5-2011

Homology of Artinian Modules Over Commutative Noetherian Rings

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HOMOLOGY OF ARTINIAN MODULES OVER COMMUTATIVE NOETHERIAN RINGS

by

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A DISSERTATION

Presented to the Faculty of

The Graduate College at the University of Nebraska

In Partial Fulfilment of Requirements

For the Degree of Doctor of Philosophy

Major: Mathematics

Under the Supervision of Professors Srikanth Iyengar and Roger Wiegand

Lincoln, Nebraska

May, 2011

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University of Nebraska, 2011

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This work is primarily concerned with the study of artinian modules over commutative noetherian rings.

We start by showing that many of the properties of noetherian modules that make homological methods work seamlessly have analogous properties for artinian modules. We prove many of these properties using Matlis duality and a recent characterization of Matlis reflexive modules. Since Matlis reflexive modules are extensions of noetherian and artinian modules many of the properties that hold for artinian and noetherian modules naturally follow for Matlis reflexive modules and more generally for mini-max modules.

In the last chapter we prove that if the Betti numbers of a finitely generated module over an equidimensional local ring are eventually non-decreasing, then the dimensions of sufficiently high syzygies are constant.

ACKNOWLEDGMENTS

The results in Chapters 2 and 3 are from joint work with Bethany Kubik and Sean Sather-Wagstaff. Working with them has been fun and expansive to my understanding of commutative algebra. I am also grateful to Luchezar Avramov, Cătălin Ciupercă, Edgar Enochs, Srikanth Iyengar, Amir Mafi, and Roger Wiegand for useful feedback about this research.

The results in Chapter 4 started at a mathematical research community in Snowbird Utah, funded by the NSF and organized by the AMS. I would like to thank my other group members, Kristen Beck, Christine Berkesch and Daniel Erman for their ideas and hard work. My group was guided by Craig Huneke, who introduced us to the problem and contributed to many of the key arguments. Additionally Srikanth Iyengar, Dan Katz and Roger Wiegand have given me valuable input on this research.

I am very grateful to the entire Math Department at UNL for being so generous with funding and for creating a supportive environment to work and learn. In particular Marilyn Johnson and Liz Youroukos have been very patient with any questions I have had about forms and in tracking me down when I overlooked e-mails that required my attention. Tom Marley and John Meakin were very generous in accepting my requests for support to travel for research, conferences and to speak in seminars. I am also grateful to the UNL Math Department and the US Department of Education for funding me with a GAANN grant for a significant portion of my graduate study. Roger Wiegand was the grant coordinator for our department, and he deserves much appreciation for making this grant possible.

I appreciate the work done by my reading committee, Luchezar Avramov and Mark Walker for their care in giving me feedback on this dissertation. My outside committee member Susan Levine should be thanked for being kind enough to oversee the legitimacy of my defense.

I feel very privileged to have been able to work under the mentorship of two great mathematicians. My advisors Srikanth and Roger have encouraged me to try and tackle problems outside of my comfort zone and to overcome many of the deficiencies in my professionalism. They have guided me from the infancy of my understanding in commutative algebra. They deserve my gratitude and my admiration. I hope that I can live up to the example that they have set.

GRANT INFORMATION

I was supported in part by a GAANN grant from the US Department of Education.

Contents

Chapter 1

Introduction

This work is primarily concerned with the study of artinian modules over commutative noetherian rings.

The work in Chapters 2 and 3 is joint with B. Kubik and Sean Sather-Wagstaff. In Chapters 2 and 3 we show that many of the properties that make homological methods work seamlessly when they are applied to finitely generated (i.e. noetherian modules) have analogous properties when applied to artinian modules. We prove many of these properties using Matlis duality and a recent characterization of Matlis reflexive modules. Matlis reflexive modules are always an extension of noetherian and artinian modules; see Fact 2.1.17 or [6, Theorem 12]. Consequently many of the properties that hold for artinian and noetherian modules also hold for Matlis reflexive modules and more generally for min-max modules. A module M is mini-max provided thaht it contains a noetherian submodule N such that the quotient M/N is artinian.

The primary way of viewing a finitely generated module is to give a list of generators and relations. This can come in the form of matrix, which defines the first map in a free resolution of the module. Alternatively, for an artinian module or another module that contains torsion elements, finding an embedding of that module in its injective hull may prove to be a better way of viewing the module. Although a direct construction of an injective resolution may seem elusive, Matlis duality offers an alternative approach to this construction. Matlis duality sends a flat resolution of a module to an injective resolution of its Matlis dual; see [12, Theorem 3.2.9]. In the case of an artinian R-module A we can use this correspondence to compute a minimal injective resolution of A. By first computing a minimal projective resolution of the Matlis dual of A, which is noetherian over a suitable semi-local complete ring and then applying Matlis duality to the free resolution, one obtains the desired minimal injective resolution. A simple consequence of this construction is that all of the Bass and hence the Betti numbers of artinian modules are finite.

When one applies $\operatorname{Ext}^i_R(-,-)$ and $\operatorname{Tor}^R_i(-,-)$ to a pair of noetherian modules the result is always noetherian. Similarly, we show that $\operatorname{Tor}^R_i(A,L)$ is artinian whenever A is artinian and for all maximal ideals $\mathfrak{m} \in \operatorname{Supp}(A)$ the ith Betti number of L with respect to $\mathfrak{m}$ is finite. Alternatively $\operatorname{Ext}^i_R(A,L)$ is noetherian over the completion with respect to $\cap_{\mathfrak{m}\in\operatorname{Supp}(A)}\mathfrak{m}$ whenever A is artinian and for all $\mathfrak{m} \in \operatorname{Supp}(A)$ the ith Bass number of L with respect to $\mathfrak{m}$ is finite. A special case occurs when $i = 0$, A and A' are artinian and N is noetherian: Both $A \otimes_R A'$ and $\operatorname{Hom}_R(A,N)$ are finite-length modules. Given a module L and a Matlis reflexive module M such that for all $\mathfrak{m} \in \text{m-Spec}(R) \cap \operatorname{Supp}(M)$ the ith Bass and Betti numbers of L with respect to $\mathfrak{m}$ are finite we show that $\operatorname{Tor}^R_i(M,L)$ and $\operatorname{Ext}^i_R(M,L)$ are Matlis reflexive modules.

When N is a noetherian R-module localization commutes with Ext, i.e., for any multiplicatively closed set $U \subseteq R$ and any R-module L we have $U^{-1}\operatorname{Ext}^i_R(N,L) \cong \operatorname{Ext}^i_{U^{-1}R}(U^{-1}N, U^{-1}L)$ for all i. Note that in general Tor always commutes with localization. When A is an artinian R-module, L is any R-module and U is a multiplicatively closed set disjoint from $\cup_{\mathfrak{m}\in\operatorname{Supp}(A)}\mathfrak{m}$ we have $\operatorname{Ext}^i_R(A,L) \cong \operatorname{Ext}^i_{U^{-1}R}(U^{-1}A, U^{-1}L)$. Artinian modules actually have a much more direct connection to localization than noetherian modules do. In particular $\mathrm{Supp}(A)$ always consists of a finite set of maximal ideals. For any maximal ideal $\mathfrak{m} \in \mathrm{Supp}(A)$ the composition $\Gamma_{\mathfrak{m}}(A) \to A \to A_{\mathfrak{m}}$ is an isomorphism, so that $A = \bigoplus_{\mathfrak{m}\in\mathrm{Supp}(A)} \Gamma_{\mathfrak{m}}(A) \cong \bigoplus_{\mathfrak{m}\in\mathrm{Supp}(A)} A_{\mathfrak{m}}$. Consequently, $\mathrm{Ext}^i_R(A, L) \cong \bigoplus_{\mathfrak{m}\in\mathrm{Supp}(A)} \mathrm{Ext}^i_{R_{\mathfrak{m}}}(A_{\mathfrak{m}}, L_{\mathfrak{m}})$, and similarly $\mathrm{Tor}^R_i(A, L) \cong \bigoplus_{\mathfrak{m}\in\mathrm{Supp}(A)} \mathrm{Tor}^{R_{\mathfrak{m}}}_i(A_{\mathfrak{m}}, L_{\mathfrak{m}})$.

We use Matlis duality to better understand of the vanishing behavior of $\mathrm{Ext}^i_R(A, A')$. Given noetherian modules N and N' the length of any maximal N'-regular sequence in $\mathrm{Ann}(N)$ equals $\inf\{i \geqslant 0 \mid \mathrm{Ext}^i_R(N, N') \neq 0\}$; see [9, Proposition 1.2.3] for details. Let $\mathfrak{a} = \bigcap_{\mathfrak{m}\in\mathrm{Supp}(A)\cup\mathrm{Supp}(A')} \mathfrak{m}$ and let $(-)^\vee$ denote the Matlis duality functor; see Definition **??**. We show that $\mathrm{Ext}^i_R(A, A') \cong \mathrm{Ext}^i_{\widehat{R}^{\mathfrak{a}}}(A'^\vee, A^\vee)$. Since $A'^\vee$ and $A^\vee$ are noetherian $\widehat{R}^{\mathfrak{a}}$-modules it follows that $\inf\{i \geqslant 0 \mid \mathrm{Ext}^i_R(A, A') \neq 0\}$ is the length of the longest $A^\vee$-regular sequence in $\mathrm{Ann}_{\widehat{R}^{\mathfrak{a}}}(A')$.

Let L and L' be R-modules. Since Matlis duality commutes with taking homology, the correspondence between a free resolution and an injective resolution of the Matlis dual implies the isomorphism $\mathrm{Tor}^R_i(L, L')^\vee \cong \mathrm{Ext}^i_R(L, L'^\vee)$. To see this isomorphism apply Hom-tensor adjointness to $(L \otimes_R F.)^\vee$ where $F.$ is a free resolution of L'. Similarly there is a map from $\mathrm{Tor}^R_i(L, L'^\vee)$ to $\mathrm{Ext}^i_R(L, L')^\vee$ given by the applying the Hom-evaluation morphism to appropriate left and right derived functors; see Remark 2.1.7. When L is a noetherian this map is an isomorphism. We show that if L is Matlis reflexive and the Bass numbers of L' are finite then the map is an isomorphism; see Theorem 2.4.8.

The results in Chapter 4 are joint work with K. Beck. Let R be a local ring with maximal ideal $\mathfrak{m}$. When M is a finitely generated R-module, the local cohomology modules $H^i_{\mathfrak{m}}(M)$ are artinian for all $i \geqslant 0$. In Chapter 4, we use local cohomology to show that if the Betti numbers of a finitely generated module over an equidimensional local ring are eventually non-decreasing, then the dimensions of sufficiently high syzygies are constant. Let $\Omega_n(M)$ denote the nth syzygy module of M. Since $\operatorname{Supp}(M)$ contains more information than just the dimension of M our results are focused on describing the asymptotic behavior of $\operatorname{Supp}(\Omega_n(M))$. In particular if the Betti numbers of M are eventually non-decreasing then $\operatorname{Supp}_n(M)$ is a 2-periodic function of n for all $n \gg 0$. Also, in this case we show that the minimal elements of $\operatorname{Supp}(\Omega_n(M))$ are actually minimal elements of $\operatorname{Spec}(R)$ for all $n \gg 0$.

All of the theorems, propositions and corollaries in this document that have not been explicitly cited from another source represent original research. Conversely, all of the statements labeled as facts in this document were previously known regardless of whether or not a citation has been included. Many of the lemmas found in the background sections in Chapters 2 and 3 require only trivial arguments or elementary observations for their proofs and we consider them to be likely known even if an explicit citation for where they can be found is not included. Our convention is that all of the Lemmas, which are not either in a background section or specifically citing another source, constitute original research.

Chapter 2

Homology of Artinian and Mini-max Modules over Local Rings

The results in this chapter are joint work with B. Kubik and S. Sather-Wagstaff. Most of the results in this chapter can be found in [17].

Throughout this chapter R will denote a unital, commutative, (noetherian) local ring with maximal ideal $\mathfrak{m}$ and residue field $k = R/\mathfrak{m}$. The $\mathfrak{m}$-adic completion of R is denoted $\widehat{R}$, the injective hull of k is $E = \mathrm{E}_R(k)$, and the Matlis duality functor is $(-)^\vee = \mathrm{Hom}_R(-, E)$. We denote the length of an R-module L by $\lambda_R(L)$.

This work is concerned, in part, with properties of the functors $\mathrm{Hom}_R(A, -)$ and $A \otimes_R -$, where A is an artinian R-module. To motivate this, recall that [13, Proposition 6.1] shows that if A and A' are artinian R-modules, then $A \otimes_R A'$ has finite length. Similarly if N is a noetherian R-module, then $\mathrm{Hom}_R(A, N)$ also has finite length; see Corollaries 2.2.12 and 2.3.9. In light of this fact, it is natural to investigate the properties of $\mathrm{Ext}^i_R(A, -)$ and $\mathrm{Tor}_i^R(A, -)$. Let $\mu^i_R(L) := \lambda_R(\mathrm{Ext}^i_R(k, L))$ denote the ith Bass number of L and $\beta_i^R(L') := \lambda_R(\operatorname{Tor}_i^R(k, L'))$ denote the ith Betti number of L'. In general, the modules $\operatorname{Ext}_R^i(A, N)$ and $\operatorname{Tor}_i^R(A, A')$ will not have finite length. However, we have the following; see Theorems 2.2.2 and 2.3.1.

Theorem 2.1. *Let A be an artinian R-module, and let $i \geqslant 0$. Let L and L' be R-modules such that $\mu_R^i(L)$ and $\beta_i^R(L')$ are finite. Then $\operatorname{Ext}_R^i(A, L)$ is a noetherian $\widehat{R}$-module, and $\operatorname{Tor}_i^R(A, L')$ is artinian.*

One should note that the Bass and Betti numbers of any artinian or noetherian module are always finite. In particular, when A and A' are artinian, Theorem 2.1 implies that $\operatorname{Ext}_R^i(A, A')$ is a noetherian $\widehat{R}$-module. The next result, contained in Theorem 2.4.3, gives another explanation for this fact.

Theorem 2.2. *Let A and A' be artinian R-modules, and let $i \geqslant 0$. Then there is an isomorphism $\operatorname{Ext}_R^i(A, A') \cong \operatorname{Ext}_{\widehat{R}}^i(A'^{\vee}, A^{\vee})$. Hence there are noetherian $\widehat{R}$-modules N and N' such that $\operatorname{Ext}_R^i(A, A') \cong \operatorname{Ext}_{\widehat{R}}^i(N, N')$.*

This result proves useful for studying the vanishing of $\operatorname{Ext}_R^i(A, A')$, since the vanishing of $\operatorname{Ext}_{\widehat{R}}^i(N, N')$ is somewhat well understood.

We say that an R-module M is *Matlis reflexive* provided that the natural biduality map $\delta_M\colon M \to M^{\vee\vee}$, given by $\delta_M(x)(\psi) = \psi(x)$, is an isomorphism. Our next result shows how extra conditions on the modules in Theorem 2.1 imply that $\operatorname{Ext}_R^i(A, L)$ and $\operatorname{Tor}_i^R(A, L')$ are Matlis reflexive; see Corollaries 2.2.4 and 2.3.3.

Theorem 2.3. *Let A, L, and L' be R-modules such that A is artinian. Assume that $R/(\operatorname{Ann}_R(A) + \operatorname{Ann}_R(L))$ and $R/(\operatorname{Ann}_R(A) + \operatorname{Ann}_R(L'))$ are complete. Given an index $i \geqslant 0$ such that $\mu_R^i(L)$ and $\beta_i^R(L')$ are finite, the modules $\operatorname{Ext}_R^i(A, L)$ and $\operatorname{Tor}_i^R(A, L')$ are Matlis reflexive.*

We say that an R-module M is *mini-max* when M has a noetherian submodule N such that M/N is artinian. In particular, noetherian modules are mini-max, as are artinian modules. A key point in the proof of the last theorem is a result of Belshoff, Enochs, and García Rozas [6]: An R-module M is Matlis reflexive if and only if it is mini-max and $R/\operatorname{Ann}_R(M)$ is complete.

A standard application of Hom-tensor adjointness shows that $\operatorname{Tor}_i^R(L, L')^\vee \cong \operatorname{Ext}_R^i(L, L'^\vee)$ for any R-modules L and L'. Similarly when N is noetherian an application of the Hom-evaluation morphism shows that $\operatorname{Ext}_R^i(N, L')^\vee \cong \operatorname{Tor}_i^R(N, L'^\vee)$. However this last isomorphism does not hold in general if we replace N by a non-noetherian module. We do, however, get the following:

Theorem 2.4. *Let A, M, M' and L be R-modules such that A is artinian and Matlis-reflexive, M is Matlis-reflexive, and M' is mini-max. We have $\operatorname{Ext}_R^i(M, M')^\vee \cong \operatorname{Tor}_i^R(M, M'^\vee)$ and $\operatorname{Ext}_R^i(M', M)^\vee \cong \operatorname{Tor}_i^R(M', M^\vee)$. Fix an index $i \geqslant 0$. If $\mu_R^i(L)$ is finite, then $\operatorname{Ext}_R^i(A, L)^\vee \cong \operatorname{Tor}_i^R(A, L^\vee)$. If $\mu_R^i(L)$ and $\mu_R^{i+1}(L)$ are finite then $\operatorname{Ext}_R^i(M, L)^\vee \cong \operatorname{Tor}_i^R(M, L^\vee)$.*

Many of our results generalize to the non-local setting. As this generalization requires additional tools, we treat it separately in Chapter 3.

2.1 Background and Preliminary Results

Torsion Modules

Definition 2.1.1. Let $\mathfrak{a}$ be a proper ideal of R. The $\mathfrak{a}$-adic completion of R, which is denoted by $\widehat{R}^{\mathfrak{a}}$, is the inverse limit of the inverse system whose objects are $R/\mathfrak{a}^n$ for all $n \geqslant 0$ and whose maps are the natural surjections $R/\mathfrak{a}^n \to R/\mathfrak{a}^m$ with $n > m \geqslant 0$. Given an R-module L, set $\Gamma_{\mathfrak{a}}(L) = \{x \in L \mid \mathfrak{a}^n x = 0 \text{ for } n \gg 0\}$. We say that L

is $\mathfrak{a}$-*torsion* if $L = \Gamma_{\mathfrak{a}}(L)$. A prime ideal $\mathfrak{p}$ of R is *associated* to L if there is an R-module monomorphism $R/\mathfrak{p} \hookrightarrow L$; the set of primes associated to L is $\mathrm{Ass}_R(L)$. The *support* of an R-module L is $\mathrm{Supp}_R(L) = \{\mathfrak{p} \in \mathrm{Spec}(R) \mid L_{\mathfrak{p}} \neq 0\}$. The set of minimal elements of $\mathrm{Supp}_R(L)$ with respect to inclusion is denoted $\mathrm{Min}_R(L)$. Also the variety of $\mathfrak{a}$ is given by $V(\mathfrak{a}) = \{\mathfrak{p} \in \mathrm{Spec}(R) | \ \mathfrak{a} \subseteq \mathfrak{p}\}$.

Fact 2.1.2. Let $\mathfrak{a}$ be a proper ideal of R, and let L be an $\mathfrak{a}$-torsion R-module.

(a) Every artinian R-module is $\mathfrak{m}$-torsion. In particular, the module E is $\mathfrak{m}$-torsion.

(b) We have $\mathrm{Supp}_R(L) \subseteq V(\mathfrak{a})$. Hence if L is $\mathfrak{m}$-torsion, then $\mathrm{Supp}_R(L) \subseteq \{\mathfrak{m}\}$.

(c) The module L has an $\widehat{R}^{\mathfrak{a}}$-module structure that is compatible with its R-module structure, as follows. For each $x \in L$, fix an exponent n such that $\mathfrak{a}^n x = 0$. For each $r \in \widehat{R}^{\mathfrak{a}}$, the isomorphism $\widehat{R}^{\mathfrak{a}}/\mathfrak{a}^n\widehat{R}^{\mathfrak{a}} \cong R/\mathfrak{a}^n$ provides an element $r_0 \in R$ such that $r - r_0 \in \mathfrak{a}^n\widehat{R}^{\mathfrak{a}}$, and we set $rx = r_0 x$.

(d) If $R/\mathfrak{a}$ is complete, then $\widehat{R}^{\mathfrak{a}}$ is naturally isomorphic to $\widehat{R}$. To see this, assume that $R/\mathfrak{a}$ is complete. Given $n \geqslant 1$ such that $R/\mathfrak{a}^n$ is complete we claim that $R/\mathfrak{a}^{n+1}$ is complete. Let $\mathfrak{a} = (f_1, \ldots, f_c)$ and let $\phi : \bigoplus_{i=1}^c R/\mathfrak{a}^n \to R/\mathfrak{a}^{n+1}$ given by multiplying the ith component by f_i. Then $\mathrm{Ker}(\phi)$ is complete since submodules of complete modules are complete. If we consider the natural map from the exact sequence

$$0 \longrightarrow \mathrm{Ker}(\phi) \longrightarrow \bigoplus_{i=1}^c R/\mathfrak{a}^n \longrightarrow R/\mathfrak{a}^{n+1} \longrightarrow R/\mathfrak{a} \longrightarrow 0$$

to the sequence we get by taking the completion, then it follows from the Five Lemma that $R/\mathfrak{a}^{n+1}$ is complete. This explains the second step in the next display:

$$\widehat{R}^{\mathfrak{a}} \cong \lim_{\leftarrow} R/\mathfrak{a}^n \cong \lim_{\leftarrow} \widehat{R}/\mathfrak{a}^n\widehat{R} = \widehat{(\widehat{R})}^{\mathfrak{a}} \cong \widehat{R}.$$

For the last step in this display, see, e.g., [1, Exercise 10.5].

Lemma 2.1.3. *Let $\mathfrak{a}$ be a proper ideal of R, and let L be an $\mathfrak{a}$-torsion R-module.*

1. *A subset $Z \subseteq L$ is an R-submodule if and only if it is an $\widehat{R}^{\mathfrak{a}}$-submodule.*
2. *The module L is noetherian (artinian, mini-max respectively) over R if and only if it is noetherian (artinian, mini-max respectively) over $\widehat{R}^{\mathfrak{a}}$.*

Proof. (1) Every $\widehat{R}^{\mathfrak{a}}$-submodule of L is an R-submodule by restriction of scalars. Conversely, fix an R-submodule $Z \subseteq L$. Since L is $\mathfrak{a}$-torsion, so is Z, and Fact 2.1.2(c) implies that Z is an $\widehat{R}^{\mathfrak{a}}$-submodule.

(2) follows immediately from (1). □

Lemma 2.1.4. *Let $\mathfrak{a}$ be a proper ideal of R, and let L be an $\mathfrak{a}$-torsion R-module.*

1. *The natural map $L \to \widehat{R}^{\mathfrak{a}} \otimes_R L$ is an isomorphism.*
2. *The left and right $\widehat{R}^{\mathfrak{a}}$-module structures on $\widehat{R}^{\mathfrak{a}} \otimes_R L$ are the same.*

Proof. The natural map $L \to \widehat{R}^{\mathfrak{a}} \otimes_R L$ is injective, as $\widehat{R}^{\mathfrak{a}}$ is faithfully flat over R. To show surjectivity, it suffices to show that each generator $r \otimes x \in \widehat{R}^{\mathfrak{a}} \otimes_R L$ is of the form $1 \otimes x'$ for some $x' \in L$. Let $n \geqslant 1$ such that $\mathfrak{a}^n x = 0$, and let $r_0 \in R$ such that $r - r_0 \in \mathfrak{a}^n\widehat{R}^{\mathfrak{a}}$. It follows that $r \otimes x = r_0 \otimes x = 1 \otimes (r_0x)$, and this yields the conclusion of part (1). This also proves (2) because $1 \otimes (r_0x) = 1 \otimes (rx)$. □

Lemma 2.1.5. *Let $\mathfrak{a}$ be a proper ideal of R, and let L and L' be R-modules such that L is $\mathfrak{a}$-torsion.*

1. *If L' is $\mathfrak{a}$-torsion, then* $\operatorname{Hom}_R(L, L') = \operatorname{Hom}_{\widehat{R}^{\mathfrak{a}}}(L, L')$*; thus* $L^\vee = \operatorname{Hom}_{\widehat{R}^{\mathfrak{a}}}(L, E)$.

2. *One has* $\operatorname{Hom}_R(L, L') \cong \operatorname{Hom}_R(L, \Gamma_{\mathfrak{a}}(L')) = \operatorname{Hom}_{\widehat{R}^{\mathfrak{a}}}(L, \Gamma_{\mathfrak{a}}(L'))$.

Proof. (1) It suffices to verify the inclusion $\operatorname{Hom}_R(L, L') \subseteq \operatorname{Hom}_{\widehat{R}^{\mathfrak{a}}}(L, L')$. Let $x \in L$ and $r \in \widehat{R}^{\mathfrak{a}}$, and fix $\psi \in \operatorname{Hom}_R(L, L')$. Let $n \geqslant 1$ such that $\mathfrak{a}^n x = 0$ and $\mathfrak{a}^n \psi(x) = 0$. Choose an element $r_0 \in R$ such that $r - r_0 \in \mathfrak{a}^n \widehat{R}^{\mathfrak{a}}$. It follows that $\psi(rx) = \psi(r_0 x) = r_0 \psi(x) = r\psi(x)$; hence $\psi \in \operatorname{Hom}_{\widehat{R}^{\mathfrak{a}}}(L, L')$. (Part (1) can also be deduced from Hom-tensor adjointness, using Lemma 2.1.4(1).)

(2) For each $f \in \operatorname{Hom}_R(L, L')$, one has $\operatorname{Im}(f) \subseteq \Gamma_{\mathfrak{a}}(L')$. This yields the desired isomorphism, and the equality is from part (1). $\square$

A Natural Map from $\mathbf{Tor_i^R(L, L'^\vee)}$ to $\mathbf{Ext_R^i(L, L')^\vee}$

Definition 2.1.6. Let L be an R-module, and let J be an R-complex. The *Hom-evaluation* morphism

$$\theta_{LJL'} \colon L \otimes_R \operatorname{Hom}_R(J, L') \to \operatorname{Hom}_R(\operatorname{Hom}_R(L, J), L')$$

is given by $\theta_{LJL'}(l \otimes \psi)(\phi) = \psi(\phi(l))$.

Remark 2.1.7. Let L and L' be R-modules, and let J be an injective resolution of L'. Using the notation $(-)^\vee$, we have $\theta_{LJE} \colon L \otimes_R J^\vee \to \operatorname{Hom}_R(L, J)^\vee$. The complex $J^\vee$ is a flat resolution of $L'^\vee$; see, e.g., [12, Theorem 3.2.16]. This explains the first isomorphism in the following sequence:

$$\operatorname{Tor}_i^R(L, L'^\vee) \xrightarrow{\cong} \operatorname{H}_i(L \otimes_R J^\vee) \xrightarrow{\operatorname{H}_i(\theta_{LJE})} \operatorname{H}_i(\operatorname{Hom}_R(L, J)^\vee) \xrightarrow{\cong} \operatorname{Ext}_R^i(L, L')^\vee.$$

For the second isomorphism, the exactness of $(-)^\vee$ implies that $\mathrm{H}_i(\mathrm{Hom}_R(L,J)^\vee) \cong \mathrm{H}^i(\mathrm{Hom}_R(L,J))^\vee \cong \mathrm{Ext}^i_R(L,L')^\vee$.

Definition 2.1.8. Let L and L' be R-modules, and let J be an injective resolution of L'. The R-module homomorphism

$$\Theta^i_{LL'}\colon\ \mathrm{Tor}^R_i(L,L'^\vee) \to \mathrm{Ext}^i_R(L,L')^\vee$$

is defined to be the composition of the the maps displayed in Remark 2.1.7.

Remark 2.1.9. Let L, L', and N be R-modules such that N is noetherian. It is straightforward to show that the map $\Theta^i_{LL'}$ is natural in L and in L'.

The fact that E is injective implies that $\Theta^i_{NL'}$ is an isomorphism; see [26, Lemma 3.60]. This explains the first of the following isomorphisms:

$$\mathrm{Ext}^i_R(N,L')^\vee \cong \mathrm{Tor}^R_i(N,L'^\vee) \qquad\qquad \mathrm{Tor}^R_i(L,L')^\vee \cong \mathrm{Ext}^i_R(L,L'^\vee).$$

The second isomorphism is a consequence of Hom-tensor adjointness,

Note that Definitions 2.1.6 and 2.1.8 will be valid in the non-local setting where E is replaced with a minimal injective cogenerator; see Definition 3.1.1. Also Remarks 2.1.7 and 2.1.9 will hold in the non-local setting.

Numerical Invariants

Definition 2.1.10. Let L be an R-module. For each integer i, the ith *Bass number* of L and the ith *Betti number* of L are respectively

$$\mu^i_R(L) = \lambda_R(\mathrm{Ext}^i_R(k,L)) \quad \text{and} \quad \beta^R_i(L) = \lambda_R(\mathrm{Tor}^R_i(k,L)),$$

where $\lambda_R(L')$ denotes the length of an R-module L'.

Remark 2.1.11. Let L be an R-module.

(a) If I is a minimal injective resolution of L, then for each index $i \geqslant 0$ such that $\mu_R^i(L) < \infty$, we have $I^i \cong E^{\mu_R^i(L)} \oplus J^i$ where J^i does not have E as a summand, that is, $\Gamma_{\mathfrak{m}}(J^i) = 0$; see, e.g., [21, Theorem 18.7]. Similarly, the Betti numbers of a noetherian module are the ranks of the free modules in a minimal free resolution. The situation for Betti numbers of non-noetherian modules is more subtle; see, e.g., Lemma 2.1.18.

(b) We have that $\mu_R^i(L) < \infty$ for all $i \geqslant 0$ if and only if $\beta_i^R(L) < \infty$ for all $i \geqslant 0$; see [19, Proposition 1.1].

When $\mathfrak{a} = \mathfrak{m}$, the next invariants can be interpreted in terms of vanishing of Bass and Betti numbers.

Definition 2.1.12. Let $\mathfrak{a}$ be an ideal of R. For each R-module L, set

$$\operatorname{grade}_R(\mathfrak{a}; L) = \inf\{i \geqslant 0 \mid \operatorname{Ext}_R^i(R/\mathfrak{a}, L) \neq 0\}$$
$$\operatorname{width}_R(\mathfrak{a}; L) = \inf\{i \geqslant 0 \mid \operatorname{Tor}_i^R(R/\mathfrak{a}, L) \neq 0\}.$$

We write $\operatorname{depth}_R(L) = \operatorname{grade}_R(\mathfrak{m}; L)$ and $\operatorname{width}_R(L) = \operatorname{width}_R(\mathfrak{m}; L)$.

Part (2) of the next result is known. We include it for ease of reference. All of the parts of the next Lemma generalize to the non-local setting with the same proof. The non-local analog of Lemma 2.1.13(2) is slightly different it can be found in Lemma 3.1.17(2).

Lemma 2.1.13. *Let L be an R-module, and let $\mathfrak{a}$ be a non-zero ideal of R.*

1. *Then* $\mathrm{width}_R(\mathfrak{a}; L) = \mathrm{grade}_R(\mathfrak{a}; L^\vee)$ *and* $\mathrm{width}_R(\mathfrak{a}; L^\vee) = \mathrm{grade}_R(\mathfrak{a}; L)$.

2. *For each index* $i \geqslant 0$ *we have* $\beta_i^R(L) = \mu_R^i(L^\vee)$ *and* $\beta_i^R(L^\vee) = \mu_R^i(L)$.

3. $L = \mathfrak{a}L$ *if and only if* $\mathrm{grade}_R(\mathfrak{a}; L^\vee) > 0$.

4. $L^\vee = \mathfrak{a}(L^\vee)$ *if and only if* $\mathrm{grade}_R(\mathfrak{a}; L) > 0$.

5. $\mathrm{grade}_R(\mathfrak{a}; L) > 0$ *if and only if* $\mathfrak{a}$ *contains a non-zero-divisor for* L.

Proof. Part (1) is from [14, Proposition 4.4], and part (2) follows directly from Remark 2.1.9.

(3)–(4) These follow from part (1) since $L = \mathfrak{a}L$ if and only if $\mathrm{width}_R(\mathfrak{a}; L) > 0$.

(5) By definition, we need to show that $\mathrm{Hom}_R(R/\mathfrak{a}, L) = 0$ if and only if $\mathfrak{a}$ contains a non-zero-divisor for L. One implication is explicitly stated in [9, Proposition 1.2.3(a)]. One can prove the converse like [9, Proposition 1.2.3(b)], using the fact that $R/\mathfrak{a}$ is finitely generated. □

The next result characterizes artinian modules in terms of Bass numbers.

Lemma 2.1.14. *Let L be an R-module. The following conditions are equivalent:*

1. *L is an artinian R-module;*

2. *L is an artinian $\widehat{R}$-module;*

3. *$\widehat{R} \otimes_R L$ is an artinian $\widehat{R}$-module; and*

4. *L is $\mathfrak{m}$-torsion and $\mu_R^0(L) < \infty$.*

Proof. (1) $\Longleftrightarrow$ (4) If L is artinian over R, then it is $\mathfrak{m}$-torsion by Fact 2.1.2(a), and we have $\mu_R^0(L) < \infty$ by [12, Theorem 3.4.3]. For the converse, assume that L is $\mathfrak{m}$-torsion and $\mu^0 = \mu_R^0(L) < \infty$. Since L is $\mathfrak{m}$-torsion, so is $\mathrm{E}_R(L)$. Thus, we have

$\mathrm{E}_R(L) \cong E^{\mu^0}$, which is artinian since $\mu^0 < \infty$. Since L is a submodule of the artinian module $\mathrm{E}_R(L)$, it is also artinian.

To show the equivalence of the conditions (1)–(3), first note that each of these conditions implies that L is $\mathfrak{m}$-torsion. (For condition (3), use the monomorphism $L \to \widehat{R} \otimes_R L$.) Thus, for the rest of the proof, we assume that L is $\mathfrak{m}$-torsion.

Because of the equivalence (1) $\iff$ (4), it suffices to show that

$$\mu^0_R(L) = \mu^0_{\widehat{R}}(L) = \mu^0_{\widehat{R}}(\widehat{R} \otimes_R L).$$

These equalities follow from the next isomorphisms

$$\mathrm{Hom}_R(k, L) \cong \mathrm{Hom}_{\widehat{R}}(k, L) \cong \mathrm{Hom}_{\widehat{R}}(k, \widehat{R} \otimes_R L)$$

which are from Lemmas 2.1.5(1) and 2.1.4, respectively. □

Lemma 2.1.15. *Let L be an R-module.*

1. *The module L is noetherian over R if and only if $L^\vee$ is artinian over R.*
2. *If $L^\vee$ is noetherian over R or over $\widehat{R}$, then L is artinian over R.*
3. *Let $\mathfrak{a}$ be a proper ideal of R such that $R/\mathfrak{a}$ is complete. If L is $\mathfrak{a}$-torsion, then L is artinian over R if and only if $L^\vee$ is noetherian over R.*

Proof. (1) This is [12, Corollary 3.4.4].

(2) If $L^\vee$ is noetherian over R, then we conclude from [12, Corollary 3.4.5] that L is artinian over R. To complete the proof of (2), we assume that $L^\vee$ is noetherian over $\widehat{R}$ and show that L is artinian. Fix a descending chain $L_1 \supseteq L_2 \supseteq \cdots$ of submodules of L. Dualize the surjections $L \twoheadrightarrow \cdots \twoheadrightarrow L/L_2 \twoheadrightarrow L/L_1$ to obtain a sequence of $\widehat{R}$-module monomorphisms $(L/L_1)^\vee \hookrightarrow (L/L_2)^\vee \hookrightarrow \cdots \hookrightarrow L^\vee$. The corresponding

ascending chain of submodules must stabilize since $L^\vee$ is noetherian over $\widehat{R}$, and it follows that the original chain $L_1 \supseteq L_2 \supseteq \cdots$ of submodules of L also stabilizes. Thus L is artinian.

(3) Assume that L is $\mathfrak{a}$-torsion. One implication is from part (2). For the converse, assume that L is artinian over R. From [21, Theorem 18.6(v)] we know that $\operatorname{Hom}_{\widehat{R}}(L, E)$ is noetherian over $\widehat{R}$, and Lemma 2.1.5(1) implies that $L^\vee = \operatorname{Hom}_{\widehat{R}}(L, E)$. Thus, Lemma 2.1.3(2) implies that $L^\vee$ is noetherian over R. □

Mini-max and Matlis Reflexive Modules

Definition 2.1.16. An R-module M is *mini-max* if there is a noetherian submodule $N \subseteq M$ such that M/N is artinian.

Fact 2.1.17. An R-module M is Matlis reflexive if and only if it is mini-max and $R/\operatorname{Ann}_R(M)$ is complete; see [6, Theorem 12]. Thus, if M is mini-max over R, then $\widehat{R} \otimes_R M$ is Matlis reflexive over $\widehat{R}$.

Lemma 2.1.18. *If M is mini-max over R, then $\beta_i^R(M) < \infty$ and $\mu_R^i(M) < \infty$ for all $i \geqslant 0$.*

Proof. We show that $\mu_R^i(M) < \infty$ for all $i \geqslant 0$; then Remark 2.1.11(b) implies that $\beta_i^R(M) < \infty$ for all $i \geqslant 0$. The noetherian case is standard. If M is artinian, then we have $\mu_R^0(M) < \infty$ by Lemma 2.1.14; since $E^{\mu_R^0(M)}$ is artinian, an induction argument shows that $\mu_R^i(M) < \infty$ for all $i \geqslant 0$. One deduces the mini-max case from the artinian and noetherian cases, using a long exact sequence. □

Lemma 2.1.19. *Let L be an R-module such that $R/\operatorname{Ann}_R(L)$ is complete. The following conditions are equivalent:*

1. *L is Matlis reflexive over R;*

2. *L is mini-max over R;*

3. *L is mini-max over $\widehat{R}$; and*

4. *L is Matlis reflexive over $\widehat{R}$.*

Proof. The equivalences (1) $\iff$ (2) and (3) $\iff$ (4) are from Fact 2.1.17. Note that conditions (3) and (4) make sense since L is an $\widehat{R}$-module; see Fact 2.1.2.

