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Sudoku

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Sudoku
A Plan for a Successful End

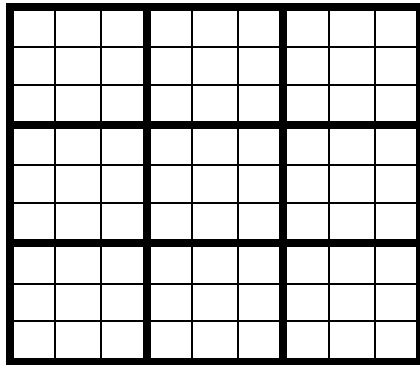
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In partial fulfillment of the requirements for the Master of Arts in Teaching with a Specialization
in the Teaching of Middle Level Mathematics in the Department of Mathematics.
Dr. Jim Lewis, Advisor

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Many people have a magazine, a crossword puzzle, a newspaper, or a book on hand to fill idle time. I, however, prefer to tease my mind with a book of Sudoku puzzles (pronounced sue-dah-coup), trying to figure out what number goes where while resisting the temptation to peek in the back at the answer to make sure I am on the right track. For those of you who are unaware of what Sudoku is, it is a highly popular number game whose notoriety may even surpass the crossword or word search puzzles. It can be found in popular newspapers such as *USA Today*, the *New York Times*, and the *LA Times*. Even the *Lincoln Journal Star* and the University of Nebraska's *Daily Nebraskan* make a daily puzzle available to their readers.

The game, originally given the name Suuji wa Dokushin ni Kagiru ("the numbers must be single") by its creator Maki Kaji, consists of a 9 x 9 grid that is broken down into nine 3 x 3 sub-grids.



The goal is to get the numbers 1 through 9 in each sub-grid without repeating any of the digits in any row or column.

1	2	3	4	5	6	7	8	9
4	5	6	7	8	9	1	2	3
7	8	9	1	2	3	4	5	6
2	3	4	5	6	7	8	9	1
5	6	7	8	9	1	2	3	4
8	9	1	2	3	4	5	6	7
3	4	5	6	7	8	9	1	2
6	7	8	9	1	2	3	4	5
9	1	2	3	4	5	6	7	8

http://farm1.static.flickr.com/79/214883343_eaaaf6e49e.jpg

A Sudoku puzzle supplies the player with clues (numbers already filled in—the amount supplied decreases with difficulty) on the grid and leaves the blanks to be figured out. Essentially it is a Latin square (described below) with an extra “box” component.

A Latin square consists of an $n \times n$ square array of n different elements such that each element does not repeat in the row or column in which it appears. Below are examples of a 3x3 and a 4x4 Latin square.

1	2	3
2	3	1
3	1	2

A	B	C	D
B	C	D	A
C	D	A	B
D	A	B	C

People have played and worked with Latin squares for hundreds of years, but the mathematician Leonhard Euler is most credited for his work with Latin squares in the 1780's. A standard Sudoku puzzle is a 9x9 Latin square consisting of 9 different elements (usually the numbers 1 through 9) that cannot be repeated in any row, column, or 3x3 sub-grid (box).

Maki Kaji, president of the Japanese puzzle company Nikoli, modeled Sudoku after the American version of the game, Numbers Place. Numbers Place, a game invented by Indianan architect Howard Garns, was first published in Dell Magazines in 1979.

Number Place original instructions:

"In this puzzle, your job is to place a number into every empty box so that each row across, each column down, and each small 9-box square within the large square (there are 9 of these) will contain each number from 1 through 9. Remember that no number may appear more than once in any row across, any column down, or within any small 9-box square; this will help you solve the puzzle. The numbers in circles below the diagram will give you a head start--each of these four numbers goes into one of the circle boxes in the diagram (not necessarily in the order given)."

