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Heron, Brahmagupta, Pythagoras, and the Law of Cosines

Kristin K. Johnson
University of Nebraska-Lincoln

Heron, Brahmagupta, Pythagoras, and the Law of Cosines
Expository Paper

Kristin K. Johnson

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David Fowler, Advisor

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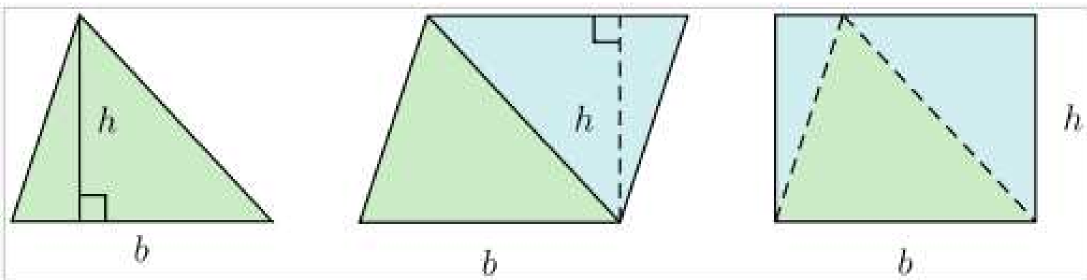
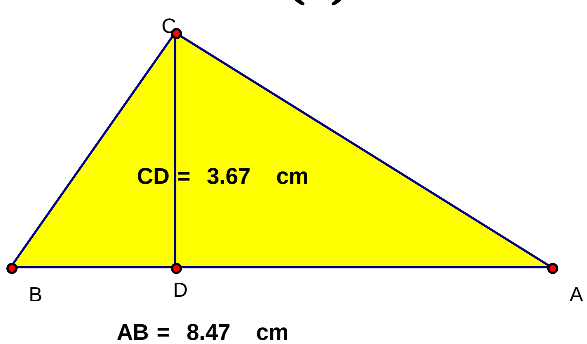
<u>Heron's Formula</u>

The formula for the area of a triangle can be developed by making an exact copy of the triangle and rotating it 180°. Then join it to the given triangle along one side to obtain a parallelogram as shown above. To form a rectangle, cut off a small triangle along the right and join it at the other side of the parallelogram. Because the area of the rectangle is the product of base (b) and height (h), the area of the given triangle must be 1⁄2bh.



The area of a triangle is A = 1⁄2bh, where b is the length of any side of the triangle (the base) and h (the altitude) is the perpendicular distance between the base and the vertex not on the base. This can be seen in the following sketch of ΔABC with base AB and altitude (height) CD. The height of a triangle is not always given so this can limit the use of this formula to calculate the area of the triangle.

Area formula = 1/2bh= $\left(\frac{1}{2}\right) AB \cdot CD = 15.53$ cm^2



Now what if you have an inquisitive student who asks, "What if you know the sides of a triangle but don't know the height?"

Enter Heron of Alexandria, a Greek mathematician who lived in the first century (c. 10 - 70). He was believed to have taught at the Museum of Alexandria. This is due to the fact that most of his writings appear as lecture notes for courses in mathematics, mechanics, physics and pneumatics. Heron's most famous invention was the first documented steam engine. He also wrote *Metrica* a book that describes how to find surfaces and volumes of various objects. In this book, there is a proof for a formula to find the area of any triangle when the lengths of the sides are all known called Heron's Formula.

To use the formula, the semi-perimeter is calculated by finding the sum of the lengths of the sides of the triangle and dividing the sum by 2.

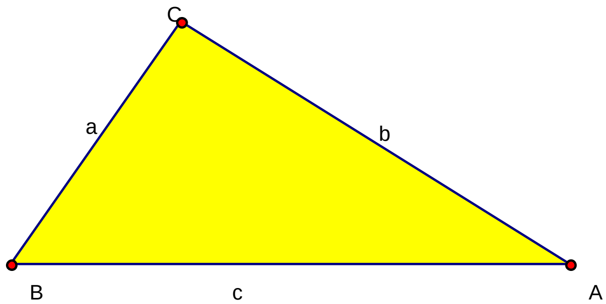
$$s = \frac{a+b+c}{2}$$

Then the area of the triangle is $\sqrt{s(s-a)(s-b)(s-c)}$. An example using this formula to find the area is shown below.