(2) $\implies$ (3) Assume that L is mini-max over R, and fix a noetherian R-submodule $N \subseteq L$ such that L/N is artinian over R. As $R/\operatorname{Ann}_R(L)$ is complete and surjects onto $R/\operatorname{Ann}_R(N)$, we conclude that $R/\operatorname{Ann}_R(N)$ is complete. Fact 2.1.2(d) and Lemma 2.1.3(1) imply that N is an $\widehat{R}$-submodule. Similarly, Lemmas 2.1.3(2) and 2.1.14 imply that N is noetherian over $\widehat{R}$, and L/N is an artinian over $\widehat{R}$. Thus L is mini-max over $\widehat{R}$.

(3) $\implies$ (2) Assume that L is mini-max over $\widehat{R}$, and fix a noetherian $\widehat{R}$-submodule $L' \subseteq L$ such that L/L' is artinian over $\widehat{R}$. Lemmas 2.1.3(2) and 2.1.14 imply that L' is noetherian over R, and L/L' is artinian over R, so L is mini-max over R. $\square$

Lemma 2.1.20. *Let L be an R-module such that $\mathfrak{m}^t L = 0$ for some integer $t \geqslant 1$. Then the following conditions are equivalent:*

1. *L is mini-max over R (equivalently, over $\widehat{R}$);*

2. *L is artinian over R (equivalently, over $\widehat{R}$);*

3. *L is noetherian over R (equivalently, over $\widehat{R}$); and*

4. *L has finite length over R (equivalently, over $\widehat{R}$).*

Proof. Lemma 2.1.19 shows that L is mini-max over R if and only if it is mini-max over $\widehat{R}$. Also, L is artinian (noetherian, resp., finite length) over R if and only if it is artinian (noetherian, resp., finite length) over $\widehat{R}$ by Lemmas 2.1.14 and 2.1.3(2).

The equivalence of conditions (2)–(4) follows from an application of [12, Proposition 2.3.20] over the artinian ring $R/\mathfrak{m}^t$. The implication (2) $\Longrightarrow$ (1) is evident. For the implication (1) $\Longrightarrow$ (2), assume that L is mini-max over R. Given a noetherian submodule $N \subseteq L$ such that L/N is artinian, the implication (3) $\Longrightarrow$ (2) shows that N is artinian; hence so is L. $\square$

Definition 2.1.21. A full subcategory of the category of R-modules is a *Serre subcategory* if it is closed under submodules, quotients, and extensions.

Lemma 2.1.22. *The category of mini-max (resp., noetherian, artinian, finite length, or Matlis reflexive) R-modules is a Serre subcategory.*

Proof. The noetherian, artinian, and finite length cases are standard, as is the Matlis reflexive case; see [12, p. 92, Exercise 2]. For the mini-max case, fix an exact sequence $0 \to L' \xrightarrow{f} L \xrightarrow{g} L'' \to 0$. Identify L' with $\mathrm{Im}(f)$. Assume first that L is mini-max, and fix a noetherian submodule N such that L/N is artinian. Then $L' \cap N$ is noetherian, and the quotient $L'/(L' \cap N) \cong (L' + N)/N$ is artinian, since it is a submodule of L/N. Thus L' is mini-max. Also, $(N + L')/L'$ is noetherian and $[L/L']/[(N + L')/L'] \cong L/(N + L')$ is artinian, so $L'' \cong L/L'$ is mini-max.

Next, assume that L' and L'' are mini-max, and fix noetherian submodules $N' \subseteq L'$ and $N'' \subseteq L''$ such that L'/N' and L''/N'' are artinian. Let $x_1, \ldots, x_h$ be coset representatives in L of a generating set for N''. Let $N = N' + Rx_1 + \ldots + Rx_h$. Then N is noetherian and the following commutative diagram has exact rows:

$$\begin{tikzcd}
0 \arrow[r] & N \cap L' \arrow[r] \arrow[d, hook] & N \arrow[r] \arrow[d, hook] & N'' \arrow[r] \arrow[d, hook] & 0 \\
0 \arrow[r] & L' \arrow[r] & L \arrow[r] & L'' \arrow[r] & 0.
\end{tikzcd}$$

The sequence $0 \to L'/(N \cap L') \to L/N \to L''/N'' \to 0$ is exact by the Snake Lemma.

The module $L'/(N \cap L')$ is artinian, being a quotient of L'/N'. Since the class of artinian modules is closed under extensions, the module L/N is artinian. It follows that L is mini-max. $\square$

The next two lemmas apply to the classes of modules from Lemma 2.1.22 and also work over non-local rings.

Lemma 2.1.23. *Let $\mathcal{C}$ be a Serre subcategory of the category of R-modules.*

1. *Given an exact sequence $L' \xrightarrow{f} L \xrightarrow{g} L''$, if $L', L'' \in \mathcal{C}$, then $L \in \mathcal{C}$.*
2. *Given an R-complex X and an integer i, if $X_i \in \mathcal{C}$, then $\mathrm{H}_i(X) \in \mathcal{C}$.*
3. *Given a noetherian R-module N, if $L \in \mathcal{C}$, then $\mathrm{Ext}^i_R(N, L), \mathrm{Tor}^R_i(N, L) \in \mathcal{C}$.*

Proof. (1) Assume that $L', L'' \in \mathcal{C}$. By assumption, $\mathrm{Im}(f), \mathrm{Im}(g) \in \mathcal{C}$. Using the exact sequence $0 \to \mathrm{Im}(f) \to L \to \mathrm{Im}(g) \to 0$, we conclude that L is in $\mathcal{C}$.

(2) The module $\mathrm{H}_i(X)$ is a subquotient of X_i, so it is in $\mathcal{C}$ by assumption.

(3) If F is a minimal free resolution of N, then the modules in the complexes $\mathrm{Hom}_R(F, L)$ and $F \otimes_R L$ are in $\mathcal{C}$, so their homologies are in $\mathcal{C}$ by part (2). $\square$

Lemma 2.1.24. *Let $R \to S$ be a local ring homomorphism, and let $\mathcal{C}$ be a Serre subcategory of the category of S-modules. Fix an S-module L, an R-module L', an R-submodule $L'' \subseteq L'$, and an index $i \geqslant 0$.*

1. *If $\mathrm{Ext}^i_R(L, L''), \mathrm{Ext}^i_R(L, L'/L'') \in \mathcal{C}$, then $\mathrm{Ext}^i_R(L, L') \in \mathcal{C}$.*
2. *If $\mathrm{Ext}^i_R(L'', L), \mathrm{Ext}^i_R(L'/L'', L) \in \mathcal{C}$, then $\mathrm{Ext}^i_R(L', L) \in \mathcal{C}$.*
3. *If $\mathrm{Tor}^R_i(L, L''), \mathrm{Tor}^R_i(L, L'/L'') \in \mathcal{C}$, then $\mathrm{Tor}^R_i(L, L') \in \mathcal{C}$.*

Proof. We prove part (1); the other parts are proved similarly. Apply $\mathrm{Ext}^i_R(L,-)$ to the exact sequence $0 \to L'' \to L' \to L'/L'' \to 0$ to obtain the next exact sequence:

$$\mathrm{Ext}^i_R(L,L'') \to \mathrm{Ext}^i_R(L,L') \to \mathrm{Ext}^i_R(L,L'/L'').$$

Since L is an S-module, the maps in this sequence are S-module homomorphisms. Now, apply Lemma 2.1.23(1). $\square$

2.2 Properties of $\mathrm{Ext}^i_R(M,-)$

This section documents properties of the functors $\mathrm{Ext}^i_R(M,-)$, where M is a minimax R-module.

Noetherianness of $\mathbf{Ext^i_R(A,L)}$

Lemma 2.2.1. *Let A and L be R-modules such that A is artinian and L is $\mathfrak{m}$-torsion.*

1. *Then* $\mathrm{Hom}_R(L,A) = \mathrm{Hom}_{\widehat{R}}(L,A) \cong \mathrm{Hom}_{\widehat{R}}(A^\vee, L^\vee)$.
2. *If L is artinian, then* $\mathrm{Hom}_R(L,A)$ *is a noetherian* $\widehat{R}$*-module.*

Proof. (1) The first equality is from Lemma 2.1.5(1). For the second equality, the fact that A is Matlis reflexive over $\widehat{R}$ explains the first step below:

$$\mathrm{Hom}_{\widehat{R}}(L,A) \cong \mathrm{Hom}_{\widehat{R}}(L,A^{vv}) \cong \mathrm{Hom}_{\widehat{R}}(A^v,L^v) \cong \mathrm{Hom}_{\widehat{R}}(A^\vee,L^\vee)$$

where $(-)^v = \mathrm{Hom}_{\widehat{R}}(-,E)$. The second step follows from Hom-tensor adjointness, and the third step is from Lemma 2.1.5(1).

(2) If L is artinian, then $L^\vee$ and $A^\vee$ are noetherian over $\widehat{R}$, so $\mathrm{Hom}_{\widehat{R}}(A^\vee,L^\vee)$ is also noetherian over $\widehat{R}$. $\square$

The next result contains part of Theorem 2.1 from the beginning of the chapter. When R is not complete, the example $\operatorname{Hom}_R(E,E) \cong \widehat{R}$ shows that $\operatorname{Ext}^i_R(A,L)$ is not necessarily noetherian or artinian over R.

Theorem 2.2.2. *Let A and L be R-modules such that A is artinian. For each index $i \geqslant 0$ such that $\mu^i_R(L) < \infty$, the module $\operatorname{Ext}^i_R(A,L)$ is a noetherian $\widehat{R}$-module.*

Proof. Let J be a minimal R-injective resolution of L. Remark 2.1.11(a) implies that $\Gamma_{\mathfrak{m}}(J)^i \cong E^{\mu^i_R(L)}$. Lemma 2.1.5(2) explains the first isomorphism below:

$$\operatorname{Hom}_R(A,J)^i \cong \operatorname{Hom}_R(A,\Gamma_{\mathfrak{m}}(J)^i) \cong \operatorname{Hom}_R(A,E)^{\mu^i_R(L)}.$$

Lemma 2.2.1 implies that these are noetherian $\widehat{R}$-modules. The differentials in the complex $\operatorname{Hom}_R(A,\Gamma_{\mathfrak{m}}(J))$ are $\widehat{R}$-linear because A is an $\widehat{R}$-module. Thus, the subquotient $\operatorname{Ext}^i_R(A,L)$ is a noetherian $\widehat{R}$-module. $\square$

Corollary 2.2.3. *Let A and M be R-modules such that A is artinian and M is mini-max. For each index $i \geqslant 0$, the module $\operatorname{Ext}^i_R(A,M)$ is a noetherian $\widehat{R}$-module.*

Proof. Apply Theorem 2.2.2 and Lemma 2.1.18. $\square$

The next result contains part of Theorem 2.3 from the beginning of the chapter.

Corollary 2.2.4. *Let A and L be R-modules such that $R/(\operatorname{Ann}_R(A) + \operatorname{Ann}_R(L))$ is complete and A is artinian. For each index $i \geqslant 0$ such that $\mu^i_R(L) < \infty$, the module $\operatorname{Ext}^i_R(A,L)$ is noetherian and Matlis reflexive over R and $\widehat{R}$.*

Proof. Theorem 2.2.2 shows that $\operatorname{Ext}^i_R(A,L)$ is noetherian over $\widehat{R}$; so, it is Matlis reflexive over $\widehat{R}$. As $\operatorname{Ann}_R(A) + \operatorname{Ann}_R(L) \subseteq \operatorname{Ann}_R(\operatorname{Ext}^i_R(A,L))$, Lemmas 2.1.3(2) and 2.1.19 imply that $\operatorname{Ext}^i_R(A,L)$ is noetherian and Matlis reflexive over R. $\square$

Corollary 2.2.5. *Let A and L be R-modules such that $R/(\mathrm{Ann}_R(A) + \mathrm{Ann}_R(L))$ is artinian and A is artinian. Given an index $i \geqslant 0$ such that $\mu_R^i(L) < \infty$, one has $\lambda_R(\mathrm{Ext}_R^i(A, L)) < \infty$.*

Proof. Apply Theorem 2.2.2 and Lemma 2.1.20. $\square$

Matlis Reflexivity of $\mathbf{Ext^i_R(M, M')}$

Theorem 2.2.6. *Let A and M be R-modules such that A is artinian and M is mini-max. For each $i \geqslant 0$, the module $\mathrm{Ext}_R^i(M, A)$ is Matlis reflexive over $\widehat{R}$.*

Proof. Fix a noetherian submodule $N \subseteq M$ such that M/N is artinian. Since A is artinian, it is an $\widehat{R}$-module. Corollary 2.2.3 implies that $\mathrm{Ext}_R^i(M/N, A)$ is a noetherian $\widehat{R}$-module. As $\mathrm{Ext}_R^i(N, A)$ is artinian, Lemma 2.1.24(2) says that $\mathrm{Ext}_R^i(M, A)$ is a mini-max $\widehat{R}$-module and hence is Matlis reflexive over $\widehat{R}$ by Fact 2.1.17. $\square$

Theorem 2.2.7. *Let M and N' be R-modules such that M is mini-max and N' is noetherian. Fix an index $i \geqslant 0$. If $R/(\mathrm{Ann}_R(M) + \mathrm{Ann}_R(N'))$ is complete, then $\mathrm{Ext}_R^i(M, N')$ is noetherian and Matlis reflexive over R and over $\widehat{R}$.*

Proof. Fix a noetherian submodule $N \subseteq M$ such that M/N is artinian. If the ring $R/(\mathrm{Ann}_R(M) + \mathrm{Ann}_R(N'))$ is complete, then so is $R/(\mathrm{Ann}_R(M/N) + \mathrm{Ann}_R(N'))$. Corollary 2.2.4 implies that $\mathrm{Ext}_R^i(M/N, N')$ is noetherian over R. Since $\mathrm{Ext}_R^i(N, N')$ is noetherian over R, Lemma 2.1.24(2) implies that $\mathrm{Ext}_R^i(M, N')$ is noetherian over R. As $R/(\mathrm{Ann}_R(\mathrm{Ext}_R^i(M, N')))$ is complete, Fact 2.1.17 implies that $\mathrm{Ext}_R^i(M, N')$ is also Matlis reflexive over R. Thus $\mathrm{Ext}_R^i(M, N')$ is noetherian and Matlis reflexive over $\widehat{R}$ by Lemmas 2.1.3(2) and 2.1.19. $\square$

Theorem 2.2.8. *Let M and M' be mini-max R-modules, and fix an index $i \geqslant 0$.*

1. *If $R/(\mathrm{Ann}_R(M)+\mathrm{Ann}_R(M'))$ is complete, then $\mathrm{Ext}^i_R(M,M')$ is Matlis reflexive over R and $\widehat{R}$.*

2. *If $R/(\mathrm{Ann}_R(M)+\mathrm{Ann}_R(M'))$ is artinian, then $\mathrm{Ext}^i_R(M,M')$ has finite length.*

Proof. Fix a noetherian submodule $N' \subseteq M'$ such that M'/N' is artinian.

(1) Assume that $R/(\mathrm{Ann}_R(M)+\mathrm{Ann}_R(M'))$ is complete. Theorem 2.2.7 implies that the module $\mathrm{Ext}^i_R(M,N')$ is Matlis reflexive over R. Theorem 2.2.6 shows that $\mathrm{Ext}^i_R(M,M'/N')$ is Matlis reflexive over $\widehat{R}$; hence it is Matlis reflexive over R by Lemma 2.1.19. Thus, Lemmas 2.1.24(1) and 2.1.19 imply that $\mathrm{Ext}^i_R(M,M')$ is Matlis reflexive over R and $\widehat{R}$.

(2) This follows from part (1), because of Fact 2.1.17 and Lemma 2.1.20. □

A special case of the next result can be found in [5, Theorem 3].

Corollary 2.2.9. *Let M and M' be R-modules such that M is mini-max and M' is Matlis reflexive. For each index $i \geqslant 0$, the modules $\mathrm{Ext}^i_R(M,M')$ and $\mathrm{Ext}^i_R(M',M)$ are Matlis reflexive over R and $\widehat{R}$.*

Proof. Apply Theorem 2.2.8(1) and Fact 2.1.17. □

Length Bounds for $\mathbf{Hom_R(A,L)}$

Lemma 2.2.10. *Let A and L be R-modules such that A is artinian and $\mathfrak{m}^n\Gamma_{\mathfrak{m}}(L)=0$ for some $n \geqslant 0$. Let t be a non-negative integer such that $\mathfrak{m}^tA=\mathfrak{m}^{t+1}A$, and let s be an integer such that $s \geqslant \min(n,t)$. Then*

$$\mathrm{Hom}_R(A,L) \cong \mathrm{Hom}_R(A/\mathfrak{m}^sA,L) \cong \mathrm{Hom}_R(A/\mathfrak{m}^sA,(0:_L \mathfrak{m}^s)).$$

Proof. Given any map $\psi \in \mathrm{Hom}_R(A/\mathfrak{m}^s A, L)$, the image of ψ is annihilated by $\mathfrak{m}^s$. That is, $\mathrm{Im}(\psi) \subseteq (0 :_L \mathfrak{m}^s)$; hence $\mathrm{Hom}_R(A/\mathfrak{m}^s A, L) \cong \mathrm{Hom}_R(A/\mathfrak{m}^s A, (0 :_L \mathfrak{m}^s))$. In the next sequence, the first and third isomorphisms are from Lemma 2.1.5(2):

$$\mathrm{Hom}_R(A, L) \cong \mathrm{Hom}_R(A, \Gamma_{\mathfrak{m}}(L)) \cong \mathrm{Hom}_R(A/\mathfrak{m}^s A, \Gamma_{\mathfrak{m}}(L)) \cong \mathrm{Hom}_R(A/\mathfrak{m}^s A, L).$$

For the second isomorphism, we argue by cases. If $s \geqslant n$, then we have $\mathfrak{m}^s \Gamma_{\mathfrak{m}}(L) = 0$ because $\mathfrak{m}^n \Gamma_{\mathfrak{m}}(L) = 0$, and the isomorphism is evident. If $s < n$, then we have $n > s \geqslant t$, so $\mathfrak{m}^t A = \mathfrak{m}^s A = \mathfrak{m}^n A$ since $\mathfrak{m}^t A = \mathfrak{m}^{t+1} A$; it follows that $\mathrm{Hom}_R(A, \Gamma_{\mathfrak{m}}(L)) \cong \mathrm{Hom}_R(A/\mathfrak{m}^n A, \Gamma_{\mathfrak{m}}(L)) \cong \mathrm{Hom}_R(A/\mathfrak{m}^s A, \Gamma_{\mathfrak{m}}(L))$. □

For the next result, the example $\mathrm{Hom}_R(E, E) \cong \widehat{R}$ shows that the condition $\mathfrak{m}^n \Gamma_{\mathfrak{m}}(L) = 0$ is necessary.

Theorem 2.2.11. *Let A and L be R-modules such that A is artinian and $\mathfrak{m}^n \Gamma_{\mathfrak{m}}(L) = 0$ for some $n \geqslant 0$. Let t be a non-negative integer such that $\mathfrak{m}^t A = \mathfrak{m}^{t+1} A$, and let s be an integer such that $s \geqslant \min(n, t)$. Then there are inequalities*

$$\lambda_R(\mathrm{Hom}_R(A, L)) \leqslant \beta_0^R(A) \lambda_R(0 :_L \mathfrak{m}^s)$$
$$\lambda_R(\mathrm{Hom}_R(A, L)) \leqslant \lambda_R(A/\mathfrak{m}^s A) \mu_R^0(L)$$

Here, we use the convention $0 \cdot \infty = 0$.

Proof. We deal with the degenerate case first. If $\beta_0^R(A) = 0$, then $A/\mathfrak{m}A = 0$, so

$$\mathrm{Hom}_R(A, L) \cong \mathrm{Hom}_R(A/\mathfrak{m}A, L) = \mathrm{Hom}_R(0, L) = 0$$

by Lemma 2.2.10. So, we assume for the rest of the proof that $\beta_0^R(A) \neq 0$. We also assume without loss of generality that $\lambda_R(0 :_L \mathfrak{m}^s) < \infty$.

Lemma 2.2.10 explains the first step in the following sequence:

$$\begin{aligned}\lambda_R(\operatorname{Hom}_R(A,L)) &= \lambda_R(\operatorname{Hom}_R(A/\mathfrak{m}^sA,(0:_L\mathfrak{m}^s)))\\ &\leqslant \beta_0^R(A/\mathfrak{m}^sA)\lambda_R(0:_L\mathfrak{m}^s)\\ &= \beta_0^R(A)\lambda_R(0:_L\mathfrak{m}^s).\end{aligned}$$

The second step can be proved by induction on $\beta_0^R(A/\mathfrak{m}^sA)$ and $\lambda_R(0:_L\mathfrak{m}^s)$. Similarly we get the sequence:

$$\begin{aligned}\lambda_R(\operatorname{Hom}_R(A,L)) &= \lambda_R(\operatorname{Hom}_R(A/\mathfrak{m}^sA,(0:_L\mathfrak{m}^s)))\\ &\leqslant \lambda(A/\mathfrak{m}^sA)\mu_R^0(0:_L\mathfrak{m}^s)\\ &= \lambda(A/\mathfrak{m}^sA)\mu_R^0(L).\end{aligned}$$

The second step can be proved by induction on $\lambda_R(A/\mathfrak{m}^sA)$ and $\mu_R^0(0:_L\mathfrak{m}^s)$. ∎

The next result can also be obtained as a corollary to [13, Proposition 6.1]. Example 2.6.3 shows that $\lambda_R(\operatorname{Ext}_R^i(A,N))$ can be infinite when $i\geqslant 1$.

Corollary 2.2.12. *If A and N are R-modules such that A is artinian and N is noetherian, then $\lambda_R(\operatorname{Hom}_R(A,N))<\infty$.*

Proof. Apply Theorem 2.2.11 and Lemma 2.1.18. ∎

2.3 Properties of $\operatorname{Tor}_i^R(M,-)$

This section focuses on properties of the functors $\operatorname{Tor}_i^R(M,-)$, where M is a mini-max R-module.

Artinianness of $\mathbf{Tor}_{\mathbf{i}}^{\mathbf{R}}(\mathbf{A}, \mathbf{L})$

The next result contains part of Theorem 2.1. Recall that a module is artinian over R if and only if it is artinian over $\widehat{R}$; see Lemma 2.1.14.

Theorem 2.3.1. *Let A and L be R-modules such that A is artinian. For each index $i \geqslant 0$ such that $\beta_i^R(L) < \infty$, the module $\operatorname{Tor}_i^R(A, L)$ is artinian.*

Proof. Lemma 2.1.13(2) implies that $\mu_R^i(L^\vee) = \beta_i^R(L) < \infty$. By Remark 2.1.9, we have $\operatorname{Ext}_R^i(A, L^\vee) \cong \operatorname{Tor}_i^R(A, L)^\vee$. Thus, $\operatorname{Tor}_i^R(A, L)^\vee$ is a noetherian $\widehat{R}$-module by Theorem 2.2.2, and we conclude that $\operatorname{Tor}_i^R(A, L)$ is artinian by Lemma 2.1.15(2). $\square$

For the next result, the example $E \otimes_R R \cong E$ shows that $\operatorname{Tor}_i^R(A, L)$ is not necessarily noetherian over R or $\widehat{R}$.

Corollary 2.3.2. *Let A and M be R-modules such that A is artinian and M minimax. For each index $i \geqslant 0$, the module $\operatorname{Tor}_i^R(A, M)$ is artinian.*

Proof. Apply Theorem 2.3.1 and Lemma 2.1.18. $\square$

The proofs of the next two results are similar to those of Corollaries 2.2.4 and 2.2.5. The first result contains part of Theorem 2.3 from the beginning of the chapter.

Corollary 2.3.3. *Let A and L be R-modules such that $R/(\operatorname{Ann}_R(A) + \operatorname{Ann}_R(L))$ is complete and A is artinian. For each index $i \geqslant 0$ such that $\beta_i^R(L) < \infty$, the module $\operatorname{Tor}_i^R(A, L)$ is artinian and Matlis reflexive over R and $\widehat{R}$.*

Corollary 2.3.4. *Let A and L be R-modules such that $R/(\operatorname{Ann}_R(A) + \operatorname{Ann}_R(L))$ is artinian and A is artinian. Given an index $i \geqslant 0$ such that $\beta_i^R(L) < \infty$, one has $\lambda_R(\operatorname{Tor}_i^R(A, L)) < \infty$.*

$\mathbf{Tor_i^R(M, M')}$ is Mini-max

Theorem 2.3.5. *Let M and M' be mini-max R-modules, and fix an index $i \geqslant 0$.*

1. *The R-module $\operatorname{Tor}_i^R(M, M')$ is mini-max over R.*

2. *If $R/(\operatorname{Ann}_R(M) + \operatorname{Ann}_R(M'))$ is complete, then $\operatorname{Tor}_i^R(M, M')$ is Matlis reflexive over R and $\widehat{R}$.*

3. *If $R/(\operatorname{Ann}_R(M) + \operatorname{Ann}_R(M'))$ is artinian, then $\operatorname{Tor}_i^R(M, M')$ has finite length.*

Proof. (1) Choose a noetherian submodule $N \subseteq M$ such that M/N is artinian. Lemmas 2.1.22 and 2.1.23(3) say that $\operatorname{Tor}_i^R(N, M')$ is mini-max. Corollary 2.3.2 implies that $\operatorname{Tor}_i^R(M/N, M')$ mini-max, so $\operatorname{Tor}_i^R(M, M')$ is mini-max by Lemma 2.1.24(3).

Parts (2) and (3) now follow from Lemmas 2.1.19 and 2.1.20. □

A special case of the next result is contained in [5, Theorem 3].

Corollary 2.3.6. *Let M and M' be R-modules such that M is mini-max and M' is Matlis reflexive. For each index $i \geqslant 0$, the module $\operatorname{Tor}_i^R(M, M')$ is Matlis reflexive over R and $\widehat{R}$.*

Proof. Apply Theorem 2.3.5(2) and Fact 2.1.17. □

Length Bounds for $\mathbf{A \otimes_R L}$

Lemma 2.3.7. *Let A be an artinian module, and let $\mathfrak{a}$ be a proper ideal of R. Fix an integer $t \geqslant 0$ such that $\mathfrak{a}^t A = \mathfrak{a}^{t+1} A$. Given an $\mathfrak{a}$-torsion R-module L, one has*

$$A \otimes_R L \cong (A/\mathfrak{a}^t A) \otimes_R L \cong (A/\mathfrak{a}^t A) \otimes_R (L/\mathfrak{a}^t L).$$

Proof. The isomorphism $(A/\mathfrak{a}^t A) \otimes_R L \cong (A/\mathfrak{a}^t A) \otimes_R (L/\mathfrak{a}^t L)$ is from the following:

$$\begin{aligned}(A/\mathfrak{a}^t A) \otimes_R L &\cong [(A/\mathfrak{a}^t A) \otimes_R (R/\mathfrak{a}^t)] \otimes_R L \\ &\cong (A/\mathfrak{a}^t A) \otimes_R [(R/\mathfrak{a}^t) \otimes_R L] \\ &\cong (A/\mathfrak{a}^t A) \otimes_R (L/\mathfrak{a}^t L).\end{aligned}$$

For the isomorphism $A \otimes_R L \cong (A/\mathfrak{a}^t A) \otimes_R L$, consider the exact sequence:

$$0 \to \mathfrak{a}^t A \to A \to A/\mathfrak{a}^t A \to 0.$$

The exact sequence induced by $- \otimes_R L$ has the form

$$(\mathfrak{a}^t A) \otimes_R L \to A \otimes_R L \to (A/\mathfrak{a}^t A) \otimes_R L \to 0. \tag{2.1}$$

The fact that L is $\mathfrak{a}$-torsion and $\mathfrak{a}^t A = \mathfrak{a}^{t+i} A$ for all $i \geqslant 1$ implies that $(\mathfrak{a}^t A) \otimes_R L = 0$, so the sequence (2.1) yields the desired isomorphism. □

Theorem 2.3.8. *Let A be an artinian R-module, and let L be an $\mathfrak{m}$-torsion R-module. Fix an integer $t \geqslant 0$ such that $\mathfrak{m}^t A = \mathfrak{m}^{t+1} A$. Then there are inequalities*

$$\lambda_R(A \otimes_R L) \leqslant \lambda_R\left(A/\mathfrak{m}^t A\right) \beta_0^R(L) \tag{2.2}$$

$$\lambda_R(A \otimes_R L) \leqslant \beta_0^R(A) \lambda_R\left(L/\mathfrak{m}^t L\right). \tag{2.3}$$

Here we use the convention $0 \cdot \infty = 0$.

Proof. From Lemma 2.3.7 we have

$$A \otimes_R L \cong (A/\mathfrak{m}^t A) \otimes_R (L/\mathfrak{m}^t L). \tag{2.4}$$

Lemmas 2.1.18 and 2.1.20 imply that $\lambda_R(A/\mathfrak{m}^t A) < \infty$ and $\beta_0^R(A) < \infty$.

For the degenerate cases, first note that $\lambda_R(A/\mathfrak{m}^t A) = 0$ if and only if $\beta_0^R(A) = 0$. When $\lambda_R(A/\mathfrak{m}^t A) = 0$, the isomorphism (2.4) implies that $A \otimes_R L = 0$; hence the desired inequalities. Thus, we assume without loss of generality that $1 \leqslant \beta_0^R(A) \leqslant \lambda_R(A/\mathfrak{m}^t A)$. Further, we assume that $\beta_0^R(L) < \infty$.

The isomorphism (2.4) provides the first step in the next sequence:

$$\lambda_R(A \otimes_R L) = \lambda_R((A/\mathfrak{m}^t A) \otimes_R (L/\mathfrak{m}^t L)) \leqslant \lambda_R(A/\mathfrak{m}^t A)\beta_0^R(L).$$

The second step in this sequence can be verified by induction on $\lambda_R(A/\mathfrak{m}^t A)$ and $\beta_0^R(L)$. This explains the inequality (2.2), and (2.3) is verified similarly. □

Notice that the condition that L is $\mathfrak{m}$-torsion from the last theorem is necessary. For instance, suppose that $\dim(R) > 0$, $A = E$ and $L = R$. Then $E \otimes_R R \cong E$ has infinite length but $\lambda_R(E/\mathfrak{m}^t E)\beta_0^R(R)$ has finite length for any t.

The next corollary recovers [13, Proposition 6.1]. Note that Example 2.6.4 shows that $\lambda_R(\operatorname{Tor}_i^R(A, A'))$ can be infinite when $i \geqslant 1$.

Corollary 2.3.9. *If A and A' are artinian R-modules, then $\lambda_R(A \otimes_R A') < \infty$.*

Proof. Apply Theorem 2.3.8 and Lemmas 2.1.18 and 2.1.20. (Alternatively, apply Corollary 2.12 and Matlis duality.) □

2.4 The Matlis Dual of $\operatorname{Ext}_R^i(L, L')$

This section contains the proof of Theorem 2.4 from the beginning of the chapter; see Corollary 2.4.13. Most of the section is devoted to technical results for use in the proof.

Lemma 2.4.1. *Let L be an R-module. If I is an R-injective resolution of L, and J is an $\widehat{R}$-injective resolution of $\widehat{R} \otimes_R L$, then there is a homotopy equivalence $\Gamma_{\mathfrak{m}}(I) \xrightarrow{\simeq} \Gamma_{\mathfrak{m}}(J) = \Gamma_{\mathfrak{m}\widehat{R}}(J)$.*

Proof. Each injective $\widehat{R}$-module J' is injective over R; this follows from the isomorphism $\operatorname{Hom}_R(-, J') \cong \operatorname{Hom}_R(-, \operatorname{Hom}_{\widehat{R}}(\widehat{R}, J')) \cong \operatorname{Hom}_{\widehat{R}}(\widehat{R} \otimes_R -, J')$ since $\widehat{R}$ is flat over R. Hence there is a lift $f \colon I \to J$ of the natural map $\xi \colon L \to \widehat{R} \otimes_R L$. This lift is a chain map of R-complexes.

We show that the induced map $\Gamma_{\mathfrak{m}}(f) \colon \Gamma_{\mathfrak{m}}(I) \to \Gamma_{\mathfrak{m}}(J) = \Gamma_{\mathfrak{m}\widehat{R}}(J)$ is a homotopy equivalence. As $\Gamma_{\mathfrak{m}}(I)$ and $\Gamma_{\mathfrak{m}}(J)$ are bounded-above complexes of injective R-modules, it suffices to show that $\Gamma_{\mathfrak{m}}(f)$ induces an isomorphism on homology in each degree. The induced map on homology is compatible with the following sequence:

$$\mathrm{H}^i(\Gamma_{\mathfrak{m}}(I)) \cong \mathrm{H}^i_{\mathfrak{m}}(L) \xrightarrow[\cong]{\mathrm{H}^i_{\mathfrak{m}}(\xi)} \mathrm{H}^i_{\mathfrak{m}}(\widehat{R} \otimes_R L) \cong \mathrm{H}^i(\Gamma_{\mathfrak{m}}(J)).$$

The map $\mathrm{H}^i_{\mathfrak{m}}(\xi) \colon \mathrm{H}^i_{\mathfrak{m}}(L) \to \mathrm{H}^i_{\mathfrak{m}}(\widehat{R} \otimes_R L)$ is an isomorphism (see the proof of [9, Proposition 3.5.4(d)]), so we have the desired homotopy equivalence. □

Lemma 2.4.2. *Let L and L' be R-modules such that L is $\mathfrak{m}$-torsion. Then for each index $i \geqslant 0$, there are $\widehat{R}$-module isomorphisms*

$$\operatorname{Ext}^i_R(L, L') \cong \operatorname{Ext}^i_R(L, \widehat{R} \otimes_R L') \cong \operatorname{Ext}^i_{\widehat{R}}(L, \widehat{R} \otimes_R L').$$

Proof. Let I be an R-injective resolution of L', and let J be an $\widehat{R}$-injective resolution of $\widehat{R} \otimes_R L'$. Because L is $\mathfrak{m}$-torsion, Lemma 2.1.5(2) explains the three isomorphisms in the next display:

$$\operatorname{Hom}_R(L, I) \cong \operatorname{Hom}_R(L, \Gamma_{\mathfrak{m}}(I)) \sim \operatorname{Hom}_R(L, \Gamma_{\mathfrak{m}}(J)) \cong \operatorname{Hom}_R(L, J)$$

$$\operatorname{Hom}_R(L, \Gamma_{\mathfrak{m}}(J)) = \operatorname{Hom}_R(L, \Gamma_{\mathfrak{m}\widehat{R}}(J)) = \operatorname{Hom}_{\widehat{R}}(L, \Gamma_{\mathfrak{m}\widehat{R}}(J)) \cong \operatorname{Hom}_{\widehat{R}}(L, J).$$

The homotopy equivalence above is from Lemma 2.4.1. The second equality above is from Lemma 2.1.5(1). Since L is $\mathfrak{m}$-torsion, it is an $\widehat{R}$-module, so the isomorphisms and the homotopy equivalence in this sequence are $\widehat{R}$-linear. In particular, the complexes $\operatorname{Hom}_R(L, I)$ and $\operatorname{Hom}_R(L, J)$ and $\operatorname{Hom}_{\widehat{R}}(L, J)$ have isomorphic cohomology over $\widehat{R}$, so one has the desired isomorphisms. □

The next result contains Theorem 2.2 from the beginning of the chapter. It shows, for instance, that, given artinian R-modules A and A', there are noetherian $\widehat{R}$-modules N and N' such that $\operatorname{Ext}^i_R(A, A') \cong \operatorname{Ext}^i_{\widehat{R}}(N, N')$; thus, it provides an alternate proof of Corollary 2.2.3.

Theorem 2.4.3. *Let A and M be R-modules such that A is artinian and M is minimax. Then, for each index $i \geqslant 0$, we have $\operatorname{Ext}^i_R(A, M) \cong \operatorname{Ext}^i_{\widehat{R}}(M^\vee, A^\vee)$.*

Proof. Case 1: R is complete. Let F be a free resolution of A. It follows that each F_i is flat, so the complex $F^\vee$ is an injective resolution of $A^\vee$; see [12, Theorem 3.2.9]. We obtain the isomorphism $\operatorname{Ext}^i_R(A, M) \cong \operatorname{Ext}^i_R(M^\vee, A^\vee)$ by taking cohomology in the next sequence:

$$\operatorname{Hom}_R(F, M) \cong \operatorname{Hom}_R(F, M^{\vee\vee}) \cong \operatorname{Hom}_R(M^\vee, F^\vee).$$

The first step follows from the fact that M is Matlis reflexive; see Fact 2.1.17. The second step is from Hom-tensor adjointness

Case 2: the general case. The first step below is from Lemma 2.4.2:

$$\operatorname{Ext}_R^i(A,M) \cong \operatorname{Ext}_{\widehat{R}}^i(A,\widehat{R}\otimes_R M) \cong \operatorname{Ext}_{\widehat{R}}^i((\widehat{R}\otimes_R M)^v, A^v) \cong \operatorname{Ext}_{\widehat{R}}^i(M^\vee, A^\vee).$$

Here $(-)^v = \operatorname{Hom}_{\widehat{R}}(-,E)$. Since M is mini-max, it follows that $\widehat{R}\otimes_R M$ is mini-max over $\widehat{R}$. Thus, the second step is from Case 1. For the third step use Hom-tensor adjointness and Lemma 2.1.5(1) to see that $(\widehat{R}\otimes_R M)^v \cong M^\vee$ and $A^v \cong A^\vee$. $\square$

Fact 2.4.4. Let L and L' be R-modules, and fix an index $i \geqslant 0$. Then the following diagram commutes, where the unlabeled isomorphism is from Remark 2.1.9:

$$\begin{tikzcd}
\operatorname{Ext}_R^i(L',L) \arrow[r, "\delta_{\operatorname{Ext}_R^i(L',L)}"] \arrow[d, "\operatorname{Ext}_R^i(L',\delta_L)"'] & \operatorname{Ext}_R^i(L',L)^{\vee\vee} \arrow[d, "(\Theta^i_{L'L})^\vee"] \\
\operatorname{Ext}_R^i(L',L^{\vee\vee}) \arrow[r, "\cong"] & \operatorname{Tor}_i^R(L',L^\vee)^\vee .
\end{tikzcd}$$

Lemma 2.4.5. *Let N and L be an R-module such that N is noetherian. Fix an index $i \geqslant 0$. Then the map $\operatorname{Ext}_R^i(N,\delta_L)\colon \operatorname{Ext}_R^i(N,L) \to \operatorname{Ext}_R^i(N,L^{\vee\vee})$ is an injection. If $\mu_R^i(L) < \infty$, then $\operatorname{Ext}_R^i(k,\delta_L)$ is an isomorphism.*

Proof. Remark 2.1.9 states that

$$\Theta^i_{NL}\colon \operatorname{Tor}_i^R(N,L^\vee) \to \operatorname{Ext}_R^i(N,L)^\vee$$

is an isomorphism. Hence $(\Theta^i_{NL})^\vee$ is also an isomorphism. Also the map

$$\delta_{\operatorname{Ext}_R^i(N,L)}\colon \operatorname{Ext}_R^i(N,L) \to \operatorname{Ext}_R^i(N,L)^{\vee\vee}$$

is an injection. Using Fact 2.4.4 with $L' = N$, we conclude that $\operatorname{Ext}_R^i(N,\delta_L)$ is an injection.

