7-2009

# THE MONTY HALL PROBLEM

Brian Johnson
*University of Nebraska-Lincoln*

# THE MONTY HALL PROBLEM

**Brian Johnson**

In partial fulfillment of the requirements for the Master of Arts in Teaching with a Specialization in the Teaching of Middle Level Mathematics in the Department of Mathematics.
Gordon Woodward, Advisor

July 2009

## The Monty Hall Problem

*In the game show "Let's Make a Deal", host Monty Hall would present a contestant with three doors. Behind one door is a car and behind the other two doors are goats. The contestant picks one door, after which Monty opens one of the other doors, showing a goat. Monty then offers the contestant the opportunity to keep the same door or switch to the other unopened door; the contestant will get to keep whatever is behind that door. Should the contestant switch?*

There are several published probability problems that are in essence of the Monty Hall problem, but it wasn't until September of 1990 that this problem really became popular. Marilyn vos Savant was known from 1986 to 1989 for having the highest IQ in the world, according to the Guinness Book of World Records (since then, Guinness does not recognize this category, as it is difficult to quantify). Marilyn writes a column for Parade Magazine called *Ask Marilyn* in which she solves puzzles and answers questions from readers. In the September 9th, 1990, issue of Parade magazine, the Monty Hall problem became mainstream. In her answer to the Monty Hall problem, she argued that the contestant should switch doors, because in switching, the contestant has moved from a 1/3 chance of winning the car to a 2/3 chance of winning the car.

Her answer sparked a national debate regarding this problem. Experienced mathematicians wrote letters to vos Savant arguing against her initial answer, and these letters invited a second column from vos Savant, causing more of a stir and leading to a front page article in *The New York Times.* In this follow-up, she further explained her assumptions and reasoning. She also called on school teachers to present this problem to their classrooms and write back to her with the results. Vos Savant's final column on the Monty Hall problem showed the results of these classroom experiments. Nearly 100% of them concluded that she was right and that the contestant should switch doors.

The column Marilyn vos Savant originally wrote is now 19 years old, but this has not stopped the attention it still receives today. More recently, this problem was addressed in a lecture setting during a 2005 episode of the CBS drama *NUMB3RS.* The problem also made a Hollywood appearance in the 2008 film *21*.

The Monty Hall problem lends itself to analysis by probability. Since the contestant in the game show has been shown one of the doors, the possible outcomes have changed. Initially the contestant had a 1 in 3 chance of picking where the car was located. Now that the host has shown one of the goats, the probability of the door the contestant has not picked has changed. This is called conditional probability. Conditional probability is the probability that some event happens, given the occurrence of some other event that already has happened. An example of conditional probability is given in the simple problem below.

*Your neighbor has 2 children. You meet one of the children, and it is a daughter. What is the probability that the other child is also a girl?*

In this case, as in the definition, there are two events to consider. We want to know if our neighbor has two daughters, given that we met one of his children and it was a girl. The typical, but wrong, answer is ½--this conclusion might be reached by reasoning that there are only two possible sexes for the other child. However, having two children, there are actually four possible combinations for our neighbor: {Boy, Boy | Boy, Girl | Girl, Boy | Girl, Girl}, with the older child listed first. Note, here, that we are counting the terms {Boy, Girl} and {Girl, Boy} as two separate terms. The girl we met may have an older or younger brother, and so we must count a girl and boy pair as two possibilities. Looking at the list of options, we can see there is only one out of four cases in which our neighbor has two daughters. This is where conditional probability comes in. We already know he has a daughter, so we can throw out the possibility that he has two sons. Now, we only have three possible outcomes for our neighbors: {Boy, Girl | Girl, Boy | Girl, Girl}. They might have a younger son, an older son or another daughter. Since we already have met one of the children, the probability that our neighbor has two girls is 1 out of 3. This problem takes a different turn if our neighbor tells us that the daughter we met is the oldest or youngest of their two children. With conditional probability, we have to make sure we're looking at all of the possible outcomes for the situation.