○	2	3			1	7		
		8	4	6			1	
9				5			4	8
5		4	3				2	○
	9		8	7		1		
1			○		4	9		5
	7				6	8		2
8		1	7		2			
	6			3	○		7	1

○ → 4 6 7 & 8

6			2	5		4		
○	1	2			9		5	
	9			4			8	7
	2		9	3		○		1
		8	1			7	3	
1	3			8	5			
		6	3	4		2	○	
5		○			7	9		6
2	4			1				8

○ → 1 4 5 & 8

The first Number Place puzzles. (Dell Pencil Puzzles & Word Games #16, page 6, 1979-05)

After a few modifications, Kaji introduced the game to Japan where it quickly gained popularity and became an international craze after retired judge, Wayne Gould, convinced *The Times* to publish it. Gould had developed a computer program, which he called Pappocom, that could create Sudoku puzzles for mass production. This invention made way for the daily puzzles found in newspapers and books upon books of Sudoku puzzles that appease the novice as well as the expert today.

This leads one to ask: exactly how many Sudoku puzzles are there? According to the work of Bertram Felgenhauer and Frazer Jarvis, there are 6,670,903,752,021,072,936,960 possible Sudoku grids (Enumerating possible Sudoku grids, 2005). Of these grids, how many of them have a unique solution? A Sudoku puzzle provides the consumer with clues, meaning some of the 81 small squares are already filled in. It is possible to have an initial grid that will yield more than one correctly completed grid, as shown below. Many Sudoku creators and fans consider puzzles that have more than one solution to be ill-posed puzzles.

9	7	2	8	1	4	6	3	5
3	6	1	7	2	5	8	4	9
5	4	8	3	9	6	1	2	7
8	9	7	2	3	1	4	5	6
2	3	6	5	4	8	9	7	1
4	1	5	6	7	9	2	8	3
7	8	4	1	6	3	5	9	2
1	5	3	9	8	2	7	6	4
6	2	9	4	5	7	3	1	8

9	7	2	8	1	4	6	3	5
3	6	1	7	2	5	8	4	9
5	4	8	3	9	6	1	2	7
8	9	7	2	3	1	4	5	6
2	3	6	5	4	8	9	7	1
4	1	5	6	7	9	3	8	2
7	5	4	1	6	3	2	9	8
1	8	3	9	5	2	7	6	4
6	2	9	4	8	7	5	1	3

So, out of the 6,670,903,752,021,072,936,960 total possible grids, how many have a unique solution? A year after Felgenhauer and Jarvis published their enumeration of Sudoku puzzles, Jarvis and Ed Russell found there to be 2,297,902,829,591,040 “essentially different grids” (Mathematics of Sudoku II, 2006).

Knowing that there are a vast number of Sudoku puzzles to be solved, the undertaker may want a strategic process to complete each puzzle in as little time as possible. For the remainder of this paper, I will refer to each 3x3 subgrid as a box numbered 1 – 9, in reading order.

1	2	3
4	5	6
7	8	9

Within the Sudoku grid there are 81 squares that will be identified according to the row and then column number (r, c) in which they lie. For instance, square (5, 8) is highlighted below.

	1	2	3	4	5	6	7	8	9
1									
2									
3									
4									
5									
6									
7									
8									
9									

One characteristic of Sudoku puzzles that make them so intriguing is that no great skill is required for a person to get started. Beginners can quickly pick up on a few simple steps from receiving these brief instructions: Fill each box with the numbers 1 – 9 without repeating any number in any row, column, or box. The first step a puzzler will take is to look for forced numbers.

Step 1: Forced Numbers

I personally begin with the number 1 and work up to 9 or start at 9 and work down to 1 (simply for organizational purposes), looking for any forced numbers that can be placed given the clues. Take the following puzzle described as “gentle” by Michael Mepham in *Solving Sudoku* (2005)

	1		4	2				5
		2		7	1		3	9
							4	
2		7	1					6
				4				
6					7	4		3
	7					3		
1	2		7	3		5		
3				8	2		7	

The number 3 is placed in square (7, 7) because threes are already present in rows 8 and 9 and columns 8 and 9. The same reasoning of forced numbers applies to 4 being placed in square (4, 2) in the grid below.

	1		4	2				5
		2		7	1		3	9
							4	
2	4	7	1					6
				4				
6					7	4		3
	7					3		
1	2		7	3		5		
3				8	2		7	

Continuing with this process will lead you to the grid shown below.

