1935

On the Summability of a Certain Class of Series of Jacobi Polynomials

A. P. Cowgill
University of Nebraska - Lincoln

Published in Bull. Amer. Math. Soc. 41 (1935) 541-549.

ON THE SUMMABILITY OF A CERTAIN CLASS OF SERIES OF JACOBI POLYNOMIALS*

BY A. P. COWGILL

1. *Introduction.* The result obtained in this paper is as follows:

The series

$$\sum_{n=1}^{\infty} n^i \frac{(p+1)(p+3)\cdots(p+2n-1)}{2^n n!} X_n^{(p-1)/2}(x),$$

where $X_n^{(p-1)/2}(x)$ *(hereafter indicated simply by* X_n*) is a symmetric Jacobi polynomial,*† $p>-1$, *and* i *a positive integer, is summable* (C, k), $k>i-1/2$, *for the range* $-1<x<1$.

The proof is limited to symmetric Jacobi polynomials because of the necessity of having the recursion formula‡ of first degree in n. Unless explicitly stated otherwise, x is confined to the range $-1<x<1$, and $p>-1$, throughout this paper.

In the proof the sum of the n first terms of the given series is transformed, following the method employed by Brenke for Hölder summability of certain series of Legendre polynomials,§ by the recursion formula for Jacobi polynomials into a new sum of n terms, plus four additional terms. Then convergence factors for summability $(C, i-1)$ are applied, followed by those of summability (C, j), $j>1/2$, necessary to evaluate the additional

* Presented to the Society, November 30, 1934. This paper is a portion of a thesis presented to the Faculty of the Graduate College, The University of Nebraska.

† This is denoted by $F(-n, n+p, (p+1)/2, (1-x)/2)$ in the notation of Darboux, *Mémoire sur l'approximation des fonctions de très grands nombres*, Journal de Mathématiques, (3), vol. 4 (1878), pp. 5–60, 377–416; p. 22. It is $G_n(p, (p+1)/2, (1-x)/2)$ in the notation of R. Courant and D. Hilbert, *Methoden der Mathematischen Physik*, vol. 1, p. 74. It is $X_n^{((p-1)/2,\,(p-1)/2)}(x)$ in the notation of G. Pólya and G. Szegö, *Aufgaben und Lehrsätze aus der Analysis*, vol. 2, pp. 93–94, where the orthogonality property is expressed by means of the equality $\int_{-1}^{1}(1-x)^{(p-1)/2}(1+x)^{(p-1)/2}X_n^{((p-1)/2,(p-1)/2)}X_m^{((p-1)/2,(p-1)/2)}dx=0(m\neq n;\, m, n=0, 1, \cdots)$.

‡ Darboux, loc. cit., p. 378.

§ W. C. Brenke, *On the summability and generalized sum of a series of Legendre polynomials*, this Bulletin, vol. 39 (1933), pp. 821–824.

terms involving n, making the equivalent of summability (C, k), $k>i-1/2$. The first application of summability causes the highest ordered part of the sum of the two additional terms involving n to take the form of a series, which is equal to the Cauchy product of two series, one of which is summable to a finite value and the other to the value zero.

Certain well known theorems of summability are used.*

THEOREM 1. *If two series are respectively summable* (C, α) *and* (C, β), *their Cauchy product is summable* $(C, \alpha+\beta+1)$ *to the product of the sums of the series.*

THEOREM 2. *If* $\sum_{n=0}^{\infty} u_n$ *is summable* (C, δ) *to the value* s, *this implies that* $\lim_{x\to 1-0} \sum_{n=0}^{\infty} u_n x^n = s$.

THEOREM 3. *A series that is bounded* $(C, \alpha), \alpha \geqq -1$, *is summable* $(C, \delta>\alpha)$ *with the sum* s, *if* $\lim_{x\to 1-0} \sum_{n=0}^{\infty} u_n x^n = s$.

THEOREM 4. *The summability* (C, δ) *of* $\sum u_n$ *implies* $u_n = o(n^\delta)$ *and* $s_n = o(n^\delta)$.