a = 4.48 cm
b = 6.94 cm
c = 8.47 cm

Semi perimeter s= $\frac{a+b+c}{2}$ = **9.94 cm**

Heron's formula
A= $\sqrt{s(s-a)(s-b)(s-c)}$ = $\sqrt{\frac{a+b+c}{2}\cdot\left(\frac{a+b+c}{2}-a\right)\left(\frac{a+b+c}{2}-b\right)\left(\frac{a+b+c}{2}-c\right)}$ = **15.53 cm^2**



Area $\triangle$BCA = 15.53 cm^2

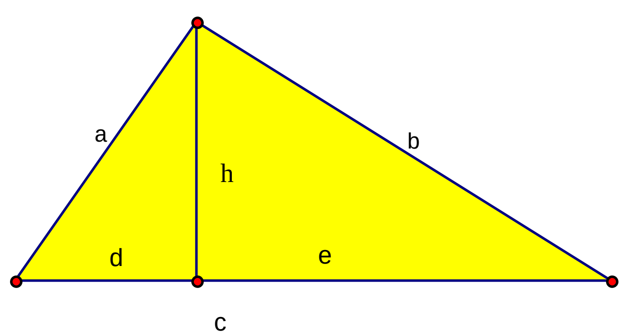
Heron's proof of this formula was found in Proposition 1.8 in his book *Metrica*. This document was lost until a fragment was found in 1894 and a full copy was found later in 1896. It is said that his proof is ingenious but complex. It is believed though that Archimedes (287 B.C. - 212 B.C.) already knew the formula. A more accessible proof, that came from http://mathpages.com/home/kmath196.htm, uses algebra.

Let A equal the area of a triangle shown below. Know that $c = d + e$ and h is an altitude.

Using the Pythagorean Theorem, a proof of which will be discussed in a later section of this paper, $d^2 + h^2 = a^2$ and $h^2 + e^2 = b^2$.

Combining the system of equations by subtraction,

$$d^2 + h^2 = a^2$$
$$-\left(h^2 + e^2 = b^2\right)$$
$$d^2 + h^2 - \left(e^2 + h^2\right) = a^2 - b^2$$
$$d^2 - e^2 = a^2 - b^2$$
$$(d+e)(d-e) = a^2 - b^2$$



Divide one side by d+e, which is equivalent to dividing the other side by c.

$$\frac{(d+e)(d-e)}{d+e} = \frac{a^2-b^2}{c}$$

Add c to both sides of the equation, then substitute d+e in for c and simplify.

$$d-e+c = \left(\frac{a^2-b^2}{c}\right)+c$$

$$d-e+d+e = \frac{a^2-b^2+c^2}{c}$$

$$2d = \frac{a^2-b^2+c^2}{c}$$

$$d = \frac{a^2-b^2+c^2}{2c}$$

Now let's look at the area formula of the triangle with base c and height *h*.

$$A = \frac{1}{2}hc$$

Using the fact $h^2 = a^2 - d^2$ then $h = \left(a^2-d^2\right)^{\frac{1}{2}}$, I can substitute in this expression for *h*.

$$A = \frac{1}{2}c\left(a^2-d^2\right)^{\frac{1}{2}}$$

$$A = \frac{1}{2}\left((ac)^2-(cd)^2\right)^{\frac{1}{2}}$$

Now substitute in the value of *d* from above and begin simplifying the expression.

$$A = \frac{1}{2}\left((ac)^2 - \left(\left(\frac{a^2-b^2+c^2}{2c}\right)c\right)^2\right)^{\frac{1}{2}}$$

$$A = \frac{1}{4}\left((2ac)^2-\left(a^2-b^2+c^2\right)^2\right)^{\frac{1}{2}}$$

$$4A = \left((2ac)^2-\left(a^2-b^2+c^2\right)^2\right)^{\frac{1}{2}}$$

$$16A^2 = (2ac)^2-\left(a^2-b^2+c^2\right)^2$$

By using the difference of squares, it can be factored.

$$16A^2 = \left(2ac+\left(a^2-b^2+c^2\right)\right)\left(2ac-\left(a^2-b^2+c^2\right)\right)$$

$$16A^2 = (a-b+c)(a+b+c)(-a+b+c)(a+b-c)$$

The semi-perimeter of a triangle is $s = \frac{a+b+c}{2}$

By using some algebra each of the following can be substituted into the above to get the next step simplified.