	1		4	2				5
4		2		7	1		3	9
							4	
2	4	7	1					6
				4				
6					7	4		3
	7			1		3		
1	2		7	3		5		
3				8	2		7	

Step 2: Isolation

This is a step I take before moving on to “mark up” which will be explained shortly. Isolation is done to keep the grid as neat as possible, pre-identify some answers prior to mark up, and take care of some of the work that marking up will require.

Once a point has been reached where no more forced numbers exist, the next step is to isolate squares where a certain number has to be placed. Below, for example, a 3 has to go in square (5, 2) or (5, 3). Therefore, I know there is a 3 in row 4, box 5. Since column 5 already contains a 3 in box 8, I know a 3 must be placed in square (4, 6).

	1		4	2				5
4		2		7	1		3	9
							4	
2	4	7	1		3			6
	3	3		4				
6					7	4		3
	7			1		3		
1	2		7	3		5		
3				8	2		7	

Continue, combining the isolation strategy with the forced number strategy to get the grid below.

	1	3	4	2				5
4		2		7	1		3	9
			3				4	
2	4	7	1		3			6
	3 ₃	3		4				
6					7	4		3
	7			1		3		
1	2		7	3		5		
3				8	2		7	

Step 3: Mark Up

When the first two simple strategies have led you to a seeming stand-still, it is time to mark up the grid. For each empty square, consider which numbers from 1 to 9 are candidates for that spot. I like to start with the box that has the most numbers completed in case a simple strategy can be used along the way to save me time from marking up unnecessary numbers in other boxes. (See grid below)

789	1	3	4	2	689	678	68	5
4	568	2	568	7	1	68	3	9
5789	5689	5689	3	569	5689	12678	4	1278
2	4	7	1	59	3	89	589	6
589	3 ₃	31589	25689	4	5689	12789	12589	1278
6	589	1589	2589	59	7	4	12589	3
589	7	45689	569	1	4569	3	2689	248
1	2	4689	7	3	469	5	689	48
3	569	4569	569	8	2	169	7	149

Step 4: Exclusive Sets

The next step is to look for what I call *exclusive sets*. Exclusive sets are groups of y amount of numbers that can only be in y amount of squares in a particular row, column, or box. The authors Crook (A Pencil-and-Paper Algorithm for Solving Sudoku Puzzles, 2009) and

Mepham (Solving Sudoku, 2005) refer to these as *pre-emptive sets* and *number sharing* respectively. For instance, in row 3 of box 3, the numbers 1 and 2 appear in squares (3, 7) and (3, 9) only. Therefore, the numbers 1 and 2 are an exclusive pair in box 3 and I can cross out the remaining numbers in the mark-up of those two squares. Although exclusive pairs are the easiest to spot, triples, quadruples, and exclusive sets of even greater size do exist and will become more apparent with practice.

789	1	3	4	2	689	678	68	5
4	568	2	568	7	1	68	3	9
5789	5689	5689	3	569	5689	12678	4	1278
2	4	7	1	59	3	89	589	6
589	3 ₃	31589	25689	4	5689	12789	12589	1278
6	589	1589	2589	59	7	4	12589	3
589	7	45689	569	1	4569	3	2689	248
1	2	4689	7	3	469	5	689	48
3	569	4569	569	8	2	169	7	149

Continue looking for sets and filling in squares until a solution is found. Remember to ask yourself what numbers each row and column need to have, which may exclude them from appearing in neighboring rows and columns (exclusive sets in the grid below are noted in color)

8 ₇₈₉	1	3	4	2	9 ₆₈₉	7 ₆₇₈	6 ₆₈	5
4	6 ₅₆₈	2	5 ₅₆₈	7	1	8 ₆₈	3	9
7 ₅₇₈₉	9 ₅₆₈₉	5 ₅₆₈₉	3	6 ₅₆₉	8 ₅₆₈₉	1 ₁₂₆₇₈	4	2 ₁₂₇₈
2	4	7	1	5 ₅₉	3	9 ₈₉	8 ₅₈₉	6
5 ₅₈₉	3 ₃	9 ₃₁₅₈₉	8 ₂₅₆₈₉	4	6 ₅₆₈₉	2 ₁₂₇₈₉	1 ₁₂₅₈₉	7 ₁₂₇₈
6	8 ₅₈₉	1 ₁₅₈₉	2 ₂₅₈₉	9 ₅₉	7	4	5 ₁₂₅₈₉	3
9 ₅₈₉	7	8 ₄₅₆₈₉	6 ₅₆₉	1	5 ₄₅₆₉	3	2 ₂₆₈₉	4 ₂₄₈
1	2	6 ₄₆₈₉	7	3	4 ₄₆₉	5	9 ₆₈₉	8 ₄₈
3	5 ₅₆₉	4 ₄₅₆₉	9 ₅₆₉	8	2	6 ₁₆₉	7	1 ₁₄₉