The assumption $\mu_R^i(L) < \infty$ says that $\operatorname{Ext}_R^i(k, L)$ is a finite dimensional k-vector space, so it is Matlis reflexive over R; that is, the map

$$\delta_{\operatorname{Ext}_R^i(k,L)} \colon \operatorname{Ext}_R^i(k, L) \to \operatorname{Ext}_R^i(k, L)^{\vee\vee}$$

is an isomorphism. Using Fact 2.4.4 with $L' = k$, we conclude that $\operatorname{Ext}_R^i(k, \delta_L)$ is an isomorphism, as desired. $\square$

Lemma 2.4.6. *Let L be an R-module such that $\mu_R^1(L)$ is finite. Then*

$$L \cong L' \oplus (\bigoplus_{\alpha \in \mathcal{S}} E),$$

where $\mathcal{S}$ is an index set, and $\mu_R^0(L')$ is finite.

Proof. Let $\mu_R^1(L) = n$. Note that any map $\phi \in \operatorname{Hom}_R(E, E) \cong \widehat{R}$ is just multiplication by some element $r \in \widehat{R}$ and hence any map in $\phi \in \operatorname{Hom}_R(E, E^n) \cong \widehat{R}^n$ is just multiplication by some vector $v \in \widehat{R}^n$. Given $v \in \widehat{R}^n$ let $\phi_v \in \operatorname{Hom}_R(E, E^n)$ denote the map which is multiplication by the vector v. Let $I^0 = J \oplus (\bigoplus_{\alpha \in \mathcal{T}} E_\alpha)$ with $\Gamma_{\mathfrak{m}}(J) = 0$, where $\mathcal{T}$ is an index set. Here $E_\alpha = E$ for every α. Let $\psi : I^0 \to I^1$ be the first map in the injective resolution I. Then $\Gamma_{\mathfrak{m}}(\psi) : \bigoplus_{\alpha \in \mathcal{T}} E_\alpha \to \bigoplus_{i=1}^n E$ can be described component wise as $(\phi_{v_\alpha})_{\alpha \in \mathcal{T}}$ with $v_\alpha \in \widehat{R}^n$. Since $\widehat{R}^n$ is a noetherian $\widehat{R}$-module, so is the submodule $N := \sum_{\alpha \in S} \widehat{R} v_\alpha$. Therefore we can choose $\alpha_1, \dots, \alpha_m \in \mathcal{T}$ such that $N = \sum_{j=1}^m \widehat{R} v_{\alpha_j}$. Given $\beta \in \mathcal{T}$ choose $r_1, \dots, r_m \in \widehat{R}$ such that $v_\beta = \sum_{i=1}^m r_i v_{\alpha_i}$. Let $X_\beta := \{[e, -r_1 e, -r_2 e, \dots, -r_m e] \in E_\beta \oplus (\bigoplus_{i=1}^m E_{\alpha_i})|\ e \in E\}$. Then the map from E to X_β defined by $e \mapsto [e, -r_1 e, -r_2 e, \dots, -r_m e]$ is an isomorphism. However, X_β is in $\operatorname{Ker}(\phi)$ so it is naturally a submodule of L. Since an injective submodule of L is a direct summand, $X_\beta \cong E$ is naturally a direct summand of L. Let $\mathcal{S} =$

$\mathcal{T} \setminus \{\alpha_1, \alpha_2, \ldots, \alpha_m\}$. Then the sum $\sum_{\beta \in \mathcal{S}} X_\beta$ is an internal direct sum of I^0, and

$$\sum_{\beta \in \mathcal{S}} X_\beta + \sum_{i=1}^{m} E_{\alpha_i} = \bigoplus_{\alpha \in \mathcal{T}} E_\alpha = \Gamma_{\mathfrak{m}}(I^0).$$

It follows that $\sum_{\beta \in \mathcal{S}} X_\beta$ is isomorphic to a direct sum of copies of E and is naturally a submodule of L. Therefore $L \cong \sum_{\beta \in \mathcal{S}} X_\beta \oplus L' \cong \sum_{\beta \in \mathcal{S}} E \oplus L'$ where the injective hull of L' is $J \oplus (\bigoplus_{i=1}^{m} E_{\alpha_i})$. Thus $\mu_R^0(L') = m$ and the result follows. □

Lemma 2.4.7. *Let A and L be R-modules such that A is artinian and $\mu_R^i(L)$ is finite for some $i \geqslant 0$. Then the map*

$$\operatorname{Ext}_R^i(A, \delta_L) \colon \operatorname{Ext}_R^i(A, L) \to \operatorname{Ext}_R^i(A, L^{\vee\vee})$$

is an isomorphism, and the map

$$\operatorname{Ext}_R^{i+1}(A, \delta_L) \colon \operatorname{Ext}_R^{i+1}(A, L) \to \operatorname{Ext}_R^{i+1}(A, L^{\vee\vee})$$

is an injection.

Proof. Case 1 suppose $i = 0$. Lemma 2.4.5 implies that the map

$$\operatorname{Hom}_R(k, \delta_L) \colon \operatorname{Hom}_R(k, L) \to \operatorname{Hom}_R(k, L^{\vee\vee})$$

is an isomorphisms and the map $\operatorname{Ext}_R^1(k, \delta_L)$ is an injection. As the biduality map δ_L is injective, we have an exact sequence

$$0 \to L \xrightarrow{\delta_L} L^{\vee\vee} \to \operatorname{Coker} \delta_L \to 0. \tag{2.5}$$

Using the long exact sequence associated to $\operatorname{Ext}_R(k, -)$, we conclude that $\operatorname{Hom}_R(k, \operatorname{Coker} \delta_L) =$

0. In other words, we have $\mu_R^0(\operatorname{Coker}\delta_L) = 0$. Thus $\mathrm{E}_R(\operatorname{Coker}\delta_L)$ does not have E as a summand by Remark 2.1.11(a). That is, we have $\Gamma_{\mathfrak{m}}(\operatorname{Coker}\delta_L) = 0$, so Lemma 2.1.5(2) implies that

$$\operatorname{Hom}_R(A, \operatorname{Coker}\delta_L) \cong \operatorname{Hom}_R(A, \Gamma_{\mathfrak{m}}(\operatorname{Coker}\delta_L)) = 0.$$

From the long exact sequence associated to $\operatorname{Ext}_R(A, -)$ with respect to (2.5) it follows that $\operatorname{Hom}_R(A, \delta_L)$ is an isomorphism and $\operatorname{Ext}_R^1(A, \delta_L)$ is an injection.

Now suppose that $i > 0$. Let J be a minimal injective resolution of L and let $L' = \operatorname{Ker}(J^{i-1} \to J^i)$. It suffices to show that the map $\operatorname{Ext}_R^1(A, \delta'_L)$ is an isomorphism and $\operatorname{Ext}_R^2(A, \delta'_L)$ is an injection. Since $\mu_R^1(L') = \mu_R^i(L) \leqslant \infty$ Lemma 2.4.6 implies that $L' \cong L'' \oplus (\bigoplus_{\alpha \in \mathcal{S}} E)$ such that $\mathcal{S}$ is an index set and $\mu^0(L'') < \infty$. Since the maps $\operatorname{Ext}_R^1(A, \delta_{(\bigoplus_{\alpha\in\mathcal{S}} E)})$ and $\operatorname{Ext}_R^2(A, \delta_{(\bigoplus_{\alpha\in\mathcal{S}} E)})$ are both just the map from the zero module to the zero module it suffices to show that $\operatorname{Ext}_R^1(A, \delta_{L''})$ is an isomorphism and $\operatorname{Ext}_R^2(A, \delta_{L''})$ is an injection.

Lemma 2.4.5 implies that for $t = 0, 1$ the map

$$\operatorname{Ext}_R^t(k, \delta_{L''})\colon \operatorname{Ext}_R^t(k, L'') \to \operatorname{Ext}_R^t(k, L''^{\vee\vee})$$

is an isomorphisms and the map $\operatorname{Ext}_R^2(k, \delta_{L''})$ is an injection. From the long exact sequence associated to $\operatorname{Ext}_R(k, -)$ with respect to (2.5) we conclude that for $t = 0, 1$ we have $\operatorname{Ext}_R^t(k, \operatorname{Coker}\delta_{L''}) = 0$. In other words, we have $\mu_R^t(\operatorname{Coker}\delta_{L''}) = 0$. Let I be a minimal injective resolution of $\operatorname{Coker}\delta_{L''}$. The previous paragraph shows that for $t = 0, 1$ the module I^t does not have E as a summand by Remark 2.1.11(a). That is, we have $\Gamma_{\mathfrak{m}}(I^t) = 0$, so Lemma 2.1.5(2) implies that

$$\operatorname{Hom}_R(A, I^t) \cong \operatorname{Hom}_R(A, \Gamma_{\mathfrak{m}}(I^t)) = 0.$$

It follows that $\operatorname{Ext}^t_R(A, \operatorname{Coker}(\delta_L)) = 0$ for $t = 0, 1$. From the long exact sequence associated to $\operatorname{Ext}_R(A, -)$ with respect to (2.5), it follows that $\operatorname{Ext}^1_R(A, \delta_{L''})$ is an isomorphism, $\operatorname{Ext}^2_R(A, \delta_{L''})$ is an injection and the result follows. $\square$

We are now ready to tackle the main results of this section.

Theorem 2.4.8. *Let A and L be R-modules such that A is artinian and $\mu^i_R(L)$ is finite for some $i \geqslant 0$.*

1. *There is an R-module isomorphism $\operatorname{Ext}^i_R(A, L)^v \cong \operatorname{Tor}^R_i(A, L^\vee)$ where $(-)^v = \operatorname{Hom}_{\widehat{R}}(-, E)$.*

2. *If $R/(\operatorname{Ann}_R(A) + \operatorname{Ann}_R(L))$ is complete, then Θ^i_{AL} provides an isomorphism $\operatorname{Tor}^R_i(A, L^\vee) \cong \operatorname{Ext}^i_R(A, L)^\vee$.*

Proof. (2) Corollary 2.2.4 and Lemma 2.4.7 show that the maps

$$
\begin{aligned}
\delta_{\operatorname{Ext}^i_R(A,L)} \colon \operatorname{Ext}^i_R(A, L) &\to \operatorname{Ext}^i_R(A, L)^{\vee\vee} \\
\operatorname{Ext}^i_R(A, \delta_L) \colon \operatorname{Ext}^i_R(A, L) &\to \operatorname{Ext}^i_R(A, L^{\vee\vee})
\end{aligned}
$$

are isomorphisms. Fact 2.4.4 implies that $(\Theta^i_{AL})^\vee$ is an isomorphism, so we conclude that Θ^i_{AL} is also an isomorphism.

(1) Lemma 2.4.2 explains the first step in the next sequence:

$$
\begin{aligned}
\operatorname{Ext}^i_R(A, L)^v &\cong \operatorname{Ext}^i_{\widehat{R}}(A, \widehat{R} \otimes_R L)^v \\
&\cong \operatorname{Tor}^{\widehat{R}}_i(A, (\widehat{R} \otimes_R L)^v) \\
&\cong \operatorname{Tor}^R_i(A, (\widehat{R} \otimes_R L)^v) \\
&\cong \operatorname{Tor}^R_i(A, L^\vee).
\end{aligned}
$$

The second step is from part (2), as $\widehat{R}$ is complete and $\mu^i_{\widehat{R}}(\widehat{R}\otimes_R L) = \mu^i_R(L) < \infty$. The fourth step is from Hom-tensor adjointness. For the third step, let P be a projective resolution of A over R. Since $\widehat{R}$ is flat over R, the complex $\widehat{R} \otimes_R P$ is a projective resolution of $\widehat{R} \otimes_R A \cong A$ over $\widehat{R}$; see Lemma 2.1.4(1). Thus, the third step follows from the isomorphism $(\widehat{R} \otimes_R P) \otimes_{\widehat{R}} (\widehat{R} \otimes_R L)^v \cong P \otimes_R (\widehat{R} \otimes_R L)^v$. □

Corollary 2.4.9. *Let A and M be R-modules such that A is artinian and M is mini-max. For each index $i \geqslant 0$, one has $\operatorname{Ext}^i_R(A, M)^v \cong \operatorname{Tor}^R_i(A, M^\vee)$, where $(-)^v = \operatorname{Hom}_{\widehat{R}}(-, E)$.*

Proof. Apply Theorem 2.4.8(1) and Lemma 2.1.18. □

Theorem 2.4.10. *Let M and L be R-modules such that M is mini-max and $\mu^i_R(M)$ and $\mu^{i+1}_R(M)$ are finite for a fixed $i \geqslant 0$. If $R/(\operatorname{Ann}_R(M) + \operatorname{Ann}_R(L))$ is complete, then Θ^i_{ML} is an isomorphism*

Proof. Since M is mini-max over R, there is an exact sequence of R-modules homomorphisms $0 \to N \to M \to A \to 0$ such that N is noetherian and A is artinian. The long exact sequences associated to $\operatorname{Tor}^R(-, L^\vee)$ and $\operatorname{Ext}_R(-, L)$ fit into the following commutative diagram:

$$\begin{tikzcd}
\cdots \arrow[r] & \operatorname{Tor}^R_i(N, L^\vee) \arrow[r] \arrow[d, "\Theta^i_{NL}"] & \operatorname{Tor}^R_i(M, L^\vee) \arrow[r] \arrow[d, "\Theta^i_{ML}"] & \operatorname{Tor}^R_i(A, L^\vee) \arrow[r] \arrow[d, "\Theta^i_{AL}"] & \cdots \\
\cdots \arrow[r] & \operatorname{Ext}^i_R(N, L)^\vee \arrow[r] & \operatorname{Ext}^i_R(M, L)^\vee \arrow[r] & \operatorname{Ext}^i_R(A, L)^\vee \arrow[r] & \cdots .
\end{tikzcd}$$

Remark 2.1.9 shows that Θ^i_{NL} and Θ^{i-1}_{NL} are isomorphisms. Theorem 2.4.8(2) implies that Θ^i_{AL} and Θ^{i+1}_{AL} are isomorphisms. Hence the Five Lemma shows that Θ^i_{ML} is an isomorphism. □

Corollary 2.4.11. *Let M and L be R-modules such that M is Matlis reflexive. Fix an index $i \geqslant 0$ such that $\mu^i(L)$ and $\mu^{i+1}(L)$ are finite. Then Θ^i_{ML} is an isomorphisms.*

Corollary 2.4.12. *Let M and M' be mini-max R-modules such that $R/(\operatorname{Ann}_R(M) + \operatorname{Ann}_R(L))$ is complete. Then*

$$\operatorname{Ext}^i_R(M, M')^v = \operatorname{Ext}^i_R(M, M')^\vee \cong \operatorname{Tor}^R_i(M, M'^\vee)$$

where $(-)^v = \operatorname{Hom}_{\widehat{R}}(-, E)$.

Proof. Theorem 2.2.8(1) implies that $\operatorname{Ext}^i_R(M, M')$ is Matlis reflexive over R, so Lemma 2.1.5(1) and Fact 2.1.17 imply that $\operatorname{Ext}^i_R(M, M')^v = \operatorname{Ext}^i_R(M, M')^\vee$. $\square$

The next result contains Theorem 2.4 from the beginning of the chapter. A special case of it can be found in [5, Theorem 3].

Corollary 2.4.13. *Let M and M' be mini-max R-modules, and fix an index $i \geqslant 0$. If either M or M' is Matlis reflexive, then $\Theta^i_{MM'}$ is an isomorphism, so one has $\operatorname{Ext}^i_R(M, M')^v = \operatorname{Ext}^i_R(M, M')^\vee \cong \operatorname{Tor}^R_i(M, M'^\vee)$, where $(-)^v = \operatorname{Hom}_{\widehat{R}}(-, E)$.*

Proof. Apply Theorem 2.4.10 and Fact 2.1.17. $\square$

The next example shows that the modules $\operatorname{Ext}^i_R(L, L')^\vee$ and $\operatorname{Tor}^R_i(L, L'^\vee)$ are not isomorphic in general.

Example 2.4.14. Assume that R is not complete. We have $\operatorname{Ann}_R(E) = 0$, so the ring $R/\operatorname{Ann}_R(E) \cong R$ is not complete, by assumption. Thus, Fact 2.1.17 implies that E is not Matlis reflexive, that is, the biduality map $\delta_E \colon E \hookrightarrow E^{\vee\vee}$ is not an isomorphism. Since $E^{\vee\vee}$ is injective, we have $E^{\vee\vee} \cong E \oplus J$ for some non-zero injective R-module J. The uniqueness of direct sum decompositions of injective R-modules implies that

$E^{\vee\vee} \ncong E$. This provides the second step below:

$$\operatorname{Hom}_R(E,E)^\vee \cong E^{\vee\vee} \ncong E \cong E \otimes_R \widehat{R} \cong E \otimes_R E^\vee.$$

The third step is from Lemma 2.1.4(1), and the remaining steps are standard.

2.5 Vanishing of Ext and Tor

In this section we describe the sets of associated primes of $\operatorname{Hom}_R(A,M)$ and attached primes of $A \otimes_R M$ over $\widehat{R}$. The section concludes with some results on the related topic of vanishing for $\operatorname{Ext}^i_R(A,M)$ and $\operatorname{Tor}^R_i(A,M)$.

Associated and Attached Primes

The following is dual to the notion of associated primes of noetherian modules; see, e.g., [20] or [21, Appendix to §6] or [24].

Definition 2.5.1. Let A be an artinian R-module. A prime ideal $\mathfrak{p} \in \operatorname{Spec}(R)$ is *attached* to A if there is a submodule $A' \subseteq A$ such that $\mathfrak{p} = \operatorname{Ann}_R(A/A')$. We let $\operatorname{Att}_R(A)$ denote the set of prime ideals attached to A.

Lemma 2.5.2. *Let A be an artinian R-module such that $R/\operatorname{Ann}_R(A)$ is complete, and let N be a noetherian R-module. There are equalities*

$$\begin{aligned}
\operatorname{Supp}_R(A^\vee) &= \cup_{\mathfrak{p}\in\operatorname{Ass}_R(A^\vee)} V(\mathfrak{p}) = \cup_{\mathfrak{p}\in\operatorname{Att}_R(A)} V(\mathfrak{p})\\
\operatorname{Att}_R(N^\vee) &= \operatorname{Ass}_R(N)\\
\operatorname{Att}_R(A) &= \operatorname{Ass}_R(A^\vee).
\end{aligned}$$

Proof. The R-module $A^\vee$ is noetherian by Lemma 2.1.15(3), so the first equality is standard, and the second equality follows from the fourth one. The third equality is from [28, (2.3) Theorem]. This also explains the second step in the next sequence

$$\operatorname{Att}_R(A) = \operatorname{Att}_R(A^{\vee\vee}) = \operatorname{Ass}_R(A^\vee)$$

since $A^\vee$ is noetherian. The first step in this sequence follows from the fact that A is Matlis reflexive; see Fact 2.1.17. $\square$

The next proposition can also be deduced from a result of Melkersson and Schenzel [22, Proposition 5.2].

Proposition 2.5.3. *Let A and L be R-modules such that $\mu_R^0(L) < \infty$ and A is artinian. Then*

$$\operatorname{Ass}_{\widehat{R}}(\operatorname{Hom}_R(A,L)) = \operatorname{Ass}_{\widehat{R}}(A^\vee) \cap \operatorname{Supp}_{\widehat{R}}(\Gamma_{\mathfrak{m}}(L)^\vee) = \operatorname{Att}_{\widehat{R}}(A) \cap \operatorname{Supp}_{\widehat{R}}(\Gamma_{\mathfrak{m}}(L)^\vee).$$

Proof. The assumption $\mu_R^0(L) < \infty$ implies that $\Gamma_{\mathfrak{m}}(L)$ is artinian. This implies that $\Gamma_{\mathfrak{m}}(L)^\vee$ is a noetherian $\widehat{R}$-module, so a result of Bourbaki [7, IV 1.4 Proposition 10] provides the third equality in the next sequence; see also [9, Exercise 1.2.27]:

$$\begin{aligned}
\operatorname{Ass}_{\widehat{R}}(\operatorname{Hom}_R(A,L)) &= \operatorname{Ass}_{\widehat{R}}(\operatorname{Hom}_R(A,\Gamma_{\mathfrak{m}}(L))) \\
&= \operatorname{Ass}_{\widehat{R}}(\operatorname{Hom}_{\widehat{R}}(\Gamma_{\mathfrak{m}}(L)^\vee, A^\vee)) \\
&= \operatorname{Ass}_{\widehat{R}}(A^\vee) \cap \operatorname{Supp}_{\widehat{R}}(\Gamma_{\mathfrak{m}}(L)^\vee) \\
&= \operatorname{Att}_{\widehat{R}}(A) \cap \operatorname{Supp}_{\widehat{R}}(\Gamma_{\mathfrak{m}}(L)^\vee).
\end{aligned}$$

The remaining equalities are from Lemmas 2.1.5(2), 2.2.1(1), and 2.5.2, respectively. $\square$

Corollary 2.5.4. *Let M and M' be mini-max R-modules such that the quotient $R/(\mathrm{Ann}_R(M)+\mathrm{Ann}_R(M'))$ is complete.*

1. *For each index $i \geqslant 0$, one has $\mathrm{Ext}^i_R(M,M') \cong \mathrm{Ext}^i_R(M'^\vee, M^\vee)$.*

2. *If M' is noetherian, then*

$$\mathrm{Ass}_{\widehat{R}}(\mathrm{Hom}_R(M,M')) = \mathrm{Att}_{\widehat{R}}(M'^\vee) \cap \mathrm{Supp}_{\widehat{R}}(\Gamma_{\mathfrak{m}}(M^\vee)^\vee).$$

Proof. (1) The first step in the next sequence comes from Theorem 2.2.8(1):

$$\mathrm{Ext}^i_R(M,M') \cong \mathrm{Ext}^i_R(M,M')^{\vee\vee} \cong (\mathrm{Tor}^R_i(M,M'^\vee))^\vee \cong \mathrm{Ext}^i_R(M'^\vee, M^\vee).$$

The remaining steps are from Theorem 2.4.10 and Remark 2.1.9, respectively.

(2) This follows from the case $i=0$ in part (1) because of Proposition 2.5.3. $\square$

Proposition 2.5.5. *Let A and L be R-modules such that A is artinian and $\beta^R_0(L)$ is finite. Then*

$$\mathrm{Att}_{\widehat{R}}(A\otimes_R L) = \mathrm{Ass}_{\widehat{R}}(A^\vee)\cap \mathrm{Supp}_{\widehat{R}}(\Gamma_{\mathfrak{m}}(L^\vee)^\vee) = \mathrm{Att}_{\widehat{R}}(A)\cap \mathrm{Supp}_{\widehat{R}}(\Gamma_{\mathfrak{m}}(L^\vee)^\vee).$$

Proof. Theorem 2.3.1 implies that $A\otimes_R L$ is artinian. Hence we have

$$\mathrm{Hom}_{\widehat{R}}(A\otimes_R L, E) \cong \mathrm{Hom}_R(A\otimes_R L, E) \cong \mathrm{Hom}_R(A, L^\vee)$$

by Lemma 2.1.5(1), and this explains the second step in the next sequence:

$$\mathrm{Att}_{\widehat{R}}(A\otimes_R L) = \mathrm{Ass}_{\widehat{R}}(\mathrm{Hom}_{\widehat{R}}(A\otimes_R L, E)) = \mathrm{Ass}_{\widehat{R}}(\mathrm{Hom}_R(A, L^\vee))$$

The first step is from Lemma 2.5.2. Since $\mu_R^0(L^\vee) < \infty$ by Lemma 2.1.13(2), we obtain the desired equalities from Proposition 2.5.3. $\square$

Next, we give an alternate description of the module $\Gamma_{\mathfrak{m}}(L)^\vee$ from the previous results. See Lemma 2.5.2 for a description of its support.

Remark 2.5.6. Let L be an R-module. There is an isomorphism $\Gamma_{\mathfrak{m}}(L)^\vee \cong \widehat{L^\vee}$. In particular, given a noetherian R-module N, one has $\Gamma_{\mathfrak{m}}(N^\vee)^\vee \cong \widehat{R} \otimes_R N$. When R is Cohen-Macaulay with a dualizing module D, Grothendieck's local duality theorem implies that $\Gamma_{\mathfrak{m}}(N)^\vee \cong \widehat{R} \otimes_R \operatorname{Ext}_R^{\dim(R)}(N, D)$; see, e.g., [9, Theorem 3.5.8]. A similar description is available when R is not Cohen-Macaulay, provided that it has a dualizing complex; see [15, Chapter V, §6].

Vanishing of Hom and Tensor Product

For the next result note that if L is noetherian, then the conditions on $\mu_R^0(L)$ and $R/(\operatorname{Ann}_R(A) + \operatorname{Ann}_R(\Gamma_{\mathfrak{m}}(L)))$ are automatically satisfied. Also, the example $\operatorname{Hom}_R(E, E) \cong R$ when R is complete shows the necessity of the condition on $R/(\operatorname{Ann}_R(A) + \operatorname{Ann}_R(\Gamma_{\mathfrak{m}}(L)))$.

Proposition 2.5.7. *Let A be an artinian R-module. Let L be an R-module such that $R/(\operatorname{Ann}_R(A) + \operatorname{Ann}_R(\Gamma_{\mathfrak{m}}(L)))$ is artinian and $\mu_R^0(L) < \infty$. Then $\operatorname{Hom}_R(A, L) = 0$ if and only if $A = \mathfrak{m}A$ or $\Gamma_{\mathfrak{m}}(L) = 0$.*

Proof. If $\Gamma_{\mathfrak{m}}(L) = 0$, then we are done by Lemma 2.1.5(2), so assume that $\Gamma_{\mathfrak{m}}(L) \neq 0$. Theorem 2.2.2 and Lemma 2.1.20 show that $\operatorname{Hom}_R(A, L)$ has finite length. Thus Proposition 2.5.3 implies that $\operatorname{Hom}_R(A, L) \neq 0$ if and only if $\mathfrak{m}\widehat{R} \in \operatorname{Ass}_{\widehat{R}}(A^\vee)$, that is, if and only if $\operatorname{depth}_{\widehat{R}}(A^\vee) = 0$. Lemma 2.1.13(3) shows that $\operatorname{depth}_{\widehat{R}}(A^\vee) = 0$ if and only if $\mathfrak{m}\widehat{R}A \neq A$, that is, if and only if $\mathfrak{m}A \neq A$. $\square$

For the next result note that the conditions on L are satisfied when L is artinian.

Proposition 2.5.8. *Let A be an artinian R-module, and let L be an $\mathfrak{m}$-torsion R-module. The following conditions are equivalent:*

1. $A \otimes_R L = 0$*;*
2. *either* $A = \mathfrak{m}A$ *or* $L = \mathfrak{m}L$*; and*
3. *either* $\operatorname{depth}_R(A^\vee) > 0$ *or* $\operatorname{depth}_R(L^\vee) > 0$.

Proof. (1) $\Longleftrightarrow$ (2) If $A \otimes_R L = 0$, then we have

$$0 = \lambda_R(A \otimes_R L) \geqslant \beta_0^R(A)\beta_0^R(L)$$

so either $\beta_0^R(A) = 0$ or $\beta_0^R(L) = 0$, that is $A/\mathfrak{m}A = 0$ or $L/\mathfrak{m}L = 0$. Conversely, if $A/\mathfrak{m}A = 0$ or $L/\mathfrak{m}L = 0$, then we have either $\beta_0^R(A) = 0$ or $\beta_0^R(L) = 0$, so Theorem 2.3.8 implies that $\lambda_R(A \otimes_R L) = 0$.

The implication (2) $\Longleftrightarrow$ (3) is from Lemma 2.1.13(3). $\square$

The next result becomes simpler when L is artinian, as $\Gamma_{\mathfrak{m}}(L) = L$ in this case.

Theorem 2.5.9. *Let A and L be R-modules such that A is artinian and $\mu_R^0(L) < \infty$. The following conditions are equivalent:*

1. $\operatorname{Hom}_R(A, L) = 0$*;*
2. $\operatorname{Hom}_R(A, \Gamma_{\mathfrak{m}}(L)) = 0$*;*
3. $\operatorname{Hom}_{\widehat{R}}(\Gamma_{\mathfrak{m}}(L)^\vee, A^\vee) = 0$*;*

4. *there is an element* $x \in \operatorname{Ann}_{\widehat{R}}(\Gamma_{\mathfrak{m}}(L))$ *such that* $A = xA$*;*

5. $\operatorname{Ann}_{\widehat{R}}(\Gamma_{\mathfrak{m}}(L))A = A$*;*

6. $\operatorname{Ann}_{\widehat{R}}(\Gamma_{\mathfrak{m}}(L))$ *contains a non-zero-divisor for* $A^{\vee}$*; and*

7. $\operatorname{Att}_{\widehat{R}}(A) \cap \operatorname{Supp}_{\widehat{R}}(\Gamma_{\mathfrak{m}}(L)^{\vee}) = \emptyset$.

Proof. The equivalence (1) $\iff$ (2) is from Lemma 2.1.5(2). The equivalence (2) $\iff$ (7) follows from Proposition 2.5.3, and the equivalence (2) $\iff$ (3) follows from Lemma 2.2.1(1). The equivalence (4) $\iff$ (6) follows from the fact that the map $A \xrightarrow{x} A$ is surjective if and only if the map $A^{\vee} \xrightarrow{x} A^{\vee}$ is injective. The equivalence (5) $\iff$ (6) follows from Lemma 2.1.13, parts (3) and (5).

The module $\Gamma_{\mathfrak{m}}(L)$ is artinian as $\mu_R^0(L) < \infty$. Since $A^{\vee}$ and $\Gamma_{\mathfrak{m}}(L)^{\vee}$ are noetherian over $\widehat{R}$, the equivalence (3) $\iff$ (6) is standard; see [9, Proposition 1.2.3]. $\square$

As with Theorem 2.5.9, the next result simplifies when L is noetherian. Also, see Remark 2.5.6 for some perspective on the module $\Gamma_{\mathfrak{m}}(L^{\vee})^{\vee}$.

Corollary 2.5.10. *Let* A *be a non-zero artinian* R*-module, and let* L *be an* R*-module such that* $\beta_0^R(L) < \infty$*. The following conditions are equivalent:*

1. $A \otimes_R L = 0$*;*

2. $\operatorname{Ann}_{\widehat{R}}(\Gamma_{\mathfrak{m}}(L^{\vee}))A = A$*;*

3. *there is an element* $x \in \operatorname{Ann}_{\widehat{R}}(\Gamma_{\mathfrak{m}}(L^{\vee}))$ *such that* $xA = A$*;*

4. $\operatorname{Ann}_{\widehat{R}}(\Gamma_{\mathfrak{m}}(L^{\vee}))$ *contains a non-zero-divisor for* $A^{\vee}$*; and*

5. $\operatorname{Att}_{\widehat{R}}(A) \cap \operatorname{Supp}_{\widehat{R}}(\Gamma_{\mathfrak{m}}(L^{\vee})^{\vee}) = \emptyset$.

Proof. For an artinian R-module A', one has $\operatorname{Att}_{\widehat{R}}(A') = \emptyset$ if and only if $A' = 0$ by Lemma 2.5.2. Thus, Proposition 2.5.5 explains the equivalence (1) $\iff$ (5); see [24, Corollary 2.3]. Since one has $A \otimes_R L = 0$ if and only if $(A \otimes_R L)^\vee = 0$, the isomorphism $(A \otimes_R L)^\vee \cong \operatorname{Hom}_R(A, L^\vee)$ from Remark 2.1.9 in conjunction with Theorem 2.5.9 shows that the conditions (1)–(4) are equivalent. □

Depth and Vanishing

Proposition 2.5.11. *Let A and L be R-modules such that A is artinian. Then $\operatorname{Ext}^i_R(A, L) = 0$ for all $i < \operatorname{depth}_R(L)$.*

Proof. Let J be a minimal R-injective resolution of L, and let $i < \operatorname{depth}_R(L)$. It follows that $\operatorname{Ext}^i_R(k, L) = 0$, that is $\mu^i_R(L) = 0$, so the module E does not appear as a summand of J^i. As in the proof of Theorem 2.2.2, this implies that $\operatorname{Hom}_R(A, J)^i = 0$, so $\operatorname{Ext}^i_R(A, L) = 0$. □

The next example shows that, in Proposition 2.5.11 one may have $\operatorname{Ext}^i_R(A, L) = 0$ when $i = \operatorname{depth}_R(L)$. See also equation (2.6).

Example 2.5.12. Assume that $\operatorname{depth}(R) \geqslant 1$. Then $\mathfrak{m}E = E$ by Lemma 2.1.13(3), so Lemma 2.2.10 implies that

$$\operatorname{Ext}^0_R(E, k) \cong \operatorname{Hom}_R(E, k) \cong \operatorname{Hom}_R(E/\mathfrak{m}E, k) = 0$$

even though $\operatorname{depth}_R(k) = 0$.

Proposition 2.5.13. *Let A and L be R-modules such that A is artinian. Then for all $i < \operatorname{depth}_R(L^\vee)$ one has $\operatorname{Tor}^R_i(A, L) = 0$.*

Proof. When $i < \operatorname{depth}_R(L^\vee)$, one has $\operatorname{Tor}_i^R(A,L)^\vee \cong \operatorname{Ext}_R^i(A,L^\vee) = 0$ by Remark 2.1.9 and Proposition 2.5.11, so $\operatorname{Tor}_i^R(A,L) = 0$. □

Theorem 2.5.14. *Let A and A' be artinian R-modules, and let N and N' be noetherian R-modules. Then one has*

$$\operatorname{grade}_{\widehat{R}}(\operatorname{Ann}_{\widehat{R}}(A'); A^\vee) = \inf\{i \geqslant 0 \mid \operatorname{Ext}_R^i(A,A') \neq 0\} \tag{2.6}$$

$$\operatorname{grade}_R(\operatorname{Ann}_R(N'); A^\vee) = \inf\{i \geqslant 0 \mid \operatorname{Ext}_R^i(A,N'^\vee) \neq 0\} \tag{2.7}$$

$$\operatorname{grade}_R(\operatorname{Ann}_R(N'); N) = \inf\{i \geqslant 0 \mid \operatorname{Ext}_R^i(N^\vee,N'^\vee) \neq 0\}. \tag{2.8}$$

Proof. We verify equation (2.6) first. For each index i, Theorem 2.4.3 implies that

$$\operatorname{Ext}_R^i(A,A') \cong \operatorname{Ext}_{\widehat{R}}^i(A'^\vee, A^\vee).$$

Since $A^\vee$ and $A'^\vee$ are noetherian over $\widehat{R}$, this explains the first equality below:

$$\inf\{i \geqslant 0 \mid \operatorname{Ext}_R^i(A,A') \neq 0\} = \operatorname{grade}_{\widehat{R}}(\operatorname{Ann}_{\widehat{R}}(A'^\vee); A^\vee) = \operatorname{grade}_{\widehat{R}}(\operatorname{Ann}_{\widehat{R}}(A'); A^\vee).$$

The second equality is standard since $A'^\vee = \operatorname{Hom}_{\widehat{R}}(A',E)$ by Lemma 2.1.5(1).

Next, we verify equation (2.7). Since $N'^\vee$ is artinian, equation (2.6) shows that we need only verify that

$$\operatorname{grade}_{\widehat{R}}(\operatorname{Ann}_{\widehat{R}}(N'^\vee); A^\vee) = \operatorname{grade}_R(\operatorname{Ann}_R(N'); A^\vee). \tag{2.9}$$

For this, we compute as follows:

$$\widehat{R} \otimes_R N' \overset{(1)}{\cong} \operatorname{Hom}_{\widehat{R}}(\operatorname{Hom}_{\widehat{R}}(\widehat{R} \otimes_R N', E), E) \overset{(2)}{\cong} \operatorname{Hom}_{\widehat{R}}(N'^\vee, E).$$

Step (1) follows from the fact that $\widehat{R} \otimes_R N'$ is noetherian (hence Matlis reflexive) over $\widehat{R}$, and step (2) is from Hom-tensor adjointness. This explains step (4) below:

$$\operatorname{Ann}_{\widehat{R}}(N'^{\vee}) \stackrel{(3)}{=} \operatorname{Ann}_{\widehat{R}}(\operatorname{Hom}_{\widehat{R}}(N'^{\vee}, E)) \stackrel{(4)}{=} \operatorname{Ann}_{\widehat{R}}(\widehat{R} \otimes_R N') \stackrel{(5)}{=} \operatorname{Ann}_R(N')\widehat{R}.$$

Steps (3) and (5) are standard. This explains step (6) in the next sequence:

$$\operatorname{grade}_{\widehat{R}}(\operatorname{Ann}_{\widehat{R}}(N'^{\vee}); A^{\vee}) \stackrel{(6)}{=} \operatorname{grade}_{\widehat{R}}(\operatorname{Ann}_R(N')\widehat{R}; A^{\vee}) \stackrel{(7)}{=} \operatorname{grade}_R(\operatorname{Ann}_R(N'); A^{\vee}).$$

Step (7) is explained by the following:

$$\begin{aligned}\operatorname{Ext}^i_{\widehat{R}}(\widehat{R}/\operatorname{Ann}_R(N')\widehat{R}, A^{\vee}) &\stackrel{(8)}{\cong} \operatorname{Ext}^i_{\widehat{R}}(\widehat{R} \otimes_R (R/\operatorname{Ann}_R(N')), A^{\vee}) \\ &\stackrel{(9)}{\cong} \operatorname{Ext}^i_R(R/\operatorname{Ann}_R(N'), A^{\vee}).\end{aligned}$$

Step (8) is standard, and step (9) is a consequence of Hom-tensor adjointness. This establishes equation (2.9) and thus equation (2.7).

