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On Morrey Spaces in the Calculus of Variations

Kyle Fey
University of Nebraska-Lincoln, feyky01@luther.edu

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by

Kyle Fey

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ON MORREY SPACES IN THE CALCULUS OF VARIATIONS

Kyle Fey, Ph. D.

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We prove some global Morrey regularity results for almost minimizers of functionals of the form

$$\mathbf{u} \mapsto \int_{\Omega} f(\mathbf{x}, \mathbf{u}, \nabla \mathbf{u}) \mathrm{d}\mathbf{x}.$$

This regularity is valid up to the boundary, provided the boundary data are sufficiently regular. The main assumption on f is that for each $\mathbf{x}$ and $\mathbf{u}$, the function $f(\mathbf{x}, \mathbf{u}, \cdot)$ behaves asymptotically like the function $h(|\cdot|)^{\alpha(\mathbf{x})}$, where h is an N-function.

Following this, we provide a characterization of the class of Young measures that can be generated by a sequence of functions $\{\mathbf{f}_j\}_{j=1}^{\infty}$ uniformly bounded in the Morrey space $L^{p,\lambda}(\Omega; \mathbb{R}^N)$ with $\{|\mathbf{f}_j|^p\}_{j=1}^{\infty}$ equiintegrable. We then treat the case that each $\mathbf{f}_j = \nabla \mathbf{u}_j$ for some $\mathbf{u}_j \in W^{1,p}(\Omega; \mathbb{R}^N)$.

Lastly, we provide applications of and connections between these results.

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Finally, it is my hope that this work brings glory to God alone, for I am nothing but what he has made me, and have nothing but what I have been given. *Soli Deo gloria.*

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Chapter 1

Introduction

1.1 Background

The functionals usually considered in the calculus of variations have the form

$$J(\mathbf{u}) := \int_\Omega f(\mathbf{x}, \mathbf{u}(\mathbf{x}), \nabla\mathbf{u}(\mathbf{x}))\mathrm{d}\mathbf{x}, \tag{1.1}$$

where $\Omega \subset \mathbb{R}^n$, the function $f : \Omega \times \mathbb{R}^N \times \mathbb{R}^{N\times n} \to \mathbb{R}$, and the mapping $\mathbf{u} : \Omega \to \mathbb{R}^N$ belongs to some admissible class $\mathcal{A}$. In the classical theory, the admissible class $\mathcal{A}$ was usually taken to be some subset of functions possesing continuous second-order derivatives. This allowed one to conclude that a minimizer would be a solution to a certain system of second order partial differential equations, namely

$$\mathrm{div}\left[\frac{\partial}{\partial\mathbf{F}} f(\mathbf{x}, \mathbf{u}(\mathbf{x}), \nabla\mathbf{u}(\mathbf{x}))\right] = \frac{\partial}{\partial\mathbf{u}} f(\mathbf{x}, \mathbf{u}(\mathbf{x}), \nabla\mathbf{u}(\mathbf{x})),$$

which is called the Euler-Lagrange system for the functional. Here and throughout, we are thinking of $\mathbf{x}$, $\mathbf{u}$ and $\mathbf{F}$ as the generic placeholders for the first, second and third arguments of f, respectively, so that the notation $\frac{\partial}{\partial\mathbf{u}}$ is the gradient of the map

$\mathbf{u} \mapsto f(\mathbf{x}, \mathbf{u}, \mathbf{F})$ (which we think of as a column vector), and $\frac{\partial}{\partial \mathbf{F}} f(\mathbf{x}, \mathbf{u}, \mathbf{F})$ is the $N \times n$ matrix of first-order partial derivatives for the map $\mathbf{F} \mapsto f(\mathbf{x}, \mathbf{u}, \mathbf{F})$.

At the beginning of the 20th century, D. Hilbert created his famous list of what he saw as the twenty-three most important problems that mathematicians should attempt to resolve within the century. The 20th problem asked if every functional of the form (1.1), where f is smooth and convex with respect to the third argument, admits a solution, provided that certain assumptions on the boundary conditions are satisfied, and that the notion of a solution is suitably relaxed, if necessary. This question has an affirmative answer, but it turns out that the notion of a solution must indeed be relaxed. Rather than consisting of a subset of twice continuously-differentiable functions, the modified admissible class is made up of functions belonging to a so-called Sobolev space, which we now introduce.

First, for each $p \in [1, \infty]$, we define the Lebesgue space $L^p(\Omega; \mathbb{R}^N)$ by

$$L^p(\Omega; \mathbb{R}^N) := \left\{ \mathbf{u} : \Omega \to \mathbb{R}^N : \int_\Omega |\mathbf{u}(\mathbf{x})|^p \, \mathrm{d}\mathbf{x} < \infty \right\}, \quad 1 \le p < \infty$$
$$L^\infty(\Omega; \mathbb{R}^N) := \left\{ \mathbf{u} : \Omega \to \mathbb{R}^N : \operatorname{ess\,sup}_{\mathbf{x} \in \Omega} |\mathbf{u}(\mathbf{x})| < \infty \right\}.$$

The norm on $L^p(\Omega; \mathbb{R}^N)$ is defined as

$$\|\mathbf{u}\|_{L^p} := \begin{cases} \left(\int_\Omega |\mathbf{u}(\mathbf{x})|^p \, \mathrm{d}\mathbf{x} \right)^{1/p}, & 1 \le p < \infty, \\ \operatorname{ess\,sup}_{\mathbf{x} \in \Omega} |\mathbf{u}(\mathbf{x})|, & p = \infty. \end{cases}$$

For each $p \in [1, \infty]$, we define the Sobolev space $W^{1,p}(\Omega; \mathbb{R}^N)$ as follows:

$$W^{1,p}(\Omega; \mathbb{R}^N) := \left\{ \mathbf{u} \in L^p(\Omega; \mathbb{R}^N) : \nabla \mathbf{u} \in L^p(\Omega; \mathbb{R}^{N \times n}) \right\}. \tag{1.2}$$

It is important to point out that in the above definition, by $\nabla\mathbf{u}$ we mean the *weak* gradient of $\mathbf{u}$. We say that a function $\mathbf{V} \in L^1_{\text{loc}}(\Omega; \mathbb{R}^{N\times n})$ is a weak gradient of $\mathbf{u}$ if

$$\int_\Omega \mathbf{u} \cdot \nabla\boldsymbol{\varphi} \mathrm{d}\mathbf{x} = -\int_\Omega \boldsymbol{\varphi} \cdot \mathbf{V} \mathrm{d}\mathbf{x} \tag{1.3}$$

for all $\boldsymbol{\varphi} \in \mathcal{C}_c^\infty(\Omega; \mathbb{R}^N)$. (Here, we are thinking of each of the above $N \times n$ matrices as a column vector with N components; each of these components is a row vector with n scalar components, so that both of the dot products defined above yield an n-dimensional row vector.) Weak gradients, when they exist, are unique up to a set of zero measure. So the definition for $W^{1,p}$ given in (1.2), when unpacked, says that $W^{1,p}(\Omega; \mathbb{R}^N)$ consists of all mappings in $L^p(\Omega; \mathbb{R}^N)$ for which the weak gradient exists and belongs to $L^p(\Omega; \mathbb{R}^{N\times n})$. (I.e., (1.3) is satisfied for some $\mathbf{V} \in L^p(\Omega; \mathbb{R}^{N\times n})$.) We define the norm on $W^{1,p}(\Omega; \mathbb{R}^N)$ by

$$\|\mathbf{u}\|_{W^{1,p}} := \|\mathbf{u}\|_{L^p} + \|\nabla\mathbf{u}\|_{L^p} ,$$

and denote by $W_0^{1,p}(\Omega; \mathbb{R}^N)$ the closure of $C_c^\infty(\Omega; \mathbb{R}^N)$ in the norm of $W^{1,p}(\Omega; \mathbb{R}^N)$. For Dirichlet problems, the relaxed admissible class for the functional J is usually defined to be

$$\tilde{\mathcal{A}} := \left\{\mathbf{u} \in W^{1,p}(\Omega; \mathbb{R}^N) : \mathbf{u} - \mathbf{u}_0 \in W_0^{1,p}(\Omega; \mathbb{R}^N)\right\} ,$$

for some fixed $\mathbf{u}_0 \in W^{1,p}(\Omega; \mathbb{R}^N)$. If the integrand f of the functional J satisfies some convexity and coercivity conditions, then using the direct method of the calculus of variations, one can show that a minimizer exists in $\mathcal{A}$.

Although relaxing the conditions for the admissible class facilitates obtaining the existence of minimizers, *a priori*, one only knows a minimizer belongs to a Sobolev space; i.e the minimizer and its gradient will belong to some Lebesgue space. If $p \leq n$, then functions belonging to $W^{1,p}(\Omega;\mathbb{R}^N)$ may even be discontinuous. Therefore, a natural and important question is whether we can expect any additional regularity (smoothness) for minimizers. In fact, this is the subject of Hilbert's 19th problem: Provided that the integrand f of the functional (1.1) is analytic, must all minimizers be analytic? This is an important question for several reasons. If the variational problem arises from a physical context, the regularity of the solution will have significant physical implications. For example, if an elastic body is deformed from its reference configuration, regularity for the minimizer might mean that no cracks or holes form as a result of this deformation. Further, additional regularity for the minimizer is often required to obtain convergence rates for numerical approximations of the solutions (see [19, 27], for example).

In the scalar case $N = 1$, the 19th problem essentially was answered in the affirmative by E. De Giorgi [23] and independently by J. Nash [63]. Unfortunately, in the general vectorial setting, one cannot expect similar results to hold, as demonstrated by De Giorgi [24] ($n, N > 2$). However, with some additional assumptions on the integrand, many different sorts of regularity results have been obtained. We provide a discussion of some of these results in Section 1.3.

1.2 Definitions and Notation

In order to facilitate the discussion of the results in the present work and how they fit into a broader context, we now introduce several definitions and notations. For the entirety of this thesis, we fix n, $N \in \mathbb{N}$, and we suppose that $\Omega \subset \mathbb{R}^n$ is open and bounded. We denote by $\mathbb{R}_+$ the interval $[0, \infty)$ and by $\mathbb{R}^*$ the set $\mathbb{R} \cup \{+\infty\}$. For a measurable set $E \subset \mathbb{R}^n$, we let $|E|$ denote the Lebesgue measure of E, and use

$\overline{E}$ to signify the closure of E in the usual Euclidean topology. If $0 < |E| < \infty$ and $\mathbf{f} \in L^1(E;\mathbb{R}^k)$, we define

$$(\mathbf{f})_E \equiv \fint_E \mathbf{f}(\mathbf{x})\mathrm{d}\mathbf{x} := \frac{1}{|E|}\int_E \mathbf{f}(\mathbf{x})\mathrm{d}\mathbf{x}.$$

We will use $\mathbf{x}$ and $\mathbf{y}$ to denote points in $\mathbb{R}^n$, and $\mathbf{F}$ to denote a point in $\mathbb{R}^{N\times n}$. The open ball of radius ρ centered at the point $\mathbf{x}$ is represented by $\mathcal{B}(\mathbf{x},\rho)$. For brevity, $\mathcal{B}$ denotes $\mathcal{B}(0,1)$. For a measurable set $E \subset \mathbb{R}^n$ not equal to $\mathcal{B}$, we use $E(\mathbf{x},\rho)$ to abbreviate $E \cap \mathcal{B}(\mathbf{x},\rho)$. (We refrain from using the previously mentioned notation when $E = \mathcal{B}$ in order to avoid ambiguity.) We define $\mathcal{H}^+ := \{(x_1,\ldots,x_n) \in \mathbb{R}^n : x_n > 0\}$, and given a set $\mathcal{U} \subset \mathbb{R}^n$, we use $\mathcal{U}^+$ to stand for $\mathcal{H}^+ \cap \mathcal{U}$. We define $Q := (0,1)^n$, and for $\mathbf{x} \in \mathbb{R}^k$ and $\rho > 0$, we use $Q_{\mathbf{x},\rho}$ to denote the open cube in $\mathbb{R}^k$ centered at $\mathbf{x}$ with edges of length ρ parallel to the coordinate axes.

We denote by $\mathcal{M}(\mathbb{R}^k)$ the set of all Radon measures supported on $\mathbb{R}^k$. Given $\mu \in \mathcal{M}(\mathbb{R}^k)$ and $\varphi \in \mathcal{C}_0(\mathbb{R}^k)$, we define

$$\langle \mu, \varphi \rangle := \int_{\mathbb{R}^k} \varphi(\mathbf{y})\mathrm{d}\mu(\mathbf{y}).$$

Definition 1.1. We say that a mapping $\nu : E \to \mathcal{M}(\mathbb{R}^k)$ is a *Young measure* if $\nu(\mathbf{x})$ is a probability measure for almost every $\mathbf{x} \in E$ and the map $\mathbf{x} \mapsto \langle \nu(\mathbf{x}), \varphi \rangle$ is measurable for every $\varphi \in \mathcal{C}_0(\mathbb{R}^k)$.

Complying with standard notation for Young measures, we will write $\nu_{\mathbf{x}}$ instead of $\nu(\mathbf{x})$, and will usually denote the map ν by $\{\nu_{\mathbf{x}}\}_{\mathbf{x}\in E}$.

We recall here the definition of an N-function.

Definition 1.2. A function $h : \mathbb{R}_+ \to \mathbb{R}_+$ is said to be an *N-function* if $h(0) = 0$ and there exists a right-continuous nondecreasing derivative h' satisfying $h'(0) = 0$ and

$h'(t) > 0$ for all $t > 0$, with $\lim_{t\to\infty} h'(t) = \infty$.

In this work, we will be working frequently with a class of N-functions satisfying some additional hypotheses. In particular, we make the following definition.

Definition 1.3. If h is an N-function satisfying $h \in W^{2,1}(0,T)$ for each $T > 0$ and

$$(p-1)h'(t) \le th''(t) \le (q-1)h'(t) \text{ for a.e. } t > 0,$$

for some $1 < p \le q < \infty$, then we will say that h has (p,q)*-structure.*

Roughly speaking, a function h with (p,q)-structure grows between the functions $t \mapsto t^p$ and $t \mapsto t^q$. (See Lemma 2.1 for some basic consequences of Definition 1.3.) One can verify that for any $1 < p \le q < \infty$ and $\beta \ge 1$, each of the following functions mapping $[0,\infty)$ into $[0,\infty)$ are examples of functions possessing $(\tilde{p},\tilde{q})$-structure for some $1 < \tilde{p} \le \tilde{q} < \infty$:

$$\begin{aligned} h_1(t) &:= t^p; \\ h_2(t) &:= t^p[\log(t+e)]^\beta; \\ h_3(t) &:= \begin{cases} t^p & \text{if } 0 \le t \le t_0, \\ t^{\frac{p+q}{2}+\frac{p-q}{2}\sin\log\log\log t} & \text{if } t > t_0. \end{cases} \end{aligned} \tag{1.4}$$

For h_3, we must choose $t_0 > 0$ large enough so that h_3 is strictly convex and $\sin\log\log\log(t_0) = 1$. (The function h_3 was first given as an example in [22].)

For a given N-function h, we define the Young conjugate as follows.

Definition 1.4. If $h : \mathbb{R}_+ \to \mathbb{R}_+$ is an N-function, then the function $h^* : \mathbb{R}_+ \to \mathbb{R}_+$ defined by

$$h^*(s) := \sup_{t\in\mathbb{R}_+} \{st - h(t)\}$$

is called the *Young conjugate* of h.

It is easily seen that h^* is also an N-function, and that for any s, $t \in \mathbb{R}_+$, we have

$$st \leq h(s) + h^*(t),$$

which is known as Young's Inequality. Furthermore, if h' is continuous and strictly increasing, as is the case if h has (p, q)-structure, then the function $t \mapsto st - h(t)$ has a unique maximum at $t = (h')^{-1}(s)$, and hence

$$h^*(s) = s(h')^{-1}(s) - h((h')^{-1}(s)). \tag{1.5}$$

We now introduce notation for certain vector spaces that are well-suited for our situation in the present paper; these spaces are special cases of Musielak-Orlicz spaces. For a development of Musielak-Orlicz spaces, see, for example, [62]. Our notation is similar to that used for Orlicz and Orlicz-Sobolev spaces in [6].

Definition 1.5. Suppose that $E \subset \mathbb{R}^n$ is open. Let $g : E \times \mathbb{R}_+ \to \mathbb{R}_+$ be such that there is a constant $c < \infty$ so that for almost every $\mathbf{x} \in E$, the function $g(\mathbf{x}, \cdot)$ is convex and nondecreasing with $g(\mathbf{x}, 0) = 0$, and $g(\mathbf{x}, 2t) \leq cg(\mathbf{x}, t)$ for every $t \in \mathbb{R}_+$. We then define

$$L_g(E; \mathbb{R}^N) := \left\{ \mathbf{u} : E \to \mathbb{R}^N : \mathbf{u} \text{ is measurable and } \int_E g(\mathbf{x}, |\mathbf{u}(\mathbf{x})|) \mathrm{d}\mathbf{x} < \infty \right\},$$

and the corresponding Sobolev-type space

$$W^1 L_g(E; \mathbb{R}^N) := \left\{ \mathbf{u} \in L_g(E; \mathbb{R}^N) : \nabla \mathbf{u} \in L_g(E; \mathbb{R}^{N \times n}) \right\},$$

where $\nabla \mathbf{u}$ denotes the weak gradient of the mapping $\mathbf{u}$.

Remark 1.1. If $h : \mathbb{R}_+ \to \mathbb{R}_+$ is a function with (p,q)-structure, we will use $L_h(E;\mathbb{R}^N)$ and $W^1L_h(E;\mathbb{R}^N)$ to denote the spaces $L_{\tilde{h}}(E;\mathbb{R}^N)$ and $W^1L_{\tilde{h}}(E;\mathbb{R}^N)$, respectively, where we have defined $\tilde{h} : E \times \mathbb{R}_+ \to \mathbb{R}_+$ by $\tilde{h}(\mathbf{x},t) := h(t)$.

The space $L_g(E;\mathbb{R}^N)$ can be equipped with the Luxemborg norm defined by

$$\|\mathbf{u}\|_{L_g} := \inf\left\{ s>0 : \int_E g\left(\mathbf{x}, \frac{|\mathbf{u}(\mathbf{x})|}{s}\right) \mathrm{d}\mathbf{x} \le 1 \right\},$$

and is complete under this norm. The norm on $W^1L_g(E;\mathbb{R}^N)$ is then defined by

$$\|\mathbf{u}\|_{W^1L_g} := \|\mathbf{u}\|_{L_g} + \|\nabla\mathbf{u}\|_{L_g}.$$

We will use $W^1_0L_g(E;\mathbb{R}^N)$ to denote the closure of $\mathcal{C}^\infty_c(E;\mathbb{R}^N)$ in $W^1L_g(E;\mathbb{R}^N)$.

We also introduce the Morrey spaces $L^{p,\lambda}(\Omega)$. We refer the reader to [47] for a development of some of the properties of Morrey spaces.

Definition 1.6. Suppose that $E \subset \mathbb{R}^n$ is measurable and bounded, that $p \in [1,\infty)$, and that $\lambda \in [0,n]$. We define the *Morrey space*

$$L^{p,\lambda}(E) := \left\{ \mathbf{u} \in L^p(E) : \|\mathbf{u}\|_{L^{p,\lambda}} := \sup_{\mathbf{x}_0\in E}\sup_{\rho>0} \rho^{-\lambda} \int_{E(\mathbf{x}_0,\rho)} |\mathbf{u}|^p \,\mathrm{d}\mathbf{x} < \infty \right\}.$$

Roughly speaking, larger values of λ imply less singular behavior of functions belonging to $L^{p,\lambda}$. Using Hölder's inequality, one can readily check that for $q \ge p$, the space L^q is contained in the Morrey space $L^{1,n(1-p/q)}$; it is also the case that $L^{p,n}$ is isomorphic to L^∞. However, for any $\lambda \in [0,n)$, the inclusion $L^{p,\lambda} \subset L^q$ does *not* hold for any $q > p$. Hence Morrey spaces provide a more precise way than the Lebesgue spaces of characterizing how bad the singularities of functions might be.

The following defines the type of almost minimizers for which we will establish regularity.

Definition 1.7. Let $\Omega \subset \mathbb{R}^n$ be open and $\mathcal{A} \subset W^{1,1}(\Omega;\mathbb{R}^N)$ be given, and suppose that $f : \Omega \times \mathbb{R}^N \times \mathbb{R}^{N\times n} \to \mathbb{R}$. Define the functional $K : \mathcal{A} \to \mathbb{R}^*$ by

$$K(\mathbf{w}) := \int_\Omega f(\mathbf{x}, \mathbf{w}, \nabla \mathbf{w})\mathrm{d}\mathbf{x}. \tag{1.6}$$

Suppose that $\mathbf{u} \in \mathcal{A}$ with $K(\mathbf{u}) < \infty$, and that there are functions $\{\nu_\varepsilon\}_{\varepsilon>0} \subset L^1(\Omega)$ and nondecreasing functions $\{\gamma_\varepsilon\}_{\varepsilon>0} \subset \mathcal{C}(\mathbb{R}_+)$ satisfying $\gamma_\varepsilon(0) = 0$ for each $\varepsilon > 0$ such that for every $\varepsilon > 0$ and $0 < \rho < \mathrm{diam}(\Omega)$, we find that

$$K(\mathbf{u}) \leq K(\mathbf{v}) + (\gamma_\varepsilon(\rho) + \varepsilon) \int_{\Omega(\mathbf{x}_0,\rho)} \{|f(\mathbf{x}, \mathbf{u}, \nabla \mathbf{u})| + |f(\mathbf{x}, \mathbf{v}, \nabla \mathbf{v})| + \nu_\varepsilon(\mathbf{x})\}\, \mathrm{d}\mathbf{x} \tag{1.7}$$

for all $\mathbf{v} \in \mathcal{A}$ such that $\mathbf{v} - \mathbf{u} \in W_0^{1,1}(\Omega(\mathbf{x}_0, \rho);\mathbb{R}^N)$. Then we say that $\mathbf{u}$ is a *$(K, \{\gamma_\varepsilon\}, \{\nu_\varepsilon\})$-minimizer over $\mathcal{A}$.*

1.3 Discussion of Results

In this section, we state the main results and discuss how they relate to the existing literature.

1.3.1 Morrey Regularity for the Gradient of Almost Minimizers

In 1977, K. Uhlenbeck [70] proved a seminal result that provided a new direction of investigation for regularity in the vectorial setting, and set off a string of regularity results over the next several decades. A consequence of her result provides Hölder continuity for the gradient of minimizers in the case of superquadratic standard growth with no dependence on $\mathbf{x}$ and $\mathbf{u}$. More precisely, one can state a version of her result as follows:

Theorem 1.1. *Suppose $f : \mathbb{R}^{N\times n} \to [0,\infty)$ is of class $\mathcal{C}^2$ and satisfies the following hypotheses for some $\beta \geq 0$, $\mu > 0$, $p \geq 2$, $M < \infty$, and $\tilde{f} : [0,\infty) \to [0,\infty)$, and for all $\mathbf{F}, \mathbf{F}_1, \mathbf{F}_2 \in \mathbb{R}^{N\times n}$:*

$$\begin{aligned}
&\text{(Growth of } f) && M^{-1}(\mu^2 + |\mathbf{F}|^2)^{p/2} \leq f(\mathbf{F}) \leq M(\mu^2 + |\mathbf{F}|^2)^{p/2}; \qquad (1.8)\\
&\text{(Growth of } \tfrac{\partial^2}{\partial \mathbf{F}^2} f) && \left|\frac{\partial^2}{\partial \mathbf{F}^2} f(\mathbf{F})\right| \leq M(\mu^2 + |\mathbf{F}|^2)^{\frac{p-2}{2}};\\
&\text{(Continuity of } \tfrac{\partial^2}{\partial \mathbf{F}^2} f) && \left|\frac{\partial^2}{\partial \mathbf{F}^2} f(\mathbf{F}_1) - \frac{\partial^2}{\partial \mathbf{F}^2} f(\mathbf{F}_2)\right|\\
& && \quad \leq M(\mu^2 + |\mathbf{F}_1|^2 + |\mathbf{F}_2|^2)^{\frac{p-2}{2}-\frac{\beta}{2}} |\mathbf{F}_1 - \mathbf{F}_2|^\beta\\
&\text{(Uniform Ellipticity)} && \frac{\partial^2}{\partial \mathbf{F}^2} f(\mathbf{F}_1) \cdot \mathbf{F}_2 \otimes \mathbf{F}_2 \geq (\mu^2 + |\mathbf{F}_1|^2)^{\frac{p-2}{2}} |\mathbf{F}_2|^2\\
&\text{(Rotational Symmetry)} && f(\mathbf{F}) = \tilde{f}(|\mathbf{F}|).
\end{aligned}$$

Define the functional J_{Uhl} by

$$J_{\mathrm{Uhl}}(\mathbf{u}) := \int_\Omega f(\nabla \mathbf{u}) \mathrm{d}x,$$

and suppose that $\mathbf{u} \in W^{1,p}_{\mathrm{loc}}(\Omega;\mathbb{R}^N)$ is a minimizer for J_{Uhl}. Then $\nabla \mathbf{u}$ is locally Hölder-continuous in Ω. Moreover, there is some $C < \infty$ so that for every $\mathbf{x}_0 \in \Omega$, it holds that

$$\sup_{\mathbf{x}\in\mathcal{B}(\mathbf{x}_0,R/2)} |\nabla \mathbf{u}|^p \leq \frac{C}{R^n} \int_{\mathcal{B}(\mathbf{x}_0,R)} (\mu^2 + |\nabla \mathbf{u}|^2)^{p/2}$$

whenever $0 < R < \mathrm{dist}(\mathbf{x}_0, \partial\Omega)$.

In 1989, E. Acerbi and N. Fusco [1] proved an analogue of K. Uhlenbeck's result in the case $1 < p < 2$. There was also a large number of other kinds of regularity results established by many researchers. These results provided various types of regularity while allowing dependence on $\mathbf{x}$ and $\mathbf{u}$ in the integrand f. In contrast with Uhlenbeck's result, most of the theorems in the vectorial case furnished what is known as partial regularity - that is, local regularity on an open set $\Omega_0 \subset \Omega$ with $\Omega \backslash \Omega_0$ a null set. For example, in [44], M. Giaquinta and G. Modica proved that if $\mathbf{u}$ is a minimizer for functionals whose model is given by $\mathbf{u} \mapsto \int_\Omega a(\mathbf{x}, \mathbf{u})\, |\nabla \mathbf{u}|^p$ for $p \geq 2$, then there is a set $\Omega_0 \subset \Omega$ such that $\Omega \setminus \Omega_0$ is a null set and $\nabla \mathbf{u}$ is locally Hölder continuous on Ω_0. Notice that in this case, we have $\mu = 0$, and the ellipticity degenerates when $\nabla \mathbf{u}$ approaches $\mathbf{0}$. Usually, partial Hölder continuity for the gradient cannot be deduced in the vectorial setting when $\mu = 0$; the reason Giaquinta and Modica could allow $\mu = 0$ is because of the additional structural assumption that the integrand only depends on $\nabla \mathbf{u}$ through $|\nabla \mathbf{u}|$. In [48], C. Hamburger produced an extension of Giaquinta's and Modica's result to minimizing differential forms, and also treated the case $1 < p < 2$ with a duality argument.

In 1989, P. Marcellini [56] provided the first regularity result in the so-called (p, q)-growth case, where the lower and upper growth exponents may differ: the assumption in (1.8) was weakened to

$$M^{-1} \sum_{j=1}^{n} |\mathbf{F}_j|^{p_j} \leq f(\mathbf{F}) \leq M \left(1 + \sum_{j=1}^{n} |\mathbf{F}_j|^{p_j} \right), \tag{1.9}$$

where $2 \leq p_j \leq 2n/(n-2)$. In this setting, working in the scalar case ($N = 1$), he was able to prove that a minimizer u is Lipschitz. Marcellini and others continued to study functionals under more general growth conditions, considering functionals with integrands possessing (p, q)-growth

$$M^{-1}\, |\mathbf{F}|^p \leq f(\mathbf{x}, \mathbf{u}, \mathbf{F}) \leq C(1 + |\mathbf{F}|^q) \tag{1.10}$$

for some $1 < p \le q$. To obtain regularity in this setting, it is often required to assume that the quantity q/p is not too large. How large q/p can be depends on the type of regularity sought and the degree of smoothness possessed by the function f (see the discussions in [32] and [61]). In the scalar case, L^∞ estimates for minimizers were proved in [14, 21], and Lipschitz regularity was demonstrated in [57]. In the vectorial setting, partial Hölder continuity for the gradient was proved in [2, 11, 10, 12, 13]. In the series of papers [28, 30, 29, 31, 32], L. Esposito, F. Leonetti, and G. Mingione proved various higher differentiability and higher integrability results, and provided counterexamples that demonstrated that their results were sharp. Regularity theory for functionals with growth conditions even more general than (1.10) has also been studied [58, 59, 55, 60].

An intermediate growth assumption, weaker than (1.8) but stronger than (1.10), is the hypothesis that

$$M^{-1}\,|\mathbf{F}|^{p(\mathbf{x})} \le f(\mathbf{x},\mathbf{F}) \le M(1 + |\mathbf{F}|^{p(\mathbf{x})}), \tag{1.11}$$

where $1 < p \le p(\mathbf{x}) \le q$ for all $\mathbf{x} \in \Omega$. Functionals possessing integrands with variable exponents arise naturally from several problems in mathematical physics, in, for example, models for thermistors [72] or electro-rheological fluids [5]. Thermistors are resistors for which the resistance depends on the temperature of the resistor. In the thermistor case, the exponent $p(\mathbf{x})$ corresponds to the temperature - as the temperature varies throughout the resistor, so does the resistance. Electro-rheological fluids are non-Newtonian fluids whose viscosity changes dramatically in the presence of an electric field. In this case, the variable exponent $p(\mathbf{x})$ results from the variability of the strength of the electric field throughout the fluid.

In Chapter 2, we provide Morrey regularity for the gradient of almost minimizers for functionals of the form (1.1), where f behaves asymptotically like a function g with more desirable properties. Let us now describe the hypotheses for g. To do so, we first introduce a function $\alpha : \Omega \to [1, \infty)$, which will play the role of an exponent in the definition for g. Suppose that α satisfies

$$1 \le \alpha(\mathbf{x}) \le \alpha_+ < \infty \text{ for every } \mathbf{x} \in \Omega. \tag{1.12}$$

With $\omega : \mathbb{R}_+ \to \mathbb{R}_+$ denoting the modulus of continuity for α, suppose also that

$$\omega(R) \left|\log(R)\right| \le M, \tag{1.13}$$

for some $M < \infty$, and moreover that

$$\lim_{R \to 0^+} \omega(R) \log(R) = 0. \tag{1.14}$$

With α as just defined, we assume that the function $g : \Omega \times \mathbb{R}_+ \to \mathbb{R}_+$ satisfies the following for some $1 < p \le q < \infty$, some $h : \mathbb{R}_+ \to \mathbb{R}_+$ with (p, q)-structure, and a nondecreasing $\delta \in \mathcal{C}(\mathbb{R}_+)$ with $\delta(0) = 0$:

$$g(\mathbf{x}, \cdot) \text{ has } (p\alpha(\mathbf{x}), q\alpha(\mathbf{x}))\text{-structure for each } \mathbf{x} \in \Omega; \tag{1.15}$$

$$h(t)^{\alpha(\mathbf{x})} \le g(\mathbf{x}, t) \le M h(t)^{\alpha(\mathbf{x})}; \tag{1.16}$$

$$\begin{aligned} |g(\mathbf{x}, t) - g(\mathbf{y}, t)| \le{}& M\omega(|\mathbf{x} - \mathbf{y}|) \left\{g(\mathbf{x}, t) + g(\mathbf{y}, t)\right\} \log(e + h(t)) \\ &+ M\delta(|\mathbf{x} - \mathbf{y}|)\{1 + g(\mathbf{x}, t) + g(\mathbf{y}, t)\}. \end{aligned} \tag{1.17}$$

To explain (1.17), with the model case $g(\mathbf{x}, t) = \beta(\mathbf{x}) h(t)^{\alpha(\mathbf{x})}$, the functions ω and δ represent the moduli of continuity for the exponent α and the function β, respectively. Note that only uniform continuity of β is required, whereas the stronger continuity assumption (1.14) is enforced for the exponent α.

To give the reader a sense of the scope of the functionals considered in Chapter 2, we give a few examples of functions $g_i := h_i^\alpha$, $i = 1, 2, 3$, where h_i is defined in (1.4):

$$g_1(\mathbf{x}, \mathbf{F}) := |\mathbf{F}|^{p(\mathbf{x})},$$
$$g_2(\mathbf{x}, \mathbf{F}) := |\mathbf{F}|^{p(\mathbf{x})} \log(e + |\mathbf{F}|)^{\gamma p(\mathbf{x})},$$
$$g_3(\mathbf{x}, \mathbf{F}) := \begin{cases} |\mathbf{F}|^{p(\mathbf{x})} & \text{if } 0 \le |\mathbf{F}| \le t_0, \\ |\mathbf{F}|^{p(\mathbf{x})\frac{1+Q}{2} + p(\mathbf{x})\frac{1-Q}{2} \sin\log\log\log|\mathbf{F}|} & \text{if } |\mathbf{F}| \ge t_0. \end{cases}$$

Here we have put $p(\mathbf{x}) = p\alpha(\mathbf{x})$, $\gamma = \beta/p$, and $Q = q/p$. The function f_1 is a particularly important case: functionals with integrands of this form have been utilized in image restoration (e.g. [16]), and as mentioned earlier, also arise from problems in mathematical physics.

We are now in a position to state the main result of Chapter 2.

Theorem 1.2. *Suppose that* $\Omega \subset \mathbb{R}^n$ *is open and bounded with* $\mathcal{C}^1$ *boundary, and that* $\alpha : \Omega \to [1, \infty)$ *satisfies* (1.12)-(1.14) *and* $g : \Omega \times \mathbb{R}_+ \to \mathbb{R}_+$ *satisfies* (1.15)-(1.17). *Let* $f : \Omega \times \mathbb{R}^N \times \mathbb{R}^{N\times n} \to \mathbb{R}$ *satisfy the following hypotheses for some* $0 \le \lambda < n$ *and* $1 < s < \min\{r_2, 1 + pr_2/n, p^*/p\}$, *where* $r_2 > 1$ *is as in Remark 2.2.*

(i) *For every* $\varepsilon > 0$, *there is a function* $\sigma_\varepsilon \in L^{1,\lambda}(\Omega)$ *and a constant* $\Sigma_\varepsilon < \infty$ *such that*

$$|f(\mathbf{x}, \mathbf{u}, \mathbf{F}) - g(\mathbf{x}, |\mathbf{F}|)| < \varepsilon g(\mathbf{x}, |\mathbf{F}|)$$

for all $(\mathbf{x}, \mathbf{u}, \mathbf{F}) \in \Omega \times \mathbb{R}^N \times \mathbb{R}^{N\times n}$ *satisfying* $g(\mathbf{x}, |\mathbf{F}|) \ge \sigma_\varepsilon(\mathbf{x}) + \Sigma_\varepsilon g(\mathbf{x}, |\mathbf{u}|)^s$;

(ii) *There is some* $\beta \in L^{1,\lambda}(\Omega)$ *such that*

$$|f(\mathbf{x}, \mathbf{u}, \mathbf{F})| \leq C(\beta(\mathbf{x}) + g(\mathbf{x}, |\mathbf{u}|)^s + g(\mathbf{x}, |\mathbf{F}|))$$

for all $(\mathbf{x}, \mathbf{u}, \mathbf{F}) \in \Omega \times \mathbb{R}^N \times \mathbb{R}^{N\times n}$.

For a fixed $\overline{\mathbf{u}} \in W^1L_g(\Omega;\mathbb{R}^N)$ *with* $g(\cdot, |\nabla \overline{\mathbf{u}}|) \in L^{1,\lambda}(\Omega)$*, define the admissible class*

$$\mathcal{A} := \left\{\mathbf{u} \in W^1L_g(\Omega;\mathbb{R}^N) : \mathbf{u} - \overline{\mathbf{u}} \in W^1_0L_g(\Omega;\mathbb{R}^N)\right\}.$$

Let the functional $K : \mathcal{A} \to \mathbb{R}$ *be as defined in* (1.6)*. If* $\mathbf{u} \in \mathcal{A}$ *and there are functions* $\{\nu_\varepsilon\}_{\varepsilon>0} \subset L^{1,\lambda}(\Omega)$ *and nondecreasing functions* $\{\gamma_\varepsilon\}_{\varepsilon>0} \subset \mathcal{C}(\mathbb{R}_+)$ *with* $\gamma_\varepsilon(0) = 0$ *so that* $\mathbf{u}$ *is a* $(K, \{\gamma_\varepsilon\}, \{\nu_\varepsilon\})$*-minimizer over* $\mathcal{A}$*, then* $g(\cdot, |\nabla \mathbf{u}|) \in L^{1,\lambda}(\Omega)$.

We now discuss how our results fit into the broader framework of regularity theory. As was previously noted, a primary special case of the functionals we consider are those whose integrands are given by $f(\mathbf{x}, \mathbf{F}) := h_1(|\mathbf{F}|)^{\alpha(\mathbf{x})} = |\mathbf{F}|^{p(\mathbf{x})}$. As one might expect, the continuity of the exponent plays an important role in the type of regularity possessed by minimizers. Working under the assumption that (1.13) was satisfied, V. Zhikov [72] provided a proof of higher integrability for the gradient of minimizers, and also provided an example that showed that if the exponent $p(\cdot)$ is merely continuous, and (1.13) is not satisfied, then the higher integrability need not hold. With the stronger assumption (1.14), which we also suppose in this work, E. Acerbi and G. Mingione [3], working within the scalar setting, showed that minimizers belong to $\mathcal{C}^{0,\alpha}$ locally for every $\alpha < 1$, and if $p(\cdot)$ is Hölder continuous, that the gradient also possesses some Hölder continuity. Again in the scalar setting, Hölder continuity of quasiminimizers, under more stringent assumptions on $p(\cdot)$, was proved in [17]. In [4], partial Hölder continuity for the gradient is obtained in the vectorial case with quasiconvex integrands. For a more extensive discussion of the case of $p(\mathbf{x})$-growth, we refer the interested reader to Section 7 of [61], and the references contained therein.

Another special case enveloped by our results is that of functionals whose integrands have growth specified by a function h with (p, q)-structure; in the notation of the present paper, this corresponds to the case $\alpha \equiv 1$. Various types of regularity have been studied in this setting; see [25], [26], and [35], for example.

The integrands of the functionals we consider are really only asymptotically convex. Functionals with asymptotically convex integrands were first considered in the case of quadratic growth ($p = q = 2$ and $\alpha(\mathbf{x}) \equiv 1$) by M. Chipot and L. Evans [18]. Asymptotically convex integrands with natural growth ($p = q \geq 2$ and $\alpha(\mathbf{x}) \equiv 1$) were subsequently treated in [36, 52, 64, 40, 42]. The case in which the integrands are asymptotically convex and possess (p, q)-growth (but still $\alpha(\mathbf{x}) \equiv 1$) was handled in [35]. In this work, we consider asymptotically convex integrands with $(p\alpha(\mathbf{x}), q\alpha(\mathbf{x}))$-growth to allow for variable exponents. The type of regularity we obtain, namely global Morrey regularity for the gradient of almost minimizers, is the same as that obtained in [35], [40], and [41]; indeed, the results in the present paper generalize those previous results to allow for variable exponent growth. A key ingredient of the proof is obtaining a local Lipschitz estimate for minimizers of functionals that have integrands with (p, q)-structure. In 2006, G. Papi and P. Marcellini established local Lipschitz regularity with the following theorem.

Theorem 1.3. *Let $h \in W^{2,\infty}_{\mathrm{loc}}((0,\infty))$ be a convex function satisfying $h(0) = h'(0) = 0$. Suppose that there exist $t_0, \mu > 0$ and $\beta \in (1/n, 2/n)$ so that for each $\alpha \in (1, n/(n-1)]$ there is a $M = M(\alpha)$ such that*

$$\mu t^{-2\beta}\left[\left(\frac{h'(t)}{t}\right)^{\frac{n-2}{n}} + \frac{h'(t)}{t}\right] \leq h''(t) \leq M\left[\frac{h'(t)}{t} + \left(\frac{h'(t)}{t}\right)^{\alpha}\right] \tag{1.18}$$

for all $t \geq t_0$*. Let* $\mathbf{u} \in W^{1,1}_{\mathrm{loc}}(\Omega;\mathbb{R}^N)$ *be a local minimizer of the functional defined by* $\mathbf{u} \mapsto \int_\Omega h(|\nabla \mathbf{u}|)\mathrm{d}x$*. Then* $\mathbf{u} \in W^{1,\infty}_{\mathrm{loc}}(\Omega;\mathbb{R}^N)$*; moreover, for each* $\varepsilon > 0$ *and* $0 < \rho < R$*, there is a constant* $C = C(\varepsilon, n, \rho, R, H, K, \sup_{0\leq t\leq t_0} h''(t))$ *such that*

$$\|\nabla \mathbf{u}\|^{2-\beta n}_{L^\infty(B_\rho;\mathbb{R}^{N\times n})} \leq C\left\{\int_{B_R}(1 + h(|\nabla \mathbf{u}|))\mathrm{d}x\right\}^{\frac{1}{1-\beta}+\varepsilon}. \tag{1.19}$$

Condition (1.18) is very mild; indeed, it allows for linear growth as well as exponential growth. Hence Theorem 1.3 establishes local Lipschitz regularity of minimizers for a very broad class of functionals. This Lipschitz regularity can then be used to establish even more regularity of the minimizers if some other hypotheses on h are assumed. However, the form of the estimate (1.19) does not lend itself to our purposes in this work. We therefore must first prove a refinement of their result assuming that h has (p,q)-structure, which is evidently a stronger assumption than (1.18). In the proof of this refinement, we use Moser's iteration technique, and obtain an estimate of the form

$$\|h(|\nabla \mathbf{u}|)\|_{L^\infty(\mathcal{B}_{x_0,\rho})} \leq \frac{C}{(R-\rho)^n}\int_{\mathcal{B}_{x_0,R}} h(|\nabla \mathbf{u}|)\mathrm{d}x, \tag{1.20}$$

which is crucial for our purposes. Until 2009, estimates of the form (1.20) (sometimes called "weak Harnack inequalities") have only been available when h satisfies natural growth conditions; i.e. when $p = q$. (Such an estimate was also obtained independently for functions with (p,q)-structure in [26].) Additionally, the proofs for these results have been separated into two cases, namely $1 < p < 2$ and $p \geq 2$, and the two cases have been proved in fairly different ways (see [1] and [70]). In contrast, the proof given in Section 2.2 is unified. Though certain growth conditions are implied by our definition of (p,q)-structure (see Lemma 2.1), our proof uses the structure intrinsic to h itself, as opposed to external growth conditions imposed on h. Therefore it seems that similar results could be shown for functions with more general growth by employing techniques similar to those used here.

One feature of the previous result is that, in contrast to many other results for functionals with (p, q)-growth, the ratio q/p can be as large as we wish. The main reason that we can relax the assumption on the ratio q/p is that we require the function h to be radial, so the growth is the same in all directions, which is not assumed for the results for which the ratio q/p does come into play.

1.3.2 Morrey Regular Young Measures

In Chapter 3, we make a transition to considering Young measures, and we restrict our focus to the case that $g(\mathbf{x}, t) = t^p$. As was previously noted, when the integrand for the functional is not convex, minimizers often fail to exist. However, if the functional is coercive, that is, if

$$\lim_{\|\mathbf{w}\|_{W^{1,p}} \to \infty} J(\mathbf{w}) = \infty,$$

then a minimizer will always exist in the sense of Young measures. Before we begin discussing our results for Young measures, we briefly recall some basic facts about Young measures.