This process will allow you to find solutions for most puzzles, but on extremely difficult puzzles, such as the one categorized by Mepham as “diabolical” (Solving Sudoku, 2005) below, another step/strategy is needed.

	9		7			8	6	
	3	1			5		2	
8		6						
		7		5				6
			3		7			
5				1		7		
						1		9
	2		6			3	5	
	5	4			8		7	

J. F. Crook (A Pencil-and-Paper Algorithm for Solving Sudoku Puzzles, 2009) has given special attention to finding a process for solving Sudoku puzzles, especially the more deviant ones, without resorting to the aid of a computer program. Crook’s algorithm, published in the Notices of the American Mathematical Society, is “a tree-based search algorithm based on backtracking in a tree until a solution is found” (Crook, 2009). If understood and accurately applied, Crook’s method should lead to a solution to any traditional Sudoku puzzle. His algorithm for concisely solving a Sudoku puzzle goes through the following steps:

1. Find all forced numbers in the puzzle.
2. Mark up the puzzle.
3. Search iteratively for preemptive sets [referred to as exclusive sets in this paper] in all rows, columns, and boxes – taking appropriate cross out action as each new preemptive set is discovered – until
4. Either
 - a. A solution is found or
 - b. A random choice must be made for continuation
5. If 4(a), the end; if 4(b), then go to step 3

In the Mepham’s diabolical puzzle, there will come a point when you reach a fork in the road and will have to choose which prong to follow. This prong may lead to a solution, to another fork, or to an error. An error occurs when a number appears twice in a row, column, or block. If an error is reached you will need to back track to the initial fork and choose a different prong. Using colored pencil at this point is suggested so you can easily undo the erroneous path that was taken if an error is reached. If the path of the prong leads to another fork, then a new prong must be chosen and the same process followed.

The following grid is the result after performing Crook’s first two steps on the ‘diabolical’ Sudoku puzzle we have been considering.

2	9	5	7	34	134	8	6	134
47	3	1	8	6	5	49	2	47
8	47	6	1249	2349	12349	459	1349	13457
13469	1468	7	249	5	249	249	13489	6
1469	146	29	3	8	7	2459	149	145
5	468	2389	249	1	6	7	3489	348
367	678	38	5	2347	234	1	48	9
179	2	89	6	479	149	3	5	48
139	5	4	19	39	8	6	7	2

Now, it is time for step three: look for exclusive sets, crossing out and writing in singles where appropriate.

2	9	5	7	34	134	8	6	134
47	3	1	8	6	5	9 ₄₉	2	47
8	47	6	1249	2349	12349	459	1349	13457
1349	148	7	249	5	249	249	13489	6
1469	146	29	3	8	7	2459	149	145
5	468	2389	249	1	6	7	3489	348
367	678	38	5	2347	234	1	48	9
179	2	89	6	479	149	3	5	48
139	5	4	19	39	8	6	7	2

For instance, there is the exclusive triple {2, 4, 9} in squares (4, 4), (4, 6), and (4, 7), so all other occurrences in that row can be crossed out. This reveals another exclusive triple: {1, 3, 8} in the remaining squares of row 4. The exclusive pair {4, 7} exists in squares (2, 1) and (2, 9), revealing the 9 in square (2, 7) as a single.

All of the exclusive sets have been found and much is still left unknown. At this point, since there are only two prongs in my 'fork', I will focus on the set {4, 7} in square (2, 1) as my 'prong' and will see what happens if I allow 7 to be entered into square (2, 1).