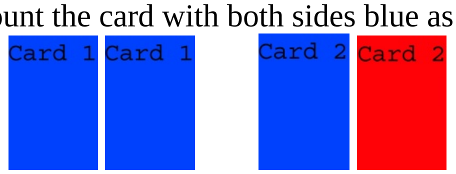
Above, we made an important assumption about the neighbors two children; we assumed that the neighbor does not reveal whether the girl we met is the youngest or oldest of the two siblings. If the neighbor had said, "I'd like you to meet my youngest child", the probabilities would be changed. We could then take out the term {Girl, Boy} as a possible outcome, and we would be left with possible outcomes of {Boy, Girl} and {Girl, Girl}. We would now have a 1 in 2 chance that the other child is also a girl. The same reasoning can be used, of course, if the neighbor says that we are meeting his youngest child.

A similar problem with which someone could really trick their friends has to do with three different colored cards:

*Suppose you have three cards: one that is blue on both sides, one that is red on both sides, and one that is red on one side and blue on the other. Suppose that the three cards are put in a hat, and one card is randomly drawn and put on a table. If the top of the drawn card is blue, what is the probability that the other side is also blue?*

In this problem, we know that the card on the table shows blue. Someone who does not know conditional probability might think this has a simple solution. They might reason that there are only two cards with the color blue on them; one of those cards has another blue side and the other card has a red side, thus giving a 50-50 chance of having blue on the other side. Such reasoning is wrong, however, and as in the previous question about the neighbor, we need to look at all of the outcomes.

First, let's count how many sides have blue—there are a total of three sides with the color blue (we have to count the card with both sides blue as two).

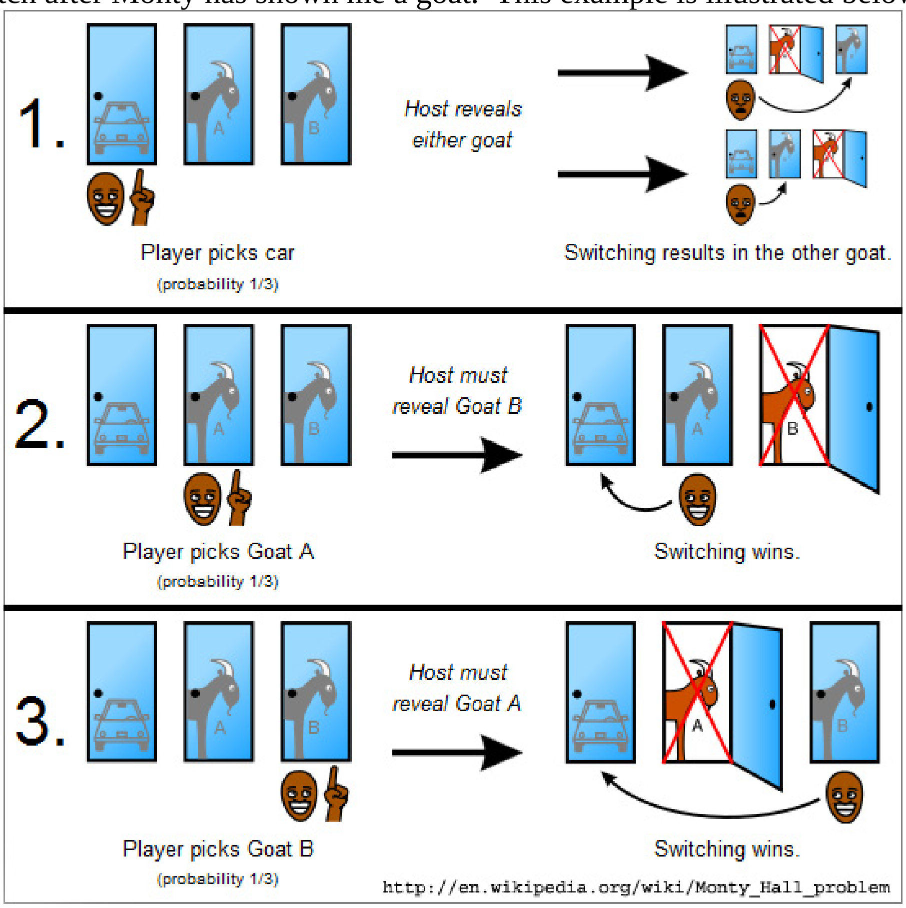



Now, we can count the outcomes. If card 1 was picked out of the hat, there are two possible sides that it could be showing. If card 2 was picked out of the hat, there is only 1 possible side it could be showing. There are three total possibilities in which the card is already showing blue. There are two out of three cases in which the card is also blue on the other side. This is a great trick for friends. I would show them the cards and explain the rules. If the same color is on the other side I win, and if the opposite color is on the other side they win. The probability of the card hiding a blue side is 2 out of 3.