Equation (2.8) follows from (2.7) because we have

$$\operatorname{grade}_R(\operatorname{Ann}_R(N'); N^{\vee\vee}) = \operatorname{width}_R(\operatorname{Ann}_R(N'); N^{\vee}) = \operatorname{grade}_R(\operatorname{Ann}_R(N'); N)$$

by Lemma 2.1.13(1). □

Corollary 2.5.15. *Let A and A' be artinian R-modules, and let N and N' be noethe-*

rian R-modules. Then

$$\operatorname{grade}_{\widehat{R}}(\operatorname{Ann}_{\widehat{R}}(A'); A^\vee) = \inf\{i \geqslant 0 \mid \operatorname{Tor}_i^R(A, A'^\vee) \neq 0\} \tag{2.10}$$

$$\operatorname{grade}_R(\operatorname{Ann}_R(N'); A^\vee) = \inf\{i \geqslant 0 \mid \operatorname{Tor}_i^R(A, N') \neq 0\} \tag{2.11}$$

$$\operatorname{grade}_R(\operatorname{Ann}_R(N'); N) = \inf\{i \geqslant 0 \mid \operatorname{Tor}_i^R(N^\vee, N') \neq 0\}. \tag{2.12}$$

Proof. We verify equation (2.10); the others are verified similarly.

Since $\operatorname{Ext}_R^i(A, A') \neq 0$ if and only if $\operatorname{Hom}_{\widehat{R}}(\operatorname{Ext}_R^i(A, A'), E) \neq 0$, the isomorphism $\operatorname{Hom}_{\widehat{R}}(\operatorname{Ext}_R^i(A, A'), E) \cong \operatorname{Tor}_i^R(A, A'^\vee)$ from Corollary 2.4.9 shows that

$$\inf\{i \geqslant 0 \mid \operatorname{Ext}_R^i(A, A') \neq 0\} = \inf\{i \geqslant 0 \mid \operatorname{Tor}_i^R(A, A'^\vee) \neq 0\}.$$

Thus equation (2.10) follows from (2.6). $\square$

2.6 Examples

This section contains some explicit computations of Ext and Tor for the classes of modules discussed in this paper. Our first example shows that $\operatorname{Ext}_R^i(A, A')$ need not be mini-max over R.

Example 2.6.1. Let k be a field, and set $R = k[X_1, \ldots, X_d]_{(X_1,\ldots,X_d)}$. We show that $\operatorname{Hom}_R(E, E) \cong \widehat{R}$ is not mini-max over R. Note that R is countably generated over k, and $\widehat{R} \cong k[\![X_1, \ldots, X_d]\!]$ is not countably generated over k. So, $\widehat{R}$ is not countably generated over R. Also, every artinian R-module A is a countable union of the finite length submodules $(0 :_A \mathfrak{m}^n)$, so A is countably generated. It follows that every mini-max R-module is also countably generated. Since $\widehat{R}$ is not countably generated, it is not mini-max over R.

Our next example describes $\operatorname{Ext}^i_R(A, A')$ for some special cases.

Example 2.6.2. Assume that $\operatorname{depth}(R) \geqslant 1$, and let A be an artinian R-module. Let $x \in \mathfrak{m}$ be an R-regular element. The map $E \xrightarrow{x} E$ is surjective since E is divisible, and the kernel $(0 :_E x)$ is artinian, being a submodule of E. Using the injective resolution $0 \to E \xrightarrow{x} E \to 0$ for $(0 :_E x)$, one can check that

$$\operatorname{Ext}^i_R(A, (0 :_E x)) \cong \begin{cases} (0 :_{A^\vee} x) & \text{if } i = 0 \\ A^\vee / xA^\vee & \text{if } i = 1 \\ 0 & \text{if } i \neq 0, 1. \end{cases}$$

For instance, in the case $A = (0 :_E x)$, the isomorphism $(0 :_E x)^\vee \cong \widehat{R}/x\widehat{R}$ implies

$$\operatorname{Ext}^i_R((0 :_E x), (0 :_E x)) \cong \begin{cases} \widehat{R}/x\widehat{R} & \text{if } i = 0, 1 \\ 0 & \text{if } i \neq 0, 1. \end{cases}$$

On the other hand, if x, y is an R-regular sequence, then $(0 :_E y)^\vee \cong \widehat{R}/y\widehat{R}$; it follows that x is $(0 :_E y)^\vee$-regular, so one has

$$\operatorname{Ext}^i_R((0 :_E y), (0 :_E x)) \cong \begin{cases} \widehat{R}/(x, y)\widehat{R} & \text{if } i = 1 \\ 0 & \text{if } i \neq 1. \end{cases}$$

The next example shows that $\operatorname{Ext}^i_R(A, N)$ need not be mini-max over R.

Example 2.6.3. Assume that R is Cohen-Macaulay with $d = \dim(R)$, and let A be an artinian R-module. Assume that R admits a dualizing (i.e., canonical) module D. (For instance, this is so when R is Gorenstein, in which case $D = R$.) A minimal injective resolution of D has the form

$$J = \quad 0 \to \bigoplus_{\mathrm{ht}(\mathfrak{p})=0} \mathrm{E}_R(R/\mathfrak{p}) \to \cdots \to \bigoplus_{\mathrm{ht}(\mathfrak{p})=d-1} \mathrm{E}_R(R/\mathfrak{p}) \to E \to 0.$$

In particular, we have $\Gamma_{\mathfrak{m}}(J) = (0 \to 0 \to 0 \to \cdots \to 0 \to E \to 0)$ where the copy of E occurs in degree d. Since $\mathrm{Hom}_R(A, J) \cong \mathrm{Hom}_R(A, \Gamma_{\mathfrak{m}}(J))$, it follows that

$$\mathrm{Ext}^i_R(A, D) \cong \begin{cases} A^\vee & \text{if } i = d \\ 0 & \text{if } i \neq d. \end{cases}$$

Assume that $d \geqslant 1$, and let $x \in \mathfrak{m}$ be an R-regular element. It follows that the map $D \xrightarrow{x} D$ is injective, and the cokernel D/xD is noetherian. Consider the exact sequence $0 \to D \xrightarrow{x} D \to D/xD \to 0$. The long exact sequence associated to $\mathrm{Ext}^i_R(A, -)$ shows that

$$\mathrm{Ext}^i_R(A, D/xD) \cong \begin{cases} (0 :_{A^\vee} x) & \text{if } i = d-1 \\ A^\vee/xA^\vee & \text{if } i = d \\ 0 & \text{if } i \neq d-1, d. \end{cases}$$

As in Example 2.6.2, we have $(0 :_E x)^\vee \cong \widehat{R}/x\widehat{R}$ and

$$\mathrm{Ext}^i_R((0 :_E x), D/xD) \cong \begin{cases} \widehat{R}/x\widehat{R} & \text{if } i = d-1, d \\ 0 & \text{if } i \neq d-1, d. \end{cases}$$

Also, if x, y is an R-regular sequence, then $(0:_E y)^\vee \cong \widehat{R}/y\widehat{R}$ and

$$\operatorname{Ext}^i_R((0:_E y), D/xD) \cong \begin{cases} \widehat{R}/(x,y)\widehat{R} & \text{if } i = d \\ 0 & \text{if } i \neq d. \end{cases}$$

Next, we show that $\operatorname{Tor}^R_i(A, A')$ need not be noetherian over R or $\widehat{R}$.

Example 2.6.4. Assume that R is Gorenstein and complete with $d = \dim(R)$. (Hence $D = R$ is a dualizing R-module.) Given two artinian R-modules A and A', Theorem 2.3.1 implies that $\operatorname{Tor}^R_i(A, A')$ is artinian, hence Matlis reflexive for each index i, since R is complete. This explains the first isomorphism below, and Remark 2.1.9 provides the second isomorphism:

$$\operatorname{Tor}^R_i(A, E) \cong \operatorname{Tor}^R_i(A, E)^{\vee\vee} \cong \operatorname{Ext}^i_R(A, E^\vee)^\vee \cong \operatorname{Ext}^i_R(A, R)^\vee \cong \begin{cases} A & \text{if } i = d \\ 0 & \text{if } i \neq d. \end{cases}$$

Example 2.6.3 explains the fourth isomorphism. Assume that $d \geqslant 1$, and let $x \in \mathfrak{m}$ be an R-regular element. Then $(0:_E x)^\vee \cong R/xR$, so Example 2.6.3 implies that

$$\operatorname{Tor}^R_i(A, (0:_E x)) \cong \operatorname{Ext}^i_R(A, (0:_E x)^\vee)^\vee \cong \begin{cases} A/xA & \text{if } i = d-1 \\ (0:_A x) & \text{if } i = d \\ 0 & \text{if } i \neq d-1, d \end{cases}$$

$$\operatorname{Tor}^R_i((0:_E x), (0:_E x)) \cong \begin{cases} (0:_E x) & \text{if } i = d-1, d \\ 0 & \text{if } i \neq d-1, d. \end{cases}$$

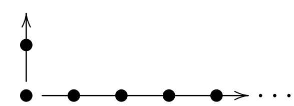
On the other hand, if x, y is an R-regular sequence, then

$$\operatorname{Tor}_i^R((0:_E y),(0:_E x)) \cong \begin{cases} (R/(x,y)R)^\vee \cong E_{R/(x,y)R}(k) & \text{if } i = d \\ 0 & \text{if } i \neq d. \end{cases}$$

Lastly, we provide an explicit computation of $E \otimes_R E$.

Example 2.6.5. Let k be a field and set $R = k[\![X, Y]\!]/(XY, Y^2)$. This is the completion of the multi-graded ring $R' = k[X, Y]/(XY, Y^2)$ with homogeneous maximal ideal $\mathfrak{m}' = (X, Y)R'$. The multi-graded structure on R' is represented in the following diagram:

R'

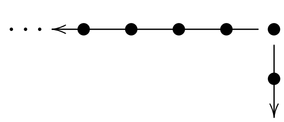


where each integer valued point, (n, m), represents the corresponding monomial, $x^n y^m$, in R'. It follows that $E \cong E_{R'}(k) \cong k[X^{-1}] \oplus kY^{-1}$ with graded module structure given by the formulas

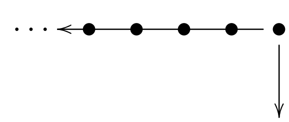
$$\begin{array}{lll} X \cdot 1 = 0 & X \cdot X^{-n} = X^{1-n} & X \cdot Y^{-1} = 0 \\ Y \cdot 1 = 0 & Y \cdot Y^{-1} = 1 & Y \cdot X^{-n} = 0 \end{array}$$

for $n \geqslant 1$. Using this grading, one can show that $\mathfrak{m}E = \mathfrak{m}'E \cong k[X^{-1}]$ and $\mathfrak{m}^2 E = \mathfrak{m}E$. These modules are represented in the next diagrams:

E



$\mathfrak{m}E$



It follows that $E/\mathfrak{m}E \cong k$, so Lemma 2.3.7 implies that

$$E \otimes_R E \cong (E/\mathfrak{m}E) \otimes_R (E/\mathfrak{m}E) \cong k \otimes_R k \cong k.$$

A similar computation shows the following: Fix positive integers a, b, c such that $c > b$, and consider the ring $S = k[\![X, Y]\!]/(X^a Y^b, Y^c)$ with maximal ideal $\mathfrak{n}$ and $E_S = E_S(k)$. Then $\mathfrak{n}^{c-b} E_S = \mathfrak{n}^{c-b+1} E_S$ and we get the following:

$$E_S/\mathfrak{n}^{c-b}E_S \cong S/(X^a, Y^{c-b})S \cong k[X, Y]/(X^a, Y^{c-b})$$

$$E_S \otimes_S E_S \cong (E_S/\mathfrak{n}^{c-b}E_S) \otimes_S (E_S/\mathfrak{n}^{c-b}E_S) \cong S/(X^a, Y^{c-b})S.$$

Chapter 3

Homology of Artinian and Matlis Reflexive Modules Over Commutative Rings

Introduction

The results in this chapter are joint work with B. Kubik and S. Sather-Wagstaff.

Throughout this chapter R will denote a commutative noetherian ring with identity. In this chapter we generalize many of the results from the previous chapter to the case where R may not be local.

3.1 Background

Definition 3.1.1. Given an R-module L we let $\mathrm{E}_R(L)$ denote the injective hull of L. Let $E_R = \bigoplus_{\mathfrak{m} \in \text{m-Spec}\,(R)} \mathrm{E}_R(R/\mathfrak{m})$ be the minimal injective cogenerator for R. The terminology minimal injective cogenerator refers to the fact that an injective

R-module I will cause $\operatorname{Hom}_R(-, I)$ to be a faithful functor if and only if E_R is a direct summand of I. To see this note that for any $\mathfrak{m} \in \text{m-Spec}(R)$ and any injective R-module I the module $\operatorname{Hom}_R(\mathrm{E}_R(R/\mathfrak{m}), I)$ is nonzero if and only if $\mathrm{E}_R(R/\mathfrak{m})$ is a direct summand of I. Let $(-)^{\vee(R)} = \operatorname{Hom}_R(-, E_R)$ be Matlis duality functor. When is clear to which ring we are referring we will simply write $(-)^{\vee}$ for $(-)^{\vee(R)}$. Set $(-)^{\vee\vee} = ((-)^{\vee})^{\vee}$. For each R-module L, let $\delta_L \colon L \to L^{\vee\vee}$ denote the natural biduality map given by $\delta_L(l)(\psi) = \psi(l)$, where $l \in L$ and $\psi \in \operatorname{Hom}_R(L, E_R)$. As in the previous chapter we say tht an R-module L is Matlis reflexive if the natural biduality map δ_L is an isomorphism.

Lemma 3.1.2. *Fix a proper ideal $\mathfrak{a} \subset R$. For each prime ideal $\mathfrak{p} \in V(\mathfrak{a})$ we have*

$$\mathrm{E}_R(R/\mathfrak{p}) \cong \mathrm{E}_{\widehat{R}^{\mathfrak{p}}}(\widehat{R}^{\mathfrak{p}}/\mathfrak{p}\widehat{R}^{\mathfrak{p}}) \cong \mathrm{E}_{\widehat{R}^{\mathfrak{a}}}(\widehat{R}^{\mathfrak{a}}/\mathfrak{p}\widehat{R}^{\mathfrak{a}}). \tag{3.1}$$

Also there is an isomorphism

$$E_{\widehat{R}^{\mathfrak{a}}} \cong \bigoplus\nolimits_{\mathfrak{m} \in \text{m-Spec}(R) \cap V(\mathfrak{a})} \mathrm{E}_R(R/\mathfrak{m}). \tag{3.2}$$

In particular, the module $E_{\widehat{R}^{\mathfrak{a}}}$ is $\mathfrak{a}$-torsion.

Proof. To see this, first recall that $\mathfrak{a}\widehat{R}^{\mathfrak{a}}$ is contained in the Jacobson radical of $\widehat{R}^{\mathfrak{a}}$, and that $\widehat{R}^{\mathfrak{a}}/\mathfrak{a}\widehat{R}^{\mathfrak{a}} \cong R/\mathfrak{a}$; see [21, Theorems 8.11 and 8.14]. From this, it is straightforward to show that there are inverse bijections

$$\begin{array}{ccc} \text{m-Spec}(R) \cap V(\mathfrak{a}) & \longleftrightarrow & \text{m-Spec}(\widehat{R}^{\mathfrak{a}}) \\ \mathfrak{m} & \longmapsto & \mathfrak{m}\widehat{R}^{\mathfrak{a}} \\ \mathfrak{n} \cap R & \longleftarrow\!\shortmid & \mathfrak{n} \end{array} \tag{3.3}$$

Using the isomorphisms $\widehat{(\widehat{R}^{\mathfrak{a}})}^{\mathfrak{p}\widehat{R}^{\mathfrak{a}}} \cong \widehat{(\widehat{R}^{\mathfrak{a}})}^{\mathfrak{p}} \cong \widehat{R}^{\mathfrak{a}+\mathfrak{p}} \cong \widehat{R}^{\mathfrak{p}}$ and the fact that $\widehat{R}^{\mathfrak{a}}/\mathfrak{p}\widehat{R}^{\mathfrak{a}} \cong R/\mathfrak{p}$ for each $\mathfrak{p} \in \operatorname{Spec}(R) \cap V(\mathfrak{a})$, we obtain the isomorphisms in (3.1). The isomorphism (3.2) now follows from (3.3) and (3.1).

For each $\mathfrak{m} \in \text{m-Spec}(R) \cap V(\mathfrak{a})$, the module $\mathrm{E}_R(R/\mathfrak{m})$ is $\mathfrak{m}$-torsion; hence it is $\mathfrak{a}$-torsion, since $\mathfrak{a} \subseteq \mathfrak{m}$. It follows that $E_{\widehat{R}^{\mathfrak{a}}}$ is $\mathfrak{a}$-torsion. □

Fact 3.1.3. Let $\mathfrak{p}_1, \dots \mathfrak{p}_n \in \operatorname{Spec}(R)$. Let $U = R \setminus \cup_{i=1}^n \mathfrak{p}_i$. Let $\mathfrak{p} \in \operatorname{Spec}(R)$. If $\mathfrak{p} \subseteq \mathfrak{p}_i$ for some i, then $\mathrm{E}_R(R/\mathfrak{p})) \cong U^{-1}\mathrm{E}_R(R/\mathfrak{p}) \cong \mathrm{E}_{U^{-1}R}(U^{-1}R/\mathfrak{p}U^{-1}R)$. Otherwise $U^{-1}\mathrm{E}_R(R/\mathfrak{p}) = 0$. If $\mathfrak{p}_i \nsubseteq \mathfrak{p}_j$ for $i \neq j$, then $E_{U^{-1}R} \cong \bigoplus_{i=1}^n \mathrm{E}_R(R/\mathfrak{p})$.

Fact 3.1.4. Let $\mathfrak{a}$ be an ideal of R. For each $\mathfrak{p} \in \operatorname{Spec}(R)$, one has

$$\Gamma_{\mathfrak{a}}(\mathrm{E}_R(R/\mathfrak{p})) = \begin{cases} \mathrm{E}_R(R/\mathfrak{p}) & \text{if } \mathfrak{a} \subseteq \mathfrak{p} \\ 0 & \text{if } \mathfrak{a} \nsubseteq \mathfrak{p}. \end{cases}$$

Lemma 3.1.5. *Let $\mathfrak{a}$ be an ideal of R. Then $\Gamma_{\mathfrak{a}}(E_R) = E_{\widehat{R}^{\mathfrak{a}}}$.*

Proof. Fact 3.1.4 explains the third equality in the next display

$$\begin{aligned}
\Gamma_{\mathfrak{a}}(E_R) &= \Gamma_{\mathfrak{a}}\left(\textstyle\bigoplus_{\mathfrak{m}\in\text{m-Spec}(R)} \mathrm{E}_R(R/\mathfrak{m})\right) \\
&= \textstyle\bigoplus_{\mathfrak{m}\in\text{m-Spec}(R)} \Gamma_{\mathfrak{a}}(\mathrm{E}_R(R/\mathfrak{m})) \\
&= \textstyle\bigoplus_{\mathfrak{m}\in\text{m-Spec}(R)\cap V(\mathfrak{a})} \mathrm{E}_R(R/\mathfrak{m}) \\
&\cong E_{\widehat{R}^{\mathfrak{a}}}
\end{aligned}$$

and the isomorphism is from Lemma 3.1.2. □

Lemma 3.1.6. *Let $\mathfrak{a}$ be an ideal of R. Let L and L' be R-modules such that L is $\mathfrak{a}$-torsion and $\mathfrak{a}^n L' = \mathfrak{a}^{n+1}L'$. Then $L \otimes_R L' \cong L \otimes_R L'/\mathfrak{a}^n L'$.*

Proof. First we prove that $L \otimes_R \mathfrak{a}^n L' = 0$. Let $x \otimes y \in L \otimes_R \mathfrak{a}^n L'$. Choose $m \in \mathbb{N}$ such that $\mathfrak{a}^m x = 0$. Since $\mathfrak{a}^n L' = \mathfrak{a}^{n+1} L'$ it follows $\mathfrak{a}^n L' = \mathfrak{a}^{n+i} L'$ for all $i \geqslant 0$. Thus $y \in \mathfrak{a}^{n+m} L'$, and we can write $y = \sum_{j=1}^{h} s_j c_j$ for some $s_j \in \mathfrak{a}^m$ and $c_j \in \mathfrak{a}^n L'$. Therefore

$$x \otimes y = x \otimes \sum_{j=1}^{h} s_j c_j = \sum_{j=1}^{h} j s_j x \otimes c_j = \sum_{j=1}^{h} 0 \otimes c_j = 0;$$

hence $L \otimes_R \mathfrak{a}^n L' = 0$. Applying $L \otimes_R (-)$ to $0 \to \mathfrak{a}^n L' \to L' \to L'/\mathfrak{a}^n L' \to 0$ we get an exact the sequence $0 \to L \otimes_R L' \to L \otimes_R L'/\mathfrak{a}^n L' \to 0$, and the result follows. $\square$

Fact 3.1.7. Let $\mathfrak{a}$ and $\mathfrak{b}$ be co-maximal ideals of R and L an $\widehat{R}^{\mathfrak{a}}$-module. Then $L = \mathfrak{b} L$.

Choose $a \in \mathfrak{a}$ and $b \in \mathfrak{b}$ so that $a + b = 1$. Then $b = 1 - a$ is a unit in $\widehat{R}^{\mathfrak{a}}$ with inverse $\sum_{i=0}^{\infty} (-a)^i$. Hence $\mathfrak{b} L = \mathfrak{b} \widehat{R}^{\mathfrak{a}} L = \widehat{R}^{\mathfrak{a}} L = L$.

Fact 3.1.8. Let $U \subseteq R$ be a multiplicatively closed set and $\mathfrak{a}$ an ideal of R such that $\mathfrak{a} \cap U \neq \emptyset$. Let L be a $U^{-1}R$-module. Then $L = \mathfrak{a} L$.

Fact 3.1.9. For each R-module L, the natural biduality map $\delta_L \colon L \to L^{\vee\vee}$ is a monomorphism.

Fact 3.1.10. [6, Theorem 12] An R-module L is Matlis reflexive if and only if L is mini-max and $R/\operatorname{Ann}_R(L)$ is semi-local and complete, that is, complete with respect to its Jacobson radical.

Lemma 3.1.11. *Let A be an artinian R-module.*

1. *The support of A consists entirely of maximal ideals of R, that is* $\operatorname{Supp}_R(A) \subseteq$ m-Spec(R).

2. *We have* $\operatorname{Min}_R(A) = \operatorname{Ass}_R(A) = \operatorname{Supp}_R(A)$.

3. *The support of A is finite.*

Proof. (1) Assume there exist $\mathfrak{p} \in \operatorname{Supp}(A) \setminus \text{m-Spec}(R)$. Let $a \in A$ be an element such that its image under the natural map $A \to A_{\mathfrak{p}}$ is non-zero. Then $\operatorname{Ann}(a) \subseteq \mathfrak{p}$. Thus Ra surjects onto $R/\mathfrak{p}$. Since $R/\mathfrak{p}$ is the homomorphic image of a submodule of an artinian module, it is artinian. However, by our assumption $R/\mathfrak{p}$ is a ring of positive dimension. Thus it cannot be artinian. From this contradiction it follows that $\operatorname{Supp}(A)$ consists of maximal ideals.

(2) From part (1) we conclude that each $\mathfrak{m} \in \operatorname{Supp}_R(A)$ is both maximal and minimal in $\operatorname{Supp}_R(A)$. This explains the inclusion $\operatorname{Supp}_R(A) \subseteq \operatorname{Min}_R(A)$, and the inclusions $\operatorname{Min}_R(A) \subseteq \operatorname{Ass}_R(A) \subseteq \operatorname{Supp}_R(A)$ hold for all modules; see [21, Theorem 6.5 (ii) and (iii)] the proof of which only uses that the module is finite for part (i).

(3) From part (2), for each $\mathfrak{m}_i \in \operatorname{Supp}(A)$ we have $\mathfrak{m}_i \in \operatorname{Ass}_R(A)$. Hence we can choose a submodule $A_i \subseteq A$ such that $A_i \cong R/\mathfrak{m}_i$. Let $A' := \sum_{\mathfrak{m}_i \in \operatorname{Supp}(A)} A_i \cong \bigoplus_{\mathfrak{m}_i \in \operatorname{Supp}(A)} R/\mathfrak{m}_i$. Since A is artinian, so is the submodule A'. Therefore the direct sum must be finite. □

Lemma 3.1.12. *Let L be an R-module.*

1. *If L is artinian over R, then $U^{-1}L$ is an artinian $U^{-1}R$-module for each multiplicatively closed subset $U \subseteq R$.*

2. *The R-module L is artinian if and only if $\operatorname{Supp}_R(L)$ is a finite set and $L_{\mathfrak{p}}$ is artinian over $R_{\mathfrak{p}}$ for each $\mathfrak{p} \in \operatorname{Supp}_R(L)$.*

Proof. (1) Each descending chain of $U^{-1}R$-submodules of $U^{-1}L$ has the form $U^{-1}L \supseteq U^{-1}L_1 \supseteq U^{-1}L_2 \supseteq \cdots$ for some descending chain $L \supseteq L_1 \supseteq L_2 \supseteq \cdots$ of R-submodules. Since L is artinian, the second chain stabilizes, hence the first chain also stabilizes.

(2) The forward implication follows from part (1) and Lemma 3.1.11.

For the reverse implication, assume that $\operatorname{Supp}_R(L)$ is finite, with $\operatorname{Supp}_R(L) := \{\mathfrak{p}_1, \dots, \mathfrak{p}_h\}$, and that $L_{\mathfrak{p}_i}$ is artinian over $R_{\mathfrak{p}_i}$ for $i = 1, \dots, h$. Let $L = L_0 \supseteq L_1 \supseteq L_2 \supseteq \cdots$ be a descending chain of R-modules. Since $L_{\mathfrak{p}_i} = (L_0)_{\mathfrak{p}_i} \supseteq (L_1)_{\mathfrak{p}_i} \supseteq (L_2)_{\mathfrak{p}_i} \supseteq \cdots$ stabilizes for $i = 1, \dots, h$, we may choose $j \in \mathbb{N}$ so that $(L_j)_{\mathfrak{p}_i} = (L_{j+n})_{\mathfrak{p}_i}$ for $i = 1, \dots, h$ and for all $n \in \mathbb{N}$. For each $\mathfrak{p} \in \operatorname{Spec}(R) \smallsetminus \operatorname{Supp}_R(L)$, we have $L_{\mathfrak{p}} = 0$, so $(L_j)_{\mathfrak{p}} = (L_{j+n})_{\mathfrak{p}}$ for all $n \in \mathbb{N}$. Hence we have $L_j = L_{j+n}$ for all $n \in \mathbb{N}$, and L is artinian. $\square$

Definition 3.1.13. Let L be an R-module, $\mathfrak{p} \in \operatorname{Spec} R$ and $\mathrm{k}(\mathfrak{p}) := R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}$. For each integer $i \geqslant 0$, the ith *Bass number* of L with respect to $\mathfrak{p}$ and the ith *Betti number* of L with respect to $\mathfrak{p}$ are as follows:

$$\mu_R^i(\mathfrak{p}, L) = \dim_{\mathrm{k}(\mathfrak{p})}(\operatorname{Ext}^i_{R_{\mathfrak{p}}}(\mathrm{k}(\mathfrak{p}), L_{\mathfrak{p}})) \qquad \beta_i^R(\mathfrak{p}, L) = \dim_{\mathrm{k}(\mathfrak{p})}(\operatorname{Tor}_i^{R_{\mathfrak{p}}}(\mathrm{k}(\mathfrak{p}), L_{\mathfrak{p}})).$$

Remark 3.1.14. Let L, L' and N be R-modules such that N is noetherian. Let $\mathfrak{m} \in \text{m-Spec}(R)$. As $\operatorname{Ext}^i_R(N, L)_{\mathfrak{p}} \cong \operatorname{Ext}^i_{R_{\mathfrak{p}}}(N_{\mathfrak{p}}, L_{\mathfrak{p}})$ and $\operatorname{Tor}_i^R(L', L)_{\mathfrak{p}} \cong \operatorname{Tor}_i^{R_{\mathfrak{p}}}(L'_{\mathfrak{p}}, L_{\mathfrak{p}})$ for all $\mathfrak{p} \in \operatorname{Spec}(R)$; see [27, Propositions 7.17 and 7.39] . It follows that

$$\mu_R^i(\mathfrak{m}, L) = \lambda_R(\operatorname{Ext}^i_R(R/\mathfrak{m}, L)) \quad \text{and} \quad \beta_i^R(\mathfrak{m}, L) = \lambda_R(\operatorname{Tor}_i^R(R/\mathfrak{m}, L)).$$

Fact 3.1.15. Let L be an R-module. For each $\mathfrak{p} \in \operatorname{Spec}(R)$ and for each integer $i \geqslant 0$, we have $\mu_R^i(\mathfrak{p}, L) = \mu^i_{R_{\mathfrak{p}}}(L_{\mathfrak{p}})$ and $\beta^i_R(\mathfrak{p}, L) = \beta^i_{R_{\mathfrak{p}}}(L_{\mathfrak{p}})$. This is immediate from the definitions. For the local definitions see Definition 2.1.10.

Definition 3.1.16. Let $\mathfrak{a}$ be an ideal of R. For each R-module L, set

$$\operatorname{depth}_R(L) = \sup\{\operatorname{grade}_R(\mathfrak{m}; L) |\ \mathfrak{m} \in \text{m-Spec}(R)\}$$

$$\operatorname{width}_R(L) = \sup\{\operatorname{width}_R(\mathfrak{m}; L) |\ \mathfrak{m} \in \text{m-Spec}(R)\};$$

see Definition 2.1.12.

Lemma 3.1.17. *Let L be an R-module, and let $\mathfrak{a}$ be an ideal of R.*

1. *One has $\lambda_R(L^\vee) = \lambda_R(L)$; in particular, $\lambda_R(L^\vee) < \infty$ if and only if $\lambda_R(L) < \infty$.*
2. *For each index $i \geqslant 0$ and $\mathfrak{m} \in \text{m-Spec}(R)$ one has $\beta_i^R(\mathfrak{m}; L) = \mu_R^i(\mathfrak{m}; L^\vee)$ and $\beta_i^R(\mathfrak{m}; L^\vee) = \mu_R^i(\mathfrak{m}; L)$.*

Proof. (1) It is straightforward to show that for each maximal ideal $\mathfrak{m} \subset R$, we have $(R/\mathfrak{m})^\vee \cong R/\mathfrak{m}$. An inductive argument using the additivity of length shows that if L has finite length, then $\lambda_R(L^\vee) = \lambda_R(L) < \infty$. Using Fact 3.1.9, we conclude that if $\lambda_R(L^\vee) < \infty$, then $\lambda_R(L) < \infty$.

Part (2) follows from Remark 2.1.9. $\square$

Remark 3.1.18. Let L be an R-module and let $\mathfrak{p} \in \operatorname{Spec} R$.

(a) One has $\mu_R^0(\mathfrak{p}, L) = \lambda_{R_\mathfrak{p}}(\operatorname{Soc}_{R_\mathfrak{p}}(L_\mathfrak{p}))$ and $\beta_0^R(\mathfrak{p}, L) = \lambda_{R_\mathfrak{p}}(L_\mathfrak{p}/\mathfrak{p}L_\mathfrak{p})$.

(b) If I is a minimal injective resolution of L, then for each index $i \geqslant 0$ we have $I^i \cong \mathrm{E}_R(R/\mathfrak{p})^{(\mu_R^i(\mathfrak{p},L))} \oplus J^i$ where J^i does not have $\mathrm{E}_R(R/\mathfrak{p})$ as a summand; see, e.g., [21, Theorem 18.7].

(c) One has $\mu_R^i(\mathfrak{p}, L) < \infty$ for all $i \geqslant 0$ if and only if $\beta_i^R(\mathfrak{p}, L) < \infty$ for all $i \geqslant 0$; see [19, Proposition 1.1] and Fact 3.1.15.

Lemma 3.1.19. *Let M be a mini-max R-module, U a multiplicatively closeed set and $\mathfrak{p} \in \operatorname{Spec} R$. Then $U^{-1}M$ is a mini-max $U^{-1}R$-module, and for all $i \geqslant 0$ we have that $\mu_R^i(\mathfrak{p}, L)$ and $\beta_i^R(\mathfrak{p}, L)$ are finite.*

Proof. The result that $U^{-1}M$ is a mini-max $U^{-1}R$-module follows from the fact that localization is exact and localizing a noetherian (artinian) R-module with resepct to U yields a noetherian (artinian) $U^{-1}R$-module. Therefore the finiteness of the Bass and Betti numbers follows from the local case and Lemma 2.1.18, by using the behavior of Bass and Betti numbers under localization. □

Corollary 3.1.20. *Let A be an artinian R-module. Let J be a minimal injective resolution for A. For $i \geqslant 0$, the module $J^i = \bigoplus_{\mathfrak{m} \in \operatorname{Supp}(A)} \operatorname{E}_R(R/\mathfrak{m})^{\mu_R^i(\mathfrak{m},A)}$ is a finite direct sum of injective hulls of residue fields.*

Proof. This follows from Fact 3.1.15, Remark 3.1.18(b) and Lemmas 3.1.11 and 3.1.19. □

We will use the next lemma to get some localization behavior for Ext.

Lemma 3.1.21. *Let $\mathfrak{m}_1, \ldots, \mathfrak{m}_n \in \text{m-Spec}(R)$, and set $\mathfrak{b} = \cap_{j=1}^n \mathfrak{m}_j$.*

1. *There are isomorphisms $\widehat{R}^{\mathfrak{b}} \cong \prod_{j=1}^n \widehat{R_{\mathfrak{m}_j}} \cong \prod_{j=1}^n \widehat{R}^{\mathfrak{m}_j}$.*
2. *Each $\widehat{R}^{\mathfrak{b}}$-module L has a unique decomposition as $L \cong \bigoplus_{j=1}^n L_j$ where each L_j is an $\widehat{R}^{\mathfrak{m}_j}$-module, specifically, with $L_j \cong L_{\mathfrak{n}_j}$ where $\mathfrak{n}_j = \mathfrak{m}_j \widehat{R}^{\mathfrak{b}}$.*

Proof. Part (1) is contained in [21, Theorem 8.15]. Part (2) is a standard consequence of (1), using the natural idempotents in $\prod_{j=1}^n \widehat{R}^{\mathfrak{m}_j}$. □

Lemma 3.1.22. *Let A be an artinian R-module. Set $\mathfrak{a} = \cap_{\mathfrak{m} \in \operatorname{Supp}_R(A)} \mathfrak{m}$. Then A is $\mathfrak{a}$-torsion.*

Proof. Let $x \in A$. The submodule $Rx \subseteq A$ is artinian because A is artinian. It is noetherian since it is finitely generated, so it has finite length, say $n = \lambda_R(Rx)$. Also, we have $\operatorname{Supp}_R(Rx) \subseteq \operatorname{Supp}_R(A)$. Let $\mathfrak{c} = \cap_{\mathfrak{m} \in \operatorname{Supp}(Rx)} \mathfrak{m}$. Since $\operatorname{Supp}(Rx) \subseteq \operatorname{Supp}(A)$, it follows that $\mathfrak{a} \subseteq \mathfrak{c}$. Since $\mathfrak{c}$ kills socle elements of Rx it follows that $\mathfrak{c}^n x = 0$ for some $n \gg 0$. Thus $\mathfrak{a}^n x = 0$. $\square$

Lemma 3.1.23. *Let $\mathfrak{a}$ be a proper ideal of R, and let L and L' be R-modules such that L is $\mathfrak{a}$-torsion. Then $L^\vee \cong L^{\vee(\widehat{R}^{\mathfrak{a}})}$.*

Proof. The result a consequence of the next display:

$$\operatorname{Hom}_R(L, E_R) \cong \operatorname{Hom}_R(L, \Gamma_{\mathfrak{a}}(E_R)) \cong \operatorname{Hom}_R(L, E_{\widehat{R}^{\mathfrak{a}}}) = \operatorname{Hom}_{\widehat{R}^{\mathfrak{a}}}(L, E_{\widehat{R}^{\mathfrak{a}}}).$$

The isomorphisms above follow from Lemma 2.1.5 (1) and (2) along with Lemma 3.1.5.

$\square$

The next result is particularly useful for artinian modules; see Lemma 3.1.22.