THEOREM 5. *The existence of* $\lim_{n\to\infty} s_n^{(\delta+\gamma)} = s$, *where* $s_n^{(\delta+\gamma)}$ *denotes the Cesàro nth partial mean of order* $\delta+\gamma$, *implies that of the double mean* $S_n^{(\delta,\gamma)}$ *of orders* δ *and* γ:

$$\lim_{n\to\infty} S_n^{(\delta,\gamma)} = \lim_{n\to\infty} S_n^{(\gamma,\delta)} = \lim_{n\to\infty} s_n^{(\delta+\gamma)},$$

and vice versa, provided $\delta, \gamma, \delta+\gamma > -1$.

2. *The Polynomial* X_n. The recursion formula is†

$$xX_n = \frac{n+p}{2n+p} X_{n+1} + \frac{n}{2n+p} X_{n-1}. \tag{1}$$

The generating function,‡ when

$$a_n = \frac{(p+1)(p+3)\cdots(p+2n-1)}{2^n n!} = \frac{\Gamma((p+1)/2+n)}{\Gamma((p+1)/2)\Gamma(n+1)}$$
$$= O(n^{(p-1)/2}),$$

is

* E. Kogbetliantz, Mémorial des Sciences Mathématiques, vol. 51, pp. 19–20, 29, 37, 30–31, and 23, respectively.

† Darboux, loc. cit., p. 378.

‡ Darboux, loc. cit., p. 23.

$$
(2)\qquad \begin{aligned} &\frac{((1-x^2)/4)^{(1-p)/2}[2tx-2+2(1-2tx+t^2)^{1/2}]^{(p-1)/2}}{(2t)^{p-1}(1-2tx+t^2)^{1/2}} \\ &= \sum_{r=0}^{\infty} a_r X_r t^r, \qquad (0<t<1). \end{aligned}
$$

I find by the method used for Legendre polynomials by Byerly* that X_r in the following formula

$$
(3)\qquad \sum_{r=0}^{\infty} \frac{\Gamma(r+p)}{\Gamma(p)\Gamma(r+1)} X_r t^r = \frac{1}{(1-2tx+t^2)^{p/2}}, \quad (0<t<1),
$$

satisfies the Jacobi recursion formula (1). We have also

$$
X_n = O(n^{-p/2}), \qquad (-1<x<1).\dagger
$$
$$
X_n(1) = 1.\ddagger
$$

3. *Transformation of the Series.* Multiplying (1) by c_n and summing from 1 to n, we get

$$
(4)\qquad \begin{aligned} x\sum_{r=1}^{n} c_r X_r &= \sum_{r=1}^{n} \frac{r+p}{2r+p} c_r X_{r+1} + \sum_{r=1}^{n} \frac{r}{2r+p} c_r X_{r-1} \\ &= \sum_{r=1}^{n}\left[\frac{r-1+p}{2r-2+p} c_{r-1} + \frac{r+1}{2r+2+p} c_{r+1}\right] X_r \\ &\quad - c_0 X_1 + \frac{n+p}{2n+p} c_n X_{n+1} + \frac{1}{2+p} c_1 X_0 \\ &\quad - \frac{n+1}{2n+2+p} c_{n+1} X_n. \end{aligned}
$$

If $c_n = n^i a_n$, the coefficient of X_r in (4) takes the form

$$
U_r \equiv [r^i + b_1(p) r^{i-2} + b_2(p) r^{i-3} + \cdots]a_r,
$$

where the coefficient of a_r is a polynomial in descending powers of r. The last term in the square brackets will be of order $O(n^\delta)$,

* W. E. Byerly, *Fourier Series*, 1893, p. 151. See also N. Nielsen, *Théorie des Fonctions Métasphériques*, 1911.

† Darboux, loc. cit., p. 378.

‡ This is shown by using equation (1), Darboux, loc. cit., p. 377, and making the transformation $x=(1-\xi)/2$, $x=0$ corresponding to $\xi=1$.

$\delta \leqq -1$, which, when multiplied by $a_r X_r$ and summed, will give a convergent series, for

$$a_r X_r O(n^\delta) = O(n^{(p-1)/2-p/2+\delta}) = O(n^{\delta'}),$$

where $\delta' = \delta - 1/2 < -1$.