## Monty Hall

Now that we have looked at how conditional probability applies to simple situations, let's take a closer look at the Monty Hall problem. We want to find the probability of a contestant winning a car if she switches doors after Monty has opened a door revealing a goat. Before we jump into this problem, we first have to understand what Monty Hall is trying to do. If I were Monty, I would make sure I knew exactly where the car and the two goats were before playing the game. With this in mind, no matter which door we initially pick, Monty is always going to reveal a goat to us. Later, we'll look at a case in which Monty does not know where the car is located.

I will label the doors X, Y and Z, and suppose that Monty knows that the car is behind door X and that the goats are behind doors Y and Z. If I choose door X and switch I will lose, no matter which door Monty opens (I have already picked the car, and if I switch I will have a goat). If I choose door Y, Monty must open door Z, because he knows the car is behind door X. Similarly, if I choose door Z, Monty will open door Y in this case. Here there are two cases in which switching doors results in my winning the car. I have found out that I will win 2 out of 3 times if I switch after Monty has shown me a goat. This example is illustrated below.



In this problem, we have made the assumption that Monty knows where the car is. From the illustration above, there is the case in which I pick the door with the car initially. In that case, Monty actually has two doors he could pick revealing a goat. If Monty were unsure of which one to pick, I would have to consider those two doors as different outcomes. Now, we have four outcomes that could happen. In this case, the contestant now has 2 out of 4 chances of winning the car, if they switch after Monty reveals a goat.

This question specifically looks at the contestant's chances of winning if the contestant switches doors. I did not count the other two possible outcomes that could occur if Monty does not know where the car is. In the illustration of step two and three above, Monty, if he did not know which door concealed a car, could have actually revealed the car instead of a goat on his turn.. Now, there are a total of six possible solutions. In these two new outcomes, the contestant loses. Since we have added two more outcomes, the chance of the contestant winning is now 1/2.

This door-switching problem doesn't have to be so simple. What if the contestant had 1,000 doors to choose from, with only one door concealing a car and all others concealing goats? Initially, the contestant has a one in a thousand chance of picking the car. From here, the host knows where the car is and reveals 998 goats, leaving only two doors closed. If the contestant knew nothing of the Monty Hall Problem, he might feel that the odds of the car sitting behind his initially chosen door improved to one out of two. This a great increase in probability from the time when all of the doors were closed. In fact the contestant should switch doors every time. If the contestant happened to have to pick the car initially, he will lose if he switches, but this is only a one out of a thousand chance that he will lose. It is far more likely (999 out of 1,000) that the contestant picked a goat initially. When the host reveals 998 goats, the contestant should definitely switch from his original door.

If the host decides to only open one door out of the 999 that are left, the contestant does not significantly increase the odds of winning a car by switching. In this case, the contestant will be losing roughly 99% of the time either way.

In 1991, a *New York Times* journalist named John Tierney went to Monty Hall to discuss this new found phenomenon that resulted from the column written by Marilyn vos Savant. Monty knew of the attention it was getting and was happy to show Mr. Tierny exactly how the problem works by playing the game in the dining room. "Sitting at the dining room table, Mr. Hall quickly conducted 10 rounds of the game as this contestant tried the non-switching strategy. The result was four cars and six goats. Then for the next 10 rounds the contestant tried switching doors, and there was a dramatic improvement: eight cars and two goats. A pattern was emerging." (Tierney, 1991) By simple trial and error, Mr. Hall was able to show the journalist that a contestant should switch their original pick if Monty knew where the car was before hand.

In my research, I was able to find an interview with Mr. Hall on an early morning talk show in which he said he always knew what was behind the doors or inside the boxes on his game show. The point of knowing what is behind the curtains or in the boxes was so that he could offer them something bigger or less than what they had already won.