Lemma 3.1.24. *Let $\mathcal{F}$ be a finite set of maximal ideals and let $\mathfrak{b} = \cap_{\mathfrak{m} \in \mathcal{F}} \mathfrak{m}$. Let L be a $\mathfrak{b}$-torsion module. We have the following:*

1. *For each $\mathfrak{m} \in \text{m-Spec}(R)$, the composition $\Gamma_{\mathfrak{m}}(L) \to L \to L_{\mathfrak{m}}$ is an isomorphism.*

2. *There is an internal direct sum $\sum_{\mathfrak{m} \in \mathcal{F}} \Gamma_{\mathfrak{m}}(L) = L \cong \bigoplus_{\mathfrak{m} \in \mathcal{F}} L_{\mathfrak{m}}$.*

Proof. Fact 2.1.2 (c) implies that L is an $\widehat{R}^{\mathfrak{b}}$-module, and we have $L \cong \bigoplus_{\mathfrak{m} \in \mathcal{F}} L_{\mathfrak{m}}$ by Lemma 3.1.21(2). Since $\mathfrak{b} R_{\mathfrak{m}} = \mathfrak{m} R_{\mathfrak{m}}$ for all $\mathfrak{m} \in \mathcal{F}$, it follows that $L_{\mathfrak{m}}$ is $\mathfrak{m}$-torsion. Choose $\mathfrak{m}_0 \in \operatorname{Supp}(L)$. Let $\mathfrak{a} = \cap_{\mathfrak{m} \in \mathcal{F} \smallsetminus \{\mathfrak{m}_0\}} \mathfrak{m}$. Suppose $x \in \bigoplus_{\mathfrak{m} \in \mathcal{F}} L_{\mathfrak{m}}$ is $\mathfrak{m}_0$-torsion. Then $x = y + z$ with $y \in L_{\mathfrak{m}_0}$ and $z \in \bigoplus_{\mathfrak{m} \in \mathcal{F} \smallsetminus \{\mathfrak{m}_0\}} L_{\mathfrak{m}}$. Since x and y are $\mathfrak{m}_0$-torsion so is z. However z is also $\mathfrak{a}$-torsion. Therefore z is $(\mathfrak{m}_0 + \mathfrak{a})$-torsion but $\mathfrak{m}_0 + \mathfrak{a} = R$ so $z = 0$. It follows that $\Gamma_{\mathfrak{m}_0}(L) \cong L_{\mathfrak{m}_0}$. Combing this with the isomorphism $L \cong \bigoplus_{\mathfrak{m}\in\mathcal{F}} L_{\mathfrak{m}}$ gives us the internal direct sum $\sum_{\mathfrak{m}\in\mathcal{F}} \Gamma_{\mathfrak{m}}(L) = L$. □

The following result is equivalent to [29, (1.4) Proposition].

Lemma 3.1.25. *If A is an artinian R-module, then $A \cong \bigoplus_{\mathfrak{m}\in\mathrm{Supp}_R(A)} A_{\mathfrak{m}}$.*

Proof. Let A be an artinian R-module. Lemma 3.1.11 implies that $\mathrm{Supp}_R(A)$ is a finite set of maximal ideals. With $\mathfrak{a} = \cap_{\mathfrak{m}\in\mathrm{Supp}_R(A)}\mathfrak{m}$, Lemma 3.1.22 implies that A is $\mathfrak{a}$-torsion, so the isomorphism $A \cong \bigoplus_{\mathfrak{m}\in\mathrm{Supp}_R(A)} A_{\mathfrak{m}}$ is from Lemma 3.1.24(2). □

Lemma 3.1.26. *Let L be an R-module such that $R/\operatorname{Ann}_R(L)$ is semi-local and complete. The set $\text{m-Spec}(R) \cap \mathrm{Supp}_R(L)$ is finite and equals $\text{m-Spec}(R) \cap V(\operatorname{Ann}_R(L))$.*

Proof. Set $\overline{R} = R/\operatorname{Ann}_R(L)$. We may Assume $L \neq 0$. Let $\pi : R \to \overline{R}$ be the natural surjection and $\pi^* : \mathrm{Spec}(\overline{R}) \to \mathrm{Spec}(R)$ the induced map given by $\pi^*(\mathfrak{p}) = \pi^{-1}(\mathfrak{p})$. Since $L_{\mathfrak{p}} = 0$ for all $\mathfrak{p}$ not containing $\operatorname{Ann}_R(L)$ we get $\mathrm{Supp}_R(L) = \pi^*(\mathrm{Supp}_{\overline{R}}(L))$. Therefore $\text{m-Spec}(R) \cap \mathrm{Supp}_R(L) = \pi^*(\text{m-Spec}(\overline{R}) \cap \mathrm{Supp}_{\overline{R}}(L))$.

The ring $\overline{R}$ is a finite product of complete local rings, say $\overline{R} \cong \prod_{i=1}^n R_i$. Since L is an $\overline{R}$-module we have $L = \prod_{i=1}^n L_i$ where L_i is an R_i-module. By construction $\operatorname{Ann}_{\overline{R}}(L) = 0$, so for all i we have $L_i \neq 0$. Thus $\text{m-Spec}(\overline{R}) \subseteq \mathrm{Supp}_{\overline{R}}(L)$. This explains the second equality in the following display. The last equality is standard.

$$
\begin{aligned}
\text{m-Spec}(R) \cap \mathrm{Supp}_R(L) &= \pi^*(\text{m-Spec}(\overline{R}) \cap \mathrm{Supp}_{\overline{R}}(L)) \\
&= \pi^*(\text{m-Spec}(\overline{R})) \\
&= \text{m-Spec}(R) \cap V(\operatorname{Ann}_R(L))
\end{aligned}
$$

As $\overline{R}$ is semi-local $|\,\text{m-Spec}(R) \cap \mathrm{Supp}_R(L)| = |\,\text{m-Spec}(\overline{R}) \cap \mathrm{Supp}_{\overline{R}}(L)| < \infty$. □

Lemma 3.1.27. *Let L be an R-module such that $R/\operatorname{Ann}_R(L)$ is semi-local and complete. Set $\mathfrak{b} = \cap_{\mathfrak{m}\in \text{m-Spec}(R)\cap \operatorname{Supp}_R(L)}\mathfrak{m}$, and let $\mathfrak{a} \subseteq \mathfrak{b}$.*

1. *L has an $\widehat{R}^{\mathfrak{a}}$-module structure that is compatible with its R-module structure via the natural map $L \to \widehat{R}^{\mathfrak{a}} \otimes_R L$.*

2. *If $\mathfrak{a}$ is a finite intersection of maximal ideals in* m-Spec(R)*, then there is an isomorphism $L^\vee \cong L^{\vee(\widehat{R}^{\mathfrak{a}})}$.*

3. *A subset $Z \subseteq L$ is an R-submodule if and only if it is an $\widehat{R}^{\mathfrak{a}}$-submodule.*

4. *L is a noetherian (artinian, minimax respectively) R-module if and only if it is a noetherian (artinian, mini-max respectively) $\widehat{R}^{\mathfrak{a}}$-module.*

Proof. Set $\overline{R} = R/\operatorname{Ann}_R(L)$. Assume without loss of generality that $L \neq 0$.

(1) There is a commutative diagram of ring homomorphisms

$$\begin{tikzcd} R \arrow[r] \arrow[d] & \widehat{R}^{\mathfrak{a}} \arrow[d] \\ R/\operatorname{Ann}_R(L) \arrow[r, "\cong"] & \widehat{R}^{\mathfrak{a}}/\operatorname{Ann}_R(L)\widehat{R}^{\mathfrak{a}}. \end{tikzcd}$$

The map in the bottom row is an isomorphism because $R/\operatorname{Ann}_R(L)$ is semi-local and complete with Jacobson radical $\mathfrak{b}/\operatorname{Ann}_R(L)$; this uses Lemma 3.1.26. Since L' has an $R/\operatorname{Ann}_R(L)$-module structure that is compatible with its R-module structure via the natural map $R \to R/\operatorname{Ann}_R(L)$, the isomorphism in the bottom row shows that L' has a compatible $\widehat{R}^{\mathfrak{a}}/\operatorname{Ann}_R(L)\widehat{R}^{\mathfrak{a}}$-module structure. It follows that L' has a compatible $\widehat{R}^{\mathfrak{a}}$-module structure.

(2) Assume in this paragraph that $\mathfrak{a}$ is a finite intersection of maximal ideals, and let $\mathcal{F}$ be a finite set of maximal ideals containing m-Spec$(R) \cap \operatorname{Supp}_R(L)$ such that $\mathfrak{a} = \cap_{\mathfrak{m}\in\mathcal{F}}\mathfrak{m}$. Note that $E_{\widehat{R}^{\mathfrak{a}}} = \bigoplus_{\mathfrak{m}\in\mathcal{F}} \operatorname{E}_R(R/\mathfrak{m})$ is the minimal injective cogenerator

of $\widehat{R}^{\mathfrak{a}}$. Part (1) explains the first step in the following sequence.

$$\begin{aligned}
\operatorname{Hom}_{\widehat{R}^{\mathfrak{a}}}(L, E_{\widehat{R}^{\mathfrak{a}}}) &= \operatorname{Hom}_{\widehat{R}^{\mathfrak{a}}}(L \otimes_R \widehat{R}^{\mathfrak{a}}, E_{\widehat{R}^{\mathfrak{a}}}) \\
&= \operatorname{Hom}_R(L, \operatorname{Hom}_{\widehat{R}^{\mathfrak{a}}}(\widehat{R}^{\mathfrak{a}}, E_{\widehat{R}^{\mathfrak{a}}})) \\
&= \operatorname{Hom}_R(L, E_{\widehat{R}^{\mathfrak{a}}}) \\
&= \operatorname{Hom}_R(L, E_R)
\end{aligned}$$

The second step is Hom-Tensor adjointness. The third step is standard and the last step follows from the fact that m-Spec$(R) \cap \operatorname{Supp}(L) \subset \mathcal{F}$.

(3) The subset $Z \subseteq L'$ is an R-submodule if and only if it is an $\overline{R}$-submodule. The isomorphisms in the diagram from part (1) show that Z is an $\overline{R}$-submodule if and only if it is an $\widehat{R}^{\mathfrak{a}}/\operatorname{Ann}_R(L)\widehat{R}^{\mathfrak{a}}$-submodule, that is, if and only if it is an $\widehat{R}^{\mathfrak{a}}$-submodule.

(4) From part (3) we have $\{R\text{-submodules of } L'\} = \{\widehat{R}^{\mathfrak{a}}\text{-submodules of } L'\}$. Thus, the first set satisfies the ascending chain condition (respectively the descending chain condition) if and only if the second one does. Lastly given a submodule N of L we have L/N is artinian as an R-module if and only if it is artinian as a $\widehat{R}^{\mathfrak{a}}$-module. It follows that L is a mini-max R-module if and only if it is a mini-max $\widehat{R}^{\mathfrak{a}}$-module. $\square$

Lemma 3.1.28. *Let $\mathfrak{a}$ be a proper ideal of R and A an artinian R-module. Let $\mathfrak{b} = \cap_{\mathfrak{m} \in \operatorname{Supp}(A) \cap V(\mathfrak{a})} \mathfrak{m}$. Then $\widehat{R}^{\mathfrak{a}} \otimes_R A \cong \Gamma_{\mathfrak{a}}(A) = \Gamma_{\mathfrak{b}}(A) \cong \bigoplus_{\mathfrak{m} \in \operatorname{Supp}(A) \cap V(\mathfrak{a})} \Gamma_{\mathfrak{m}}(A)$, and these modules are artinian both as R-modules and as $\widehat{R}^{\mathfrak{a}}$-modules.*

Proof. Let $\mathfrak{c} = \cap_{\mathfrak{m} \in \operatorname{Supp}(A) \setminus V(\mathfrak{a})} \mathfrak{m}$. Then $A \cong \Gamma_{\mathfrak{b}}(A) \oplus \Gamma_{\mathfrak{c}}(A)$. Since $\mathfrak{c}$ and $\mathfrak{a}$ are co-maximal by Fact 3.1.7 we have $\widehat{R}^{\mathfrak{a}} = \mathfrak{c}\widehat{R}^{\mathfrak{a}}$. Therefore by Lemma 3.1.6 we have $\widehat{R}^{\mathfrak{a}} \otimes_R \Gamma_{\mathfrak{c}}(A) = \widehat{R}^{\mathfrak{a}}/\mathfrak{c}^0\widehat{R}^{\mathfrak{a}} \otimes_R \Gamma_{\mathfrak{c}}(A) = 0$. Since $\mathfrak{a} \subset \mathfrak{b}$, it follows that $\Gamma_{\mathfrak{b}}(A)$ is $\mathfrak{a}$-torsion. Therefore by Lemma 2.1.4 (1) we have that $\widehat{R}^{\mathfrak{a}} \otimes_R \Gamma_{\mathfrak{b}}(A) \cong \Gamma_{\mathfrak{b}}(A)$. Therefore $\widehat{R}^{\mathfrak{a}} \otimes_R A \cong \Gamma_{\mathfrak{b}}(A)$. Since $\mathfrak{a} \subseteq \mathfrak{b}$ the R-module $\Gamma_{\mathfrak{b}}(A)$ is $\mathfrak{a}$-torsion. It follows from

Lemma 2.1.3 (2) that $\Gamma_{\mathfrak{b}}(A)$ is an artinian $\widehat{R}^{\mathfrak{a}}$-module.

The isomorphism $\Gamma_{\mathfrak{b}}(A) \cong \bigoplus_{\mathfrak{m}\in \mathrm{Supp}(A)\cap V(\mathfrak{a})} \Gamma_{\mathfrak{m}}(L)$ follows from Lemma 3.1.24.

Since $\mathfrak{a} \subseteq \mathfrak{b}$ it follows that $\Gamma_{\mathfrak{b}}(A) \subseteq \Gamma_{\mathfrak{a}}(A)$. Since $\mathfrak{c} + \mathfrak{a} = R$ we get the following:

$$0 = \Gamma_R(A) = \Gamma_{\mathfrak{c}+\mathfrak{a}}(A) = \Gamma_{\mathfrak{c}}(A) \cap \Gamma_{\mathfrak{a}}(A)$$

Since $A = \Gamma_{\mathfrak{b}}(A) + \Gamma_{\mathfrak{c}}(A)$ is an internal direct sum, we get the reverse inclusion $\Gamma_{\mathfrak{a}}(A) \subseteq \Gamma_{\mathfrak{b}}(A)$, and the result follows. $\square$

Lemma 3.1.29. *Let $U \subset R$ be a multiplicatively closed set and A an artinian R-module. Let $\mathcal{F} = \{\mathfrak{m} \in \mathrm{Supp}(A) |\ \mathfrak{m} \cap U = \emptyset\}$, $V = R \setminus \cup_{\mathfrak{m}\in\mathcal{F}} \mathfrak{m}$ and $\mathfrak{b} = \cap_{\mathfrak{m}\in\mathcal{F}}\mathfrak{m}$. Then $U^{-1}A \cong V^{-1}A \cong \Gamma_{\mathfrak{b}}(A) \cong \bigoplus_{\mathfrak{m}\in\mathcal{F}} A_{\mathfrak{m}}$, and these modules are artinian both as R-modules and as $U^{-1}R$-modules.*

Proof. Let $\mathcal{G} = \{\mathfrak{m} \in \mathrm{Supp}(A) |\ \mathfrak{m} \cap U \neq \emptyset\}$. Let $\mathfrak{m} \in \mathcal{G}$, $c \in \mathfrak{m} \cap U$ and $a \in A_{\mathfrak{m}}$. Choose $n \in \mathbb{N}$ such that $\mathfrak{m}^n a = 0$. Since U is multiplicatively closed $c^n \in U$. Since $c^n a = 0$ and $a \in A_{\mathfrak{m}}$ was arbitrary it follows that $U^{-1}A_{\mathfrak{m}} = 0$.

Let $\mathfrak{n} \in \mathcal{F}$. Since $\mathfrak{n}\cap U = \emptyset$ it follows that $\mathfrak{n}U^{-1}R$ is a maximal ideal in $U^{-1}R$. Let $W = R\setminus\mathfrak{n}$ and $W' = U^{-1}R\setminus\mathfrak{n}U^{-1}R$. Since $\mathrm{Supp}_{U^{-1}R}(U^{-1}A_{\mathfrak{n}}) = \{\mathfrak{n}U^{-1}R\}$, Lemma 3.1.24 explains the first step in the following sequence $U^{-1}A_{\mathfrak{n}} \cong (U^{-1}A_{\mathfrak{n}})_{\mathfrak{n}U^{-1}R} = W'^{-1}U^{-1}A_{\mathfrak{n}} \cong W^{-1}A_{\mathfrak{n}} = A_{\mathfrak{n}}$. Thus the result follows from the decomposition in Lemma 3.1.25. $\square$

Lemma 3.1.30. *Let $\mathfrak{a}$ be a proper ideal of R. If M is a mini-max R-module, then $\widehat{R}^{\mathfrak{a}} \otimes_R M$ is a mini-max $\widehat{R}^{\mathfrak{a}}$-module.*

Proof. If M is mini-max over R, then there is an exact sequence of R-module homomorphisms

$$0 \to N \to M \to A \to 0$$

where N is noetherian over R and A is artinian over R. The ring $\widehat{R}^{\mathfrak{a}}$ is flat over R, so the base-changed sequence

$$0 \to \widehat{R}^{\mathfrak{a}} \otimes_R N \to \widehat{R}^{\mathfrak{a}} \otimes_R M \to \widehat{R}^{\mathfrak{a}} \otimes_R A \to 0$$

is an exact sequence of $\widehat{R}^{\mathfrak{a}}$-module homomorphisms. The $\widehat{R}^{\mathfrak{a}}$-module $\widehat{R}^{\mathfrak{a}} \otimes_R N$ is noetherian. Lemma 3.1.28 implies that the $\widehat{R}^{\mathfrak{a}}$-module $\widehat{R}^{\mathfrak{a}} \otimes_R A$ is artinian, so $\widehat{R}^{\mathfrak{a}} \otimes_R M$ is mini-max over $\widehat{R}^{\mathfrak{a}}$. $\square$

Lemma 3.1.31. *Let L be an R-module, and let $\mathcal{F}$ be a finite subset of* $\text{m-Spec}(R)$*. Set $\mathfrak{a} = \cap_{\mathfrak{m} \in \mathcal{F}} \mathfrak{m}$. Then the following conditions are equivalent:*

1. *L is artinian over R and* $\operatorname{Supp}_R(L) \subseteq \mathcal{F}$*;*
2. *L has an $\widehat{R}^{\mathfrak{a}}$-module structure, compatible with its R-module structure, such that L is an artinian $\widehat{R}^{\mathfrak{a}}$-module;*
3. *$\widehat{R}^{\mathfrak{a}} \otimes_R L$ is an artinian $\widehat{R}^{\mathfrak{a}}$-module and* $\operatorname{Supp}_R(L) \subseteq \mathcal{F}$*;*
4. *L is $\mathfrak{a}$-torsion and $\mu^0_R(\mathfrak{m}, L) < \infty$ for all $\mathfrak{m} \in$* $\text{m-Spec}(R)$*; and*
5. *L is $\mathfrak{a}$-torsion and $\mu^0_R(\mathfrak{m}, L) < \infty$ for all $\mathfrak{m} \in \mathcal{F}$.*

Proof. (1) $\implies$ (2) Assume that L is artinian over R such that $\operatorname{Supp}_R(L) \subseteq \mathcal{F}$. Lemma 3.1.22 implies that L is $\mathfrak{a}$-torsion, so Fact 2.1.2 (c) and Lemma 2.1.3 (2) imply (2).

(2) $\implies$ (4) Assume that L has an $\widehat{R}^{\mathfrak{a}}$-module structure, compatible with its R-module structure, such that L is an artinian $\widehat{R}^{\mathfrak{a}}$-module. The Jacobson radical of $\widehat{R}^{\mathfrak{a}}$ is $\mathfrak{a}\widehat{R}^{\mathfrak{a}}$. Since L is artinian over $\widehat{R}^{\mathfrak{a}}$, we know that L is $\mathfrak{a}\widehat{R}^{\mathfrak{a}}$-torsion, so it is $\mathfrak{a}$-torsion. By Lemma 2.1.3 (2) it follows that L is an artinian R-module. Thus by Lemma 3.1.19 we get that $\mu^0_R(\mathfrak{m}, L) < \infty$ for all $\mathfrak{m} \in \text{m-Spec}(R)$.

(4) $\implies$ (5) This is evident.

(5) $\implies$ (1) Assume that L is $\mathfrak{a}$-torsion and $\mu_R^0(\mathfrak{m}, L) < \infty$ for all $\mathfrak{m} \in \mathcal{F}$. By Lemma 3.1.24 (2) we have that $L \cong \bigoplus_{\mathfrak{m}\in\mathcal{F}} L_{\mathfrak{m}}$. Therefore the indecomposable injective summand of $\mathrm{E}_R(L)$ are all of the form $\mathrm{E}_R(R/\mathfrak{m})$ for some $\mathfrak{m} \in \mathcal{F}$. Since $\mu_R^0(\mathfrak{m}, L) < \infty$ for all $\mathfrak{m} \in \mathcal{F}$, it follows that $\mathrm{E}_R(L)$ is a finite co-product of artinian modules. Therefore $\mathrm{E}_R(L)$ is artian. Since L injects into $\mathrm{E}_R(L)$ we have that L is artinian.

(1) $\implies$ (3) This is follows immediately from Lemma 3.1.28.

(3) $\implies$ (2) Assume that $\widehat{R}^{\mathfrak{a}} \otimes_R L$ is an artinian $\widehat{R}^{\mathfrak{a}}$-module and $\mathrm{Supp}_R(L) \subseteq \mathcal{F}$. Since $L_{\mathfrak{p}} = 0$ for all $\mathfrak{p} \notin \mathcal{F}$ we have $\mu_R^0(\mathfrak{p}, L) = 0$ for all $\mathfrak{p} \notin \mathcal{F}$. Since $\mathrm{E}_R(R/\mathfrak{m})$ is $\mathfrak{a}$-torsion for all $\mathfrak{m} \in \mathcal{F}$ it follows that $\mathrm{E}_R(L) = \bigoplus_{\mathfrak{m}\in\mathcal{F}} \mathrm{E}_R(R/\mathfrak{m})^{\mu_R^0(\mathfrak{m},L)}$ is $\mathfrak{a}$-torsion. Since L injects into $\mathrm{E}_R(L)$ it is $\mathfrak{a}$-torsion. By Lemma 2.1.4 (1) we know that L is isomorphic to $\widehat{R}^{\mathfrak{a}} \otimes_R L$. Therefore L is an artinian $\widehat{R}^{\mathfrak{a}}$-module. Hence by Lemma 2.1.3 (2) it follows that L is an artinian R-module. $\square$

Lemma 3.1.32. *Let L be an R-module such that $R/\operatorname{Ann}_R(L)$ is semi-local and complete. Set $\mathfrak{b} = \bigcap_{\mathfrak{m}\in \text{m-Spec}(R)\cap \mathrm{Supp}_R(L)} \mathfrak{m}$, and let $\mathfrak{a} \subseteq \mathfrak{b}$. Then the following conditions are equivalent:*

1. *L is mini-max as an R-module;*
2. *L is mini-max as an $\widehat{R}^{\mathfrak{a}}$-module;*
3. *L is Matlis reflexive as an R-module; and*
4. *L is Matlis reflexive as an $\widehat{R}^{\mathfrak{a}}$-module.*

Proof. Assume without loss of generality that $L \neq 0$.

(1) $\iff$ (2) This is stated in Lemma 3.1.27 (4).

(1) $\iff$ (3) This is an immediate consequence of Fact 3.1.10.

(2) $\iff$ (4) Since $R/\operatorname{Ann}_R(L) \cong \widehat{R}^{\mathfrak{a}}/\operatorname{Ann}_R(L)\widehat{R}^{\mathfrak{a}} \cong \widehat{R}^{\mathfrak{a}}/\operatorname{Ann}_{\widehat{R}^{\mathfrak{a}}}(L)$. The equivalence is a consequence of Fact 3.1.10. $\square$

Lemma 3.1.33. *Let L be an R-module such that $R/\operatorname{Ann}_R(L)$ is artinian. Then L has finite length if and only if L is mini-max.*

Proof. If L has finite length then clearly it is mini-max. Conversely assume that L is mini-max. Then L is mini-max as an $R/\operatorname{Ann}_R(L)$-module. Over an artinian ring artinian and noetherian modules have finite length; hence so do mini-max modules, and the result follows. $\square$

Lemma 3.1.34. *Given an R-module L, there is a inclusion*

$$\operatorname{Supp}_R(L) \cap \text{m-Spec}(R) \subseteq \operatorname{Supp}_R(L^\vee) \cap \text{m-Spec}(R).$$

Proof. Let $\mathfrak{m} \in \operatorname{Supp}_R(L) \cap \text{m-Spec}(R)$. Since $L_\mathfrak{m} \neq 0$, there is an element $x \in L$ such that $x/1 \neq 0$ in $L_\mathfrak{m}$. Thus, the submodule $L' = Rx \subseteq L$ is finitely generated and $L'_\mathfrak{m} \neq 0$. It follows that

$$(L'^\vee)_\mathfrak{m} \cong (L'_\mathfrak{m})^{\vee(R_\mathfrak{m})} \neq 0.$$

The inclusion $L' \subseteq L$ yields an epimorphism $(L^\vee)_\mathfrak{m} \twoheadrightarrow (L'^\vee)_\mathfrak{m} \neq 0$, implying that $(L^\vee)_\mathfrak{m} \neq 0$. This shows that $\mathfrak{m} \in \operatorname{Supp}_R(L^\vee) \cap \text{m-Spec}(R)$, as desired. $\square$

The next example shows that the inclusion in Lemma 3.1.34 can be strict.

Example 3.1.35. Let $R = k[X]$, $\mathfrak{n} = RX$ and $L = \bigoplus_{\mathfrak{m} \in \text{m-Spec}(R) \setminus \{\mathfrak{n}\}} R/\mathfrak{m}$. The maximal ideal $\mathfrak{n}$ is not in $\operatorname{Supp}_R(L)$. We claim, however, that $\mathfrak{n} \in \operatorname{Supp}_R(L^\vee)$. To see this, observe that

$$L^\vee \cong \prod_{\mathfrak{m}\in\text{m-Spec}(R)\smallsetminus\{\mathfrak{n}\}}(R/\mathfrak{m})^\vee \cong \prod_{\mathfrak{m}\in\text{m-Spec}(R)\smallsetminus\{\mathfrak{n}\}} R/\mathfrak{m}.$$

The natural map $R \to \prod_{\mathfrak{m}\neq\mathfrak{n}} R/\mathfrak{m} \cong L^\vee$ given by $1 \mapsto \{1+\mathfrak{m}\}$ is a monomorphism since its kernel is $\cap_{\mathfrak{m}\neq\mathfrak{n}}\mathfrak{m} = 0$. It follows that $\mathfrak{n} \in \operatorname{Supp}_R(R) \subseteq \operatorname{Supp}_R(L^\vee)$.

3.2 Properties of $\operatorname{Ext}^i_R(M,-)$ and $\operatorname{Tor}^R_i(M,-)$

Theorem 3.2.1. *Let A and L be R-modules such that A is artinian. Let $\mathcal{F}$ be a finite subset of* m-Spec(R) *containing* $\operatorname{Supp}(A)\cap\operatorname{Supp}(L)$. *Let $\mathfrak{b} = \cap_{\mathfrak{m}\in\mathcal{F}}\mathfrak{m}$. Given any $i \geqslant 0$ such that $\mu^i_R(\mathfrak{m}, L) < \infty$ for all $\mathfrak{m} \in \operatorname{Supp}(A)\cap\operatorname{Supp}(L)$ we have that* $\operatorname{Ext}^i_R(A,L)$ *is a noetherian $\widehat{R}^{\mathfrak{b}}$-module.*

Proof. Let $J^\cdot$ be the minimal R-injective resolution of L. For $\mathfrak{m} \in$ m-Spec(R), we have $\Gamma_{\mathfrak{b}}(J^i) \cong \bigoplus_{\mathfrak{m}\in\mathcal{F}}\Gamma_{\mathfrak{m}}(J^i) \cong \bigoplus_{\mathfrak{m}\in\mathcal{F}} \mathrm{E}_R(R/\mathfrak{m})^{\mu^i_R(\mathfrak{m},L)}$. This explains the second step in the next sequence.

$$\begin{aligned}
\operatorname{Hom}_R(A,J^i) &\cong \textstyle\bigoplus_{\mathfrak{m}\in\operatorname{Supp}(A)} \operatorname{Hom}_R(A_{\mathfrak{m}},J^i)\\
&\cong \textstyle\bigoplus_{\mathfrak{m}\in\operatorname{Supp}(A)} \operatorname{Hom}_R(A_{\mathfrak{m}},\Gamma_{\mathfrak{m}}(J^i))\\
&\cong \textstyle\bigoplus_{\mathfrak{m}\in\operatorname{Supp}(A)} \operatorname{Hom}_R(A_{\mathfrak{m}},\mathrm{E}_R(R/\mathfrak{m}))^{\mu^i_R(\mathfrak{m},L)}\\
&\cong \textstyle\bigoplus_{\mathfrak{m}\in\operatorname{Supp}(A)\cap\operatorname{Supp}(L)} \operatorname{Hom}_R(A_{\mathfrak{m}},\mathrm{E}_R(R/\mathfrak{m}))^{\mu^i_R(\mathfrak{m},L)}\\
&\cong \textstyle\bigoplus_{\mathfrak{m}\in\operatorname{Supp}(A)\cap\operatorname{Supp}(L)} \operatorname{Hom}_{\widehat{R}^{\mathfrak{m}}}(A_{\mathfrak{m}},\mathrm{E}_R(R/\mathfrak{m}))^{\mu^i_R(\mathfrak{m},L)}.
\end{aligned}$$

The second and last steps are from Lemma 2.1.5. The first step is from Lemma 3.1.24. The fourth step follows from the fact that $\mu^i_R(\mathfrak{m},L) = 0$ whenever $\mathfrak{m} \notin \operatorname{Supp}(L)$.

By Matlis duality, $\mathrm{Hom}_{\widehat{R}^{\mathfrak{m}}}(A_{\mathfrak{m}}, \mathrm{E}_R(R/\mathfrak{m}))$ is a noetherian $\widehat{R}^{\mathfrak{m}}$-module and hence a noetherian $\widehat{R}^{\mathfrak{b}}$ module. Therefore $\mathrm{Hom}_R(A, J^{\cdot})$ is a complex of noetherian $\widehat{R}^{\mathfrak{b}}$-modules. (Also, the differentials in $\mathrm{Hom}_R(A, J^{\cdot})$ are $\widehat{R}^{\mathfrak{b}}$-linear.) Thus, the subquotient $\mathrm{Ext}^i_R(A, L)$ is a noetherian $\widehat{R}^{\mathfrak{b}}$-module by Lemma 2.1.23 (2). $\square$

Corollary 3.2.2. *Let A be an artinian R-module and M a mini-max R-module. Let $\mathcal{F}$ be a finite subset of* m-Spec(R) *containing* $\mathrm{Supp}(A) \cap \mathrm{Supp}(M)$*. Let $\mathfrak{b} = \cap_{\mathfrak{m} \in \mathcal{F}} \mathfrak{m}$. Then* $\mathrm{Ext}^i_R(A, M)$ *is a noetherian $\widehat{R}^{\mathfrak{b}}$-module.*

Theorem 3.2.3. *Let A and L be R-modules so that A is artinian. For any $i \geqslant 0$ with $\beta^R_i(\mathfrak{m}, L) < \infty$ for all $\mathfrak{m} \in \mathrm{Supp}(A) \cap \mathrm{Supp}(L)$, the module $\mathrm{Tor}^R_i(A, L)$ is artinian.*

Proof. Since A is artinian, Lemma 3.1.11 implies that $\mathrm{Supp}_R(A)$ is finite. Thus, the inclusion

$$\mathrm{Supp}_R(\mathrm{Tor}^R_i(A, L)) \subseteq \mathrm{Supp}_R(A) \cap \mathrm{Supp}_R(L) \tag{3.4}$$

implies that $\mathrm{Supp}_R(\mathrm{Tor}^R_i(A, L))$ is finite. For each $\mathfrak{p} \in \mathrm{Supp}_R(\mathrm{Tor}^R_i(A, L))$, the $R_{\mathfrak{p}}$-module $A_{\mathfrak{p}}$ is artinian. Furthermore, we have $\beta^{R_{\mathfrak{p}}}_i(L_{\mathfrak{p}}) = \beta^R_i(\mathfrak{p}, L) < \infty$ by Lemma 3.1.19. Hence by [17, Theorem 3.1], the $R_{\mathfrak{p}}$-module $\mathrm{Tor}^{R_{\mathfrak{p}}}_i(A_{\mathfrak{p}}, L_{\mathfrak{p}}) \cong \mathrm{Tor}^R_i(A, L)_{\mathfrak{p}}$ is artinian. Thus, Lemma 3.1.12 implies that $\mathrm{Tor}^R_i(A, L)$ is artinian. $\square$

Corollary 3.2.4. *Let A be an artinian R-module and M a mini-max R-module. Then for all $i \geqslant 0$, the R-module $\mathrm{Tor}^R_i(A, M)$ is artinian.*

Theorem 3.2.5. *Let M and M' be mini-max R-modules. Then for all $i \geqslant 0$, the R-module $\mathrm{Tor}^R_i(M, M')$ is mini-max.*

Proof. Let N be a noetherian submodule of M such that the quotient M/N is artinian. Lemma 2.1.23(3) and Corollary 3.2.4 imply that $\mathrm{Tor}^R_i(N, M')$ and $\mathrm{Tor}^R_i(A, M')$ are mini-max. Thus, $\mathrm{Tor}^R_i(M, M')$ is mini-max by Lemma 2.1.24(2). $\square$

Proposition 3.2.6. *Let A be an artinian R-module and M a mini-max R-module. Let $\mathcal{F}$ be a finite subset of* m-Spec(R) *containing* $\operatorname{Supp}(A) \cap \operatorname{Supp}(M)$. *Let $\mathfrak{b} = \cap_{\mathfrak{m}\in\mathcal{F}}\mathfrak{m}$. Then $\operatorname{Ext}^i_R(M,A)$ is a Matlis reflexive $\widehat{R}^{\mathfrak{b}}$-module.*

Proof. Fix a noetherian submodule $N \subseteq M$ such that M/N is artinian. Lemma 2.1.23(3) implies that $\operatorname{Ext}^i_R(N,A)$ is an artinian R-module. Since N is noetherian, we have

$$\operatorname{Supp}(\operatorname{Ext}^i_R(N,A)) \subseteq \operatorname{Supp}_R(N) \cap \operatorname{Supp}_R(A) \subseteq \operatorname{Supp}_R(M) \cap \operatorname{Supp}_R(A) \subseteq \mathcal{F},$$

so we conclude from Lemma 3.1.31 that $\operatorname{Ext}^i_R(N,A)$ is an artinian $\widehat{R}^{\mathfrak{b}}$-module. Corollary 3.2.2 implies that $\operatorname{Ext}^i_R(M/N,A)$ is a noetherian $\widehat{R}^{\mathfrak{b}}$-module. Since $\mathcal{F}$ is a finite set of maximal ideals, the ring $\widehat{R}^{\mathfrak{b}}$ is semi-local and complete. Hence Fact 3.1.10 implies that the $\widehat{R}^{\mathfrak{b}}$-modules $\operatorname{Ext}^i_R(N,A)$ and $\operatorname{Ext}^i_R(M/N,A)$ are Matlis reflexive. Therefore $\operatorname{Ext}^i_R(M,A)$ is a Matlis reflexive $\widehat{R}^{\mathfrak{b}}$-module by Lemma 2.1.24(2). $\square$

Theorem 3.2.7. *Let M and M' be mini-max R-modules.*

1. *If the quotient ring $R/(\operatorname{Ann}_R(M)+\operatorname{Ann}_R(M'))$ is semi-local and complete then $\operatorname{Tor}_i^R(M,M')$ is a Matlis reflexive R-module.*
2. *If $R/(\operatorname{Ann}_R(M)+\operatorname{Ann}_R(M'))$ is artinian then $\operatorname{Tor}_i^R(M,M')$ has finite length.*

Proof. (1) Since $\operatorname{Ann}_R(M)+\operatorname{Ann}_R(M') \subseteq \operatorname{Ann}_R(\operatorname{Tor}_i^R(M,M'))$ we get an epimorphism:

$$R/(\operatorname{Ann}_R(M)+\operatorname{Ann}_R(M')) \twoheadrightarrow R/\operatorname{Ann}_R(\operatorname{Tor}_i^R(M,M')).$$

Therefore $R/\operatorname{Ann}_R(\operatorname{Tor}_i^R(M,M'))$ is semi-local and complete. It follows from Fact 3.1.10 and Theorem 3.2.5 that $\operatorname{Tor}_i^R(M,M')$ is Matlis reflexive over R.

(2) Part (1) and Lemma 3.1.33 imply that $\operatorname{Tor}_i^R(M,M')$ has finite length. $\square$

Corollary 3.2.8. *Let M be a mini-max R-module and M' a Matlis reflexive R-module. Then $\operatorname{Tor}_i^R(M, M')$ is a Matlis reflexive R-module.*

Theorem 3.2.9. *Let M and M' be mini-max R-modules.*

1. *If the quotient ring $R/(\operatorname{Ann}_R(M)+\operatorname{Ann}_R(M'))$ is semi-local and complete, then $\operatorname{Ext}_R^i(M, M')$ is a Matlis reflexive R-module.*

2. *If $R/(\operatorname{Ann}_R(M) + \operatorname{Ann}_R(M'))$ is artinian then $\operatorname{Ext}_R^i(M, M')$ has finite length.*

Proof. (1) Fix a noetherian submodule $N \subseteq M$ such that M/N is artinian. The inclusions

$$\operatorname{Ann}_R(M) + \operatorname{Ann}_R(M') \subseteq \operatorname{Ann}_R(N) + \operatorname{Ann}_R(M') \subseteq \operatorname{Ann}_R(\operatorname{Ext}_R^i(N, M'))$$

provide an epimorphism:

$$R/(\operatorname{Ann}_R(M) + \operatorname{Ann}_R(M')) \twoheadrightarrow R/\operatorname{Ann}_R(\operatorname{Ext}_R^i(N, M')).$$

Therefore $R/\operatorname{Ann}_R(\operatorname{Ext}_R^i(N, M'))$ is semi-local and complete. Thus, Fact 3.1.10 and Lemma 2.1.23 (3) imply that $\operatorname{Ext}_R^i(N, M')$ is Matlis reflexive over R.

