = 0.5r_{22}$. 
Figure 3.8  Associated gain matrix: $U_{11} = 0.6r_{11}$ and $U_{22} = 0.4r_{22}$

Figure 3.9  Associated gain matrix: $U_{11} = 0.8r_{11}$ and $U_{22} = 0.2r_{22}$
Figure 3.10  Associated gain matrix: $U_{11} = 0.1r_{11}$ and $U_{22} = 0.9r_{22}$.

Figure 3.11  Associated gain matrix: $U_{11} = 0.1r_{11}$ and $U_{22} = 0.1r_{22}$.
Figures 3.6 through 3.12 indicate that the G matrix has large values and sharp peaks. It is also noted that the larger values are exhibited on the side of $h_{22}$ as the ratio of $U_{22}$ over $r_{22}$ get high even if the ratio of $U_{11}$ and $r_{11}$ stays low. On the other hand increasing the ratio of $U_{11}$ and $r_{11}$ increases very slightly the gain $h_{11}$ as will be shown in Table 1 where a comparison of the maximum peaks in H matrices by using different ratio can be deduced.

3.2.3 Responses of system with feedback control

In principle, in order to verify the desired response reductions, the time varying feedback control gain matrix resulting from each set of reduction factors can now be used in the feedback control of the system and the computation of the mean
square response or covariance matrix can be performed. However, because of the
very irregular time histories of the elements of the associated gain matrix it is
difficult to practically design a feedback gain with such time varying
characteristics. Computationally, because of the sharp peaks and troughs appeared
in the time histories, evaluation of the covariance responses or mean squares of
responses by the RK4 algorithm of Equation (2.29) is impossible since the
responses do not converge no matter how small the time step was applied. This
observation was confirmed during the course of computation by the fact that some
real parts of the eigenvalues of matrix \( A - H \) are positive. This occurs when any one
of the elements of the associated gain matrix \( \mathbf{H} \) becomes negative. In order to have
negative real parts of all the eigenvalues of the system and therefore stable
responses one needs to have positive values for all elements of \( \mathbf{H} \). To circumvent
the difficulty of designing a system having time varying gains as those presented in
Figures 3.6 through 3.12 it is convenient and practical to choose a constant
feedback gain for the system. That is, one can choose positive constant values of all
the elements of matrix \( \mathbf{H} \) in the response computations.

Representative computed results for the controlled responses of the system are
presented in Figures 3.13 through 3.16.

Figures 3.13 represent the controlled system responses in displacement covariances
and (\( \mathbf{R}_{11} \)) velocity covariances (\( \mathbf{R}_{22} \)) by using a feedback control matrix with
\( h_{11} = 21 \) and \( h_{22} = 56 \). It is noted that this control affects much more the
responses in displacement than the responses in velocity covariances. In this case
the maximum peak of covariance of displacements was reduced from around 0.23
to around 0.00005 while the maximum peak of velocity covariances was reduced
from around 0.18 to 0.05. When the gains are changed to \( h_{11} = 21 \) and \( h_{22} = 56 \)
(responses represented by Figure 3.15) the displacement covariances peak is
reduced from 0.23 to around 0.00008 and the velocity covariances maximum peak
is reduced from 0.18 to 0.045. Hence we may postulate that an increase in \( \mathbf{H} \)
matrix element will cause a reduction in element in \( \mathbf{R} \) matrix of the same index.
These two figures also show us that the displacement of the first mass is now
higher than the displacement covariance on the second mass. This is because we
used a very high gain $h_{22}$ to control the second mass displacement while using low gain $h_{11}$ to control the first mass displacement.

The figures 3.14 and the figures 3.16 are representing the covariances of displacement with velocity ($R_{12}$ and $R_{21}$). It is verified by observation on the figures that one is the transpose of the other. This time the responses are turned out to be always positive and the maximum peaks were reduced from around 0.025 to 0.0015.

Table 3.1 includes the largest peak values of the mean squares of displacement responses for the uncontrolled and typically controlled systems as well as the used constant values of $H$ matrix, and the corresponding relative reduction ratios ($\gamma_{11}$ and $\gamma_{22}$). This table gives a clear picture on of our previous observations through different pictures that greater reduction in maximum peak is more likely to be achieved by using higher gain of $H$ matrix element corresponding to the state. This means to have a higher reduction on $r_{11}$, higher value of $h_{11}$ is need and to have a greater reduction on $r_{22}$ larger $h_{22}$ is needed. Using equal values of $h_{11}$ and $h_{22}$ results higher response in $r_{22}$ than $r_{11}$. This is in accordance to the responses of the uncontrolled system where the second mass displacement covariance was found to be the highest.
Figure 3.13  Controlled displacement and velocity response mean squares of two
dof system: $h_{11} = 21$, $h_{22} = 56$. 

$h_{11} = 21; h_{22} = 56; \Delta t = 0.001; \gamma_{11} = 0.5; \gamma_{22} = 0.75$
Figure 3.14  Controlled displacements response mean squares of two dof system:

\[ h_{11} = 21, \quad h_{22} = 56. \]
Figure 3.15 Controlled displacement (above) and velocity (below) response mean squares of two dof system: $h_{11} = 17$, $h_{22} = 30$. 

$h_{11} = 17; h_{22} = 30; \Delta t = 0.001; \gamma_{11} = 0.6; \gamma_{22} = 0.4$
Figure 3.16  Controlled displacements response mean squares of two dof system:

\[ h_{11} = 17, \ h_{22} = 30. \]
<table>
<thead>
<tr>
<th>Reduction factors</th>
<th>10% of max. peak of $h_{11}$</th>
<th>1% of max. peak of $h_{22}$</th>
<th>Max. peak of $r_{11}^{(c)} \times 10^{-3}$</th>
<th>Max. peak of $r_{22}^{(c)} \times 10^{-3}$</th>
<th>Max. peak of $r_{11}$</th>
<th>Max. peak of $r_{22}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\gamma_{11}$</td>
<td>$\gamma_{22}$</td>
<td>21</td>
<td>56</td>
<td>0.0529</td>
<td>0.0166</td>
<td>0.0160</td>
</tr>
<tr>
<td>0.50</td>
<td>0.75</td>
<td>21</td>
<td>56</td>
<td>0.0529</td>
<td>0.0166</td>
<td></td>
</tr>
<tr>
<td>0.10</td>
<td>0.90</td>
<td>10</td>
<td>66</td>
<td>0.1730</td>
<td>0.0165</td>
<td></td>
</tr>
<tr>
<td>0.30</td>
<td>0.80</td>
<td>16</td>
<td>59</td>
<td>0.0827</td>
<td>0.0168</td>
<td></td>
</tr>
<tr>
<td>0.60</td>
<td>0.40</td>
<td>17</td>
<td>30</td>
<td>0.0758</td>
<td>0.0508</td>
<td></td>
</tr>
<tr>
<td>0.80</td>
<td>0.20</td>
<td>15</td>
<td>15</td>
<td>0.0945</td>
<td>0.1529</td>
<td>0.0160</td>
</tr>
<tr>
<td>0.50</td>
<td>0.50</td>
<td>17</td>
<td>37</td>
<td>0.0755</td>
<td>0.0361</td>
<td></td>
</tr>
<tr>
<td>0.001</td>
<td>0.50</td>
<td>1</td>
<td>37</td>
<td>3.1790</td>
<td>0.0737</td>
<td></td>
</tr>
<tr>
<td>0.10</td>
<td>0.10</td>
<td>4</td>
<td>7</td>
<td>0.6735</td>
<td>0.7177</td>
<td></td>
</tr>
<tr>
<td>0.01</td>
<td>0.01</td>
<td>1</td>
<td>1</td>
<td>3.5437</td>
<td>9.7756</td>
<td></td>
</tr>
</tbody>
</table>