$2s = a+b+c$ and $2(s-a) = -a+b+c$ and $2(s-b) = a-b+c$ and $2(s-c) = a+b-c$

$16A^2 = 16s(s-a)(s-b)(s-c)$

$A^2 = s(s-a)(s-b)(s-c)$

$A = \sqrt{s(s-a)(s-b)(s-c)}$

Therefore, Heron's formula is proved.

A student might ask then, "How would you find the area of other polygons? Does this formula work for quadrilaterals?"

Brahmagupta's Formula

This question can be answered by looking at Brahmagupta's Formula. Brahmagupta lived in India (c. 598 - 670). He wrote *Brahma Sputa Siddhanta* of which covered many topics about astronomy and mathematics. He is attributed to the finding of zero. This is the earliest known text to treat zero as a number all its own. It also describes the use of positive numbers as fortunes and negatives as debts. In this text he describes a formula for finding the area of cyclic quadrilaterals which is named after Brahmagupta. It is an extension of Heron's Formula to a four-sided polygon whose vertices all lie on the same circle. Every triangle is cyclic, which means it can be inscribed in a circle, and a triangle can be regarded as a quadrilateral with one of its four edges set equal to zero.

Brahmagupta's Formula says that the semi-perimeter of a cyclic quadrilateral is $s = \frac{a+b+c+d}{2}$ so the area of the cyclic quadrilateral is equal to $\sqrt{(s-a)(s-b)(s-c)(s-d)}$.

The area of quadrilateral ABCD is equal to the sum of the areas of ΔABD and ΔBCD.

$$Area = \frac{1}{2}bc\sin(A) + \frac{1}{2}ad\sin(C)$$

Since it's a cyclic quadrilateral, $\angle DAB = 180^\circ - \angle DCB$ and the $\sin(A) = \sin(C)$.
Then using this relationship the area becomes:

$$Area = \frac{1}{2}bc\sin(A) + \frac{1}{2}ad\sin(A)$$

Factoring out a common factor of ½ sin(A) gives:

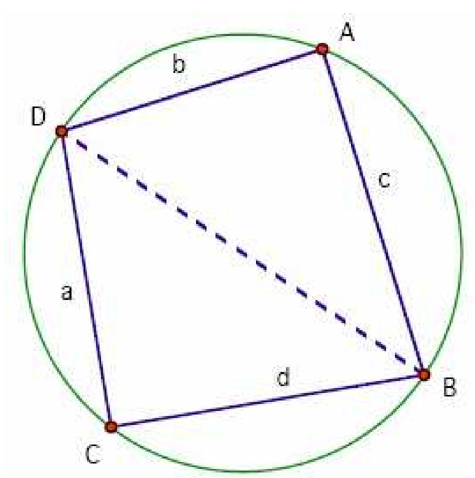
$$Area = \frac{1}{2}\sin(A)(bc+ad)$$

Square both sides and use the relationship that $\sin^2(A) = 1-\cos^2(A)$.

$$Area^2 = \frac{1}{4}\sin^2(A)(bc+ad)^2$$

$$Area^2 = \frac{1}{4}(1-\cos^2(A))(bc+ad)^2$$

Multiply both sides of the equation by 4 and then use the distributive property to rewrite the equation.



$$4Area^2 = (bc + ad)^2 - \cos^2(A)(bc + ad)^2$$

Using the Law of Cosines, which will be proved later in this paper, I can write the length of DB in two ways.

$$b^2 + c^2 - 2bc\cos(A) = a^2 + d^2 - 2ad\cos(C)$$

Since A and C are supplementary, then cos (C) = -cos(A). Substitute this into the length of DB and solve for cos(A).