The *New York Times* article continued with Monty explaining how none of these assumptions or statistics really had much to do with the game show anyway, how it works out much differently in actuality. Monty read Ms. vos Savant's article and noticed that she did not take human emotion into account, unlike in the actual game show. Monty then asked the reporter to play a few more rounds, but this time, they would play the game like the actual show. The contestant picked door number one and Monty opened one of the remaining two doors to reveal a goat. Just before the contestant was going to switch doors— knowing he has a two-thirds chance of winning if he switches—Monty pulled out a roll of money. This time, Mr. Hall used a little psychology on the contestant and offered him three thousand dollars not to switch doors. The journalist tried to stick with the odds of him winning the car by switching and wanted to switch the door and not take the money. The two gentlemen negotiated back and forth until Monty finished by offering him five thousand dollars not to switch. The journalist declined the offer and switched doors, only to see Monty reveal a goat behind the door he had just switched to.

Monty explained that by increasing the dollar amount and trying to get the contestant not to switch, he made the contestant believe even more that the car would be behind the door he initially wanted to switch to. Monty and Mr. Tierney continued to play the game as it was intended to be played, with this psychological aspect: "He [Monty] proceeded to prove his case by winning the next eight rounds. Whenever the contestant began with the wrong door, Mr. Hall promptly opened it and awarded the goat; whenever the contestant started out with the correct door, Mr. Hall allowed him to switch doors and get another goat. The only way to win a car would have been to disregard Ms. vos Savant's advice and stick with the original door." (Tierney, 1991)

In conclusion, we have seen how the famous Monty Hall problem works and what a contestant's best option is, given the conditions of the game. If the host has no extra cash prize with which to entice the contestant, it is no advantage for the host to know the position of the car, but if the host does have such a cash enticement for the contestant, then knowing the position of the car allows him to use psychological knowledge about people to direct the contestant toward a smaller prize.

## Bayes' Theorem

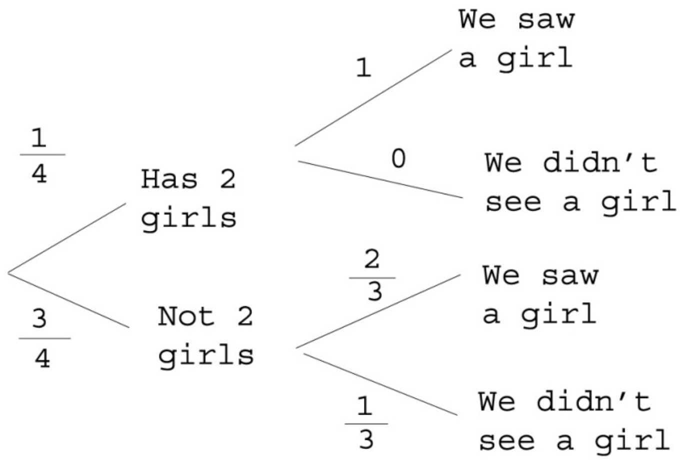
Bayes' Theorem, named after Rev. Thomas Bayes, can be used when solving conditional probability problems using algebra. In the previous examples, we looked at how such problems can be solved by talking through the problem and looking at possible outcomes. However, we can also use Bayes' Theorem to analyze these situations. This is Bayes' Theorem:

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)}$$

The left-hand-side of this the formula is read, "The probability of event A given event B", and gives us a way to talk about conditional probability: The probability of some event A happening given the occurrence of some other event B already having happened. Let's take a look at the application of Bayes' Theorem to the Neighbor Problem, stated previously but repeated here:

> *Your neighbor has 2 children. You meet one of the children, and it is a daughter. What is the probability that the other child is also a girl?*

We begin by looking at the following counting chart:




Here, we can see all of the possibilities. The question considered here is, "what is the probability that our neighbor has two daughters, given that he has two children, and one of them is a daughter". This is a conditional probability, so we can define variables for Bayes' Theorem. Let "A" be the event that our neighbor has two girls and "B" be the event, or fact, that one of them is a girl. An interesting thing happens with the top portion of our picture. If our neighbor has two girls and we happen to meet one of our neighbor's children, there is a 100% chance it will be a girl and a 0% chance that we will meet a boy. Now, we'll use Bayes' Theorem to solve our problem.