Figures 3.17 and 3.18 include three dimensional (3D) plots of the largest peak values of the mean squares of displacement responses of the controlled systems with different values of elements of the associated gain matrix $H$. Such 3D plots are particularly useful for the designer in the sense that once a design specification in term of the largest peak values are chosen the corresponding values of $h_{11}$ and $h_{22}$ can be determined from the 3D plots. Note that in these figures $r_{ii}$ are the controlled mean squares of responses.
Figure 3.17 Variation of maximum peak of $r_{11}$ against variation of gains.

Figure 3.18 Variation of maximum peak of $r_{22}$ against variation of gains.
3.3 Simplified Two Degree-of-freedom Hysteretic Structure

The linear counter-part of the hysteretic structure approximated as a two dof system [3.3] is considered in this section. This system is a model for the base-isolated building. The properties of the superstructure are as follows: mass, $m_2 = 600 \times 10^3 \, kg$; pre-yielding stiffness, $k_2 = 65797 \, \frac{kN}{m}$; damping coefficient, $c_2 = 37.8 \, \frac{kNs}{m}$. The properties of the rubber-bearing base isolation system are: mass, $m_1 = 156 \times 10^3 \, kg$; pre-yielding stiffness, $k_1 = 6000 \, \frac{kN}{m}$; damping coefficient, $c_1 = 20.0 \, kNs/m$. The natural frequencies of the entire building are $0.44$ and $3.77 \, Hz$ and the damping ratios are $0.6$ and $0.8\%$. The envelope or modulating function used is given as follows: $\eta(t) = \left(\frac{t}{t_1}\right)^2$ for $0 \leq t \leq t_1$, $\eta(t) = 1$ for $t_1 \leq t \leq t_2$ and $\eta(t) = \exp \left[-\theta(t - t_2)\right]$ for $t > t_2$ with specific parameters associated with the earthquake model given as: $t_3 = 3 \, s$; $\theta = 0.26 \, s^{-1}$; $S_0 = 10.82 \, cm^2/s^4$.

Note that for simplicity, the filter considered in [3.3] was disregarded in the present investigation. The particular modulating function is included in Figure 3.19. The system covariance or mean squares of responses without control are represented in Figures 3.20 through 3.23.
Figure 3.19  Modulating function of earthquake excitation.

Figure 3.20  Covariances of displacements of uncontrolled system.
Figure 3.21 Covariances of velocities of uncontrolled system.

Figure 3.22 Covariances of displacements and velocities of uncontrolled system.
3.3.1 Gain matrix of controlled simplified structure

In the last sub-section the covariances or mean squares of responses were obtained and therefore, the associated gain matrices $H$ can be evaluated by equation (2.28), and, in turn, the gain matrix $G$ can be deduced. Again, for simplicity one assumes that $G = H$.

Representative computed results for $H$ are presented in Figures 3.24 and 3.25. The time step size used in the computations was $\Delta t = 0.01$ s.
Figure 3.24  Associated gain matrix: $U_{11} = 0.1r_{11}, U_{22} = 0.9r_{22}$

Figure 3.25  Associated gain matrix: $U_{11} = 0.1r_{11}, U_{22} = 0.1r_{22}$. 
3.3.2 Responses of controlled simplified structure

Similar to the approach described in Sub-section 3.2.3, positive constant values of are selected with reference to the largest positive values of $h_{11}$ and $h_{22}$ so that stable responses of the system may be determined. Typical controlled responses after the incorporation of gain matrix $H$ in the system are presented in Figures 3.26 through 3.29. Note that for all the responses obtained the time step size $\Delta t = 0.01s$ was applied. Table 2 includes typical responses with corresponding elements of gain matrix. For design purpose, 3D plots of controlled responses of the present system are included in Figures 3.30 and 3.31.

Similar observations to the results obtained in Sub-section 3.2.3 are obtained in Figures 3.26 through 3.29. The increase in $h_{11}$ gives greater reduction on the response of $r_{11}$ while affecting slightly the decrease of the response $r_{22}$ and the increase in $h_{22}$ gives higher reduction in the response of $r_{22}$ while affecting the decrease of $r_{11}$. It is also noted that the control is highly affecting the reduction of displacements responses than on velocity responses (Figure 3.26 and Figure 3.28).

The Table 3.2 compares the maximum peaks of the controlled displacement mean square responses, $r^{(c)}_{11}$ and $r^{(c)}_{22}$ with the uncontrolled responses $r_{11}$ and $r_{22}$ in two set of controls by using two different pairs of gains $h_{11}$ and $h_{22}$. The observation discussed above is again verified in this table.

The Figures 3.30 and 3.31 gives D-D plots of the overall view of how maximum peaks of $r^{(c)}_{11}$ varies against the variation of the gains pair $h_{11}$ and $h_{22}$ and $r^{(c)}_{22}$ against the gains pair $h_{11}$ and $h_{22}$. It is observed that both $r^{(c)}_{11}$ and $r^{(c)}_{22}$ decrease exponentially as the gains $h_{11}$ and $h_{22}$ increase. Again, these plots can be used for engineer reference to control design.
Figure 3.26  Top: displacement;  Bottom: velocity mean squares and responses of controlled simplified structure: $h_{11} = 9, h_{11} = 36$. 