$$b^2 + c^2 - 2bc\cos(A) = a^2 + d^2 + 2ad\cos(A)$$

$$b^2 + c^2 - a^2 - d^2 = 2ad\cos(A) + 2bc\cos(A)$$

$$b^2 + c^2 - a^2 - d^2 = 2\cos(A)(ad + bc)$$

$$\frac{b^2 + c^2 - a^2 - d^2}{2(ad + bc)} = \cos(A)$$

Use the value of the cos(A) and substitute it into the area formula. Through factoring it can be written as shown,

$$4Area^2 = (bc + ad)^2 - \left(\frac{b^2 + c^2 - a^2 - d^2}{2(ad + bc)}\right)^2 (bc + ad)^2$$

$$16Area^2 = 4(bc + ad)^2 - (b^2 + c^2 - a^2 - d^2)^2$$

$$16Area^2 = (b + c + d - a)(a + c + d - b)(a + b + d - c)(a + b + c - d)$$

The semi-perimeter is $s = \dfrac{a + b + c + d}{2}$, so $2s = a + b + c + d$.

This relationship can be rewritten in the following ways:

$$2(s - a) = b + c + d - a$$

$$2(s - b) = a + c + d - b$$

$$2(s - c) = a + b + d - c$$

$$2(s - d) = a + b + c - d$$

Now each of the above can be substituted into the area formula and then simplified in terms of the semi-perimeter.

$$16Area^2 = 2(s - a)2(s - b)2(s - c)2(s - d)$$

$$Area^2 = (s - a)(s - b)(s - c)(s - d)$$

$$Area = \sqrt{(s - a)(s - b)(s - c)(s - d)}$$

Therefore, Brahmagupta's Formula has been proved.

Brahmagupta's Formula can be extended to finding the area of other quadrilaterals by adding another term. A and C are any two pairs of opposite angles. In a cyclic quadrilateral each pair of opposite angles sum to π, so cos(π/2)=0 and the final term reduces to 0.

$$\sqrt{(s - a)(s - b)(s - c)(s - d) - (abcd)\left(\cos\left(\frac{A + C}{2}\right)\right)}$$

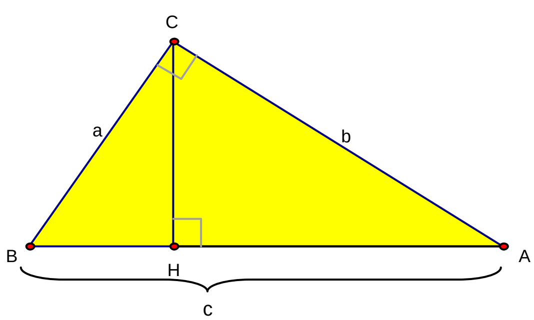
Heron's Formula is the special case of Brahmagupta's Formula where the length d is equal to zero. This is similar to how the Pythagorean Theorem is a special case of the Law of Cosines, which will be discussed next.

Pythagorean Theorem

The Pythagorean Theorem states that the sum of the squares of the legs of a right triangle is equal to the square of the hypotenuse. It can be used to find the length of a side of a right triangle if the other two sides are known. Pythagoras lived around 560 B.C. - 480 B.C. He was a Greek mathematician and philosopher. He founded a society based on mystic, religious, and scientific ideas. This society lived by a strict code of silence. It is said a Pythagorean would be put to death if they discussed ideas with people outside their circle. Whether Pythagoras or someone else from his school was the first to discover a proof of the theorem can't be determined. A statement of the theorem was discovered on a Babylonian tablet around 1900 B.C. - 1600 B.C. Euclid's (c. 300 B.C.) *Elements* furnished the first standard reference in Geometry and he was the first to prove that the theorem was "reversible".

Hundreds of proofs have been published, and may possibly have more known proofs than any other theorem. United States President James Garfield devised an original proof in 1876. *Pythagorean Proposition* by Elisha Scott Loomis contains over 350 proofs. An interesting note about the use of this formula in Hollywood is that after receiving his brains from the wizard in the 1939 film *The Wizard of Oz*, the scarecrow incorrectly states, "the sum of the square roots of any two sides of an isosceles triangle is equal to the square root of the remaining side."

I found two proofs that I particularly like. The first is based on the use of proportions.



$\frac{AC}{AB} = \frac{AH}{AC}$ and $\frac{BC}{AB} = \frac{HB}{BC}$

Using cross products to rewrite the two proportions result in the following equations.