Similarly, since

$$\operatorname{Ann}_R(M) + \operatorname{Ann}_R(M') \subseteq \operatorname{Ann}_R(M/N) + \operatorname{Ann}_R(M') \subseteq \operatorname{Ann}_R(\operatorname{Ext}_R^i(M/N, M')),$$

it follows that $R/\operatorname{Ann}_R(\operatorname{Ext}_R^i(M/N, M'))$ is semi-local and complete. Let

$$\mathfrak{b} = \cap_{\mathfrak{m} \in \operatorname{Supp}(\operatorname{Ext}_R^i(M/N, M'))} \mathfrak{m} \quad \text{and} \quad \mathfrak{a} = \cap_{\mathfrak{m} \in \operatorname{Supp}(M/N) \cap \operatorname{Supp}(M')} \mathfrak{m}.$$

Then $\mathfrak{a} \subseteq \mathfrak{b}$. Corollary 3.2.2 implies that $\operatorname{Ext}^i_R(M/N, M')$ is mini-max as an $\widehat{R}^{\mathfrak{a}}$-module, so it is mini-max as an R-module by Lemma 3.1.32. Using Fact 3.1.10, we conclude that $\operatorname{Ext}^i_R(M/N, M')$ is Matlis reflexive over R. Thus, Lemma 2.1.24(2) implies that $\operatorname{Ext}^i_R(M, M')$ is also Matlis reflexive over R.

(2) Part (1) and Lemma 3.1.33 imply that $\operatorname{Ext}^i_R(M, M')$ has finite length. $\square$

Corollary 3.2.10. *Let M be a mini-max R-module and M' a Matlis reflexive R-module. Then $\operatorname{Ext}^i_R(M, M')$ and $\operatorname{Ext}^i_R(M', M)$ are Matlis reflexive R-modules.*

Proposition 3.2.11. *Let M be a mini-max R-module and N' a noetherian R-module such that $R/(\operatorname{Ann}_R(M) + \operatorname{Ann}_R(N'))$ is semi-local and complete. Let $\mathcal{F}$ be a finite subset of $\operatorname{m-Spec} R$ containing the finite set $\operatorname{m-Spec}(R) \cap \operatorname{Supp}(M) \cap \operatorname{Supp}(N')$. Let $\mathfrak{b} = \cap_{\mathfrak{m} \in \mathcal{F}} \mathfrak{m}$. Then $\operatorname{Ext}^i_R(M, N')$ is noetherian as an R-module and as an $\widehat{R}^{\mathfrak{b}}$-module.*

Proof. The set $\operatorname{m-Spec}(R) \cap \operatorname{Supp}_R(M) \cap \operatorname{Supp}_R(N')$ is finite, because the quotient $R/(\operatorname{Ann}_R(M) + \operatorname{Ann}_R(N'))$ is semi-local. Let N be a noetherian submodule of M such that M/N is artinian. Lemma 2.1.23(3) implies that $\operatorname{Ext}^i_R(N, N')$ is a noetherian R-module. Since $\operatorname{m-Spec}(R) \cap \operatorname{Supp}_R(M/N) \cap \operatorname{Supp}_R(N') \subseteq \mathcal{F}$, it follows by Corollary 3.2.2 that $\operatorname{Ext}^i_R(M/N, N')$ is a noetherian $\widehat{R}^{\mathfrak{b}}$-module. As $R/\operatorname{Ann}_R(\operatorname{Ext}^i_R(M/N, N'))$ is semi-local and complete, Lemma 3.1.27(4) implies that $\operatorname{Ext}^i_R(M/N, N')$ is a noetherian R-module. Therefore the R-module $\operatorname{Ext}^i_R(M, N')$ is also noetherian, by Lemma 2.1.24(2). Since $R/\operatorname{Ann}(\operatorname{Ext}^i_R(M, N'))$ is semi-local and complete, Lemma 3.1.27(4) implies that $\operatorname{Ext}^i_R(M, N')$ is a noetherian $\widehat{R}^{\mathfrak{b}}$-module. $\square$

3.3 Change of Rings Results for $\operatorname{Ext}^i_R(A, L)$

Lemma 3.3.1. *Let I be an injective R-module, and $\mathcal{G}$ be a finite subset of $\operatorname{m-Spec}(R)$. Let $\mathfrak{b} = \cap_{\mathfrak{m} \in \mathcal{G}} \mathfrak{m}$, $V = R \smallsetminus \cup_{\mathfrak{m} \in \mathcal{G}} \mathfrak{m}$ and U be a multiplicatively closed set contained in*

V. *Then the natural map* $\Gamma_{\mathfrak{b}}(I) \to \Gamma_{\mathfrak{b}}(U^{-1}I)$ *is bijective.*

Proof. It is straightforward to show that if $\mathfrak{p} \cap U = \emptyset$, then $U^{-1}\,\mathrm{E}_R(R/\mathfrak{p}) \cong \mathrm{E}_R(R/\mathfrak{p})$. Also, if $\mathfrak{p} \cap U \neq \emptyset$, then $U^{-1}\,\mathrm{E}_R(R/\mathfrak{p}) = 0$.

Write $I = \bigoplus_{\mathfrak{p}\in\mathrm{Spec}(R)} \mathrm{E}_R(R/\mathfrak{p})^{(\mu_p)}$. From the previous paragraph, the localization map $\rho\colon I \to U^{-1}I$ is a split surjection with $\mathrm{Ker}(\rho) = \bigoplus_{\mathfrak{p}\cap U\neq\emptyset} \mathrm{E}_R(R/\mathfrak{p})^{(\mu_p)}$. Since ρ is a split surjection, it follows that $\Gamma_{\mathfrak{b}}(\rho)\colon \Gamma_{\mathfrak{b}}(I) \to \Gamma_{\mathfrak{b}}(U^{-1}I)$ is a split surjection with $\mathrm{Ker}(\Gamma_{\mathfrak{b}}(\rho)) = \bigoplus_{\mathfrak{p}\cap U\neq\emptyset} \Gamma_{\mathfrak{b}}(\mathrm{E}_R(R/\mathfrak{p}))^{(\mu_p)}$. Thus, it remains to show that $\Gamma_{\mathfrak{b}}(\mathrm{E}_R(R/\mathfrak{p})) = 0$ when $\mathfrak{p} \cap U \neq \emptyset$.

Assume that $\mathfrak{p} \cap U \neq \emptyset$. Then $\mathfrak{p} \cap V \neq \emptyset$, so $\mathfrak{p} \nsubseteq \mathfrak{m}$ for any $\mathfrak{m} \in \mathcal{G}$. Since $\mathcal{G}$ is a set of maximal ideals it follows that $\mathfrak{m} \nsubseteq \mathfrak{p}$ for any $\mathfrak{m} \in \mathcal{G}$ and $\mathfrak{b} = \cap_{\mathfrak{m}\in\mathcal{G}}\mathfrak{m} \nsubseteq \mathfrak{p}$. Therefore $\Gamma_{\mathfrak{b}}(\mathrm{E}_R(R/\mathfrak{p})) = 0$ and the result follows. □

Lemma 3.3.2. *Let I be an injective R-module and $\mathcal{F}$ be a finite subset of* $\text{m-Spec}(R)$. *Let $\mathfrak{b} = \cap_{\mathfrak{m}\in\mathcal{F}}\mathfrak{m}$ and let $\mathfrak{a}$ be a proper ideal such that $\mathcal{F} \subseteq V(\mathfrak{a})$. Then the natural map $\Gamma_{\mathfrak{b}}(I) \to \Gamma_{\mathfrak{b}}(\widehat{R}^{\mathfrak{a}} \otimes_R I)$ is bijective.*

Proof. By Lemma 2.1.4 (1) $\Gamma_{\mathfrak{b}}(\widehat{R}^{\mathfrak{a}} \otimes \mathrm{E}_R(R/\mathfrak{m})) \cong \Gamma_{\mathfrak{b}}(\mathrm{E}_R(R/\mathfrak{m}))$ for all $\mathfrak{m} \in \mathcal{F}$. Also for all $\mathfrak{m} \in \text{m-Spec}(R)$ and $\mathfrak{p} \in \mathrm{Spec}(R)$ with $\mathfrak{p} \neq \mathfrak{m}$ we have that $\Gamma_{\mathfrak{m}}(\widehat{R}^{\mathfrak{a}} \otimes_R \mathrm{E}_R(R/\mathfrak{p})) = 0$. Therefore for all $\mathfrak{p} \notin \mathcal{F}$ we have

$$\Gamma_{\mathfrak{b}}(\widehat{R}^{\mathfrak{a}} \otimes_R \mathrm{E}_R(R/\mathfrak{p})) = \bigoplus_{\mathfrak{m}\in\mathcal{F}} \Gamma_{\mathfrak{m}}(\widehat{R}^{\mathfrak{a}} \otimes_R \mathrm{E}_R(R/\mathfrak{p})) = 0 = \Gamma_{\mathfrak{b}}(\mathrm{E}_R(R/\mathfrak{p})).$$

□

Theorem 3.3.3. *Let A and L be R-modules such that A is artinian. Let $\mathcal{F} = \mathrm{Supp}_R(A) \cap \mathrm{Supp}_R(L)$. Let $\mathcal{G}$ be a finite set of maximal ideals containing $\mathcal{F}$. Let $\mathfrak{a}$ be a proper ideal such that $\mathcal{F} \subseteq V(\mathfrak{a})$. Let $U \subset R$ be a multiplicatively closed set such* *that* $U \subseteq R \setminus \cup_{\mathfrak{m}\in\mathcal{F}} \mathfrak{m}$. *Then*

$$\begin{aligned}\operatorname{Ext}^i_R(A,L) &\cong \operatorname{Ext}^i_{U^{-1}R}(U^{-1}A,U^{-1}L) \cong \textstyle\prod_{\mathfrak{m}\in\mathcal{G}} \operatorname{Ext}^i_{R_\mathfrak{m}}(A_\mathfrak{m},L_\mathfrak{m})\\ &\cong \operatorname{Ext}^i_{\widehat{R}^\mathfrak{a}}(\Gamma_\mathfrak{a}(A),\widehat{R}^\mathfrak{a}\otimes_R L) \cong \textstyle\prod_{\mathfrak{m}\in\mathcal{G}} \operatorname{Ext}^i_{\widehat{R}^\mathfrak{m}}(A_\mathfrak{m},\widehat{R}^\mathfrak{m}\otimes_R L).\end{aligned}$$

Proof. Let $I^\cdot$ be an injective resolution of L. Let $\mathfrak{b} = \cap_{\mathfrak{m}\in\mathcal{G}}\mathfrak{m}$ and let $V = R \setminus \cup_{\mathfrak{m}\in\mathcal{G}} \mathfrak{m}$. For all $\mathfrak{m} \notin \mathcal{F}$ either $A_\mathfrak{m} = 0$ or $\Gamma_\mathfrak{m}(I^\cdot) = 0$. It follows that $\operatorname{Hom}_R(A_\mathfrak{m}, I^\cdot) \cong \operatorname{Hom}_R(A_\mathfrak{m}, \Gamma_\mathfrak{m}(I^\cdot)) = 0$ for all $\mathfrak{m} \notin \mathcal{F}$. Since $\operatorname{Supp}_R(A)$ and $\mathcal{G}$ both contain $\mathcal{F}$ this explains the third isomorphism in the next display.

$$\begin{aligned}\operatorname{Hom}_R(A,I^\cdot) &\cong \operatorname{Hom}_R(\textstyle\bigoplus_{\mathfrak{m}\in\operatorname{Supp}_R(A)} A_\mathfrak{m}, \Gamma_\mathfrak{a}(I^\cdot))\\ &\cong \textstyle\prod_{\mathfrak{m}\in\operatorname{Supp}_R(A)} \operatorname{Hom}_R(A_\mathfrak{m},I^\cdot)\\ &\cong \textstyle\prod_{\mathfrak{m}\in\mathcal{G}} \operatorname{Hom}_R(A_\mathfrak{m},I^\cdot)\\ &\cong \textstyle\prod_{\mathfrak{m}\in\mathcal{G}} \operatorname{Hom}_R(A_\mathfrak{m},\Gamma_\mathfrak{m}(I^\cdot))\\ &\cong \textstyle\prod_{\mathfrak{m}\in\mathcal{G}} \operatorname{Hom}_R(A_\mathfrak{m},\Gamma_\mathfrak{m}(I^\cdot_\mathfrak{m}))\\ &\cong \textstyle\prod_{\mathfrak{m}\in\mathcal{G}} \operatorname{Hom}_R(A_\mathfrak{m},\operatorname{Hom}_{R_\mathfrak{m}}(R_\mathfrak{m},I^\cdot_\mathfrak{m}))\\ &\cong \textstyle\prod_{\mathfrak{m}\in\mathcal{G}} \operatorname{Hom}_{R_\mathfrak{m}}(A_\mathfrak{m}\otimes_R R_\mathfrak{m},I^\cdot_\mathfrak{m})\\ &\cong \textstyle\prod_{\mathfrak{m}\in\mathcal{G}} \operatorname{Hom}_{R_\mathfrak{m}}(A_\mathfrak{m},I^\cdot_\mathfrak{m})\end{aligned}$$

The second, sixth and last steps are standard. The seventh step is from Hom-tensor adjointness. The first, fourth and fifth steps come from Lemmas 3.1.25, 2.1.5(2) and 3.3.1 respectively. Taking cohomology we have $\operatorname{Ext}^i_R(A,L) \cong \prod_{\mathfrak{m}\in\mathcal{G}} \operatorname{Ext}^i_{R_\mathfrak{m}}(A_\mathfrak{m},L_\mathfrak{m})$.

We may assume $\mathcal{G} = \{\mathfrak{m} \in \operatorname{Supp}_R(A) \mid \mathfrak{m} \cap U = \emptyset\}$. Let $\mathfrak{b} = \cap_{\mathfrak{m}\in\mathcal{G}}\mathfrak{m}$. Continuing from the third line in the previous display we get the first step in the next display.

$$\begin{aligned}
\operatorname{Hom}_R(A, I^{\cdot}) &\cong \textstyle\prod_{\mathfrak{m}\in\mathcal{G}} \operatorname{Hom}_R(A_{\mathfrak{m}}, I^{\cdot}) \\
&\cong \operatorname{Hom}_R(\textstyle\bigoplus_{\mathfrak{m}\in\mathcal{G}} A_{\mathfrak{m}}, I^{\cdot}) \\
&\cong \operatorname{Hom}_R(\Gamma_{\mathfrak{b}}(A), I^{\cdot}) \\
&\cong \operatorname{Hom}_R(\Gamma_{\mathfrak{b}}(A), \Gamma_{\mathfrak{b}}(I^{\cdot})) \\
&\cong \operatorname{Hom}_R(\Gamma_{\mathfrak{b}}(A), \Gamma_{\mathfrak{b}}(U^{-1}I^{\cdot})) \\
&\cong \operatorname{Hom}_R(\Gamma_{\mathfrak{b}}(A), U^{-1}I^{\cdot}) \\
&\cong \operatorname{Hom}_R(U^{-1}A, U^{-1}I^{\cdot}) \\
&\cong \operatorname{Hom}_R(U^{-1}A, \operatorname{Hom}_{U^{-1}R}(U^{-1}R, U^{-1}I^{\cdot})) \\
&\cong \operatorname{Hom}_{U^{-1}R}(U^{-1}A \otimes_R U^{-1}R, U^{-1}I^{\cdot}) \\
&\cong \operatorname{Hom}_{U^{-1}R}(U^{-1}A, U^{-1}I^{\cdot})
\end{aligned}$$

The second, eigth and last steps are standard. The ninth step is Hom-tensor adjointness. Steps four and six are from Lemma 2.1.5 (2). Steps three, five and seven are from Lemmas 3.1.24, 3.3.1 and 3.1.29 respectively. Taking cohomology we have $\operatorname{Ext}^i_R(A, L) \cong \operatorname{Ext}^i_{U^{-1}R}(U^{-1}A, U^{-1}L)$.

We may assume that $\mathcal{G} = V(\mathfrak{a}) \cap \operatorname{Supp}(A)$. Let $\mathfrak{b} = \cap_{\mathfrak{m}\in\mathcal{G}}\mathfrak{m}$. Then the first four lines of the previous display hold so that we get the first step in the following display

$$\begin{aligned}
\operatorname{Hom}_R(A, I^{\cdot}) &\cong \operatorname{Hom}_R(\Gamma_{\mathfrak{b}}(A), \Gamma_{\mathfrak{b}}(I^{\cdot})) \\
&\cong \operatorname{Hom}_R(\Gamma_{\mathfrak{b}}(A), \Gamma_{\mathfrak{b}}(\widehat{R}^{\mathfrak{a}} \otimes_R I^{\cdot})) \\
&\cong \operatorname{Hom}_R(\Gamma_{\mathfrak{b}}(A), \widehat{R}^{\mathfrak{a}} \otimes_R I^{\cdot}) \\
&\cong \operatorname{Hom}_R(\Gamma_{\mathfrak{a}}(A), \widehat{R}^{\mathfrak{a}} \otimes_R I^{\cdot}) \\
&\cong \operatorname{Hom}_R(\Gamma_{\mathfrak{a}}(A), \operatorname{Hom}_{\widehat{R}^{\mathfrak{a}}}(\widehat{R}^{\mathfrak{a}}, \widehat{R}^{\mathfrak{a}} \otimes_R I^{\cdot})) \\
&\cong \operatorname{Hom}_{\widehat{R}^{\mathfrak{a}}}(\Gamma_{\mathfrak{a}}(A) \otimes_R \widehat{R}^{\mathfrak{a}}, \widehat{R}^{\mathfrak{a}} \otimes_R I^{\cdot}) \\
&\cong \operatorname{Hom}_{\widehat{R}^{\mathfrak{a}}}(\Gamma_{\mathfrak{a}}(A), \widehat{R}^{\mathfrak{a}} \otimes_R I^{\cdot})
\end{aligned}$$

Step five is standard and step six is from Hom-tensor adjointness. Steps two and three are from Lemmas 3.3.2 and 2.1.5 (2) respectively. Lastly steps steps four and seven are from Lemma 3.1.28. Taking cohomology we have $\operatorname{Ext}^i_R(A, L) \cong \operatorname{Ext}^i_{\widehat{R}^{\mathfrak{a}}}(\Gamma_{\mathfrak{a}}A, \widehat{R}^{\mathfrak{a}} \otimes_R L)$.

The isomorphism $\operatorname{Ext}^i_{R_{\mathfrak{m}}}(A_{\mathfrak{m}}, L_{\mathfrak{m}}) \cong \operatorname{Ext}^i_{\widehat{R}^{\mathfrak{m}}}(A_{\mathfrak{m}}, \widehat{R}^{\mathfrak{m}} \otimes_{R_m} L_{\mathfrak{m}})$ is immediate from [17, Lemma 4.2], which explains the final isomorphism in the Theorem. □

Corollary 3.3.4. *Let A and L be R-modules such that A is artinian. Let $U \subset R$ be a multiplicatively closed set and let $\mathfrak{a}$ be a proper ideal of R. Then* $\operatorname{Ext}^i_R(U^{-1}A, L) \cong \operatorname{Ext}^i_{U^{-1}R}(U^{-1}A, U^{-1}L)$ *and* $\operatorname{Ext}^i_R(\Gamma_{\mathfrak{a}}(A), L) \cong \operatorname{Ext}^i_{\widehat{R}^{\mathfrak{a}}}(\Gamma_{\mathfrak{a}}(A), \widehat{R}^{\mathfrak{a}} \otimes_R L)$

Proof. This follows from Lemmas 3.1.28, 3.1.29 and Theoerem 3.3.3. □

The following result shows that, when A is artinian and L is mini-max, the module $\operatorname{Ext}^i_R(A, L)$ can be computed as an extension module over a semi-local complete ring with a Matlis reflexive module in the first component and a noetherian module in the second component. Alternatively, it can be computed as a finite coproduct of extension modules over complete local rings with Matlis reflexive modules in the first component and noetherian modules in the second component.

Lemma 3.3.5. *Let $\mathcal{F}$ be a finite subset of* m-Spec(R). *Set $\mathfrak{b} = \cap_{\mathfrak{m}\in\mathcal{F}}\mathfrak{m}$ and $U = R \smallsetminus \cup_{\mathfrak{m}\in\mathcal{F}}\mathfrak{m}$. Then $U^{-1}E_R \cong E_{\widehat{R}^{\mathfrak{b}}}$.*

Proof. Fact 3.1.3 explains the second isomorphism in the next display:

$$U^{-1}E_R \cong \bigoplus_{\mathfrak{m}\in\text{m-Spec}(R)} U^{-1}\,\mathrm{E}_R(R/\mathfrak{m}) \cong \bigoplus_{\mathfrak{m}\in\mathcal{F}} \mathrm{E}_R(R/\mathfrak{m}) \cong E_{\widehat{R}^{\mathfrak{b}}}.$$

The first isomorphism is by definition, and the third one is from Lemma 3.1.2. □

Theorem 3.3.6. *Let A be an artinian R-module and let M be a mini-max R-module. Let $\mathcal{F}$ be a finite subset of* m-Spec(R) *containing* $\mathrm{Supp}_R(A) \cap \mathrm{Supp}_R(M)$. *Let $\mathfrak{b} = \cap_{\mathfrak{m}\in\mathcal{F}}\mathfrak{m}$ and let $U = R \smallsetminus \cup_{\mathfrak{m}\in\mathcal{F}}\mathfrak{m}$. Then*

$$\begin{aligned} \mathrm{Ext}^i_R(A, M) &\cong \mathrm{Ext}^i_{\widehat{R}^{\mathfrak{b}}}(\mathrm{Hom}_R(M, U^{-1}E_R), (U^{-1}A)^\vee) && (3.5)\\ &\cong \bigoplus_{\mathfrak{m}\in\mathcal{F}} \mathrm{Ext}^i_{\widehat{R}^{\mathfrak{m}}}(\mathrm{Hom}_R(M, \mathrm{E}_R(R/\mathfrak{m})), (A_{\mathfrak{m}})^\vee). && (3.6)\end{aligned}$$

Note that $\mathrm{Hom}_R(M, U^{-1}E_R) \cong (\widehat{R}^{\mathfrak{b}} \otimes_R M)^{\vee(\widehat{R}^{\mathfrak{b}})}$ is a Matlis reflexive $\widehat{R}^{\mathfrak{b}}$-module and $(U^{-1}A)^\vee$ is a noetherian $\widehat{R}^{\mathfrak{b}}$-module.

Proof. Lemma 3.1.30 implies that $\widehat{R}^{\mathfrak{b}} \otimes_R M$ is mini-max over $\widehat{R}^{\mathfrak{b}}$. Since $\widehat{R}^{\mathfrak{b}}$ is semi-local and complete, Fact 3.1.10 shows that $\widehat{R}^{\mathfrak{b}} \otimes_R M$ is Matlis reflexive over $\widehat{R}^{\mathfrak{b}}$. Theorem 3.3.3 implies that

$$\mathrm{Ext}^i_R(A, M) \cong \mathrm{Ext}^i_{\widehat{R}^{\mathfrak{b}}}(U^{-1}A, \widehat{R}^{\mathfrak{b}} \otimes_R M). \tag{3.7}$$

Let F be a free resolution of $U^{-1}A$ over $\widehat{R}^{\mathfrak{b}}$.

Lemma 3.3.5 shows that $U^{-1}E_R \cong E_{\widehat{R}^{\mathfrak{b}}}$. In particular $U^{-1}E_R$ is injective over $\widehat{R}^{\mathfrak{b}}$, so the complex $\mathrm{Hom}_{\widehat{R}^{\mathfrak{b}}}(F, U^{-1}E_R)$ is an injective resolution of $\mathrm{Hom}_{\widehat{R}^{\mathfrak{b}}}(U^{-1}A, U^{-1}E_R)$ over $\widehat{R}^{\mathfrak{b}}$. Thus, the isomorphism

$$\mathrm{Ext}^i_{\widehat{R}^{\mathfrak{b}}}(U^{-1}A, \widehat{R}^{\mathfrak{b}} \otimes_R M) \cong \mathrm{Ext}^i_{\widehat{R}^{\mathfrak{b}}}(\mathrm{Hom}_R(M, U^{-1}E_R), (U^{-1}A)^{\vee(\widehat{R}^{\mathfrak{b}})}) \tag{3.8}$$

follows from taking cohomology in the next sequence:

$$\begin{aligned}
\mathrm{Hom}_{\widehat{R}^{\mathfrak{b}}}(F, \widehat{R}^{\mathfrak{b}} \otimes_R M) &\cong \mathrm{Hom}_{\widehat{R}^{\mathfrak{b}}}(F, \mathrm{Hom}_{\widehat{R}^{\mathfrak{b}}}(\mathrm{Hom}_{\widehat{R}^{\mathfrak{b}}}(\widehat{R}^{\mathfrak{b}} \otimes_R M, U^{-1}E_R), U^{-1}E_R)) \\
&\cong \mathrm{Hom}_{\widehat{R}^{\mathfrak{b}}}(\mathrm{Hom}_{\widehat{R}^{\mathfrak{b}}}(\widehat{R}^{\mathfrak{b}} \otimes_R M, U^{-1}E_R), \mathrm{Hom}_{\widehat{R}^{\mathfrak{b}}}(F, U^{-1}E_R)) \\
&\cong \mathrm{Hom}_{\widehat{R}^{\mathfrak{b}}}(\mathrm{Hom}_R(M, \mathrm{Hom}_{\widehat{R}^{\mathfrak{b}}}(\widehat{R}^{\mathfrak{b}}, U^{-1}E_R)), \mathrm{Hom}_{\widehat{R}^{\mathfrak{b}}}(F, U^{-1}E_R)) \\
&\cong \mathrm{Hom}_{\widehat{R}^{\mathfrak{b}}}(\mathrm{Hom}_R(M, U^{-1}E_R), \mathrm{Hom}_{\widehat{R}^{\mathfrak{b}}}(F, U^{-1}E_R)).
\end{aligned}$$

The first step is from the fact that $\widehat{R}^{\mathfrak{b}} \otimes_R M$ is Matlis reflexive over $\widehat{R}^{\mathfrak{b}}$. The second and third steps follow by Hom-tensor adjointness, and the fourth step is standard.

Lemma 3.1.12(1) implies that $U^{-1}A$ is artinian over $U^{-1}R$. It follows from Lemma 3.1.22 that $U^{-1}A$ is torsion with respect to the Jacobson radical of $U^{-1}R$. In particular $U^{-1}A$ is $\mathfrak{b}$-torsion, so Lemma 3.1.23 shows that $(U^{-1}A)^{\vee(\widehat{R}^{\mathfrak{b}})} \cong (U^{-1}A)^{\vee}$. Combining this with (3.7) and (3.8), we have the isomorphism (3.5).

To verify (3.6), argue similarly, using the isomorphism

$$\mathrm{Ext}^i_R(A, M) \cong \textstyle\bigoplus_{\mathfrak{m}\in\mathcal{F}} \mathrm{Ext}^i_{\widehat{R}^{\mathfrak{m}}}(A_{\mathfrak{m}}, \widehat{R}^{\mathfrak{m}} \otimes_R M)$$

from Theorem 3.3.3. $\square$

The following result shows that extension functors applied to two artinian modules over arbitrary noetherian rings can be computed as a finite direct sum of extension functors applied to pairs of noetherian modules over complete local rings.

Corollary 3.3.7. *Let A and L be R-modules such that A is artinian and L is either artinian or Matlis reflexive. Let $\mathcal{F}$ be a finite subset of* m-Spec(R) *containing* $\mathrm{Supp}_R(A) \cap \mathrm{Supp}_R(L)$. *Setting $\mathfrak{b} = \cap_{\mathfrak{m}\in\mathcal{F}}\mathfrak{m}$ and $U = R \smallsetminus \cup_{\mathfrak{m}\in\mathcal{F}}\mathfrak{m}$, we have*

$$\mathrm{Ext}^i_R(A, L) \cong \mathrm{Ext}^i_{\widehat{R}^{\mathfrak{b}}}((U^{-1}L)^\vee, (U^{-1}A)^\vee) \cong \textstyle\bigoplus_{\mathfrak{m}\in\mathcal{F}} \mathrm{Ext}^i_{\widehat{R}^{\mathfrak{m}}}((L_\mathfrak{m})^\vee, (A_\mathfrak{m})^\vee).$$

Fact 3.3.8. Let L and L' be R-modules, and fix an index $i \geqslant 0$. Then the following diagram commutes where δ_L and $\delta_{\mathrm{Ext}^i_R(L',L)}$ are the natural biduality maps

$$\begin{tikzcd}
\mathrm{Ext}^i_R(L', L) \arrow[r, "\delta_{\mathrm{Ext}^i_R(L',L)}"] \arrow[d, "\mathrm{Ext}^i_R(L',\delta_L)"'] & \mathrm{Ext}^i_R(L', L)^{\vee\vee} \arrow[d, "(\Theta^i_{L'L})^\vee"] \\
\mathrm{Ext}^i_R(L', L^{\vee\vee}) \arrow[r, "\cong"] & \mathrm{Tor}^R_i(L', L^\vee)^\vee.
\end{tikzcd}$$

The unlabeled isomorphism is from Remark 2.1.9, and $\Theta^i_{L'L}$ is from Definition 2.1.8.

Lemma 3.3.9. *Let N and L be R-modules such that N is noetherian. Let $\mathfrak{m} \in$* m-Spec(R) *and fix an index $i \geqslant 0$. The map $\mathrm{Ext}^i_R(N, \delta_L)\colon \mathrm{Ext}^i_R(N, L) \to \mathrm{Ext}^i_R(N, L^{\vee\vee})$ is an injection. If $\mu^i_R(\mathfrak{m}; L) < \infty$, then $\mathrm{Ext}^i_R(R/\mathfrak{m}, \delta_L)$ is an isomorphism.*

Proof. Remark 2.1.9 implies that

$$\Theta^i_{NL}\colon \mathrm{Tor}^R_i(N, L^\vee) \to \mathrm{Ext}^i_R(N, L)^\vee$$

is an isomorphism. Hence $(\Theta^i_{NL})^\vee$ is also an isomorphism. The map

$$\delta_{\mathrm{Ext}^i_R(N,L)}\colon \mathrm{Ext}^i_R(N, L) \to \mathrm{Ext}^i_R(N, L)^{\vee\vee}$$

is an injection. Using Fact 3.3.8 with $L' = N$, we conclude that $\mathrm{Ext}^i_R(N, \delta_L)$ is an injection.

The assumption $\mu^i_R(\mathfrak{m}, L) < \infty$ implies that $\mathrm{Ext}^i_R(R/\mathfrak{m}, L)$ is a finite dimensional $R/\mathfrak{m}$-vector space, so it is Matlis reflexive over R; hence $\delta_{\mathrm{Ext}^i_R(R/\mathfrak{m},L)}$ is an isomorphism. Again, using Fact 3.3.8 we conclude that $\mathrm{Ext}^i_R(R/\mathfrak{m}, \delta_L)$ is an isomorphism, as desired. $\square$

Lemma 3.3.10. *Let A and L be R-modules such that A is artinian. Fix an index $i \geqslant 0$ such that the Bass numbers $\mu^i_R(\mathfrak{m}; L)$ are finite for all $\mathfrak{m} \in \mathrm{Supp}_R(A)$. Then the map*

$$\mathrm{Ext}^i_R(A, \delta_L) \colon \mathrm{Ext}^i_R(A, L) \to \mathrm{Ext}^i_R(A, L^{\vee\vee})$$

is an isomorphism, and the map

$$\mathrm{Ext}^{i+1}_R(A, \delta_L) \colon \mathrm{Ext}^{i+1}_R(A, L) \to \mathrm{Ext}^{i+1}_R(A, L^{\vee\vee})$$

is an injection.

Proof. Since $A \cong \bigoplus_{\mathfrak{m} \in \mathrm{Supp}(A)} A_{\mathfrak{m}}$ is a finite direct sum, the maps $\mathrm{Ext}^i_R(A, \delta_L)$ decomposes into a direct sum of maps $\bigoplus_{\mathfrak{m} \in \mathrm{Supp}(A)} \mathrm{Ext}^i_R(A_{\mathfrak{m}}, \delta_L)$. The proof that each of these maps are isomorphisms parallels that of the local case, Lemma 2.4.7. The same reasoning also shows that $\mathrm{Ext}^{i+1}_R(A, \delta_L)$ is an injection. $\square$

Lemma 3.3.11. *Let A, I and L be R-modules such that A is artinian and I is injective. Let $\mathfrak{a}$ be an ideal contained in $\mathfrak{b} = \cap_{\mathfrak{m} \in \mathrm{Supp}(A) \cap \mathrm{Supp}(I)} \mathfrak{m}$. Then*

$$A \otimes_R \mathrm{Hom}_R(I, L) \cong A \otimes_R \mathrm{Hom}_R(\Gamma_{\mathfrak{a}}(I), L).$$

Proof. There is an isomorphism $I \cong \bigoplus_{\mathfrak{p} \in \mathrm{Supp}_R(I)} \mathrm{E}_R(R/\mathfrak{p})^{(\mu_{\mathfrak{p}})}$ where $\{\mu_{\mathfrak{p}}\}_{\mathfrak{p} \in \mathrm{Spec}(R)}$ is a sequence of sets. If $\mathfrak{p} \in \operatorname{Supp}_R(I) \smallsetminus \operatorname{Supp}(A)$, then $\mathrm{E}_R(R/\mathfrak{p})^{(\mu_\mathfrak{p})}$ is an $R_\mathfrak{p}$-module; hence so is $\operatorname{Hom}_R(\mathrm{E}_R(R/\mathfrak{p})^{(\mu_\mathfrak{p})}, L')$. In this case since $\mathfrak{b} \not\subseteq \mathfrak{p}$ Fact 3.1.8 explains the second step in the following display:

$$\begin{aligned}
&\operatorname{Hom}_R(\textstyle\bigoplus_{\mathfrak{p}\in\operatorname{Supp}_R(I)\smallsetminus\operatorname{Supp}_R(A)} \mathrm{E}_R(R/\mathfrak{p})^{(\mu_\mathfrak{p})}, L) \\
=&\textstyle\prod_{\mathfrak{p}\in\operatorname{Supp}_R(I)\smallsetminus\operatorname{Supp}_R(A)} \operatorname{Hom}_R(\mathrm{E}_R(R/\mathfrak{p})^{(\mu_\mathfrak{p})}, L) \\
=&\textstyle\prod_{\mathfrak{p}\in\operatorname{Supp}_R(I)\smallsetminus\operatorname{Supp}_R(A)} \mathfrak{b}\operatorname{Hom}_R(\mathrm{E}_R(R/\mathfrak{p})^{(\mu_\mathfrak{p})}, L) \\
=&\mathfrak{b}\textstyle\prod_{\mathfrak{p}\in\operatorname{Supp}_R(I)\smallsetminus\operatorname{Supp}_R(A)} \operatorname{Hom}_R(\mathrm{E}_R(R/\mathfrak{p})^{(\mu_\mathfrak{p})}, L)
\end{aligned}$$

The first step above is standard and the third step follows from the fact $\mathfrak{b}$ is finitely generated. Let $X := \bigoplus_{\mathfrak{p}\in\operatorname{Supp}_R(I)\smallsetminus\operatorname{Supp}_R(A)} \mathrm{E}_R(R/\mathfrak{m})^{(\mu_\mathfrak{m})}$. Since A is $\mathfrak{b}$-torsion it follows from Lemma 3.1.6 that $A \otimes_R \operatorname{Hom}_R(X, L) = 0$. Also we have that

$$I \cong (\textstyle\bigoplus_{\mathfrak{m}\in\operatorname{Supp}_R(A)\cap\operatorname{Supp}_R(I)} \mathrm{E}_R(R/\mathfrak{m})^{(\mu_\mathfrak{m})}) \oplus X \cong \Gamma_\mathfrak{b}(I) \oplus X.$$

Therefore

$$A \otimes_R \operatorname{Hom}_R(I, L) \cong A \otimes_R \operatorname{Hom}_R(\Gamma_\mathfrak{b}(I) \oplus X, L) \cong A \otimes_R \operatorname{Hom}_R(\Gamma_\mathfrak{b}(I), L).$$

Since $\Gamma_\mathfrak{a}(I)$ is injective and $\Gamma_\mathfrak{b}(\Gamma_\mathfrak{a}(I)) = \Gamma_\mathfrak{b}(I)$ a similar argument shows that $A \otimes_R \operatorname{Hom}_R(\Gamma_\mathfrak{a}(I), L) \cong A \otimes_R \operatorname{Hom}_R(\Gamma_\mathfrak{b}(I), L)$ and the result follows. $\square$

Lemma 3.3.12. *Let A and L be R-modules such that A is artinian. Let $\mathfrak{a}$ be an ideal contained in $\bigcap_{\mathfrak{m}\in\operatorname{Supp}(A)\cap\operatorname{Supp}(L)}\mathfrak{m}$. For each index $i \geqslant 0$, there is an isomorphism*

$$\operatorname{Tor}_i^R(A, \operatorname{Hom}_R(L, E_{\widehat{R}^\mathfrak{a}})) \cong \operatorname{Tor}_i^R(A, L^\vee).$$

Proof. Lemma 3.3.11 explains the first and fourth steps in the following display:

$$\begin{aligned}
A \otimes_R \operatorname{Hom}_R(I, E_R) &\cong A \otimes_R \operatorname{Hom}_R(\Gamma_{\mathfrak{a}}(I), E_R) \\
&\cong A \otimes_R \operatorname{Hom}_R(\Gamma_{\mathfrak{a}}(I), \Gamma_{\mathfrak{a}}(E_R)) \\
&\cong A \otimes_R \operatorname{Hom}_R(\Gamma_{\mathfrak{a}}(I), E_{\widehat{R}^{\mathfrak{a}}}) \\
&\cong A \otimes_R \operatorname{Hom}_R(I, E_{\widehat{R}^{\mathfrak{a}}}).
\end{aligned}$$

The second and third isomorphism are from Lemmas 2.1.5 (2) and 3.1.5 respectively. Since E_R is injective, the complex $\operatorname{Hom}_R(I, E_R)$ is a flat resolution of $\operatorname{Hom}_R(L, E_R) = L^\vee$; see [12, Theorem 3.2.16]. Similarly, the complex $\operatorname{Hom}_R(I, E_{\widehat{R}^{\mathfrak{a}}})$ is a flat resolution of $\operatorname{Hom}_R(L, E_{\widehat{R}^{\mathfrak{a}}})$. Therefore the result follows by taking homology. $\square$

Theorem 3.3.13. *Let A and L be R-modules such that A is artinian. Let $\mathcal{F}$ be a finite set of maximal ideals containing $\operatorname{Supp}_R(A) \cap \operatorname{Supp}_R(L)$, and set $\mathfrak{b} = \cap_{\mathfrak{m} \in \mathcal{F}} \mathfrak{m}$. Fix an index $i \geqslant 0$ such that the Bass numbers $\mu_R^i(\mathfrak{m}; L)$ are finite for all $\mathfrak{m} \in \operatorname{Supp}_R(A) \cap \operatorname{Supp}_R(L)$. Then we have the following:*

1. *There is an isomorphism $\operatorname{Ext}_R^i(A, L)^{\vee(\widehat{R}^{\mathfrak{b}})} \cong \operatorname{Tor}_i^R(A, L^\vee)$; and*
2. *If $R/(\operatorname{Ann}_R(A) + \operatorname{Ann}_R(L))$ is semi-local and complete, then Θ_{AL}^i provides an isomorphism $\operatorname{Ext}_R^i(A, L)^\vee \cong \operatorname{Tor}_i^R(A, L^\vee)$.*

Proof. (2) Assume that $R/(\operatorname{Ann}_R(A) + \operatorname{Ann}_R(L))$ is semi-local and complete. Theorem 3.2.9(2) and Lemma 3.3.10 show that the maps

$$\delta_{\operatorname{Ext}_R^i(A,L)} \colon \operatorname{Ext}_R^i(A, L) \to \operatorname{Ext}_R^i(A, L)^{\vee\vee}$$
$$\operatorname{Ext}_R^i(A, \delta_L) \colon \operatorname{Ext}_R^i(A, L) \to \operatorname{Ext}_R^i(A, L^{\vee\vee})$$

are isomorphisms. Fact 3.3.8 implies that $(\Theta^i_{AL})^\vee$ is an isomorphism, so we conclude that Θ^i_{AL} is also an isomorphism.