$h_{11} = 9; h_{22} = 36; \Delta t = 0.01 \text{ seconds}; \gamma_{11} = 0.1; \gamma_{22} = 0.9$
Figure 3.27  Top: ensemble averages of displacements and velocities;  Bottom: ensembles of velocities and displacements of controlled simplified structure for $h_{11} = 9, h_{11} = 36$. 
Figure 3.28  Top: displacements; Bottom: velocity mean squares of controlled simplified structure for \( h_{11} = 3, h_{11} = 4 \).
Figure 3.29  Top: ensemble averages of displacements and velocities; Bottom: ensemble of velocities and displacements of controlled simplified structure for $h_{11} = 3, h_{11} = 4.$
Table 3.2  Representative uncontrolled \( r_{ii} \) and controlled \( r_{ii}^{(c)} \).

<table>
<thead>
<tr>
<th>Reduction factors</th>
<th>0.1% of max. peak of ( h_{11} )</th>
<th>0.1% of max. peak of ( h_{22} )</th>
<th>Max. peak of ( r_{11}^{(c)} (mm^2) )</th>
<th>Max. peak of ( r_{11} ) (mm²)</th>
<th>Max. peak of ( r_{22} ) (mm²)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \gamma_{11} )</td>
<td>0.1</td>
<td>9.0</td>
<td>36.0</td>
<td>1.2987</td>
<td>1.4791</td>
</tr>
<tr>
<td>( \gamma_{22} )</td>
<td>0.1</td>
<td>3.0</td>
<td>4.0</td>
<td>10.875</td>
<td>12.399</td>
</tr>
</tbody>
</table>

Figure 3.30  Variation of maximum peak of \( r_{11}^{(c)} \) against \( h_{11} \) and \( h_{22} \).
3.4 Closing Remarks

In this chapter uncontrolled responses of two linear systems under nonstationary random excitations were obtained. Elements of gain matrices were computed by using equation (2.28) for the controlled systems in accordance with the optimal control strategy introduced in Chapter 2. By selecting various constant values of elements of the gain matrices covariances or mean squares of controlled responses were obtained and presented in the foregoing. Three dimensional (3D) plots of variations of maximum peaks of responses were determined for various constant values of elements of the gain matrices of the two systems. These 3D plots are very useful for the stochastic response control of the systems studied in that for a
specific maximum peak of mean square or variance of controlled response the corresponding gain elements can be deduced.

In the next chapter the control strategy introduced in Chapter 2 will be applied to two nonlinear two dof systems under nonstationary and stationary random excitations.
4.1 Introduction

The optimal control strategy introduced in Chapter 2 has been applied to the studies of two two-dof linear systems in Chapter 3. The same optimal control strategy is applied to two two-dof nonlinear systems in this chapter. The first two-dof nonlinear system studied in this chapter is the one examined by Kimura and Sakata [4.1], To and Liu [4.2], and Liu and To [4.3]. In these latter references no control was applied. The computed results of this system with or without the application of the optimal control strategy are presented in Section 4.2. Section 4.3 is concerned with the studies of the second two-dof nonlinear system. Computed results of this system with and without optimal control are included. Closing remarks are presented in the last section of this chapter.

4.2 System of Primary Building and Secondary Equipment

The system considered in this section is the two-dof nonlinear system that studied by Kimura and Sakata [4.1], To and Liu [4.2, 4.3]. The schematic diagram for this model is shown in Figure 4.1. Sub-section 4.2.1 includes the equations of motion and other relations. Sub-section 4.2.2 presents the computed responses of the system while Sub-section 4.2.3 deals with the gain matrix. Sub-section 4.2.4 is concerned with controlled responses.
The equations of motion for the system shown in Figure 4.1 and expressed in matrix form are [4.1-4.3]:

\[
\begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}
\begin{bmatrix}
\ddot{x}_1 \\
\ddot{x}_2
\end{bmatrix}
+ \begin{bmatrix}
2\zeta_1 W & -2\mu\zeta_2 \\
-2\zeta_1 W & 2(1 + \mu)\zeta_2
\end{bmatrix}
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2
\end{bmatrix}

+ \begin{bmatrix}
W^2 & -\mu \\
W^2 & 1 + \mu
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix}
+ \begin{bmatrix}
(1 - \mu)\eta x_2^2 + (1 + \mu)\varepsilon x_2^3 \\
\mu\eta x_2^2 - \mu\varepsilon x_2^3
\end{bmatrix}

= \begin{bmatrix}
\ddot{y}_0 \\
0
\end{bmatrix}
= \begin{bmatrix}
e(\bar{T})w(\bar{T}) \\
0
\end{bmatrix},
\]

(4.1)

in which the relative displacements \( x_1 = y_1 - y_0 \) and \( x_2 = y_2 - y_1 \) where \( y_i \) with \( i = 0, 1, 2 \) are the absolute displacements. The other symbols are
\[ \omega_1^2 = \frac{k_1}{m_1}, \quad \omega_2^2 = \frac{k_2}{m_2}, \quad 2\zeta_1 \omega_1 = \frac{c_1}{m_1}, \]

\[ 2\zeta_2 \omega_2 = \frac{c_2}{m_2}, \quad \eta = \frac{\eta'}{m_2 \omega_2^2}, \quad \varepsilon = \frac{\varepsilon'}{m_2 \omega_2^2}, \]

\[ \mu = \frac{m_2}{m_1}, \quad W = \frac{\omega_1}{\omega_2}, \quad \bar{\tau} = \omega_2 t \]

with the latter, \( \bar{\tau} \) being the dimensionless time.

The modulating function of the nonstationary random earthquake excitation shown in Figure 4.2 is given as

\[ e(\bar{\tau}) = 4(e^{-0.125\bar{\tau}} - e^{-0.25\bar{\tau}}). \quad (4.2) \]

Other pertinent parameters are: \( \mu = 1.0, \quad \zeta_1 = \zeta_2 = 0.1, \) and \( S_0 = 0.0012 \).
For equation (4.1) to be expressed in the form of equation (2.35), the equivalent linearized stiffness matrix is required [4.2]

\[ \mathbf{K}_e = \mathbf{K} + \mathbf{K}_n \]  

(4.3)

where

\[ \mathbf{K}_e = \mathbf{K} + \begin{bmatrix} 0 & 2\mu \eta \langle x_2(\bar{\tau}) \rangle - 3\mu \varepsilon \langle (x_2(\bar{\tau}))^2 \rangle \\ 0 & 2(1 - \mu) \eta \langle x_2(\bar{\tau}) \rangle + 3(1 + \mu) \varepsilon \langle (x_2(\bar{\tau}))^2 \rangle \end{bmatrix} \]
and \( \mathbf{K} \) is the linear stiffness matrix given by

\[
\mathbf{K} = \begin{bmatrix}
  w^2 & -\mu \\
-\mu w^2 & 1 + \mu
\end{bmatrix}.
\]