The terms in (4) free from n are

$$R_0 \equiv -c_0 X_1 + \frac{1}{2+p} c_1 X_0.$$

These carry over without change in the process of summation.

The *remainder* terms in (4) will be

$$R_n \equiv \frac{n+p}{2n+p} c_n X_{n+1} - \frac{n+1}{2n+2+p} c_{n+1} X_n. \tag{5}$$

The relation (4) can now be written in the form

$$x \sum_{r=1}^{n} c_r X_r = R_0 + \sum_{r=1}^{n} U_r X_r + R_n, \qquad (c_r = r^i a_r). \tag{6}$$

4. *Application of Summation* (C, k) *to* (6). Apply Cesàro summation of order k to both sides of (6). Representing by $S_{n,i}^{(k)}$ the kth Cesàro mean of $S_{n,i} = \sum_{r=0}^{n} r^i a_r X_r$, we find after transposing the sum $S_{n,i}^{(k)}$ from the right to the left side of equation (6),

$$(x-1)S_{n,i}^{(k)} = R_0 + b_1(p)S_{n,i-2}^{(k)} + \cdots + b_{i-1}(p)S_{n,0}^{(k)}$$
$$+ [S_n^{(k)} \text{ of a convergent series}] + (C, k) \text{ of } R_n.$$

The order of the terms of R_n is $O(n^{i-1/2})$, so, by Theorem 4, Cesàro summability of order $< i - 1/2$ could not be expected.

Then, applying Cesàro summation of order k, so chosen that $(C, k)R_n \to 0$ as $n \to \infty$, and writing $\lim_{n\to\infty} S_{n,h}^{(k)} = S_{\infty,h}^{(k)}$, we have $S_{\infty,i}^{(k)}$ expressed in terms of R_0, $S_{\infty,i-2}^{(k)}$, $S_{\infty,i-3}^{(k)}$, $\cdots$, $S_{\infty,0}^{(k)}$ and a convergent series. Values of $S_{\infty,i}^{(k)}$ must be calculated successively; we begin with $S_{\infty,0}^{(k)}$, which can be obtained from the generating function (2), and take successive integral values of i, beginning with $i=1$. As stated in the introduction, two applications of Cesàro summation, which are equivalent to summation (C, k), are then used to make (C, k) $R_n \to 0$ as $n \to \infty$.

5. *Summation* $(C, i-1)$ *Applied to* R_n. The convergence factors for summability (C, k) have the form

$$\frac{\Gamma(k+n-r+1)}{n^k\Gamma(n-r+1)},$$

where n represents the total number of terms in the sum under consideration and r the rank of the particular term to which the convergence factor is applied. One groups the remainder R_n with the nth term of the sum in the right-hand member of (6) so that the nth Cesàro convergence factor will be applied to R_n as well as to U_nX_n. The remainder R_n becomes, after application of summation $(C, i-1)$,

$$\begin{aligned} R_n^{(i-1)} &= \Gamma(i)\Bigg[\frac{(n+p)n}{2n+p}X_{n+1} \\ &\qquad - \frac{(n+1)^i}{(2n+2+p)}\frac{(p+2n+1)}{2\cdot n^{i-1}}X_n\Bigg]a_n, \end{aligned} \tag{7}$$

$$R_n^{(i-1)} = \frac{1}{2}a_n\Gamma(i)\left[1+O\left(\frac{1}{n}\right)\right](n+p)(X_{n+1}-X_n). \tag{8}$$

To evaluate this expression as $n\to\infty$, the Christoffel-Darboux identity is used.