We say that a sequence of measurable functions $\{\mathbf{f}_j\}_{j=1}^{\infty}$, each mapping Ω into $\mathbb{R}^d$, generates the Young measure $\nu = \{\nu_{\mathbf{x}}\}_{\mathbf{x}\in\Omega}$ if

$$\varphi(\mathbf{f}_j(\cdot)) \rightharpoonup^* \int_{\mathbb{R}^d} \varphi(\mathbf{y}) \mathrm{d}\nu_{(\cdot)}(y) \text{ in } L^{\infty}(\Omega)$$

for every $\varphi \in \mathcal{C}_0(\mathbb{R}^d)$. It is a standard fact that if $g : \Omega \times \mathbb{R}^d \to \mathbb{R}$ is measurable with respect to the first argument and continuous with respect to the second, then if $g(\cdot, \mathbf{f}_j(\cdot))$ converges weakly in $L^1(\Omega)$ and $\{\mathbf{f}_j\}$ generates the Young measure ν, it follows that

$$g(\cdot, \mathbf{f}_j(\cdot)) \rightharpoonup \int_{\mathbb{R}^d} g(\cdot, \mathbf{y}) \mathrm{d}\nu_{(\cdot)}(\mathbf{y}) \text{ in } L^1(\Omega),$$

and thus the Young measure identifies the weak limit whenever the weak limit exists. Another important fact is that a sequence that is bounded in $L^p(\Omega; \mathbb{R}^d)$ always will have a subsequence which generates some Young measure ν.

In [67], L. Tartar characterized Young measures generated by p-equiintegrable sequence $\{\mathbf{f}_j\}_{j=1}^{\infty}$ bounded in $L^p(\Omega; \mathbb{R}^N)$ as precisely those Young measures for which the map $\mathbf{x} \mapsto \int_{\mathbb{R}^N} |\mathbf{y}|^p \, \mathrm{d}\nu_{\mathbf{x}}(\mathbf{y})$ belongs to $L^1(\Omega)$. Here, by p-equiintegrable, we mean that the sequence $\{|\mathbf{f}_j|^p\}_{j=1}^{\infty}$ is equiintegrable. The first part of Chapter 3 is devoted to providing an extension of this theorem. Namely, we prove that a Young measure $\nu = \{\nu_{\mathbf{x}}\}_{\mathbf{x} \in \Omega}$ can be generated by a p-equiintegrable sequence of functions bounded in $L^{p,\lambda}(\Omega; \mathbb{R}^N)$ if and only if the map $\mathbf{x} \mapsto \int_{\mathbb{R}^N} |\mathbf{y}|^p \, \mathrm{d}\nu_{\mathbf{x}}(\mathbf{y})$ belongs to $L^{1,\lambda}(\Omega)$. The proof for this result is entirely constructive.

In the calculus of variations, often the Young measures that are important are those that can be generated by a sequence of (weak) gradients. In [51], P. Pedregal and D. Kinderlehrer provided the following characterization of Young measures generated by a p-equiintegrable sequence of gradients, where for notational convenience we define $\psi_\nu : \Omega \to [0, \infty]$ by

$$\psi_\nu(\mathbf{x}) := \int_{\mathbb{R}^{N \times n}} |\mathbf{F}|^p \, \mathrm{d}\nu_{\mathbf{x}}(\mathbf{F}).$$

Theorem 1.4. *Let $p \in [1, \infty)$. A Young measure $\nu = \{\nu_{\mathbf{x}}\}_{\mathbf{x} \in \Omega}$ can be generated by a p-equiintegrable sequence of weak gradients if and only if there is a function $\mathbf{u} \in W^{1,p}(\Omega; \mathbb{R}^N)$ such that*

(i) $\psi_\nu \in L^1(\Omega)$;

(ii) $\int_{\mathbb{R}^{N \times n}} \mathbf{F} \mathrm{d}\nu_{\mathbf{x}}(\mathbf{F}) = \nabla \mathbf{u}(\mathbf{x})$ *for almost every* $\mathbf{x} \in \Omega$;

(iii) $\int_{\mathbb{R}^{N\times n}} \varphi(\mathbf{F})\mathrm{d}\nu_{\mathbf{x}}(\mathbf{F}) \geq \varphi(\nabla\mathbf{u}(\mathbf{x}))$ *for all quasiconvex* $\varphi : \mathbb{R}^{N\times n} \to [0,\infty)$ *with* $\varphi(\mathbf{F}) \leq C(1+|\mathbf{F}|)^p$.

In Chapter 3, we extend their result to characterize those Young measures generated by sequences of weak gradients that are uniformly bounded in $L^{p,\lambda}$. Rather than only characterizing the integrability of the generating sequence (that is, in which L^p spaces the sequence is bounded), this extension also characterizes how much Morrey regularity that generating sequence can be expected to possess. The result is as follows:

Theorem 1.5. *Let* $p \in [1,\infty)$*. A Young measure* $\nu = \{\nu_{\mathbf{x}}\}_{\mathbf{x}\in\Omega}$ *can be generated by a p-equiintegrable sequence of gradients bounded in* $L^{p,\lambda}(\Omega)$ *if and only if there is a function* $\mathbf{u} \in W^{1,p}(\Omega;\mathbb{R}^N)$ *such that*

(i) $\psi_\nu \in L^{1,\lambda}(\Omega)$*;*

(ii) $\int_{\mathbb{R}^{N\times n}} \mathbf{F}\mathrm{d}\nu_{\mathbf{x}}(\mathbf{F}) = \nabla\mathbf{u}(\mathbf{x})$ *for almost every* $\mathbf{x} \in \Omega$*;*

(iii) $\int_{\mathbb{R}^{N\times n}} \varphi(\mathbf{F})\mathrm{d}\nu_{\mathbf{x}}(\mathbf{F}) \geq \varphi(\nabla\mathbf{u}(\mathbf{x}))$ *for all quasiconvex* $\varphi : \mathbb{R}^{N\times n} \to [0,\infty)$ *with* $\varphi(\mathbf{F}) \leq C(1+|\mathbf{F}|)^p$.

While the results in [51] provide optimal integrability results for the sequence of generating functions, our results allow one to refine the regularity properties the generating sequence can be expected to possess. For example, if a Young measure ν is homogeneous and can be generated by a sequence of gradients $\{\nabla\mathbf{u}_j\}$ bounded in $L^p(\Omega;\mathbb{R}^{N\times n})$, by the results obtained here, it can also be generated by a sequence of gradients $\{\nabla\mathbf{v}_j\}$ that is uniformly bounded in $L^{p,\lambda}(\Omega;\mathbb{R}^{N\times n})$ for each $0 \leq \lambda < n$. So even when it may not be possible to get any higher integrability for a sequence of gradients generating ν, it is nevertheless possible to get much more Morrey regularity on the gradients. If the boundary of Ω is Lipschitz, the Morrey regularity on the gradients translates to Hölder continuity of the potential functions $\mathbf{v}_j$. In fact, in this case we would have that $\{\mathbf{v}_j\}$ is uniformly bounded in $\mathcal{C}^{0,\alpha}(\Omega;\mathbb{R}^N)$ for each $0 \leq \alpha < 1$.

I. Fonseca and S. Müller [38] have generalized the result of D. Kinderlehrer and P. Pedregal [51] in a different direction than is carried out here; they characterize Young measures generated by sequences $\{\mathbf{v}_j\}$ bounded in L^p that satisfy $\mathrm{A}\mathbf{v}_j = 0$ for some constant rank partial differential operator A. The gradient case $\mathbf{v}_j = \nabla \mathbf{u}_j$ addressed in [51] corresponds to the case A = curl. This generalization, among other things, allows one to characterize Young measures generated by sequences of higher gradients. It may be possible to combine the ideas in [38] with those in Chapter 3 to obtain a characterization of Young measures generated by sequences $\{\mathbf{v}_j\}$ bounded in $L^{p,\lambda}$ that satisfy $\mathrm{A}\mathbf{v}_j = 0$. The main obstacle to employing the methods used here to obtain such a characterization is the absence of a way to approximate an A-free function by essentially bounded A-free functions (cf. Theorem 3.2). We would like to mention here that one can look to [8, 15, 33, 37, 50, 51, 65, 66, 69, 68, 71] for some applications of and supplementary material regarding Young measures.

1.3.3 Some Applications of the Main Results

In Chapter 4, we present several applications of our results in Chapters 2 and 3. Our first application characterizes the space $W^1L_{g,\lambda}(\Omega;\mathbb{R}^N)$ as all those functions $\mathbf{u} : \Omega \to \mathbb{R}^N$ which are almost minimizers (in an appropriate sense) for the functional

$$\mathbf{u} \mapsto \int_\Omega g(\mathbf{x}, \nabla \mathbf{u}) \mathrm{d}\mathbf{x}. \tag{1.21}$$

We then give an application that provides Morrey regularity for the gradient of solutions of systems of partial differential equations of the form

$$\mathbf{A}(\mathbf{x}, \mathbf{u}, \nabla\mathbf{u}) = \mathbf{b}(\mathbf{x}, \mathbf{u}, \nabla\mathbf{u}),$$

where the map $\mathbf{F} \mapsto \mathbf{A}(\mathbf{x}, \mathbf{u}, \mathbf{F})$ behaves asymptotically like $\mathbf{F} \mapsto \frac{\partial}{\partial \mathbf{F}} g(\mathbf{x}, |\mathbf{F}|)$, and $\mathbf{b}$ satisfies some growth conditions. The main idea is to show that solutions of the system will be almost minimizers for the functional in (1.21).

Next, we show the existence of Morrey-regular minimizing sequences for functionals using our results from Chapter 2. As a corollary of this result and the characterization of Young measures generated by Morrey-regular sequences of gradients (provided in Chapter 3), we conclude that for functionals with natural growth (when $g(\mathbf{x}, t) = t^p$), there exists a Morrey-regular minimizing Young measure. Lastly, we show that if the minimizing Young measure is Morrey regular, then there exists a Morrey-regular minimizing sequence.

Chapter 2

Morrey Regularity Results for the Gradient of Almost Minimizers

In this chapter, we will prove Morrey regularity for the gradient of almost minimizers for functionals of the form

$$\mathbf{u} \mapsto \int_{\Omega} f(\mathbf{x}, \mathbf{u}(\mathbf{x}), \nabla \mathbf{u}(\mathbf{x})) \mathrm{d}\mathbf{x}, \tag{2.1}$$

where for each $\mathbf{x} \in \Omega$ and $\mathbf{u} \in \mathbb{R}^N$, the integrand $\mathbf{F} \mapsto f(\mathbf{x}, \mathbf{u}, \mathbf{F})$ behaves asymptotically like the function $\mathbf{F} \mapsto h(|\mathbf{F}|)^{\alpha(\mathbf{x})}$, where h is an N-function such that th'' is comparable to h' and α satisfies a continuity assumption. Provided that there is a function h with (p,q)-structure such that the map $\mathbf{F} \mapsto f(\mathbf{x}, \mathbf{u}, \mathbf{F})$ behaves like $\mathbf{F} \mapsto h(|\mathbf{F}|)^{\alpha(\mathbf{x})}$ when $|\mathbf{F}|$ is large, and provided that $\mathbf{u}$ is an almost minimizer for the functional defined in (2.1), we prove that $\mathbf{x} \mapsto h(|\nabla \mathbf{u}(\mathbf{x})|)^{\alpha(\mathbf{x})}$ belongs to the Morrey space $L^{1,\lambda}(\Omega)$. The regularity we obtain is global; i.e. we have $\|h(|\nabla \mathbf{u}|)^{\alpha}\|_{L^{1,\lambda}(\Omega)} < \infty$.

Before we move on to the proofs of the results, let us first discuss the basic strategy we will employ. The first step is to prove a refinement of the result due to P. Marcellini and G. Papi, which yields an estimate of the form (1.20) (Theorems 2.8 and 2.9). Using this estimate for the Lipschitz regularity, in combination with the techniques employed in [3], we prove that if $\mathbf{v}$ is a minimizer for a functional with integrand that looks like $(\mathbf{x}, \mathbf{F}) \mapsto h(|\mathbf{F}|)^{\alpha(\mathbf{x})}$, then $h(|\nabla \mathbf{v}|)^{\alpha} \in L^{1,\kappa}_{\text{loc}}(\Omega)$ for every $\kappa \in [0, n)$. Next, under the assumption that $\mathbf{u}$ is an almost minimizer for the functional, we make a comparison between $\mathbf{u}$ and $\mathbf{v}$ to obtain some Morrey regularity for $\mathbf{u}$. It is at this stage where we also incorporate the boundary values into the functional to show that $\mathbf{u}$ is in fact globally Morrey regular. All of the aforementioned work is done in the setting of functionals with convex integrands that only depend on $\mathbf{x}$ and $|\nabla \mathbf{u}|$, and constitutes Section 2.3. In Section 2.4, we extend the scope of these results to include functionals that are merely asymptotically convex and possess integrands that can depend on $\mathbf{x}$, $\mathbf{u}$, and $\nabla \mathbf{u}$. The main idea required to treat this more general case is embodied in Lemma 2.10, where we show that an almost minimizer for the asymptotically convex functional with dependence on $\mathbf{u}$ will also be an almost minimizer for an appropriate convex functional with no dependence on $\mathbf{u}$. Hence the regularity obtained for almost minimizers of convex functionals is passed on to almost minimizers of asymptotically convex functionals.

2.1 Preliminaries

We first collect some basic properties of functions $g : \Omega \times \mathbb{R}_+ \to \mathbb{R}_+$ satisfying (1.15) and (1.16).

Lemma 2.1. *Let* $\alpha : \Omega \to [1, \infty)$ *and* $g : \Omega \times \mathbb{R}_+ \to \mathbb{R}_+$ *satisfy* (1.12) *and* (1.15)-(1.16). *Then for every* $\mathbf{x} \in \Omega$ *and every* $s, t \in \mathbb{R}_+$, *the following hold:*

(i) $p\alpha(\mathbf{x})g(\mathbf{x}, t) \leq tg_t(\mathbf{x}, t) \leq q\alpha(\mathbf{x})g(\mathbf{x}, t)$;

(ii) $c^{p\alpha(\mathbf{x})}g(\mathbf{x},t) \leq g(\mathbf{x},ct) \leq c^{q\alpha(\mathbf{x})}g(\mathbf{x},t)$ *for every* $c \geq 1$*;*

(iii) $\varepsilon^{q\alpha(\mathbf{x})}g(\mathbf{x},t) \leq g(\mathbf{x},\varepsilon t) \leq \varepsilon^{p\alpha(\mathbf{x})}g(\mathbf{x},t)$ *for every* $\varepsilon \in (0,1]$*;*

(iv) $\varepsilon^{q\alpha(\mathbf{x})-1}g_t(\mathbf{x},t) \leq g_t(\mathbf{x},\varepsilon t) \leq \varepsilon^{p\alpha(\mathbf{x})-1}g_t(\mathbf{x},t)$ *for every* $\varepsilon \in (0,1]$*;*

(v) $g(\mathbf{x},s+t) \leq 2^{q\alpha(\mathbf{x})-1}(g(\mathbf{x},s)+g(\mathbf{x},t))$*;*

(vi) $tg_t(\mathbf{x},s) \leq g(\mathbf{x},t)+(q-1)g(\mathbf{x},s)$*;*

(vii) $g_t(\mathbf{x},|\mathbf{F}_1|)\,|\mathbf{F}_2-\mathbf{F}_1| \leq C\left(g(\mathbf{x},|\mathbf{F}_2|)-g_t(\mathbf{x},|\mathbf{F}_1|)\frac{\mathbf{F}_1}{|\mathbf{F}_1|}\cdot(\mathbf{F}_2-\mathbf{F}_1)\right)$ *for some* $C = C(p,q\alpha_+)$ *and all* $\mathbf{F}_1$*,* $\mathbf{F}_2 \in \mathbb{R}^{N\times n}$*;*

Proof. Since $g(\mathbf{x},\cdot)$ has $(p\alpha(\mathbf{x}),q\alpha(\mathbf{x}))$-structure for every $\mathbf{x} \in \Omega$, we have that $g(\mathbf{x},0)=0$, so that we may write

$$g(\mathbf{x},t) = \int_0^t g_t(\mathbf{x},s)\mathrm{d}s.$$

But $g_t(\mathbf{x},s) \geq \frac{1}{p-1}sg_{tt}(\mathbf{x},s)$; putting this inequality into the above integral and integrating by parts yields

$$g(\mathbf{x},t) \leq \frac{1}{p-1}[tg_t(\mathbf{x},t)-g(\mathbf{x},t)].$$

Solving the inequality for $tg_t(\mathbf{x},t)$ gives the left inequality in part (i). A similar argument gives the other inequality, concluding our proof for the first statement of the lemma. Part (ii) is an immediate consequence of (i) and Propositions 2.1 and 2.3 in [22]. Part (iii) follows quickly from (ii) letting $c=\varepsilon^{-1}$. To prove (iv), we note that since $g(\mathbf{x},\cdot)$ has $(p\alpha(\mathbf{x}),q\alpha(\mathbf{x}))$-structure, we have

$$\frac{p\alpha(\mathbf{x})-1}{s} \leq \frac{g_{tt}(\mathbf{x},s)}{g_t(\mathbf{x},s)} \leq \frac{q\alpha(\mathbf{x})-1}{s}$$

for almost every $s > 0$. Integrating the above inequality from εt to t and using properties of logarithms gives (iv). For part (v), we define the probability measure μ on $\{s,t\}$ to be one-half the counting measure, so that we have by (ii) and Jensen's inequality that

$$g(\mathbf{x}, s+t) = g\left(\mathbf{x}, 2\int_{\{s,t\}} y \mathrm{d}\mu(y)\right) \leq 2^{q\alpha(\mathbf{x})} g\left(\mathbf{x}, \int_{\{s,t\}} y \mathrm{d}\mu(y)\right) \leq 2^{q\alpha(\mathbf{x})} \int_{\{s,t\}} g(\mathbf{x}, y) \mathrm{d}\mu(y)$$
$$= 2^{q\alpha(\mathbf{x})-1} \left(g(\mathbf{x}, s) + g(\mathbf{x}, t)\right),$$

which is (v). To prove (vi), we let $g^*(\mathbf{x}, \cdot)$ denote the Young conjugate of $g(\mathbf{x}, \cdot)$ (Definition 1.4). Then by (1.5) and (i), we have

$$tg_t(\mathbf{x}, s) \leq g(\mathbf{x}, s) + g^*(\mathbf{x}, g_t(\mathbf{x}, s)) = g(\mathbf{x}, t) + sg_t(\mathbf{x}, s) - g(\mathbf{x}, s) \leq g(\mathbf{x}, t) + (q-1)g(\mathbf{x}, s),$$

which proves (vi).

The proof of (vii) is more involved. We consider two cases.

Case 1: $|\mathbf{F}_2| \leq 5\,|\mathbf{F}_1|$.

In this case, we have by (i) that

$$g_t(\mathbf{x}, |\mathbf{F}_1|)\,|\mathbf{F}_2 - \mathbf{F}_1| \leq 6g_t(\mathbf{x}, |\mathbf{F}_1|)\,|\mathbf{F}_1| \leq 6q\alpha_+ g(\mathbf{x}, |\mathbf{F}_1|).$$

But by the convexity of $\mathbf{F} \mapsto g(\mathbf{x}, |\mathbf{F}|)$, we have that

$$g(\mathbf{x}, |\mathbf{F}_1|) \leq g(\mathbf{x}, |\mathbf{F}_2|) - \frac{\partial}{\partial \mathbf{F}} g(\mathbf{x}, |\mathbf{F}_1|) \cdot [\mathbf{F}_2 - \mathbf{F}_1]$$
$$= g(\mathbf{x}, |\mathbf{F}_2|) - g_t(\mathbf{x}, |\mathbf{F}_1|) \tfrac{\mathbf{F}_1}{|\mathbf{F}_1|} \cdot (\mathbf{F}_2 - \mathbf{F}_1).$$

Combining the above inequalities gives the desired estimate, and thus concludes the proof for Case 1.

Case 2: $|\mathbf{F}_2| > 5\,|\mathbf{F}_1|$.

In this case, note that we have that $|\mathbf{F}_1| \leq \frac{1}{4}\,|\mathbf{F}_2 - \mathbf{F}_1|$, and hence for $t \in [1/2, 3/4]$, we obtain the inequality

$$\tfrac{1}{4}\,|\mathbf{F}_2 - \mathbf{F}_1| \leq |\mathbf{F}_1 + t[\mathbf{F}_2 - \mathbf{F}_1]| \leq |\mathbf{F}_2 - \mathbf{F}_1|\,. \tag{2.2}$$

Now we will show that there is a constant C such that for all $x \in \mathcal{V}$ and a.e. $t \in [1/2, 3/4]$, we have

$$g\left(\mathbf{x}, \tfrac{1}{4}\,|\mathbf{F}_2 - \mathbf{F}_1|\right) \leq C\frac{\mathrm{d}^2}{\mathrm{d}t^2} g\left(\mathbf{x}, |\mathbf{F}_1 + t[\mathbf{F}_2 - \mathbf{F}_1]|\right). \tag{2.3}$$

A routine computation shows that

$$\begin{aligned}\frac{\mathrm{d}^2}{\mathrm{d}t^2}[g(\mathbf{x}, |\mathbf{F}_1+t[\mathbf{F}_2-\mathbf{F}_1]|)] = {} & \frac{g_t(\mathbf{x}, |\mathbf{F}_1 + t[\mathbf{F}_2 - \mathbf{F}_1]|)}{|\mathbf{F}_1 + t[\mathbf{F}_2 - \mathbf{F}_1]|}\,|\mathbf{F}_2 - \mathbf{F}_1|^2 \\ & + \frac{g_{tt}(\mathbf{x}, |\mathbf{F}_1+t[\mathbf{F}_2-\mathbf{F}_1]|)}{|\mathbf{F}_1+t[\mathbf{F}_2-\mathbf{F}_1]|^2}\big([\mathbf{F}_1+t(\mathbf{F}_2-\mathbf{F}_1)]\cdot[\mathbf{F}_2-\mathbf{F}_1]\big)^2 \\ & - \frac{g_t(|\mathbf{F}_1 + t[\mathbf{F}_2 - \mathbf{F}_1]|}{|\mathbf{F}_1 + t[\mathbf{F}_2 - \mathbf{F}_1]|^3}\big([\mathbf{F}_1 + t(\mathbf{F}_2 - \mathbf{F}_1)]\cdot[\mathbf{F}_2 - \mathbf{F}_1]\big)^2.\end{aligned} \tag{2.4}$$

To obtain (2.3), we need to consider two cases. First, we suppose that $1 < p < 2$. Using (2.2), part (i), and (2.2) again, we obtain

$$\begin{aligned} g\left(\mathbf{x}, \tfrac{1}{4}\,|\mathbf{F}_2 - \mathbf{F}_1|\right) &\leq g(\mathbf{x}, |\mathbf{F}_1 + t[\mathbf{F}_2 - \mathbf{F}_1]|) \\ &\leq \frac{1}{p}\frac{g_t(\mathbf{x}, |\mathbf{F}_1 + t[\mathbf{F}_2 - \mathbf{F}_1]|)}{|\mathbf{F}_1 + t[\mathbf{F}_2 - \mathbf{F}_1]|}\,|\mathbf{F}_1 + t[\mathbf{F}_2 - \mathbf{F}_1]|^2 \\ &\leq \frac{1}{p}\frac{g_t(\mathbf{x}, |\mathbf{F}_1 + t[\mathbf{F}_2 - \mathbf{F}_1]|)}{|\mathbf{F}_1 + t[\mathbf{F}_2 - \mathbf{F}_1]|}\,|\mathbf{F}_2 - \mathbf{F}_1|^2\,. \end{aligned} \tag{2.5}$$

We rewrite the right side of the previous inequality as follows:

$$\frac{1}{p}\frac{g_t(\mathbf{x},|\mathbf{F}_1+t[\mathbf{F}_2-\mathbf{F}_1]|)}{|\mathbf{F}_1+t[\mathbf{F}_2-\mathbf{F}_1]|}|\mathbf{F}_2-\mathbf{F}_1|^2 = \frac{p-2}{p(p-1)}\frac{g_t(\mathbf{x},|\mathbf{F}_1+t[\mathbf{F}_2-\mathbf{F}_1]|)}{|\mathbf{F}_1+t[\mathbf{F}_2-\mathbf{F}_1]|}|\mathbf{F}_2-\mathbf{F}_1|^2 + \frac{1}{p(p-1)}\frac{g_t(\mathbf{x},|\mathbf{F}_1+t[\mathbf{F}_2-\mathbf{F}_1]|)}{|\mathbf{F}_1+t[\mathbf{F}_2-\mathbf{F}_1]|}|\mathbf{F}_2-\mathbf{F}_1|^2.$$

Since we are assuming for the moment that $1<p<2$, obviously $p-2<0$; therefore, from the equality above and the Cauchy-Schwartz inequality, we have that

$$\begin{aligned}\frac{1}{p}\frac{g_t(\mathbf{x},|\mathbf{F}_1+t[\mathbf{F}_2-\mathbf{F}_1]|)}{|\mathbf{F}_1+t[\mathbf{F}_2-\mathbf{F}_1]|}|\mathbf{F}_2-\mathbf{F}_1|^2 &\le \frac{p-2}{p(p-1)}\frac{g_t(\mathbf{x},|\mathbf{F}_1+t[\mathbf{F}_2-\mathbf{F}_1]|)}{|\mathbf{F}_1+t[\mathbf{F}_2-\mathbf{F}_1]|^3}\left([\mathbf{F}_1+t[\mathbf{F}_2-\mathbf{F}_1]]\cdot[\mathbf{F}_2-\mathbf{F}_1]\right)^2 \\ &\quad + \frac{1}{p(p-1)}\frac{g_t(\mathbf{x},|\mathbf{F}_1+t[\mathbf{F}_2-\mathbf{F}_1]|)}{|\mathbf{F}_1+t[\mathbf{F}_2-\mathbf{F}_1]|}|\mathbf{F}_2-\mathbf{F}_1|^2. \end{aligned} \tag{2.6}$$

Now, we note that by (i) and by the fact that $g(\mathbf{x},\cdot)$ has $(p\alpha(\mathbf{x}),q\alpha(x))$-structure, we have that

$$p(p-1)g(\mathbf{x},t)\le t^2 g_{tt}(\mathbf{x},t)\le q\alpha_+(q\alpha_+-1)g(\mathbf{x},t) \tag{2.7}$$

for almost every $t\in(0,\infty)$. Using this inequality along with the computation in (2.4), we get

$$\frac{1}{p}\frac{g_t(\mathbf{x},|\mathbf{F}_1+t[\mathbf{F}_2-\mathbf{F}_1]|)}{|\mathbf{F}_1+t[\mathbf{F}_2-\mathbf{F}_1]|}|\mathbf{F}_2-\mathbf{F}_1|^2 \le \frac{1}{p(p-1)}\frac{\mathrm{d}^2}{\mathrm{d}t^2}g(\mathbf{x},|\mathbf{F}_1+t[\mathbf{F}_2-\mathbf{F}_1]|).$$

Combining this with (2.5), we see that

$$g\left(\mathbf{x},\tfrac{1}{4}|\mathbf{F}_2-\mathbf{F}_1|\right)\le\frac{1}{p(p-1)}\frac{\mathrm{d}^2}{\mathrm{d}t^2}g\left(\mathbf{x},|\mathbf{F}_1+t[\mathbf{F}_2-\mathbf{F}_1]|\right) \tag{2.8}$$

for $1<p<2$.

Now assume $p \geq 2$. Then $g_t(\mathbf{x}, s)/s$ is increasing with respect to s; using this fact along with (i) and (2.2), we find that

$$g\left(\mathbf{x}, \tfrac{1}{4}\left|\mathbf{F}_2 - \mathbf{F}_1\right|\right) \leq \frac{1}{16p}\frac{g_t(\mathbf{x}, \frac{1}{4}\left|\mathbf{F}_2 - \mathbf{F}_1\right|)}{\frac{1}{4}\left|\mathbf{F}_2 - \mathbf{F}_1\right|}\left|\mathbf{F}_2 - \mathbf{F}_1\right|^2 \qquad (2.9)$$
$$\leq \frac{1}{16p}\frac{g_t(\mathbf{x}, |\mathbf{F}_1 + t[\mathbf{F}_2 - \mathbf{F}_1]|)}{|\mathbf{F}_1 + t[\mathbf{F}_2 - \mathbf{F}_1]|}\left|\mathbf{F}_2 - \mathbf{F}_1\right|^2.$$

Since $p \geq 2$, we have

$$\frac{p-2}{16p}\frac{g_t(\mathbf{x}, |\mathbf{F}_1 + t[\mathbf{F}_2 - \mathbf{F}_1]|)}{|\mathbf{F}_1 + t[\mathbf{F}_2 - \mathbf{F}_1]|^3}\left([\mathbf{F}_1 + t[\mathbf{F}_2 - \mathbf{F}_1]] \cdot [\mathbf{F}_2 - \mathbf{F}_1]\right)^2 \geq 0,$$

and therefore we can add it to the right side of (2.9) and use (2.6) and (2.4) to obtain

$$g\left(\mathbf{x}, \tfrac{1}{4}\left|\mathbf{F}_2 - \mathbf{F}_1\right|\right) \leq \frac{1}{16p}\frac{\mathrm{d}^2}{\mathrm{d}t^2}g\left(\mathbf{x}, |\mathbf{F}_1 + t[\mathbf{F}_2 - \mathbf{F}_1]|\right) \qquad (2.10)$$

when $p \geq 2$. Combining our estimates for the case $1 < p < 2$ and the case $p \geq 2$, we have established (2.3) for any $p > 1$ and almost every $t \in [1/2, 3/4]$.

We now proceed with the original estimate. Using part (vi) and (2.3), we obtain

$$g_t(\mathbf{x}, |\mathbf{F}_1|)\,|\mathbf{F}_2 - \mathbf{F}_1| = 4g_t(\mathbf{x}, |\mathbf{F}_1|)\left(\tfrac{1}{4}\left|\mathbf{F}_2 - \mathbf{F}_1\right|\right)$$
$$\leq Cg(\mathbf{x}, |\mathbf{F}_1|) + C\int_{1/2}^{3/4}(1-t)\frac{\mathrm{d}^2}{\mathrm{d}t^2}g(\mathbf{x}, |\mathbf{F}_1 + t[\mathbf{F}_2 - \mathbf{F}_1]|)\mathrm{d}t\mathrm{d}\mathbf{x}.$$

Recalling that $\frac{\mathrm{d}^2}{\mathrm{d}t^2}g(|\mathbf{F}_1 + t[\mathbf{F}_2 - \mathbf{F}_1]|) \geq 0$ for a.e. $t \in [0, 1]$, since the map $t \mapsto g(|\mathbf{F}_1 + t[\mathbf{F}_2 - \mathbf{F}_1]|)$ is convex, we can expand the domain of integration in the right side of the previous inequality to get

$$\begin{aligned} g_t(\mathbf{x},|\mathbf{F}_1|)\,|\mathbf{F}_2-\mathbf{F}_1| &\le Cg(\mathbf{x},|\mathbf{F}_1|)+C\int_0^1(1-t)\frac{\mathrm{d}^2}{\mathrm{d}t^2}g(\mathbf{x},|\mathbf{F}_1+t[\mathbf{F}_2-\mathbf{F}_1]|)\mathrm{d}t \\ &= Cg(\mathbf{x},|\mathbf{F}_1|)+C\left[g(\mathbf{x},|\mathbf{F}_2|)-g(\mathbf{x},|\mathbf{F}_1|)-g_t(\mathbf{x},|\mathbf{F}_1|)\tfrac{\mathbf{F}_1}{|\mathbf{F}_1|}\cdot[\mathbf{F}_2-\mathbf{F}_1]\right] \\ &= C\left\{g(\mathbf{x},|\mathbf{F}_2|)-g_t(\mathbf{x},|\mathbf{F}_1|)\tfrac{\mathbf{F}_1}{|\mathbf{F}_1|}\cdot[\mathbf{F}_2-\mathbf{F}_1]\right], \end{aligned}$$

which finishes the proof. □

The following lemma establishes that the Euler-Lagrange system of equations holds in the weak sense for minimizers of appropriate functionals. In our proof, we use the same method that Evans uses to prove Theorem 4 on page 451 in [34]. The main modification that we need to make is to use part (vi) of Lemma 2.1 instead of the standard version of Young's inequality.

Lemma 2.2. *Suppose that* $\alpha : \Omega \to [1,\infty)$ *satisfies* (1.12)-(1.14) *and* $g : \Omega\times\mathbb{R}_+ \to \mathbb{R}_+$ *satisfies* (1.15)-(1.17). *Let* $\mathbf{G}_0 \in \mathbb{R}^{n\times n}$ *be an invertible matrix, and suppose that* $\mathbf{w} \in W^1L_g(\Omega;\mathbb{R}^N)$ *is the minimizer for the functional*

$$\mathbf{u} \mapsto \int_\Omega g(\mathbf{x}, |\nabla\mathbf{u}\mathbf{G}_0|)\mathrm{d}\mathbf{x}$$

among all mappings $\mathbf{u} \in \mathbf{w} + W^1_0L_g(\Omega;\mathbb{R}^N)$. *Then*

$$\int_\Omega g_t(\mathbf{x}, |\nabla\mathbf{w}\mathbf{G}_0|)\frac{\nabla\mathbf{w}\mathbf{G}_0}{|\nabla\mathbf{w}\mathbf{G}_0|}\cdot\nabla\boldsymbol{\varphi}\mathbf{G}_0\mathrm{d}\mathbf{x} = 0$$

for every $\boldsymbol{\varphi} \in W^1_0L_g(\Omega;\mathbb{R}^N)$.

Proof. Let $\boldsymbol{\varphi} \in W^1_0L_g(\Omega;\mathbb{R}^N)$ be given. For every $\varepsilon > 0$, since $\mathbf{w}$ is a minimizer, we have that

$$\begin{aligned}0 &\le \frac{1}{\varepsilon}\int_\Omega \{g(\mathbf{x}, |[\nabla \mathbf{w} + \varepsilon\nabla\boldsymbol{\varphi}]\mathbf{G}_0|) - g(\mathbf{x}, |\nabla \mathbf{w}\mathbf{G}_0|)\}\,\mathrm{d}\mathbf{x} \\ &= \frac{1}{\varepsilon}\int_\Omega\int_0^1 \frac{\mathrm{d}}{\mathrm{d}s} g(\mathbf{x}, |[\nabla \mathbf{w} + s\varepsilon\nabla\boldsymbol{\varphi}]\mathbf{G}_0|)\mathrm{d}s\mathrm{d}\mathbf{x} \\ &= \int_\Omega\int_0^1 g_t(\mathbf{x}, |[\nabla \mathbf{w} + s\varepsilon\nabla\boldsymbol{\varphi}]\mathbf{G}_0|)\frac{[\nabla \mathbf{w} + s\varepsilon\nabla\boldsymbol{\varphi}]\mathbf{G}_0}{|[\nabla \mathbf{w} + s\varepsilon\nabla\boldsymbol{\varphi}]\mathbf{G}_0|}\cdot\nabla\boldsymbol{\varphi}\mathbf{G}_0\mathrm{d}s\mathrm{d}\mathbf{x}.\end{aligned} \tag{2.11}$$

Note that for any $\varepsilon \in (0,1)$, by parts (vi), (ii), and (v) of Lemma 2.1, we have

$$\begin{aligned}&\left|g_t(\mathbf{x}, |[\nabla \mathbf{w} + s\varepsilon\nabla\boldsymbol{\varphi}]\mathbf{G}_0|)\frac{[\nabla \mathbf{w} + s\varepsilon\nabla\boldsymbol{\varphi}]\mathbf{G}_0}{|[\nabla \mathbf{w} + s\varepsilon\nabla\boldsymbol{\varphi}]\mathbf{G}_0|}\cdot\nabla\boldsymbol{\varphi}\mathbf{G}_0\right| \\ &\qquad\le g_t(\mathbf{x}, |[\nabla \mathbf{w} + s\varepsilon\nabla\boldsymbol{\varphi}]\mathbf{G}_0|)\,|\nabla\boldsymbol{\varphi}\mathbf{G}_0| \\ &\qquad\le Cg(\mathbf{x}, |[\nabla \mathbf{w} + s\varepsilon\nabla\boldsymbol{\varphi}]\mathbf{G}_0|) + g(\mathbf{x}, |\nabla\boldsymbol{\varphi}\mathbf{G}_0|) \\ &\qquad\le C[g(\mathbf{x}, |\nabla \mathbf{w}|) + g(\mathbf{x}, |\nabla\boldsymbol{\varphi}|)].\end{aligned}$$

So for $\varepsilon \in (0,1)$, the integrand for the last integral in (2.11) is uniformly bounded by a function belonging to $L^1(\Omega)$, so that we may use Lebesgue's dominated convergence theorem along with the continuity of g_t to conclude that

$$\begin{aligned}0 &\le \lim_{\varepsilon\to 0^+}\int_\Omega\int_0^1 g_t(\mathbf{x}, |[\nabla \mathbf{w} + s\varepsilon\nabla\boldsymbol{\varphi}]\mathbf{G}_0|)\frac{[\nabla \mathbf{w} + s\varepsilon\nabla\boldsymbol{\varphi}]\mathbf{G}_0}{|[\nabla \mathbf{w} + s\varepsilon\nabla\boldsymbol{\varphi}]\mathbf{G}_0|}\cdot\nabla\boldsymbol{\varphi}\mathbf{G}_0\mathrm{d}s\mathrm{d}\mathbf{x} \\ &= \int_\Omega g_t(\mathbf{x}, |\nabla \mathbf{w}\mathbf{G}_0|)\frac{\nabla \mathbf{w}\mathbf{G}_0}{|\nabla \mathbf{w}\mathbf{G}_0|}\cdot\nabla\boldsymbol{\varphi}\mathbf{G}_0\mathrm{d}\mathbf{x}.\end{aligned}$$

We can repeat the same argument with $-\boldsymbol{\varphi}$ replacing $\boldsymbol{\varphi}$, which gives the desired conclusion. □

Lemma 2.3. *Suppose that h is a function with (p,q)-structure, and let $\beta \ge 0$. Then there is a positive constant $C = C(p,q)$, independent of β, such that*

$$\left[\int_0^t s^\beta h'(s)^\beta\sqrt{h''(s)}\mathrm{d}s\right]^2 \ge \frac{1}{C(2\beta+1)^2}t^{2\beta+1}h'(t)^{2\beta+1}.$$

Proof. Using the definition of (p,q)-structure, integrating by parts, then using that h has (p,q)-structure again, we obtain

$$\int_0^t s^\beta h'(s)^\beta \sqrt{h''(s)}\mathrm{d}s \\ \geq \frac{2\sqrt{p-1}}{2\beta+1} t^{\beta+\frac{1}{2}} h'(t)^{\beta+\frac{1}{2}} - \sqrt{(p-1)(q-1)} \int_0^t s^\beta h'(s)^\beta \sqrt{h''(s)}\mathrm{d}s.$$

Solving the inequality for the integral and squaring both sides yields the result. □

The following theorem gives a type of Sobolev-Poincaré inequality for functions in $W^1L_h(\Omega;\mathbb{R}^N)$.

Theorem 2.1. *Suppose $h : \mathbb{R}_+ \to \mathbb{R}_+$ is a function with (p,q)-structure and that $\Omega \subset \mathbb{R}^n$ is open and bounded with no external cusps. If $r_0 \in (1, n/(n-1))$, then there exists a constant C, which depends on n, N, p, q, r_0, and Ω, such that if $0 < R < \mathrm{diam}(\Omega)$, then*

$$\fint_{\Omega(\mathbf{x}_0,R)} h\left(\frac{|\mathbf{u}-\boldsymbol{\xi}|}{R}\right)^{r_0} \mathrm{d}\mathbf{x} \leq C\left(\fint_{\Omega(\mathbf{x}_0,R)} h(|\nabla \mathbf{u}|)\mathrm{d}\mathbf{x}\right)^{r_0}$$

for all $\mathbf{u} \in W^1L_h(\Omega;\mathbb{R}^N)$, where $\boldsymbol{\xi} = (\mathbf{u})_{\Omega(\mathbf{x}_0,R)}$. If $\mathbf{u} \in W^1_0L_h(\Omega(\mathbf{x}_0,R);\mathbb{R}^N)$, then the above inequality also holds with $\boldsymbol{\xi} = \mathbf{0}$.

Remark 2.1. The manner in which C depends on Ω is only by the quantity

$$\sup_{\mathbf{x}_0\in\Omega} \sup_{R\in(0,\mathrm{diam}(\Omega)]} \frac{|\mathcal{B}(\mathbf{x}_0,R)|}{|\Omega(\mathbf{x}_0,R)|}.$$

Proof. We initially suppose that $\mathbf{w} \in W^1L_{h^{r_0}}(\Omega;\mathbb{R}^N)$. By Lemma 7.14 in [46] (for $\boldsymbol{\xi} = \mathbf{0}$ and $\mathbf{w} \in W^1_0L_{h^{r_0}}(\Omega;\mathbb{R}^N) \subset W^{1,1}_0(\Omega;\mathbb{R}^N)$) and Lemma 1.50 in [54] (for $\boldsymbol{\xi} =$

$(\mathbf{w})_{\Omega(\mathbf{x}_0,R)})$, there is a constant $C = C(n, N)$ so that

$$|\mathbf{w}(\mathbf{x}) - \boldsymbol{\xi}| \leq C \int_{\Omega(\mathbf{x}_0,R)} \frac{|\nabla \mathbf{w}(\mathbf{y})|}{|\mathbf{x} - \mathbf{y}|^{n-1}} \mathrm{d}\mathbf{y},$$

for almost every $\mathbf{x} \in \Omega(\mathbf{x}_0, R)$. Following the proof of Theorem 7 in [25], we obtain

$$\fint_{\Omega(\mathbf{x}_0,R)} h\left(\frac{|\mathbf{w} - \boldsymbol{\xi}|}{R}\right)^{r_0} \mathrm{d}\mathbf{x} \leq C \left(\fint_{\Omega(\mathbf{x}_0,R)} h(|\nabla \mathbf{w}|) \mathrm{d}\mathbf{x}\right)^{r_0}. \tag{2.12}$$

So we have obtained the desired estimate under the additional hypothesis that $\mathbf{u} \in W^1 L_{h^{r_0}}(\Omega; \mathbb{R}^N)$. To remove this extra assumption, we use an approximation scheme. First assume that $\boldsymbol{\xi} = (\mathbf{u})_{\Omega(\mathbf{x}_0,R)}$. By Theorem 8.31(e) in [6], there is a sequence $\{\mathbf{u}_j\}_{j=1}^{\infty} \subset C^{\infty}(\overline{\Omega(\mathbf{x}_0, R)})$ with $\mathbf{u}_j$ converging to $\mathbf{u}$ in $W^1 L_h(\Omega(\mathbf{x}_0, R); \mathbb{R}^N)$. Without loss of generality, we can assume that $(\mathbf{u}_j)_{\Omega(\mathbf{x}_0,R)} = \boldsymbol{\xi}$. We also note here that $\{\mathbf{u}_j\} \subset W^1 L_{h^{r_0}}(\Omega(\mathbf{x}_0, R); \mathbb{R}^N)$ Putting $\mathbf{w} = \mathbf{u}_j - \mathbf{u}_k$ into (2.12) gives

$$\fint_{\Omega(\mathbf{x}_0,R)} h\left(\frac{|\mathbf{u}_j - \mathbf{u}_k|}{R}\right)^{r_0} \mathrm{d}\mathbf{x} \leq C \left(\fint_{\Omega(\mathbf{x}_0,R)} h(|\nabla \mathbf{u}_j - \nabla \mathbf{u}_k|) \mathrm{d}\mathbf{x}\right)^{r_0}.$$

We therefore see that the sequence $\{\mathbf{u}_j\}_{j=1}^{\infty}$ is Cauchy in $L_{h^{r_0}}(\Omega(\mathbf{x}_0, R); \mathbb{R}^N)$. Using this and that $\mathbf{u}_j$ converges to $\mathbf{u}$ in $L^1(\Omega(\mathbf{x}_0, R); \mathbb{R}^N)$, we must have that actually $\mathbf{u}_j$ converges to $\mathbf{u}$ in $L_{h^{r_0}}(\Omega(\mathbf{x}_0, R); \mathbb{R}^N)$. But by (2.12) with $\mathbf{w} = \mathbf{u}_j$, we have that

$$\fint_{\Omega(\mathbf{x}_0,R)} h\left(\frac{|\mathbf{u}_j - \boldsymbol{\xi}|}{R}\right)^{r_0} \mathrm{d}\mathbf{x} \leq C \left(\fint_{\Omega(\mathbf{x}_0,R)} h(|\nabla \mathbf{u}_j|) \mathrm{d}\mathbf{x}\right)^{r_0}.$$

Taking limits yields the inequality which was to be shown.