Since \( \mathbf{K}_n \) contains the time dependent recursive ensemble average or simply mean vectors as defined by equation (2.38) in which the ensemble average of the nonlinear vector \( g(\mathbf{X}(\bar{\tau})) \) is required therefore for completeness it is included in the following [4.2]

\[
\langle g(\mathbf{X}(\bar{\tau})) \rangle = \left\{ \begin{array}{l}
\mu \eta \langle x_2^2(\bar{\tau}) \rangle - \mu \varepsilon a_1 \\
(1 - \mu) \eta \langle (x_2(\bar{\tau}))^2 \rangle + (1 + \mu) \varepsilon a_1
\end{array} \right.
\]

\[\text{with } a_1 = 3 \langle x_2(\bar{\tau}) \rangle \langle (x_2(\bar{\tau}))^2 \rangle - 2 \langle x_2(\bar{\tau}) \rangle^3.\]

Before the RK4 algorithm in equation (2.29) can be applied to the present system expressed by equation (2.41), the starting or initial conditions for the system are required. For the present approach they are

\[
\Lambda(0) = \Lambda(1) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad R_{11}(0) = R_{11}(1) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}. \tag{4.5a,b}
\]

Now that the equations and pertinent parameters are available the responses of the uncontrolled system can be expressed in the form of equation (2.41).

### 4.2.1 Responses of uncontrolled system

With Equation (2.41) and initial conditions (4.5) the responses of the system can be computed by applying the RK4 algorithm given in Equation (2.29).

In the course of computational experiments many cases were performed in the present investigation. At the end of every computer run the computed results are divided by the spectral density \( S_0 \) of the white noise process so that direct comparison can be made to those presented in [4.2]. The constant dimensionless
time step size applied in the RK4 scheme was 0.50 which gave convergent results. Only representative computed results are included in Figures 4.3 through 4.8.

It is observed that in general the mean squares of displacement responses agree with those obtained by various techniques in [4.2] in which no ensembles of displacements and velocities were provided. For the cases presented in Figures 4.5 and 4.7 the mean squares $r_{11}$ obtained in the present study are lower than those in [4.2]. For example, in Figure 4.7, the maximum peak of $r_{11}/S_o$ is about 24.0 whereas the corresponding value in [4.2] is about 33.0 resulting in a discrepancy of about 27% with respect to the value in [4.2] which has been compared very well with those evaluated by the Monte Carlo simulation technique.

Before leaving this sub-section it should be mentioned that responses of the uncontrolled system have been computed with variable dimensionless time step sizes but no observable difference has been found between those of the constant and variable dimensionless time step sizes and therefore they are not included here.
Figure 4.3  Top: displacement-displacement mean square matrix $R_{11}$; Bottom: velocity-velocity mean square matrix $R_{22}$.
Figure 4.4  Top: ensemble average of displacement and velocity $R_{12}$; 
Bottom: ensemble average of velocity and displacement $R_{21}$. 
Figure 4.5  Top: displacement-displacement mean square matrix $R_{11}$; 
Bottom: velocity-velocity mean square matrix $R_{22}$. 
Figure 4.6  Top: ensemble average of displacement and velocity $R_{12}$; 
Bottom: ensemble average of velocity and displacement $R_{21}$.
Figure 4.7  Top: displacement-displacement mean square matrix $R_{11}$;
Bottom: velocity-velocity mean square matrix $R_{22}$. 
Figure 4.8  Top: ensemble average of displacement and velocity $R_{12}$;  
Bottom: ensemble average of velocity and displacement $R_{21}$. 
4.2.2 Gain matrix of controlled system

Using the constant dimensionless time step size computed results for the uncontrolled system and application of Equations (2.28a) and (2.28b) the elements of the gain matrix H are obtained. Representative computed results are presented in Figures 4.9, 4.10 and 4.11. It is observed that the elements of the gain matrix H exhibit trenches similar to those for the linear cases presented in Chapter 3. That is, the time dependent elements of the gain matrix contain relatively large positive and negative peaks. The negative peaks render the system unstable since they result in positive eigenvalues. Therefore, the gain matrix H for displacement control is obtained as to provide a guideline for selecting the positive values of the elements of matrix H.

Figure 4.9 Elements of gain matrix: $\eta = -0.6, \varepsilon = 0.9, U_{11} = 0.1 r_{11}, U_{22} = 0.9 r_{22}$

$U_{22} = 0.9 r_{22}$
Figure 4.10 Elements of gain matrix H:

\[ \eta = -0.6, \varepsilon = 0.9, U_{11} = 0.9r_{11}, \quad U_{22} = 0.1r_{22} \]

Figure 4.11 Elements of gain matrix H:

\[ \eta = -0.6, \varepsilon = 0.9, U_{11} = 0.5r_{11}, \quad U_{22} = 0.5r_{22} \]
4.2.3 Responses of optimally controlled system

Now that once the elements of the gain matrix \( H \) are obtained, one is able to select particular positive values of these elements so that the desired control displacements can be achieved. In order to examine the correctness and usefulness of such an approach various pairs of control gains were used and corresponding results were obtained. However, only results for the first two pairs are presented in this sub-section.

In the first pair range of \( h_{11} = 3 \) and \( h_{22} = 6 \) units and the second pair \( h_{11} = 4 \) and \( h_{22} = 30 \) units were chosen. Computed results are included in Figures 4.12 through 4.15. It is observed that by using the first pair to control the system displacements, the controlled mean squares were found to be reduced to a very small value with the peak of the normalized mean square of displacement response for the first mass reducing from about 24.5 in Figure 4.3 to less than 0.9 in Figure 4.12. Similarly, by using the second pair \( h_{11} = 4 \) and \( h_{22} = 30 \) units, the normalized maximum peak in the controlled system is less than 0.55 in Figure 4.14. The responses in both pairs attenuate to values close to zero faster than those of the uncontrolled system. This is because by introducing positive values of elements associated with the displacement control of the gain matrix it amounts to adding positive damping to the system.

It should be mentioned that in the controlled system the computed responses by employing the RK4 algorithm outlined in Chapter 2 has to use a dimensionless time step size \( \Delta \bar{\tau} \leq 0.03 \) to maintain computationally stable solutions. For the computed responses presented in Figures 4.12 through 4.15, \( \Delta \bar{\tau} = 0.01 \) has been applied.

Furthermore, the control gains have much more influence on the system displacement mean squares than on the system velocity mean squares. This is logical in the sense that in the present investigation only displacement controls were applied.
To provide a direct and simple illustration of the differences between uncontrolled $r_{ii}$ and controlled $r_{ii}^{(c)}$ selected computed results are presented in Table 4.1. It is noted that even for the relatively positive small values of $h_{ii}$ used the reduction in magnitudes of $r_{ii}^{(c)}$ is very large.

