7-2008

The Kaprekar Routine

Emy Jones
University of Nebraska-Lincoln

The Kaprekar Routine

Emy Jones

In partial fulfillment of the requirements for the Master of Arts in Teaching with a Specialization in the Teaching of Middle Level Mathematics in the Department of Mathematics.

David Fowler, Advisor

July 2008

The Kaprekar Routine

There is a story about a man named "Joe" whose wife sent him to the supermarket. Joe was never a very good listener – he tended to pick up on the major points of a conversation, but never seemed to get things in the right order. So when he arrived at the store, he headed straight for the garlic section (for he was sure that his wife had mentioned garlic). However, when he got there, he stared at the $10 bill in his hand. She had asked him to buy $4.95 worth of garlic, or was it $9.54, or maybe $5.94??? He was positive that he had the correct digits, but for the life of him could not remember their order.

Knowing he would probably not choose his wife's desired amount, he immediately started to consider his predicament. How much difference could it possibly make? What would the biggest difference – or error – be that he could make? He created the largest and the smallest numbers that he could make with those digits, and then found that difference:

954 – 459 = 495

He surmised that the biggest mistake that he could make would be \$4.95. That didn't seem too bad, and he was sure that she would forgive him for such a small mistake, but now something else caught his attention. His "largest mistake" happened to consist of the same three original digits. This seemed rather odd, so on the way home he thought about this. What if his wife had given him different digits to work with?

So he tried the number 3.92. The largest error in this case could be found by taking 239(smallest possible number) from 932(the largest number).

932 – 239 = 693

At this point, Joe was already home and had much bigger problems to deal with, so he decided this incident with the digits 4, 9, and 5 was a coincidence and never pursued the matter any further.

What Joe had accidentally stumbled across was a problem typical of a special branch of mathematics called *recreational mathematics*. Recreational mathematics refers to mathematical puzzles and games which often inspire people to further study more serious mathematics.

In this particular mathematical puzzle, specifically known as the Kaprekar (kuh PREE ker) Routine, the digits 4, 5, and 9 seem to be magic numbers for the difference of three-digit numbers that are made up of these digits. According to http://mathworld.wolfram.com/KaprekarRoutine.html, the Kaprekar Routine is an algorithm discovered in 1949 by D. R. Kaprekar. To apply the Kaprekar Routine to a number n, arrange the digits in descending (n') and ascending (n'') order. Then compute the difference n' - n'' and repeat. The difference will eventually reach zero, or go into a cycle. For an example of a cycle, consider the digits 9, 6 and 3 from the example above. The greatest difference between numbers formed by these digits is

963 – 369 = 594 (Here are those magic numbers again!)

We repeat this process to get

954 – 459 = 495

Now we can see that we are in an unending cycle, i.e. repeating the algorithm will just continue to yield 495. But why??

While I had many questions at this point, my first instinct was to use problem-solving strategies, particularly the strategy entitled "solve a simpler problem." I began by considering two-digit numbers. Is there also a magical pair of digits for these differences?

Let's try the digits 2 and 3

	Largest number	32
–	<u>Smallest number</u>	<u>23</u>
		9

Then if we continue the process, we have 9 – 9 = 0.

What about the digits 4 and 7?

	Largest	74
–	<u>Smallest number</u>	<u>47</u>
		27

Continuing the process gives 72 – 27 = 45 → 54 – 45 = 9 → 9 – 9 = 0 (which looks familiar!)

<u>Conjecture 1</u>
Applying the Kaprekar Routine to a two-digit number will always result in a zero.

To set out to prove this, I used a table. After looking at the table, I had some questions about why some numbers took longer to cycle to nine, and then to zero, than others. The evidence suggests that Conjecture 1 is true.

Digits	Max Number	Min Number	Difference	Number of Times To Converge to Zero
1 and 2	21	12	9	
	9	9	0	2
1 and 3	31	13	18	
	81	18	63	
	63	36	27	
	72	27	45	
	54	45	9	
	9	9	0	6
1 and 4	41	14	27	
	72	27	45	
	54	45	9	
	9	9	0	4
1 and 5	51	15	36	
	63	36	27	
	72	27	45	
	54	45	9	
	9	9	0	5
1 and 6	61	16	45	
	54	45	9	
	9	9	0	3
1 and 7	71	17	54	
	54	45	9	
	9	9	0	3

Then I decided to study the number of steps needed to get to the magical number zero for all two digit numbers to see if there were any apparent patterns.