$AC^2 = AB \cdot AH$ and $BC^2 = AB \cdot HB$

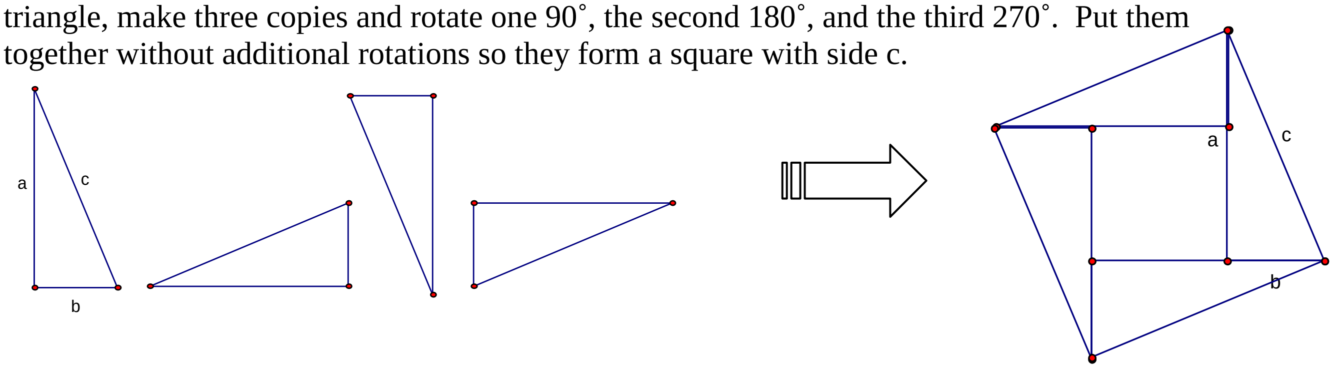
These can be combined using the addition property of equality and then simplify.

$AC^2 + BC^2 = AB \cdot AH + AB \cdot HB$

$AC^2 + BC^2 = AB(AH + HB)$

Noticing the fact that $AB = AH + HB$, the equation can be rewritten to $AC^2 + BC^2 = AB \cdot AB$ or $AC^2 + BC^2 = AB^2$. Since AC = b, BC = a, and AB = c, then $b^2 + a^2 = c^2$.

A visual proof, credited to Bhaskara, may be more accessible to my own students. Using a right triangle, make three copies and rotate one 90˚, the second 180˚, and the third 270˚. Put them together without additional rotations so they form a square with side c.



Area of the large outer square is which is equal to the sum of the four triangles and the small inner square.

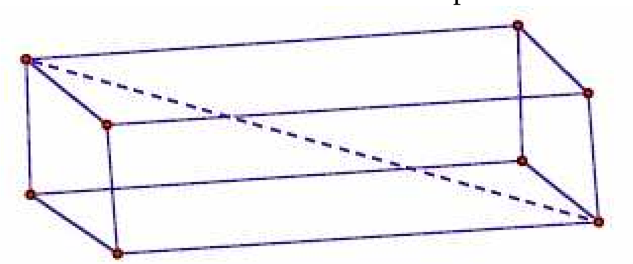
$$c^2 = (a-b)^2 + 4\left(\frac{ab}{2}\right)$$

$$c^2 = a^2 - 2ab + b^2 + 2ab$$

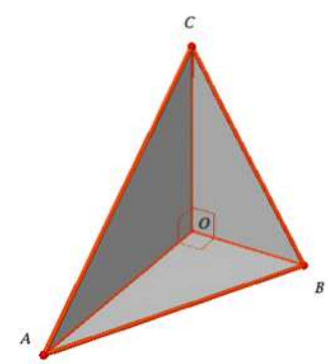
$$c^2 = a^2 + b^2$$

The Pythagorean theorem is used in more advanced mathematics. The applications that use the Pythagorean theorem include computing the distance between points on a plane, converting between polar and rectangular coordinates, computing perimeters, surface areas and volumes of geometric shapes.

The following is an example that might be used in an application problem of the Pythagorean Theorem. A removal truck comes to pick up a pole of length 6.5m. The dimensions of the truck are 3m, 3.5m, and 4m. Will the pole fit in the truck? Students are encouraged to draw a diagram such as the one shown below to help them visualize the situation.



It was generalized to a tri-rectangular tetrahedron and another theorem called de Gua's Theorem and another connection to the use of Heron's Formula to calculate the surface area.