Figure 4.12 Top: mean square matrix of displacements $R_{11}$; 
Bottom: mean square matrix of velocities $R_{22}$. 
Figure 4.13  Top: ensemble average of displacement and velocity $R_{12}$; 
Bottom: ensemble average of velocity and displacement $R_{21}$. 
Figure 4.14 Top: mean square matrix of displacements $R_{11}$; 
Bottom: mean square matrix of velocities $R_{22}$. 
Figure 4.15  Top: ensemble average of displacement and velocity $R_{12}$; 

Bottom: ensemble average of velocity and displacement $R_{21}$. 
Table 4.1  Representative uncontrolled \( r_{ii} \) and controlled \( r_{ii}^{(c)} \).

<table>
<thead>
<tr>
<th>Reduction factors</th>
<th>0.1% of max. peak of ( h_{11} )</th>
<th>0.1% of max. peak of ( h_{22} )</th>
<th>Max. peak of ( r_{11}^{(c)}/S_o )</th>
<th>Max. peak of ( r_{22}^{(c)}/S_o )</th>
<th>Max. peak of ( r_{11}/S_o )</th>
<th>Max. peak of ( r_{22}/S_o )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \gamma_{11} )</td>
<td>( \gamma_{22} )</td>
<td>3</td>
<td>53</td>
<td>0.7692</td>
<td>0.0022</td>
<td>24.4470</td>
</tr>
<tr>
<td>0.1</td>
<td>0.9</td>
<td>3</td>
<td>53</td>
<td>0.7692</td>
<td>0.0022</td>
<td>24.4470</td>
</tr>
<tr>
<td>0.9</td>
<td>0.1</td>
<td>3</td>
<td>6</td>
<td>0.8463</td>
<td>0.0853</td>
<td>9.0189</td>
</tr>
<tr>
<td>0.5</td>
<td>0.5</td>
<td>4</td>
<td>30</td>
<td>0.5232</td>
<td>0.0056</td>
<td>9.0189</td>
</tr>
</tbody>
</table>

As briefly mentioned earlier in this sub-section, in order to be able to obtain computationally stable responses for the controlled system smaller dimensionless time step sizes have to be employed. It seems that the reduced dimensionless time step size \( \Delta \bar{\tau} \) depends on the magnitude of the control gains. Every positive value of \( h_{ii} \) corresponds to a maximum dimensionless time step size \( \Delta \bar{\tau} \). In order to reduce the computational time and develop an efficient strategy for the determination of the maximum dimensionless time step size for every \( h_{ii} \) the maximum dimensionless time step size corresponding to the highest value of gains are applied. Of course, this time step size is relatively very small and the computation of the results for the 3D plots of responses against elements of the gain matrix is very time consuming. Figures 4.16 and 4.17 present the computed results in 3D. In these figures \( r_{ii}/S_o \) are in fact \( r_{11}^{(c)}/S_o \).

In Figure 4.16, one observes that by keeping \( h_{22} \) constant the peaks of the controlled normalized mean square \( r_{11}^{(c)}/S_o \) decrease exponentially to nearly zero as \( h_{11} \) increases. On the other hand, keeping \( h_{11} \) constant \( r_{11}^{(c)}/S_o \) decreases exponentially near the origin of both gains and reaches an approximately constant
value $h_{22}$ increases. In Figure 4.17, $r_{22}^{(c)}/S_o$ approximately varies with $h_{11}$ with $h_{22}$ exponentially.

Figure 4.16  Variation of maximum peak of $r_{11}^{(c)}/S_o$ with positive $h_{11}$ and $h_{22}$

$$(\eta = -0.6, \, \varepsilon = 0.9).$$
Figure 4.17  Variation of maximum peak of $r_{22}^{(c)}/S_0$ with positive $h_{11}$ and $h_{22}$

$(\eta = -0.6, \varepsilon = 0.9)$.

4.3 Optimal Control of Tractor System under Stationary Random Excitations

A two-dof agricultural tractor system operating on rough surface is investigated in this section. It is considered that the bounce is large and the pitching angle is small such that the system is nonlinear with the rough surface providing the stationary random excitations. Sub-section 4.3.1 presents the matrix equation of motion with various system parameters. The computed mean square responses of this two-dof nonlinear system are included in Sub-section 4.3.2 while the brief discussion on the elements of the gain matrix for displacement control is presented in Sub-section 4.3.3. Controlled mean squares of responses are studied in Sub-section 4.3.4.
4.3.1 Equations of motion of tractor system

In order to reduce the complexity of the model it is assumed that the tractor travels at a constant horizontal speed. A similar model in which the excitations are deterministic was employed by Sakai et. al [4.4] to the study of chaotic vibrations. It should be mentioned that in the latter reference the governing equations of motion contain typographical mistakes and the weight of the tractor was incorrectly included. The correct equations of motion for the system under stationary random excitations have been investigated by To [4.5] and its schematic diagram is included in Figure 4.18.

\[
\begin{align*}
M \dddot{y} + C \dddot{\theta} + K_1 \dot{\dot{y}} + K_2 \dot{\theta} &= F_1 w_1(t) + w_2(t), \\
&= F_2 w_2(t). 
\end{align*}
\]  
(4.6)

Figure 4.18   Schematic diagram of tractor system.

The matrix equation of motion for the system in Figure 4.18 is given by [4.5]
where \( \mathbf{M} = \begin{bmatrix} m & 0 \\ 0 & I \end{bmatrix} \),  \( \mathbf{C} = \begin{bmatrix} c_1 + c_2 & c_2 b - c_1 a \\ c_2 b - c_1 a & c_1 a^2 + c_2 b^2 \end{bmatrix} \),  
\( \mathbf{K}_1 = \begin{bmatrix} y_{11} + y_{21} & y_{21} b - y_{11} a \\ y_{21} b - y_{11} a & y_{11} a^2 + y_{21} b^2 \end{bmatrix} \),  \( \mathbf{K}_2 = y^2 \begin{bmatrix} y_{12} + y_{22} & y_{22} b - y_{12} a \\ y_{22} b - y_{12} a & y_{12} a^2 + y_{22} b^2 \end{bmatrix} \),  
\( \mathbf{F}_1 = \begin{bmatrix} -y_{11} \\ y_{11} a \end{bmatrix} \),  \( \mathbf{F}_2 = y^2 \begin{bmatrix} -y_{12} \\ y_{12} a \end{bmatrix} \).  

\( w_1(\nu t) = F_0 w(t) \),  \( w_2(\nu t) = F_0 w(t - \frac{L}{v}) \),  with  

\[ \langle w(\nu)^2 \rangle = \langle w(t)w(t + \tau) \rangle = 2\pi S_x \delta(\tau) \]  

where \( \delta(\tau) \) is the Dirac delta function.