2	9	5	7	34	434	8	6	1 ₁₃₄
7 ₄₇	3	1	8	6	5	9 ₄₉	2	4 ₄₇
8	4 ₄₇	6	1 ₁₂₄₉	2 ₂₃₄₉	9 ₁₂₃₄₉	5 ₄₅₉	3 ₁₃₄₉	7 ₁₃₄₅₇
3 ₁₃₄₉	148	7	249	5	4 ₂₄₉	249	13489	6
1469	146	29	3	8	7	2459	149	5 ₁₄₅
5	48	2389	249	1	6	7	3489	3 ₃₄₈
6 ₃₆₇	8 ₆₇₈	3 ₃₈	5	7 ₂₃₄₇	2 ₂₃₄	1	4 ₄₈	9
7 ₁₇₉	2	9 ₈₉	6	4 ₄₇₉	1 ₁₄₉	3	5	8 ₄₈
1 ₁₃₉	5	4	9 ₁₉	3 ₃₉	8	6	7	2

Choosing 7 in square (2, 1) seemed to be leading to a solution until an error appeared in squares (4, 4) and (6, 4). The mark up states that 2 should be placed in both squares, leaving box 5 without a 9. The entire path I took with this prong (shown in gray) will have to be erased and 4 accepted as the correct "prong" instead.

2	9	5	7	34	134	8	6	134
4 ₄₇	3	1	8	6	5	9 ₄₉	2	7 ₄₇
8	7 ₄₇	6	1249	2349	12349	459	1349	13457
1349	148	7	249	5	249	249	13489	6
1469	146	29	3	8	7	2459	149	145
5	48	2389	249	1	6	7	3489	348
367	678	38	5	2347	234	1	48	9
179	2	89	6	479	149	3	5	48
139	5	4	19	39	8	6	7	2

Placing a 4 in square (2, 1) did not get me far, nor did it reveal any additional exclusive sets. However, I already know that a 7 in square (2, 1) is incorrect, so I will leave the changes that were made and find a new pair to act as another fork. In box 9, there is the pair {4, 8}, so I will choose a 4 in square (7, 8) as the prong to follow.

2	9	5	7	3 ₄	1 ₃₄	8	6	1 ₁₃₄
4 ₄₇	3	1	8	6	5	9 ₄₉	2	7 ₄₇
8	7 ₄₇	6	1 ₁₂₄₉	2 ₂₃₄₉	9 ₁₂₃₄₉	4 ₅₉	3 ₁₃₄₉	1 ₃₄₅₇
3 ₁₃₄₉	1 ₁₄₈	7	9 ₄₉	5	9 ₄₉	2 ₂₄₉	8 ₁₃₄₈₉	6
9 ₁₄₆₉	6 ₁₄₆	2 ₂₉	3	8	7	2 ₄₅₉	1 ₁₄₉	4 ₄₅
5	4 ₄₈	8 ₂₃₈₉	2 ₂₄₉	1	6	7	9 ₃₄₈₉	3 ₃₄₈
6 ₃₆₇	8 ₆₇₈	3 ₃₈	5	7 ₂₃₄₇	2 ₂₃₄	1	4 ₄₈	9
7 ₁₇₉	2	9 ₈₉	6	4 ₄₇₉	1 ₁₄₉	3	5	8 ₄₈
1 ₁₃₉	5	4	9 ₄₉	3 ₃₉	8	6	7	2

Towards the end, in choosing 4 as my prong in square (7, 8), a few errors arose. Highlighted in yellow, there is already a 4 and 3 present in column 5. However, the only options for square (1, 5) are a 3 or a 4. In orange, you see that columns 4 and 6 already have nines and the exclusive pair {4, 9} forces a 9 to be in either square (4, 4) or (4, 6). Erasing my path to get back to the initial fork I now know that 8 is the correct prong for the fork in square (7, 8).