(1) Set $U = R \smallsetminus \cup_{\mathfrak{m}\in\mathcal{F}}\mathfrak{m}$. We first verify that

$$\operatorname{Tor}_i^{\widehat{R}^{\mathfrak{b}}}(U^{-1}A, (\widehat{R}^{\mathfrak{b}} \otimes_R L)^{\vee(\widehat{R}^{\mathfrak{b}})}) \cong \operatorname{Tor}_i^R(A, (\widehat{R}^{\mathfrak{b}} \otimes_R L)^{\vee(\widehat{R}^{\mathfrak{b}})}). \tag{3.9}$$

For this, let P be a projective resolution of A over R. Since $\widehat{R}^{\mathfrak{b}}$ is flat over R, the complex $\widehat{R}^{\mathfrak{b}} \otimes_R P$ is a projective resolution of $\widehat{R}^{\mathfrak{b}} \otimes_R A \cong U^{-1}A$ over $\widehat{R}^{\mathfrak{b}}$; see Lemmas 3.1.28 and 3.1.29. Thus, we have

$$(\widehat{R}^{\mathfrak{b}} \otimes_R P) \otimes_{\widehat{R}^{\mathfrak{b}}} (\widehat{R}^{\mathfrak{b}} \otimes_R L)^{\vee(\widehat{R}^{\mathfrak{b}})} \cong P \otimes_R (\widehat{R}^{\mathfrak{b}} \otimes_R L)^{\vee(\widehat{R}^{\mathfrak{b}})}$$

and the isomorphism (3.9) follows by taking homology.

Theorem 3.3.3 explains the first step below:

$$\begin{aligned}
\operatorname{Ext}^i_R(A, L)^{\vee(\widehat{R}^{\mathfrak{b}})} &\cong \operatorname{Ext}^i_{\widehat{R}^{\mathfrak{b}}}(U^{-1}A, \widehat{R}^{\mathfrak{b}} \otimes_R L)^{\vee(\widehat{R}^{\mathfrak{b}})} \\
&\cong \operatorname{Tor}_i^{\widehat{R}^{\mathfrak{b}}}(U^{-1}A, (\widehat{R}^{\mathfrak{b}} \otimes_R L)^{\vee(\widehat{R}^{\mathfrak{b}})}) \\
&\cong \operatorname{Tor}_i^R(A, (\widehat{R}^{\mathfrak{b}} \otimes_R L)^{\vee(\widehat{R}^{\mathfrak{b}})}) \\
&\cong \operatorname{Tor}_i^R(A, \operatorname{Hom}_R(L, E_{\widehat{R}^{\mathfrak{b}}})) \\
&\cong \operatorname{Tor}_i^R(A, L^\vee).
\end{aligned}$$

The second step is from part (2); this uses the fact that $\widehat{R}^{\mathfrak{b}}$ is semi-local and complete, and the equality $\mu^i_{\widehat{R}^{\mathfrak{b}}}(\mathfrak{m}\widehat{R}^{\mathfrak{b}}; \widehat{R}^{\mathfrak{b}} \otimes_R L) = \mu^i_R(\mathfrak{m}; L) < \infty$ for all $\mathfrak{m} \in \operatorname{Supp}_R(A) \cap \operatorname{Supp}_R(L)$. The third step is from (3.9), and the fourth step is from Hom-tensor adjointness. The fifth step is from Lemma 3.3.12. $\square$

Corollary 3.3.14. *Let A and M be R-modules such that A is artinian and M is* *mini-max. Let* $\mathcal{F}$ *be a finite set of maximal ideals containing* $\operatorname{Supp}_R(A) \cap \operatorname{Supp}_R(M)$*. Let* $\mathfrak{b} = \cap_{\mathfrak{m} \in \mathcal{F}} \mathfrak{m}$*. For each index* $i \geqslant 0$*, one has* $\operatorname{Ext}^i_R(A, M)^{\vee(\widehat{R}^{\mathfrak{b}})} \cong \operatorname{Tor}^R_i(A, M^{\vee})$*.*

Theorem 3.3.15. *Let* M *and* L *be* R*-modules such that* M *is mini-max and the quotient* $R/(\operatorname{Ann}_R(M) + \operatorname{Ann}_R(L))$ *is semi-local and complete. Fix an index* $i \geqslant 0$ *such that* $\mu^i_R(\mathfrak{m}; L)$ *and* $\mu^{i+1}_R(\mathfrak{m}; L)$ *are finite for all* $\mathfrak{m} \in \operatorname{Supp}_R(M) \cap \operatorname{Supp}_R(L) \cap$ m-Spec(R)*. Then* Θ^i_{ML} *is an isomorphism, so*

$$\operatorname{Ext}^i_R(M, L)^{\vee} \cong \operatorname{Tor}^R_i(M, L^{\vee}).$$

Proof. Since M is mini-max over R, there is an exact sequence of R-modules homomorphisms $0 \to N \to M \to A \to 0$ such that N is noetherian and A is artinian. The long exact sequences associated to $\operatorname{Tor}^R(-, L^{\vee})$ and $\operatorname{Ext}_R(-, L)$ fit into the following commutative diagram:

$$\begin{tikzcd}
\cdots \arrow[r] & \operatorname{Tor}^R_i(N, L^{\vee}) \arrow[r] \arrow[d, "\Theta^i_{NL}"] & \operatorname{Tor}^R_i(M, L^{\vee}) \arrow[r] \arrow[d, "\Theta^i_{ML}"] & \operatorname{Tor}^R_i(A, L^{\vee}) \arrow[r] \arrow[d, "\Theta^i_{AL}"] & \cdots \\
\cdots \arrow[r] & \operatorname{Ext}^i_R(N, L)^{\vee} \arrow[r] & \operatorname{Ext}^i_R(M, L)^{\vee} \arrow[r] & \operatorname{Ext}^i_R(A, L)^{\vee} \arrow[r] & \cdots .
\end{tikzcd}$$

By Remark 2.1.9, the maps Θ^i_{NL} and Θ^{i-1}_{NL} are isomorphisms. Theorem 3.3.13(2) implies that Θ^i_{AL} and Θ^{i+1}_{AL} are isomorphisms. Hence the map Θ^i_{ML} is an isomorphism by the Five Lemma. $\square$

Corollary 3.3.16. *Let* M*,* M' *and* L *be* R*-modules such that* M *is Matlis reflexive and* M' *is mini-max. Then for all* $i \geqslant 0$*,* $\Theta^i_{M'M}$ *is an isomorphism, so*

$$\operatorname{Ext}^i_R(M', M)^{\vee} \cong \operatorname{Tor}^R_i(M', M^{\vee})$$

Fix an index $i \geqslant 0$ *such that* $\mu^i_R(\mathfrak{m}; L)$ *and* $\mu^{i+1}_R(\mathfrak{m}; L)$ *are finite for all* $\mathfrak{m} \in \operatorname{Supp}_R(M) \cap$

$\mathrm{Supp}_R(L) \cap \text{m-Spec}(R)$. *Then* Θ^i_{ML} *is an isomorphism, so*

$$\mathrm{Ext}^i_R(M, L)^\vee \cong \mathrm{Tor}^R_i(M, L^\vee).$$

Corollary 3.3.17. *Let* M *and* M' *be mini-max* R*-modules such that the quotient* $R/(\mathrm{Ann}_R(M) + \mathrm{Ann}_R(M'))$ *is semi-local and complete. Let* $\mathcal{F}$ *be a finite set of maximal ideals containing* $\mathrm{Supp}_R(M) \cap \mathrm{Supp}_R(M') \cap \text{m-Spec}(R)$*, and set* $\mathfrak{b} = \cap_{\mathfrak{m} \in \mathcal{F}} \mathfrak{m}$*. Then for all* $i \geqslant 0$ *the map* $\Theta^i_{MM'}$ *is an isomorphism, so*

$$\mathrm{Ext}^i_R(M, M')^{\vee(\widehat{R}^{\mathfrak{b}})} \cong \mathrm{Ext}^i_R(M, M')^\vee \cong \mathrm{Tor}^R_i(M, M'^\vee).$$

Proof. Theorem 3.2.9(1) implies that $\mathrm{Ext}^i_R(M, M')$ is Matlis reflexive over R. Therefore by Lemma 3.1.27(2) it follows that $\mathrm{Ext}^i_R(M, M')^\vee \cong \mathrm{Ext}^i_R(M, M')^{\vee(\widehat{R}^{\mathfrak{b}})}$. Also it follows from Theorem 3.3.15 that $\Theta^i_{MM'}$ is an isomorphism for all $i \geqslant 0$; hence $\mathrm{Ext}^i_R(M, M')^\vee \cong \mathrm{Tor}^R_i(M, M'^\vee)$. □

3.4 Length of $\mathrm{Hom}_R(L, L')$ and $L \otimes_R L'$

Fact 3.4.1. Let A be an artinian R-module, and let $\mathfrak{b}$ be an ideal of R. Fix an integer $t \geqslant 0$ such that $\mathfrak{b}^t A = \mathfrak{b}^{t+1} A$. Given a $\mathfrak{b}$-torsion R-module L, one has

$$A \otimes_R L \cong (A/\mathfrak{b}^t A) \otimes_R L \cong (A/\mathfrak{b}^t A) \otimes_R (L/\mathfrak{b}^t L).$$

This is proved as in Lemma 3.1.6 or [17, Lemma 3.5].

Lemma 3.4.2. *Let* $\mathfrak{a}$ *be a finite intersection of maximal ideals. Let* A *and* L *be* R*-modules such that* A *is artinian and* L *is* $\mathfrak{a}$*-torsion. Let* $\mathcal{F} = \mathrm{Supp}_R(A) \cap \mathrm{Supp}_R(L)$ *and* $\mathfrak{b}$ *be an ideal contained in* $\cap_{\mathfrak{m} \in \mathcal{F}} \mathfrak{m}$*. Choose* $t \geqslant 0$ *such that* $\mathfrak{b}^t A = \mathfrak{b}^{t+1} A$*. For* *each* $\mathfrak{m} \in \mathcal{F}$ *choose* $\alpha_{\mathfrak{m}} \geqslant 0$ *such that either* $\mathfrak{m}^{\alpha_{\mathfrak{m}}} A = \mathfrak{m}^{\alpha_{\mathfrak{m}}+1} A$ *or* $\mathfrak{m}^{\alpha_{\mathfrak{m}}} L = \mathfrak{m}^{\alpha_{\mathfrak{m}}+1} L$. *Then there are isomorphisms*

$$\begin{aligned} A \otimes_R L &\cong (A/\mathfrak{b}^t A) \otimes_R (L/\mathfrak{b}^t L) \\ &\cong \textstyle\bigoplus_{\mathfrak{m}\in\mathcal{F}} (A/\mathfrak{m}^{\alpha_{\mathfrak{m}}} A) \otimes_R (L/\mathfrak{m}^{\alpha_{\mathfrak{m}}} L) \end{aligned}$$

Proof. The isomorphism $A \otimes_R L \cong \bigoplus_{\mathfrak{m}\in\mathcal{F}} A_{\mathfrak{m}} \otimes_R L_{\mathfrak{m}}$ follows from Lemma 3.1.24 along with the fact that $A_{\mathfrak{m}} \otimes_R L_{\mathfrak{n}} = 0$ for $\mathfrak{m} \neq \mathfrak{n} \in$ m-Spec(R). Since $\mathfrak{m}^{\alpha_{\mathfrak{m}}} A = \mathfrak{m}^{\alpha_{\mathfrak{m}}+1} A$ implies $\mathfrak{m}^{\alpha_{\mathfrak{m}}} A_{\mathfrak{m}} = \mathfrak{m}^{\alpha_{\mathfrak{m}}+1} A_{\mathfrak{m}}$ and we have $A/\mathfrak{m}^{\alpha_{\mathfrak{m}}} A \cong A_{\mathfrak{m}}/\mathfrak{m}^{\alpha_{\mathfrak{m}}} A_{\mathfrak{m}}$ the isomorphism $A \otimes_R A \cong \bigoplus_{\mathfrak{m}\in\mathcal{F}} (A/\mathfrak{m}^{\alpha_{\mathfrak{m}}} A) \otimes_R (L/\mathfrak{m}^{\alpha_{\mathfrak{m}}} L)$ follows from Fact 3.4.1.

Since $\mathfrak{b}^t A = \mathfrak{b}^{t+1} A$ we have $\mathfrak{b}^t A_{\mathfrak{m}} = \mathfrak{b}^{t+1} A_{\mathfrak{m}}$. By Fact 3.4.1 we have $A \otimes_R L \cong \bigoplus_{\mathfrak{m}\in\mathcal{F}} A_{\mathfrak{m}}/\mathfrak{b}^t A_{\mathfrak{m}} \otimes_R L_{\mathfrak{m}}/\mathfrak{b}^t L$. Since $A_{\mathfrak{m}}/\mathfrak{b}^t A_{\mathfrak{m}} \otimes_R L_{\mathfrak{n}}/\mathfrak{b}^t L_{\mathfrak{n}} = 0$ when either $\mathfrak{m} \neq \mathfrak{n}$ or $\mathfrak{m} = \mathfrak{n} \notin \mathcal{F}$ we get the first step in the next display:

$$\begin{aligned} A \otimes_R L &\cong \textstyle\bigoplus_{\mathfrak{m}\in \mathrm{Supp}(A)} \bigoplus_{\mathfrak{n}\in \mathrm{Supp}(L)} A_{\mathfrak{m}}/\mathfrak{b}^t A_{\mathfrak{m}} \otimes_R L_{\mathfrak{n}}/\mathfrak{b}^t L_{\mathfrak{n}} \\ &\cong A/\mathfrak{b}^t A \otimes_R L/\mathfrak{b}^t L. \end{aligned}$$

The second step above follows from Lemma 3.1.24. □

Theorem 3.4.3. *Let* $\mathfrak{a}$ *be a finite intersection of maximal ideals. Let* A *and* L *be* R*-modules such that* A *is artinian and* L *is* $\mathfrak{a}$*-torsion. Let* $\mathcal{F} = \mathrm{Supp}_R(A) \cap \mathrm{Supp}_R(L)$ *and* $\mathfrak{b}$ *be an ideal contained in* $\cap_{\mathfrak{m}\in\mathcal{F}} \mathfrak{m}$. *Choose* $t \geqslant 0$ *such that* $\mathfrak{b}^t A = \mathfrak{b}^{t+1} A$. *For each* $\mathfrak{m} \in \mathcal{F}$ *choose* $\alpha_{\mathfrak{m}} \geqslant 0$ *such that either* $\mathfrak{m}^{\alpha_{\mathfrak{m}}} A = \mathfrak{m}^{\alpha_{\mathfrak{m}}+1} A$ *or* $\mathfrak{m}^{\alpha_{\mathfrak{m}}} L = \mathfrak{m}^{\alpha_{\mathfrak{m}}+1} L$.

Then there are inequalities

$$\begin{aligned}\lambda_R(A \otimes_R L) &\leqslant \textstyle\sum_{\mathfrak{m}\in\mathcal{F}} \min\left\{\lambda_R\left(A/\mathfrak{m}^{\alpha_\mathfrak{m}}A\right)\beta_0^R(\mathfrak{m}, L),\ \beta_0^R(\mathfrak{m}, A)\lambda_R\left(L/\mathfrak{m}^{\alpha_\mathfrak{m}}L\right)\right\}\\ &\leqslant \lambda_R\left(A/\mathfrak{b}^tA\right)\max\{\beta_0^R(\mathfrak{m}, L) \mid \mathfrak{m}\in\mathcal{F}\}\\ &\leqslant \lambda_R\left(A/\mathfrak{b}^tA\right)\lambda_R(L/\mathfrak{b}L).\end{aligned}$$

Here we use the convention $0 \cdot \infty = 0$*.*

Proof. By Lemma 3.1.24 we get the first step in the next display:

$$A \otimes_R L \cong \textstyle\bigoplus_{\mathfrak{m}\in\mathrm{Supp}(A)} \bigoplus_{\mathfrak{n}\in\mathrm{Supp}(L)} A_\mathfrak{m} \otimes_R L_\mathfrak{n} \cong \bigoplus_{\mathfrak{m}\in\mathcal{F}} A_\mathfrak{m} \otimes_R L_\mathfrak{m}$$

The second step above follows from that fact that $A_\mathfrak{m} \otimes_R L_\mathfrak{n} = 0$ when either $\mathfrak{m} \neq \mathfrak{n}$ or $\mathfrak{m} = \mathfrak{n} \notin \mathcal{F}$. This explains the first step in the next display:

$$\begin{aligned}\lambda_R(A \otimes_R L) &= \textstyle\sum_{\mathfrak{m}\in\mathcal{F}} \lambda_R(A_\mathfrak{m} \otimes_R L_\mathfrak{m})\\ &\leqslant \textstyle\sum_{\mathfrak{m}\in\mathcal{F}} \min\{\lambda_R(A/\mathfrak{m}^{\alpha_\mathfrak{m}}A)\beta_0^R(L_\mathfrak{m}),\ \beta_0^R(A_\mathfrak{m})\lambda_R(L/\mathfrak{m}^{\alpha_\mathfrak{m}}L)\}\end{aligned}$$

Since a tensor of $\mathfrak{m}$-torsion modules is the same whether the tensor is over R or $\widehat{R}^\mathfrak{m}$ the second step in the last display follows from Theorem 2.3.8 and Lemma 2.1.3 (1).

Since $\mathfrak{b}^tA = \mathfrak{b}^{t+1}A$ it follows that $\mathfrak{b}^tA_\mathfrak{m} = \mathfrak{b}^{t+1}A_\mathfrak{m}$ for all $\mathfrak{m} \in \text{m-Spec}(R)$. Notice that $\mathfrak{b}^tA_\mathfrak{m} = \mathfrak{b}^{t+\alpha_\mathfrak{m}}A_\mathfrak{m} \subseteq \mathfrak{m}^{t+\alpha_\mathfrak{m}}A_\mathfrak{m} = \mathfrak{m}^{\alpha_\mathfrak{m}}A_\mathfrak{m}$. This explains the second step in the next display:

$$\begin{aligned}\lambda_R(A/\mathfrak{b}^tA) &\geqslant \textstyle\sum_{\mathfrak{m}\in\mathcal{F}} \lambda_R(A_\mathfrak{m}/\mathfrak{b}^tA_\mathfrak{m})\\ &\geqslant \textstyle\sum_{\mathfrak{m}\in\mathcal{F}} \lambda_R(A_\mathfrak{m}/\mathfrak{m}^{\alpha_\mathfrak{m}}A_\mathfrak{m})\\ &= \textstyle\sum_{\mathfrak{m}\in\mathcal{F}} \lambda_R(A/\mathfrak{m}^{\alpha_\mathfrak{m}}A)\end{aligned}$$

Also we have the following:

$$\begin{aligned}\lambda_R(L/\mathfrak{b}L) &\geqslant \lambda_R(L/(\cap_{\mathfrak{m}\in\mathcal{F}}\mathfrak{m})L)\\ &= \textstyle\sum_{\mathfrak{m}\in\mathcal{F}} \lambda_R(L/\mathfrak{m}L)\\ &= \textstyle\sum_{\mathfrak{m}\in\mathcal{F}} \beta_0^R(\mathfrak{m}, L).\end{aligned}$$

The first step in the last display follows from the assumption $\mathfrak{b} \subseteq \cap_{\mathfrak{m}\in\mathcal{F}}\mathfrak{m}$. The second step follows from the Chinese remainder Theorem. It is elementary to show that $\beta_0^R(\mathfrak{m}, L) = \lambda_R(L/\mathfrak{m}L)$ and the third step follows. The inequlities in the last two displays imply the last two inequlities in the Theorem; hence the result follows. $\square$

From Theorem 3.4.3 one easily recovers [13, Proposition 6.1]

Corollary 3.4.4. *If A and A' be artinian R-modules, then $\lambda_R(A \otimes_R A') < \infty$.*

The next result provides conditions equivalent to the vanishing of $A \otimes_R A'$.

Proposition 3.4.5. *Let A and A' be artinian R-modules. Let $\mathcal{F} = \operatorname{Supp}(A) \cap \operatorname{Supp}(A')$ and $\mathfrak{b} = \cap_{\mathfrak{m}\in\mathcal{F}}\mathfrak{m}$. Then the following are equivalent:*

1. $A \otimes_R A' = 0$;
2. $\operatorname{Supp}(A/\mathfrak{b}A) \cap \operatorname{Supp}(A'/\mathfrak{b}A') = \emptyset$;
3. *For all $\mathfrak{m} \in \mathcal{F}$, either $A = \mathfrak{m}A$ or $A' = \mathfrak{m}A'$; and*
4. *For all $\mathfrak{m} \in \mathcal{F}$, either $\operatorname{grade}_R(\mathfrak{m}; A^\vee) > 0$ or $\operatorname{grade}(\mathfrak{m}; A'^\vee) > 0$.*

Proof. The equivalence (3) $\iff$ (4) follows from the next sequence of equivalences:

$$\begin{aligned}
\mathfrak{m}A = A &\iff R/\mathfrak{m} \otimes_R A = 0 \\
&\iff \operatorname{Hom}_R(R/\mathfrak{m} \otimes_R A, E) = 0 \\
&\iff \operatorname{Hom}_R(R/\mathfrak{m}, \operatorname{Hom}_R(A, E)) = 0 \\
&\iff \operatorname{grade}_R(\mathfrak{m}; \operatorname{Hom}_R(A, E)) > 0
\end{aligned}$$

(1) $\implies$ (3): Assume that $A \otimes A' = 0$, and let $\mathfrak{m} \in \operatorname{Supp}_R(A) \cap \operatorname{Supp}_R(A')$. The natural map $R \to R/\mathfrak{m}$ yields the surjection in the next sequence

$$\begin{aligned}
A \otimes_R A' \twoheadrightarrow R/\mathfrak{m} \otimes_R (A \otimes_R A') &\cong (R/\mathfrak{m} \otimes_R A) \otimes_{R/\mathfrak{m}} (R/\mathfrak{m} \otimes_R A') \\
&\cong (A/\mathfrak{m}A) \otimes_{R/\mathfrak{m}} (A'/\mathfrak{m}A').
\end{aligned}$$

The isomorphisms are standard. Hence we have $(A/\mathfrak{m}A) \otimes_{R/\mathfrak{m}} (A'/\mathfrak{m}A') = 0$. Since $A/\mathfrak{m}A$ and $A'/\mathfrak{m}A'$ are vector spaces over $R/\mathfrak{m}$, it follows that either $A/\mathfrak{m}A = 0$ or $A'/\mathfrak{m}A' = 0$, as desired.

(3) $\implies$ (1): Assume that for each $\mathfrak{m} \in \operatorname{Supp}_R(A) \cap \operatorname{Supp}_R(A')$, either $A = \mathfrak{m}A$ or $A' = \mathfrak{m}A'$. Then by Lemma 3.4.2

$$A \otimes_R A' \cong \textstyle\bigoplus_{\mathfrak{m} \in \mathcal{F}} (A_\mathfrak{m}/\mathfrak{m}^0 A_\mathfrak{m}) \otimes_R (A'_\mathfrak{m}/\mathfrak{m}^0 A'_\mathfrak{m}) = 0.$$

(3) $\implies$ (2): Assume that for each $\mathfrak{m} \in \mathcal{F}$, either $A = \mathfrak{m}A$ or $A' = \mathfrak{m}A'$. In general for $\mathfrak{n} \neq \mathfrak{m} \in \text{m-Spec}(R)$ we have $\mathfrak{n}A_\mathfrak{m} = A_\mathfrak{m}$. Let $\mathfrak{m} \in \mathcal{F}$ and suppose that $A = \mathfrak{m}A$. Then $A_\mathfrak{m} = \mathfrak{m}A_\mathfrak{m} = \prod_{\mathfrak{n} \in \mathcal{F}} \mathfrak{n}A_\mathfrak{m} = \mathfrak{b}A_\mathfrak{m}$. So $(A/\mathfrak{b}A)_\mathfrak{m} = A_\mathfrak{m}/\mathfrak{b}A_\mathfrak{m} = A_\mathfrak{m}/\mathfrak{m}A_\mathfrak{m} = 0$. Thus $\mathfrak{m} \notin \operatorname{Supp}(A/\mathfrak{b}A)$. Since $\operatorname{Supp}(A/\mathfrak{b}A) \cap \operatorname{Supp}(A'/\mathfrak{b}A') \subseteq \operatorname{Supp}(A) \cap \operatorname{Supp}(A')$ it follows that $\operatorname{Supp}(A/\mathfrak{b}A) \cap \operatorname{Supp}(A'/\mathfrak{b}A') = \emptyset$.

(2) $\Longrightarrow$ (3): Assume that $\mathrm{Supp}(A/\mathfrak{b}A)\cap\mathrm{Supp}(A'/\mathfrak{b}A')=\emptyset$. Let $\mathfrak{m}\in\mathcal{F}$. Without loss of generality suppose $\mathfrak{m}\notin\mathrm{Supp}(A/\mathfrak{b}A)$. Therefore $0=(A/\mathfrak{b}A)_\mathfrak{m}=A_\mathfrak{m}/\mathfrak{b}A_\mathfrak{m}=A_\mathfrak{m}/\mathfrak{m}A_\mathfrak{m}$; hence $A_\mathfrak{m}=\mathfrak{m}A_\mathfrak{m}$. Since $A\cong\bigoplus_{\mathfrak{n}\in\mathrm{Supp}(A)}A_\mathfrak{n}$ and $A_\mathfrak{n}=\mathfrak{m}A_\mathfrak{n}$ for all maximal ideals $\mathfrak{n}\neq\mathfrak{m}$ it follows that $A=\mathfrak{m}A$. □

Proposition 3.4.6. *Let A and L be R-modules such that A is artinian. Let $\mathcal{F}=\{\mathfrak{m}\in\mathrm{Supp}(A)|\ (0:_L\mathfrak{m})\neq 0\}$. Let $\mathfrak{a}$ be an ideal contained in $\mathfrak{b}=\cap_{\mathfrak{m}\in\mathcal{F}}\mathfrak{m}$. Then*

$$\mathrm{Hom}_R(A,L)\cong\mathrm{Hom}_R(\Gamma_\mathfrak{a}(A),\Gamma_\mathfrak{a}(L))\cong\bigoplus_{\mathfrak{m}\in\mathcal{F}}\mathrm{Hom}_R(A_\mathfrak{m},\Gamma_\mathfrak{m}(L))$$

If in addition $\lambda_R(0:_L\mathfrak{b})<\infty$ then

$$\mathrm{Hom}_R(A,L)\cong\mathrm{Hom}_{\widehat{R}^\mathfrak{b}}(\Gamma_\mathfrak{b}(L)^\vee,\Gamma_\mathfrak{a}(A)^\vee)\cong\bigoplus_{\mathfrak{m}\in\mathcal{F}}\mathrm{Hom}_{\widehat{R}^\mathfrak{m}}(\Gamma_\mathfrak{m}(L)^\vee,A_\mathfrak{m}^\vee).$$

Note that $\Gamma_\mathfrak{b}(L)^\vee$ and $\Gamma_\mathfrak{b}(A)^\vee$ are noetherian $\widehat{R}^\mathfrak{b}$-modules while $\Gamma_\mathfrak{m}(L)^\vee$ and $A_\mathfrak{m}^\vee$ are noetherian $\widehat{R}^\mathfrak{m}$-modules.

Proof. The first sequence of isomorphisms above follows from We get the first step and last step in the next display follow from Lemma 3.1.24 (2) and the second step is from Lemma 2.1.5 (2).

$$\begin{aligned}\mathrm{Hom}_R(A,L)&\cong\bigoplus_{\mathfrak{m}\in\mathrm{Supp}(A)}\mathrm{Hom}_R(A_\mathfrak{m},L)\\&\cong\bigoplus_{\mathfrak{m}\in\mathrm{Supp}(A)}\mathrm{Hom}_R(A_\mathfrak{m},\Gamma_\mathfrak{m}(L))\\&\cong\bigoplus_{\mathfrak{m}\in\mathcal{F}}\mathrm{Hom}_R(A_\mathfrak{m},\Gamma_\mathfrak{m}(L))\\&\cong\bigoplus_{\mathfrak{m}\in\mathcal{F}}\bigoplus_{\mathfrak{n}\in\mathcal{F}}\mathrm{Hom}_R(A_\mathfrak{m},\Gamma_\mathfrak{n}(L))\\&\cong\mathrm{Hom}_R(\Gamma_\mathfrak{b}(A),\Gamma_\mathfrak{b}(L))\end{aligned}$$

The third step above follows from the fact that for all maximal ideals $\mathfrak{m}\notin\mathcal{F}$ either

$A_{\mathfrak{m}} = 0$ or $\Gamma_{\mathfrak{m}}(L) = 0$. The fourth step follows from the fact that for $\mathfrak{m} \neq \mathfrak{n} \in$ m-Spec(R) we have $\mathrm{Hom}_R(A_{\mathfrak{m}}, \Gamma_{\mathfrak{n}}(L)) = \mathrm{Hom}_R(A_{\mathfrak{m}}, \Gamma_{\mathfrak{m}+\mathfrak{n}}(L)) = 0$. Similarly since $\Gamma_{\mathfrak{b}}(A) = \Gamma_{\mathfrak{b}}(\Gamma_{\mathfrak{a}}(A))$ and $\Gamma_{\mathfrak{b}}(L) = \Gamma_{\mathfrak{b}}(\Gamma_{\mathfrak{a}}(L))$ it follows that $\mathrm{Hom}_R(\Gamma_{\mathfrak{a}}(A), \Gamma_{\mathfrak{a}}(L)) \cong \mathrm{Hom}_R(\Gamma_{\mathfrak{b}}(A), \Gamma_{\mathfrak{b}}(L)) \cong \mathrm{Hom}_R(A, L)$.

Assume that $\lambda_R(0 :_L \mathfrak{b}) < \infty$. Since $(0 :_L \mathfrak{b}) \cong \bigoplus_{\mathfrak{m}\in\mathcal{F}}(0 :_L \mathfrak{m})$ it follows that $E(\Gamma_{\mathfrak{m}}(L))$ is a finite direct sum of copies of $E(R/\mathfrak{m})$ for all $\mathfrak{m} \in \mathcal{F}$. Hence $\Gamma_{\mathfrak{b}}(L)$ is artinian and so is $\Gamma_{\mathfrak{m}}(L)$ for all $\mathfrak{m} \in \mathcal{F}$. The isomorphisms

$$\mathrm{Hom}_R(A, L) \cong \mathrm{Hom}_{\widehat{R}^{\mathfrak{b}}}(\Gamma_{\mathfrak{b}}(L)^{\vee}, A_{\mathfrak{u}}^{\vee}) \cong \bigoplus_{\mathfrak{m}\in\mathcal{F}} \mathrm{Hom}_{\widehat{R}^{\mathfrak{m}}}(\Gamma_{\mathfrak{m}}(L)^{\vee}, A_{\mathfrak{m}}^{\vee}).$$

follow from Corollary 3.3.7. The note about the modules being noetherian over the rings $\widehat{R}^{\mathfrak{b}}$ and $\widehat{R}^{\mathfrak{m}}$ follows from Lemma 2.1.5 (1). $\square$

Proposition 3.4.7. *Let A be an artinian R-module and let L be an R-module. Let $\mathcal{F} = \{\mathfrak{m} \in \mathrm{Supp}(A) | \ (0 :_L \mathfrak{m}) \neq 0\}$ and $\mathfrak{b} = \cap_{\mathfrak{m}\in\mathcal{F}}\mathfrak{m}$. Suppose there exists $x \geqslant 0$ such that $\mathfrak{b}^x\Gamma_{\mathfrak{b}}(L) = 0$. Choose $y \geqslant 0$ such that $\mathfrak{b}^y A = \mathfrak{b}^{y+1}A$. Let $n = \min\{x, y\}$. For all $\mathfrak{m} \in \mathcal{F}$ choose $\alpha_{\mathfrak{m}} \geqslant 0$ such that either $\mathfrak{m}^{\alpha_{\mathfrak{m}}}A = \mathfrak{m}^{\alpha_{\mathfrak{m}}+1}A$ or $\mathfrak{m}^{\alpha_{\mathfrak{m}}}\Gamma_{\mathfrak{m}}(L) = 0$. Then*

$$\mathrm{Hom}_R(A, L) \cong \bigoplus_{\mathfrak{m}\in\mathcal{F}} \mathrm{Hom}_R(A/\mathfrak{m}^{\alpha_{\mathfrak{m}}}A, (0 :_L \mathfrak{m}^{\alpha_{\mathfrak{m}}})) \cong \mathrm{Hom}_R(A/\mathfrak{b}^n A, (0 :_L \mathfrak{b}^n))$$

Proof. The first step in the following display is from Proposition 3.4.6:

$$\begin{aligned}
\mathrm{Hom}_R(A, L) &\cong \mathrm{Hom}_R(\Gamma_{\mathfrak{b}}(A), \Gamma_{\mathfrak{b}}(L)) \\
&\cong \mathrm{Hom}_R(\Gamma_{\mathfrak{b}}(A)/\mathfrak{b}^x\Gamma_{\mathfrak{b}}(A), \Gamma_{\mathfrak{b}}(L)) \\
&\cong \mathrm{Hom}_R(A/\mathfrak{b}^x A, \Gamma_{\mathfrak{b}}(L)) \\
&\cong \mathrm{Hom}_R(A/\mathfrak{b}^n A, \Gamma_{\mathfrak{b}}(L)) \\
&\cong \mathrm{Hom}_R(A/\mathfrak{b}^n A, (0 :_L \mathfrak{b}^n)).
\end{aligned}$$

The second step follows from the assumption $\mathfrak{b}^x\Gamma_{\mathfrak{b}}(L) = 0$. The third step follows by noticing that $\Gamma_{\mathfrak{b}}(A) = \bigoplus_{\mathfrak{m}\in\mathcal{F}} A_{\mathfrak{m}}$ and $\mathfrak{b}A_{\mathfrak{m}} = A_{\mathfrak{m}}$ for all $\mathfrak{m} \in \text{m-Spec}(R)\diagdown\mathcal{F}$.