When the RK4 algorithm is applied in the computation of mean squares of responses the above matrix equation has to be discretized in the time domain such that the discrete counterparts of \( w_1(\nu t) \) and \( w_2(\nu t) \) become \( \langle w_{1D}^2(\nu) \rangle = 2\pi S_x = \sigma_{i1}^2 \) and \( \langle w_{2D}^2(\nu) \rangle = \sigma_{i1}^2 e^{-\alpha|\mu|} \), where the second subscript D denotes the discrete value, \( \alpha \) is the roughness parameter for the road, \( \mu = \frac{L}{v} \) is the so called special lag, \( L = a + b \), and \( v \) is the constant horizontal speed of the tractor. The remaining notations are  

\( c \) : damping coefficient in N/(m/s),  
\( k \) : stiffness in N/m,  
\( a \) : distance between the front wheel axle and the gravity point in m,  
\( b \) : distance between the real wheel axle and the gravity point in m,  
\( m \) : mass of the tractor in kg,  
\( I \) : moment of inertia in kg m\(^2\),  
\( w(\nu t) \) : stationary white noise random process,  
\( \dot{w}(\nu t) \) : being equal to zero since \( w(\nu t) \) is assumed to be a white noise,  
\( y \) : vertical displacement of the gravity point or bounce in m, and
\( \theta \): pitching angle or angular displacement about an axis perpendicular to the plane of the diagram in Figure 4.18 and through the center of gravity in radians.

### 4.3.1.1 Equivalent linearized system equations

Consider the Equation (4.6). Let

\[
X = \begin{pmatrix} y \\ \theta \end{pmatrix}, \quad G(X) = K_1 \begin{pmatrix} y \\ \theta \end{pmatrix} + K_2 \begin{pmatrix} y \\ \theta \end{pmatrix}, \quad \text{and} \quad P = F_1 \begin{pmatrix} w_1(t) \\ w_2(t) \end{pmatrix} + F_2 \begin{pmatrix} w_1(t) \\ w_2(t) \end{pmatrix}
\]

so that it can be rewritten as

\[
M\ddot{X} + C\dot{X} + G(X) = P. \quad (4.7)
\]

Similar to the nonlinear system in Chapter 2 the equivalent linearized matrix equation can be shown to be [4.5]

\[
M\ddot{X} + C\dot{X} + K_eX = P \quad (4.8)
\]

in which the elements of the equivalent linearized stiffness matrix are

\[
(k_{11})_e = (\gamma_{11} + \gamma_{21}) + 3(\gamma_{21}b - \gamma_{11}a)(y^2) + 2(\gamma_{22}b - \gamma_{12}a)(y\theta),
\]

\[
(k_{12})_e = (\gamma_{21}b - \gamma_{11}a) + (\gamma_{22}b - \gamma_{12}a)(y^2),
\]

\[
(k_{21})_e = (\gamma_{21}b - \gamma_{11}a) + 3(\gamma_{22}b - \gamma_{12}a)(y^2) + 2(\gamma_{12}a^2 + \gamma_{22}b^2)(y\theta),
\]

\[
(k_{22})_e = (\gamma_{11}a^2 + \gamma_{21}b^2) + (\gamma_{12}a^2 + \gamma_{22}b^2)(y^2).
\]

Note that the resulting equivalent stiffness matrices are dependant variables of the mean squares. Therefore for implementation of this elements in RK4 algorithm they needs to be discretized in time and the they are then updated at each time step. The ensemble average or mean vector can be calculated by following the procedure that was described in the Section 2.4.
4.3.1.2 State matrix equation of system

Equation (4.8) can be cast into the state variable matrix equation as that in equation (2.8),

\[
\dot{Z} = AZ + Q_w
\]

(4.9)

in which now

\[
A = \begin{bmatrix}
0 & I \\
-M^{-1}K_e & -M^{-1}C
\end{bmatrix}, \quad Q_w = D\mu, \quad D = \begin{bmatrix}
I & 0 \\
0 & M^{-1}
\end{bmatrix},
\]

\[
\mu = \begin{bmatrix}
0 \\
P
\end{bmatrix} = \begin{bmatrix}
0 \\
F_w
\end{bmatrix}, \text{ where } F = F_0 + F_1 \text{ and } w = \begin{bmatrix} w_1(t) \\ w_2(t) \end{bmatrix}.
\]

4.3.1.3 System Lyapunov equation

With equation (4.9) one can derive the Lyapunov equation for the system similar to equation (2.16) as

\[
\dot{R} = AR^T + A^T R + \langle Q_w Q_w^T \rangle
\]

(4.10)

where \( R = \langle ZZ^T \rangle \), and the over dot designates the time derivative. The third term on the right-hand side of equation (4.10) is the mean square matrix of the random excitation vector \( Q_w \). The detail of this term is given in the following. First, one has

\[
Q_w Q_w^T = D\begin{bmatrix}
0 \\
F_w
\end{bmatrix}\begin{bmatrix}
0 \\
w^T F^T
\end{bmatrix}D^T = \begin{bmatrix}
I & 0 \\
0 & M^{-1}
\end{bmatrix}\begin{bmatrix}
0 \\
F_w w^T F^T
\end{bmatrix}\begin{bmatrix}
I \\
0
\end{bmatrix} (M^{-1})^T
\]

\[
= \begin{bmatrix}
0 \\
0 & M^{-1}Q (M^{-1})^T
\end{bmatrix}, \text{ where } Q = F w w^T F^T.
\]

Upon taking the ensemble average one obtains

\[
\langle Q_w Q_w^T \rangle = \begin{bmatrix}
0 \\
0 & M^{-1} \langle F w w^T F^T \rangle (M^{-1})^T
\end{bmatrix}.
\]

(4.11)
Before the explicit expressions of equation (4.11) can be derived one writes

\[
Q = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix}, \quad F_0 = \begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix}, \quad \text{and} \quad F_1 = y^2 \begin{bmatrix} f_{110} & f_{120} \\ f_{210} & f_{220} \end{bmatrix}.
\]

Recall that \(w_1\) and \(w_2\) are zero mean Gaussian processes, and \(y\) is assumed Gaussian at every time step although for nonlinear system \(y\), in general, is not Gaussian in the entire time domain. In other words,

\[
\langle w_1 \rangle = \langle w_2 \rangle = 0, \quad \langle w_1 w_2 \rangle = \langle w_1 \rangle \langle w_2 \rangle = 0, \quad \langle y^4 \rangle = 3(\langle y^2 \rangle)^2. \quad (4.12)
\]