6. *Use of the Christoffel-Darboux Identity.* This identity, when modified to conform to the notation of this paper,* becomes

$$\begin{aligned} &\frac{\Gamma(n+p)}{\Gamma(p)\Gamma(n+1)}(n+p)\frac{X_{n+1}-X_n}{x-1} \\ &\qquad = \sum_{r=0}^{n}\frac{\Gamma(r+p)}{\Gamma(p)\Gamma(r+1)}(2r+p)X_r. \end{aligned} \tag{9}$$

Differentiate both sides of (3), multiply throughout by $2t$, and add to (3); we get

$$\sum_{r=0}^{\infty}\frac{\Gamma(r+p)}{\Gamma(p)\Gamma(r+1)}(2r+p)X_rt^r = \frac{p(1-t^2)}{(1-2tx+t^2)^{p/2+1}},$$

* Darboux, loc. cit.; one substitutes (44), p. 46, in (14), p. 381. After changing x and z into $(1-x)/2$ and $(1-z)/2$, respectively, let $\alpha=p$ and $\gamma=(p+1)/2$. Then let $z=1$, so that $Z_n=Z_{n+1}=1$, and simplify.

which one can write in the form

$$\sum_{r=0}^{\infty} \frac{\Gamma(r+p)}{\Gamma(r+1)} (2r+p)X_r t^r = \frac{\Gamma(p+1)}{(1-2tx+t^2)^{p/2}} \frac{1-t^2}{1-2tx+t^2}, \tag{10}$$

which incidentally checks with well known relations for Tchebychef (trigonometric) polynomials, where $p=0$, and Legendre polynomials, where $p=1$.

From an article by Fejér,* we have

$$\frac{1-t^2}{1-2tx+t^2} = 2\left(\frac{1}{2} + \sum_{r=1}^{\infty} t^r \cos rv\right), \qquad (x = \cos v),$$

which is the generating function of the trigonometric polynomials. Chapman proved† that $(1/2+\sum_{r=1}^{\infty} \cos rv)$ is summable (C, k), $k>0$, to the value zero.

To obtain the order of summability of

$$\sum_{r=0}^{\infty} \frac{\Gamma(r+p)}{\Gamma(p)\Gamma(r+1)} X_r,$$

(which is formula (3) for $t=1$), we may use the method of proof given by Chapman‡ for Legendre polynomials, based on obtaining an asymptotic expression for $S_n^{(k)}$ for the above series by the method of Darboux. Let $x=\cos\theta$. Since

$$\frac{1}{(1-2z\cos\theta+z^2)^{p/2}} = \sum_{r=0}^{\infty} \frac{\Gamma(r+p)}{\Gamma(p)\Gamma(r+1)} X_r(\cos\theta) z^r,$$

one has

$$\frac{1}{(1-z)^{k+1}(1-2z\cos\theta+z^2)^{p/2}} = \sum_{n=0}^{\infty} S_n^{(k)} z^n.$$

* L. Fejér, *Über die Laplacesche Reihe*, Mathematische Annalen, vol. 67 (1909), pp. 76–109; p. 81.

† S. Chapman, *The general principle of summability*, Quarterly Journal of Mathematics, vol. 43 (1912), pp. 1–52; p. 27.

‡ Ibid., p. 45.

The generating function of the sequence $\{S_n^{(k)}\}$ is consequently

$$\frac{1}{(1-z)^{k+1}(1-2z\cos\theta+z^2)^{p/2}}$$
$$=\frac{1}{(1-z)^{k+1}(1-e^{i\theta}z)^{p/2}(1-e^{-i\theta}z)^{p/2}}.$$

This function may (for $0<\theta<\pi$) be developed into a power series in z with unit radius of convergence. If $k+1>p/2$, the predominant singularity on the circle of convergence is at $z=1$. Therefore, the leading term in the asymptotic expression of $S_n^{(k)}$ is given by the coefficient of z^n in the expansion of

$$\frac{1}{(1-e^{i\theta})^{p/2}(1-e^{-i\theta})^{p/2}(1-z)^{k+1}}\equiv\frac{1}{[2(1-\cos\theta)]^{p/2}}\sum_{n=0}^{\infty}A_n^{(k)}z^n.$$

Hence

$$S_n^{(k)}=\frac{1}{[2(1-x)]^{p/2}}A_n^{(k)}(1+o(1)),\quad (x=\cos\theta),$$
$$\lim_{n\to\infty}\frac{S_n^{(k)}}{A_n^{(k)}}=\frac{1}{[2(1-x)]^{p/2}},\quad (-1<x<1),$$

so that the series is summable (C, k) for $k+1>p/2$.