For the fourth step, we need to show that $\mathfrak{m}^n A = \mathfrak{m}^x A$. If $n = x$, this is clear. If $n \neq x$, then $n = y < x$. From the assumption $\mathfrak{b}^y A = \mathfrak{b}^{y+1}A$ it follows that $\mathfrak{m}^n A = \mathfrak{m}^y A = \mathfrak{m}^x A$. The last step is a consequence of the fact that $\mathfrak{b}^n \in \mathrm{Ann}(A/\mathfrak{b}^n A)$. Since $\mathfrak{b}^x\Gamma_{\mathfrak{b}}(L) = 0$ it follows that $\mathfrak{m}^x\Gamma_{\mathfrak{m}}(L) = 0$ for all $\mathfrak{m} \in \mathcal{F}$. A similar sequence of isomorphisms to the one above shows the first isomorphism in the Proposition and the result follows. $\square$

Proposition 3.4.8. *Let A be an artinian R-module and let L be an R-module. Let $\mathcal{F} = \{\mathfrak{m} \in \mathrm{Supp}(A) | \ (0 :_L \mathfrak{m}) \neq 0\}$ and $\mathfrak{b} = \cap_{\mathfrak{m}\in\mathcal{F}}\mathfrak{m}$. Suppose there exists $x \geqslant 0$ such that $\mathfrak{b}^x\Gamma_{\mathfrak{b}}(L) = 0$. Choose $y \geqslant 0$ such that $\mathfrak{b}^y A = \mathfrak{b}^{y+1}A$. Let $n = \min\{x, y\}$. For all $\mathfrak{m} \in \mathcal{F}$ choose $\alpha_{\mathfrak{m}} \geqslant 0$ such that either $\mathfrak{m}^{\alpha_{\mathfrak{m}}}A = \mathfrak{m}^{\alpha_{\mathfrak{m}}+1}A$ or $\mathfrak{m}^{\alpha_{\mathfrak{m}}}\Gamma_{\mathfrak{m}}(L) = 0$. Then*

$$\begin{aligned}
\lambda_R(\mathrm{Hom}_R(A, L)) &\leqslant \textstyle\sum_{\mathfrak{m}\in\mathcal{F}} \lambda_R(A/\mathfrak{m}A)\lambda_R(0 :_L \mathfrak{m}^{\alpha_{\mathfrak{m}}}) \\
&\max\{\lambda_R(A/\mathfrak{m}A) | \ \mathfrak{m} \in \mathcal{F}\}\lambda_R(0 :_L \mathfrak{b}^n) \\
&\leqslant \lambda_R(A/\mathfrak{b}A)\lambda_R(0 :_L \mathfrak{b}^n).
\end{aligned}$$

Here, we follow the convention $0 \cdot \infty = 0$.

Proof. An inductive argument on $\lambda(A/\mathfrak{m})$ and $\lambda(0 :_L \mathfrak{m}^{\alpha_\mathfrak{m}})$ shows that

$$\lambda_R \operatorname{Hom}_R(A/\mathfrak{m}^{\alpha_\mathfrak{m}} A, (0 :_L \mathfrak{m}^{\alpha_\mathfrak{m}})) \leqslant \lambda_R(A/\mathfrak{m}A)\lambda_R(0 :_L \mathfrak{m}^{\alpha_\mathfrak{m}})$$

Therefore by Proposition 3.4.7 and the additivity of length we get the first inequality in the proposition.

By the Chinese remainder Theorem $A/\mathfrak{b}A \cong \bigoplus_{\mathfrak{m}\in\mathcal{F}} A/\mathfrak{m}A$. Similarly, $\Gamma_\mathfrak{b}(L) \cong \bigoplus_{\mathfrak{m}\in\mathcal{F}} \Gamma_\mathfrak{m}(L)$. From these isomorphisms we deduce that $n \geqslant \max\{\alpha_\mathfrak{m} |\ \mathfrak{m} \in \mathcal{F}\}$ and that $(0 :_L \mathfrak{b}^n) = \bigoplus_{\mathfrak{m}\in\mathcal{F}}(0 :_L \mathfrak{m}^n)$. From the injections $(0 :_L \mathfrak{m}^{\alpha_\mathfrak{m}}) \hookrightarrow (0 :_L \mathfrak{m}^n)$ and the additivity of length of direct sums we conclude that

$$\lambda_R(A/\mathfrak{b}A) = \textstyle\sum_{\mathfrak{m}\in\mathcal{F}} \lambda_R(A/\mathfrak{m}A) \quad \text{and} \quad \lambda_R(0 :_L \mathfrak{b}^z) \geqslant \textstyle\sum_{\mathfrak{m}\in\mathcal{F}} \lambda_R(0 :_L \mathfrak{m}^{\alpha_\mathfrak{m}}).$$

The last two inequalities in the proposition follow. □

Corollary 3.4.9. *Let A and N be R-modules such that A is artinian and N is noetherian. Then* $\lambda_R(\operatorname{Hom}_R(A, N)) < \infty$.

Definition 3.4.10. Given an R-module L we say that $\mathfrak{p} \in \operatorname{Spec}(R)$ is an attached prime of L if there exists a submodule L' of L such that $\mathfrak{p} = \operatorname{Ann}_R(L/L')$. We denote by $\operatorname{Att}_R(L)$ the set of attached primes of L.

Proposition 3.4.11. *Let A and L be R-modules such that A is artinian. Let* $\mathcal{F} = \{\mathfrak{m} \in \operatorname{Supp}(A) |\ (0 :_L \mathfrak{m}) \neq 0\}$. *Let* $\mathfrak{b} = \cap_{\mathfrak{m}\in\mathcal{F}}\mathfrak{m}$ *and suppose that* $\lambda_R(0 :_L \mathfrak{b}) < \infty$

(equivalently $\mu^0_R(\mathfrak{m}, L) < \infty$ *for all* $\mathfrak{m} \in \mathcal{F}$*). Then*

$$\begin{aligned}\mathrm{Ass}_{\widehat{R}^{\mathfrak{b}}}(\mathrm{Hom}_R(A, L)) &= \mathrm{Ass}_{\widehat{R}^{\mathfrak{b}}}(\Gamma_{\mathfrak{b}}(A)^{\vee}) \cap \mathrm{Supp}_{\widehat{R}^{\mathfrak{b}}}(\Gamma_{\mathfrak{b}}(L)^{\vee}) \\ &= \mathrm{Att}_{\widehat{R}^{\mathfrak{b}}}(\Gamma_{\mathfrak{b}}(A)) \cap \mathrm{Supp}_{\widehat{R}^{\mathfrak{b}}}(\Gamma_{\mathfrak{b}}(L)^{\vee})\end{aligned}$$

Proof. By Lemma 3.4.6 $\mathrm{Hom}_R(A, L) \cong \mathrm{Hom}_{\widehat{R}^{\mathfrak{b}}}(\Gamma_{\mathfrak{b}}(L)^{\vee}, \Gamma_{\mathfrak{b}}(A)^{\vee})$. Since $\Gamma_{\mathfrak{b}}(L)^{\vee}$ is a noetherian $\widehat{R}^{\mathfrak{b}}$-module we can apply a result from Bourbaki [7, IV 1.4 Proposition 10] to get the first equality in the Proposition above. Also by [24, Proposition 2.7] $\mathrm{Ass}_{\widehat{R}^{\mathfrak{b}}}(\Gamma_{\mathfrak{b}}(A)^{\vee}) = \mathrm{Att}_{\widehat{R}^{\mathfrak{b}}}(\Gamma_{\mathfrak{b}}(A)^{\vee})^{\vee(\widehat{R}^{\mathfrak{b}})}) = \mathrm{Att}_{\widehat{R}^{\mathfrak{b}}}(\Gamma_{\mathfrak{b}}(A))$. $\square$

Proposition 3.4.12. *Let* A *and* L *be* R*-modules such that* A *is artinian. Let* $\mathcal{F} = \{\mathfrak{m} \in \mathrm{Supp}(A) | \ (0 :_L \mathfrak{m}) \neq 0\}$*. Let* $\mathfrak{b} = \cap_{\mathfrak{m}\in\mathcal{F}}\mathfrak{m}$ *and suppose that* $\lambda_R(0 :_L \mathfrak{b}) < \infty$*. Then the following conditions are equivalent:*

1. $\mathrm{Hom}_R(A, L) = 0$;
2. $\mathrm{Hom}_R(\Gamma_{\mathfrak{b}}(A), \Gamma_{\mathfrak{b}}(L)) = 0$;
3. $\mathrm{Hom}_{\widehat{R}^{\mathfrak{b}}}(\Gamma_{\mathfrak{b}}(L)^{\vee}, \Gamma_{\mathfrak{b}}(A)^{\vee}) = 0$;
4. $\mathrm{Ann}_{\widehat{R}^{\mathfrak{b}}}(\Gamma_{\mathfrak{b}}(L))\Gamma_{\mathfrak{b}}(A) = \Gamma_{\mathfrak{b}}(A)$;
5. $\mathrm{Ann}_{\widehat{R}^{\mathfrak{b}}}(\Gamma_{\mathfrak{b}}(L))$ *contains an* $\Gamma_{\mathfrak{b}}(A)^{\vee}$*-regular element;*
6. $\mathrm{Ass}_{\widehat{R}^{\mathfrak{b}}}(\Gamma_{\mathfrak{b}}(A)^{\vee}) \cap \mathrm{Supp}_{\widehat{R}^{\mathfrak{b}}}(\Gamma_{\mathfrak{b}}(L)^{\vee}) = \emptyset$; *and*
7. $\mathrm{Att}_{\widehat{R}^{\mathfrak{b}}}(\Gamma_{\mathfrak{b}}(A)) \cap \mathrm{Supp}_{\widehat{R}^{\mathfrak{b}}}(\Gamma_{\mathfrak{b}}(L)^{\vee}) = \emptyset$.

Proof. Lemma 3.4.6 gives the equivalence of (1)-(3). Since $\Gamma_{\mathfrak{b}}(L)^{\vee}$ and $\Gamma_{\mathfrak{b}}(A)^{\vee}$ are noetherian $\widehat{R}^{\mathfrak{b}}$-module the equivalence of (3) and (5) is standard; see [9, Proposition

1.2.3]. Let $I := \mathrm{Ann}_{\widehat{R}^{\mathfrak{b}}}(\Gamma_{\mathfrak{b}}(L))$. The isomorphism $\Gamma_{\mathfrak{b}}(A)^{\vee} \cong \Gamma_{\mathfrak{b}}(A)^{\vee(\widehat{R}^{\mathfrak{b}})}$ along with Lemma 2.1.13 (3) gives the equivalence of (4) and (5). The equivalence of (3), (6) and (7) follows from Proposition 3.4.11 and the fact that the $\widehat{R}^{\mathfrak{b}}$-module $\mathrm{Hom}_R(A, L) = 0$ iff $\mathrm{Ass}_{\widehat{R}^{\mathfrak{b}}}(\mathrm{Hom}_R(A, L)) = \emptyset$. $\square$

Chapter 4

Asymptotic Behavior of Dimensions of Syzygy Modules

The results in this chapter are joint work with K. Beck. Throughout this chapter let R be a commutative (noetherian) local ring with maximal ideal $\mathfrak{m}$ and residue field $k := R/\mathfrak{m}$. Let M be a non-zero finitely generated R-module with length $\lambda(M)$. The nth Betti number of M is given by $\beta_n(M) := \dim_k(\mathrm{Tor}_n^R(k, M))$. A minimal free resolution of M then has the form

$$\cdots \xrightarrow{\delta_3} R^{\beta_2(M)} \xrightarrow{\delta_2} R^{\beta_1(M)} \xrightarrow{\delta_1} R^{\beta_0(M)} \xrightarrow{\delta_0} M \xrightarrow{\delta_{-1}} 0\ .$$

The nth syzygy module of M is $\Omega_n(M) := \mathrm{Im}(\delta_n) = \mathrm{Ker}(\delta_{n-1})$. In particular $\Omega_0(M) = M$. Recall that $\mathrm{Min}(M)$ denotes the set of minimal elements of $\mathrm{Supp}(M)$. The projective dimension of M is given by $\mathrm{pd}(M) := \inf\{n|\ \beta_n(M) = 0\}$. Given an ideal $\mathfrak{a} \subset R$ the ith local cohomology functor with respect to $\mathfrak{a}$ is denoted $\mathrm{H}^i_{\mathfrak{a}}(-)$ and is defined by $\mathrm{H}^i_{\mathfrak{a}}(M) = \mathrm{H}^i(\Gamma_{\mathfrak{a}}(I))$, where I is an injective resolution of M, and $\mathrm{H}^i(-)$ is the ith homology functor. In particular, the functor $\mathrm{H}^i_{\mathfrak{a}}(-)$ are the right derived functors of the functor $\Gamma_{\mathfrak{a}}(-)$. For background on local cohomology see [16].

The asymptotic behavior of the depths of syzygy modules is known. If $\mathrm{pd}(M) = \infty$, then $\mathrm{depth}(\Omega_n(M)) \geqslant \mathrm{depth}(R)$ for $n \geqslant \max\{0, \mathrm{depth}(R) - \mathrm{depth}(M)\}$, with at most one strict inequality at either $n = 0$ or $n = \mathrm{depth}(M) - \mathrm{depth}(R) + 1$; see [23]. However, the asymptotic behavior of $\dim(\Omega_n(M))$ is not known in general. Many of our results are motivated by trying to answer the following open question.

Question 4.1. *Is* $\dim(\Omega_n(M))$ *constant for* $n \gg 0$*?*

Several instances in which this question is known to have an affirmative answer are given in [11, Remark 5.2].

We prove new instances in which Question 4.1 has an affirmative answer. All of our results are for modules whose Betti numbers are eventually non-decreasing. Therefore finding an affirmative answer to the following open question first asked by L. Avramov would improve our results.

Question 4.2. *[3] Are the Betti numbers of a finitely generated module over a local ring always eventually non-decreasing?*

In [3] and [10] several instances are given for which this question has an affirmative answer.

Whenever the Betti numbers of a module are eventually strictly increasing it is known that the the dimension of a sufficiently high syzygy will have the dimension of the ring.

Remark 4.3. *If* $\beta_i(M) > \beta_{i-1}(M)$ *for some* $i > 0$*, then* $\mathrm{Supp}(\Omega_{i+1}(M)) = \mathrm{Spec}(R)$*; hence* $\dim(\Omega_{i+1}(M)) = \dim(R)$.

Proof. We prove the contrapositive. Suppose $\mathrm{Supp}(\Omega^R_{i+1}(M)) \neq \mathrm{Spec}(R)$. Then, there exist a minimal prime $\mathfrak{p}$ such that $\Omega^R_{i+1}(M)_{\mathfrak{p}} = 0$. Let $n = \lambda_{R_{\mathfrak{p}}}(R_{\mathfrak{p}})$. Localizing the exact sequence $0 \to \Omega_{i+1}^R(M) \to R^{\beta_i(M)} \to R^{\beta_{i-1}(M)}$ at $\mathfrak{p}$ we obtain an injection $R_\mathfrak{p}^{\beta_i(M)} \to R_\mathfrak{p}^{\beta_{i-1}(M)}$ from an module of length $n\beta_i(M)$ to a module of length $n\beta_{i-1}(M)$. It follows that $\beta_i \leqslant \beta_{i-1}$. $\square$

Lemma 4.4. *Let n be a positive integer. Suppose that $\operatorname{Supp}(\Omega_{2n}(M)) \neq \operatorname{Spec}(R)$ and that $\beta_0(M) \leqslant \beta_1(M) \leqslant \ldots \leqslant \beta_{2n-1}(M)$. Then we have the following:*

1. *$\beta_{2i}(M) = \beta_{2i+1}(M)$ for $i = 0, \ldots, n-1$;*
2. *$\operatorname{Supp}(\Omega_{2n}(M)) \subseteq \operatorname{Supp}(\Omega_{2n-2}(M)) \subseteq \ldots \subseteq \operatorname{Supp}(\Omega_2(M)) \subseteq \operatorname{Supp}(M)$; and*
3. *$\operatorname{Supp}(\Omega_{2n}(M)) \cap \operatorname{Min}(R) = \operatorname{Supp}(M) \cap \operatorname{Min}(R)$.*

Proof. Choose $\mathfrak{p} \in \operatorname{Min}(R) \setminus \operatorname{Supp}(\Omega_{2n}(M))$. Localizing part of a minimal free resolution of M at $\mathfrak{p}$, we get an exact sequence of finite-length $R_\mathfrak{p}$-modules:

$$0 \longrightarrow R_\mathfrak{p}^{\beta_{2n-1}(M)} \xrightarrow{\phi_{2n-1}} R_\mathfrak{p}^{\beta_{2n-2}(M)} \xrightarrow{\phi_{2n-2}} \cdots \xrightarrow{\phi_1} R_\mathfrak{p}^{\beta_0(M)} \xrightarrow{\phi_0} M_\mathfrak{p} \longrightarrow 0\,.$$

Since ϕ_{2n-1} is an injection, $\lambda(R_\mathfrak{p}^{\beta_{2n-2}(M)}) \geqslant \lambda(R_\mathfrak{p}^{\beta_{2n-1}(M)})$; hence $\beta_{2n-2}(M) \geqslant \beta_{2n-1}(M)$. It follows that $\beta_{2n-2}(M) = \beta_{2n-1}(M)$ and, since $R_\mathfrak{p}$ has finite length, that ϕ_{2n-1} is an isomorphism. Therefore ϕ_{2n-2} is the zero map. By an inductive argument it follows that $\beta_{2i}(M) = \beta_{2i+1}(M)$, ϕ_{2i+1} is an isomorphism, and ϕ_{2i} is the zero map for $i = 0, \ldots n-1$.

Since ϕ_0 is the zero map $M_\mathfrak{p} = 0$; hence $\mathfrak{p} \notin \operatorname{Supp}(M)$. It follows that

$$\operatorname{Supp}(M) \cap \operatorname{Min}(R) \subseteq \operatorname{Supp}(\Omega_{2n}(M)) \cap \operatorname{Min}(R). \tag{4.1}$$

Let $\mathfrak{q} \in \operatorname{Spec}(R) \setminus \operatorname{Supp}(\Omega_{2i}(M))$ for some i with $0 \leqslant i \leqslant n-1$. Localizing the exact sequence $0 \to \Omega_{2i+2}(M) \to R^{\beta_{2i}(M)} \to R^{\beta_{2i}(M)} \to \Omega_{2i}(M)$ at $\mathfrak{q}$ we obtain an exact sequence $0 \to \Omega_{2i+2}(M)_\mathfrak{q} \to R_\mathfrak{q}^{\beta_{2i}(M)} \to R_\mathfrak{q}^{\beta_{2i}(M)} \to 0$. It follows that

$\Omega_{2i+2}(M)_{\mathfrak{q}} = 0$ and $\mathfrak{q} \notin \operatorname{Supp}(\Omega_{2i+2}(M))$. Thus $\operatorname{Supp}(\Omega_{2i+2}(M)) \subseteq \operatorname{Supp}(\Omega_{2i}(M))$ for $i = 0, \ldots, n-1$. In particular we have $\operatorname{Supp}(\Omega_{2n}(M)) \cap \operatorname{Min}(R) \subseteq \operatorname{Supp}(M) \cap \operatorname{Min}(R)$ and statement (3) from the lemma follows from display (4.1). □

Fact 4.5. *Let B be a square n by n matrix with entries in R defining a map from R^n to R^n. Then using invertible row and column operations one can transform B into a matrix $A = I_m \oplus B'$ where I_m is the m by m identity matrix for some integer m and B' has entries in $\mathfrak{m}$. Indeed, the following row and column operations are invertible and, hence each represent a change of basis for R^n:*

1. *swapping two rows or columns;*
2. *multiplying a row or column by a unit; and*
3. *adding a multiple of a row (column) to another row (column).*

The proof of our original statement follows using standard techniques in linear algebra.

Theorem 4.6. *Suppose that the sequence $(\beta_i(M))_{i=0}^{\infty}$ is eventually non-decreasing. Then we have the following:*

1. *For all $i \gg 0$, $\operatorname{Min}(\Omega_i(M)) \subseteq \operatorname{Min}(R)$;*
2. *The sequences $(\operatorname{Supp}(\Omega_{2i}(M)))_{i=0}^{\infty}$ and $(\operatorname{Supp}(\Omega_{2i+1}(M)))_{i=0}^{\infty}$ stabilize;*
3. *Either $\operatorname{Supp}(\Omega_{2i}(M)) = \operatorname{Spec}(R)$ for all $i \gg 0$, or $\beta_{2i}(M) = \beta_{2i+1}(M)$ for all $i \gg 0$; and*
4. *Either $\operatorname{Supp}(\Omega_{2i+1}(M)) = \operatorname{Spec}(R)$ for all $i \gg 0$, or $\beta_{2i}(M) = \beta_{2i-1}(M)$ for all $i \gg 0$.*

Proof. We may assume that $\operatorname{pd}(M) = \infty$. By replacing M by a sufficiently high syzygy, one may assume that $\beta_{i+1}(M) \geqslant \beta_i(M)$ for all $i \geqslant 0$. If $\operatorname{Supp}(\Omega_{2i}(M)) =$

Spec(R) for $i \gg 0$ then all of the statements hold for even (odd) syzygies, that is assuming M was replaced by an even (odd) syzygy. Therefore we may suppose that there exist infinitely many $i \in \mathbb{N}$ such that $\mathrm{Supp}(\Omega_{2i}(M)) \neq \mathrm{Spec}(R)$. Since we could have replaced M by either an odd or an even syzygy it suffices to show that $\beta_{2i}(M) = \beta_{2i+1}(M)$ for all $i \gg 0$, the sequence $(\mathrm{Supp}(\Omega_{2i}(M)))_{i=0}^{\infty}$ stabilizes, and $\mathrm{Min}(\Omega_{2i}(M)) \subseteq \mathrm{Min}(R)$ for $i \gg 0$.

Since $\mathrm{Min}(R)$ is a finite set we may choose $\mathfrak{p} \in \mathrm{Min}(R)$ such that there are infinitely many $i \in \mathbb{N}$ for which $\mathfrak{p} \notin \mathrm{Supp}(\Omega_{2i}(M))$. For each positive integer c such that $\mathfrak{p} \notin \mathrm{Supp}(\Omega_{2c}(M))$ Lemma 4.4 applied to $\mathrm{Supp}(\Omega_{2c}(M))$ implies that we have $\beta_{2i}(M) = \beta_{2i+1}(M)$ and $\mathrm{Supp}(\Omega_{2i+2}(M)) \subseteq \mathrm{Supp}(\Omega_{2i}(M))$ for all $0 \leqslant i < c$. Since, c can be chosen to be arbitrarily large we have $\beta_{2i}(M) = \beta_{2i+1}(M)$ and $\mathrm{Supp}(\Omega_{2i+2}(M)) \subseteq \mathrm{Supp}(\Omega_{2i}(M))$ for all $i \geqslant 0$. Since closed sets in the Zariski topology satisfy DCC it follows that we may choose $m \gg 0$ such that $\mathrm{Supp}(\Omega_{2m+2i}(M))$ is constant for all $i \geqslant 0$.

Therefore it remains to show that $\mathrm{Min}(\Omega_{2i}(M)) \subseteq \mathrm{Min}(R)$ for $i \gg 0$. Choose $\mathfrak{q} \in \mathrm{Min}(\Omega_{2m}(M))$. Let $S := R_{\mathfrak{q}}$, $M_i := (\Omega_{2m+2i}(M))_{\mathfrak{q}}$ for all $i \geqslant 0$ and $\mathfrak{n} := \mathfrak{q}R_{\mathfrak{q}}$ be the maximal ideal of S. For all $i \geqslant 0$ we have an exact sequence of the form

$$0 \longrightarrow M_{i+1} \longrightarrow S^{b_i} \xrightarrow{B_i} S^{b_i} \longrightarrow M_i \longrightarrow 0\,.$$

If the matrix B_i defining the map $S^{b_i} \to S^{b_i}$ has some entries which are units, then by Fact 4.5 we can reduce this sequence by taking away free summands; hence we may assume that B_i has all of its entries in $\mathfrak{n}$. For all $i \geqslant 0$ let $N_i = \mathrm{Im}(B_i)$. Since M_{i+1} has finite length, a minimal injective resolution I of M_{i+1} is $\mathfrak{m}$-torsion. It follows that $0 = \mathrm{H}^j(I) = \mathrm{H}^j(\Gamma_{\mathfrak{m}}(I)) = \mathrm{H}^j_{\mathfrak{m}}(M_{i+1})$ for all $j > 0$. From the exact sequence

$0 \longrightarrow M_{i+1} \longrightarrow S^{b_i} \xrightarrow{\phi_i} N_i \longrightarrow 0$ we get an exact sequence

$$0 \to M_{i+1} \to \mathrm{H}^0_{\mathfrak{n}}(S^{b_i}) \to \mathrm{H}^0_{\mathfrak{n}}(N_i) \to 0 \tag{4.2}$$

and isomorphisms $\mathrm{H}^j_{\mathfrak{n}}(\phi_i) : \mathrm{H}^j_{\mathfrak{n}}(S^{b_i}) \to \mathrm{H}^j_{\mathfrak{n}}(N_i)$ for all $j \geqslant 1$. Similarly the exact sequence $0 \longrightarrow N_i \xrightarrow{\psi_i} S^{b_i} \longrightarrow M_i \longrightarrow 0$ yields an exact sequence

$$0 \to \mathrm{H}^0_{\mathfrak{n}}(N_i) \to \mathrm{H}^0_{\mathfrak{n}}(S^{b_i}) \to M_i \to \mathrm{H}^1_{\mathfrak{n}}(N_i) \to \mathrm{H}^1_{\mathfrak{n}}(S^{b_i}) \to 0 \tag{4.3}$$

and isomorphims $\mathrm{H}^j_{\mathfrak{n}}(\psi_i) : \mathrm{H}^j_{\mathfrak{n}}(N_i) \to \mathrm{H}^j_{\mathfrak{n}}(S^{b_i})$ for all $j \geqslant 2$. By the additivity of length we get the first and third steps in the next display from sequences (4.3) and (4.2) respectively.

$$\begin{aligned}
\lambda(M_i) &= \lambda(\mathrm{H}^0_{\mathfrak{n}}(S^{b_i})) - \lambda(\mathrm{H}^0_{\mathfrak{n}}(N_i)) + \lambda(\mathrm{Im}(M_i \to \mathrm{H}^1_{\mathfrak{n}}(N_i))) \\
&\geqslant \lambda(\mathrm{H}^0_{\mathfrak{n}}(S^{b_i})) - \lambda(\mathrm{H}^0_{\mathfrak{n}}(N_i)) \\
&= \lambda(M_{i+1}).
\end{aligned}$$

Since the sequence $(\lambda(M_i))_{i=0}^{\infty}$ is positive and non-increasing it is eventually constant.

Choose $t \in \mathbb{N}$ such that $\lambda(M_t) = \lambda(M_{t+1})$. Then $\lambda(\mathrm{Im}(M_t \to \mathrm{H}^1_{\mathfrak{n}}(N_t))) = 0$. Therefore the map $M_t \to \mathrm{H}^1_{\mathfrak{n}}(N_t)$ is the zero map. From sequence (4.3), it follows that the map $\mathrm{H}^1_{\mathfrak{n}}(\psi_t) : \mathrm{H}^1_{\mathfrak{n}}(N_t) \to \mathrm{H}^1_{\mathfrak{n}}(S^{b_t})$ is an isomorphism. Thus

$$\mathrm{H}^j_{\mathfrak{n}}(B_t) = \mathrm{H}^j_{\mathfrak{n}}(\psi_t) \circ \mathrm{H}^j_{\mathfrak{n}}(\phi_t) : \mathrm{H}^j_{\mathfrak{n}}(S^{b_t}) \to \mathrm{H}^j_{\mathfrak{n}}(S^{b_t})$$

is an isomorphism for all $j \geqslant 1$. Since $\mathrm{H}^j_{\mathfrak{n}}(-)$ is an additive functor the map $\mathrm{H}^j_{\mathfrak{n}}(B_t)$ is just matrix multiplication by the matrix B_t applied to the components of $\mathrm{H}^j_{\mathfrak{n}}(S^{b_t})$. Since B_t has entries in $\mathfrak{n}$ it must kill socle elements of $\mathrm{H}^j_{\mathfrak{n}}(S^{b_t})$. Therefore $\mathrm{H}^j_{\mathfrak{n}}(S^{b_t})$ has no socle elements. Since $\mathrm{H}^j_{\mathfrak{n}}(S^{b_t})$ is n-torsion it follows that $\mathrm{H}^j_{\mathfrak{n}}(S^{b_t}) = 0$ for all $j \geqslant 1$. By [16, Theorem 9.3] we get the first equality in the next display:

$$\dim(S) = \sup\{n| \ \mathrm{H}^j_{\mathfrak{n}}(S) \neq 0\} = 0;$$

hence $\mathfrak{q} \in \mathrm{Min}(R)$. Thus $\mathrm{Min}(\Omega_{2i}(M)) \subseteq \mathrm{Min}(R)$ for all $i \gg 0$, and the result follows. □

Corollary 4.7. *Suppose the sequence $(\beta_i(M))_{i=0}^{\infty}$, is non-decreasing for $i \gg 0$. Then $(\dim(\Omega_{2i}(M)))_{i=1}^{\infty}$ and $(\dim(\Omega_{2i+1}(M)))_{i=0}^{\infty}$ are constant for $i \gg 0$. If $\mathrm{pd}(M) = \infty$ then one sequence stabilizes to $\dim(R)$ and the other sequence stabilizes to $\dim(R/\mathfrak{p})$ for some $\mathfrak{p} \in \mathrm{Min}(R)$.*

Corollary 4.8. *Suppose the sequence $(\beta_i(M))_{i=0}^{\infty}$, is non-decreasing for $i \gg 0$. If R satisfies one of the following conditions then $\dim(\Omega_i(M))$ is constant for $i \gg 0$:*

1. *R is equidimensional (i.e. $\dim(R/\mathfrak{p})$ is constant for all $\mathfrak{p} \in \mathrm{Min}(R)$);*

2. *R is a domain; or*

3. *$\dim(R) \leqslant 1$.*

Proof. (1) Follows from Corollary 4.7. (2) and (3) are special cases of (1). □

Remark 4.9. *It should be noted that [11, Remark 5.6] claims that using [11, Proposition 5.5] one can show that if R is equidimensional and Question 4.2 has an affirmative answer, then $\dim(\Omega_n(M))$ is constant for $n \gg 0$. However, [11, Proposition 5.5] requires the assumption that $\dim(R) \geqslant 2$. Therefore although the conclusions of [11, Remark 5.6] are correct, the justification given for these conclusions is invalid. One should note the justification uses a localization argument, so it is invalid in every positive dimension, not just dimension 1.*

The goal of the next two results, Lemma 4.10 and Proposition 4.12, is to show how quickly the sequence $(\operatorname{Supp}(\Omega_{2i}(M)))_{i=0}^{\infty}$ stabilizes when the Betti numbers of M are non-decreasing.

Lemma 4.10. *Given $i > 0$, if $\beta_i(M) = \beta_{i+1}(M)$, then $\operatorname{Supp}(\Omega_i(M)) = \operatorname{Supp}(\Omega_{i+2}(M))$. Suppose $\beta_0(M) = \beta_1(M)$. Then we have the following:*

1. *If $\operatorname{Supp}(M) \setminus \operatorname{Supp}(\Omega_2(M)) \neq \emptyset$, then M is not a first syzygy of any module.*

2. *If $\mathfrak{p} \in \operatorname{Min}(M) \setminus \operatorname{Supp}(\Omega_2(M))$ then $\operatorname{height}(\mathfrak{p}) = 1$.*

Proof. Suppose that $b := \beta_0(M) = \beta_1(M)$ and $\operatorname{Supp}(M) \setminus \operatorname{Supp}(\Omega_2(M)) \neq \emptyset$. Choose $\mathfrak{p} \in \operatorname{Min}(M) \setminus \operatorname{Supp}(\Omega_2(M))$. As $M_\mathfrak{p}$ has finite length $0 \to R_\mathfrak{p}^{\beta_1(M)} \to R_\mathfrak{p}^{\beta_0(M)} \to 0$ has non-zero finite length homology. By the New Intersection Theorem [25] $\dim(R_\mathfrak{p}) \leqslant 1$. If we suppose that $\dim(R_\mathfrak{p}) = 0$, then $0 \neq \lambda(M_\mathfrak{p}) = \lambda(R_\mathfrak{p}^{\beta_1(M)}) - \lambda(R_\mathfrak{p}^{\beta_1(M)}) = 0$, which is a contradiction. Thus, $\operatorname{height}(\mathfrak{p}) = \dim(R_\mathfrak{p}) = 1$.

Since $0 \to R_\mathfrak{p}^b \to R_\mathfrak{p}^b \to M_\mathfrak{p} \to 0$ is exact, Fact 4.5 implies that there exists a minimal $R_\mathfrak{p}$-free resolution of $M_\mathfrak{p}$ of the form $0 \to R_\mathfrak{p}^n \to R_\mathfrak{p}^n \to M_\mathfrak{p} \to 0$ for some $n > 0$. Therefore $\operatorname{pd}_{R_\mathfrak{p}}(M_\mathfrak{p}) = 1$; hence $\operatorname{depth}_{R_\mathfrak{p}}(M_\mathfrak{p}) = \operatorname{depth}(R_\mathfrak{p}) - 1$.

Assume that $M = \Omega_1(L)$ for some R-module L we will obtain a contradiction. Since $M_\mathfrak{p}$ is finite length and $\dim(R_\mathfrak{p}) = 1$ it follows that $M_\mathfrak{p}$ has no $R_\mathfrak{p}$-free summands. Therefore $M_\mathfrak{p} = \Omega_1^{R_\mathfrak{p}}(L_\mathfrak{p})$. Since

$$0 \leqslant \operatorname{depth}_{R_\mathfrak{p}}(M_\mathfrak{p}) = \operatorname{depth}(R_\mathfrak{p}) - 1 \leqslant \dim(R_\mathfrak{p}) - 1 = 1 - 1 = 0$$

it follows that $\operatorname{depth}(R_\mathfrak{p}) = 1$ and $\operatorname{depth}(M_\mathfrak{p}) = 0$. However, since $M_\mathfrak{p}$ is a first syzygy $\operatorname{depth}(M_\mathfrak{p}) \geqslant \min\{1, \operatorname{depth}(R_\mathfrak{p})\} = 1$. This is a contradiction; hence M is not a first syzygy of any module.

Now suppose that $\beta_i(M) = \beta_{i+1}(M)$ for some $i > 0$. Since $\Omega_i(M)$ is a first syzygy of $\Omega_{i-1}(M)$ it follows that $\mathrm{Supp}(\Omega_i(M)) \subseteq \mathrm{Supp}(\Omega_{i+2}(M))$. By Lemma 4.4 we get the opposite inclusion and the result follows. $\square$

The following is an example where $\beta_1(M) = \beta_0(M)$ but $\mathrm{Supp}(M) \nsubseteq \mathrm{Supp}(\Omega_2(M))$.

Example 4.11. *Let $S = k[x, y, z]$ and $\mathfrak{m} = (x, y, z)$. Let $R = S_{\mathfrak{m}}/yzS_{\mathfrak{m}}$ and let $M = R/xyR$. The complex* $\ldots \xrightarrow{z} R \xrightarrow{y} R \xrightarrow{z} R \xrightarrow{xy} R \longrightarrow M \longrightarrow 0$ *is a minimal free resolution of M. We have $\Omega_2(M) \cong zR \cong R/(y)$. The prime ideal $\mathfrak{p} = (x, z)$ of height 1 is in $\mathrm{Supp}_R(M) \setminus \mathrm{Supp}_R(\Omega_2(M))$.*

Proposition 4.12. *Suppose that M is a non-zero finitely generated R-module with non-decreasing Betti numbers. Then, either $\mathrm{Supp}(\Omega_{2i}(M))$ is constant for all $i \geqslant 1$, or there exists $n \geqslant 0$ such that $\mathrm{Supp}(\Omega_{2i}(M)) = \mathrm{Spec}(R)$ for all $i > n$ and $\mathrm{Supp}(\Omega_{2j}(M))$ is constant for $1 \leqslant j \leqslant n$.*

Proof. Suppose $\mathrm{Supp}(\Omega_{2i}(M)) \neq \mathrm{Spec}(R)$ for for some $n \geqslant 2$. By Lemma 4.4 it follows that $\beta_{2j}(M) = \beta_{2j+1}(M)$ for all j with $0 \leqslant j < n$. From Lemma 4.10 we get that $\mathrm{Supp}(\Omega_{2j}(M))$ is constant for $1 \leqslant j \leqslant n$, and the result follows. $\square$

The following example is due to Hamid Rahmati and can be found in [11]

Example 4.13. *Let $R = k[[x, y]]/(x^2, xy)$ and $M = R/(y)$. Then we have a minimal free resolution of the form*

$$\cdots \xrightarrow{\left[\begin{smallmatrix} x & y & 0 \\ 0 & 0 & x \end{smallmatrix}\right]} R^2 \xrightarrow{[x,y]} R \xrightarrow{x} R \xrightarrow{y} R \longrightarrow M \longrightarrow 0\,.$$

Then $\dim(M) = \dim(\Omega_2(M)) = 0$ and $\dim(\Omega_i(M)) = 1 = \dim(R)$ for $i \neq 0, 2$.

The following example shows that the even syzygies can have support equal to $\mathrm{Spec}(R)$ while the odd syzygies do not.

Example 4.14. *Let* $R = [a, b, c, d, e]/(ade - bce)$*. Let* M *be the cokernel of the first map in the following matrix factorization:*

$$\cdots \xrightarrow{\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]} R^2 \xrightarrow{\left[\begin{smallmatrix} de & -be \\ -ce & ae \end{smallmatrix}\right]} R^2 \xrightarrow{\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]} M \longrightarrow 0\,.$$

Then $\operatorname{Supp}(\Omega_{2i}(M)) = \operatorname{Spec}(R)$ *and* $\operatorname{Supp}(\Omega_{2i+1}(M)) = \operatorname{Supp}(R/(e)) \neq \operatorname{Spec}(R)$ *for all* $i \geqslant 0$*.*

The following example is due to Craig Huneke and can be found in [11].

Example 4.15. *Let* $S = \mathbb{Q}[x, y, z, u, v]$ *and let* $I \subseteq S$ *be the ideal*

$$I = \langle x^2, xz, z^2, xu, zv, u^2, v^2, zu + xv + uv, yu, yv, yx - zu, yz - xv\rangle.$$

Consider $R = S/I$*, which is a* 1*-dimensional ring of depth* 0*. A computation using Macaulay2 yields that* y *is a parameter,* $(0 : y) = (u, v, z^2)$ *and* $(y) = (0 :_R (0 :_R y))$*.*

$$\cdots \longrightarrow R^3 \xrightarrow{[u \;\; v \;\; z^2]} R \xrightarrow{y} R \xrightarrow{\left[\begin{smallmatrix} u \\ v \\ z^2 \end{smallmatrix}\right]} R^3 \longrightarrow 0$$

Let M *be the cokernel of the rightmost map. Then the first and third syzygy modules of* M *are* $R/(y)$ *and* $(0 : y)$ *respectively. These are both finite length since* y *is a parameter, but all other syzygies have dimension 1.*

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