It should be emphasized that in Equation (4.12) \(y\) is understood to be the discrete value. Without introducing more notations the following quantities are understood to be the discrete values. Thus, after some algebraic manipulation it can be shown that \([4.5]\)

\[
\langle Q_{11} \rangle = f_{11}^2 \langle w_1^2 \rangle + f_{12}^2 \langle w_2^2 \rangle + 2\langle y^2 \rangle (f_{11} f_{110} \langle w_1^2 \rangle + f_{12} f_{120} \langle w_2^2 \rangle) + 3(\langle y^2 \rangle)^2 (f_{110}^2 \langle w_1^2 \rangle + f_{120}^2 \langle w_2^2 \rangle),
\]

\( (4.13a) \)

\[
\langle Q_{12} \rangle = f_{11} f_{21} \langle w_1^2 \rangle + f_{12} f_{22} \langle w_2^2 \rangle + \langle y^2 \rangle (f_{11} f_{210} \langle w_1^2 \rangle) + f_{21} f_{110} \langle w_1^2 \rangle + f_{12} f_{220} \langle w_2^2 \rangle + f_{22} f_{120} \langle w_2^2 \rangle + 3(\langle y^2 \rangle)^2 (f_{110} f_{210} \langle w_1^2 \rangle + f_{120} f_{220} \langle w_2^2 \rangle),
\]

\( 4.13b) \)

\[
\langle Q_{21} \rangle = \langle Q_{12} \rangle, \quad (4.13c)
\]

\[
\langle Q_{22} \rangle = f_{21}^2 \langle w_1^2 \rangle + f_{22}^2 \langle w_2^2 \rangle + 3(\langle y^2 \rangle)^2 (f_{210}^2 \langle w_1^2 \rangle + f_{220}^2 \langle w_2^2 \rangle) + 2\langle y^2 \rangle (f_{21} f_{210} \langle w_1^2 \rangle + f_{22} f_{220} \langle w_2^2 \rangle).
\]

\( 4.13d) \)
4.3.2 Responses of uncontrolled system

Now all the quantities in equation (4.10) are defined explicitly one is ready to apply the RK4 algorithm in Chapter 2 for the computation of system responses. The pertinent system parameters for the present investigation are given below.

The nonlinear stiffness

\[ k_1 = \gamma_{11} + \gamma_{12}y^2, \]

where \( \gamma_{11} = 200 \frac{kN}{m}, \quad \gamma_{12} = 0.3\gamma_{11} \quad \text{or} \quad \gamma_{12} = 60000 \frac{N}{m^3} \). Similarly, \( k_2 = \gamma_{21} + \gamma_{22}y^2, \) where \( \gamma_{21} = 260 \frac{kN}{m}, \quad \gamma_{22} = 0.3\gamma_{21} \quad \text{or} \quad \gamma_{22} = 7800 \frac{N}{m^3} \). Other system parameters are: \( \sigma_{f1}^2 = 2.89 \times 10^{-4} \, m^2, \) \( \alpha = 0.32, \quad a = 0.7 \, m, \quad b = 0.64 \, m, \)
\( L = 1.34 \, m, \quad m = 988.0 \, kg, \quad I = 700 \, kg \, m^2, \quad c_1 = 5500.0 \, N/(m/s), \) and \( c_2 = 6690.0 \, N/(m/s). \)

Representative responses computed by applying the RK4 scheme in Chapter 2 are presented in Figures 4.19 through 4.22. All these plots exhibit nonstationary random responses up to about 0.5s and after this time the responses are stationary. This explains the fact that the ensemble averages of different stationary random displacements, and stationary random displacements and velocities are zero. Note that the terms \( \langle y^2 \rangle, \langle y\theta \rangle, \langle \theta^2 \rangle \) represent respectively the elements \( r_{11}, r_{12}, r_{22} \) of the matrix in Equation (2.27) in which \( r_{12} = r_{21}. \)
Figure 4.19  Mean square of vertical displacement or bounce.

Figure 4.20  Mean square of angular displacement or pitch.
Figure 4.21  Mean square of vertical or bouncing velocity.

Figure 4.22  Mean square of angular or pitching velocity.
4.3.3 Design of H matrix

Applying Equations (2.28a) and (2.28b) the elements of the associated gain matrix H are calculated. Representative computed results of the time-dependent elements of H for various reductions in bounce and pitch are included in Figure 4.23 through 4.25. It is observed that the elements of H have extremely large values at about 1.0s from the origin of the time axis. This is not difficult to explain if reference is made of the denominator terms in Equations (2.28a) and (2.28b). In theory the ensemble averages of the different displacements, displacements and velocities at stationary values are zero. Numerically, however, they are close to zero which are the denominator terms in Equations (2.28a) and (2.28b), resulting in extremely high values around 1.0 s.

Figure 4.23 Elements of associated gain matrix: $U_{11} = 0.1 r_{11}, U_{22} = 0.9 r_{22}$. 
Figure 4.24  Elements of associated gain matrix: $U_{11} = 0.9r_{11}$, $U_{22} = 0.1r_{22}$.

Figure 4.25  Elements of associated gain matrix: $U_{11} = 0.5r_{11}$, $U_{22} = 0.5r_{22}$.
The task to control responses in the present investigation is to find the feedback gains that reduce the mean squares of vehicle vertical displacement or bounce and the vehicle pitch angle. It is therefore useful, as was performed in previous sections, to provide 3D plots of controlled responses versus elements of the associated gain matrix. Figures 4.26 through 4.27 contain representative 3D plots.

Figure 4.26  Variation of maximum peak of $r_{11}^{(c)}$ with positive $h_{11}$ and $h_{22}$. 
With reference to Figure 4.26, one observes that mean square of the controlled vertical displacement is more influenced by the elements of the gain, $h_{11}$ than $h_{22}$. On the other hand, Figure 4.27 indicates that the mean square of the controlled pitch angle is influenced more by the positive peak value of $h_{11}$.
4.3.4 Responses of feedback controlled system

The computed results for three cases with different elements of gain matrix are presented in Figures 4.28 through 4.39. These are representative results that convey the influence of the gain matrix on the controlled responses. For specific controlled mean squares of displacement responses, one applied the 3D plots in the last subsection to determine the particular element or elements of the associated gain matrix required in the system.