To obtain the desired inequality with $\boldsymbol{\xi} = \mathbf{0}$ for $\mathbf{u} \in W_0^1 L_h(\Omega(\mathbf{x}_0, R); \mathbb{R}^N)$, we just note that in this case it is possible to take $\{\mathbf{u}_j\}_{j=1}^{\infty} \subset \mathcal{C}_c^{\infty}(\Omega(\mathbf{x}_0, R); \mathbb{R}^N)$. Hence $\{\mathbf{u}_j\}_{j=1}^{\infty} \subset W_0^1 L_{h^{r_0}}(\Omega(\mathbf{x}_0, R); \mathbb{R}^N)$, and so the same arguments can be employed, using (2.12) with $\boldsymbol{\xi} = \mathbf{0}$. □

Next we prove a version of a Sobolev-Poincaré inequality for functions belonging to $W^1L_g(\Omega;\mathbb{R}^N)$; our method of proof follows that of Zhikov [72], with suitable modifications.

Theorem 2.2. *Suppose that $\Omega \subset \mathbb{R}^n$ is open and bounded and has no external cusps. Suppose also that $\alpha : \Omega \to [1,\infty)$ satisfies* (1.12) *and* (1.13), *and that $g : \Omega \times \mathbb{R}_+ \to \mathbb{R}_+$ satisfies* (1.15) *and* (1.16). *Then there is some $r_1 > 1$ and $R_0 > 0$, both of which depend on p, q, α_+, and ω, so that for any $\mathbf{u} \in W^1L_g(\Omega;\mathbb{R}^N)$, it follows that $\mathbf{u} \in L_{g^{r_1}}(\Omega;\mathbb{R}^N)$. Furthermore, if $\boldsymbol{\xi} := (\mathbf{u})_{\mathcal{B}(\mathbf{x}_0,R)}$, then*

$$\fint_{\Omega(\mathbf{x}_0,R)} g\left(\mathbf{x}, \frac{|\mathbf{u}-\boldsymbol{\xi}|}{R}\right) \mathrm{d}\mathbf{x} \leq C\left\{1 + \left(\fint_{\Omega(\mathbf{x}_0,R)} g(\mathbf{x},|\nabla\mathbf{u}|)^{\frac{1}{r_1}} \mathrm{d}\mathbf{x}\right)^{r_1}\right\}$$

whenever $0 < R < R_0$. The constant C depends on n, N, α_+, M, ω, Ω, and $\int_\Omega (1 + g(\mathbf{x},|\nabla\mathbf{u}|))\mathrm{d}\mathbf{x}$. If $\mathbf{u} \in W_0^1L_g(\mathcal{B}(\mathbf{x}_0,R);\mathbb{R}^N)$, then the same inequality holds with $\boldsymbol{\xi} = \mathbf{0}$.

Remark 2.2. By examining the proof, we see that we also have the inequality

$$\left(\fint_{\Omega(\mathbf{x}_0,R)} g\left(\mathbf{x}, \frac{|\mathbf{u}-\boldsymbol{\xi}|}{R}\right)^{r_2} \mathrm{d}\mathbf{x}\right)^{\frac{1}{r_2}} \leq C\fint_{\Omega(\mathbf{x}_0,R)} \{1 + g(\mathbf{x},|\nabla\mathbf{u}|)\}\,\mathrm{d}\mathbf{x},$$

for some $r_2 \geq r_1$ and for all $R \in (0,R_0]$, where again $\boldsymbol{\xi} = (\mathbf{u})_{\mathcal{B}(\mathbf{x}_0,R)}$. If $\mathbf{u} \in W_0^1L_g(\mathcal{B}(\mathbf{x}_0,R);\mathbb{R}^N)$, the same inequality holds with $\boldsymbol{\xi} = \mathbf{0}$.

Proof. Fix $\mathbf{x}_0 \in \Omega$, and define

$$\alpha_1(R) := \inf_{\mathbf{x}\in\Omega(\mathbf{x}_0,R)} \alpha(\mathbf{x}),$$

$$\alpha_2(R) := \sup_{\mathbf{x}\in\Omega(\mathbf{x}_0,R)} \alpha(\mathbf{x}).$$

For ease of notation, we will henceforth write α_1 and α_2 for $\alpha_1(R)$ and $\alpha_2(R)$, respectively, keeping in mind that α_1 and α_2 vary as R varies. Find $\mathbf{x}_1$ and $\mathbf{x}_2$ in $\overline{\Omega(\mathbf{x}_0, R)}$ such that $\alpha(\mathbf{x}_1) = \alpha_1$ and $\alpha(\mathbf{x}_2) = \alpha_2$. Let $r_0 \in (1, n/(n-1))$ be such that $1 > r_0^{-1/2} \geq 1 - (p-1)/(2q\alpha_+)$, so that $h^{\alpha_1/\sqrt{r_0}}$ has $((p+1)/2, q\alpha_+)$-structure. Then select $r_1 > 1$ and $R_0 > 0$ so that for $0 < R \leq R_0$, we have

$$1 \leq \frac{\alpha_2 r_1}{\alpha_1} \leq \sqrt{r_0}, \quad \text{and} \quad \frac{r_1\sqrt{r_0}(\alpha_2 - \alpha_1)}{\alpha_1} \leq C\omega(2R) \leq 1. \tag{2.13}$$

We use (2.13), Hölder's Inequality and Theorem 2.1 to obtain

$$\begin{aligned} \fint_{\Omega(\mathbf{x}_0,R)} h\left(\frac{|\mathbf{u}-\boldsymbol{\xi}|}{R}\right)^{\alpha_2 r_1} \mathrm{d}\mathbf{x} &\leq C\left(\fint_{\Omega(\mathbf{x}_0,R)} \left(h\left(\frac{|\mathbf{u}-\boldsymbol{\xi}|}{R}\right)^{\frac{\alpha_1}{\sqrt{r_0}}}\right)^{r_0} \mathrm{d}\mathbf{x}\right)^{\frac{\alpha_2 r_1}{\alpha_1\sqrt{r_0}}} \\ &\leq C\left(\fint_{\Omega(\mathbf{x}_0,R)} h\left(|\nabla\mathbf{u}|\right)^{\frac{\alpha_1}{\sqrt{r_0}}} \mathrm{d}\mathbf{x}\right)^{\frac{\alpha_2 r_1\sqrt{r_0}}{\alpha_1}}. \end{aligned}$$

After writing $\frac{\alpha_2 r_1\sqrt{r_0}}{\alpha_1} = \frac{(\alpha_2-\alpha_1)r_1\sqrt{r_0}}{\alpha_1} + r_1\sqrt{r_0}$, Hölder's inequality and (2.13) yield

$$\begin{aligned} &\left(\fint_{\Omega(\mathbf{x}_0,R)} h\left(|\nabla\mathbf{u}|\right)^{\frac{\alpha_1}{\sqrt{r_0}}} \mathrm{d}\mathbf{x}\right)^{\frac{\alpha_2 r_1\sqrt{r_0}}{\alpha_1}} \\ &\qquad \leq CR^{-Cn\omega(2R)}\left(\int_{\Omega(\mathbf{x}_0,R)} h(|\nabla\mathbf{u}|)^{\alpha_1}\mathrm{d}\mathbf{x}\right)\left(\fint_{\Omega(\mathbf{x}_0,R)} h(|\nabla\mathbf{u}|)^{\frac{\alpha_1}{\sqrt{r_0}}}\mathrm{d}\mathbf{x}\right)^{r_1\sqrt{r_0}}. \end{aligned}$$

By (1.13), we have that $R^{-Cn\omega(2R)}$ is uniformly bounded, so by setting

$$\tilde{C} := 1 + \sup_{R>0} CR^{-Cn\omega(2R)} \int_\Omega (1 + g(\mathbf{x}, |\nabla\mathbf{u}|))\mathrm{d}\mathbf{x}$$

and combining our previous inequalities, then using Hölder's inequality, we find that

$$\begin{aligned} \fint_{\Omega(\mathbf{x}_0,R)} h\left(\frac{|\mathbf{u}-\boldsymbol{\xi}|}{R}\right)^{\alpha_2 r_1} \mathrm{d}\mathbf{x} &\le \tilde{C}\left(1+\fint_{\Omega(\mathbf{x}_0,R)} h(|\nabla\mathbf{u}|)^{\frac{\alpha(\mathbf{x})}{\sqrt{r_0}}} \mathrm{d}\mathbf{x}\right)^{r_1\sqrt{r_0}} \\ &\le \tilde{C}\left(1+\fint_{\Omega(\mathbf{x}_0,R)} h(|\nabla\mathbf{u}|)^{\frac{\alpha(\mathbf{x})}{r_1}} \mathrm{d}\mathbf{x}\right)^{r_1^2}. \end{aligned} \tag{2.14}$$

Because Ω is bounded, this implies that $\mathbf{u} \in L_{g^{r_1}}(\Omega;\mathbb{R}^N)$. Furthermore, the estimate in the statement of the theorem follows from (2.14), (1.16), and Hölder's inequality.

□

We also have the following Caccioppoli inequality.

Theorem 2.3. *Suppose that* $\alpha : \Omega \to [1,\infty)$ *satisfies* (1.12)-(1.14) *and that* $g : \Omega \times \mathbb{R}_+ \to \mathbb{R}_+$ *satisfies* (1.15) *and* (1.16). *Let* $\mathbf{G}_0 \in \mathbb{R}^{n\times n}$ *be an invertible matrix, and suppose that* $\mathbf{w} \in W^1L_g(\Omega;\mathbb{R}^N)$ *is a minimizer for*

$$\mathbf{v} \mapsto \int_\Omega g(\mathbf{x}, |\nabla\mathbf{v}(\mathbf{x})\mathbf{G}_0|)\mathrm{d}\mathbf{x}$$

among all mappings $\mathbf{v} \in \mathbf{w}+W_0^1L_g(\Omega;\mathbb{R}^N)$. *There is a constant* $C = C(p, q\alpha_+, |\mathbf{G}_0^{-1}|)$ *such that*

$$\int_{\mathcal{B}(\mathbf{x}_0,\frac{R}{2})} g(\mathbf{x}, |\nabla\mathbf{w}|)\mathrm{d}\mathbf{x} \le C\int_{\mathcal{B}(\mathbf{x}_0,R)} g\left(\mathbf{x}, \frac{|\mathbf{w}-\boldsymbol{\xi}|}{R}\right)\mathrm{d}\mathbf{x}$$

for all $\boldsymbol{\xi} \in \mathbb{R}^N$ *and all balls* $\mathcal{B}(\mathbf{x}_0,R) \subset \Omega$.

Proof. First, we show that for every $\mu \in (0,1]$, it holds that

$$g^*(\mathbf{x}, \mu g_t(\mathbf{x},t)) \le q\alpha_+\mu^{\frac{q\alpha_+}{q\alpha_+-1}} g(\mathbf{x},t), \tag{2.15}$$

where $g^*(\mathbf{x},\tau)$ denotes the Young conjugate of $g(\mathbf{x},\cdot)$ evaluated at τ. To demonstrate (2.15), we first note that for every $\tau \geq 0$, we have

$$g^*(\mathbf{x},\tau) = \tau g_t^{-1}(\mathbf{x},\tau) - g(\mathbf{x}, g_t^{-1}(\mathbf{x},\tau)) \leq \tau g_t^{-1}(\mathbf{x},\tau),$$

where $g_t^{-1}(\mathbf{x},\tau)$ denotes the inverse of $g_t(\mathbf{x},\cdot)$ evaluated at τ. Putting $\tau = \mu g_t(\mathbf{x},t)$ in the above inequality, we have that

$$g^*(\mathbf{x}, \mu g_t(\mathbf{x},t)) \leq \mu g_t(\mathbf{x},t) g_t^{-1}(\mathbf{x}, \mu g_t(\mathbf{x},t)). \tag{2.16}$$

To estimate $g_t^{-1}(\mathbf{x}, \mu g_t(\mathbf{x},t))$, we apply part (iv) of Lemma 2.1 with $\varepsilon = \mu^{1/(q\alpha_+-1)}$ and use that $\alpha(\mathbf{x}) \leq \alpha_+$ to obtain $\mu g_t(\mathbf{x},t) \leq g_t(\mathbf{x}, \mu^{1/(q\alpha_+-1)}t)$. Applying $g_t^{-1}(\mathbf{x},\cdot)$ to both sides of this inequality yields $g_t^{-1}(\mathbf{x}, \mu g_t(\mathbf{x},t)) \leq \mu^{\frac{1}{q\alpha_+-1}} t$. Putting this inequality into (2.16) and using part (i) of Lemma 2.1, we obtain (2.15).

With (2.15) available, we can now prove the result. By the Euler-Lagrange equations, we have that

$$\int_\Omega g_t(\mathbf{x}, |\nabla \mathbf{w}\mathbf{G}_0|) \frac{\nabla \mathbf{w}\mathbf{G}_0}{|\nabla \mathbf{w}\mathbf{G}_0|} \cdot \nabla \boldsymbol{\varphi} \mathbf{G}_0 \mathrm{d}\mathbf{x} = 0, \tag{2.17}$$

for any $\boldsymbol{\varphi} \in W_0^1 L_g(\Omega; \mathbb{R}^N)$. For a ball $\mathcal{B}(\mathbf{x}_0, R) \subset \Omega$, let $\eta \in C_c^\infty(\mathcal{B}(\mathbf{x}_0,R))$ be such that $\chi_{\mathcal{B}(\mathbf{x}_0,R/2)} \leq \eta \leq \chi_{\mathcal{B}(\mathbf{x}_0,R)}$ and $|\nabla \eta| \leq \frac{4}{R}$. For a fixed $\boldsymbol{\xi} \in \mathbb{R}^N$, putting $\boldsymbol{\varphi} = (\mathbf{w} - \boldsymbol{\xi})\eta^{q\alpha_+}$ in (2.17), we have

$$\begin{aligned} &\int_{\mathcal{B}(\mathbf{x}_0,R)} \eta^{q\alpha_+} g_t(\mathbf{x}, |\nabla \mathbf{w}\mathbf{G}_0|)\, |\nabla \mathbf{w}\mathbf{G}_0| \,\mathrm{d}\mathbf{x} \\ &\qquad = -q\alpha_+ \int_{\mathcal{B}(\mathbf{x}_0,R)} \eta^{q\alpha_+-1} g_t(\mathbf{x}, |\nabla \mathbf{w}\mathbf{G}_0|) \frac{\nabla \mathbf{w}\mathbf{G}_0}{|\nabla \mathbf{w}\mathbf{G}_0|} \cdot (\mathbf{w} - \boldsymbol{\xi}) \otimes \nabla \eta \mathrm{d}\mathbf{x}. \end{aligned}$$

Using Lemma 2.1 part (i), Young's inequality, and the bound $|\nabla\eta| \le 4/R$, we have

$$p\int_{\mathcal{B}(\mathbf{x}_0,R)} \eta^{q\alpha_+} g(\mathbf{x}, |\nabla \mathbf{w}\mathbf{G}_0|)\mathrm{d}\mathbf{x} \\ \le 4q\alpha_+\varepsilon \int_{\mathcal{B}(\mathbf{x}_0,R)} \left\{ g^*\left(\mathbf{x}, \eta^{q\alpha_+-1} g_t(\mathbf{x}, |\nabla \mathbf{w}\mathbf{G}_0|)\right) + g\left(\mathbf{x}, \frac{|\mathbf{w}-\boldsymbol{\xi}|}{R\varepsilon}\right)\right\} \mathrm{d}\mathbf{x},$$

for any $\varepsilon > 0$. Now employing (2.15) with $\mu = \eta^{q\alpha+-1}$ yields

$$p\int_{\mathcal{B}(\mathbf{x}_0,R)} \eta^{q\alpha_+} g(\mathbf{x}, |\nabla \mathbf{w}\mathbf{G}_0|)\mathrm{d}\mathbf{x} \\ \le 4(q\alpha_+)^2\varepsilon \int_{\mathcal{B}(\mathbf{x}_0,R)} \eta^{q\alpha_+} g(\mathbf{x}, |\nabla \mathbf{w}\mathbf{G}_0|)\mathrm{d}\mathbf{x} + 4q\alpha_+\varepsilon \int_{\mathcal{B}(\mathbf{x}_0,R)} g\left(\mathbf{x}, \frac{|\mathbf{w}-\boldsymbol{\xi}|}{R\varepsilon}\right) \mathrm{d}\mathbf{x}.$$

Choosing $\varepsilon = (p-1)/(4q^2\alpha_+^2)$, subtracting the first integral on the right from both sides of the inequality, and utilizing parts (ii) and (iii) of Lemma 2.1, we obtain the desired result. □

Combining the results of Theorems 2.2 and 2.3, we use Proposition 5.1 in [43] to obtain the following.

Theorem 2.4. *Suppose that* $\alpha : \Omega \to [1,\infty)$ *satisfies* (1.12)-(1.14) *and that* $g : \Omega \times \mathbb{R}_+ \to \mathbb{R}_+$ *satisfies* (1.15) *and* (1.16). *Let* $\mathbf{G}_0 \in \mathbb{R}^{n\times n}$ *be an invertible matrix, and suppose that* $\mathbf{w} \in W^1L_g(\Omega;\mathbb{R}^N)$ *is a minimizer for*

$$\mathbf{v} \mapsto \int_\Omega g(\mathbf{x}, |\nabla \mathbf{v}(\mathbf{x})\mathbf{G}_0|)\mathrm{d}\mathbf{x}$$

among all mappings $\mathbf{v} \in \mathbf{w} + W_0^1L_g(\Omega;\mathbb{R}^N)$. *Then there is a constant* $C < \infty$ *and* $r_3 > 1$ *such that*

$$\left(\fint_{\mathcal{B}(\mathbf{x}_0,\frac{R}{2})} g(\mathbf{x}, |\nabla \mathbf{w}|)^{r_3}\mathrm{d}\mathbf{x}\right)^{\frac{1}{r_3}} \le C\left(\fint_{\mathcal{B}(\mathbf{x}_0,R)} \{1 + g(\mathbf{x}, |\nabla \mathbf{w}|)\}\,\mathrm{d}\mathbf{x}\right)$$

for all balls $\mathcal{B}(\mathbf{x}_0, R) \subset \Omega$. *The constants* C *and* r_3 *depend only on* n, N, p, q, M, α_+, ω, $\left|\mathbf{G}_0^{-1}\right|$, *and* $\int_\Omega (1 + g(\mathbf{x}, |\nabla \mathbf{w}|))\mathrm{d}\mathbf{x}$.

Using the argument provided in [20], along with Theorem 2.3 and Lemma 2.1, we have the following boundary version of the above result.

Theorem 2.5. *Suppose that* $\alpha : \Omega \to [1, \infty)$ *satisfies* (1.12)-(1.14) *and that* $g : \Omega \times \mathbb{R}_+ \to \mathbb{R}_+$ *satisfies* (1.15)-(1.17). *Let* $\mathbf{G}_0 \in \mathbb{R}^{n\times n}$ *be an invertible matrix, and assume* $\mathcal{B}(\mathbf{x}_0, R) \subset \Omega$. *Suppose that* $\mathbf{w}$ *is a minimizer for*

$$\mathbf{u} \mapsto \int_{\mathcal{B}(\mathbf{x}_0, R/4)} g(\mathbf{x}, |\nabla \mathbf{u}(\mathbf{x})\mathbf{G}_0|)\mathrm{d}\mathbf{x}$$

satisfying $\mathbf{w} - \mathbf{v} \in W_0^1 L_g(\mathcal{B}(\mathbf{x}_0, R/4); \mathbb{R}^N)$ *for some function* $\mathbf{v} \in W^1 L_g(\Omega; \mathbb{R}^N)$ *with* $g(\cdot, |\nabla \mathbf{v}|) \in L^r(\mathcal{B}(\mathbf{x}_0, R/2)$, *where* $r > 1$. *Then there is a constant* $C < \infty$ *and* $r_4 > 1$ *such that*

$$\left(\fint_{\mathcal{B}(\mathbf{x}_0, R/4)} g(\mathbf{x}, |\nabla \mathbf{w}|)^{r_4}\mathrm{d}\mathbf{x}\right)^{\frac{1}{r_4}} \leq C \left(\fint_{\mathcal{B}(\mathbf{x}_0, R/2)} \{1 + g(\mathbf{x}, |\nabla \mathbf{v}|)^r\}\, \mathrm{d}\mathbf{x}\right)^{\frac{1}{r}}.$$

Here, the constants C *and* r_4 *only depend on* n, N, p, q, M, α_+, ω, $\left|\mathbf{G}_0^{-1}\right|$, r, *and* $\int_\Omega (1 + g(\mathbf{x}, |\nabla \mathbf{u}|))\mathrm{d}\mathbf{x}$.

We also have the following theorem that gives Morrey regularity for the function itself if the gradient possesses Morrey regularity. The proof of this theorem uses some ideas from Lemma 2.2 and Proposition 2.3 in [47].

Theorem 2.6. *Suppose that* $\Omega \subset \mathbb{R}^n$ *is open and bounded, and has no external cusps. Suppose that* $\alpha : \Omega \to [1, \infty)$ *satisfies* (1.12) *and* (1.13), *and that* $g : \Omega \times \mathbb{R}_+ \to \mathbb{R}_+$ *satisfies* (1.15) *and* (1.16). *Suppose also that* $\mathbf{u} \in W^1 L_g(\Omega; \mathbb{R}^N)$ *and* $g(\cdot, |\nabla \mathbf{u}|) \in$

$L^{1,\lambda}(\Omega)$ for some $0 \leq \lambda < n$. Then, if r_2 is as in Remark 2.2 and $1 \leq s < r_2$, we have that $g(\cdot, |\mathbf{u}|)^s \in L^{1,\kappa}(\Omega)$ for every $0 \leq \kappa < \min\{n + s(p+\lambda-n), n\}$.

Proof. Put $\mu := \min\{n + s(p+\lambda-n), n\}$, fix $\kappa < \mu$, and let $\gamma \in (\kappa, \mu)$. Since $\gamma < \mu \leq n + s(p+\lambda-n)$, there is some $s' \in (s, r_2)$ such that $n + s'(p+\lambda-n) \geq \gamma$. With R_0 as in Theorem 2.2, find $R_1 \in (0, R_0]$ so that $\omega(2R_1) \leq \frac{s'}{s} - 1$. Since Ω has no external cusps, there is some constant $A < \infty$ such that

$$\frac{1}{A}\left|\Omega(\mathbf{x}_0, a)\right|\left(\frac{b}{a}\right)^n \leq \left|\Omega(\mathbf{x}_0, b)\right| \leq A\left|\Omega(\mathbf{x}_0, a)\right|\left(\frac{b}{a}\right)^n \tag{2.18}$$

for any $a, b \in (0, \operatorname{diam}(\Omega))$ and $\mathbf{x}_0 \in \Omega$, and hence to show that $g(\cdot, |\mathbf{u}|) \in L^{1,\kappa}(\Omega)$, it suffices to show that

$$\sup_{\substack{\mathbf{x}_0 \in \Omega \\ 0<\rho<R_1}} \left|\Omega(\mathbf{x}_0, \rho)\right|^{-\frac{\kappa}{n}} \int_{\Omega(\mathbf{x}_0,\rho)} g(\mathbf{x}, |\mathbf{u}|)\mathrm{d}\mathbf{x} < \infty.$$

With $\mathbf{x}_0 \in \Omega$ and $\rho \in (0, R_1)$ fixed for the remainder of the proof, we denote by Ω_r the set $\Omega(\mathbf{x}_0, r)$, by α_1 and α_2 the quantities $\inf_{\mathbf{x}\in\Omega_{R_1}} \alpha(\mathbf{x})$ and $\sup_{\mathbf{x}\in\Omega_{R_1}} \alpha(\mathbf{x})$, respectively, and by $\mathbf{u}_t$ the average $(\mathbf{u})_{\Omega_t}$. We observe that our choice of R_1 ensures that $\alpha_2 s \leq \alpha_1 s'$, since $\alpha_2/\alpha_1 \leq \omega(2R_1)/\alpha_1 + 1 \leq s'/s$. In what follows, we write C for any constant that does not depend on either $\mathbf{x}_0$ or ρ; in particular, we allow C to depend on $\mathbf{u}$, κ, and A. We have

$$\begin{aligned} |\Omega_\rho|^{-\frac{\kappa}{n}} \int_{\Omega_\rho} g(\mathbf{x}, |\mathbf{u}|)^s \mathrm{d}\mathbf{x} \leq C\, |\Omega_\rho|^{-\frac{\kappa}{n}} \Bigg\{ &\int_{\Omega_\rho} g(\mathbf{x}, |\mathbf{u} - \mathbf{u}_\rho|)^s \mathrm{d}\mathbf{x} \\ &+ \int_{\Omega_\rho} g(\mathbf{x}, |\mathbf{u}_{R_0} - \mathbf{u}_\rho|)^s \mathrm{d}\mathbf{x} + \int_{\Omega_\rho} g(\mathbf{x}, |\mathbf{u}_{R_0}|)^s \mathrm{d}\mathbf{x} \Bigg\} \\ = C\, |\Omega_\rho|^{-\frac{\kappa}{n}} \{I_1 + I_2 + I_3\}, \end{aligned} \tag{2.19}$$

where I_1, I_2, and I_3 are defined naturally. We first note that for any $R < R_0$, by

Theorem 2.2 and the previously noted inequality $\alpha_2 s \leq \alpha_1 s'$, it holds that

$$\begin{aligned}\int_{\Omega_R} h(|\mathbf{u}-\mathbf{u}_R|)^{\alpha_2 s}\mathrm{d}\mathbf{x} &\leq C\int_{\Omega_R}\left\{1+h\left(|\mathbf{u}-\mathbf{u}_R|\right)^{\alpha(\mathbf{x})s'}\right\}\mathrm{d}\mathbf{x}\\ &\leq C\left(|\Omega_R|+|\Omega_R|^{\frac{n+ps'}{n}}\fint_{\Omega_R}\left\{h\left(\frac{|\mathbf{u}-\mathbf{u}_R|}{R}\right)^{\alpha(\mathbf{x})s'}\right\}\mathrm{d}\mathbf{x}\right)\\ &\leq C\left(|\Omega_R|+|\Omega_R|^{\frac{n+ps'}{n}}\left(\fint_{\Omega_R}\left\{1+h(|\nabla\mathbf{u}|)^{\alpha(\mathbf{x})}\right\}\mathrm{d}\mathbf{x}\right)^{s'}\right)\\ &\leq C\left(|\Omega_R|+|\Omega_R|^{\frac{n+ps'}{n}}+|\Omega_R|^{\frac{n+s'(p+\lambda-n)}{n}}\|g(\cdot,|\nabla\mathbf{u}|)\|_{L^{1,\lambda}}^{s'}\right).\end{aligned}$$

But $\gamma < n$, and clearly $n+ps' \geq n$; also, by the selection of s', we have that $n+s'(p+\lambda-n) \geq \gamma$, so defining the finite constant G by $G := 1+\|g(\cdot,|\nabla\mathbf{u}|)\|_{L^{1,\lambda}}^{s'}$, the above string of inequalities yields

$$|\Omega_R|^{-\frac{\gamma}{n}}\int_{\Omega_R} h(|\mathbf{u}-\mathbf{u}_R|)^{\alpha_2 s}\mathrm{d}\mathbf{x} \leq CG, \tag{2.20}$$

for $0 < R < R_1$. With this work, we quickly observe

$$I_1 \leq C\int_{\Omega_\rho}\left\{1+h(|\mathbf{u}-\mathbf{u}_\rho|)^{\alpha_2 s}\right\}\mathrm{d}\mathbf{x} \leq CG\,|\Omega_\rho|^{\frac{\gamma}{n}}.$$

To estimate I_2, we first let $0 < a < b \leq R_1$ be given, and define $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ to be the inverse of $h^{\alpha_2 s}$. Then by Jensen's inequality and (2.20), we find

$$\begin{aligned}h(|\mathbf{u}_b-\mathbf{u}_a|)^{\alpha_2 s} &\leq h\left(|\Omega_a|^{-1}\int_{\Omega_a}|\mathbf{u}-\mathbf{u}_b|\,\mathrm{d}\mathbf{x}\right)^{\alpha_2 s} \leq |\Omega_a|^{-1}\int_{\Omega_a} h(|\mathbf{u}-\mathbf{u}_b|)^{\alpha_2 s}\mathrm{d}\mathbf{x}\\ &\leq |\Omega_a|^{-1}\,|\Omega_b|^{\frac{\gamma}{n}}\,G.\end{aligned}$$

For each $i = 0, 1, \ldots,$, put $r_i = 2^{-i}R_1$. Putting $a = r_i$ and $b = r_{i-1}$ in the above inequality and using that φ is the inverse of $h^{\alpha_2 s}$, we have

$$\left|\mathbf{u}_{r_{i-1}} - \mathbf{u}_{r_i}\right| \leq \varphi\left(\left|\Omega_{r_i}\right| \left|\Omega_{r_{i-1}}\right|^{\frac{\gamma}{n}} G\right). \tag{2.21}$$

Now, if a nonnegative integer k is selected so that $2^{-k-1}R_1 \leq \rho \leq 2^{-k}R_1$, then it follows that $k \leq \frac{1}{\log(2)} \log\left(\frac{R_1}{\rho}\right)$, and so, using the triangle inequality and (2.21), we obtain

$$|\mathbf{u}_{R_0} - \mathbf{u}_\rho| \leq \sum_{i=1}^{k} \varphi\left(\left|\Omega_{r_i}\right| \left|\Omega_{r_{i-1}}\right|^{\frac{\gamma}{n}} G\right) + \varphi\left(\left|\Omega_\rho\right|^{-1} \left|\Omega_{r_k}\right|^{\frac{\gamma}{n}} G\right) \sum_{i=1}^{k+1} \varphi\left(\left|\Omega_{r_i}\right| \left|\Omega_{r_{i-1}}\right|^{\frac{\gamma}{n}} G\right).$$

Hence, employing Lemma 2.1 and Jensen's inequality, along with the two preceding inequalities, we have

$$\begin{aligned} h(|\mathbf{u}_{R_0} - \mathbf{u}_\rho|)^{\alpha_2 s} &\leq (k+1)^{qs\alpha_+} h\left(\frac{1}{k+1} \sum_{i=1}^{k+1} \varphi\left(\left|\Omega_{r_i}\right| \left|\Omega_{r_{i-1}}\right|^{\frac{\gamma}{n}} G\right)\right)^{\alpha_2 s} \\ &\leq (k+1)^{qs\alpha_+ - 1} \sum_{i=1}^{k+1} \left|\Omega_{r_i}\right| \left|\Omega_{r_{i-1}}\right|^{\frac{\gamma}{n}} G \\ &\leq \left(\frac{1}{\log(2)} \log\left(\frac{R_1}{\rho}\right) + 1\right)^{qs\alpha_+ - 1} G \sum_{i=1}^{k+1} \left|\Omega_{r_i}\right|^{-1} \left|\Omega_{r_{i-1}}\right|^{\frac{\gamma}{n}}. \end{aligned}$$

But (2.18) gives that

$$\left|\Omega_{r_i}\right|^{-1} \left|\Omega_{r_{i-1}}\right|^{\frac{\gamma}{n}} \leq C\left(2^{(n-\gamma)(i-k)}\right) \left|\Omega_{r_{k+1}}\right|^{\frac{\gamma-n}{n}} \leq C\left(2^{(n-\gamma)(i-k)}\right) \left|\Omega_\rho\right|^{\frac{\gamma-n}{n}},$$

and hence

$$
\begin{aligned}
h(|\mathbf{u}_{R_1} - \mathbf{u}_\rho|)^{\alpha_2 s} &\leq \left(\frac{1}{\log(2)} \log\left(\frac{R_1}{\rho}\right) + 1\right)^{qs\alpha_+ - 1} G\,|\Omega_\rho|^{\frac{\gamma-n}{n}} \sum_{i=1}^{k+1} \left(2^{n-\gamma}\right)^{i-k} \\
&\leq \left(\frac{1}{\log(2)} \log\left(\frac{R_1}{\rho}\right) + 1\right)^{qs\alpha_+ - 1} G\,|\Omega_\rho|^{\frac{\gamma-n}{n}} \sum_{i=-1}^{k-1} \left(2^{n-\gamma}\right)^{-i} \\
&\leq \left(\frac{1}{\log(2)} \log\left(\frac{R_1}{\rho}\right) + 1\right)^{qs\alpha_+ - 1} G\,|\Omega_\rho|^{\frac{\gamma-n}{n}} .
\end{aligned}
$$

Using this estimate and that $\gamma > \kappa$, we have that

$$
\begin{aligned}
I_2 &\leq C \int_{\Omega_\rho} \{1 + h(|\mathbf{u}_{R_1} - \mathbf{u}_\rho|)^{\alpha_2}\}\, \mathrm{d}\mathbf{x} \leq C \left(\frac{1}{\log(2)} \log\left(\frac{R_1}{\rho}\right) + 1\right)^{qs\alpha_+ - 1} G\,|\Omega_\rho|^{\frac{\gamma}{n}} \\
&\leq CG\,|\Omega_\rho|^{\frac{\kappa}{n}} .
\end{aligned}
$$

Turning now to I_3, we have by Jensen's inequality and the inequality $\alpha_2 s \leq \alpha_1 s' \leq \alpha_1 r_2$ that

$$
\begin{aligned}
I_3 &\leq |\Omega_\rho| \left(1 + h(|\mathbf{u}_{R_1}|)^{\alpha_2 s}\right) \leq |\Omega_\rho| \left(1 + |\Omega_{R_1}|^{-1} \int_{\Omega_{R_1}} h(|\mathbf{u}|)^{\alpha_2 s} \mathrm{d}\mathbf{x}\right) \\
&\leq |\Omega_\rho| \left(1 + |\Omega_{R_1}|^{-1} \int_{\Omega} \{g(\mathbf{x}, |\mathbf{u}|)^{r_2} \mathrm{d}\mathbf{x}\right) .
\end{aligned}
$$

By Remark 2.2, we have that $g(\cdot, |\mathbf{u}|)^{r_2} \in L^1(\Omega)$, and so we have

$$
I_3 \leq C\left(1 + \|g(\cdot, |\mathbf{u}|)^{r_2}\|_{L^1}\right) |\Omega_\rho| .
$$

Putting our estimates for I_1, I_2, and I_3 into (2.19) and using that $\gamma > \kappa$, we have

$$
|\Omega_\rho|^{-\frac{\kappa}{n}} \int_{\Omega_\rho} g(\mathbf{x}, |\mathbf{u}|) \mathrm{d}\mathbf{x} \leq C\left(1 + G + |\Omega_{R_1}|^{-1} \|g(\cdot, |\mathbf{u}|)^{r_2}\|_{L^1(\Omega)}\right)
$$

for all $0 < \rho \leq R_1$, which gives that $g(\cdot, |\mathbf{u}|) \in L^{1,\kappa}(\Omega)$, as desired. $\square$

The next lemma and its proof are essentially taken from Lemma 1 in [39].

Lemma 2.4. *Let* $\varphi : (0, \infty) \to \mathbb{R}$ *be nondecreasing, and suppose that there exist* $A \geq 1$*,* $B \geq 0$*,* $R_0 > 0$*, and* $\alpha > \beta \geq 0$ *such that for some* $0 \leq \varepsilon \leq \left(\frac{1}{2A}\right)^{\frac{2\alpha}{\alpha-\beta}}$*, the inequality*

$$\varphi(\rho) \leq A\left[\left(\frac{\rho}{R}\right)^{\alpha} + \varepsilon\right]\varphi(R) + BR^{\beta}$$

holds for each $0 < \rho \leq R \leq R_0$*. Then there is some finite constant* $C = C(A, \alpha, \beta)$ *such that*

$$\varphi(\rho) \leq C\left(\frac{\rho}{R}\right)^{\beta}\varphi(R) + CB\rho^{\beta}$$

for all $0 < \rho \leq R \leq R_0$*.*

Proof. First, we define $\gamma = (\alpha + \beta)/2$ and $\tau = (2A)^{-2/(\alpha-\beta)}$. Note that $\gamma \in (\beta, \alpha)$ and $2A\tau^{\alpha} = \tau^{\gamma}$, so that the assumption on ε gives

$$A(\tau^{\alpha} + \varepsilon) \leq 2A\tau^{\alpha} = \tau^{\gamma}. \tag{2.22}$$

Using (2.22) and the assumption on φ, we can employ a straightforward induction argument to show that

$$\varphi(\tau^j R) \leq \tau^{j\gamma}\varphi(R) + B(\tau^{j-1}R)^{\beta}\sum_{i=0}^{j-1}(\tau^{\gamma-\beta})^i \tag{2.23}$$

for every nonnegative integer j. For fixed $0 < \rho \leq R$, we can find j such that $\tau^{j+1}R \leq \rho \leq \tau^j R$. Using that φ is increasing and (2.23), we find

$$\begin{aligned}\varphi(\rho) \leq \varphi(\tau^j R) &\leq \tau^{j\gamma}\varphi(R) + B\tau^{-2\beta}(\tau^{j+1}R)^{\beta}\sum_{i=0}^{j-1}(\tau^{\gamma-\beta})^i \\ &\leq \tau^{-\gamma}\left(\frac{\rho}{R}\right)^{\gamma}\varphi(R) + \frac{B\tau^{-2\beta}}{1-\tau^{\gamma-\beta}}\rho^{\beta},\end{aligned}$$

which establishes the lemma. □

2.2 Lipschitz Regularity Results

We now prove a refinement of the local Lipschitz regularity result established in [60]; our strategy is similar to the one used there. We consider the functional

$$J(\mathbf{v}) := \int_\Omega h(|\nabla \mathbf{v}|)\mathrm{d}\mathbf{x}, \tag{2.24}$$

where h has (p,q)-structure. We temporarily make the assumption that there are positive constants μ and M such that for all $\mathbf{F}_1, \mathbf{F}_2 \in \mathbb{R}^{N\times n}$, the following holds:

$$\mu \left|\mathbf{F}_1\right|^2 \leq \frac{\partial^2}{\partial \mathbf{F}^2} h(|\mathbf{F}_2|) \cdot (\mathbf{F}_1 \otimes \mathbf{F}_1) \leq M \left|\mathbf{F}_1\right|^2. \tag{2.25}$$

Here, by $\frac{\partial^2}{\partial \mathbf{F}^2} h(|\mathbf{F}_2|)$, we mean the $(N\times n)\times(N\times n)$ matrix (i.e. an $N\times n$ matrix with $N\times n$ matrices as the entries) for which the (i,j,k,l) entry is given by $\frac{\partial^2}{\partial F^{i,j}\partial F^{k,l}} h(|\mathbf{F}_2|)$, and by $\mathbf{F}_1 \otimes \mathbf{F}_1$, we mean the $(N\times n)\times(N\times n)$ matrix for which the (i,j,k,l) entry is $F_1^{(i,j)} F_1^{(k,l)}$. Thus, (2.25) is just shorthand for the inequalities

$$\mu \left|\mathbf{F}_1\right|^2 \leq \sum_{i,j,k,l} \frac{\partial^2}{\partial F^{i,j}\partial F^{k,l}} h(|\mathbf{F}_2|) F_1^{i,j} F_1^{k,l} \leq M \left|\mathbf{F}_1\right|^2,$$

and is asserting the uniform positive-definiteness (left inequality) and boundedness (right inequality) of the Hessian $\frac{\partial^2}{\partial \mathbf{F}^2} h(|\cdot|)$. A routine computation shows that

$$\frac{\partial^2}{\partial F^{i,j}\partial F^{k,l}} h(|\mathbf{F}_2|) = \left(\frac{h''(|\mathbf{F}_2|)}{\left|\mathbf{F}_2\right|^2} - \frac{h'(|\mathbf{F}_2|)}{\left|\mathbf{F}_2\right|^3} \right) F_2^{i,j} F_2^{k,l} + \frac{h'(|\mathbf{F}_2|)}{|\mathbf{F}_2|} \delta_{(i,j)(k,l)},$$

where we have defined $\delta_{(i,j)(k,l)}$ to be 1 if $(i,j) = (k,l)$ and 0 otherwise. Using this, we see that

$$\begin{aligned} \frac{\partial^2}{\partial \mathbf{F}^2} h(|\mathbf{F}_2|) \cdot (\mathbf{F}_1 \otimes \mathbf{F}_1) &= \left(\frac{h''(|\mathbf{F}_2|)}{|\mathbf{F}_2|^2} - \frac{h'(|\mathbf{F}_2|)}{|\mathbf{F}_2|^3} \right) \sum_{i,j,k,l} F_2^{i,j} F_2^{k,l} F_1^{i,j} F_1^{k,l} \\ &\quad + \frac{h'(|\mathbf{F}_2|)}{|\mathbf{F}_2|} \sum_{i,j} (F_1^{i,j})^2 \\ &= \left(\frac{h''(|\mathbf{F}_2|)}{|\mathbf{F}_2|^2} - \frac{h'(|\mathbf{F}_2|)}{|\mathbf{F}_2|^3} \right) \left(\sum_{i,j} F_2^{i,j} F_1^{i,j} \right)^2 + \frac{h'(|\mathbf{F}_2|)}{|\mathbf{F}_2|} |\mathbf{F}_1|^2 . \end{aligned} \tag{2.26}$$

By the Cauchy-Schwartz inequality, we have that

$$\left(\sum_{i,j} F_2^{i,j} F_1^{i,j} \right)^2 \leq |\mathbf{F}_2|^2 |\mathbf{F}_1|^2 ,$$

so that if $h''(|\mathbf{F}_2|) \leq h'(|\mathbf{F}_2|)/|\mathbf{F}_2|$, then from (2.26), we see that

$$h''(|\mathbf{F}_2|) |\mathbf{F}_1|^2 \leq \frac{\partial^2}{\partial \mathbf{F}^2} h(|\mathbf{F}_2|) \cdot (\mathbf{F}_1 \otimes \mathbf{F}_1) \leq \frac{h'(|\mathbf{F}_2|)}{|\mathbf{F}_2|} |\mathbf{F}_1|^2 .$$

On the other hand, if $h''(|\mathbf{F}_2|) \geq h'(|\mathbf{F}_2|)/|\mathbf{F}_2|$, then again using Cauchy-Schwartz and (2.26), we have

$$\frac{h'(|\mathbf{F}_2|)}{|\mathbf{F}_2|} |\mathbf{F}_1|^2 \leq \frac{\partial^2}{\partial \mathbf{F}^2} h(|\mathbf{F}_2|) \cdot (\mathbf{F}_1 \otimes \mathbf{F}_1) \leq h''(|\mathbf{F}_2|) |\mathbf{F}_1|^2 .$$

In either case, we have shown that

$$|\mathbf{F}_1|^2 \min\left\{ h''(|\mathbf{F}_2|), \frac{h'(|\mathbf{F}_2|)}{|\mathbf{F}_2|} \right\} \leq \frac{\partial^2}{\partial \mathbf{F}^2} h(|\mathbf{F}_2|) \cdot (\mathbf{F}_1 \otimes \mathbf{F}_1) \leq |\mathbf{F}_1|^2 \max\left\{ h''(|\mathbf{F}_2|), \frac{h'(|\mathbf{F}_2|)}{|\mathbf{F}_2|} \right\}$$

for all $\mathbf{F}_1, \mathbf{F}_2 \in \mathbb{R}^{N\times n}$. Therefore (2.25) is satisfied if both $h''(t)$ and $h'(t)/t$ are bounded below by μ and above by M for $t \in (0,\infty)$. Assumption (2.25) gives quadratic growth of h, which in turn forces any minimizer to be of class $W^{2,2}_{\mathrm{loc}} \cap W^{1,\infty}_{\mathrm{loc}}(\Omega;\mathbb{R}^N)$. This is a well-known result, and is found in, for example, Theorems 8.1 and 8.2 of [47]. Theorem 8.1 from [47] gives the local square-integrability of the second-order derivatives; the overall strategy of the proof of this part of the result is to consider difference quotients of the gradient and show that these difference quotients are uniformly locally bounded in L^2 using the Euler-Lagrange equations with a particular test function involving difference quotients of $\mathbf{u}$. The uniform local boundedness in L^2 of the difference quotients implies that the weak second-order derivatives exist and belong to L^2_{loc}. The local essential boundedness of $\nabla\mathbf{u}$ is the content of Theorem 8.2 in [47]; for the proof, one essentially iterates an estimate that comes from an inequality contained in the proof of the existence of square-integrable second derivatives and the Sobolev embedding theorem.

The following lemma provides an estimate of the form (1.20) for minimizers of (2.24) under the additional assumptions that (2.25) holds and that h'' is continuous. The estimate we obtain is independent of the constants μ and M, which allows us to eventually remove both the assumption in (2.25) and the continuity assumption on h'' using an approximation argument.