$$P(B|A) = \textit{Probability we saw a girl given that he has two girls} = 1$$
$$P(A) = \textit{Probability that he has two girls} = \frac{1}{4}$$
$$P(B) = \textit{Probability that we saw a girl} = \frac{1}{4}(1) + \frac{3}{4}\left(\frac{2}{3}\right) = \frac{3}{4}$$
$$P(A|B) = \frac{1\left(\frac{1}{4}\right)}{\frac{3}{4}} = \frac{1}{3}$$

Our conclusion using Bayes' Theorem agrees with our previous conclusion based on just totaling up the outcomes. Though I would not use this theorem on a simple type of problem as such as this, there are more difficult situations where Bayes' Theorem would be most useful.

Now that we have an example of a use of Bayes' Theorem, we can use this formula for the Blue and Red Card Problem and The Monty Hall Problem. First, let's look at the Blue and Red Card problem. Recall that one of the three double-sided cards is pulled out of a hat and placed on a table. Let "A" be the event that the color on the bottom side of the card is blue, and let "B" be the event that the color showing is blue. I will show that in this case, using Bayes' Theorem does not help much. In the first line below, we see P(B|A) is equal to 2/3. This is the probability that we had a blue side on top given there is a blue side on the bottom. Finding this probability is symmetric to our original question.

$$P(B|A) = \textit{Probability we had blue given blue is on the bottom} = \frac{2}{3}$$

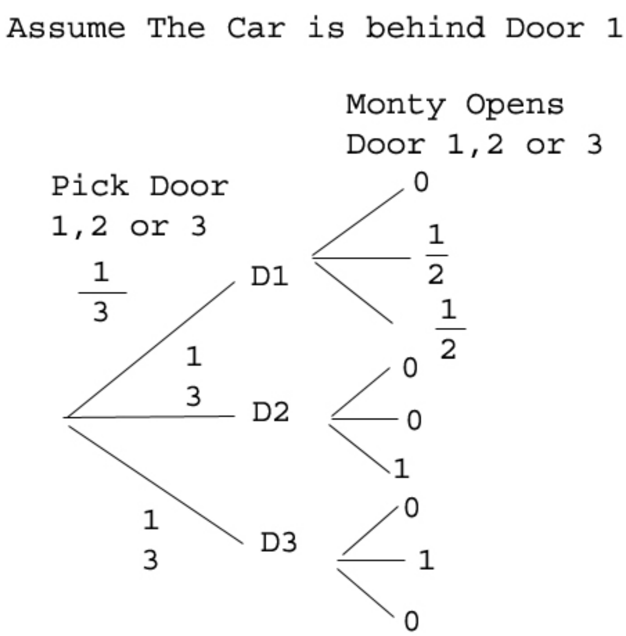
$P(A) =$ Probability that blue is on the bottom $= \frac{1}{2}$

$P(B) =$ Probability the card on the table shows blue $= \frac{1}{2}\left(\frac{2}{3}\right) + \frac{1}{2}\left(\frac{2}{3}\right) = \frac{1}{2}$

$$P(A|B) = \frac{\frac{2}{3}\left(\frac{1}{2}\right)}{\frac{1}{2}} = \frac{2}{3}$$

A two-thirds chance of having blue on the bottom of the drawn card was our original answer for the card problem. Here, finding P(B|A) was just as difficult as finding P(B|A), so Bayes Formula was not so useful. This problem was very similar to the two daughters problem.

With the Monty Hall problem, there are three doors, and the car could be behind any one of them. Clearly, there is a one-third possibility that the car is behind door #1. After the contestant opens a door, Monty will then open a different door, revealing a goat. Using the diagram below, we can see what the probability is, given different assumptions. First, let's assume the car is behind door #1. Since we're assuming the car is behind door #1, we know that if the contestant opens either door #2 or door #3, Monty has only one other option to open a goat door.. Let's take a look at the probability that the car is behind #1 and Monty opens door #2.




Here, we can name our variables A and B. Let A be the event that the contestant picked door #1 and B the event that Monty opens door #2.