Number of Steps Until Difference Reaches to Zero

Digit One→ / Digit Two↓	1	2	3	4	5	6	7	8	9
0	2	6	4	5	3	3	5	4	6
1	1	2	6	4	5	3	3	5	4
2	2	1	2	6	4	5	3	3	5
3	6	2	1	2	6	4	5	3	3
4	4	6	2	1	2	6	4	5	3
5	5	4	6	2	1	2	6	4	3
6	3	5	4	6	2	1	2	6	4
7	3	3	5	4	6	2	1	2	6
8	5	3	3	5	4	6	2	1	2
9	4	5	3	3	5	4	6	2	1

I noticed that the entries along diagonal rows are equal (see the table *Number of Steps Before Difference Converges to Zero* below). Specifically, it appears that if the two digits used to form the numbers have the same difference, we might hypothesize that applying the Kaprekar Routine would result in the same number of steps until reaching zero.

I created yet another chart to portray this same information in addition to the difference of the two digits. The resulting conjecture and accompanying chart are below.

<u>Conjecture 2</u>

There is a relationship between the *difference of the digits* and the *number of steps* that it takes to reach zero.

Difference Between Digits	Digit Pairs	Number of Steps Until Zero
1	(1,2),(2,3),(3,4)	2
2	(1,3)	6
	(2,4)	6
	(5,7)	6
3	(1,4)	4
	(2,5)	4
4	(3,7)	5
	(4,8)	5
5	(7,2)	3
	(4,9)	3

This chart provides evidence that my conjecture is true.

Since I wasn't able to use the chart to find any further connections to Kaprekar's Routine, I set out to explain it algebraically. Below we have maximum and minimum numbers and their difference for the example 73 – 37 and for a general two-digit number with digits a and b (where $a \geq b$). We can use place value to rename the maximum number as $10a + 1b$ and the minimum number as $10b + 1a$.

$$10(7) + 1(3) - [10(3) + 1(7)] \qquad (10a + 1b) - (10b + 1a)$$

$$= 9(7) + -9(3) \qquad = 9a + -9b$$

$$= 9(7+-3) \qquad = 9(a - b)$$

$$=9(4) = 36$$

This shows that when applying the Kaprekar Routine to any two-digit number, the difference will always be nine multiplied by the difference of the two digits. Now, in order to show that the first conjecture is true, we can test the product of nine and each digit 0 through 8 to show that Kaprekar's Routine will lead to zero in each case. (We don't have to worry about nine because a difference of nine would imply that one of the digits is zero, which would only give you a one-digit number, since the zero would have to be in the tens place.)

9(0) = 0

9(1) = 9 – 9 = 0

9(2) = 18; 81 – 18 = 63; 63 – 36 = 27; 72 – 27 = 45; 54 – 45 = 9; 9 – 9 = 0

9(3)= 27; 72 – 27 = 45; 54 – 45 = 9; 9 – 9 = 0

9(4) = 36; 63 – 36 = 27; 72 – 27 = 45; 54 – 45 = 9; 9 – 9 = 0

9(5) = 45; 54 – 45 = 9; 9 – 9 = 0

9(6) = 54; 54 – 45 = 9; 9 – 9 = 0

9(7) = 63; 63 – 36 = 27; 72 – 27 = 45; 54 – 45 = 9; 9 – 9 = 0

9(8) = 72; 72 – 27 = 45; 54 – 45 = 9; 9 – 9 = 0

Thus the first conjecture (the difference between the maximum and minimum numbers with any two digits will eventually converge down to zero) is true.

Now let's take a look at three-digit numbers. We will consider three digit numbers for which the digits are distinct, starting with the digits 6, 2 and 1 as an example. In this case, the difference between the maximum and minimum would be

$$\begin{array}{r} 621 \\ -\ \underline{126} \\ 495 \end{array}$$

Then continuing this process, we have the same digits that Joe used. So we are now stuck in a cycle.

$$\begin{array}{r} 954 \\ -\underline{459} \\ 495 \end{array}$$

Given a three-digit number with digits *a*, *b*, and *c* (where $a > b > c$ since the digits are distinct), I can write the first step in the Kaprekar Routine as

$$100a + 10b + 1c - (100c + 10b + 1a)$$

$$= 99a + -99c$$

$$= 99(a - c)$$

This makes sense because the middle terms will always cancel out to leave you with the hundreds and ones. We also know that the $(a - c)$ must have a difference of at least two, since for this case no two of the digits a, b, and c are identical to each other.