Lemma 2.5. *Let $\Omega \subset \mathbb{R}^n$ be open, and let $h \in \mathcal{C}^2([0,\infty))$ be a function with (p,q)-structure that satisfies (2.25). Suppose that $\mathbf{v} \in W^{1,1}(\Omega;\mathbb{R}^N)$ is a minimizer of (2.24). Then there is a constant $C = C(n,p,q)$ such that*

$$\|h(|\nabla\mathbf{v}|)\|_{L^\infty(\mathcal{B}_{\mathbf{x}_0,\rho})} \le \frac{C}{(R-\rho)^n}\int_{\mathcal{B}_{\mathbf{x}_0,R}} h(|\nabla\mathbf{v}|)\mathrm{d}\mathbf{x}$$

whenever $\mathcal{B}_{\mathbf{x}_0,R} \subset \Omega$ and $0 < \rho < R$.

Proof. First, we establish that

$$\|h(|\nabla \mathbf{v}|)\|_{L^\infty(\mathcal{B}_{\mathbf{x}_0,\frac{r}{2}})} \leq \frac{C}{r^n}\int_{\mathcal{B}_{\mathbf{x}_0,r}} h(|\nabla \mathbf{v}|)\mathrm{d}\mathbf{x} \tag{2.27}$$

for any $\mathbf{x}_0 \in \Omega$ and $r > 0$ such that $B_{\mathbf{x}_0,r} \subset \Omega$. Using a rescaling argument, we see that it suffices to show (2.27) when $\mathbf{x}_0 = 0$ and $r = 1$. As was just discussed above, it can be shown that $\mathbf{v} \in W^{2,2}_{\mathrm{loc}}(\mathcal{B};\mathbb{R}^N) \cap W^{1,\infty}_{\mathrm{loc}}(\mathcal{B}\mathbb{R}^N)$. We now follow the first part of the proof for Lemma 4.1 in [60] to show that

$$\int_{\mathcal{B}} \eta^2 \Phi(|\nabla \mathbf{v}|)h''(|\nabla \mathbf{v}|)\,|\nabla(|\nabla \mathbf{v}|)|^2\,\mathrm{d}\mathbf{x} \leq C\int_{\mathcal{B}} \Phi(|\nabla \mathbf{v}|)\max\left\{h''(|\nabla \mathbf{v}|), \frac{h'(|\nabla \mathbf{v}|)}{|\nabla \mathbf{v}|}\right\}|\nabla \eta|^2\,|\nabla \mathbf{v}|^2\,\mathrm{d}\mathbf{x}$$

for every $\eta \in \mathcal{C}^1_c(\mathcal{B})$ and Φ that is nondecreasing, continuous on $[0,\infty)$, and Lipschitz continuous on $[\varepsilon, T]$ for all $T > \varepsilon > 0$. In fact, this is exactly (4.19) in [60]. Since h has (p,q)-structure, we obtain from the above inequality that

$$\int_{\mathcal{B}} \eta^2 \Phi(|\nabla \mathbf{v}|)h''(|\nabla \mathbf{v}|)\,|\nabla(|\nabla \mathbf{v}|)|^2\,\mathrm{d}\mathbf{x} \leq C\int_{\mathcal{B}} \Phi(|\nabla \mathbf{v}|)h''(|\nabla \mathbf{v}|)\,|\nabla \eta|^2\,|\nabla \mathbf{v}|^2\,\mathrm{d}\mathbf{x} \tag{2.28}$$

Thus, for fixed $\beta \geq 0$, we can define $\Phi(t) = t^{2\beta}h'(t)^{2\beta}$; with this definition of Φ, the above inequality becomes

$$\int_{\mathcal{B}} \eta^2\,|\nabla \mathbf{v}|^{2\beta}\,h'(|\nabla \mathbf{v}|)^{2\beta}h''(|\nabla \mathbf{v}|)\,|\nabla(|\nabla \mathbf{v}|)|^2\,\mathrm{d}\mathbf{x} \leq C\int_{\mathcal{B}} |\nabla \mathbf{v}|^{2\beta+2}\,h'(|\nabla \mathbf{v}|)^{2\beta}h''(|\nabla \mathbf{v}|)\,|\nabla \eta|^2\,\mathrm{d}\mathbf{x}.$$

By the definition of h having (p,q)-structure, we get

$$\int_{\mathcal{B}} \eta^2 \left|\nabla \mathbf{v}\right|^{2\beta} h'(|\nabla \mathbf{v}|)^{2\beta} h''(|\nabla \mathbf{v}|) \left|\nabla(|\nabla \mathbf{v}|)\right|^2 \mathrm{d}\mathbf{x} \leq C \int_{\mathcal{B}} \left|\nabla \mathbf{v}\right|^{2\beta+1} h'(|\nabla \mathbf{v}|)^{2\beta+1} \left|\nabla \eta\right|^2 \mathrm{d}\mathbf{x}. \quad (2.29)$$

Now define $G : [0,\infty) \to [0,\infty)$ by $G(t) := \int_0^t s^\beta h'(s)^\beta \sqrt{h''(s)} \mathrm{d}s$. Since h' is increasing, by Hölder's inequality we obtain

$$[G(t)]^2 \leq t^{2\beta+1} h'(t)^{2\beta} \int_0^t h''(s) \mathrm{d}s = t^{2\beta+1} h'(t)^{2\beta+1}.$$

Hence we see that

$$\begin{aligned} \left|\nabla(\eta G(|\nabla \mathbf{v}|))\right|^2 &= \left|(\nabla \eta) G(|\nabla \mathbf{v}|) + \eta G'(|\nabla \mathbf{v}|) \nabla(|\nabla \mathbf{v}|)\right|^2 \\ &\leq 2 \left|\nabla \eta\right|^2 \left|\nabla \mathbf{v}\right|^{2\beta+1} h'(|\nabla \mathbf{v}|)^{2\beta+1} \\ &\quad + 2\eta^2 \left|\nabla \mathbf{v}\right|^{2\beta} h'(|\nabla \mathbf{v}|)^{2\beta} h''(|\nabla \mathbf{v}|) \left|\nabla(|\nabla \mathbf{v}|)\right|^2 . \end{aligned}$$

Note that the assumption in (2.25) implies that $\nabla \mathbf{v}$ is locally bounded. Integrating the above inequality over $\mathcal{B}$, using (2.29) and Sobolev's inequality, we deduce that there is a constant C depending only upon n such that

$$\left\{ \int_{\mathcal{B}} \eta^{2^*} \left[G(|\nabla \mathbf{v}|)^2 \right]^{\frac{2^*}{2}} \mathrm{d}\mathbf{x} \right\}^{\frac{2}{2^*}} \leq C \int_{\mathcal{B}} \left|\nabla \eta\right|^2 \left[\left|\nabla \mathbf{v}\right| h'(|\nabla \mathbf{v}|) \right]^{2\beta+1} \mathrm{d}\mathbf{x}. \quad (2.30)$$

If $n = 2$, we select 2^* to be any finite number strictly larger than 2. Recalling the definition of G and using Lemma 2.3, from (2.30) we obtain

$$\left\{ \int_{\mathcal{B}} \eta^{2^*} \left[\left|\nabla \mathbf{v}\right| h'(|\nabla \mathbf{v}|) \right]^{\frac{2^*}{2}(2\beta+1)} \mathrm{d}\mathbf{x} \right\}^{\frac{2}{2^*}} \leq C(2\beta+1)^2 \int_{\mathcal{B}} \left|\nabla \eta\right|^2 \left[\left|\nabla \mathbf{v}\right| h'(|\nabla \mathbf{v}|) \right]^{2\beta+1} \mathrm{d}\mathbf{x}. \quad (2.31)$$

Now let $0 < \rho < R \leq 1$ be given, and let η be a non-negative test function that is equal to 1 in $\mathcal{B}_\rho$, has support contained in $\mathcal{B}_R$, and is such that $|\nabla\eta| \leq \frac{C}{R-\rho}$; then from (2.31), we see that

$$\left\{\int_{\mathcal{B}_\rho}\left[\,|\nabla\mathbf{v}|\,h'(|\nabla\mathbf{v}|)\right]^{\frac{2^*}{2}(2\beta+1)}\mathrm{d}\mathbf{x}\right\}^{\frac{2}{2^*}} \leq \frac{C(2\beta+1)^2}{(R-\rho)^2}\int_{\mathcal{B}_R}\left[\,|\nabla\mathbf{v}|\,h'(|\nabla\mathbf{v}|)\right]^{2\beta+1}\mathrm{d}\mathbf{x}.$$

Now putting $\gamma = 2\beta + 1$ (note that $\gamma \geq 1$, since $\beta \geq 0$), we can rewrite the above inequality as

$$\left\{\int_{\mathcal{B}_\rho}\left[\,|\nabla\mathbf{v}|\,h'(|\nabla\mathbf{v}|)\right]^{\frac{2^*}{2}\beta}\mathrm{d}\mathbf{x}\right\}^{\frac{2}{2^*}} \leq \frac{C\gamma^2}{(R-\rho)^2}\int_{\mathcal{B}_R}\left[\,|\nabla\mathbf{v}|\,h'(|\nabla\mathbf{v}|)\right]^{\gamma}\mathrm{d}\mathbf{x}. \qquad (2.32)$$

Define the decreasing sequence $\{\rho_i\}_{i=0}^\infty$ by $\rho_i = \frac{1}{2}(1 + 2^{-i})$. Then $\rho_0 = 1$ and ρ_i decreases to $\frac{1}{2}$ as $i \to \infty$. Also define an increasing sequence $\{\gamma_i\}_{i=0}^\infty$ by $\gamma_i = \left(\frac{2^*}{2}\right)^i$. Thus we can rewrite (2.32) with $R = \rho_i$, $\rho = \rho_{i+1}$, and $\gamma = \gamma_i$. Upon iterating the result and substituting in the expression for γ_i, we obtain

$$\left\{\int_{\mathcal{B}_{\rho_{i+1}}}\left[\,|\nabla\mathbf{v}|\,h'(|\nabla\mathbf{v}|)\right]^{\left(\frac{2^*}{2}\right)^{i+1}}\mathrm{d}\mathbf{x}\right\}^{\left(\frac{2}{2^*}\right)^{i+1}} \leq \prod_{k=0}^{i}\left[C\,(2^*)^{2k}\right]^{\left(\frac{2}{2^*}\right)^k}\int_{\mathcal{B}_1}|\nabla\mathbf{v}|\,h'(|\nabla\mathbf{v}|)\mathrm{d}\mathbf{x}. \qquad (2.33)$$

Now we verify that the product occurring in the above inequality remains bounded. For each i, put $A_i := \prod_{k=0}^{i} C^{\left(\frac{2}{2^*}\right)^k}$ and $B_i := \prod_{k=0}^{i} (2^*)^{2k\left(\frac{2}{2^*}\right)^k}$. We will estimate A_i and B_i separately. If $n \geq 3$, then we can bound A_i as follows:

$$A_i \leq \prod_{k=0}^{\infty} C^{\left(\frac{2}{2^*}\right)^k} = C^{\sum_{k=0}^{\infty}\left(\frac{2}{2^*}\right)^k} = C^{\frac{n}{2}}.$$

Similarly, if $n \geq 3$, we get that

$$B_i \leq (2^*)^{\frac{n(n-2)}{2}} = \left(\frac{2n}{n-2}\right)^{\frac{n(n-2)}{2}}.$$

If $n = 2$, then 2^* is a fixed number larger than 2 and we obtain similar estimates for A_i and B_i. Introducing the estimates for A_i and B_i into (2.33), we find that

$$\left\{\int_{\mathcal{B}_{\rho_{i+1}}} \left[\,|\nabla \mathbf{v}|\, h'(|\nabla \mathbf{v}|)\right]^{\left(\frac{2^*}{2}\right)^{i+1}} \mathrm{d}\mathbf{x}\right\}^{\left(\frac{2}{2^*}\right)^{i+1}} \leq C \int_{\mathcal{B}_1} |\nabla \mathbf{v}|\, h'(|\nabla \mathbf{v}|) \mathrm{d}\mathbf{x}. \tag{2.34}$$

Taking the limit as $i \to \infty$ in (2.34) yields

$$\begin{aligned}\big\|\,|\nabla \mathbf{v}|\, h'(|\nabla \mathbf{v}|)\big\|_{L^\infty(\mathcal{B}_{\frac{1}{2}})} &\leq \lim_{i\to\infty} \left\{\int_{\mathcal{B}_{\rho_{i+1}}} \left[\,|\nabla \mathbf{v}|\, h'(|\nabla \mathbf{v}|)\right]^{\left(\frac{2^*}{2}\right)^{i+1}} \mathrm{d}\mathbf{x}\right\}^{\left(\frac{2}{2^*}\right)^{i+1}} \\ &\leq C \int_{\mathcal{B}} |\nabla \mathbf{v}|\, h'(|\nabla \mathbf{v}|) \mathrm{d}\mathbf{x}.\end{aligned}$$

Using part (i) of Lemma 2.1 in both sides of the above inequality gives

$$\|h(|\nabla \mathbf{v}|)\|_{L^\infty(\mathcal{B}_{\frac{1}{2}})} \leq C \int_{\mathcal{B}_1} h(|\nabla \mathbf{v}|) \mathrm{d}\mathbf{x}. \tag{2.35}$$

As was mentioned at the beginning of the proof, using a rescaling argument and (2.35), we obtain (2.27).

Now we use (2.27) to finish the proof. Fix $0 < \rho < R$ and $\mathbf{x}_0 \in \Omega$ satisfying $B_{\mathbf{x}_0,R} \subset \Omega$, and let $y \in \mathcal{B}_{\mathbf{x}_0,\rho}$. Then $\mathcal{B}_{y,R-\rho} \subset \mathcal{B}_{\mathbf{x}_0,R}$, and so taking $r = R - \rho$ in (2.27) yields

$$\|h(|\nabla \mathbf{v}|)\|_{L^\infty(\mathcal{B}_{y,\frac{R-\rho}{2}})} \leq \frac{C}{(R-\rho)^n} \int_{\mathcal{B}_{y,R-\rho}} h(|\nabla \mathbf{v}|) \mathrm{d}\mathbf{x} \leq \frac{C}{(R-\rho)^n} \int_{\mathcal{B}_{\mathbf{x}_0,R}} h(|\nabla \mathbf{v}|) \mathrm{d}\mathbf{x}.$$

Since the above inequality holds for all $y \in \mathcal{B}_{\mathbf{x}_0,\rho}$, we conclude that

$$\|h(|\nabla \mathbf{v}|)\|_{L^\infty(\mathcal{B}_{\mathbf{x}_0,\rho})} \leq \frac{C}{(R-\rho)^n} \int_{\mathcal{B}_{\mathbf{x}_0,R}} h(|\nabla \mathbf{v}|) \mathrm{d}\mathbf{x},$$

which was to be shown. □

Now we will assume that h has (p,q)-structure, but does not necessarily satisfy (2.25), and also is not necessarily of class C^2. Our strategy is the same as that in [60]. We define a sequence of functions $\{h_k\}_{k=1}^{\infty}$ that approximate h and satisfy (2.25); we also define a corresponding sequence of integral functionals $\{J_k\}_{k=1}^{\infty}$. The conclusion of Lemma 2.5 holds for minimizers of J_k; we show that we can pass to the limit to get the result for the minimizer of the original functional.

Since we are assuming h has (p,q)-structure, $h(t) > 0$ for all positive t. Let $\{\varepsilon_k\}_{k=1}^{\infty}$ be a sequence of positive numbers decreasing to 0, choosing $\varepsilon_1 < 1$ sufficiently small so that $h'(\frac{1}{\varepsilon_1}) \geq 1$. We define $h'_k : [0,+\infty) \to [0,+\infty)$ by

$$h'_k(t) = \begin{cases} \frac{h'(\varepsilon_k)}{\varepsilon_k} t, & 0 \leq t \leq \varepsilon_k \\ h'(t), & \varepsilon_k < t \leq \frac{1}{\varepsilon_k} \\ \min\left\{\varepsilon_k h'\left(\frac{1}{\varepsilon_k}\right) t, h'(t) + \varepsilon_k t - 1\right\}, & t > \frac{1}{\varepsilon_k}. \end{cases} \tag{2.36}$$

Now we define h_k as

$$h_k(t) = \int_0^t h'_k(s) \mathrm{d}s, \tag{2.37}$$

where h'_k is defined in (2.36). Then $h_k \in W^{2,\infty}_{\mathrm{loc}}([0,\infty))$ is an N-function and satisfies (2.25) for some positive constants μ_k and M_k. We compute h_k for $t \leq \frac{1}{\varepsilon_k}$, and find that

$$h_k(t) = \begin{cases} \frac{h'(\varepsilon_k)}{2\varepsilon_k} t^2, & 0 \le t \le \varepsilon_k \\ h(t) + \frac{h'(\varepsilon_k)(\varepsilon_k)}{2} - h(\varepsilon_k), & \varepsilon_k \le t \le \frac{1}{\varepsilon_k}. \end{cases} \tag{2.38}$$

For the remainder of the section, h will be a function with (p, q)-structure and $\{h_k\}_{k=1}^{\infty}$ will be the approximating functions defined in (2.37).

Lemma 2.6. *Fix $k \in \mathbb{N}$, and assume that $\mathbf{v} \in W^{1,1}(\Omega; \mathbb{R}^N)$ is a minimizer for the functional*

$$\mathbf{u} \mapsto \int_{\Omega} h_k(|\nabla \mathbf{u}|) \mathrm{d}\mathbf{x}.$$

Then there is a constant $C = C(n, p, q)$ such that

$$\|h_k(|\nabla \mathbf{v}|)\|_{L^\infty(\mathcal{B}_{\mathbf{x}_0,\rho})} \le \frac{C}{(R-\rho)^n} \int_{\mathcal{B}_{\mathbf{x}_0,R}} h_k(|\nabla \mathbf{v}|) \mathrm{d}\mathbf{x}$$

whenever $\mathcal{B}_{\mathbf{x}_0,R} \subset \Omega$ and $0 < \rho < R$.

Proof. Note that h_k is only of class $W^{2,\infty}_{\mathrm{loc}}$, so we may not simply apply Lemma 2.5, which would require h_k to be $\mathcal{C}^2$. Our strategy is to mollify h_k, apply Lemma 2.5 to the minimizers of the functionals involving the mollifications of h_k, then pass to the limit to obtain the result for the original minimizer. Before we perform the mollification, let us extend h_k to an even function on all of $\mathbb{R}$. Now, for every $0 < \delta < \varepsilon_k^2/4$, let h_k^δ denote a standard mollification of h_k, where the support of the mollifier is contained in $[-\delta, \delta]$. Then $(h_k^\delta)'(0) = 0$, but $h_k^\delta(0) > 0$. Define $h_\delta : [0, \infty) \to [0, \infty)$ by

$$h_\delta(t) := h_k^\delta(t) - h_k^\delta(0).$$

Then h_δ is an N-function. Recall that $\{\varepsilon_k\}_{k=1}^{\infty}$ was chosen to be a decreasing sequence with $\varepsilon_1 < 1$ selected small enough so that $h'(1/\varepsilon_k) \ge 1$; keeping this in mind, it is straightforward to show that h_k has $(\overline{p}, \overline{q})$-structure, where $\overline{p} := \min\{p, 2\}$ and $\overline{q} := \max\{q+1, 3\}$. Using this and the fact that $\delta < 1/4$, we can show that h_δ has $(\tilde{p}, \tilde{q})$-structure, where we have put $\tilde{p} := \min\left\{\frac{5}{3}, \frac{1}{3} + \frac{2p}{3}\right\}$ and $\tilde{q} := \max\{2q+1, 5\}$. We also find that h_δ satisfies (2.25) for the same μ_k, M_k as h_k. Suppose $\mathcal{B}_{\mathbf{x}_0,R} \subset \Omega$, and let $\mathbf{v}_\delta \in W^{1,1}(\Omega; \mathbb{R}^N)$ be a minimizer of the functional

$$u \mapsto \int_{\mathcal{B}_{\mathbf{x}_0,R}} h_\delta(|\nabla \mathbf{u}|)\mathrm{d}\mathbf{x}$$

satisfying $\mathbf{v}_\delta = v$ on $\partial\mathcal{B}_{\mathbf{x}_0,R}$. Using Lemma 2.5 and the minimality of $\mathbf{v}_\delta$, we obtain

$$\|h_\delta(|\nabla \mathbf{v}_\delta|)\|_{L^\infty(\mathcal{B}_{\mathbf{x}_0,\rho})} \le \frac{C}{(R-\rho)^n} \int_{\mathcal{B}_{\mathbf{x}_0,R}} h_\delta(|\nabla \mathbf{v}_\delta|)\mathrm{d}\mathbf{x} \le \frac{C}{(R-\rho)^n} \int_{\mathcal{B}_{\mathbf{x}_0,R}} h_\delta(|\nabla \mathbf{v}|)\mathrm{d}\mathbf{x}, \tag{2.39}$$

for every $0 < \rho < R$. Using the convexity of h_k, it is not difficult to see that

$$h_k(t) - h_k^\delta(0) \le h_\delta(t) \le h_k(t+\delta) + h_k^\delta(0), \tag{2.40}$$

for all $t \ge 0$. Using (2.40) in (2.39), we find that

$$\|h_k(|\nabla \mathbf{v}_\delta|)\|_{L^\infty(\mathcal{B}_{\mathbf{x}_0,\rho})} \le \frac{C}{(R-\rho)^n} \int_{\mathcal{B}_{\mathbf{x}_0,R}} \{h_k(|\nabla \mathbf{v}|+\delta) + h_k^\delta(0)\}\mathrm{d}\mathbf{x} + h_k^\delta(0) \le c_1, \tag{2.41}$$

where c_1 depends on n, p, q, k, ρ, and R. Hence $h(|\nabla \mathbf{v}_\delta|)$ is equibounded with respect to δ in $\mathcal{B}_{\mathbf{x}_0,\rho}$. Since h is an N-function, we deduce that $\big\| \, |\nabla \mathbf{v}_\delta| \, \big\|_{L^\infty(\mathcal{B}_{\mathbf{x}_0,\rho})}$ is equibounded, and so, up to a subsequence, $\nabla \mathbf{v}_\delta$ converges to some $\nabla \mathbf{w}$ in the weak* topology of $L^\infty(\mathcal{B}_{\mathbf{x}_0,\rho}; \mathbb{R}^{N\times n})$ for every $\rho < R$. Passing to the limit in (2.41), we obtain

$$\|h_k(|\nabla \mathbf{w}|)\|_{L^\infty(\mathcal{B}_{\mathbf{x}_0,\rho})} \le \liminf_{\delta\to 0^+} \|h_k(|\nabla \mathbf{v}_\delta|)\|_{L^\infty(\mathcal{B}_{\mathbf{x}_0,\rho})} \le \frac{C}{(R-\rho)^n} \int_{\mathcal{B}_{\mathbf{x}_0,R}} h_k(|\nabla \mathbf{v}|)\mathrm{d}\mathbf{x}. \tag{2.42}$$

Using (2.40), the minimality of $\mathbf{v}_\delta$, and the dominated convergence theorem, we estimate that

$$\limsup_{\delta\to 0^+}\int_{\mathcal{B}_{\mathbf{x}_0,R}} h_k(|\nabla\mathbf{v}_\delta|)\mathrm{d}\mathbf{x} \le \limsup_{\delta\to 0^+}\int_{\mathcal{B}_{\mathbf{x}_0,R}} h_\delta(|\nabla\mathbf{v}_\delta|)\mathrm{d}\mathbf{x} \le \lim_{\delta\to 0^+}\int_{\mathcal{B}_{\mathbf{x}_0,R}} h_\delta(|\nabla\mathbf{v}|)\mathrm{d}\mathbf{x}$$
$$= \int_{\mathcal{B}_{\mathbf{x}_0,R}} h_k(|\nabla\mathbf{v}|)\mathrm{d}\mathbf{x}. \tag{2.43}$$

Part (i) of Lemma 2.1 gives the inequality $\overline{p}/t \le h_k'(t)/h_k(t)$; integrating this inequality from 1 to t and using properties of logarithms gives that $h_k(t) \ge h_k(1)t^{\overline{p}}$ for $t \ge 1$. Thus (2.43) implies that $\|\nabla\mathbf{v}_\delta\|_{L^{\overline{p}}(\mathcal{B}_{\mathbf{x}_0,R})}$ is uniformly bounded, so $\nabla\mathbf{v}_\delta$ converges in the weak topology of $L^{\overline{p}}(\mathcal{B}_{\mathbf{x}_0,R};\mathbb{R}^{N\times n})$ to some function $\nabla\mathbf{w}$. Therefore by weak lower semicontinuity and (2.43), we have

$$\int_{\mathcal{B}_{\mathbf{x}_0,R}} h_k(|\nabla\mathbf{w}|)\mathrm{d}\mathbf{x} \le \liminf_{\delta\to 0^+}\int_{\mathcal{B}_{\mathbf{x}_0,R}} h_k(|\nabla\mathbf{v}_\delta|)\mathrm{d}\mathbf{x} \le \int_{\mathcal{B}_{\mathbf{x}_0,R}} h_k(|\nabla\mathbf{v}|)\mathrm{d}\mathbf{x}.$$

Hence $\mathbf{w}$ is also a minimizer for the functional $\mathbf{u} \mapsto \int_\Omega h_k(|\nabla\mathbf{u}|)\mathrm{d}\mathbf{x}$. Since $h_k(|\cdot|)$ is strictly convex on $\mathbb{R}^{N\times n}$, the minimizer for the Dirichlet problem is unique, and so $\mathbf{w} = \mathbf{v}$. Therefore we can replace $\mathbf{w}$ with $\mathbf{v}$ in (2.42) and obtain the result. □

Lemma 2.7. *There are decreasing sequences* $\{\beta_k\}_{k=1}^\infty$ *and* $\{\gamma_k\}_{k=1}^\infty$ *converging to 0 such that* $h_k(t) \le h(t) + \beta_k t^2 + \gamma_k$ *for all* $t \ge 0$ *and* $k \in \mathbb{N}$.

Proof. If $0 \le t \le \frac{1}{\varepsilon_k}$, then we can use (2.38) to get $h_k(t) \le h(t) + \frac{1}{2}\varepsilon_k h'(\varepsilon_k)$. If $t > \frac{1}{\varepsilon_k}$, then $h_k'(s) \le h'(s) + \varepsilon_k s$ for all $s > \frac{1}{\varepsilon_k}$, so by (2.38) we have

$$h_k(t) = h_k\left(\tfrac{1}{\varepsilon_k}\right) + \int_{\frac{1}{\varepsilon_k}}^t h_k'(s)\mathrm{d}s \le h_k\left(\tfrac{1}{\varepsilon_k}\right) + \int_{\frac{1}{\varepsilon_k}}^t (h'(s)+\varepsilon_k s)\mathrm{d}s \le h(t) + \tfrac{1}{2}\varepsilon_k t^2 + \tfrac{1}{2}\varepsilon_k h'(\varepsilon_k).$$

We see that the lemma is proved upon taking $\beta_k = \frac{1}{2}\varepsilon_k$ and $\gamma_k = \frac{1}{2}\varepsilon_k h'(\varepsilon_k)$. □

Equipped with these lemmas, we can prove the following theorem.

Theorem 2.7. *Let $\Omega \subset \mathbb{R}^n$ be open and h be a function with (p, q)-structure. Suppose that $\mathbf{v} \in W^{1,1}(\Omega; \mathbb{R}^N)$ is a minimizer for the functional in (2.24). Then there is a constant $C = C(n, p, q)$ such that*

$$\|h(|\nabla \mathbf{v}|)\|_{L^\infty(\mathcal{B}_{\mathbf{x}_0,\rho})} \leq \frac{C}{(R-\rho)^n} \int_{\mathcal{B}_{\mathbf{x}_0,R}} h(|\nabla \mathbf{v}|) \mathrm{d}\mathbf{x}$$

whenever $\mathcal{B}_{\mathbf{x}_0,R} \subset \Omega$ and $0 < \rho < R$.

Proof. First assume that $\mathcal{B}_{\mathbf{x}_0,2R} \subset \Omega$. For each $k \in \mathbb{N}$, define the integral functional

$$J_k(\mathbf{u}) = \int_{\mathcal{B}_{\mathbf{x}_0,R}} h_k(|\nabla \mathbf{u}|) \mathrm{d}\mathbf{x},$$

where h_k is as defined in (2.37). For each $0 < \sigma < \min\{1, R\}$, let $\mathbf{v}_\sigma$ be a smooth function defined from $\mathbf{v}$ using a standard mollifier. Then $\mathbf{v}_\sigma \in W^{1,2}(\mathcal{B}_{\mathbf{x}_0,R}; \mathbb{R}^N)$. Let $\mathbf{v}_{k,\sigma}$ be a minimizer of J_k that satisfies $\mathbf{v}_{k,\sigma} = \mathbf{v}_\sigma$ on $\partial\mathcal{B}_{\mathbf{x}_0,R}$. Then by Lemma 2.6, there is a constant C, independent of k and σ, such that

$$\|h_k(|\nabla \mathbf{v}_{k,\sigma}|)\|_{L^\infty(\mathcal{B}_{\mathbf{x}_0,\rho})} \leq \frac{C}{(R-\rho)^n} \int_{\mathcal{B}_{\mathbf{x}_0,R}} h_k(|\nabla \mathbf{v}_{k,\sigma}|) \mathrm{d}\mathbf{x}. \tag{2.44}$$

Since $\mathbf{v}_{k,\sigma}$ is a minimizer for J_k, we have that

$$\int_{\mathcal{B}_{\mathbf{x}_0,R}} h_k(|\nabla \mathbf{v}_{k,\sigma}|) \mathrm{d}\mathbf{x} \leq \int_{\mathcal{B}_{\mathbf{x}_0,R}} h_k(|\nabla \mathbf{v}_\sigma|) \mathrm{d}\mathbf{x}. \tag{2.45}$$

By Lemma 2.7, we obtain decreasing sequences $\{\beta_k\}_{k=1}^\infty$ and $\{\gamma_k\}_{k=1}^\infty$ converging to 0 such that

$$\int_{\mathcal{B}_{\mathbf{x}_0,R}} h_k(|\nabla \mathbf{v}_\sigma|)\mathrm{d}\mathbf{x} \le \int_{\mathcal{B}_{\mathbf{x}_0,R}} \left\{h(|\nabla \mathbf{v}_\sigma|) + \beta_k \left|\nabla \mathbf{v}_\sigma\right|^2 + \gamma_k\right\} \mathrm{d}\mathbf{x}. \tag{2.46}$$

By properties of mollifiers,

$$\begin{aligned}\int_{\mathcal{B}_{\mathbf{x}_0,R}} &\left\{h(|\nabla \mathbf{v}_\sigma|) + \beta_k \left|\nabla \mathbf{v}_\sigma\right|^2 + \gamma_k\right\}\mathrm{d}\mathbf{x} \\ &\le \int_{\mathcal{B}_{\mathbf{x}_0,R+\sigma}} h(|\nabla \mathbf{v}|)\mathrm{d}\mathbf{x} + \int_{\mathcal{B}_{\mathbf{x}_0,R}} \left\{\beta_k \left|\nabla \mathbf{v}_\sigma\right|^2 + \gamma_k\right\} \mathrm{d}\mathbf{x}.\end{aligned} \tag{2.47}$$

Combining (2.44)-(2.47), we have

$$\begin{aligned}&\|h_k(|\nabla \mathbf{v}_{k,\sigma}|)\|_{L^\infty(B_{\mathbf{x}_0,\rho})} \\ &\le \frac{C}{(R-\rho)^n}\left[\int_{\mathcal{B}_{\mathbf{x}_0,R+\sigma}} h(|\nabla \mathbf{v}|)\mathrm{d}\mathbf{x} + \int_{\mathcal{B}_{\mathbf{x}_0,R}} \left\{\beta_k \left|\nabla \mathbf{v}_\sigma\right|^2 + \gamma_k\right\} \mathrm{d}\mathbf{x}\right] \le c_{1,\sigma},\end{aligned} \tag{2.48}$$

where, in addition to the explicit dependence on σ, $c_{1,\sigma}$ also depends on n, p, q, R, and ρ. It follows that $\left\|\,|\nabla \mathbf{v}_{k,\sigma}|\,\right\|_{L^\infty(\mathcal{B}_{\mathbf{x}_0,\rho})}$ is uniformly bounded in k by some $M_\sigma < \infty$. Hence there is a subsequence of $\mathbf{v}_{k,\sigma}$ that converges in the weak* topology of $W^{1,\infty}(\mathcal{B}_{\mathbf{x}_0,\rho};\mathbb{R}^N)$ to some function $\mathbf{w}_\sigma$. Also, since $|\nabla \mathbf{v}_{k,\sigma}| \le M_\sigma$ in $\mathcal{B}_{\mathbf{x}_0,\rho}$, for k large enough so that $\frac{1}{\varepsilon_k} \ge M_\sigma$, the computation in (2.38) gives

$$\|h_k(|\nabla \mathbf{v}_{k,\sigma}|) - h(|\nabla \mathbf{v}_{k,\sigma}|)\|_{L^\infty(\mathcal{B}_{\mathbf{x}_0,\rho})} \le \frac{h'(\varepsilon_k)\varepsilon_k}{2} + h(\varepsilon_k). \tag{2.49}$$

Using (2.49) and going to the limit in (2.48), we obtain

$$\begin{aligned}\liminf_{k\to\infty} \|h(|\nabla \mathbf{v}_{k,\sigma}|)\|_{L^\infty(\mathcal{B}_{\mathbf{x}_0,\rho})} &\le \liminf_{k\to\infty} \|h_k(|\nabla \mathbf{v}_{k,\sigma}|)\|_{L^\infty(\mathcal{B}_{\mathbf{x}_0,\rho})} \\ &\le \frac{C}{(R-\rho)^n}\int_{\mathcal{B}_{1+\sigma}} h(|\nabla \mathbf{v}|)\mathrm{d}\mathbf{x}.\end{aligned} \tag{2.50}$$

By properties of weak* convergent sequences, we have

$$\|h(|\nabla \mathbf{w}_\sigma|)\|_{L^\infty(\mathcal{B}_{\mathbf{x}_0,\rho})} \leq \liminf_{k\to\infty} \|h(|\nabla \mathbf{v}_{k,\sigma}|)\|_{L^\infty(\mathcal{B}_{\mathbf{x}_0,\rho})}. \tag{2.51}$$

Combining (2.51) and (2.50), we get

$$\|h(|\nabla \mathbf{w}_\sigma|)\|_{L^\infty(\mathcal{B}_{\mathbf{x}_0,\rho})} \leq \frac{C}{(R-\rho)^n} \int_{\mathcal{B}_{R+\sigma}} h(|\nabla \mathbf{v}|)\mathrm{d}\mathbf{x} \leq c_2, \tag{2.52}$$

where $c_2 := \frac{C}{(R-\rho)^n} \int_{\mathcal{B}_{\mathbf{x}_0,2R}} h(|\nabla \mathbf{v}|)\mathrm{d}\mathbf{x}$. Therefore, by part (v) of Lemma 2.1, we have that $\nabla \mathbf{w}_\sigma$ is uniformly bounded in $L^\infty(\mathcal{B}_{\mathbf{x}_0,\rho}; \mathbb{R}^{N\times n})$, and so we can extract a subsequence that converges weak* in $L^\infty(\mathcal{B}_{\mathbf{x}_0,\rho}; \mathbb{R}^{N\times n})$ to a function $\nabla \mathbf{w}$ for some $\mathbf{w}$. We will show that $\mathbf{w} = \mathbf{v}$. By lower semicontinuity, we have

$$\int_{\mathcal{B}_{\mathbf{x}_0,\rho}} h(|\nabla \mathbf{w}_\sigma|)\mathrm{d}\mathbf{x} \leq \liminf_{k\to\infty} \int_{\mathcal{B}_{\mathbf{x}_0,\rho}} h(|\nabla \mathbf{v}_{k,\sigma}|)\mathrm{d}\mathbf{x}. \tag{2.53}$$

Using (2.49), we obtain

$$\liminf_{k\to\infty} \int_{\mathcal{B}_{\mathbf{x}_0,\rho}} h(|\nabla \mathbf{v}_{k,\sigma}|)\mathrm{d}\mathbf{x} \leq \liminf_{k\to\infty} \int_{\mathcal{B}_{\mathbf{x}_0,\rho}} h_k(|\nabla \mathbf{v}_{k,\sigma}|)\mathrm{d}\mathbf{x} \leq \liminf_{k\to\infty} \int_{\mathcal{B}_{\mathbf{x}_0,R}} h_k(|\nabla \mathbf{v}_{k,\sigma}|)\mathrm{d}\mathbf{x} \tag{2.54}$$

But by combining (2.45)-(2.47) and taking the limit as $k \to \infty$, we find that

$$\liminf_{k\to\infty} \int_{\mathcal{B}_{\mathbf{x}_0,R}} h_k(|\nabla \mathbf{v}_{k,\sigma}|)\mathrm{d}\mathbf{x} \leq \int_{\mathcal{B}_{\mathbf{x}_0,R+\sigma}} h(|\nabla \mathbf{v}|)\mathrm{d}\mathbf{x}. \tag{2.55}$$

Collecting the inequalities in (2.53)-(2.55), we have

$$\int_{\mathcal{B}_{\mathbf{x}_0,\rho}} h(|\nabla \mathbf{w}_\sigma|)\mathrm{d}\mathbf{x} \leq \int_{\mathcal{B}_{\mathbf{x}_0,R+\sigma}} h(|\nabla \mathbf{v}|)\mathrm{d}\mathbf{x}.$$

Since the inequality above holds for every $\rho < R$, we conclude that

$$\int_{\mathcal{B}_{\mathbf{x}_0,R}} h(|\nabla \mathbf{w}_\sigma|)\mathrm{d}\mathbf{x} \le \int_{\mathcal{B}_{\mathbf{x}_0,R+\sigma}} h(|\nabla \mathbf{v}|)\mathrm{d}\mathbf{x}. \tag{2.56}$$

By lower semicontinuity and (2.56), we get

$$\int_{\mathcal{B}_{\mathbf{x}_0,R}} h(|\nabla \mathbf{w}|)\mathrm{d}\mathbf{x} \le \liminf_{\sigma\to 0} \int_{\mathcal{B}_{\mathbf{x}_0,R}} h(|\nabla \mathbf{w}_\sigma|)\mathrm{d}\mathbf{x} \le \int_{\mathcal{B}_{\mathbf{x}_0,R}} h(|\nabla \mathbf{v}|)\mathrm{d}\mathbf{x}. \tag{2.57}$$

Since $h''(t) > 0$ for all $t > 0$, we see that $h(|\cdot|)$ is strictly convex on $\mathbb{R}^{N\times n}$. Thus the minimizer to the Dirichlet problem is unique, and so we can conclude from (2.57) that $\mathbf{w} = \mathbf{v}$. Passing to the limit in (2.52) yields

$$\|h(|\nabla \mathbf{v}|)\|_{\mathcal{B}_{\mathbf{x}_0,\rho}} \le \liminf_{\sigma\to 0} \|h(|\nabla \mathbf{w}_\sigma|)\|_{\mathcal{B}_{\mathbf{x}_0,\rho}} \le \frac{C}{(R-\rho)^n} \int_{\mathcal{B}_{\mathbf{x}_0,R}} h(|\nabla \mathbf{v}|)\mathrm{d}\mathbf{x}.$$

Thus we have shown the result if $\mathcal{B}_{\mathbf{x}_0,2R} \subset \Omega$. Now suppose only that $\mathcal{B}_{\mathbf{x}_0,R} \subset \Omega$, and $0 < \rho < R$. Then $\mathcal{B}_{\mathbf{y},R-\rho} \subset \Omega$ for every $\mathbf{y} \in \mathcal{B}_{\mathbf{x}_0,\rho}$, so by the argument above, we have that

$$\|h(|\nabla \mathbf{v}|)\|_{L^\infty(\mathcal{B}_{\mathbf{y},\frac{R-\rho}{4}})} \le \frac{C}{(R-\rho)^n} \int_{\mathcal{B}_{\mathbf{y},\frac{R-\rho}{2}}} h(|\nabla \mathbf{v}|)\mathrm{d}\mathbf{x} \le \frac{C}{(R-\rho)^n} \int_{\mathcal{B}_{\mathbf{x}_0,R}} h(|\nabla \mathbf{v}|)\mathrm{d}\mathbf{x}.$$

Since the above inequality holds for every $\mathbf{y} \in \mathcal{B}_{\mathbf{x}_0,\rho}$, we see that

$$\|h(|\nabla \mathbf{v}|)\|_{L^\infty(\mathcal{B}_{\mathbf{x}_0,\rho})} \le \frac{C}{(R-\rho)^n} \int_{\mathcal{B}_{\mathbf{x}_0,R}} h(|\nabla \mathbf{v}|)\mathrm{d}\mathbf{x},$$

which is the desired result. □

We can change variables and use Theorem 2.7 to establish the apparently more general result that follows.

Theorem 2.8. *Let $\Omega \subset \mathbb{R}^n$ be open, and let $\mathbf{v} \in W^{1,1}(\Omega; \mathbb{R}^N)$ be a minimizer of the functional*

$$\mathbf{u} \mapsto \int_\Omega h(|\nabla \mathbf{u} \mathbf{G}_0|) \mathrm{d}\mathbf{x},$$

where h is a function with (p,q)-structure and $\mathbf{G}_0$ is an invertible $n \times n$ constant matrix. Then there is a constant $C = C(n, p, q, \left|\mathbf{G}_0^{-1}\right|, |\mathbf{G}_0|)$ such that

$$\|h(|\nabla \mathbf{v} \mathbf{G}_0|)\|_{L^\infty(\mathcal{B}_{\mathbf{x}_0,\rho})} \leq \frac{C}{(R-\rho)^n} \int_{\mathcal{B}_{\mathbf{x}_0,R}} h(|\nabla \mathbf{v} \mathbf{G}_0|) \mathrm{d}\mathbf{x}$$

whenever $\mathcal{B}_{\mathbf{x}_0,R} \subset \Omega$ and $0 < \rho < R$.

Using a reflection argument and Theorem 2.8, we can show the following version of the result for the half-ball.

Theorem 2.9. *Let h be a function with (p,q)-structure, and suppose that $\mathbf{G}_0 \in \mathbb{R}^{n \times n}$ is invertible. Let $\mathbf{v} \in W^{1,1}(\Omega; \mathbb{R}^N)$ be a minimizer of the functional*

$$u \mapsto \int_{\mathcal{B}^+} h(|\nabla \mathbf{u} \mathbf{G}_0|) \mathrm{d}\mathbf{x},$$

satisfying $\mathbf{v} = \mathbf{0}$ on $\mathcal{B} \cap \partial \mathcal{H}^+$ in the sense of trace. Then there exists a constant $C = C(n, p, q, \left|\mathbf{G}_0^{-1}\right|, |\mathbf{G}_0|)$ such that

$$\|h(|\nabla \mathbf{v} \mathbf{G}_0|)\|_{L^\infty(\mathcal{B}^+_{\mathbf{x}_0,\rho})} \leq \frac{C}{(R-\rho)^n} \int_{\mathcal{B}^+_{\mathbf{x}_0,R}} h(|\nabla \mathbf{v} \mathbf{G}_0|) \mathrm{d}\mathbf{x}$$

for any $\mathbf{x}_0 \in \mathcal{B}^+$ and $0 < \rho < R \leq 1 - |\mathbf{x}_0|$.

2.3 Results for Convex Functionals

The proof of the following theorem uses many of the ideas from the proof for Proposition 3.1 in [3].