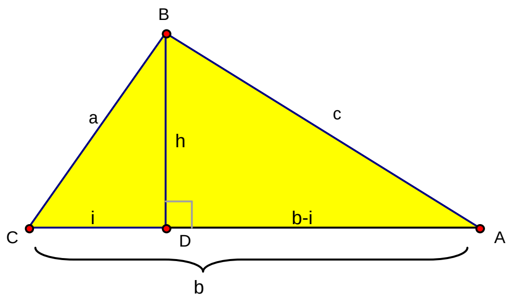


De Gua's Theorem then the square of the area of the face opposite the right angle corner is the sum of the squares of the areas of the other three faces.

The Pythagorean Theorem is limited to right triangles. So another bright student might ask, "What happens to the formula if the we had an acute or obtuse triangle?"

The Law of Cosines

The Law of Cosines enters the picture. The Law of Cosines relates the cosine of an angle in a triangle to its sides. For any triangle with sides of lengths a, b, c, and with C the angle opposite the side with length c, $c^2 = a^2 + b^2 - 2ab\cos(C)$. The Law of Cosines can be written for the other two sides in a triangle: $a^2 = b^2 + c^2 - 2bc\cos(A)$ and $b^2 = a^2 + c^2 - 2ac\cos(B)$.

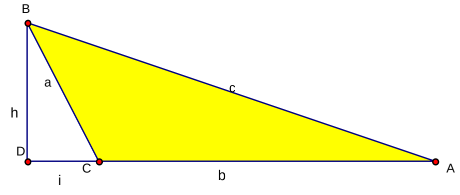


This law generalizes to the Pythagorean Theorem, which holds only for right triangles, so when C is equal to 90˚ its cosine is 0, so $c^2 = a^2 + b^2 - 2ab\cos(90^\circ)$. Since the cos(90˚) = 0, then the term -2ab(0) = 0 and it simplifies to $c^2 = a^2 + b^2$. It is useful for finding the length of the third side of a triangle when two sides and the included angle are know. It also allows you to calculate angle measures in a triangle when the lengths of the three sides are given.

This law is proved looking at two cases, the first is when the triangle is acute and the second is when it is obtuse.

Case 1- acute triangle using the figure above:
To prove the $c^2 = a^2 + b^2 - 2ab\cos(C)$, look at ΔBDA.
Using the Pythagorean Theorem $c^2 = h^2 + (b-i)^2$. This can be rewritten through multiplication to get $c^2 = h^2 + b^2 - 2bi + i^2$. In ΔCBD, $a^2 = h^2 + i^2$ so $h^2 = a^2 - i^2$. This allows $(a^2 - i^2)$ to be substituted in for h^2 which results in $c^2 = (a^2 - i^2) + b^2 - 2bi + i^2$. By simplifying, $c^2 = a^2 + b^2 - 2bi$. In ΔBCD, the trigonometric ratio for cos(C) is $\frac{i}{a}$. Solving for *i* in the cosine ratio leaves $i = a\cos(C)$.
Thus through substitution, $c^2 = a^2 + b^2 - 2ba\cos(C)$ or $c^2 = a^2 + b^2 - 2ab\cos(C)$.

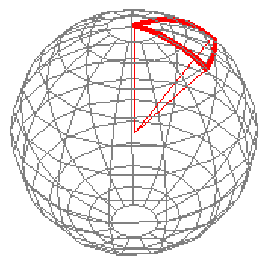
Case 2- obtuse triangle using the figure below:

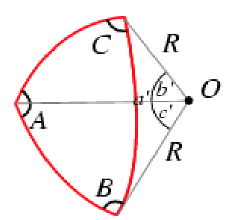


To prove the $c^2 = a^2 + b^2 - 2ab\cos(C)$, look at ΔBDA.
Using the Pythagorean Theorem, $c^2 = h^2 + (b+i)^2$. This can be rewritten through multiplication to get $c^2 = h^2 + b^2 + 2bi + i^2$. In ΔBCD, $a^2 = h^2 + i^2$ so $h^2 = a^2 - i^2$. This allows $(a^2 - i^2)$ to be substituted in for h^2 which results in $c^2 = (a^2 - i^2) + b^2 + 2ci + i^2$. By simplifying, $c^2 = a^2 + b^2 + 2ci$. In ΔBCD, the trigonometric ratio for cos(180°-C) is $\frac{i}{a}$. Solving for *i* in the cosine ratio leaves $i = a\cos(180^\circ - C)$ or $i = -a\cos(C)$
Thus, through substitution, $c^2 = a^2 + b^2 + 2b(-a\cos(C))$ or $c^2 = a^2 + b^2 - 2ab\cos(C)$.