2	9	5	7	4 ₃₄	3 ₁₃₄	8	6	1 ₁₃₄
4 ₄₇	3	1	8	6	5	9 ₄₉	2	7 ₄₇
8	7 ₄₇	6	1 ₁₂₄₉	9 ₂₃₄₉	2 ₁₂₃₄₉	5 ₄₅₉	4 ₁₃₄₉	3 ₁₃₄₅₇
3 ₁₃₄₉	8 ₁₄₈	7	4 ₂₄₉	5	9 ₂₄₉	2 ₂₄₉	1 ₁₃₄₈₉	6
6 ₁₄₆₉	1 ₁₄₆	2 ₂₉	3	8	7	4 ₂₄₅₉	9 ₁₄₉	5 ₁₄₅
5	4 ₄₈	9 ₂₃₈₉	2 ₂₄₉	1	6	7	3 ₃₄₈₉	8 ₃₄₈
7 ₃₆₇	6 ₆₇₈	3 ₃₈	5	2 ₂₃₄₇	4 ₂₃₄	1	8 ₄₈	9
9 ₁₇₉	2	8 ₈₉	6	7 ₄₇₉	1 ₁₄₉	3	5	4 ₄₈
1 ₁₃₉	5	4	9 ₄₉	3 ₃₉	8	6	7	2

Once a stand still was reached with steps 1 through 3 of Crooks algorithm, a solution to this diabolical Sudoku puzzle was found by the decisions of 4 in square (2, 1) and 8 in square (7, 8).

Crook's algorithm resulted in finding a solution to a Sudoku puzzle. Those, like me, who like the challenge and feared that the algorithm would render the game void of its "puzzle" aspect can rest in knowing that at some point, a good guess will have to be implemented. The algorithm simply leads to educated guessing that takes place later rather than sooner on difficult puzzles and for guessing not to occur at all on less challenging ones. With a solid understanding of Crook's algorithm and expert shortcuts learned from steady practice, a Sudoku puzzle solver is on his/her way to bring the 6,670,903,752,021,072,936,960 possible Sudoku grids into submission. Once this has been accomplished, puzzle makers have created a plethora of other versions of Sudoku to whet the appetite. (See below for examples available at www.brainfreeze puzzles.com)

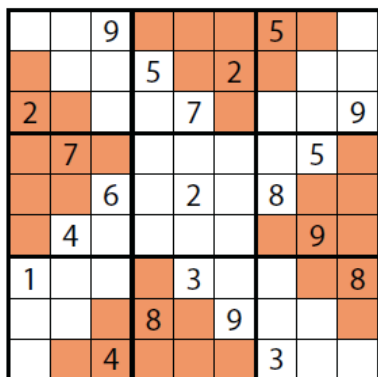


Figure 10. *Pyramids:* Each row, column, block, and pyramid region must contain 1–9 exactly once.

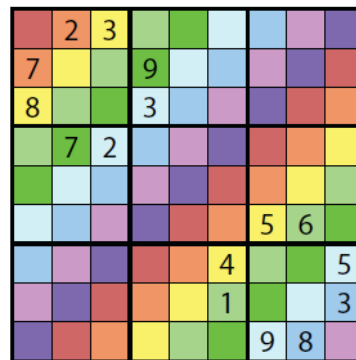


Figure 12. *Wrap Up:* Each row, column, block, and color-region must contain 1–9 exactly once. Note that the color-regions wrap around the square in "torus" fashion.

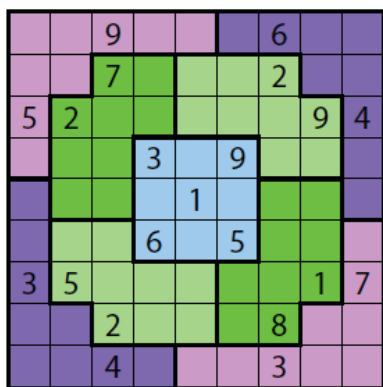


Figure 11. *Jigsaw:* Each row, column, and jigsaw-region must contain 1–9 exactly once.

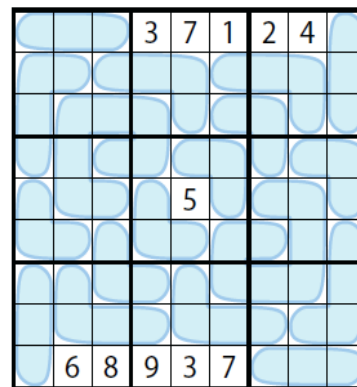


Figure 14. *Worms:* Each row, column, and block must contain 1–9 exactly once, and the entries in each worm either increase or decrease monotonically (although not necessary sequentially; e.g., 2, 3, 6, 8 is allowable).

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