Theorem 2.10. *Suppose that* $\alpha : \Omega \to [1, \infty)$ *satisfies* (1.12)-(1.14)*, and that* $g : \Omega \times \mathbb{R}_+ \to \mathbb{R}_+$ *satisfies* (1.15)-(1.17)*. Let* $\mathbf{G}_0 \in \mathbb{R}^{n \times n}$ *be an invertible matrix, and suppose* $\mathbf{v}$ *is a minimizer for the functional* J *defined by*

$$\mathbf{w} \mapsto \int_\Omega g(\mathbf{x}, |\nabla \mathbf{w} \mathbf{G}_0|) \mathrm{d}\mathbf{x},$$

among all mappings $\mathbf{w} \in \mathbf{v} + W_0^1 L_g(\Omega; \mathbb{R}^N)$*. Then for every* $0 \leq \kappa < n$*, there are constants* C_κ *and* R_κ*, which, in addition to* κ*, also depend on* n*,* N*,* p*,* q*,* α_+*,* $|\mathbf{G}_0|$*,* $\left|\mathbf{G}_0^{-1}\right|$*,* M*,* ω*,* δ*,* κ*, and* $L := \int_\Omega (1 + g(\mathbf{x}, |\nabla \mathbf{w} \mathbf{G}_0|)) \mathrm{d}\mathbf{x}$*, such that*

$$\int_{\mathcal{B}(\mathbf{x}_0, \rho)} (1 + g(\mathbf{x}, |\nabla \mathbf{v}|))\, \mathrm{d}\mathbf{x} \leq C_\kappa \left(\frac{\rho}{R}\right)^\kappa \int_{\mathcal{B}(\mathbf{x}_0, R)} (1 + g(\mathbf{x}, |\nabla \mathbf{v}|))\, \mathrm{d}\mathbf{x}$$

whenever $\mathcal{B}(\mathbf{x}_0, R) \subset \Omega$ *and* $0 < \rho \leq R \leq R_\kappa$*.*

Proof. Fix $\kappa \in [0, n)$. Throughout the proof, C will denote a constant that may depend only on the parameters listed in the statement of the theorem, and its value may change from line to line. We define the functions $\alpha_1, \alpha_2 : \Omega \times (0, \infty) \to [1, \infty)$ by

$$\alpha_1(\mathbf{x}, r) := \min \left\{ \alpha(\mathbf{y}) : \mathbf{y} \in \overline{\Omega(\mathbf{x}, r)} \right\},$$
$$\alpha_2(\mathbf{x}, r) := \max \left\{ \alpha(\mathbf{y}) : \mathbf{y} \in \overline{\Omega(\mathbf{x}, r)} \right\}.$$

By Theorem 2.4 and Hölder's inequality, we have that there are constants $C > 0$ and

$r_3 > 1$ such that

$$\left(\fint_{\mathcal{B}(\mathbf{x},r)} \{1 + g(\mathbf{x}, |\nabla \mathbf{v}|)^s\} \,\mathrm{d}\mathbf{x}\right)^{\frac{1}{s}} \leq C \fint_{\mathcal{B}(\mathbf{x},2r)} \{1 + g(\mathbf{x}, |\nabla \mathbf{v}|)\} \,\mathrm{d}\mathbf{x} \tag{2.58}$$

whenever $\mathcal{B}(\mathbf{x}, 2r) \subset \Omega$ and $s \in [1, r_3]$. Fix $r' \in (1, \min\{2, r_3\})$, and select $R_0 \in (0, 1/2)$ so that $\omega(2R_0) < \frac{r_3}{r'} - 1$. Now we suppose that $0 < 8\rho < R \leq R_0$ and $\mathbf{x}_0 \in \Omega$ are such that $\mathcal{B}(\mathbf{x}_0, R) \subset \Omega$. With $\mathbf{x}_0$ and R fixed, we will use for convenience α_1 and α_2 to denote $\alpha_1(\mathbf{x}_0, R)$ and $\alpha_2(\mathbf{x}_0, R)$, respectively, and $\mathcal{B}_r := \mathcal{B}(\mathbf{x}_0, r)$ for $r > 0$. Note that $\alpha_2 \leq \alpha_1 + \omega(2R)$, and hence the choices for R_0 and r' above imply that

$$\alpha_2 r' \leq \alpha(\mathbf{x}) r'(1 + \omega(2R)) \leq \alpha(\mathbf{x}) r_3 \tag{2.59}$$

for all $\mathbf{x} \in \mathcal{B}_R$.

We can select $\mathbf{x}_2 \in \overline{\mathcal{B}_R}$ so that $\alpha(\mathbf{x}_2) = \alpha_2$. Let $\mathbf{w} \in W^1 L_{h^{\alpha_2}}(\mathcal{B}_{R/4}; \mathbb{R}^N)$ be the minimizer for the functional $J_{\mathbf{x}_2} : W^1 L_{h^{\alpha_2}}(\mathcal{B}_{R/4}; \mathbb{R}^N) \to \mathbb{R}$ defined by

$$J_{\mathbf{x}_2}(\mathbf{u}) := \int_{\mathcal{B}_{R/4}} g(\mathbf{x}_2, |\nabla \mathbf{u} \mathbf{G}_0|) \mathrm{d}\mathbf{x},$$

satisfying $\mathbf{w} = \mathbf{v}$ on $\partial \mathcal{B}_{R/4}$. (Note that by Theorem 2.4 and (2.59), we have $\mathbf{v} \in W^1 L_{h^{\alpha_2}}(\mathcal{B}_{R/4}; \mathbb{R}^N)$.) We clearly have

$$\begin{aligned}
\int_{\mathcal{B}_\rho} g(\mathbf{x}_2, |\nabla \mathbf{v} \mathbf{G}_0|) \mathrm{d}\mathbf{x} = \int_{\mathcal{B}_\rho} g(\mathbf{x}_2, |\nabla \mathbf{w} g_0|) \mathrm{d}\mathbf{x} + \int_{\mathcal{B}_\rho} \Big\{ & g(\mathbf{x}_2, |\nabla \mathbf{v} \mathbf{G}_0|) - g(\mathbf{x}_2, |\nabla \mathbf{w} \mathbf{G}_0|) \\
& - g_t(\mathbf{x}_2, |\nabla \mathbf{w} \mathbf{G}_0|) \frac{\nabla \mathbf{w} \mathbf{G}_0}{|\nabla \mathbf{w} \mathbf{G}_0|} \cdot (\nabla \mathbf{v} - \nabla \mathbf{w}) \mathbf{G}_0 \Big\} \mathrm{d}\mathbf{x} \\
& + \int_{\mathcal{B}_\rho} g_t(\mathbf{x}_2, |\nabla \mathbf{w} \mathbf{G}_0|) \frac{\nabla \mathbf{w} \mathbf{G}_0}{|\nabla \mathbf{w} \mathbf{G}_0|} \cdot (\nabla \mathbf{v} - \nabla \mathbf{w}) \mathbf{G}_0 \mathrm{d}\mathbf{x} \\
= I_1 + I_2 + I_3, &
\end{aligned} \tag{2.60}$$

where I_1, I_2, and I_3 are defined to be the first, second, and third integrals, respectively. By part (vii) of Lemma 2.1, there is a constant C depending only on n, p, and $q\alpha_+$ such that $I_3 \le C(I_1 + I_2)$, so that it remains only to estimate I_1 and I_2.

For I_1, we have from Theorem 2.8 and the minimality of $\mathbf{w}$ for $J_{\mathbf{x}_2}$ that

$$I_1 \le C\left(\frac{\rho}{R}\right)^n \int_{\mathcal{B}_{R/4}} g(\mathbf{x}_2, |\nabla \mathbf{w}|)\mathrm{d}\mathbf{x} \le C\left(\frac{\rho}{R}\right)^n \int_{\mathcal{B}_{R/4}} g(\mathbf{x}_2, |\nabla \mathbf{v}|)\mathrm{d}\mathbf{x}.$$

To estimate I_2, we note that the integrand is nonnegative because of the convexity of $g(\mathbf{x}_2, \cdot)$, so we can expand the domain of integration to $\mathcal{B}_{R/4}$. Then using the Euler-Lagrange equations for $\mathbf{w}$ (Lemma 2.2), the minimality of $\mathbf{v}$ for J, and (1.17), we have

$$\begin{aligned} I_2 \le C \int\limits_{\mathcal{B}_{R/4}} & \{g(\mathbf{x}_2, |\nabla \mathbf{v}\mathbf{G}_0|) - g(\mathbf{x}, |\nabla \mathbf{v}\mathbf{G}_0|)\}\, \mathrm{d}\mathbf{x} \\ &+ C \int\limits_{\mathcal{B}_{R/4}} \{g(\mathbf{x}, |\nabla \mathbf{w}\mathbf{G}_0|) - g(\mathbf{x}_2\, |\nabla \mathbf{w}\mathbf{G}_0|)\}\, \mathrm{d}\mathbf{x} \\ \le C\int\limits_{\mathcal{B}_{R/4}} & \{\omega(R)\,(1+h(|\nabla \mathbf{v}|)^{\alpha_2})\log(e+h(|\nabla \mathbf{v}|))+\delta(R)\,(1+h(|\nabla \mathbf{v}|)^{\alpha_2})\}\, \mathrm{d}\mathbf{x} \\ &+ C\int\limits_{\mathcal{B}_{R/4}} \{\omega(R)\,(1+h(|\nabla \mathbf{w}|)^{\alpha_2})\log(e+h(|\nabla \mathbf{w}|))+\delta(R)(1+h(|\nabla \mathbf{w}|)^{\alpha_2})\}\, \mathrm{d}\mathbf{x}. \end{aligned}$$

Now we use the minimality of $\mathbf{w}$ for $J_{\mathbf{x}_2}$ in the above inequality to conclude that

$$\begin{aligned} I_2 &\le C\omega(R) \int\limits_{\mathcal{B}_{R/4}} \left(1 + h(|\nabla \mathbf{v}|)^{\alpha_2}\right) \log\left(e + h(|\nabla \mathbf{v}|)\right) \mathrm{d}\mathbf{x} \\ &\quad + C\omega(R) \int\limits_{\mathcal{B}_{R/4}} \left(1 + h(|\nabla \mathbf{w}|)^{\alpha_2}\right) \log\left(e + h(|\nabla \mathbf{w}|)\right) \mathrm{d}\mathbf{x} \\ &\quad + C\delta(R) \int\limits_{\mathcal{B}_{R/4}} \left(1 + h(|\nabla \mathbf{v}|)^{\alpha_2}\right) \mathrm{d}\mathbf{x} \\ &= I_{2,1} + I_{2,2} + I_{2,3}. \end{aligned} \tag{2.61}$$

Before we estimate $I_{2,1}$, $I_{2,2}$, and $I_{2,3}$, we introduce some notation and ideas that we will use in these estimates. As in [3], for each $s \in [1, \infty)$, we use the norm $\|\cdot\|_s$ on $L^s(\mathcal{B}_{R/4})$ by

$$\left\|\hat{h}\right\|_s := \left(\fint_{\mathcal{B}_{R/4}} \left|\hat{h}\right|^s \right)^{\frac{1}{s}}.$$

We recall the following from [3], which follows from a result in [49]: for each $s > 1$, there is a constant $c(s)$, which does not depend on R or the mapping $\hat{h}$, such that

$$\fint_{\mathcal{B}_{R/4}} \left|\hat{h}\right| \log\left(e + \frac{\left|\hat{h}\right|}{\left\|\hat{h}\right\|_1} \right) \mathrm{d}\mathbf{x} \le c(s) \left\|\hat{h}\right\|_s. \tag{2.62}$$

Using the inequality $\log(e+ab) \le \log(e+a) + \log(e+b)$, which is valid for all $a, b \ge 0$, we have that

$$\begin{aligned} I_{2,1} &\le C\omega(R) \int_{\mathcal{B}_{R/4} \cap \{h(|\nabla \mathbf{v}|) \ge e\}} h(|\nabla \mathbf{v}|)^{\alpha_2} \log(h(|\nabla \mathbf{v}|)^{\alpha_2}) \mathrm{d}\mathbf{x} + C\omega(R) R^n \\ &\le C\omega(R) R^n \fint_{\mathcal{B}_{R/4}} h(|\nabla \mathbf{v}|)^{\alpha_2} \log\left(e + \frac{h(|\nabla \mathbf{v}|)^{\alpha_2}}{\|h(|\nabla \mathbf{v}|)^{\alpha_2}\|_1} \right) \mathrm{d}\mathbf{x} \\ &\quad + C\omega(R) R^n \log\left(e + \|h(|\nabla \mathbf{v}|)^{\alpha_2}\|_1\right) \fint_{\mathcal{B}_{R/4}} h(|\nabla \mathbf{v}|)^{\alpha_2} \mathrm{d}\mathbf{x} + C\omega(R) R^n. \end{aligned} \tag{2.63}$$

By Hölder's inequality and (2.59), we find that

$$\begin{aligned}\fint_{\mathcal{B}_{R/4}} h(|\nabla \mathbf{v}|)^{\alpha_2} \mathrm{d}\mathbf{x} &\le \left(\fint_{\mathcal{B}_{R/4}} h(|\nabla \mathbf{v}|)^{\alpha_2 r'} \mathrm{d}\mathbf{x} \right)^{\frac{1}{r'}} \\ &\le \left(\fint_{\mathcal{B}_{R/4}} \left\{ 1 + h(|\nabla \mathbf{v}|)^{\alpha(\mathbf{x}) r'(1+\omega(2R))} \right\} \mathrm{d}\mathbf{x} \right)^{\frac{1}{r'}} .\end{aligned} \tag{2.64}$$

Employing (2.59) and (2.58), we have that

$$\log \left(e + \left\| h(|\nabla \mathbf{v}|)^{\alpha_2} \right\|_1 \right) \le C \log \left(\frac{1}{R} \right). \tag{2.65}$$

Putting (2.64) and (2.65) into (2.63), and using (2.62) in the first integral of the right side of (2.63) yields

$$\begin{aligned}I_{2,1} \le{}& C\omega(R) R^n \left(\fint_{\mathcal{B}_{R/4}} \left\{ 1 + h(|\nabla \mathbf{v}|)^{\alpha(\mathbf{x}) r'(1+\omega(2R)} \right\} \mathrm{d}\mathbf{x} \right)^{\frac{1}{r'}} \\ &+ C\omega(R) \log \left(\frac{1}{R} \right) R^n \left(\fint_{\mathcal{B}_{R/4}} \left\{ 1 + h(|\nabla \mathbf{v}|)^{\alpha(\mathbf{x}) r'(1+\omega(2R))} \right\} \mathrm{d}\mathbf{x} \right)^{\frac{1}{r'}} + C\omega(R) R^n.\end{aligned}$$

Using (2.58) in the above inequality with $s = r'(1 + \omega(2R))$, we obtain

$$\begin{aligned}I_{2,1} &\le C\omega(R) \log \left(\frac{1}{R} \right) R^n \left(\fint_{\mathcal{B}_{R/2}} \left\{ 1 + h(|\nabla \mathbf{v}|)^{\alpha(\mathbf{x})} \right\} \mathrm{d}\mathbf{x} \right)^{1+\omega(2R)} + C\omega(R) R^n \\ &\le C\omega(R) \log \left(\frac{1}{R} \right) R^{-n\omega(2R)} \int_{\mathcal{B}_{R/2}} \left\{ 1 + h(|\nabla \mathbf{v}|)^{\alpha(\mathbf{x})} \right\} \mathrm{d}\mathbf{x} + C\omega(R) R^n \\ &\le C\omega(R) \log \left(\frac{1}{R} \right) \int_{\mathcal{B}_{R/2}} (1 + g(\mathbf{x}, |\nabla \mathbf{v}|)) \mathrm{d}\mathbf{x}.\end{aligned} \tag{2.66}$$

We note here that we used (1.13) to conclude that $R^{-n\omega(2R)} \le C$ in the last line.

Now we turn to the task of providing an estimate for $I_{2,2}$. In the same way that we obtained (2.63), we find that

$$I_{2,2} \leq C\omega(R)R^n \fint_{\mathcal{B}_{R/4}} h(|\nabla \mathbf{w}|)^{\alpha_2} \log\left(e + \frac{h(|\nabla \mathbf{w}|)^{\alpha_2}}{\|h(|\nabla \mathbf{w}|)^{\alpha_2}\|_1}\right) \mathrm{d}\mathbf{x} + C\omega(R)R^n \log\left(e + \|h(|\nabla \mathbf{w}|)^{\alpha_2}\|_1\right) \fint_{\mathcal{B}_{R/4}} h(|\nabla \mathbf{w}|)^{\alpha_2} \mathrm{d}\mathbf{x} + C\omega(R)R^n.$$

Using the minimality of $\mathbf{w}$ for $J_{\mathbf{x}_2}$, we can bound $\log(e + \|h(|\nabla \mathbf{w}|)^{\alpha_2}\|_1)$ in the same way that we bounded the analogous term for $\mathbf{v}$, so that the above inequality gives

$$I_{2,2} \leq C\omega(R)R^n \fint_{\mathcal{B}_{R/4}} h(|\nabla \mathbf{w}|)^{\alpha_2} \log\left(e + \frac{h(|\nabla \mathbf{w}|)^{\alpha_2}}{\|h(|\nabla \mathbf{v}|)^{\alpha_2}\|_1}\right) \mathrm{d}\mathbf{x} + C\omega(R) \log\left(\frac{1}{R}\right) R^n \fint_{\mathcal{B}_{R/4}} h(|\nabla \mathbf{w}|)^{\alpha_2} \mathrm{d}\mathbf{x} + C\omega(R)R^n. \tag{2.67}$$

Now, by Theorem 2.5, we have that there is some $r_4 > 1$ such that

$$\left(\fint_{\mathcal{B}_{R/4}} h(|\nabla \mathbf{w}|)^{\alpha_2 r_4} \mathrm{d}\mathbf{x}\right)^{\frac{1}{r_4}} \leq C \left(\fint_{\mathcal{B}_{R/2}} \left\{1 + h(|\nabla \mathbf{v}|)^{\alpha_2 r'}\right\} \mathrm{d}\mathbf{x}\right)^{\frac{1}{r'}}. \tag{2.68}$$

Utilizing (2.62) in the first term of (2.67) and Hölder's inequality in the second, we find that

$$\begin{aligned} I_{2,2} &\leq C\omega(R)R^n \left(\fint_{\mathcal{B}_{R/4}} h(|\nabla \mathbf{w}|)^{\alpha_2 r_4} \mathrm{d}\mathbf{x}\right)^{\frac{1}{r_4}} \\ &\quad + C\omega(R) \log\left(\frac{1}{R}\right) R^n \left(\fint_{\mathcal{B}_{R/4}} h(|\nabla \mathbf{w}|)^{\alpha_2 r_4} \mathrm{d}\mathbf{x}\right)^{\frac{1}{r_4}} + C\omega(R)R^n \\ &\leq C\omega(R) \log\left(\frac{1}{R}\right) R^n \left(\fint_{\mathcal{B}_{R/4}} h(|\nabla \mathbf{w}|)^{\alpha_2 r_4} \mathrm{d}\mathbf{x}\right)^{\frac{1}{r_4}} + C\omega(R)R^n. \end{aligned} \tag{2.69}$$

We use (2.68) and (2.59) in (2.69) to arrive at the inequality

$$I_{2,2} \leq C\omega(R)\log\left(\frac{1}{R}\right)R^n\left(\fint_{\mathcal{B}_{R/2}}\left\{1+h(|\nabla \mathbf{v}|)^{\alpha(\mathbf{x})r'(1+\omega(2R))}\right\}\mathrm{d}\mathbf{x}\right)^{\frac{1}{r'}}+C\omega(R)R^n.$$

Now we employ (2.58) and conclude as in (2.66) that

$$\begin{aligned}
I_{2,2} &\leq C\omega(R)\log\left(\frac{1}{R}\right)R^n\left(\fint_{\mathcal{B}_R}\left\{1+h(|\nabla \mathbf{v}|)^{\alpha(\mathbf{x})}\right\}\mathrm{d}\mathbf{x}\right)^{1+\omega(2R)}+C\omega(R)R^n\\
&\leq C\omega(R)\log\left(\frac{1}{R}\right)R^{-n\omega(2R)}\int_{\mathcal{B}_R}\left\{1+h(|\nabla \mathbf{v}|)^{\alpha(\mathbf{x})}\right\}\mathrm{d}\mathbf{x}+C\omega(R)R^n \qquad (2.70)\\
&\leq C\omega(R)\log\left(\frac{1}{R}\right)\int_{\mathcal{B}_R}(1+g(\mathbf{x},|\nabla \mathbf{v}|))\,\mathrm{d}\mathbf{x}.
\end{aligned}$$

The estimate for $I_{2,3}$ is easier. By (2.64) and (2.58), we have

$$\begin{aligned}
I_{2,3} &\leq C\delta(R)R^n\fint_{\mathcal{B}_{R/4}}h(|\nabla \mathbf{v}|)^{\alpha_2}\mathrm{d}\mathbf{x}+C\delta(R)R^n\\
&\leq C\delta(R)R^n\left(\fint_{\mathcal{B}_{R/4}}\left\{1+h(|\nabla \mathbf{v}|)^{\alpha(\mathbf{x})r'(1+\omega(2R))}\right\}\mathrm{d}\mathbf{x}\right)^{\frac{1}{r'}}+C\delta(R)R^n\\
&\leq C\delta(R)R^n\left(\fint_{\mathcal{B}_{R/2}}\left\{1+h(|\nabla \mathbf{v}|)^{\alpha(\mathbf{x})}\right\}\mathrm{d}\mathbf{x}\right)^{1+\omega(2R)}+C\delta(R)R^n.
\end{aligned}$$

Similarly to (2.66), we can conclude

$$I_{2,3} \leq C\delta(R)\int_{\mathcal{B}_{R/2}}(1+g(\mathbf{x},|\nabla \mathbf{v}|))\,\mathrm{d}\mathbf{x}. \qquad (2.71)$$

Putting (2.66), (2.70), and (2.71) into (2.61) yields

$$I_2 \leq C\left(\omega(R)\log\left(\frac{1}{R}\right)+\delta(R)\right)\int_{\mathcal{B}_{R/2}}(1+g(\mathbf{x},|\nabla \mathbf{v}|))\,\mathrm{d}\mathbf{x}. \qquad (2.72)$$

We have already noted that $I_3 \leq C(I_1+I_2)$, so putting our estimates for I_1 and I_2

into (2.60) gives

$$\int\limits_{\mathcal{B}_\rho} (1+g(\mathbf{x}, |\nabla \mathbf{v}|))\,\mathrm{d}\mathbf{x} \leq C\left(\left(\frac{\rho}{R}\right)^n + \omega(R)\log\left(\frac{1}{R}\right) + \delta(R)\right)\int\limits_{\mathcal{B}_R} (1+g(\mathbf{x}, |\nabla \mathbf{v}|))\,\mathrm{d}\mathbf{x} \tag{2.73}$$

for all $0 < 8\rho < R \leq R_0$. But by enlarging C if necessary, we clearly have that (2.73) holds for $0 < \rho \leq R \leq 8\rho \leq R_0$ as well, so that, in fact, the inequality in (2.73) holds for all $0 < \rho \leq R \leq R_0$. By (1.14), for each $\kappa \in [0, n)$, we can find $R_\kappa \in (0, R_0)$ so that

$$\omega(R)\log\left(\frac{1}{R}\right) + \delta(R) \leq \varepsilon_0 := \left(\frac{1}{2C}\right)^{\frac{n}{n-\kappa}}$$

for all $0 < R < R_\kappa$. Then by Lemma 2.4, we conclude that

$$\int_{\mathcal{B}_\rho} (1+g(\mathbf{x}, |\nabla \mathbf{v}|))\mathrm{d}\mathbf{x} \leq C_\kappa \left(\frac{\rho}{R}\right)^\kappa \int_{\mathcal{B}_R} (1+g(\mathbf{x}, |\nabla \mathbf{v}|))\mathrm{d}\mathbf{x}$$

for all $0 < \rho \leq R \leq R_\kappa$, which concludes the proof. □

Using a reflection argument and Theorem 2.10, we can show the following version of the result for the half-ball.

Theorem 2.11. *Suppose that* $\alpha : \mathcal{B}^+ \to [1, \infty)$ *satisfies* (1.12)-(1.14), *and that* $g : \mathcal{B}^+ \times \mathbb{R}_+ \to \mathbb{R}_+$ *satisfies* (1.15)-(1.17). *Let* $\mathbf{G}_0 \in \mathbb{R}^{n\times n}$ *be an invertible matrix, and suppose that* $\mathbf{v}$ *is a minimizer for the functional* $J : W^{1,1}(\mathcal{B}^+; \mathbb{R}^N) \to \mathbb{R}^*$ *defined by*

$$J(\mathbf{w}) := \int_{\mathcal{B}^+} g(\mathbf{x}, |\nabla \mathbf{w}\mathbf{G}_0|)\mathrm{d}\mathbf{x},$$

satisfying $\mathbf{v} = \mathbf{0}$ *on* $\mathcal{B} \cap \partial\mathcal{H}^+$ *in the sense of trace. Then for every* $0 \leq \kappa < n$, *there are constants* C_κ *and* R_κ, *which, in addition to* κ, *also depend on* n, N, p, q, α_+,

$|\mathbf{G}_0|$, $\left|\mathbf{G}_0^{-1}\right|$, *$M$, ω, δ, κ, and* $L := \int_{\mathcal{B}^+}(1+g(\mathbf{x},|\nabla\mathbf{w}\mathbf{G}_0|))\mathrm{d}\mathbf{x}$, *such that*

$$\int_{\mathcal{B}(\mathbf{x}_0,\rho)^+}(1+g(\mathbf{x},|\nabla\mathbf{v}|))\mathrm{d}\mathbf{x} \leq C\left(\frac{\rho}{R}\right)^{\kappa}\int_{\mathcal{B}(\mathbf{x}_0,R)^+}(1+g(\mathbf{x},|\nabla\mathbf{v}|))\mathrm{d}\mathbf{x}$$

whenever $\mathcal{B}(\mathbf{x}_0,R)^+ \subset \mathcal{B}^+$ *and* $0 < \rho \leq R$.

Now we can prove the following lemma.

Lemma 2.8. *Suppose that* $\alpha : \mathcal{B}^+ \times [1,\infty)$ *satisfies* (1.12)-(1.14) *and that* $g : \mathcal{B}^+ \times \mathbb{R}_+ \to \mathbb{R}_+$ *satisfies* (1.15)-(1.17). *Assume* $0 \leq \lambda < n$. *Let*

$$\mathcal{A} := \left\{\mathbf{u} \in W^{1,1}(\mathcal{B}^+;\mathbb{R}^N) : \mathbf{u} = \mathbf{0} \text{ on } \mathcal{B}\cap\partial\mathcal{H}^+ \text{in the sense of trace}\right\},$$

and define the functional $K : \mathcal{A} \to \mathbb{R}^*$ *by*

$$K(\mathbf{w}) := \int_{\mathcal{B}^+} g(\mathbf{x},|[\nabla\mathbf{w}+\mathbf{A}]\mathbf{G}|)\mathrm{d}\mathbf{x},$$

where $g(\cdot,|\mathbf{A}|) \in L^{1,\lambda}(\mathcal{B}^+)$ *and* $\mathbf{G} \in \mathcal{C}(\overline{\mathcal{B}^+};\mathbb{R}^{n\times n})$ *has continuous matrix inverse* $\mathbf{G}^{-1} \in \mathcal{C}(\overline{\mathcal{B}^+};\mathbb{R}^{n\times n})$. *If* $\mathbf{u} \in \mathcal{A}$ *and there are functions* $\{\nu_\varepsilon\}_{\varepsilon>0} \subset L^{1,\lambda}(\mathcal{B}^+)$ *and* $\{\gamma_\varepsilon\}_{\varepsilon>0} \subset \mathcal{C}(\mathbb{R}_+)$ *satisfying* $\gamma_\varepsilon(0) = 0$ *such that* $\mathbf{u}$ *is a* $(K,\{\gamma_\varepsilon\},\{\nu_\varepsilon\})$*-minimizer over* $\mathcal{A}$, *then* $g(\cdot,|\nabla\mathbf{u}|) \in L^{1,\lambda}_{\mathrm{loc}}(\mathcal{B}\cap\overline{\mathcal{H}^+})$.

Proof. Fix $\mathbf{x}_0 \in \mathcal{B}^+$ and $R > 0$ so that $\mathcal{B}^+(\mathbf{x}_0,R) \subset \mathcal{B}^+$. For ease of notation, let $\mathbf{G}_0 = \mathbf{G}(\mathbf{x}_0)$. Also, for each $r > 0$, define

$$\mu(r) := \sup_{|\mathbf{x}-\mathbf{y}|\leq r}|\mathbf{G}(\mathbf{x})-\mathbf{G}(\mathbf{y})|,$$

and set $\mathcal{B}_r^+ := \mathcal{B}(\mathbf{x}_0,r)^+$. Fix $R > 0$ such that $\mathcal{B}_R^+ \subset \mathcal{B}^+$ and $R \leq R_{(n+\lambda)/2}$, where $R_{(n+\lambda)/2}$ is the value of R_κ given by Theorem 2.11 for $\kappa = (n+\lambda)/2$ and $L =$

$\int_{\mathcal{B}^+}(1+g(\mathbf{x},|\nabla\mathbf{u}\mathbf{G}_0|)\mathrm{d}\mathbf{x}$. Let $\mathbf{v}\in W^1L_g(\mathcal{B}_R^+;\mathbb{R}^N)$ be the minimizer of the functional $J: W^1L_g(\mathcal{B}_R^+;\mathbb{R}^N)\to\mathbb{R}$ defined by

$$J(\mathbf{w}) := \int_{\mathcal{B}_R^+} g(\mathbf{x},|\nabla\mathbf{w}\mathbf{G}_0|)\mathrm{d}\mathbf{x},$$

satisfying $\mathbf{v}=\mathbf{u}$ on $\partial\mathcal{B}_R^+$ in the sense of trace. Then for $0<\rho<R/2$, we have

$$\begin{aligned}\int_{\mathcal{B}_\rho^+} g(\mathbf{x},|\nabla\mathbf{u}\mathbf{G}_0|)\mathrm{d}\mathbf{x} &= \int_{\mathcal{B}_\rho^+} g(\mathbf{x},|\nabla\mathbf{v}\mathbf{G}_0|)\mathrm{d}\mathbf{x} + \int_{\mathcal{B}_\rho^+}\Big\{g(\mathbf{x},|\nabla\mathbf{u}\mathbf{G}_0|) - g(\mathbf{x},|\nabla\mathbf{v}\mathbf{G}_0|)\\ &\qquad - g_t(\mathbf{x},|\nabla\mathbf{v}\mathbf{G}_0|)\frac{\nabla\mathbf{v}\mathbf{G}_0}{|\nabla\mathbf{v}\mathbf{G}_0|}\cdot[\nabla\mathbf{u}-\nabla\mathbf{v}]\mathbf{G}_0\Big\}\mathrm{d}\mathbf{x}\\ &\qquad + \int_{\mathcal{B}_\rho^+} g_t(\mathbf{x},|\nabla\mathbf{v}\mathbf{G}_0|)\frac{\nabla\mathbf{v}\mathbf{G}_0}{|\nabla\mathbf{v}\mathbf{G}_0|}\cdot[\nabla\mathbf{u}-\nabla\mathbf{v}]\mathbf{G}_0\mathrm{d}\mathbf{x}\\ &= I_1+I_2+I_3,\end{aligned} \tag{2.74}$$

where I_1, I_2, and I_3 are defined to be the respective integrals. By Theorem 2.11 and the minimality of $\mathbf{v}$, we have that

$$I_1 \le C\left(\frac{\rho}{R}\right)^{(\lambda+n)/2}\int_{\mathcal{B}_R^+}(1+g(\mathbf{x},|\nabla\mathbf{u}|))\,\mathrm{d}\mathbf{x}.$$

We now consider I_2. By the convexity of g, the integrand in I_2 is nonnegative, so we can expand the domain of integration to $\mathcal{B}_R^+$ and use Lemma 2.2 to arrive at

$$I_2 \le \int_{\mathcal{B}_R^+}\{g(\mathbf{x},|\nabla\mathbf{u}\mathbf{G}_0|) - g(\mathbf{x},|\nabla\mathbf{v}\mathbf{G}_0|)\}\,\mathrm{d}\mathbf{x}.$$

Since $\mathbf{u}$ is a $(K, \{\gamma_\varepsilon\}, \{\nu_\varepsilon\})$-minimizer, we have

$$\begin{aligned}
I_2 \leq & \int_{\mathcal{B}_R^+} \{g(\mathbf{x}, |\nabla \mathbf{u} \mathbf{G}_0|) - g(\mathbf{x}, |[\nabla \mathbf{u} + \mathbf{A}]\mathbf{G}|)\} \mathrm{d}\mathbf{x} \\
& + \int_{\mathcal{B}_R^+} \{g(\mathbf{x}, |[\nabla \mathbf{v} + \mathbf{A}]\mathbf{G}|) - g(\mathbf{x}, |\nabla \mathbf{v} \mathbf{G}_0|)\} \mathrm{d}\mathbf{x} \\
& + (\gamma_\varepsilon(R) + \varepsilon) \int_{\mathcal{B}_R^+} \{g(\mathbf{x}, |\nabla \mathbf{u}|) + g(\mathbf{x}, |\nabla \mathbf{v}|)\} \mathrm{d}\mathbf{x} + \int_{\mathcal{B}_R^+} |\nu_\varepsilon| \, \mathrm{d}\mathbf{x} \\
\leq & \; I_{2,1} + I_{2,2} + I_{2,3} + R^\lambda \|\nu_\varepsilon\|_{L^{1,\lambda}}.
\end{aligned} \tag{2.75}$$

First we estimate $I_{2,1}$ using part (vi) of Lemma 2.1.

$$\begin{aligned}
I_{2,1} = & \int_{\mathcal{B}_R^+} \int_0^1 \frac{\partial}{\partial \mathbf{F}} g(\mathbf{x}, \nabla \mathbf{u} \mathbf{G}_0 + t([\nabla \mathbf{u} + \mathbf{A}]\mathbf{G} - \nabla \mathbf{u} \mathbf{G}_0)) \cdot (\nabla \mathbf{u}[\mathbf{G} - \mathbf{G}_0] + \mathbf{A}\mathbf{G}) \mathrm{d}t \mathrm{d}\mathbf{x} \\
\leq & \; C\mu(R) \int_{\mathcal{B}_R^+} g_t(\mathbf{x}, |\nabla \mathbf{u}| + |\mathbf{A}|) \, |\nabla \mathbf{u}| \, \mathrm{d}\mathbf{x} + C\varepsilon \int_{\mathcal{B}_R^+} g_t(\mathbf{x}, |\nabla \mathbf{u}| + |\mathbf{A}|) \frac{|\mathbf{A}|}{\varepsilon} \mathrm{d}\mathbf{x} \\
\leq & \; C(\varepsilon + \mu(R)) \int_{\mathcal{B}_R^+} g(\mathbf{x}, |\nabla \mathbf{u}|) + C_\varepsilon \int_{\mathcal{B}_R^+} g(\mathbf{x}, |\mathbf{A}|) \mathrm{d}\mathbf{x} \\
\leq & \; C(\varepsilon + \mu(R)) \int_{\mathcal{B}_R^+} g(\mathbf{x}, |\nabla \mathbf{u}|) + C_\varepsilon R^\lambda \|g(\cdot, |\mathbf{A}|)\|_{L^{1,\lambda}}.
\end{aligned}$$

Because $\mathbf{v}$ is a minimizer for J, a similar computation yields

$$I_{2,2} \leq C(\varepsilon + \mu(R)) \int_{\mathcal{B}_R^+} g(\mathbf{x}, |\nabla \mathbf{u}|) \mathrm{d}\mathbf{x} + C_\varepsilon R^\lambda \|g(\cdot, |\mathbf{A}|)\|_{L^{1,\lambda}}.$$

For $I_{2,3}$, we again use the fact that $\mathbf{v}$ is a minimizer for J to conclude that

$$I_{2,3} \leq C(\gamma_\varepsilon(R) + \varepsilon) \int_{\mathcal{B}^+(\mathbf{x}_0, R)} g(\mathbf{x}, |\nabla \mathbf{u}|) \mathrm{d}\mathbf{x}.$$

Collecting our estimates for $I_{2,1}$, $I_{2,2}$, and $I_{2,3}$, by (2.75) we have

$$I_2 \leq C(\varepsilon + \mu(R) + \gamma_\varepsilon(R)) \int_{\mathcal{B}_R^+} g(\mathbf{x}, |\nabla \mathbf{u}|)\mathrm{d}\mathbf{x} + (C_\varepsilon \left\| g(\cdot, |\mathbf{A}|) \right\|_{L^{1,\lambda}} + \left\| \nu_\varepsilon \right\|_{L^{1,\lambda}}) R^\lambda.$$

Using (vii) from Lemma 2.1 as we did in the proof of Theorem 2.10, we see that $I_3 \leq C(I_1 + I_2)$, and so our estimates for I_1 and I_2, along with (2.74), give

$$\begin{aligned} \int_{\mathcal{B}^+(\mathbf{x}_0,\rho)} g(\mathbf{x}, |\nabla \mathbf{u}|)\mathrm{d}\mathbf{x} \leq C \left(\left(\frac{\rho}{R}\right)^{\frac{n+\lambda}{2}} + \varepsilon + \mu(R) + \gamma_\varepsilon(R) \right) \int_{\mathcal{B}^+(\mathbf{x}_0,R)} g(\mathbf{x}, |\nabla \mathbf{u}|)\mathrm{d}\mathbf{x} \\ + (C_\varepsilon \left\| g(\cdot, |\mathbf{A}|) \right\|_{L^{1,\lambda}} + \left\| \nu_\varepsilon \right\|_{L^{1,\lambda}} + 1) R^\lambda \end{aligned} \tag{2.76}$$

Let $\varepsilon_0 = (2C)^{-\frac{2(n+\lambda)}{n-\lambda}}/2$. Find $0 < R^* < 1$ such that $\mu(R^*) < \varepsilon_0/4$ and $\gamma_{\varepsilon_0/2}(R^*) < \varepsilon_0/4$. Let $R_0 = \min\{R^*, 1 - |\mathbf{x}_0|, R_{(n+\lambda)/2}\}$. Then for $0 < \rho \leq R \leq R_0$, putting $\varepsilon = \varepsilon_0/2$ in (2.76) we have

$$\begin{aligned} \int_{\mathcal{B}_\rho^+} g(\mathbf{x}, |\nabla \mathbf{u}|)\mathrm{d}\mathbf{x} \leq C \left(\left(\frac{\rho}{R}\right)^{\frac{n+\lambda}{2}} + \varepsilon_0 \right) \int_{\mathcal{B}_R^+} g(\mathbf{x}, |\nabla \mathbf{u}|)\mathrm{d}\mathbf{x} \\ + (C_{\frac{\varepsilon_0}{2}} \left\| g(\cdot, |\mathbf{A}|) \right\|_{L^{1,\lambda}} + \left\| \nu_{\frac{\varepsilon_0}{2}} \right\|_{L^{1,\lambda}} + 1) R^\lambda \end{aligned} \tag{2.77}$$

By Lemma 2.4, we have that

$$\begin{aligned} \int_{\mathcal{B}_\rho^+} g(\mathbf{x}, |\nabla \mathbf{u}|)\mathrm{d}\mathbf{x} &\leq C \left(\frac{\rho}{R}\right)^\lambda \left[\int_{\mathcal{B}_R^+} g(\mathbf{x}, |\nabla \mathbf{u}|)\mathrm{d}\mathbf{x} + 1 \right] \\ &\leq C \left(\frac{\rho}{R}\right)^\lambda \left[\int_{\mathcal{B}^+} g(\mathbf{x}, |\nabla \mathbf{u}|)\mathrm{d}\mathbf{x} + 1 \right] \end{aligned} \tag{2.78}$$

whenever $0 < \rho \leq R \leq R_0$. Now if $\mathcal{U} \subset\subset \mathcal{B} \cap \overline{\mathcal{H}^+}$, letting $d = \mathrm{dist}(\mathcal{U}; \partial\mathcal{B})$ and

$c(\mathcal{U}) = C/\min\{R_0^\lambda, d^\lambda\}$, we have from (2.78) that

$$\rho^{-\lambda}\int_{\mathcal{B}^+(\mathbf{x}_0,\rho)} g(\mathbf{x}, |\nabla\mathbf{u}|)\mathrm{d}\mathbf{x} \le c(\mathcal{U})\left[\int_{\mathcal{B}^+} g(\mathbf{x}, |\nabla\mathbf{u}|)\mathrm{d}\mathbf{x} + 1\right]$$

for all $\rho < \min\{R_0, d\}$ and $\mathbf{x}_0 \in \mathcal{U}$, and hence $g(\cdot, |\nabla\mathbf{u}|) \in L^{1,\lambda}_{\mathrm{loc}}(\mathcal{B}^+ \cap \overline{\mathcal{H}^+}; \mathbb{R}^{N\times n})$, which completes the proof. □

Using Theorem 2.10 instead of Theorem 2.11, we can demonstrate the following lemma in the same way that we proved Lemma 2.8.

Lemma 2.9. *Suppose that* $\alpha : \mathcal{B} \to [1, \infty)$ *satisfies* (1.12)-(1.14) *and that* $g : \mathcal{B} \times \mathbb{R}_+ \to \mathbb{R}_+$ *satisfies* (1.15)-(1.17). *Assume* $0 \le \lambda < n$. *Let* $\mathcal{A} := W^{1,1}(\mathcal{B}; \mathbb{R}^N)$, *and define the functional* $K : \mathcal{A} \to \mathbb{R}^*$ *by*

$$K(\mathbf{w}) := \int_{\mathcal{B}} g(\mathbf{x}, |[\nabla\mathbf{w} + \mathbf{A}]\mathbf{G}|)\mathrm{d}\mathbf{x},$$

where $g(\cdot, |\mathbf{A}|) \in L^{1,\lambda}(\mathcal{B})$ *and* $\mathbf{G} \in \mathcal{C}(\overline{\mathcal{B}}; \mathbb{R}^{n\times n})$ *has continuous matrix inverse* $\mathbf{G}^{-1} \in \mathcal{C}(\overline{\mathcal{B}^+}; \mathbb{R}^{n\times n})$. *If* $\mathbf{u} \in \mathcal{A}$ *and there are functions* $\{\nu_\varepsilon\}_{\varepsilon>0} \subset L^{1,\lambda}(\mathcal{B}^+)$ *and* $\{\gamma_\varepsilon\}_{\varepsilon>0} \subset \mathcal{C}(\mathbb{R}_+)$ *satisfying* $\gamma_\varepsilon(0) = 0$ *such that* $\mathbf{u}$ *is a* $(K, \{\gamma_\varepsilon\}, \{\nu_\varepsilon\})$*-minimizer over* $\mathcal{A}$, *then* $g(\cdot, |\nabla\mathbf{u}|) \in L^{1,\lambda}_{\mathrm{loc}}(\mathcal{B})$.

Using Lemmas 2.8 and 2.9, we prove the following result.

Theorem 2.12. *Suppose that* $\Omega \subset \mathbb{R}^n$ *is open and bounded with* $\mathcal{C}^1$ *boundary, and that* $\alpha : \Omega \to [1, \infty)$ *satisfies* (1.12)-(1.14) *and* $g : \Omega \times \mathbb{R}_+ \to \mathbb{R}_+$ *satisfies* (1.15)-(1.17). *Let* $\overline{\mathbf{u}} \in W^1L_g(\Omega; \mathbb{R}^N)$ *with* $g(\cdot, |\nabla\mathbf{u}|) \in L^{1,\lambda}(\Omega)$ *be given, and define*

$$\mathcal{A} := \left\{\mathbf{w} \in W^{1,1}(\Omega; \mathbb{R}^N) : \mathbf{w} - \overline{\mathbf{u}} \in W^{1,1}_0(\Omega; \mathbb{R}^N)\right\}.$$

Define the functional $J : \mathcal{A} \to \mathbb{R}^*$ *by*

$$J(\mathbf{w}) := \int_\Omega g(\mathbf{x}, |\nabla \mathbf{w}(\mathbf{x})|)\mathrm{d}\mathbf{x}.$$

Let $\mathbf{u} \in \mathcal{A}$ *be given. If there are functions* $\{\nu_\varepsilon\}_{\varepsilon>0} \subset L^{1,\lambda}(\Omega)$ *and nondecreasing functions* $\{\gamma_\varepsilon\}_{\varepsilon>0} \subset \mathcal{C}(\mathbb{R}_+)$ *with* $\gamma_\varepsilon(0) = 0$ *such that* $\mathbf{u}$ *is a* $(J, \{\gamma_\varepsilon\}, \{\nu_\varepsilon\})$*-minimizer over* $\mathcal{A}$*, then* $g(\cdot, |\nabla \mathbf{u}|) \in L^{1,\lambda}(\Omega)$.

Proof. We use a standard argument to incorporate the boundary values into the functional and straighten out the boundary, and then use a covering argument along with Lemmas 2.8 and 2.9 to obtain that $g(\cdot, |\nabla \mathbf{u}|) \in L^{1,\lambda}(\Omega)$. □

2.4 Results for Asymptotically Convex Functionals

As mentioned in the introduction, this section is devoted to extending the results of the previous section to almost minimizers of functionals of the form (1.6), where for each $\mathbf{x}$ and $\mathbf{u}$, the function $\mathbf{F} \mapsto f(\mathbf{x}, \mathbf{u}, \mathbf{F})$ looks like $\mathbf{F} \mapsto g(\mathbf{x}, |\mathbf{F}|)$ when $|\mathbf{F}|$ is large. Define the functional J by

$$J(\mathbf{w}) := \int_\Omega g(\mathbf{x}, |\nabla \mathbf{w}|)\mathrm{d}\mathbf{x}. \tag{2.79}$$

The following lemma establishes that almost minimizers for K will also be almost minimizers for J. In this lemma and in the sequel, we denote by p^* the Sobolev-conjugate of p; i.e., if $p < n$, then we put $p^* = np/(n-p)$, and if $p \geq n$, we set $p^* = +\infty$.