Figure 4.28 Mean square of vertical displacement or bounce in controlled system \((h_1 = 1, h_2 = 20)\).
Figure 4.29  Mean square of pitch angle in controlled system

\[ (h_{11} = 1, \ h_{22} = 20). \]

Figure 4.30  Mean square of vertical velocity in controlled system

\[ (h_{11} = 1, \ h_{22} = 20). \]
Figure 4.31 Mean square of angular velocity in controlled system
\(( h_{11} = 1, \ h_{22} = 20). \)

Figure 4.32 Mean square of vertical displacement or bounce in controlled system
\(( h_{11} = 20, \ h_{22} = 1). \)
Figure 4.33  Mean square of pitch angle in controlled system

\[ h_{11} = 20, \quad h_{22} = 1. \]

Figure 4.34  Mean square of vertical velocity in controlled system

\[ h_{11} = 20, \quad h_{22} = 1. \]
Figure 4.35  Mean square of angular velocity in controlled system

\(( h_{11} = 20, \ h_{22} = 1 ) . \)

Figure 4.36  Mean square of vertical displacement or bounce in controlled system

\(( h_{11} = 5, \ h_{22} = 5 ) . \)
Figure 4.37 Mean square of angular displacement in controlled system
\( (h_{11} = 5, \ h_{22} = 5) \).

Figure 4.38 Mean square of vertical velocity in controlled system
\( (h_{11} = 5, \ h_{22} = 5) \).
The Figures 4.28 through 4.39 represent the controlled responses including the mean squares of vertical displacement, pitch angle, vertical displacement velocity, and the pitch angular velocity. With reference to uncontrolled systems responses it was found that mean squares stabilize at $3 \times 10^{-3} m^2$ of displacement, $4.5 \times 10^{-8} rad^2$ of pitch angle, $1.4 m^2/s^2$ of velocity and $1.6 rad^2/s^2$. By using the control of $h_{11} = 1$, $h_{22} = 20$ (Figures 4.28 through 4.31) the mean squares reduced to around $2.7 \times 10^{-3} m^2$ of displacement, $1.25 \times 10^{-8} rad^2$ of pitch angle, $1.3 m^2/s^2$ of velocity and $0.49 rad^2/s^2$. This shows that using higher gain $h_{22}$ will influence more the reduction on the pitch angle and its velocity. The pitch angle is reduced by a factor of less than 1/3 while the displacement is reduced by a factor greater than 0.5. By using the controls $h_{11} = 20$, $h_{22} = 1$ (Figures 4.32 through Figure 4.35) we see that the influence of reversesh $h_{11}$ and $h_{22}$ on the responses reverses. The mean squares are reduced to around $0.71 \times 10^{-3} m^2$ of displacement, $4 \times 10^{-8} rad^2$ of pitch angle, $0.8 m^2/s^2$ of velocity and $1.55 rad^2/s^2$. Here the vertical displacement and velocity mean squares are more reduced than the pitch angle and pitch angular velocity mean squares. By using
equal amount of the gains elements $h_{11} = 5,\ h_{22} = 5$ the system quite similar influence is observed: the mean squares reduce to around $1.8 \times 10^{-3} m^2$ of displacement, $2.7 \times 10^{-8} rad^2$ of pitch angle, $1.0 m^2/s^2$ of velocity and $1.1 rad^2/s^2$.

4.4 Closing Remarks

In this chapter two nonlinear two-dof systems were studied. The first system was subjected to a nonstationary random excitation while the second system was under stationary random excitations. The main objective was to provide computed results of the elements of the associated gain matrix. For the optimal stochastic control of nonlinear systems the 3D plots of mean squares of displacement responses were presented. These 3D plots are provided to enable the designer to achieve the particular optimal control required in the sense that by specifying the mean square of displacement response the corresponding elements of the associated gain matrix can be directly determined from these 3D plots.
CHAPTER 5  CONCLUSION AND RECOMMENDATIONS

5.1 Introduction

In this investigation existing stochastic optimal control strategies in linear systems under nonstationary random excitations and nonlinear systems under stationary and nonstationary random excitations were studied. In particular, existing stochastic optimal control strategies are mainly based on the linear quadratic Gaussian (LQG) technique. These strategies cannot be applied to control a specific state or transient response. Skelton and associates [5.1-5.4] introduced the state covariance assignment (SCA) method that eliminates the limitation of approaches employing the LQG technique. However, their strategy only applies to linear systems under stationary random excitations.

More recently, To [5.5] has introduced a strategy in dealing with the optimal stochastic control of linear and nonlinear systems under nonstationary random excitations. However, no computed results were provided. Consequently, the main thrust of the present investigation were concerned with the application of the strategy proposed in [5.5] and its implementation as well as providing computed results for mdof systems. Computed results for two linear two degree-of-freedom (dof) systems under nonstationary random excitations, and two nonlinear two-dof systems under nonstationary and stationary random excitations were presented in this thesis.

The linear multi-dof (mdof) systems studied include that previously investigated by Masri [5.6], and To and Orisamolu [5.7], and the linear counterpart of the inelastic hysteretic hybrid two dof system provided by Yang and Vongchavalitkul [5.8]. In the case of nonlinear mdof systems, the building structure and secondary equipment model under nonstationary random excitation previously investigated by Kimura and Sakata [5.9], and To and Liu [5.10] was considered. The second mdof nonlinear system studied in the present investigation is the two dof nonlinear tractor model with stationary random excitation.
5.2 Conclusion

In all the systems studied in this investigation the control gains have much more influence on the mean squares of displacements than on the mean squares of velocities. This observation has to do with the fact that all optimal controls studied were of displacement feedback types. Even for the relatively small positive values of the elements of the associated gain matrix very large reduction in magnitudes of the mean squares of responses were achieved. This confirms the usefulness of the strategy proposed by To [5.5]. By applying this new strategy, elements of the associated gain matrices $H$ for linear and nonlinear two-dof systems under stationary and nonlinear random excitations were evaluated. These time-dependent elements were applied as guides to the selection of positive constant values for the specific feedback controls. Three dimensional (3D) plots of mean squares of controlled responses versus the chosen constant values of the elements of associated gain matrix were constructed. These 3D plots were applied to the optimal control of the corresponding system mean squares of responses. The graphical representation is very important for the designers of control systems as it allows them to select the values of the control matrix that closely meet the desired individual objectives of the expected states. Therefore, the computational procedure presented in this thesis is highly suited for control of mdof systems that need to operate within individual state ranges or that better performance is achieved if a certain state is operating within the expected range of values regardless of the range of values of other states.
5.3 Recommendations

In this thesis the computed results were for the optimal stochastic control of displacements of linear and nonlinear mdof systems under stationary and nonstationary random excitations. Therefore, future studies should be pursued on optimal stochastic control of velocities of linear and nonlinear mdof systems. Physical experimentation to confirm the efficiency of the control strategy should constitute another venue for future investigation.
References

References in Chapter 1


**References in Chapter 2**


**References in Chapter 3**


**References in Chapter 4**


References in Chapter 5