$P(B|A) =$ Probability Monty opens Door 2 given we picked Door 1 $= \frac{1}{2}$

$P(A) =$ Probability that we pick Door 1 $= \frac{1}{3}$

$$P(B) = \Pr obability\ that\ Monty\ opens\ Door\ 2 = \frac{1}{3}\left(\frac{1}{2}\right) + \frac{1}{3}(0) + \frac{1}{3}(1) = \frac{1}{2}$$

$$P(A|B) = \frac{\frac{1}{2}\left(\frac{1}{3}\right)}{\frac{1}{2}} = \frac{\frac{1}{6}}{\frac{1}{2}} = \frac{1}{3}$$

One-third is what we expected to find—we have a one-third chance of winning the car given that Monty opens door number 2 to reveal a goat. Let's look at what happens if we do switch doors after he shows the contestant a goat. In this case, we just figure out the exhaustive event.

The probability that we pick door #2 and Monty opens door #2 is zero. This is our second case. Our third case is simply one minus the previous two cases.

$$1 - \frac{1}{3} - 0 = \frac{2}{3}$$

This shows us that, given the car is behind door #1, if the contestant initially picks door #3 and Monty opens door #2, we have a two-thirds chance of winning the car by switching doors. If we were to continue this analysis and consider the case where the car is behind door #2, for example, we would find results nearly identical to those above.

## Classroom

For my classroom project, I will have the class test and research the Monty Hall problem.

Day 1

**Introduce**: On the first day, I will play the blue/red card game with a student a few times as a demonstration. I will take the same color that is shown, and the student will always take the opposite color. After this, I will have the Monty Hall problem set up on the computer and we'll play it a few times. I will not call it the Monty Hall problem, however, for research reasons.

**Test:** After we have played the two games enough to get the idea—but not enough to see the pattern—I will have them split up into groups of 2 or 3 to test these games at least 20 times. Half of the class will play the card game, while the other half will play the Monty Hall game.

**Results:** I will have the students display their results on the two whiteboards in my room.

**Discuss:** With this many students playing the game a total of over 100 times, we should be able to see a pattern. I'll lead the discussion and ask students what their guess is about the actual probability of switching doors or picking a certain color. After the discussion, I'll have them work on probability homework.

Day 2

**Research:** Because most students use the internet quite frequently, I have only shown them two interesting problems. They do not know the name Monty Hall problem, but I would like them to figure this out on their own by using the internet. During their research time in the computer lab, the students will be asked to write down any notes they feel are interesting. I will also ask them to copy down websites that they found interesting. They will be asked to find information about the history of the Monty Hall problem, especially any regarding Ms. vos Savant. I will also put some questions on their research guide, such as:

What recent movie used this problem in one of their scenes?

What magazine was this problem published in that made it famous?

What TV show is this problem based off of?

The questions I create will all be easy enough to find, once they figure out that the common name for the problem is The Monty Hall Problem.

This test and research process is to be used at the beginning of a study of conditional probability. In my curriculum, we go through basic probability and a small amount of conditional probability. Adding these two days prior to the more difficult problems should help more students understand the material. Prior to the Monty Hall problem, I didn't have something that I could use to spark interest in probability. Now, with my understanding of these more difficult problems, I should be better prepared.

## References

Devlin, K. (2003, July). Devlin's Angle. *The Mathematical Association of America Online.* Retrieved from http://www.maa.org/devlin/devlin_07_03.html

Tierney, J., (1991, July 21). Behind Monty Hall's Doors: Puzzle, Debate and Answer. *New York Times*. Retrieved from http://www.nytimes.com/1991/07/21/us/behind-monty-hall-s-doors-puzzle-debate-and-answer.html

Wikipedia. the free encyclopedia, "Monty Hall Problem," Retrieved on June 25, 2009 from http://en.wikipedia.org/wiki/Monty_Hall_problem

Wikipedia. the free encyclopedia, "Conditional Probability," Retrieved on June 25, 2009 from http://en.wikipedia.org/wiki/Conditional_probability

Wikipedia. the free encyclopedia, "Bayes' Theorem," Retrieved on June 25, 2009 from http://en.wikipedia.org/wiki/Bayes'_theorem

## Websites of Interest

http://www.letsmakeadeal.com/problem.htm

Monty Hall interview - http://www.youtube.com/watch?v=GFrZ24H23Lg

NUMB3RS & 21 - http://www.youtube.com/watch?v=5e_NKJD7msg