Lemma 2.10. *Suppose that $\Omega \subset \mathbb{R}^n$ is open and bounded with $\mathcal{C}^1$ boundary, and suppose also that $\alpha : \Omega \to [1, \infty)$ satisfies (1.12)-(1.14) and $g : \Omega \times \mathbb{R}_+ \to \mathbb{R}_+$ satisfies (1.15)-(1.17). Let $f : \Omega \times \mathbb{R}^N \times \mathbb{R}^{N\times n} \to \mathbb{R}$, and assume the following hypotheses hold for some $0 \leq \lambda < n$ and $1 < s < \min\{r_2, 1 + pr_2/n, p^*/p\}$, where r_2 is as in Remark 2.2.*

(i) *For every $\varepsilon > 0$, there is a function $\sigma_\varepsilon \in L^{1,\lambda}(\Omega)$ and a constant $\Sigma_\varepsilon < \infty$ such that*

$$|f(\mathbf{x}, \mathbf{u}, \mathbf{F}) - g(\mathbf{x}, |\mathbf{F}|)| < \varepsilon g(\mathbf{x}, |\mathbf{F}|)$$

for all $(\mathbf{x}, \mathbf{u}, \mathbf{F}) \in \Omega \times \mathbb{R}^N \times \mathbb{R}^{N\times n}$ satisfying $g(\mathbf{x}, |\mathbf{F}|) \geq \sigma_\varepsilon(\mathbf{x}) + \Sigma_\varepsilon g(\mathbf{x}, |\mathbf{u}|)^s$.

(ii) *There is some $\beta \in L^{1,\lambda}(\Omega)$ such that*

$$|f(\mathbf{x}, \mathbf{u}, \mathbf{F})| \leq C(\beta(\mathbf{x}) + g(\mathbf{x}, |\mathbf{u}|)^s + g(\mathbf{x}, |\mathbf{F}|))$$

for all $(\mathbf{x}, \mathbf{u}, \mathbf{F}) \in \Omega \times \mathbb{R}^N \times \mathbb{R}^{N\times n}$.

For a fixed $\overline{\mathbf{u}} \in W^1L_g(\Omega; \mathbb{R}^N)$ with $g(\cdot, |\nabla\overline{\mathbf{u}}|) \in L^{1,\lambda}(\Omega)$, define the admissible class

$$\mathcal{A} := \left\{\mathbf{u} \in W^1L_g(\Omega; \mathbb{R}^N) : \mathbf{u} - \overline{\mathbf{u}} \in W^1_0L_g(\Omega; \mathbb{R}^N)\right\}.$$

Let the functionals J and K, each mapping $\mathcal{A}$ into $\mathbb{R}$, be defined by (2.79) and (1.6), respectively. Let $\mathbf{u} \in \mathcal{A}$, and suppose that there are functions $\{\nu_\varepsilon\}_{\varepsilon>0} \subset L^{1,\lambda}(\Omega)$ and nondecreasing functions $\{\gamma_\varepsilon\}_{\varepsilon>0} \subset \mathcal{C}(\mathbb{R}_+)$ satisfying $\gamma_\varepsilon(0) = 0$, along with constants $\{T_\varepsilon\}_{\varepsilon>0} \subset \mathbb{R}_+$ such that

$$\begin{aligned} K(\mathbf{u}) \leq K(\mathbf{v}) + (\gamma_\varepsilon(\rho) + \varepsilon) &\int_{\Omega(\mathbf{x}_0,\rho)} \{\nu_\varepsilon(\mathbf{x}) + g(\mathbf{x}, |\nabla\mathbf{u}|) + g(\mathbf{x}, |\nabla\mathbf{v}|)\}\,\mathrm{d}\mathbf{x} \\ &+ T_\varepsilon \int_{\Omega(\mathbf{x}_0,\rho)} \{g(\mathbf{x}, |\mathbf{u}|)^s + g(\mathbf{x}, |\mathbf{v}|)^s\}\,\mathrm{d}\mathbf{x} \end{aligned} \tag{2.80}$$

for all $\mathbf{v} \in \mathcal{A}$ *with* $\mathbf{u} - \mathbf{v} \in W_0^{1,1}(\Omega(\mathbf{x}_0, \rho); \mathbb{R}^N)$. *Then there are functions* $\{\tilde{\nu}_\varepsilon\}_{\varepsilon>0} \subset L^{1,\lambda}(\Omega)$ *and nondecreasing functions* $\{\tilde{\gamma}_\varepsilon\}_{\varepsilon>0} \subset C(\mathbb{R}_+)$ *with* $\tilde{\gamma}_\varepsilon(0) = 0$, *as well as constants* $\{\tilde{T}_\varepsilon\}_{\varepsilon>0} \subset \mathbb{R}_+$, *such that* $\mathbf{u}$ *is a* $(J, \{\tilde{\gamma}_\varepsilon\}, \{\tilde{\nu}_\varepsilon + \tilde{T}_\varepsilon g(\cdot, |\mathbf{u}|)^s\})$*-minimizer.*

Proof. It suffices to show that

$$J(\mathbf{u}) \leq J(\mathbf{v}) + (\tilde{\gamma}_\varepsilon(\rho) + \varepsilon) \int_{\Omega(\mathbf{x}_0,\rho)} \left\{ \tilde{\nu}_\varepsilon(\mathbf{x}) + \tilde{T}_\varepsilon g(\mathbf{x}, |\mathbf{u}|)^s + g(\mathbf{x}, |\nabla \mathbf{u}|) \right\} \mathrm{d}\mathbf{x} \tag{2.81}$$

for all $\mathbf{v} \in \mathcal{A}$ such that $\mathbf{u} - \mathbf{v} \in W_0^1 L_g(\Omega(\mathbf{x}_0, \rho); \mathbb{R}^N)$. To this end, we let $\mathbf{w} \in W^1 L_g(\Omega(\mathbf{x}_0, \rho); \mathbb{R}^N)$ be the minimizer of the functional $J_{\mathbf{x}_0,\rho}$ defined by

$$J_{\mathbf{x}_0,\rho}(\mathbf{v}) = \int_{\Omega(\mathbf{x}_0,\rho)} g(\mathbf{x}, |\nabla \mathbf{v}|) \mathrm{d}\mathbf{x},$$

satisfying $\mathbf{w} - \mathbf{u} \in W_0^1 L_g(\Omega(\mathbf{x}_0, \rho); \mathbb{R}^N)$. Then for any $\mathbf{v} \in \mathcal{A}$ satisfying $\mathbf{u} - \mathbf{v} \in W_0^1 L_g(\Omega(\mathbf{x}_0, \rho); \mathbb{R}^N)$, we have by the minimality of $\mathbf{w}$ that

$$\begin{aligned} J(\mathbf{u}) - J(\mathbf{v}) &\leq J_{\mathbf{x}_0,\rho}(\mathbf{u}) - J_{\mathbf{x}_0,\rho}(\mathbf{w}) \\ &= \int_{\Omega(\mathbf{x}_0,\rho)} \{g(\mathbf{x}, |\nabla \mathbf{u}|) - f(\mathbf{x}, \mathbf{u}, \nabla \mathbf{u})\} \, \mathrm{d}\mathbf{x} \\ &\quad + \int_{\Omega(\mathbf{x}_0,\rho)} \{f(\mathbf{x}, \mathbf{u}, \nabla \mathbf{u}) - f(\mathbf{x}, \mathbf{w}, \nabla \mathbf{w})\} \, \mathrm{d}\mathbf{x} \\ &\quad + \int_{\Omega(\mathbf{x}_0,\rho)} \{f(\mathbf{x}, \mathbf{w}, \nabla \mathbf{w}) - g(\mathbf{x}, |\nabla \mathbf{w}|)\} \, \mathrm{d}\mathbf{x} \\ &= I_1 + I_2 + I_3. \end{aligned} \tag{2.82}$$

To estimate I_1, we partition $\Omega(\mathbf{x}_0, \rho)$ into the set on which $g(\mathbf{x}, |\nabla \mathbf{u}|) \leq \sigma_\varepsilon(\mathbf{x}) + \Sigma_\varepsilon g(\mathbf{x}, |\mathbf{u}|)^s$ (call this set $\mathcal{S}$), and the set on which the opposite inequality holds (call this set $\mathcal{T}$). By the growth conditions on f and g, we have

$$\begin{aligned}\int_{\mathcal{S}} \{g(\mathbf{x}, |\nabla \mathbf{u}|) - f(\mathbf{x}, \mathbf{u}, \nabla \mathbf{u})\}\, \mathrm{d}\mathbf{x} &\leq C \int_{\mathcal{S}} \{1 + \beta + g(\mathbf{x}, |\mathbf{u}|)^s + g(\mathbf{x}, |\nabla \mathbf{u}|)\}\, \mathrm{d}\mathbf{x} \\ &\leq C \int_{\Omega(\mathbf{x}_0,\rho)} \{1 + \beta + \sigma_\varepsilon + (1 + \Sigma_\varepsilon) g(\mathbf{x}, |\mathbf{u}|)^s\}\, \mathrm{d}\mathbf{x}.\end{aligned}$$

To estimate the integral over $\mathcal{T}$, we use the assumption in (i) to conclude that

$$\int_{\mathcal{T}} \{g(\mathbf{x}, |\nabla \mathbf{u}|) - f(\mathbf{x}, \mathbf{u}, \nabla \mathbf{u})\}\, \mathrm{d}\mathbf{x} \leq \varepsilon \int_{\Omega(\mathbf{x}_0,\rho)} g(\mathbf{x}, |\nabla \mathbf{u}|) \mathrm{d}\mathbf{x}.$$

Combining the estimates for the integrals over $\mathcal{S}$ and $\mathcal{T}$ yields

$$I_1 \leq C \int_{\Omega(\mathbf{x}_0,\rho)} \{1 + \alpha + \sigma_\varepsilon + (1 + \Sigma_\varepsilon) g(\mathbf{x}, |\mathbf{u}|)^s + \varepsilon g(\mathbf{x}, |\nabla \mathbf{u}|)\}\, \mathrm{d}\mathbf{x}.$$

We estimate I_3 in a similar fashion, keeping in mind the minimality of $\mathbf{w}$ for $J_{\mathbf{x}_0,\rho}$, to obtain

$$I_3 \leq C \int_{\Omega(\mathbf{x}_0,\rho)} \{1 + \alpha + \sigma_\varepsilon + (1 + \Sigma_\varepsilon) g(\mathbf{x}, |\mathbf{w}|)^s + \varepsilon g(\mathbf{x}, |\nabla \mathbf{u}|)\}\, \mathrm{d}\mathbf{x}. \tag{2.83}$$

We have

$$\int_{\Omega(\mathbf{x}_0,\rho)} g(\mathbf{x}, |\mathbf{w}|)^s \mathrm{d}\mathbf{x} \leq C \rho^{n+ps} \fint_{\Omega(\mathbf{x}_0,\rho)} g\left(\mathbf{x}, \frac{|\mathbf{u} - \mathbf{w}|}{\rho}\right)^s \mathrm{d}\mathbf{x} + C \int_{\Omega(\mathbf{x}_0,\rho)} g(\mathbf{x}, |\mathbf{u}|)^s \mathrm{d}\mathbf{x} \tag{2.84}$$

By Remark 2.2, the minimality of $\mathbf{w}$ for $J_{\mathbf{x}_0,\rho}$, and (2.84), there is a constant C, which does not depend on $\mathbf{w}$, $\mathbf{x}_0$, or ρ, such that

$$\begin{aligned}\int_{\Omega(\mathbf{x}_0,\rho)} g(\mathbf{x},|\mathbf{w}|)^s \mathrm{d}\mathbf{x} \leq C\rho^{n+ps} &\left(⨍_{\Omega(\mathbf{x}_0,\rho)} \{1+g\left(\mathbf{x},|\nabla\mathbf{u}-\nabla\mathbf{w}|\right)\} \mathrm{d}\mathbf{x}\right)^s \\ &+ C\int_{\Omega(\mathbf{x}_0,\rho)} g(\mathbf{x},|\mathbf{u}|)^s \mathrm{d}\mathbf{x} \\ \leq C\rho^{n+ps-ns} &\left(\int_{\Omega(\mathbf{x}_0,\rho)} \{1+g\left(\mathbf{x},|\nabla\mathbf{u}|\right)\} \mathrm{d}\mathbf{x}\right)^s \\ &+ C\int_{\Omega(\mathbf{x}_0,\rho)} g(\mathbf{x},|\mathbf{u}|)^s \mathrm{d}\mathbf{x}.\end{aligned} \tag{2.85}$$

Define $\Delta : \mathbb{R}_+ \to \mathbb{R}_+$ by

$$\Delta(r) := r^{n+ps-ns} \sup_{\mathbf{y}\in\Omega} \left(\int_{\Omega(\mathbf{y},r)} \{1+g(\mathbf{x},|\nabla\mathbf{u}|)\} \mathrm{d}\mathbf{x}\right)^{s-1}.$$

Note that the exponent on r is positive, since we have assumed that $1 < s < p^*/p$, so we have that Δ is continuous with $\Delta(0) = 0$. With this notation in place and the estimates in (2.83) and (2.85), we now have that

$$I_3 \leq C(\varepsilon + \Sigma_\varepsilon \Delta(\rho)) \int_{\Omega(\mathbf{x}_0,\rho)} \{1+g(\mathbf{x},|\nabla\mathbf{u}|)\} \mathrm{d}\mathbf{x} + C\int_{\Omega(\mathbf{x}_0,\rho)} \{1+\alpha+\sigma_\varepsilon\} \mathrm{d}\mathbf{x}.$$

Finally, to estimate I_2, we use the fact that $\mathbf{u}$ satisfies (2.80) to get

$$\begin{aligned}I_2 \leq (\varepsilon + \gamma_\varepsilon(\rho)) &\int_{\Omega(\mathbf{x}_0,\rho)} \{\nu_\varepsilon + g(\mathbf{x},|\nabla\mathbf{u}|) + g(\mathbf{x},|\nabla\mathbf{w}|)\} \mathrm{d}\mathbf{x} \\ + T_\varepsilon &\int_{\Omega(\mathbf{x}_0,\rho)} \{g(\mathbf{x},|\mathbf{u}|)^s + g(\mathbf{x},|\mathbf{w}|)^s\} \mathrm{d}\mathbf{x}.\end{aligned}$$

Using (2.85) and the definition of Δ, along with the minimality of $\mathbf{w}$ for $J_{\mathbf{x}_0,\rho}$, we have

$$I_2 \leq C\left(\varepsilon + \gamma_\varepsilon(\rho) + T_\varepsilon\Delta(\rho)\right) \int_{\Omega(\mathbf{x}_0,\rho)} \left\{\nu_\varepsilon(\mathbf{x}) + 1 + \varepsilon^{-1}T_\varepsilon g(\mathbf{x},|\mathbf{u}|)^s + g(\mathbf{x},|\nabla\mathbf{u}|)\right\} \mathrm{d}\mathbf{x}.$$

Inserting our estimates for I_1, I_2, and I_3 into (2.82), we see that (2.81) holds with $\tilde{\nu}_\varepsilon$, $\tilde{\gamma}_\varepsilon$, and $\tilde{T}_\varepsilon$ defined by

$$\begin{aligned}
\tilde{\nu}_\varepsilon &:= C(\varepsilon/C)^{-1}\left(1+\frac{\varepsilon}{C}+\alpha+\sigma_\varepsilon+\frac{\varepsilon}{C}\nu_{\varepsilon/C}\right), \\
\tilde{\gamma}_\varepsilon &:= C\left\{\gamma_{\varepsilon/C}+(T_{\varepsilon/C}+\Sigma_{\varepsilon/C}+1)\Delta\right\}, \\
\tilde{T}_\varepsilon &:= C(\varepsilon/C)^{-1}\left(1+\Sigma_{\varepsilon/C}+T_{\varepsilon/C}\right).
\end{aligned}$$

Note that clearly $\{\tilde{\nu}_\varepsilon\}_{\varepsilon>0} \subset L^{1,\lambda}(\Omega)$, and $\{\tilde{\gamma}_\varepsilon\}_{\varepsilon>0} \subset \mathcal{C}(\mathbb{R}_+)$ satisfies $\tilde{\gamma}_\varepsilon = 0$ for each $\varepsilon > 0$. Furthermore, it is manifest that $\{\tilde{T}_\varepsilon\}_{\varepsilon>0} \subset \mathbb{R}_+$, so the lemma is proved. □

We are now in a position to prove the main theorem.

Theorem 2.13. *Suppose that* $\Omega \subset \mathbb{R}^n$ *is open and bounded with* $\mathcal{C}^1$ *boundary, and that* $\alpha : \Omega \to [1,\infty)$ *satisfies* (1.12) *and* $g : \Omega \times \mathbb{R}_+ \to \mathbb{R}_+$ *satisfies* (1.15)-(1.17). *Let* $f : \Omega \times \mathbb{R}^N \times \mathbb{R}^{N\times n} \to \mathbb{R}$ *satisfy the following hypotheses for some* $0 \le \lambda < n$ *and* $1 < s < \min\{r_2, 1 + pr_2/n, p^*/p\}$, *where* $r_2 > 1$ *is as in Remark 2.2.*

(i) *For every* $\varepsilon > 0$, *there is a function* $\sigma_\varepsilon \in L^{1,\lambda}(\Omega)$ *and a constant* $\Sigma_\varepsilon < \infty$ *such that*

$$|f(\mathbf{x},\mathbf{u},\mathbf{F}) - g(\mathbf{x},|\mathbf{F}|)| < \varepsilon g(\mathbf{x},|\mathbf{F}|)$$

for all $(\mathbf{x},\mathbf{u},\mathbf{F}) \in \Omega \times \mathbb{R}^N \times \mathbb{R}^{N\times n}$ *satisfying* $g(\mathbf{x},|\mathbf{F}|) \ge \sigma_\varepsilon(\mathbf{x}) + \Sigma_\varepsilon g(\mathbf{x},|\mathbf{u}|)^s$;

(ii) *There is some* $\beta \in L^{1,\lambda}(\Omega)$ *such that*

$$|f(\mathbf{x},\mathbf{u},\mathbf{F})| \le C(\beta(\mathbf{x}) + g(\mathbf{x},|\mathbf{u}|)^s + g(\mathbf{x},|\mathbf{F}|))$$

for all $(\mathbf{x},\mathbf{u},\mathbf{F}) \in \Omega \times \mathbb{R}^N \times \mathbb{R}^{N\times n}$.

For a fixed $\overline{\mathbf{u}} \in W^1L_g(\Omega;\mathbb{R}^N)$ *with* $g(\cdot,|\nabla\overline{\mathbf{u}}|) \in L^{1,\lambda}(\Omega)$*, define the admissible class*

$$\mathcal{A} := \left\{\mathbf{u} \in W^1L_g(\Omega;\mathbb{R}^N) : \mathbf{u} - \overline{\mathbf{u}} \in W_0^1L_g(\Omega;\mathbb{R}^N)\right\}.$$

Let the functional $K : \mathcal{A} \to \mathbb{R}$ *be as defined in* (1.6)*. If* $\mathbf{u} \in \mathcal{A}$ *and there are functions* $\{\nu_\varepsilon\}_{\varepsilon>0} \subset L^{1,\lambda}(\Omega)$ *and nondecreasing functions* $\{\gamma_\varepsilon\}_{\varepsilon>0} \subset \mathcal{C}(\mathbb{R}_+)$ *with* $\gamma_\varepsilon(0) = 0$ *such that* $\mathbf{u}$ *is a* $(K,\{\gamma_\varepsilon\},\{\nu_\varepsilon\})$*-minimizer over* $\mathcal{A}$*, then* $g(\cdot,|\nabla\mathbf{u}|) \in L^{1,\lambda}(\Omega)$.

Before we prove this theorem, we make a few remarks for later convenience.

Remark 2.3. If $K(\mathbf{v}) = +\infty$ for any function $\mathbf{v} \in W^{1,1}(\Omega;\mathbb{R}^N)$ with $g(\cdot,|\nabla\mathbf{v}|) \notin L^1(\Omega;\mathbb{R}^{N\times n})$, then clearly we can enlarge the admissible class $\mathcal{A}$ to

$$\mathcal{A}' := \left\{\mathbf{w} \in W^{1,1}(\Omega;\mathbb{R}^N) : \mathbf{w} - \overline{\mathbf{u}} \in W_0^{1,1}(\Omega;\mathbb{R}^N)\right\},$$

and the same result holds.

Remark 2.4. Examining the proof of Theorem 2.12 and Lemma 2.10, we see that we do not actually need the inequality in (1.7) to hold for all $\varepsilon > 0$, but only for $\varepsilon \geq \varepsilon_0$, where $\varepsilon_0 > 0$ depends on n, N, p, q, α, $\overline{\mathbf{u}}$, Ω, and f.

Remark 2.5. By analyzing the proofs of Theorem 2.12 and Lemma 2.10, we see that the bound on the Morrey norm $\|g(\cdot,|\nabla\mathbf{u}|)\|_{L^{1,\lambda}}$ stays uniformly bounded if the quantity $L := \int_\Omega g(\mathbf{x},|\nabla\mathbf{u}|)$ stays bounded. That is, if $\{\mathbf{u}_t\}_{t\in\Lambda}$ is a collection of $(K,\{\gamma_\varepsilon\},\{\nu_\varepsilon\})$-minimizers with $\int_\Omega g(\mathbf{x},|\nabla\mathbf{u}_t|)\mathrm{d}\mathbf{x} \leq L$ for some $L < \infty$ and all $t \in \Lambda$, then there is a finite constant $\tilde{L}$ such that $\|g(\cdot,|\nabla\mathbf{u}_t|)\|_{L^{1,\lambda}} \leq \tilde{L}$ for all $t \in \Lambda$.

Proof. Since $\mathbf{u}$ is a $(K,\{\gamma_\varepsilon\},\{\nu_\varepsilon\})$-minimizer over $\mathcal{A}$, by the growth conditions on f and Lemma 2.10, we have that there are functions $\{\tilde{\nu}_\varepsilon\}_{\varepsilon>0} \subset L^{1,\lambda}(\Omega)$ and nondecreasing functions $\{\tilde{\gamma}_\varepsilon\}_{\varepsilon>0} \subset C(\mathbb{R}_+)$ with $\tilde{\gamma}_\varepsilon(0) = 0$, as well as constants $\{\tilde{T}_\varepsilon\}_{\varepsilon>0} \subset \mathbb{R}_+$, such that $\mathbf{u}$ is a $(J, \{\tilde{\gamma}_\varepsilon\}, \{\tilde{\nu}_\varepsilon + \tilde{T}_\varepsilon g(\cdot, |\mathbf{u}|)^s\})$-minimizer. In view of Theorem 2.12, it therefore suffices to prove that $\mu_\varepsilon := \tilde{\nu}_\varepsilon + \tilde{T}_\varepsilon g(\cdot, |\mathbf{u}|)^s \in L^{1,\lambda}(\Omega)$. By hypothesis, we have that $\mathbf{u} \in W^1 L_g(\Omega; \mathbb{R}^N)$. Therefore, $g(\cdot, |\mathbf{u}|)^s \in L^{r_2/s}(\Omega)$ by Remark 2.2, and hence we see by Hölder's inequality that $g(\cdot, |\mathbf{u}|)^s \in L^{1,n-ns/r_2}(\Omega)$. Therefore, letting $\lambda_1 := n\,(1 - s/r_2)$, we have that $\mu_\varepsilon \in \min\{\lambda_1, \lambda\}$, and hence Theorem 2.12 implies that $g(\cdot, |\nabla \mathbf{u}|) \in L^{1,\min\{\lambda,\lambda_1\}}(\Omega)$. If $\lambda_1 \geq \lambda$, the proof is therefore complete.

So suppose that $\lambda_1 < \lambda$. Since $g(\cdot, |\nabla \mathbf{u}|) \in L^{1,\lambda_1}(\Omega)$, we can use Theorem 2.6 to conclude that $g(\cdot, |\mathbf{u}|)^s \in L^{1,\kappa}(\Omega)$ for every $\kappa < \min\{n + s(p + \lambda_1 - n), n\}$. Hence, if $n + s(p + \lambda_1 - n) > \lambda$, then $g(\cdot, |\mathbf{u}|)^s \in L^{1,\lambda}(\Omega)$, whence $\mu_\varepsilon \in L^{1,\lambda}(\Omega)$, and the proof is finished. If $n + s(p + \lambda_1 - n) \leq \lambda$, set $\lambda_2 = n + s(p + \lambda_1 - n)$. Arguing as before, we have that $\mu_\varepsilon \in L^{1,\kappa}(\Omega)$ for every $\kappa < \lambda_2$. Thus Theorem 2.12 implies that $g(\cdot, |\nabla \mathbf{u}|) \in L^{1,\kappa}$ for every $0 \leq \kappa < \lambda_2$. Recursively defining

$$\lambda_{j+1} := n + s(p + \lambda_j - n)$$

and continuing to bootstrap as above, we have that $\{\mu_\varepsilon\}_{\varepsilon>0} \subset L^{1,\kappa}(\Omega)$ for every $\kappa < \lambda_j$ if $\lambda_j \leq \lambda$, and $\{\mu_\varepsilon\}_{\varepsilon>0} \subset L^{1,\lambda}(\Omega)$ if $\lambda_j > \lambda$. We claim that λ_j increases without bound. Indeed, inductively, one can easily show that

$$\lambda_j = n - \frac{ps}{s-1} + \left(\frac{p}{s-1} - \frac{n}{r_2} \right) s^j.$$

Since $1 < s < 1 + pr_2/n$, we have that the coefficient in front of s^j is positive, and since first two terms are constant in j, we see that indeed $\lim_{j\to\infty} \lambda_j = \infty$. Hence, if $j_0 \in \mathbb{N}$ is selected so that $\lambda_{j_0} > \lambda$, then we can use a bootstrap argument as above with j_0 iterations to get that $\{\mu_\varepsilon\}_{\varepsilon>0} \subset L^{1,\lambda}(\Omega)$. Applying Theorem 2.12 one last time gives $g(\cdot, |\nabla \mathbf{u}|) \in L^{1,\lambda}(\Omega)$, and the proof is complete. $\square$

Chapter 3

Young Measures Generated by Sequences Bounded in Morrey and Sobolev-Morrey Spaces

In this chapter, we provide a characterization of Young measures that are generated by a p-equiintegrable sequence $\{\mathbf{f}_j\}_{j=1}^{\infty}$ bounded in the Morrey space $L^{p,\lambda}(\Omega;\mathbb{R}^N)$. By p-equiintegrability, we mean that the sequence $\{|\mathbf{f}_j|^p\}_{j=1}^{\infty}$ is equiintegrable. After first considering the easier case where no additional constraints are placed on the generating sequence $\{\mathbf{f}_j\}$, we then investigate the case that $\{\mathbf{f}_j\}$ is a sequence of weak gradients.

3.1 Some Preliminary Results for Young Measures

We first state a version of the fundamental theorem for Young measures; more general versions of this theorem are available (see [7, 9, 37, 38], for instance), but the one given here suffices for our purposes. We recall that a function $f : E \times \mathbb{R}^N \to \mathbb{R}$ is said to be Carathéodory if f is Borel measurable and $f(\mathbf{x}, \cdot)$ is continuous for almost every $\mathbf{x} \in E$.

Theorem 3.1 (Fundamental Theorem for Young Measures)**.** *Let $E \subset \mathbb{R}^n$ be a measurable set with finite measure, and let $\{\mathbf{z}_j\}_{j=1}^{\infty}$ be a sequence of measurable functions mapping E into $\mathbb{R}^N$ that generates the Young measure $\nu = \{\nu_{\mathbf{x}}\}_{\mathbf{x}\in E}$. Suppose $f : E \times \mathbb{R}^N \to [0, \infty)$ is Carathéodory. Then*

$$\liminf_{j\to\infty} \int_E f(\mathbf{x}, \mathbf{z}_j(\mathbf{x}))\mathrm{d}\mathbf{x} \geq \int_E \int_{\mathbb{R}^N} f(\mathbf{x}, \mathbf{y})\mathrm{d}\nu_{\mathbf{x}}(\mathbf{y})\mathrm{d}\mathbf{x}.$$

Furthermore, if $\{f(\cdot, \mathbf{z}_j(\cdot))\} \subset L^1(E)$, then $\{f(\cdot, \mathbf{z}_j(\cdot))\}$ is equiintegrable if and only if

$$\lim_{j\to\infty} \int_E f(\mathbf{x}, \mathbf{z}_j(\mathbf{x}))\mathrm{d}\mathbf{x} = \int_E \int_{\mathbb{R}^N} f(\mathbf{x}, \mathbf{y})\mathrm{d}\nu_{\mathbf{x}}(\mathbf{y})\mathrm{d}\mathbf{x} < \infty.$$

In this case,

$$f(\cdot, \mathbf{z}_j(\cdot)) \rightharpoonup \int_{\mathbb{R}^N} f(\cdot, \mathbf{y})\mathrm{d}\nu_{(\cdot)}(\mathbf{y}) \text{ in } L^1(E).$$

The following theorem, which is essentially Theorem 4 on page 203 of [45], provides a tool for approximating functions in $W^{1,p}$ by Lipschitz functions.

Theorem 3.2. *Let $\mathbf{u} \in W^{1,p}(\mathbb{R}^n; \mathbb{R}^N)$ with $1 \leq p < \infty$. For $T \geq 0$, define the closed set A_T by*

$$A_T := \{\mathbf{x} \in \mathbb{R}^n : M(|\nabla \mathbf{u}|)(\mathbf{x}) \leq T\},$$

where $M(f)$ denotes the maximal function of f. Then there exists a Lipschitz function $\mathbf{v}_T : \mathbb{R}^n \to \mathbb{R}^N$ such that

(i) *$\mathbf{v}_T(\mathbf{x}) = \mathbf{u}(\mathbf{x})$ and $\nabla \mathbf{v}_T(\mathbf{x}) = \nabla \mathbf{u}(\mathbf{x})$ for almost every $\mathbf{x} \in A_T$;*

(ii) *$\|\nabla \mathbf{v}_T\|_{L^\infty} \leq c(N, n)T$;*

(iii) $m(\mathbb{R}^n \setminus A_T) \leq c(n)T^{-p} \int_{\{|\nabla u|>T/2\}} |\nabla \mathbf{u}|^p \, \mathrm{d}\mathbf{x}$.

The following theorem and its proof can be found in [37].

Theorem 3.3. *Let $E \subset \mathbb{R}^n$ be a Lebesgue measurable set with finite measure and let $\{\mathbf{f}_j\}_{j=1}^\infty$ and $\{\mathbf{g}_j\}$ be sequences of measurable functions mapping E into $\mathbb{R}^N$. If $\{\mathbf{f}_j\}$ generates the Young measure $\nu = \{\nu_\mathbf{x}\}_{\mathbf{x}\in E}$ and $\{\mathbf{g}_j\}$ converges in measure to a measurable function $\mathbf{g} : E \to \mathbb{R}^N$, then $\{\mathbf{f}_j + \mathbf{g}_j\}$ generates the translated Young measure $\tilde{\nu} = \{\tilde{\nu}_\mathbf{x}\}_{\mathbf{x}\in E}$ defined by $\langle \tilde{\nu}_\mathbf{x}, \varphi\rangle := \langle \nu_\mathbf{x}, \varphi(\cdot + \mathbf{g}(\mathbf{x}))\rangle$ for every $\varphi \in \mathcal{C}_0(\mathbb{R}^N)$. In particular, if $\mathbf{g}_j \to 0$ in measure, then $\{\mathbf{f}_j + \mathbf{g}_j\}$ generates ν.*

3.2 The General Case

In this section, we determine which Young measures can be generated by a sequence bounded in $L^{p,\lambda}(\Omega;\mathbb{R}^N)$, for $1 \leq p < \infty$ and $0 \leq \lambda < n$. We first consider the homogeneous case when $\nu_\mathbf{x} \equiv \nu$ for some probability measure ν supported on $\mathbb{R}^N$; using a similar strategy as is found in [51] and [65], we use the homogeneous result to prove the more general theorem in the nonhomogenous case. We would like to point out that the arguments in this section are entirely constructive, though it is possible to give a shorter, nonconstructive argument using some of the same techniques that are utilized in Section 3.3.

3.2.1 Homogeneous Measures

Let $\nu \in \mathcal{M}(\mathbb{R}^N)$ be a probability measure. We construct a sequence of functions $\{\mathbf{f}_j\}_{j=1}^\infty$ mapping Q into $\mathbb{R}^N$ that generates the homogeneous Young measure ν. Furthermore, we demonstrate that this sequence of functions is uniformly bounded in

$L^{p,\lambda}(Q;\mathbb{R}^N)$ for every $0 \leq \lambda < n$ if ν satisifies the condition

$$\int_{\mathbb{R}^N} |\mathbf{y}|^p \, \mathrm{d}\nu(\mathbf{y}) < \infty.$$

We will use the following lemma.

Lemma 3.1. *Suppose that $\nu \in \mathcal{M}(\mathbb{R}^N)$ is a probability measure and that $\{\mathbf{f}_j\}_{j=1}^{\infty}$ is a sequence of measurable functions mapping a measurable set $E \subset \mathbb{R}^n$ into $\mathbb{R}^N$. If the equality*

$$\nu(G)m(D \cap E) = \lim_{j\to\infty} m(D \cap E \cap \mathbf{f}_j^{-1}(G)) \tag{3.1}$$

holds for each Borel set $G \subset \mathbb{R}^N$ and every cube $D \subset \mathbb{R}^n$, then the sequence $\{\mathbf{f}_j\}_{j=1}^{\infty}$ generates ν.

Proof. Recall that the sequence $\{\mathbf{f}_j\}$ generates ν if and only if

$$\lim_{j\to\infty} \int_E \xi(\mathbf{x})\varphi(\mathbf{f}_j(\mathbf{x}))\mathrm{d}\mathbf{x} = \int_E \xi(\mathbf{x})\mathrm{d}\mathbf{x} \int_{\mathbb{R}^N} \varphi(\mathbf{y})\mathrm{d}\nu(\mathbf{y}) \tag{3.2}$$

for every $\varphi \in \mathcal{C}_0(\mathbb{R}^N)$ and $\xi \in L^1(E)$; it actually suffices to show that (3.2) holds for all $\varphi \in \mathcal{S}$ and $\xi \in \mathcal{T}$, where $\mathcal{S}$ and $\mathcal{T}$ are dense subsets of $\mathcal{C}_0(\mathbb{R}^N)$ and $L^1(E)$, respectively. To this end, fix $\varphi \in \mathcal{C}_0(\mathbb{R}^N)$, and suppose that $\xi = \chi_{D\cap E}$ for some cube $D \subset \mathbb{R}^n$. We will show that (3.2) holds; i.e.,

$$\lim_{j\to\infty} \int_{D\cap E} \varphi(\mathbf{f}_j(\mathbf{x}))\mathrm{d}\mathbf{x} = m(D \cap E) \int_{\mathbb{R}^N} \varphi(\mathbf{y})\mathrm{d}\nu(\mathbf{y}).$$

Since $\varphi \in \mathcal{C}_0(\mathbb{R}^N)$, for any $\varepsilon > 0$, we can find a compact $K \subset \mathbb{R}^N$ such that $|\varphi(\mathbf{z})| < \varepsilon$ for all $\mathbf{z} \in \mathbb{R}^N \setminus K$ and a $\delta > 0$ so that $|\varphi(\mathbf{x}) - \varphi(\mathbf{y})| < \varepsilon$ whenever $|\mathbf{x} - \mathbf{y}| < \delta$. We select a finite number of disjoint cubes $\{Q_k\}_{k=1}^{M}$ that cover K and that each have diameter less than δ. Denote by $A_{k,j}$ the sets

$$A_{k,j} := D \cap E \cap \mathbf{f}_j^{-1}(Q_k).$$

Lastly, we choose $\mathbf{a}_k \in Q_k$ for each $1 \leq k \leq M$, and introduce the simple function $\mathbf{g}_j : E \to \mathbb{R}^N$ defined by

$$\mathbf{g}_j := \sum_{k=1}^{M} \varphi(\mathbf{a}_k)\chi_{A_{k,j}}.$$

We begin by adding and subtracting $\mathbf{g}_j$ inside the integral:

$$\begin{aligned}
\limsup_{j\to\infty} &\left| \int_{D\cap E} \varphi(\mathbf{f}_j(\mathbf{x}))\mathrm{d}\mathbf{x} - m(D\cap E)\int_{\mathbb{R}^N} \varphi(\mathbf{y})\mathrm{d}\nu(\mathbf{y}) \right| \\
&\leq \limsup_{j\to\infty} \int_{D\cap E} \left|\varphi(\mathbf{f}_j(\mathbf{x})) - \mathbf{g}_j(\mathbf{x})\right| \mathrm{d}\mathbf{x} \\
&\quad + \limsup_{j\to\infty} \left| \int_{D\cap E} \mathbf{g}_j(\mathbf{x})\mathrm{d}\mathbf{x} - m(D\cap E)\int_{\mathbb{R}^N} \varphi(\mathbf{y})\mathrm{d}\nu(\mathbf{y}) \right|.
\end{aligned}$$

By the way we defined $\mathbf{g}_j$, the integrand in the first integral is bounded by ε for each j. Also, by (3.1), we see that

$$\lim_{j\to\infty} m\left(A_{k,j}\right) = m(D\cap E)\nu(Q_k).$$

Using this and the definition of $\mathbf{g}_j$ in the inequality obtained above, we see that

$$\begin{aligned}
\limsup_{j\to\infty} &\left| \int_{D\cap E} \varphi(\mathbf{f}_j(\mathbf{x}))\mathrm{d}\mathbf{x} - m(D\cap E)\int_{\mathbb{R}^N} \varphi(\mathbf{y})\mathrm{d}\nu(\mathbf{y}) \right| \\
&\leq m(D\cap E)\varepsilon + \left| \sum_{k=1}^{M} \varphi(\mathbf{a}_k) m(D\cap E)\nu(Q_k) - m(D\cap E)\int_{\mathbb{R}^N} \varphi(\mathbf{y})\mathrm{d}\nu(\mathbf{y}) \right| \\
&\leq m(D\cap E)\left\{ \varepsilon + \sum_{k=1}^{M} \int_{Q_k} \left|\varphi(\mathbf{a}_k) - \varphi(\mathbf{y})\right| \mathrm{d}\nu(\mathbf{y}) + \int_{\mathbb{R}^N\setminus\cup_{k=1}^{\infty} Q_k} \left|\varphi(\mathbf{y})\right| \mathrm{d}\nu(\mathbf{y}) \right\} \\
&\leq 2m(D\cap E)\varepsilon.
\end{aligned}$$

Letting $\varepsilon \to 0^+$, we obtain the equality in (3.2) when $\xi = \chi_{D\cap E}$ for some cube $D \subset \mathbb{R}^n$. Letting $\mathcal{T}$ be the set of all finite linear combinations of functions of the form $\chi_{D\cap E}$, we see that (3.2) also holds for every $\xi \in \mathcal{T}$ and $\varphi \in \mathcal{C}_0(\mathbb{R}^N)$. Since $\mathcal{T}$ is dense in $L^1(E)$, it follows that $\{\mathbf{f}_j\}$ generates ν. □

Lemma 3.2. *Let $\nu \in \mathcal{M}(\mathbb{R}^N)$ be a probability measure. There is a measurable function $\mathbf{g} : (0,1) \to \mathbb{R}^N$ such that*

$$\nu(E) = m(\mathbf{g}^{-1}(E))$$

for every Borel set $E \subset \mathbb{R}^N$. Moreover, $\mathbf{g}$ satisfies

$$\int_{(0,1)} |\mathbf{g}(x)|^p \,\mathrm{d}x = \int_{\mathbb{R}^N} |\mathbf{y}|^p \,\mathrm{d}\nu(\mathbf{y})$$

for every $1 \le p < \infty$.

Proof. First we construct the function $\mathbf{g}$. Let $\{\mathbf{a}_{1,j}\}_{j=1}^\infty$ be an enumeration of $\mathbb{Z}^N$, and for each $j \in \mathbb{N}$, we define $D_{1,j} := \mathbf{a}_{1,j} + [0,1)^N$. Note that $\{D_{1,j}\}_{j=1}^\infty$ partitions $\mathbb{R}^N$. Assuming that the cubes $\{D_{k,j}\}_{j=1}^\infty$ have been chosen for some $k \in \mathbb{N}$, we partition each of these cubes into 2^N subcubes, each subcube having the form $D_{k+1,j} = \mathbf{a}_{k+1,j} + [0, 2^{-k})^N$ for some $\mathbf{a}_{k+1,j} \in 2^{-k}\mathbb{Z}^N$. We thus obtain a collection of dyadic cubes $\{D_{k+1,j}\}_{j=1}^\infty$ that partitions $\mathbb{R}^N$, and each cube has edges of length 2^{-k}. Furthermore, we stipulate that we first partition $D_{k,1}$ into subcubes, then $D_{k,2}$, and so on. That is, we require

$$\bigcup_{l=(j-1)2^N+1}^{j2^N} D_{k+1,l} = D_{k,j}, \tag{3.3}$$

for every k, $j \in \mathbb{N}$.

For each k, we partition the interval $[0,1)$ into a family of intervals $\{I_{k,j}\}_{j=1}^\infty$ as follows. Define

$$I_{k,j} := \left[\nu\left(\cup_{l=1}^{j-1} D_{k,l}\right), \nu\left(\cup_{l=1}^{j} D_{k,l}\right)\right). \tag{3.4}$$

Note that the length of $I_{k,j}$ is $\nu(D_{k,j})$. (Here we are using the convention that the interval $[a, a) := \emptyset$.)

Now we define a sequence $\{\mathbf{g}_k\}_{k=1}^{\infty}$ of functions mapping $[0, 1)$ into $\mathbb{R}^N$ by

$$\mathbf{g}_k(x) := \sum_{j=1}^{\infty} \mathbf{a}_{k,j} \chi_{I_{k,j}}(x). \tag{3.5}$$

We will show this sequence is Cauchy in the uniform norm. To this end, fix $\varepsilon > 0$, and find $K \in \mathbb{N}$ such that $\operatorname{diam}(D_{K,j}) < \varepsilon$. Now fix $x \in (0, 1)$. Then there is a unique $j_1 \in \mathbb{N}$ such that

$$x \in I_{1,j_1} = \left[\nu\left(\cup_{l=1}^{j_1-1} D_{1,l}\right), \nu\left(\cup_{l=1}^{j_1} D_{1,l}\right)\right).$$

Recalling the way that $\mathbf{g}_1$ is defined, we see that $\mathbf{g}_1(x) = \mathbf{a}_{1,j_1} \in D_{1,j_1}$. By (3.3) and (3.4), we have that

$$I_{1,j_1} = \bigcup_{j=(j_1-1)2^N+1}^{j_1 2^N} I_{2,j},$$

whence $x \in I_{2,j_2}$ for some $(j_1 - 1)2^N + 1 \leq j_2 \leq j_1 2^N$ and $\mathbf{g}_2(x) \in D_{2,j_2} \subset D_{1,j_1}$. Proceeding inductively, we obtain $\{j_k\}_{k=1}^{\infty} \subset \mathbb{N}$ such that $\mathbf{g}_k(x) \in D_{k,j_k}$ and $D_{1,j_1} \supset \cdots \supset D_{k,j_k} \supset \cdots$. Recalling that we chose $K \in \mathbb{N}$ so that $\operatorname{diam}(D_{K,j}) < \varepsilon$, we see that if k_1 and k_2 are both at least K, then $\mathbf{g}_{k_1}(x)$ and $\mathbf{g}_{k_2}(x)$ belong to the same cube $D_{K,j}$ for some $j \in \mathbb{N}$, and hence $|\mathbf{g}_{k_1}(x) - \mathbf{g}_{k_2}(x)| < \varepsilon$. Hence $\{\mathbf{g}_k\}$ is Cauchy in the uniform norm, and so there is some $\mathbf{g} : (0, 1) \to \mathbb{R}^N$ such that $\mathbf{g}_k \to \mathbf{g}$ uniformly as $k \to \infty$.