The concepts behind the Law of Cosines in the book *Elements* by Euclid (c 300 B.C.) show up before the general use of the word "cosine" by mathematicians. Proposition 12 of Euclid's *Elements* describes the property for an obtuse triangle and Proposition 13 for an acute triangle. Al-Battani (c 10 A.D.) generalized these results to spherical geometry. Al-Kashi (c. 1380-1429 A.D.) wrote the theorem in a form suitable for triangulation. Its applications extend to calculating the necessary aircraft heading to counter a wind velocity and still proceed along a desired bearing to a destination. It has a practical use in land surveying.

Looking further into connections to the Law of Cosines one would find out how it works for spherical triangles.





Since this is a unit sphere, the lengths a, b, and c are equal to the angles (in radians) extended opposite an angle by those sides from the center of the sphere. For a non-unit sphere, they are the distances divided by the radius. When looking at small spherical triangles, that is in which the distances a, b, and c are small, the spherical law of cosines is about the same as the law of cosines in a plane. In spherical triangles both sides and angles are usually treated by their angle measure since sides are arc lengths of a great circle. There is a Law of Cosines for the sides and another for the angles. Using capital letters to represent angles, and lower case to represent the opposite sides, the law for sides is given as:

$\cos(a) = \cos(b)\cos(c) + \sin(b)\sin(c)\cos(A)$
$\cos(b) = \cos(a)\cos(c) + \sin(a)\sin(c)\cos(B)$
$\cos(c) = \cos(a)\cos(b) + \sin(a)\sin(b)\cos(C)$

and the law for angles is given by:

$\cos(A) = -\cos(B)\cos(C) + \sin(B)\sin(C)\cos(a)$
$\cos(B) = -\cos(A)\cos(C) + \sin(A)\sin(C)\cos(b)$
$\cos(C) = -\cos(A)\cos(B) + \sin(A)\sin(B)\cos(c)$

The National Council of Teachers of Mathematics discusses the need to make connections between mathematical ideas in the publication *Principles and Standards for School Mathematics*. When my students are encouraged to think mathematically, they begin to look for and make connections increasing their mathematical understanding. By creating these connections, they can build new knowledge on previous understandings. The types of questions that I need to ask my students are: What made you think of that? Why does this make sense? How are these ideas related? Did anyone think about this in a different way? Students can develop new connections by listening to their classmate's thinking as these answers are discussed. New ideas can then be seen as extensions of other mathematical ideas they already know.

As has been discussed in this paper, the idea that one can generalize one mathematical concept to a new or more complex problem is a very powerful and often utilized tool for mathematicians. It was suggested to me to find another pair of concepts that bear the same relationship as those I have discussed. The following information appears on the website cut-the-knot, by Andrew Bogomolny about generalizations.

"Pairs of statements in which one is a clear generalization of another whereas in fact the two are equivalent.

1. The Intermediate Value Theorem - The Location Principle (Bolzano Theorem)
2. Rolle's Theorem - The Mean Value Theorem
3. Existence of a tangent parallel to a chord - existence of a tangent parallel to the x-axis.
4. $\int \frac{dx}{1+x^2}$ and $\int \frac{dx}{p^2+x^2}$
5. The Maclaurin and Taylor series.
6. Two properties of Greatest Common Divisor
7. Pythagoras' Theorem and the Cosine Rule
8. Pythagoras' Theorem and its particular case of an isosceles right triangle
9. Pythagoras' Theorem and Larry Hoehn's generalization
10. Combining pieces of 2 and N squares into a single square
11. Measurement of inscribed and (more generally) secant angles"

I leave it to the reader to determine if these are really proven.

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http://www.answers.com/topic/brahmagupta - after_ad1
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http://www.cut-the-knot.org/pythagoras/cosine.shtml
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