7. *Second Application of Cesàro Summation to R_n.* After application of summation $(C, i-1)$, the remainder terms (5) take the form (8), of which the highest ordered part is, from (9),

$$(11)\quad \begin{aligned}&(1/2)\Gamma(i)a_n(n+p)(X_{n+1}-X_n)\\&=(1/2)\Gamma(i)a_n\frac{\Gamma(n+1)}{\Gamma(n+p)}(x-1)\sum_{r=0}^{n}\frac{\Gamma(r+p)}{\Gamma(r+1)}(2r+p)X_r.\end{aligned}$$

From (10), if we let $t=1$, and make use of the equation (3) and the equation below (10), we find

$$\sum_{r=0}^{\infty}\frac{\Gamma(r+p)}{\Gamma(r+1)}(2r+p)X_r=\text{formal Cauchy product}$$
$$\left[\Gamma(p+1)\sum_{r=0}^{\infty}\frac{\Gamma(r+p)}{\Gamma(p)\Gamma(r+1)}X_r\right]\cdot\left[1+2\sum_{r=1}^{\infty}\cos rv\right],$$

and this is summable (C, f) by Theorem 1 (combined with the above result of Chapman) to the value

$$\frac{1}{[2(1-x)]^{p/2}}\cdot 0 = 0,$$

where $f=(p/2-1)+k+1>p/2$, (for $k>0$). Hence, from (11),

$$(1/2)\Gamma(i)a_n \frac{\Gamma(n+1)}{\Gamma(n+p)}(x-1)\sum_{r=0}^{n}\frac{\Gamma(r+p)}{\Gamma(r+1)}(2r+p)X_r$$
$$= O(n^{(p-1)/2+1-p})o(n^f) = o(n^{f+(1-p)/2}),$$

by Theorem 4, $f>p/2$. Now apply summation (C, j). The convergence factor multiplying $R_n^{(i-1)}$ is

$$\frac{\Gamma(j+1)}{n^j}(1+o(1)),$$

so that the remainder terms (5) now become of order

$$o(n^{f+(1-p)/2})O(n^{-j}) = o(n^{f+(1-p)/2-j}) = o(1),$$

if $f+(1-p)/2-j\leqq 0$, that is, $j>1/2$.

Thus the two applications of Cesàro summability $(C, i-1)$ and $(C, j>1/2)$, which are seen to be equivalent to summability $(C, k>i-1/2)$ by Theorem 5, cause the highest ordered part of the remainder terms (5), R_n, to approach zero as $n\rightarrow\infty$. The other parts of (8), being of lesser order, likewise approach zero by application of summability $(C, k>i-1/2)$. The value of $S_{\infty,i}^{(k>i-1/2)}$ can now be calculated as indicated in §4.

8. *Legendre Polynomials*. By letting $p=1$, one can easily obtain the values of $S_{\infty,i}^{(k)}$. In this case $a_n=1$ and $\sum_0^\infty X_n = 1/(2-2x)^{1/2}$. The remainder terms are handled by the use of Christoffel's formula

$$\sum_{0}^{n}(2i+1)X_i = (n+1)\frac{X_{n+1}-X_n}{x-1}, \tag{12}$$

the series $\sum_0^\infty(2n+1)X_n$ having been previously proved summable $(C, k>1/2)$ to the value zero by Chapman.* The results obtained by this method check with those obtained by Brenke†

* Chapman, loc. cit., p. 46.

† Brenke, loc. cit.

with the Hölder method of summability, where $i=1, 2, 3$. In the latter case, for example,

$$(x - 1)S_{\infty,3}^{(k)} = 2S_{\infty,1}^{(k)} + \frac{1}{4}S_{\infty,0}^{(k)} + \frac{1}{3}$$
$$- \frac{1}{4}\left[(C, k) \text{ of } \sum_{1}^{\infty} \frac{1}{(2r - 1)(2r + 3)} X_r\right], \qquad (k > 5/2).$$

THE UNIVERSITY OF NEBRASKA