We claim that the function $\mathbf{g}$ so constructed satisfies the conclusions of the lemma. First note that if $D = \cup_{i=1}^{J} D_{k_i,j_i}$ is a finite (disjoint) union of cubes of the form $D_{k,j}$, then for $l \geq \max\{k_i\}$, we have by (3.5) that

$$\mathbf{g}_l^{-1}(D) = \bigcup_{i=1}^{J} I_{k_i,j_i}.$$

Since $\mathbf{g}_l \to \mathbf{g}$ uniformly, we therefore have

$$\mathbf{g}^{-1}(\overline{D}) \supset \bigcup_{i=1}^{J} I_{k_i,j_i}. \tag{3.6}$$

Note that, by (3.4), we have $m(I_{k_i,j_i}) = \nu(D_{k_i,j_i})$, so from (3.6) we obtain

$$m(\mathbf{g}^{-1}(\overline{D})) \geq \nu(D). \tag{3.7}$$

Now, for fixed $k, j \in \mathbb{N}$, we can find a sequence of cubes $\{D_i\}_{i=1}^{\infty}$ such that for each $i \in \mathbb{N}$, the cube D_i is a finite union of cubes of the form $D_{s,t}$ with $D_i \subset D_{i+1} \subset D_{k,j}$, and

$$\bigcup_{i=1}^{\infty} \overline{D_i} = \bigcup_{i=1}^{\infty} D_i = D_{k,j}. \tag{3.8}$$

Using (3.7) along with (3.8), we thus obtain

$$m(\mathbf{g}^{-1}(D_{k,j})) = \lim_{i\to\infty} m(\mathbf{g}^{-1}(\overline{D_i})) \geq \lim_{i\to\infty} \nu(D_i) = \nu(D_{k,j}).$$

Since the cubes $\{D_{k,j}\}_{k,j\in\mathbb{N}}$ are a generating set for the Borel σ-algebra on $\mathbb{R}^N$, the preceding inequality implies

$$m(\mathbf{g}^{-1}(E)) \geq \nu(E) \tag{3.9}$$

for all Borel sets $E \subset \mathbb{R}^N$. Since ν and the image of m under $\mathbf{g}$ are both probability measures on $\mathbb{R}^N$, the inequality in (3.9) must in fact be an equality; that is, we must have

$$m(\mathbf{g}^{-1}(E)) = \nu(E) \tag{3.10}$$

for all Borel sets $E \subset \mathbb{R}^N$, which establishes the first statement of the lemma.

If $\psi : \mathbb{R}^N \to \mathbb{R}$ is a simple function, then (3.10) gives

$$\int_{(0,1)} \psi(\mathbf{g}(x))\mathrm{d}x = \int_{\mathbb{R}^N} \psi(\mathbf{y})\mathrm{d}\nu(\mathbf{y}).$$

By taking a sequence of simple functions $\{\psi_j\}_{j=1}^{\infty}$ increasing to the function $\mathbf{y} \mapsto |\mathbf{y}|^p$ and using the above equality and the monotone convergence theorem, we find that

$$\int_{(0,1)} |\mathbf{g}(x)|^p \, \mathrm{d}x = \int_{\mathbb{R}^N} |\mathbf{y}|^p \, \mathrm{d}\nu(\mathbf{y}),$$

which concludes the proof of the lemma. □

Now we use the function $\mathbf{g}$ given by Lemma 3.2 to build a sequence of functions uniformly bounded in $L^{p,\lambda}(Q;\mathbb{R}^N)$ that generates the measure ν.

Theorem 3.4. *Suppose that $\nu \in \mathcal{M}(\mathbb{R}^N)$ is a probability measure that satisfies*

$$\int_{\mathbb{R}^N} |\mathbf{y}|^p \, \mathrm{d}\nu(\mathbf{y}) < \infty.$$

For each $0 \leq \lambda < n$, there is a p-equiintegrable sequence of functions $\{\mathbf{f}_j\}_{j=1}^{\infty}$ that generates ν, is uniformly bounded in $L^{p,\mu}(Q,\mathbb{R}^N)$ for every $0 \leq \mu < n$, and satisfies

$$\|\mathbf{f}_j\|_{L^{p,\lambda}}^p \leq 2^n \int_{\mathbb{R}^N} |\mathbf{y}|^p \, \mathrm{d}\nu(\mathbf{y}).$$

Proof. Let $\mathbf{g}$ be the function given by Lemma 3.2, and extend it by periodicity to all of $\mathbb{R}$. Let $\{\mathbf{g}_j\}_{j=1}^{\infty}$ be the sequence of functions mapping Q into $\mathbb{R}^N$ defined by

$$\mathbf{g}_j(\mathbf{x}) = \mathbf{g}(jx_1).$$

Then using a change of variables, the periodicity of $\mathbf{g}$, and Lemma 3.2, we see that

$$\|\mathbf{g}_j\|^p_{L^p(Q;\mathbb{R}^N)} = \int_{(0,1)} |\mathbf{g}(jx_1)|^p \, \mathrm{d}x_1 = \frac{1}{j} \int_{(0,j)} |\mathbf{g}(x_1)|^p \, \mathrm{d}x_1 = \int_{(0,1)} |\mathbf{g}(x_1)|^p \, \mathrm{d}x_1 = \int_{\mathbb{R}^N} |\mathbf{y}|^p \, \mathrm{d}\nu(\mathbf{y}). \tag{3.11}$$

If $D \subset \mathbb{R}^n$ is a cube and $G \subset \mathbb{R}^N$ is a Borel set, then

$$\lim_{j\to\infty} m(D \cap Q \cap \mathbf{g}_j^{-1}(G)) = m(D \cap Q) m(\mathbf{g}^{-1}(G)) = m(D \cap Q)\nu(G),$$

where we have employed Lemma 3.2 to obtain the last equality. Therefore, by Lemma 3.1, the sequence $\{\mathbf{g}_j\}$ generates the Young measure ν.

We now define a new sequence of functions $\{\mathbf{f}_j\}_{j=1}^{\infty}$ that are truncations of the functions $\mathbf{g}_j$:

$$\mathbf{f}_j(\mathbf{x}) := \operatorname{sgn}(\mathbf{g}_j(\mathbf{x})) \min\left\{ \frac{\log(j)}{C_\lambda}, |\mathbf{g}_j(\mathbf{x})| \right\},$$

where for each $0 \le \mu < n$ we have put

$$C_\mu := \left(\max_{j\in\mathbb{N}} \left\{ \frac{\{\log(j)\}^p}{2^n j^{n-\mu} \int_{\mathbb{R}^N} |\mathbf{y}|^p \, \mathrm{d}\nu(\mathbf{y})} \right\} \right)^{\frac{1}{p}}.$$

Then $\{\mathbf{f}_j\}$ generates the measure ν by Theorem 3.3, since the measure of the set where $\mathbf{f}_j \neq \mathbf{g}_j$ tends to 0 as $j \to \infty$. It is easily seen that $\{\mathbf{g}_j\}$, and hence also $\{\mathbf{f}_j\}$, is p-equiintegrable. Furthermore, $\{\mathbf{f}_j\}$ is uniformly bounded in $L^{p,\mu}$ for every $0 \le \mu < n$. To see this, fix $0 < \rho \le 1$, $\mathbf{x}_0 \in Q$, and $j \in \mathbb{N}$. We consider two cases.

Case 1: $0 < \rho \leq 1/j$.

In this case, using the bound $|\mathbf{f}_j| \leq \log(j)/C_\lambda$, we have

$$\rho^{-\mu} \int_{Q \cap Q_{\mathbf{x}_0,\rho}} |\mathbf{f}_j(\mathbf{x})|^p \, \mathrm{d}\mathbf{x} \leq \rho^{n-\mu} \left\{ \frac{\log(j)}{C_\lambda} \right\}^p \leq \frac{\{\log(j)\}^p}{C_\lambda^p j^{n-\mu}} \leq 2^n \left(\frac{C_\mu}{C_\lambda} \right)^p \int_{\mathbb{R}^N} |\mathbf{y}|^p \, \mathrm{d}\nu(\mathbf{y}).$$

Case 2: $k/j < \rho \leq (k+1)/j$ for some $k \in \{1, 2, \cdots, j-1\}$.

Using the inequality $|\mathbf{f}_j| \leq |\mathbf{g}_j|$ and the periodicity of $\mathbf{g}_j$, we obtain

$$\begin{aligned} \rho^{-\mu} \int_{Q \cap Q_{\mathbf{x}_0,\rho}} |\mathbf{f}_j(\mathbf{x})|^p \, \mathrm{d}\mathbf{x} &\leq \rho^{-\mu} \int_{Q \cap Q_{\mathbf{x}_0,\rho}} |\mathbf{g}_j(\mathbf{x})|^p \, \mathrm{d}\mathbf{x} \leq \rho^{-\mu} (k+1)^n \int_{Q \cap Q_{\mathbf{x}_0,\frac{1}{j}}} |\mathbf{g}_j(\mathbf{x})|^p \, \mathrm{d}\mathbf{x} \\ &\leq \rho^{-\mu} \left(\frac{k+1}{j} \right)^n \|\mathbf{g}_j\|^p_{L^p(Q;\mathbb{R}^N)} . \end{aligned}$$

But $\rho^{-\mu} \leq \rho^{-n} \leq (j/k)^n$; using this and (3.11) in the above inequality, we see that

$$\rho^{-\mu} \int_{Q \cap Q_{\mathbf{x}_0,\rho}} |\mathbf{f}_j(\mathbf{x})|^p \, \mathrm{d}\mathbf{x} \leq 2^n \int_{\mathbb{R}^N} |\mathbf{y}|^p \, \mathrm{d}\nu(\mathbf{y}).$$

Collecting the estimates we obtained in each case, we have shown that

$$\|\mathbf{f}_j\|^p_{L^{p,\mu}} \leq 2^n \max \left\{ \left(\frac{C_\mu}{C_\lambda} \right)^p, 1 \right\} \int_{\mathbb{R}^N} |\mathbf{y}|^p \, \mathrm{d}\nu(\mathbf{y}),$$

which implies that $\{\mathbf{f}_j\}$ is uniformly bounded in $L^{p,\mu}(Q; \mathbb{R}^N)$ for each $0 \leq \mu < n$; taking $\mu = \lambda$ yields

$$\|\mathbf{f}_j\|^p_{L^{p,\lambda}} \leq 2^n \int_{\mathbb{R}^N} |\mathbf{y}|^p \, \mathrm{d}\nu(\mathbf{y}),$$

which concludes the proof. □

3.2.2 Nonhomogeneous Measures

We now extend the result just proved to the nonhomogeneous case.

Theorem 3.5. *Let* $\nu = \{\nu_{\mathbf{x}}\}_{\mathbf{x}\in\Omega}$ *be a Young measure on* $\mathbb{R}^N$, *and let* $0 \le \lambda < n$. *There is a p-equiintegrable sequence of functions* $\{\mathbf{f}_j\}_{j=1}^{\infty}$ *that is uniformly bounded in* $L^{p,\lambda}(\Omega;\mathbb{R}^N)$ *and generates the measure* ν *if and only if* ν *satisfies*

$$\sup_{\substack{\mathbf{x}_0\in\Omega \\ \rho>0}} \rho^{-\lambda} \int_{\Omega\cap Q_{\mathbf{x}_0,\rho}} \int_{\mathbb{R}^N} |\mathbf{y}|^p \, \mathrm{d}\nu_{\mathbf{x}}(\mathbf{y})\mathrm{d}\mathbf{x} < \infty. \tag{3.12}$$

Proof. The necessity follows from Theorem 3.1. Indeed, for fixed $\mathbf{x}_0 \in \Omega$ and $\rho > 0$, define $f : \Omega\times\mathbb{R}^N \to \mathbb{R}$ by $f(\mathbf{x},\mathbf{y}) = \rho^{-\lambda}\chi_{\Omega\cap Q_{\mathbf{x}_0,\rho}}(\mathbf{x})\,|\mathbf{y}|^p$. The aforementioned theorem now yields

$$\liminf_{j\to\infty} \rho^{-\lambda} \int_{\Omega\cap Q_{\mathbf{x}_0,\rho}} |\mathbf{f}_j|^p \, \mathrm{d}\mathbf{x} \ge \rho^{-\lambda} \int_{\Omega\cap Q_{\mathbf{x}_0,\rho}} \int_{\mathbb{R}^N} |\mathbf{y}|^p \, \mathrm{d}\nu_{\mathbf{x}}(\mathbf{y})\mathrm{d}\mathbf{x}.$$

Since the sequence $\{\mathbf{f}_j\}$ is uniformly bounded in $L^{p,\lambda}(\Omega;\mathbb{R}^N)$, it follows that ν satisfies (3.12).

We now turn to the sufficiency. For each $k \in \mathbb{N}$, let $\{\mathbf{a}_{k,i}\}_{i=1}^{\infty}$ be an enumeration of the set $2^{-k}\mathbb{Z}^n$, and define

$$D_{k,i} = \mathbf{a}_{k,i} + 2^{-k}[0,1)^n.$$

Note that for each k, $\{D_{k,i}\}_{i=1}^{\infty}$ is a set of dyadic cubes that partition $\mathbb{R}^n$ and have edges of length 2^{-k}. For each k, there will be finitely many of these cubes that are entirely contained in Ω. We will define $A_k \subset \mathbb{N}$ to be the collection of the indices of such cubes:

$$A_k := \{i \in \mathbb{N} : D_{k,i} \subset \Omega\}.$$

Since Ω is open, it is easy to see that

$$\bigcup_{k=1}^{\infty} \bigcup_{i \in A_k} D_{k,i} = \Omega.$$

For each $k \in \mathbb{N}$ and i in A_k, we define the probability measure $\nu_{k,i} \in \mathcal{M}(\mathbb{R}^N)$ to be the "average" of the measures $\{\nu_{\mathbf{x}}\}_{\mathbf{x} \in D_{k,i}}$. That is, we select $\nu_{k,i}$ so that for every $\varphi \in \mathcal{C}_0(\mathbb{R}^N)$, we have

$$\int_{\mathbb{R}^N} \varphi(\mathbf{y}) \mathrm{d}\nu_{k,i}(\mathbf{y}) = \fint_{D_{k,i}} \int_{\mathbb{R}^N} \varphi(\mathbf{y}) \mathrm{d}\nu_{\mathbf{x}}(\mathbf{y}) \mathrm{d}\mathbf{x}. \tag{3.13}$$

The existence of such a measure is guaranteed by Theorem 7.1 in [65]. Note that this measure is homogeneous, and that

$$\int_{\mathbb{R}^N} |\mathbf{y}|^p \, \mathrm{d}\nu_{k,i}(\mathbf{y}) = \fint_{D_{k,i}} \int_{\mathbb{R}^N} |\mathbf{y}|^p \, \mathrm{d}\nu_{\mathbf{x}}(\mathbf{y}) \mathrm{d}\mathbf{x} = 2^{kn} \int_{D_{k,i}} \int_{\mathbb{R}^N} |\mathbf{y}|^p \, \mathrm{d}\nu_{\mathbf{x}}(\mathbf{y}) \mathrm{d}\mathbf{x} < \infty. \tag{3.14}$$

Using Theorem 3.4, we find a p-equiintegrable sequence $\{\mathbf{f}_j^{k,i}\}_{j=1}^{\infty} \subset L^{p,\lambda}(Q; \mathbb{R}^N)$ that satisfies

$$\left\| \mathbf{f}_j^{k,i} \right\|_{L^{p,\lambda}}^p \leq 2^n \int_{\mathbb{R}^N} |\mathbf{y}|^p \, \mathrm{d}\nu_{k,i}(\mathbf{y}) \tag{3.15}$$

and generates the measure $\nu_{k,i}$. For a given $\varphi \in \mathcal{C}_0(\mathbb{R}^N)$, we denote by $\overline{\varphi}$ and $\overline{\varphi_k}$ the functions defined by

$$\overline{\varphi}(\mathbf{x}) := \int_{\mathbb{R}^N} \varphi(\mathbf{y}) \mathrm{d}\nu_{\mathbf{x}}(\mathbf{y});$$
$$\overline{\varphi_k}(\mathbf{x}) := \sum_{i \in A_k} \chi_{D_{k,i}}(\mathbf{x}) \int_{\mathbb{R}^N} \varphi(\mathbf{y}) \mathrm{d}\nu_{k,i}(\mathbf{y}).$$

Notice that both $\overline{\varphi}$ and $\overline{\varphi_k}$ are functions belonging to $L^\infty(\Omega)$, with essential supremums at most $\|\varphi\|_{L^\infty(\mathbb{R}^N)}$. If $\mathbf{x} \in \Omega$, then for each k sufficiently large, there is a unique $i(k) \in A_k$ such that $\mathbf{x} \in D_{k,i(k)}$. And if $\mathbf{x}$ is a Lebesgue point of $\overline{\varphi}$, we use (3.13) to compute

$$\lim_{k\to\infty} \overline{\varphi_k}(\mathbf{x}) = \lim_{k\to\infty} \int_{\mathbb{R}^N} \varphi(\mathbf{y}) \mathrm{d}\nu_{k,i(k)}(\mathbf{y}) = \lim_{k\to\infty} \fint_{D_{k,i(k)}} \int_{\mathbb{R}^N} \varphi(\mathbf{y}) \mathrm{d}\nu_{\hat{\mathbf{x}}}(\mathbf{y}) \mathrm{d}\hat{\mathbf{x}}$$
$$= \lim_{k\to\infty} \fint_{D_{k,i(k)}} \overline{\varphi}(\hat{\mathbf{x}}) \mathrm{d}\hat{\mathbf{x}} = \overline{\varphi}(\mathbf{x}).$$

Therefore $\overline{\varphi_k}$ converges to $\overline{\varphi}$ pointwise almost everywhere. We have already noted that $|\overline{\varphi_k}| \leq \|\varphi\|_{L^\infty}$, so we can use Lebesgue's Dominated Convergence Theorem to show that $\overline{\varphi_k}$ converges to $\overline{\varphi}$ in $L^1(\Omega)$. Therefore we have that

$$\lim_{k\to\infty} \int_\Omega \xi(\mathbf{x}) \overline{\varphi_k}(\mathbf{x}) \mathrm{d}\mathbf{x} = \int_\Omega \xi(\mathbf{x}) \overline{\varphi}(\mathbf{x}) \mathrm{d}\mathbf{x} \tag{3.16}$$

for every $\varphi \in \mathcal{C}_0(\mathbb{R}^N)$ and $\xi \in L^\infty(\Omega)$.

Let $\{\xi_k\}_{k=1}^\infty \subset L^\infty(\Omega)$ and $\{\varphi_j\}_{j=1}^\infty \subset \mathcal{C}_0(\mathbb{R}^N)$ be countable dense subsets of $L^1(\Omega)$ and $\mathcal{C}_0(\mathbb{R}^N)$, respectively. Let $\varphi_0 \in \mathcal{C}(\mathbb{R}^N)$ and $\xi_0 \in L^\infty(\Omega)$ be defined by $\varphi_0(\mathbf{y}) := |\mathbf{y}|^p$ and $\xi_0 \equiv 1$. For each $k \in \mathbb{N}$ and $i \in A_k$, since the p-equiintegrable sequence $\{\mathbf{f}_j^{k,i}\}_{j=1}^\infty$ generates the measure $\nu_{k,i}$, we can choose $j = j(k,i)$ such that

$$\left| \int_Q \xi_s(\mathbf{a}_{k,i} + 2^{-k}\mathbf{x}) \varphi_t(\mathbf{f}_j^{k,i}(\mathbf{x})) \mathrm{d}\mathbf{x} - \int_Q \xi_s(\mathbf{a}_{k,i} + 2^{-k}\mathbf{x}) \mathrm{d}\mathbf{x} \int_{\mathbb{R}^N} \varphi_t(\mathbf{y}) \mathrm{d}\nu_{k,i}(\mathbf{y}) \right| \leq \frac{2^{nk-i}}{k}$$

for $0 \leq s, t \leq k$. With j chosen in this way, we define the sequence of functions $\{\mathbf{f}_k\}_{k=1}^\infty$ in the following way:

$$\mathbf{f}_k(\mathbf{x}) := \begin{cases} \mathbf{f}_j^{k,i}\left(\frac{\mathbf{x} - \mathbf{a}_{k,i}}{2^{-k}}\right) & \text{if } \mathbf{x} \in D_{k,i} \text{ for some } i \in A_k \\ 0 & \text{if } \mathbf{x} \in \Omega \setminus \cup_{i \in A_k} D_{k,i} \end{cases}. \tag{3.17}$$

Fix $s, t \in \mathbb{N}$, and suppose that $k \geq \max\{s, t\}$. Using the definition of $\mathbf{f}_k$ and changing variables in the preceding inequality yields

$$\left| \int_{D_{k,i}} \xi_s(\mathbf{x}) \varphi_t(\mathbf{f}_k(\mathbf{x})) \mathrm{d}\mathbf{x} - \int_{D_{k,i}} \xi_s(\mathbf{x}) \mathrm{d}\mathbf{x} \int_{\mathbb{R}^N} \varphi_t(\mathbf{y}) \mathrm{d}\nu_{k,i}(\mathbf{y}) \right| \leq \frac{2^{-i}}{k}$$

whenever $i \in A_k$. Hence

$$\left| \int_{\cup_{i \in A_k} D_{k,i}} \xi_s(\mathbf{x}) \varphi_t(\mathbf{f}_k(\mathbf{x})) \mathrm{d}\mathbf{x} - \int_\Omega \xi_s(\mathbf{x}) \overline{(\varphi_t)_k}(\mathbf{x}) \mathrm{d}\mathbf{x} \right| \leq \frac{1}{k}. \tag{3.18}$$

So for each $s, t \in \mathbb{N}$, using (3.18) and (3.16), we have

$$\begin{aligned} \lim_{k \to \infty} \int_\Omega \xi_s(\mathbf{x}) \varphi_t(\mathbf{f}_k(\mathbf{x})) \mathrm{d}\mathbf{x} &= \lim_{k \to \infty} \int_\Omega \xi_s(\mathbf{x}) \overline{(\varphi_t)_k}(\mathbf{x}) \mathrm{d}\mathbf{x} = \int_\Omega \xi_s(\mathbf{x}) \overline{\varphi_t}(\mathbf{x}) \mathrm{d}\mathbf{x} \\ &= \int_\Omega \xi_s(\mathbf{x}) \int\limits_{\mathbb{R}^N} \varphi_t(\mathbf{y}) \mathrm{d}\nu_{\mathbf{x}}(\mathbf{y}) \mathrm{d}\mathbf{x}. \end{aligned}$$

This implies that the sequence $\{\mathbf{f}_k\}_{k=1}^\infty$ generates $\{\nu_{\mathbf{x}}\}_{\mathbf{x} \in \Omega}$.

Using the same steps we used to arrive at (3.18), this time taking $s = t = 0$ and recalling that $\mathbf{f}_k = 0$ in $\Omega \setminus \cup_{i \in A_k} D_{k,i}$, we obtain

$$\left| \int_\Omega |\mathbf{f}_k(\mathbf{x})|^p \, \mathrm{d}\mathbf{x} - \sum_{i \in A_k} \int_{D_{k,i}} \int_{\mathbb{R}^N} |\mathbf{y}|^p \, \mathrm{d}\nu_{k,i}(\mathbf{y}) \mathrm{d}\mathbf{x} \right| \leq \frac{1}{k}. \tag{3.19}$$

Using (3.14), we have

$$\begin{aligned} \sum_{i \in A_k} \int_{D_{k,i}} \int_{\mathbb{R}^N} |\mathbf{y}|^p \, \mathrm{d}\nu_{k,i}(\mathbf{y}) \mathrm{d}\mathbf{x} &= \sum_{i \in A_k} \int_{D_{k,i}} \int_{\mathbb{R}^N} |\mathbf{y}|^p \, \mathrm{d}\nu_{\mathbf{x}}(\mathbf{y}) \mathrm{d}\mathbf{x} \\ &= \int_{\cup_{i \in A_k} D_{k,i}} \int_{\mathbb{R}^N} |\mathbf{y}|^p \, \mathrm{d}\nu_{\mathbf{x}}(\mathbf{y}) \mathrm{d}\mathbf{x}. \end{aligned} \tag{3.20}$$

By (3.19) and (3.20), we have

$$\lim_{k\to\infty}\int_\Omega |\mathbf{f}_k(\mathbf{x})|^p\,\mathrm{d}\mathbf{x} = \int_\Omega\int_{\mathbb{R}^N} |\mathbf{y}|^p\,\mathrm{d}\nu_{\mathbf{x}}(\mathbf{y})\mathrm{d}\mathbf{x} < \infty,$$

and hence the p-equiintegrability of $\{\mathbf{f}_k\}$ follows from Theorem 3.1. We only have yet to show that this sequence is uniformly bounded in $L^{p,\lambda}(\Omega;\mathbb{R}^N)$.

To this end, fix $\mathbf{x}_0 \in \Omega$, $\rho > 0$, and $k \in \mathbb{N}$. We consider two cases.

Case 1: $0 < \rho \leq 2^{-k}$.

Let $J := \{i \in A_k : D_{k,i} \cap Q_{\mathbf{x}_0,\rho} \neq \emptyset\}$. Note that there are at most 2^n elements in J, since $\rho \leq 2^{-k}$. By changing variables and using the fact that $\mathbf{f}_k = 0$ outside of $\cup_{i\in A_k} D_{k,i}$, we obtain

$$\rho^{-\lambda}\int_{\Omega\cap Q_{\mathbf{x}_0,\rho}} |\mathbf{f}_k|^p\,\mathrm{d}\mathbf{x} = \rho^{-\lambda}\sum_{i\in J} 2^{-kn}\int_{Q\cap Q_{2^k(\mathbf{x}_0-\mathbf{a}_{k,i}),2^k\rho}} \left|\mathbf{f}_j^{k,i}\right|^p\mathrm{d}\mathbf{x}.$$

By (3.15), we have

$$\int_{Q\cap Q_{2^k(\mathbf{x}_0-\mathbf{a}_{k,i}),2^k\rho}} \left|\mathbf{f}_j^{k,i}\right|^p\mathrm{d}\mathbf{x} \leq 2^n(2^k\rho)^\lambda\int_{\mathbb{R}^N} |\mathbf{y}|^p\,\mathrm{d}\nu_{k,i}(\mathbf{y}).$$

Using this inequality in the one preceding it and then employing (3.14), we find that

$$\begin{aligned}\rho^{-\lambda}\int_{\Omega\cap Q_{\mathbf{x}_0,\rho}} |\mathbf{f}_k|^p\,\mathrm{d}\mathbf{x} &\leq 2^n\sum_{i\in J}(2^k)^{\lambda-n}\int_{\mathbb{R}^N}|\mathbf{y}|^p\,\mathrm{d}\nu_{k,i}(\mathbf{y})\\ &= 2^n\sum_{i\in J}(2^{-k})^{-\lambda}\int_{D_{k,i}}\int_{\mathbb{R}^N}|\mathbf{y}|^p\,\mathrm{d}\nu_{\mathbf{x}}(\mathbf{y})\mathrm{d}\mathbf{x}.\end{aligned}$$

But $D_{k,i} = Q_{\mathbf{y}_{k,i},2^{-k}}$ for some $\mathbf{y}_{k,i} \in \Omega$, so the above inequality gives

$$\rho^{-\lambda}\int_{\Omega\cap Q_{\mathbf{x}_0,\rho}} |\mathbf{f}_k|^p\,\mathrm{d}\mathbf{x} \leq 2^n\sum_{i\in J} M,$$

where

$$M := \sup_{\substack{\mathbf{x}_0 \in \Omega \\ R>0}} R^{-\lambda} \int_{\Omega \cap Q_{\mathbf{x}_0,R}} \int_{\mathbb{R}^N} |\mathbf{y}|^p \, \mathrm{d}\nu_{\mathbf{x}}(\mathbf{y}) \mathrm{d}\mathbf{x} < \infty;$$

therefore, since $|J| \leq 2^n$,

$$\rho^{-\lambda} \int_{\Omega \cap Q_{\mathbf{x}_0,\rho}} |\mathbf{f}_k|^p \, \mathrm{d}\mathbf{x} \leq 4^n M.$$

Case 2: $\rho > 2^{-k}$.

In this case, we can find a cube Q_0 containing $Q_{\mathbf{x}_0,\rho}$ that is comprised of cubes of the form $D_{k,i}$, and such that the sides of Q_0 have length of at most 2ρ. Letting $\mathbf{y}_0$ denote the center of Q_0, we can break up $Q_{\mathbf{x}_0,\rho}$ into cubes $D_{k,i}$ as we did in Case 1 and perform a similar computation to obtain

$$\rho^{-\lambda} \int_{\Omega \cap Q_{\mathbf{x}_0,\rho}} |\mathbf{f}_k|^p \, \mathrm{d}\mathbf{x} \leq \rho^{-\lambda} \int_{\Omega \cap Q_{\mathbf{y}_0,2\rho}} \int_{\mathbb{R}^N} |\mathbf{y}|^p \, \mathrm{d}\nu_{\mathbf{x}}(\mathbf{y}) \mathrm{d}\mathbf{x} \leq 2^\lambda M.$$

From the estimates we obtained in each case, we see that

$$\|\mathbf{f}_k\|^p_{L^{p,\lambda}(\Omega;\mathbb{R}^N)} \leq 4^n M,$$

which finishes the proof. □

3.3 Gradient Young Measures

We now turn our attention to Young measures generated by sequences of gradients bounded in $L^{p,\lambda}$. To simplify the statements of the theorems, for this section we will assume that Ω has Lipschitz-continuous boundary. The following lemma can be deduced from Theorem 8.16 and Lemma 8.3 in [65].

Lemma 3.3. *Suppose that* $\nu \in \mathcal{M}(\mathbb{R}^{N\times n})$ *is a probability measure that satisfies*

(i) $\int_{\mathbb{R}^{N\times n}} \mathbf{F} \mathrm{d}\nu(\mathbf{F}) = \mathbf{0}$;

(ii) $\int_{\mathbb{R}^{N\times n}} \varphi(\mathbf{F}) \mathrm{d}\nu(\mathbf{F}) \geq \varphi(\mathbf{0})$ *for every quasiconvex* $\varphi : \mathbb{R}^{N\times n} \to [0,\infty)$ *satisfying* $\varphi(\mathbf{F}) \leq C(1+|\mathbf{F}|^p)$;

(iii) $\int_{\mathbb{R}^{N\times n}} |\mathbf{F}|^p \, \mathrm{d}\nu(\mathbf{F}) < \infty$.

Then there exists a sequence of functions $\{\mathbf{u}_j\}_{j=1}^{\infty} \subset W_0^{1,p}(Q;\mathbb{R}^N)$ *such that the sequence* $\{\nabla \mathbf{u}_j\}_{j=1}^{\infty}$ *generates the measure* ν *and is p-equiintegrable.*

To find Morrey regular sequences generating the measure, we will need the following lemma, which allows us to generate certain Young measures by p-equiintegrable gradients of Lipschitz functions.

Lemma 3.4. *Let* $\{\mathbf{u}_j\}_{j=1}^{\infty}$ *be a bounded sequence in* $W_0^{1,p}(Q;\mathbb{R}^N)$ *for some* $1 \leq p < \infty$, *and suppose that* $\{\nabla \mathbf{u}_j\}_{j=1}^{\infty}$ *generates the measure* $\nu = \{\nu_{\mathbf{x}}\}_{\mathbf{x}\in Q}$ *and is p-equiintegrable. Let* $\{T_j\}_{j=1}^{\infty}$ *be a sequence of non-negative numbers such that* $\lim_{j\to\infty} T_j = +\infty$. *Then there is a sequence of functions* $\{\mathbf{v}_j\}_{j=1}^{\infty} \subset W_0^{1,\infty}(Q;\mathbb{R}^N)$ *such that*

(i) $\{\nabla \mathbf{v}_j\}_{j=1}^{\infty}$ *generates* ν *and is p-equiintegrable;*

(ii) $\|\nabla \mathbf{v}_j\|_{L^\infty} \leq T_j$;

Proof. Since $\{\mathbf{u}_j\}_{j=1}^{\infty} \subset W_0^{1,p}(Q;\mathbb{R}^N)$, we can extend each $\mathbf{u}_j$ by zero to all of $\mathbb{R}^n$ so that $\mathbf{u}_j \in W^{1,p}(\mathbb{R}^n;\mathbb{R}^N)$. For each j, we define the set A_j by

$$A_j := \{\mathbf{x} \in \mathbb{R}^n : M(|\nabla \mathbf{u}_j|)(\mathbf{x}) \leq T_j\};$$

by Theorem 3.2, there is a sequence of Lipschitz functions $\{\mathbf{w}_j\}_{j=1}^{\infty}$ and a constant $c = c(n,N,p)$ such that

(i) $\mathbf{u}_j = \mathbf{w}_j$ and $\nabla \mathbf{u}_j = \nabla \mathbf{w}_j$ almost everywhere on A_j;

(ii) $\|\nabla \mathbf{w}_j\|_{L^\infty} \leq T_j$;

(iii) $m(\mathbb{R}^n \setminus A_j) \leq c T_j^{-p} \int_{\{|\nabla \mathbf{u}_j| > T_j/c\}} |\nabla \mathbf{u}_j|^p \, \mathrm{d}\mathbf{x}$.

Combining (ii) and (iii) gives

$$\int_{\mathbb{R}^n \setminus A_j} |\nabla \mathbf{w}_j|^p \, \mathrm{d}\mathbf{x} \leq c \int_{|\nabla \mathbf{u}_j| > T_j/c} |\nabla \mathbf{u}_j|^p \, \mathrm{d}\mathbf{x}.$$

Therefore, since $\nabla \mathbf{w}_j = \nabla \mathbf{u}_j$ almost everwhere on A_j, the above inequality and the p-equiintegability of $\{\nabla \mathbf{u}_j\}_{j=1}^\infty$ yield the p-equiintegrability of $\{\nabla \mathbf{w}_j\}_{j=1}^\infty$. If we simply restrict $\mathbf{w}_j$ to Q, we see that $\{\nabla \mathbf{w}_j\}$ generates ν, is p-equiintegrable, and satisfies the appropriate L^∞ estimates, but we do not necessarily have that $\mathbf{w}_j$ has zero trace on ∂Q. However, using the definitions of A_j and the maximal function, we have

$$A_j \supset \left\{ \mathbf{x} \in \mathbb{R}^n : \frac{L}{\operatorname{dist}(\mathbf{x}, Q)^n} \leq T_j \right\}$$

for some $L < \infty$; since $\mathbf{w}_j = \mathbf{u}_j$ almost everywhere on A_j and $\mathbf{u}_j = 0$ outside Q, we obtain a sequence $\{r_j\}_{j=1}^\infty \subset [1, \infty)$ and a sequence of cubes $\{Q_j\}_{j=1}^\infty$ such that $\lim_{j\to\infty} r_j = 1$, $Q \subset Q_j$, the sides of Q_j have length r_j, and $\mathbf{w}_j \in W_0^{1,\infty}(Q_j; \mathbb{R}^N)$. Furthermore, we can assume that Q_j has sides parallel to the axes and has center at $(1/2, 1/2, \cdots, 1/2)$. Letting $\mathbf{x}_j := ((1-r_j)/2, (1-r_j)/2, \cdots, (1-r_j)/2)$ (i.e. $\mathbf{x}_j$ is the corner of Q_j in which each coordinate is minimized), we can define $\mathbf{v}_j \subset W_0^{1,\infty}(Q; \mathbb{R}^N)$ to be a rescaled version of $\mathbf{w}_j$:

$$\mathbf{v}_j(\mathbf{x}) := \frac{1}{r_j} \mathbf{w}_j \left(\mathbf{x}_j + r_j \mathbf{x} \right).$$

Note that $\nabla \mathbf{v}_j(\mathbf{x}) = \nabla \mathbf{w}_j(\mathbf{x}_j + r_j \mathbf{x})$. Thus the p-equiintegrability of $\{\nabla \mathbf{v}_j\}_{j=1}^\infty$ follows from the p-equiintegrability of $\{\nabla \mathbf{w}_j\}_{j=1}^{\infty}$, and $\|\nabla \mathbf{v}_j\|_{L^\infty} \leq T_j$. Therefore the only thing we have yet to show is that $\{\nabla \mathbf{v}_j\}_{j=1}^{\infty}$ generates the Young measure ν. It suffices to show that

$$\lim_{j\to\infty} \int_{Q_0} \{\varphi(\nabla \mathbf{w}_j(\mathbf{x})) - \varphi(\nabla \mathbf{v}_j(\mathbf{x}))\} \mathrm{d}\mathbf{x} = 0 \tag{3.21}$$

for every cube $Q_0 \subset Q$ and every $\varphi \in \mathcal{C}_0(\mathbb{R}^{N\times n})$, since from this it follows that $\{\nabla \mathbf{w}_j\}$ and $\{\nabla \mathbf{v}_j\}$ generate the same Young measure. Using a change of variables and letting $\tilde{Q}_j := \mathbf{x}_j + r_j Q_0$, we obtain

$$\int\limits_{Q_0} \varphi(\nabla \mathbf{w}_j(\mathbf{x}))\mathrm{d}\mathbf{x} - \int\limits_{Q_0} \varphi(\nabla \mathbf{v}_j(\mathbf{x}))\mathrm{d}\mathbf{x} = \int\limits_{Q_0} \varphi(\nabla \mathbf{w}_j(\mathbf{x}))\mathrm{d}\mathbf{x} - r_j^{-n} \int\limits_{\tilde{Q}_j} \varphi(\nabla \mathbf{w}_j(\mathbf{x}))\mathrm{d}\mathbf{x}.$$

Since $r_j \to 1$ and $\mathbf{x}_j \to (0, 0, \cdots, 0)$, it is easily seen that

$$m((Q_0 \setminus \tilde{Q}_j) \cup (\tilde{Q}_j \setminus Q_0)) \to 0.$$

Using this and that $r_j^{-n} \to 1$ in the above equality yields (3.21), which finishes the proof. $\square$

Lemma 3.5. *Suppose that $\nu \in \mathcal{M}(\mathbb{R}^{N\times n})$ is a probability measure that satisfies*

(i) $\int_{\mathbb{R}^{N\times n}} \mathbf{F} \mathrm{d}\nu(\mathbf{F}) = \mathbf{0}$;

(ii) $\int_{\mathbb{R}^{N\times n}} \varphi(\mathbf{F}) \mathrm{d}\nu(\mathbf{F}) \geq \varphi(\mathbf{0})$ *for every quasiconvex* $\varphi : \mathbb{R}^{N\times n} \to [0, \infty)$ *satisfying* $\varphi(\mathbf{F}) \leq C(1 + |\mathbf{F}|^p)$;

(iii) $\int_{\mathbb{R}^{N\times n}} |\mathbf{F}|^p \, \mathrm{d}\nu(\mathbf{F}) < \infty$.

For each $0 \leq \lambda < n$, there is a sequence of functions $\{\mathbf{u}_j\}_{j=1}^{\infty} \subset W_0^{1,p}(\Omega; \mathbb{R}^N)$ uniformly bounded in $W^{1,(p,\mu)}(Q; \mathbb{R}^N)$ for every $0 \leq \mu < n$ such that $\{\nabla \mathbf{u}_j\}$ is p-equiintegrable, *generates the measure ν, and satisfies*

$$\|\nabla \mathbf{u}_j\|^p_{L^{p,\lambda}(Q;\mathbb{R}^{N\times n})} \leq 2^{n+1} \int_{\mathbb{R}^{N\times n}} |\mathbf{F}|^p \, \mathrm{d}\nu(\mathbf{F}).$$

Remark 3.1. Using the continuous embedding

$$W^{1,(p,\mu)}(Q;\mathbb{R}^N) \hookrightarrow \mathcal{C}^{0,1-(n-\mu)/p}(Q;\mathbb{R}^N),$$

where $\mathcal{C}^{0,\alpha}(Q;\mathbb{R}^N)$ denotes the space of Hölder-continuous functions with exponent α, we see that $\{\mathbf{u}_j\}$ is uniformly bounded in $\mathcal{C}^{0,\alpha}(Q;\mathbb{R}^N)$ for each $0 \leq \alpha < 1$.

Proof. By Lemma 3.3, there is a sequence $\{\mathbf{w}_j\}_{j=1}^\infty \subset W_0^{1,p}(Q;\mathbb{R}^N)$ such that the sequence $\{\nabla \mathbf{w}_j\}$ is p-equiintegrable and generates ν. For each $0 \leq \mu < n$, we define

$$C_\mu := \left(\max_{j\in\mathbb{N}} \left\{ \frac{\{\log(j)\}^p}{2^{n+1} j^{n-\mu} \int_{\mathbb{R}^{N\times n}} |\mathbf{F}|^p \, \mathrm{d}\nu(\mathbf{F})} \right\} \right)^{\frac{1}{p}},$$

and apply Lemma 3.4 with $T_j := \log(j)/C_\lambda$ to obtain $\{\mathbf{v}_j\}_{j=1}^\infty \subset W_0^{1,\infty}(Q;\mathbb{R}^N)$ such that the sequence $\{\nabla \mathbf{v}_j\}_{j=1}^\infty$ generates the Young measure ν, is p-equiintegrable, and satisfies

$$\|\nabla \mathbf{v}_j\|_{L^\infty} \leq \log(j)/C_\lambda.$$

We extend $\mathbf{v}_j$ by periodicity so that it is defined on all of $\mathbb{R}^n$ and define the new sequence $\{\mathbf{u}_j\}_{j=1}^\infty \subset W_0^{1,p}(Q;\mathbb{R}^N)$ by

$$\mathbf{u}_j(\mathbf{x}) = j^{-1}\mathbf{v}_j(j\mathbf{x}).$$

Note that

$$\|\nabla \mathbf{u}_j\|_{L^p(Q;\mathbb{R}^{N\times n})} = \|\nabla \mathbf{v}_j\|_{L^p(Q;\mathbb{R}^{N\times n})}$$

and that $\{\nabla \mathbf{u}_j\}$ is also p-equiintegrable. Furthermore, it is not too difficult to see that, for any cube $D \subset Q$ and $\varphi \in \mathcal{C}_0(\mathbb{R}^{N\times n})$,

$$\lim_{j\to\infty}\left(\int_D \varphi(\nabla \mathbf{u}_j)\mathrm{d}\mathbf{x} - \int_D \varphi(\nabla \mathbf{v}_j)\mathrm{d}\mathbf{x}\right) = 0,$$

from which it follows that $\{\nabla \mathbf{u}_j\}$ generates the same Young measure as $\{\nabla \mathbf{v}_j\}$, namely ν. Since $\{\nabla \mathbf{u}_j\}$ is p-equiintegrable, Theorem 3.1 gives

$$\lim_{j\to\infty} \|\nabla \mathbf{u}_j\|^p_{L^p} = \int_{\mathbb{R}^{N\times n}} |\mathbf{F}|^p \,\mathrm{d}\nu(\mathbf{F});$$

therefore, upon taking the tail end of the sequence if necessary, we can assume without loss of generality that

$$\|\nabla \mathbf{u}_j\|^p_{L^p} \leq 2\int_{\mathbb{R}^{N\times n}} |\mathbf{F}|^p \,\mathrm{d}\nu(\mathbf{F}).$$

With this observation in mind, the proof that $\{\nabla \mathbf{u}_j\}$ is uniformly bounded in $L^{p,\mu}$ for every $0 \leq \mu < n$ and that $\|\nabla \mathbf{u}_j\|^p_{L^{p,\lambda}}$ satisfies the appropriate estimate proceeds in the same way as the proof of Theorem 3.4; it follows from Proposition 3.7 in [47] that $\{\mathbf{u}_j\}$ is uniformly bounded in $W^{1,(p,\mu)}(\Omega;\mathbb{R}^N)$ for each $0 \leq \mu < n$. □

Now we consider nonhomogeneous measures.

Theorem 3.6. *Suppose that Ω has Lipschitz-continuous boundary. Let $0 \leq \lambda < n$ be given, and suppose that $\nu = \{\nu_{\mathbf{x}}\}_{\mathbf{x}\in\Omega}$ is a Young measure on $\mathbb{R}^{N\times n}$ that satisfies*

(i) $\int_{\mathbb{R}^{N\times n}} \mathbf{F}\mathrm{d}\nu_{\mathbf{x}}(\mathbf{F}) = \nabla \mathbf{u}(\mathbf{x})$ *for some* $\mathbf{u} \in W^{1,p}(\Omega;\mathbb{R}^N)$*;*

(ii) $\int_{\mathbb{R}^{N\times n}} \varphi(\mathbf{F})\mathrm{d}\nu_{\mathbf{x}}(\mathbf{F}) \geq \varphi(\nabla \mathbf{u}(\mathbf{x}))$ *for almost every* $\mathbf{x} \in \Omega$ *and every quasiconvex* $\varphi : \mathbb{R}^{N\times n} \to [0,\infty)$ *satisfying* $\varphi(\mathbf{F}) \leq C(1 + |\mathbf{F}|^p)$*;*

(iii) $\sup_{\substack{\mathbf{x}_0\in\Omega \\ \rho>0}} \rho^{-\lambda} \int_{\Omega\cap Q_{\mathbf{x}_0,\rho}} \int_{\mathbb{R}^{N\times n}} |\mathbf{F}|^p \,\mathrm{d}\nu_{\mathbf{x}}(\mathbf{F})\mathrm{d}\mathbf{x} < \infty.$

Then there is a sequence $\{\mathbf{u}_j\}_{j=1}^{\infty}$ *uniformly bounded in* $W^{1,(p,\lambda)}(\Omega;\mathbb{R}^N)$ *such that the sequence of gradients* $\{\nabla\mathbf{u}_j\}_{j=1}^{\infty}$ *generates the Young measure* ν *and is* p*-equiintegrable, and* $\mathbf{u}_j - \mathbf{u} \in W_0^{1,p}(\Omega;\mathbb{R}^N)$*.*

Remark 3.2. It is easily seen that conditions (i), (ii), and (iii) are also necessary, and therefore these conditions characterize Young measures generated by p-equiintegrable sequences of gradients uniformly bounded in $L^{p,\lambda}(\Omega;\mathbb{R}^{N\times n})$.

Remark 3.3. If Ω does not have Lipschitz-continuous boundary, then the conclusion is weakened slightly; even though $\{\nabla\mathbf{u}_j\}$ is still uniformly bounded in $L^{p,\lambda}(\Omega;\mathbb{R}^{N\times n})$, we do not necessarily have that $\{\mathbf{u}_j\}$ is uniformly bounded in $W^{1,(p,\lambda)}(\Omega;\mathbb{R}^N)$. The rest of the conclusion remains unchanged.

Proof. First, assume that the function u appearing in (i) and (ii) is identically 0. In this case, the proof is similar to the proof of Theorem 3.5. Using the notation found there, we see that each $\nu_{k,i}$ satisfies (i), (ii), and (iii) of Lemma 3.5, and hence can be generated by a sequence of gradients $\{\nabla\mathbf{u}_j^{k,i}\}_{j=1}^{\infty}$ where $\{\mathbf{u}_j^{k,i}\} \subset W_0^{1,p}(Q;\mathbb{R}^N)$ is uniformly bounded in $W^{1,(p,\lambda)}(Q;\mathbb{R}^N)$. Since each $\mathbf{u}_j^{k,i} \in W_0^{1,p}(Q;\mathbb{R}^N)$, rescaling and "patching together" the gradients $\nabla\mathbf{u}_j^{k,i}$ (cf. (3.17) in Theorem 3.5) yields a function that is still the weak gradient of some function $\mathbf{u}_k \in W_0^{1,p}(\Omega;\mathbb{R}^N)$. We can also show that $\{\nabla\mathbf{u}_k\}$ generates ν, is uniformly bounded in $L^{p,\lambda}(\Omega;\mathbb{R}^{N\times n})$, and is p-equiintegrable in the exact same way as we did in the proof of Theorem 3.5. Since Ω has Lipschitz-continuous boundary, it follows from Proposition 3.7 in [47] that $\{\mathbf{u}_j\}$ is uniformly bounded in $W^{1,(p,\lambda)}(\Omega;\mathbb{R}^N)$.

If we do not assume that $\mathbf{u} \equiv 0$, but instead that $\mathbf{u} \in W^{1,p}(\Omega;\mathbb{R}^N)$, then we first note that we must in fact have that $\mathbf{u} \in W^{1,(p,\lambda)}(\Omega;\mathbb{R}^N)$. Indeed, by (i) and Jensen's inequality, we have

$$\sup_{\substack{\mathbf{x}_0\in\Omega\\ \rho>0}} \rho^{-\lambda}\int_{\Omega\cap Q_{\mathbf{x}_0,\rho}} |\nabla\mathbf{u}|^p\,\mathrm{d}\mathbf{x} = \sup_{\substack{\mathbf{x}_0\in\Omega\\ \rho>0}} \rho^{-\lambda}\int_{\Omega\cap Q_{\mathbf{x}_0,\rho}} \left|\int_{\mathbb{R}^{N\times n}} \mathbf{F}\mathrm{d}\nu_{\mathbf{x}}(\mathbf{F})\right|^p \mathrm{d}\mathbf{x}$$
$$\leq \sup_{\substack{\mathbf{x}_0\in\Omega\\ \rho>0}} \rho^{-\lambda}\int_{\Omega\cap Q_{\mathbf{x}_0,\rho}} \int_{\mathbb{R}^{N\times n}} |\mathbf{F}|^p\,\mathrm{d}\nu_{\mathbf{x}}(\mathbf{F})\mathrm{d}\mathbf{x},$$

which is finite by (iii). Hence $\nabla\mathbf{u}\in L^{p,\lambda}(\Omega;\mathbb{R}^{N\times n})$; again employing Proposition 3.7 in [47], we find that $\mathbf{u}\in W^{1,(p,\lambda)}(\Omega;\mathbb{R}^N)$.

Now define the translated Young measure $\tilde{\nu}=\{\tilde{\nu}_{\mathbf{x}}\}_{\mathbf{x}\in\Omega}$ by

$$\langle\tilde{\nu}_{\mathbf{x}},\varphi\rangle := \langle\nu_{\mathbf{x}},\varphi(\cdot-\nabla\mathbf{u}(\mathbf{x}))\rangle.$$

It is easy to check that the function $\tilde{\nu}$ is in the previous situation; i.e. $\tilde{\nu}$ satisfies (i), (ii), and (iii) with $\mathbf{u}\equiv 0$, so we obtain a sequence $\{\mathbf{v}_j\}_{j=1}^{\infty}\subset W_0^{1,p}(\Omega;\mathbb{R}^N)$ that is uniformly bounded in $W^{1,(p,\lambda)}(\Omega;\mathbb{R}^N)$ and such that $\{\nabla\mathbf{v}_j\}$ generates the measure $\tilde{\nu}$ and is p-equiintegrable. Now let $\mathbf{u}_j:=\mathbf{v}_j+\mathbf{u}$; then $\{\mathbf{u}_j\}$ is uniformly bounded in $W^{1,(p,\lambda)}$ and $\mathbf{u}_j-\mathbf{u}=\mathbf{v}_j\in W_0^{1,p}(\Omega;\mathbb{R}^N)$. The sequence $\{\nabla\mathbf{u}_j\}$ is p-equiintegrable, and generates the original measure ν by Theorem 3.3, which completes the proof. □

Chapter 4

Connections and Applications

In this section, we present some applications of Theorem 2.13 to various problems.

4.1 A Characterization of the Spaces $W^1L_{g,\lambda}(\Omega;\mathbb{R}^N)$

In this section, given a function $f : \Omega \times \mathbb{R}^{N\times n} \to [0,\infty)$, we define the functional $K_f : W^{1,1}(\Omega;\mathbb{R}^N) \to \mathbb{R}^*$ by

$$K_f(\mathbf{w}) := \int_\Omega f(\mathbf{x}, \nabla\mathbf{w})\mathrm{d}x.$$

With this notation, we have the following theorem.

Theorem 4.1. *Suppose that $\Omega \subset \mathbb{R}^n$ is open and bounded with $\mathcal{C}^1$ boundary, and that $\alpha : \Omega \to [1,\infty)$ satisfies* (1.12)-(1.14) *and $g : \Omega \times \mathbb{R}_+ \to \mathbb{R}_+$ satisfies* (1.15)-(1.17)*. Fix $\mathbf{u} \in W^1L_g(\Omega;\mathbb{R}^N)$, and let $0 \le \lambda < n$. The following are equivalent:*

(i) $g(\cdot, \nabla\mathbf{u}) \in L^{1,\lambda}(\Omega;\mathbb{R}^N)$*;*

(ii) $\mathbf{u}$ *is a* $(K_g, \{\gamma_\varepsilon\}, \{\nu_\varepsilon\})$*-minimizer for some* $\{\nu_\varepsilon\}_{\varepsilon>0} \subset L^{1,\lambda}(\Omega)$ *and* $\{\gamma_\varepsilon\}_{\varepsilon>0} \subset \mathcal{C}([0,\infty))$ *satisfying* $\gamma_\varepsilon(0) = 0$*; furthermore,* $\mathbf{u} - \overline{\mathbf{u}} \in W_0^1 L_g(\Omega; \mathbb{R}^N)$ *for some* $\overline{\mathbf{u}}$ *with* $g(\cdot, \nabla\overline{\mathbf{u}}) \in L^{1,\lambda}(\Omega)$*;*

(iii) *For every* $f : \Omega \times \mathbb{R}^{N\times n} \to [0,\infty)$ *that satisfies*

$$|f(\mathbf{x}, \mathbf{F})| \leq C(\beta(\mathbf{x}) + g(\mathbf{x}, |\mathbf{F}|))$$

for some $\beta \in L^{1,\lambda}(\Omega)$ *and also satisfies for some* $\sigma_\varepsilon \in L^{1,\lambda}(\Omega)$ *the inequality*

$$|f(\mathbf{x}, \mathbf{F}) - g(\mathbf{x}, |\mathbf{F}|)| < \varepsilon g(\mathbf{x}, |\mathbf{F}|)$$

whenever $g(\mathbf{x}, |\mathbf{F}|) > \sigma_\varepsilon(\mathbf{x})$*, it holds that* $\mathbf{u}$ *is a* $(K_f, \{\gamma_\varepsilon\}, \{\nu_\varepsilon\})$*-minimizer for some* $\{\nu_\varepsilon\}_{\varepsilon>0} \subset L^{1,\lambda}(\Omega)$ *and* $\{\gamma_\varepsilon\}_{\varepsilon>0} \subset \mathcal{C}([0,\infty))$ *satisfying* $\gamma_\varepsilon(0) = 0$*; furthermore,* $\mathbf{u} - \overline{\mathbf{u}} \in W_0^{1,p(\cdot)}(\Omega; \mathbb{R}^N)$ *for some* $\overline{\mathbf{u}} \in W^1 L_g(\Omega; \mathbb{R}^N)$ *with* $g(\cdot, |\nabla\overline{\mathbf{u}}|) \in L^{1,\lambda}(\Omega)$.

Remark 4.1. As a corollary, we have that $\mathbf{u}$ is either a $(K_f, \{\gamma_\varepsilon\}, \{\nu_\varepsilon\})$-minimizer for every f asymptotically related to g, or it is not a $(K_f, \{\gamma_\varepsilon\}, \{\nu_\varepsilon\})$-minimizer for any f asymptotically related to g.

Proof. That (iii) implies (ii) is trivial, and that (ii) implies (i) is follows from Theorem 2.13. We only need to show that (i) implies (iii). To this end, suppose that $g(\cdot, |\nabla\mathbf{u}|) \in L^{1,\lambda}(\Omega)$, and that $f : \Omega \times \mathbb{R}^{N\times n} \to \mathbb{R}$ satisfies the hypotheses of (iii). We can put $\overline{\mathbf{u}} := \mathbf{u}$ and $\nu_\varepsilon := \varepsilon^{-1} f(\cdot, \nabla\mathbf{u})$. Note that by the growth conditions on f, we have that $\nu_\varepsilon \in L^{1,\lambda}(\Omega)$. If $\mathbf{x}_0 \in \Omega$ and $\rho > 0$ are fixed, and $\boldsymbol{\varphi} \in W_0^1 L_g(\Omega(\mathbf{x}_0, \rho); \mathbb{R}^N)$, then we have, since f is nonnegative, that

$$K_f(\mathbf{u}) - K_f(\mathbf{u}+\boldsymbol{\varphi}) = \int_{\Omega(\mathbf{x}_0,\rho)} \{f(\mathbf{x},\nabla\mathbf{u}) - f(\mathbf{x},\nabla\mathbf{u}+\nabla\boldsymbol{\varphi})\}\,\mathrm{d}\mathbf{x} \le \varepsilon \int_{\Omega(\mathbf{x}_0,\rho)} \nu_\varepsilon(\mathbf{x})\mathrm{d}\mathbf{x},$$

so that $\mathbf{u}$ is a $(K_f, \{0\}, \{\nu_\varepsilon\})$-minimizer. Thus (i) implies (iii), and the proof is complete. □

4.2 Partial Differential Equations

We now present an application of Theorem 2.13 to partial differential equations.

Theorem 4.2. *Suppose that $\Omega \subset \mathbb{R}^n$ is open and bounded with $\mathcal{C}^1$ boundary, and that $\alpha : \Omega \to [1,\infty)$ satisfies (1.12)-(1.14) and $g : \Omega \times \mathbb{R}_+ \to \mathbb{R}_+$ satisfies (1.15)-(1.17). Suppose also that $\overline{\mathbf{u}} \in W^1L_g(\Omega;\mathbb{R}^N)$ satisfies $g(\cdot,|\nabla u|) \in L^{1,\lambda}(\Omega)$ for some $0 \le \lambda < n$. Fix $1 < s < \min\{r_2, 1+pr_2/n, p^*/p\}$, where r_2 is as in Remark 2.2, and suppose that the mappings $\mathbf{A} : \Omega \times \mathbb{R}^N \times \mathbb{R}^{N\times n} \to \mathbb{R}^{N\times n}$ and $\mathbf{b} : \Omega \times \mathbb{R}^N \times \mathbb{R}^{N\times n} \to \mathbb{R}^N$ satisfy the following properties:*

(i) *For each $\varepsilon > 0$, there is a function $\sigma_\varepsilon \in L_g(\Omega)$ with $g(\cdot,\sigma_\varepsilon(\cdot)) \in L^{1,\lambda}(\Omega)$ and a constant $\Sigma_\varepsilon < \infty$ such that*

$$\left|\mathbf{A}(\mathbf{x},\mathbf{u},\mathbf{F}) - g_t(\mathbf{x},|\mathbf{F}|)\frac{\mathbf{F}}{|\mathbf{F}|}\right| < \varepsilon g_t(\mathbf{x},|\mathbf{F}|)$$

for all $(\mathbf{x},\mathbf{u},\mathbf{F}) \in \Omega \times \mathbb{R}^N \times \mathbb{R}^{N\times n}$ satisfying $|\mathbf{F}| > \sigma_\varepsilon(\mathbf{x}) + \Sigma_\varepsilon\,|\mathbf{u}|$.

(ii) *There is a constant $M \ge 1$ and a function $\beta \in L_g(\Omega)$ with $g(\cdot,\beta(\cdot)) \in L^{1,\lambda}(\Omega)$*

such that

$$|\mathbf{A}(\mathbf{x},\mathbf{u},\mathbf{F})| \le M g_t(\mathbf{x},\beta(\mathbf{x})+|\mathbf{u}|+|\mathbf{F}|);$$
$$|\mathbf{b}(\mathbf{x},\mathbf{u},\mathbf{F})| \le M g_t(\mathbf{x},\beta(\mathbf{x})+|\mathbf{u}|+|\mathbf{F}|);$$

for all $(\mathbf{x},\mathbf{u},\mathbf{F}) \in \Omega \times \mathbb{R}^N \times \mathbb{R}^{N\times n}$.

Suppose that $\mathbf{u} \in W^1L_g(\Omega;\mathbb{R}^N)$ *is a weak solution to the system*

$$\operatorname{div}[\mathbf{A}(\mathbf{x},\mathbf{u}(\mathbf{x}),\nabla\mathbf{u}(\mathbf{x}))] = \mathbf{b}(\mathbf{x},\mathbf{u}(\mathbf{x}),\nabla\mathbf{u}(\mathbf{x})) \text{ in } \Omega,$$
$$\mathbf{u}(\mathbf{x}) = \overline{\mathbf{u}}(\mathbf{x}) \text{ on } \partial\Omega;$$

i.e. $\mathbf{u}-\overline{\mathbf{u}} \in W_0^1L_g(\Omega;\mathbb{R}^N)$, *and for each* $\boldsymbol{\varphi} \in W_0^1L_g(\Omega;\mathbb{R}^N)$,

$$\int_\Omega \{\mathbf{A}(\mathbf{x},\mathbf{u},\nabla\mathbf{u})\cdot\nabla\boldsymbol{\varphi} + \mathbf{b}(\mathbf{x},\mathbf{u},\nabla\mathbf{u})\cdot\boldsymbol{\varphi}\}\,\mathrm{d}\mathbf{x} = 0. \tag{4.1}$$

Then $g(\cdot,|\nabla\mathbf{u}|) \in L^{1,\lambda}(\Omega)$.

Proof. The proof given here is similar to the proofs given for the analogous theorems in [35, 41]; our overall strategy is to show that $\mathbf{u}$ is an almost minimizer for the functional $J : \mathcal{A} \to \mathbb{R}$ defined by

$$J(\mathbf{w}) := \int_\Omega g(\mathbf{x},|\nabla\mathbf{w}(\mathbf{x})|)\mathrm{d}\mathbf{x},$$

where

$$\mathcal{A} := \left\{\mathbf{w} \in W^1L_g(\Omega;\mathbb{R}^N) : \mathbf{w}-\overline{\mathbf{u}} \in W_0^1L_g(\Omega;\mathbb{R}^N)\right\}.$$

We will allow the constant C to depend on s, along with all the other usual parameters. Fix $\mathbf{x}_0 \in \Omega$, $0 < \rho < \operatorname{diam}(\Omega)$, and $\mathbf{v} \in \mathcal{A}$ with $\mathbf{v}-\mathbf{u} \in W^{1,1}(\Omega(\mathbf{x}_0,\rho);\mathbb{R}^N)$. Then using the convexity of $g(\mathbf{x},\cdot)$ and (4.1), we have

$$
\begin{aligned}
K(\mathbf{u}) &\le K(\mathbf{v}) + \int_{\Omega(\mathbf{x}_0,\rho)} g_t(\mathbf{x},\nabla\mathbf{u})\frac{\nabla\mathbf{u}}{|\nabla\mathbf{u}|}\cdot[\nabla\mathbf{u}-\nabla\mathbf{v}] \\
&\le K(\mathbf{v}) + \int_{\Omega(\mathbf{x}_0,\rho)} \left(g_t(\mathbf{x},\nabla\mathbf{u})\frac{\nabla\mathbf{u}}{|\nabla\mathbf{u}|} - \mathbf{A}(\mathbf{x},\mathbf{u},\nabla\mathbf{u})\right)\cdot[\nabla\mathbf{u}-\nabla\mathbf{v}]\mathrm{d}\mathbf{x} \\
&\qquad - \int_{\Omega(\mathbf{x}_0,\rho)} \mathbf{b}(\mathbf{x},\mathbf{u},\nabla\mathbf{u})\cdot(\mathbf{u}-\mathbf{v})\mathrm{d}\mathbf{x}. \\
&= K(\mathbf{v}) + I_1 + I_2.
\end{aligned}
\tag{4.2}
$$

To estimate I_1, for $0<\varepsilon<1$ we split $\Omega(\mathbf{x}_0,\rho)$ into the set on which $|\nabla\mathbf{u}(\mathbf{x})| \le \sigma_\varepsilon(\mathbf{x}) + \Sigma_\varepsilon\,|\mathbf{u}(\mathbf{x})|$ (call this set $\mathcal{S}$), and the set on which the reverse inequality holds (call this set $\mathcal{T}$); using the growth conditions on f and $\mathbf{A}$, followed by part (iv) of Lemma 2.1, gives

$$
\begin{aligned}
I_1 &\le C\int_{\mathcal{S}} g_t(\mathbf{x},\sigma_\varepsilon+\beta+(1+\Sigma_\varepsilon)\,|\mathbf{u}|)\,|\nabla\mathbf{u}-\nabla\mathbf{v}|\,\mathrm{d}\mathbf{x} + \varepsilon\int_{\mathcal{T}} g_t(\mathbf{x},|\nabla\mathbf{u}|)\,|\nabla\mathbf{u}-\nabla\mathbf{v}|\,\mathrm{d}\mathbf{x} \\
&\le C\varepsilon\int_{\Omega(\mathbf{x}_0,\rho)} g_t\left(\mathbf{x},\varepsilon^{-1/(p\alpha(\mathbf{x})-1)}(\sigma_\varepsilon+\beta+(1+\Sigma_\varepsilon)\,|\mathbf{u}|)\right)|\nabla\mathbf{u}-\nabla\mathbf{v}|\,\mathrm{d}\mathbf{x} \\
&\quad + \varepsilon\int_{\Omega(\mathbf{x}_0,\rho)} g_t(\mathbf{x},|\nabla\mathbf{u}|)\,|\nabla\mathbf{u}-\nabla\mathbf{v}|\,\mathrm{d}\mathbf{x}.
\end{aligned}
$$

Now utilizing (vi) in Lemma 2.1 yields

$$
\begin{aligned}
I_1 &\le C\varepsilon\int_{\Omega(\mathbf{x}_0,\rho)}\{g(\mathbf{x},|\nabla\mathbf{u}|)+g(\mathbf{x},|\nabla\mathbf{v}|)\}\,\mathrm{d}\mathbf{x} \\
&\quad + C_\varepsilon\int_{\Omega(\mathbf{x}_0,\rho)}\{g(\mathbf{x},\beta)+g(\mathbf{x},\sigma_\varepsilon)+(1+\Sigma_\varepsilon)g(\mathbf{x},|\mathbf{u}|)\}\,\mathrm{d}\mathbf{x}.
\end{aligned}
$$

To estimate I_2, we use the growth constraints on $\mathbf{b}$ and again employ (vi) in Lemma 2.1

to obtain

$$I_2 \le \varepsilon \int_{\Omega(\mathbf{x}_0,\rho)} \{g(\mathbf{x},\beta) + g(\mathbf{x},|\mathbf{u}|) + g(\mathbf{x},|\nabla\mathbf{u}|)\}\,\mathrm{d}\mathbf{x} + C_\varepsilon \int_{\Omega(\mathbf{x}_0,\rho)} \{g(\mathbf{x},|\mathbf{u}|) + g(\mathbf{x},|\mathbf{v}|)\}\,\mathrm{d}\mathbf{x}.$$

Defining

$$\nu_\varepsilon(\mathbf{x}) := C_\varepsilon(\varepsilon/C_\varepsilon)^{-1}\left\{g(\mathbf{x},\beta(\mathbf{x})) + g(\mathbf{x},\sigma_{\varepsilon/C_\varepsilon}(\mathbf{x}))\right\} + g(\mathbf{x},\beta(\mathbf{x}))$$
$$T_\varepsilon := C_\varepsilon(1+\varepsilon+\Sigma_\varepsilon),$$

we have shown that

$$I_1 + I_2 \le \varepsilon \int_{\Omega(\mathbf{x}_0,\rho)} \{\nu_\varepsilon + g(\mathbf{x},|\nabla\mathbf{u}|) + g(\mathbf{x},|\nabla\mathbf{v}|)\}\,\mathrm{d}\mathbf{x} + T_\varepsilon \int_{\Omega(\mathbf{x}_0,\rho)} \{g(\mathbf{x},|\mathbf{u}|) + g(\mathbf{x},|\mathbf{v}|)\}\,\mathrm{d}\mathbf{x}.$$

Note that $\{\nu_\varepsilon\}_{\varepsilon>0} \subset L^{1,\lambda}(\Omega)$ and $\{T_\varepsilon\}_{\varepsilon>0} \subset \mathbb{R}_+$, so putting this estimate into (4.2) and employing Lemma 2.10 and Theorem 2.12 then bootstrapping as we did in Theorem 2.13 gives the desired result. $\square$

4.3 Regularity for Minimizing Sequences and Minimizing Young Measures

To prove the existence of Morrey regular minimizing sequences, we will use the following version of Ekeland's variational principle. For the proof of this result, see, for example, [47].

Theorem 4.3. *Let $(\mathcal{V}, d)$ be a complete metric space, and let $J : \mathcal{V} \to \mathbb{R}^*$ be a lower* *semicontinuous functional that is finite at some point in* $\mathcal{V}$. *Assume that for some* $\mathbf{v} \in \mathcal{V}$ *and some* $\varepsilon > 0$, *we have*

$$J(\mathbf{v}) \leq \inf_{\mathbf{w} \in \mathcal{V}} J(\mathbf{w}) + \varepsilon.$$

Then there exists a point $\mathbf{u} \in \mathcal{V}$ *such that*

$$J(\mathbf{u}) \leq J(\mathbf{v}) \text{ and } J(\mathbf{u}) \leq J(\mathbf{w}) + \varepsilon d(\mathbf{u}, \mathbf{w}) \text{ for all } \mathbf{w} \in \mathcal{V}.$$

We use Theorem 4.3 to prove the following, which supplies uniform regularity for minimizing sequences.

Theorem 4.4. *Suppose that* $\Omega \subset \mathbb{R}^n$ *is open and bounded with* $\mathcal{C}^1$ *boundary, and that* $\alpha : \Omega \to [1, \infty)$ *satisfies* (1.12)-(1.14) *and* $g : \Omega \times \mathbb{R}_+ \to \mathbb{R}_+$ *satisfies* (1.15)-(1.17). *Suppose that* $f : \Omega \times \mathbb{R}^N \times \mathbb{R}^{N \times n} \to \mathbb{R}$ *is a measurable function that is lower semicontinuous with respect to the second and third arguments and satisfies the following hypotheses for some* $0 \leq \lambda < n$, $r < 1 < s < \min\{r_2, 1 + pr_2/n, p^*/p\}$, $\beta \in L^1(\Omega)$, *and* $\gamma \in L^{1,\lambda}(\Omega)$:

$$\frac{1}{M} g(\mathbf{x}, |\mathbf{F}|) - M g(\mathbf{x}, |\mathbf{u}|)^r - \beta(\mathbf{x}) \leq f(\mathbf{x}, \mathbf{u}, \mathbf{F}) \leq M(\gamma(\mathbf{x}) + g(\mathbf{x}, |\mathbf{u}|)^s + g(\mathbf{x}, |\mathbf{F}|).$$

Here, r_2 *is as in Remark 2.2. Suppose further that* $\overline{\mathbf{u}} \in W^1 L_g(\Omega; \mathbb{R}^N)$ *with* $g(\cdot, |\nabla \overline{\mathbf{u}}|) \in L^{1,\lambda}(\Omega)$ *is given and define the admissible class by*

$$\mathcal{A} := \left\{ \mathbf{u} \in W^{1,1}(\Omega; \mathbb{R}^N) : \mathbf{u} - \overline{\mathbf{u}} \in W_0^{1,1}(\Omega; \mathbb{R}^N) \right\}.$$

If the functional $K : \mathcal{A} \to \mathbb{R}^*$ *is defined by* (1.6), *then there is a minimizing sequence* $\{\mathbf{u}_k\}_{k=1}^{\infty} \subset \mathcal{A}$ *for* K *such that the sequence* $\{g(\cdot, |\nabla \mathbf{u}_k|)\}_{k=1}^{\infty}$ *is uniformly bounded in*

$L^{1,\lambda}(\Omega)$.

Proof. By the growth condition imposed on f, we have that $K(\overline{\mathbf{u}}) < \infty$, so that K is finite at some point in $\mathcal{A}$. Let $\{\mathbf{v}_k\}_{k=1}^{\infty} \subset \mathcal{A}$ be a minimizing sequence for K, and let ε_k be defined by

$$\varepsilon_k := K(\mathbf{v}_k) - \inf_{\mathbf{w}\in\mathcal{V}} K(\mathbf{w}).$$

Without loss of generality, we assume that $\varepsilon_k \leq \varepsilon_0$, where ε_0 is as in Remark 2.4. By the coercivity condition on g, we have that K is bounded from below and we have that the sequence $\{g(\cdot, |\nabla \mathbf{v}_k|)\}_{k=1}^{\infty}$ is bounded in $L^1(\Omega)$. Notice that $\mathcal{A}$ equipped with the metric

$$d(\mathbf{u}, \mathbf{v}) := \|\nabla \mathbf{u} - \nabla \mathbf{v}\|_{L^1}$$

is a complete metric space, and that by Fatou's Lemma and the lower semicontinuity of f with respect to the second and third arguments, K is lower semicontinuous with respect to this metric. Therefore, by Theorem 4.3, we have that there is a sequence $\{\mathbf{u}_k\}_{k=1}^{\infty} \subset \mathcal{A}$ such that $K(\mathbf{u}_k) \leq K(\mathbf{v}_k)$ and $K(\mathbf{u}_k) \leq K(\mathbf{w}) + \varepsilon_k \|\nabla \mathbf{u}_k - \nabla \mathbf{w}\|_{L^1}$ for every $\mathbf{w} \in \mathcal{A}$. Since $K(\mathbf{u}_k)$ is dominated by $K(\mathbf{v}_k)$, it is clear that $\{\mathbf{u}_k\}_{k=1}^{\infty}$ is a minimizing sequence for K. Also, if $\boldsymbol{\varphi} \in W_0^{1,1}(\Omega(\mathbf{x}_0, \rho); \mathbb{R}^N)$, then from the above inequality, we have that

$$\begin{aligned} K(\mathbf{u}_k) &\leq K(\mathbf{u}_k + \boldsymbol{\varphi}) + \varepsilon_k \int_{\Omega(\mathbf{x}_0,\rho)} |\nabla \boldsymbol{\varphi}| \,\mathrm{d}\mathbf{x} \\ &\leq K(\mathbf{u}_k + \boldsymbol{\varphi}) + \varepsilon_k \int_{\Omega(\mathbf{x}_0,\rho)} C\,(1 + g(\mathbf{x}, |\nabla \boldsymbol{\varphi}|))\,\mathrm{d}\mathbf{x}. \end{aligned}$$

Recall that $\varepsilon_k \leq \varepsilon_0$, so that we have

$$K(\mathbf{u}_k) \leq K(\mathbf{u}_k + \boldsymbol{\varphi}) + \varepsilon \int_{\Omega(\mathbf{x}_0,\rho)} C\,(1 + g(\mathbf{x}, |\nabla \boldsymbol{\varphi}|))$$

for all $\varepsilon \geq \varepsilon_0$. Therefore, by Theorem 2.13 and Remarks 2.3 and 2.4, we have that $\{g(\cdot, |\nabla \mathbf{u}_k|)\}_{k=1}^{\infty} \subset L^{1,\lambda}(\Omega)$. In fact, since $\{\mathbf{u}_k\}_{k=1}^{\infty}$ is a minimizing sequence for K, the coercivity assumption on f implies that the quantities $\int_\Omega g(\mathbf{x}, |\nabla \mathbf{u}_k|)\mathrm{d}\mathbf{x}$ are uniformly bounded, so by Remark 2.5, the Morrey norms $\|g(\cdot, |\nabla \mathbf{u}_k|)\|_{L^{1,\lambda}}$ are uniformly bounded also, as desired. □

As a corollary to the above result, we have the existence of a minimizing Young measure that is Morrey regular in the case of natural growth (i.e. $g(\mathbf{x}, t) \equiv t^p$). Suppose that $\overline{\mathbf{u}} \in W^{1,p}(\Omega; \mathbb{R}^N)$ is given, and set

$$\mathcal{A} := \left\{ \mathbf{u} \in W^{1,p}(\Omega; \mathbb{R}^N) : \mathbf{u} - \overline{\mathbf{u}} \in W_0^{1,p}(\Omega) \right\}.$$

For a fixed function $f : \Omega \times \mathbb{R}^N \times \mathbb{R}^{N\times n}$, define the functional $K : \mathcal{A} \to \mathbb{R}$ by

$$K(w) := \int_\Omega f(\mathbf{x}, \mathbf{w}, \nabla \mathbf{w})\mathrm{d}\mathbf{x}.$$

It is well known that K need not admit a minimizer in $\mathcal{A}$ if g is not quasiconvex. One way to deal with this is to expand the admissible class to include Young measures as follows. Define $\mathcal{Y}_{\mathcal{A}}$ to be all those $\nu \in \mathcal{Y}(\Omega; \mathbb{R}^{N\times n})$ that are generated by $\{\nabla \mathbf{u}_j\}_{j=1}^{\infty}$ for a sequence $\{\mathbf{u}_j\}_{j=1}^{\infty} \subset \mathcal{A}$. It can be shown that for each $\nu \in \mathcal{Y}_{\mathcal{A}}$, there is a unique $\mathbf{u}_\nu \in \mathcal{A}$ with $\nabla \mathbf{u}_\nu(\mathbf{x}) = \int_{\mathbb{R}^{N\times n}} \mathbf{F}\mathrm{d}\nu_{\mathbf{x}}(\mathbf{F})$ almost everywhere in Ω. With this notation in place, we expand the admissible class $\mathcal{A}$ to the set $\tilde{\mathcal{A}}$, which we define to be the collection of all pairs $(\mathbf{u}_\nu, \nu) \in \mathcal{A} \times \mathcal{Y}_{\mathcal{A}}$. Then we define $\tilde{K} : \tilde{\mathcal{A}} \to \mathbb{R}$ by

$$\tilde{K}(\mathbf{u}_\nu, \nu) := \int_\Omega \int_{\mathbb{R}^{N\times n}} g(\mathbf{x}, \mathbf{u}_\nu(x), \mathbf{F})\mathrm{d}\nu_{\mathbf{x}}(\mathbf{F})\mathrm{d}\mathbf{x}.$$

We note that for every $\mathbf{u} \in \mathcal{A}$, we can define $\delta_{\nabla \mathbf{u}} = \{\delta_{\nabla \mathbf{u}(\mathbf{x})}\}_{\mathbf{x}\in\Omega}$ to be the Young measure that maps $\mathbf{x}$ to the Dirac mass centered at $\nabla\mathbf{u}(\mathbf{x})$. Then, for $\mathbf{u} \in \mathcal{A}$, we have that the pair $(\mathbf{u}, \delta_{\nabla\mathbf{u}}) \in \tilde{A}$, and $\tilde{K}(\mathbf{u}, \delta_{\nabla\mathbf{u}}) = K(\mathbf{u})$, so that we can think of $\mathcal{A} \subset \tilde{\mathcal{A}}$ and $\tilde{K}$ as simply extending K to the larger domain $\tilde{\mathcal{A}}$. Furthermore, it can be shown (see [53]) that if K is coercive, then $\tilde{K}$ admits a minimizer $(\mathbf{u}_\nu, \nu)$ and

$$\tilde{K}(\mathbf{u}_\nu, \nu) = \inf_{\mathbf{u}\in\mathcal{A}} K(\mathbf{u}).$$

With this notation and background, we now state the following corollary to Theorem 4.4.

Corollary 4.1. *Suppose that all the hypotheses of Theorem 4.4 hold with* $g(\mathbf{x}, t) \equiv t^p$ *for some* $p > 1$. *Let the functionals* K *and its extension* $\tilde{K}$, *as well as the admissible class* $\mathcal{A}$ *and* $\tilde{\mathcal{A}}$, *be defined as above. Then there is a minimizer* $(\mathbf{u}_\nu, \nu) \in \tilde{\mathcal{A}}$ *for* $\tilde{K}$ *such that the function*

$$\mathbf{x} \mapsto \int_{\mathbb{R}^{N\times n}} |\mathbf{F}|^p \, \mathrm{d}\nu_{\mathbf{x}}(\mathbf{F})$$

belongs to $L^{1,\lambda}(\Omega)$.

Proof. By Theorem 4.4, there is a minimizing sequence $\{\mathbf{u}_k\}_{k=1}^\infty \subset \mathcal{A}$ for K such that $\{\nabla\mathbf{u}_k\}_{k=1}^\infty$ is uniformly bounded in $L^{p,\lambda}(\Omega; \mathbb{R}^{N\times n}$. Let ν be the Young measure generated by (possibly a subsequence of) $\{\nabla\mathbf{u}_k\}_{k=1}^\infty$. Then ν will be a minimizing Young measure, and furthermore, for any $\mathbf{x}_0 \in \Omega$ and $\rho > 0$, we have, by the fundamental theorem for Young measures (Theorem 3.1), that

$$\begin{aligned}\rho^{-\lambda} \int_{\Omega(x_0,\rho)} \int_{\mathbb{R}^{N\times n}} |\mathbf{F}|^p \, \mathrm{d}\nu_{\mathbf{x}}(\mathbf{F}) \mathrm{d}\mathbf{x} &\leq \liminf_{k\to\infty} \rho^{-\lambda} \int_{\Omega(\mathbf{x}_0,\rho)} |\nabla\mathbf{u}_k|^p \, \mathrm{d}\mathbf{x} \\ &\leq \sup_{k\in\mathbb{N}} \| |\nabla\mathbf{u}_k|^p \|_{L^{1,\lambda}} < \infty,\end{aligned}$$

which immediately yields the desired result. ∎

We also have the following, which is a sort of converse to the above Corollary.

Theorem 4.5. *Suppose that* Ω *has Lipschitz-continuous boundary, and let* $f : \Omega \times \mathbb{R}^N \times \mathbb{R}^{N\times n} \to \mathbb{R}$ *be a measurable function such that* $f(\mathbf{x}, \cdot, \cdot)$ *is continuous for almost every* $\mathbf{x} \in \Omega$*. Also assume that* f *satisfies*

$$\frac{1}{c}\left|\mathbf{F}\right|^p - c\left|\mathbf{y}\right|^q - \alpha(\mathbf{x}) \leq f(\mathbf{x}, \mathbf{y}, \mathbf{F}) \leq c\left|\mathbf{F}\right|^p + c\left|\mathbf{y}\right|^r + \alpha(\mathbf{x})$$

for some $1 < p < \infty$*,* $0 \leq q < p$*,* $0 \leq r < p^*$*,* $c > 0$*, and* $\alpha \in L^1(\Omega)$*. Let* $\mathbf{u}_0 \in W^{1,p}(\Omega; \mathbb{R}^N)$ *be given, and suppose that* $\{\mathbf{u}_j\}_{j=1}^{\infty} \subset \mathcal{A} := \mathbf{u}_0 + W_0^{1,p}(\Omega; \mathbb{R}^N)$ *is a minimizing sequence for the functional* $J : \mathcal{A} \to \mathbb{R}$ *defined by*

$$J(\mathbf{v}) := \int_\Omega f(\mathbf{x}, \mathbf{v}(\mathbf{x}), \nabla\mathbf{v}(\mathbf{x}))\mathrm{d}\mathbf{x}.$$

Let $\nu = \{\nu_\mathbf{x}\}_{\mathbf{x}\in\Omega}$ *be the Young measure generated by* $\{\nabla\mathbf{u}_j\}_{j=1}^{\infty}$ *(or possibly a subsequence). If* ν *satisfies*

$$\sup_{\substack{\mathbf{x}_0\in\Omega \\ \rho>0}} \rho^{-\lambda} \int_{\Omega\cap Q_{\mathbf{x}_0,\rho}} \int_{\mathbb{R}^{N\times n}} \left|\mathbf{F}\right|^p \mathrm{d}\nu_\mathbf{x}(\mathbf{F})\mathrm{d}\mathbf{x} < \infty \tag{4.3}$$

for some $0 \leq \lambda < n$*, then there is a minimizing sequence* $\{\mathbf{v}_j\}_{j=1}^{\infty} \subset \mathcal{A}$ *such that* $\{\mathbf{v}_j\}_{j=1}^{\infty}$ *is uniformly bounded in* $W^{1,(p,\lambda)}(\Omega; \mathbb{R}^N)$ *and* $\{\nabla\mathbf{v}_j\}_{j=1}^{\infty}$ *is* p*-equiintegrable. In particular, if* $p + \lambda > n$*, then* $\{\mathbf{v}_j\}_{j=1}^{\infty}$ *is uniformly bounded in* $\mathcal{C}^{0,1-(n-\lambda)/p}(\Omega; \mathbb{R}^N)$*, the space of Hölder-continuous functions with exponent* $1 - (n - \lambda)/p$*.*

Remark 4.2. The conclusion of the theorem also holds in the case $p = 1$ if we additionally assume that $\{\nabla\mathbf{u}_j\}_{j=1}^{\infty}$ is equiintegrable.

Proof. Because of the coercivity condition on f, any minimizing sequence will be bounded in $W^{1,p}(\Omega; \mathbb{R}^N)$. Therefore, taking a subsequence if necessary, we may assume that $\mathbf{u}_j \rightharpoonup \mathbf{u}$ in $W^{1,p}(\Omega;\mathbb{R}^N)$ and that $\{\nabla\mathbf{u}_j\}_{j=1}^{\infty}$ generates the Young measure ν. Since $\{\nabla\mathbf{u}_j\}$ is equiintegrable, it follows from Theorem 3.1 that

$$\nabla\mathbf{u}_j \rightharpoonup \int_{\mathbb{R}^{N\times n}} \mathbf{F}\mathrm{d}\nu_{(\cdot)}(\mathbf{F}) \text{ in } L^1(\Omega;\mathbb{R}^N),$$

and hence

$$\nabla\mathbf{u}(\mathbf{x}) = \int_{\mathbb{R}^{N\times n}} \mathbf{F}\mathrm{d}\nu_{\mathbf{x}}(\mathbf{F})$$

for almost every $\mathbf{x} \in \Omega$. Thus condition (i) in Theorem 3.6 is satisfied. As ν is the Young measure generated by a sequence of gradients bounded in $L^p(\Omega;\mathbb{R}^{N\times n})$, it follows that (ii) in the same theorem is also fulfilled. Seeing that (4.3) is precisely (iii), Theorem 3.6 implies that there exists a sequence $\{\mathbf{v}_j\}_{j=1}^{\infty} \subset \mathcal{A}$ uniformly bounded in $W^{1,(p,\lambda)}(\Omega;\mathbb{R}^N)$ such that the sequence of gradients $\{\nabla\mathbf{v}_j\}$ is p-equiintegrable and generates ν.

To see that $\{\mathbf{v}_j\}$ is a minimizing sequence for the functional J, we define the Young measure $\mu = \{\mu_{\mathbf{x}}\}_{\mathbf{x}\in\Omega} \subset \mathcal{M}(\mathbb{R}^N \times \mathbb{R}^{N\times n})$ by

$$\mu_{\mathbf{x}} := \delta_{\mathbf{u}(\mathbf{x})} \times \nu_{\mathbf{x}},$$

where $\delta_{\mathbf{u}(\mathbf{x})}$ denotes the Dirac mass centered at $\mathbf{u}(\mathbf{x})$. Define the sequences of functions $\{\mathbf{w}_j\}$ and $\{\mathbf{z}_j\}$, with $\mathbf{w}_j, \mathbf{z}_j : \Omega \to \mathbb{R}^N \times \mathbb{R}^{N\times n}$, by

$$\mathbf{w}_j(\mathbf{x}) := (\mathbf{u}_j(\mathbf{x}), \nabla\mathbf{u}_j(\mathbf{x}));$$
$$\mathbf{z}_j(\mathbf{x}) := (\mathbf{v}_j(\mathbf{x}), \nabla\mathbf{v}_j(\mathbf{x})).$$

Both $\{\mathbf{u}_j\}$ and $\{\mathbf{v}_j\}$ converge weakly to $\mathbf{u}$ in $W^{1,p}(\Omega;\mathbb{R}^N)$, and hence also converge strongly to $\mathbf{u}$ in $L^p(\Omega;\mathbb{R}^N)$. Because of this strong convergence and because the sequences $\{\nabla \mathbf{u}_j\}$ and $\{\nabla \mathbf{v}_j\}$ generate ν, we have that each of the sequences $\{\mathbf{w}_j\}$ and $\{\mathbf{z}_j\}$ generate the measure μ. Furthermore, using the growth conditions on f, we see that the sequence $\{f(\cdot, \mathbf{z}_j(\cdot))\}$ is equiintegrable, so using Theorem 1, we obtain

$$\begin{aligned}\lim_{j\to\infty} J(\mathbf{v}_j) = \lim_{j\to\infty} \int_\Omega f(\mathbf{x}, \mathbf{z}_j(\mathbf{x}))\mathrm{d}\mathbf{x} &= \int_\Omega \int_{\mathbb{R}^{N\times n}} f(\mathbf{x}, \mathbf{u}(\mathbf{x}), \mathbf{F})\mathrm{d}\nu_{\mathbf{x}}(\mathbf{F}) \\ &\le \liminf_{j\to\infty} \int_\Omega \int_\Omega f(\mathbf{x}, \mathbf{w}_j(\mathbf{x}))\mathrm{d}\mathbf{x} = \liminf_{j\to\infty} J(\mathbf{u}_j).\end{aligned}$$

Since $\{\mathbf{u}_j\}$ is a minimizing sequence for J, it follows that $\{\mathbf{v}_j\}$ is also a minimizing sequence. This concludes the proof. $\square$

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