<u>Conjecture 3:</u> When applying the Kaprekar Routine to 3-digit numbers, the difference will eventually converge to 495.

Assuming that $(a - c)$ can be any number 2 through 9, we can apply the Kaprekar Routine and study the results:

99(2) = 198; 981 - 189 = 792; 972 – 279 = 693; 963 – 369 =594; 954 – 459 = 495

99(3) = 297; 972 – 279 – 693; 963 – 369 = 594; 954 – 459 = 495

99(4) = 396; 963 – 369 = 594; 954 – 459 = 495

99(5) = 495; 954 – 459 = 495

99(6) = 594; 954 – 459 = 495

99(7) = 693; 963 – 369 = 594; 954 – 459 = 495

99(8) = 792; 972 – 279 = 693; 963 – 369 = 594; 954 – 459 = 495

99(9) = 891; 981 – 189 = 792; 972 – 279 = 693; 963 – 369 = 594; 954 – 459 = 495

(Notice this table is symmetrical due to the repetition of the same digits in the difference.)

This proves Conjecture 3 to be true.

Using this concept, we can apply it to our previous problem:

$$100(9) + 10(5) + 1(4) - [100(4) + 10(5) + 1(9)]$$

$$= 99(9) + -99(4)$$

$$=99(9 - 4)$$

$$=99(5) = 495$$

This now left me curious to see what would happen with four digits. So I applied Kaprekar's Routine to the digits 3, 4, 5, 7 and the digits 1, 2, 7, 9:

7543 – 3457 = 4086
8640 – 0468 = 8172
8721 – 1278 = 7443
7443 – 3447 = 3996
9963 – 3699 = 6264
6642 – 2466 = 4176
7641 – 1467 = 6174
(begins to cycle)

9721 – 1279 = 8842
8842 – 2488 = 6354
6543 – 3456 = 3087
8730 – 0378 = 8352
8532 – 2358 =6174
7641 – 1467 = 6174
(begins to cycle)

This was fascinating. Even though it took more steps to begin its cycle than the three-digit numbers, it did cycle, nonetheless. But now my question was "Why these digits?

Once again, I set out to explore this algebraically. Using the digits a, b, c, and d where $a \geq b \geq c \geq d$ we have

$$(1000a + 100b + 10c + d) - (1000d + 100c + 10b + 1a)$$
$$= 999a + 90b + -90c + -999d$$
$$= 999(a + -d) + 90(b + -c).$$

In summary, the difference will always be the product of 999 and the difference between the largest and smallest digits, plus 90 multiplied by the difference of the middle two digits. By factoring out 9, we could also write this as

$$9[111(a - d) + 10(b - c)].$$

Reviewing my findings for two- and three- digit numbers, I formed a conjecture for four-digit numbers. I was sure that nine had something to do with these differences, so I factored out a nine from each of the numerical differences computed above. Then I noticed a unique pattern between the digits of the differences and the digits of the numbers that we were subtracting. After factoring out a nine, the other factor always had three digits. The first and last digits were the same (the difference of a and d). The middle digit was always $(a - d) + (b-c)$.

For instance, using the above example, 6421 – 1246 = 5175; 5175 = 9(575)

- The first and last digit of the three-digit number are the same ($a - d$)
- The middle digit is the sum of (a-d) and (b-c)

This observation was verified by the previous algebraic factoring. After factoring out the 9, we have [111($a - d$) + 10($b - c$)] = 100(a-d) + 10[(a-d) + (b-c)] + 1(a-d).

The Kaprekar Routine has not been solved for all numbers, however a computer generated program has made it possible to find solutions for extremely large numbers. Some of the numbers that we have explored in this paper so far are:

1 digit –> 0, where the symbol –> means "converges to"

2 digit –> 0

3 digit –> 495 4 digit –> 6174

Now we are ready to explore five-digit numbers.

Try the digits 6, 9, 5, 3, and 2

72963 > 73953 > 63954 > 61974 > 82962 > 75933

This is a cycle of length four.

Try the digits 2,7,1,4, and 8

74943 > 62964 > 71973 > 83952 > 74943

This is another cycle of length four.

<u>Try the digits 5, 3, 9, 5, and 5</u>

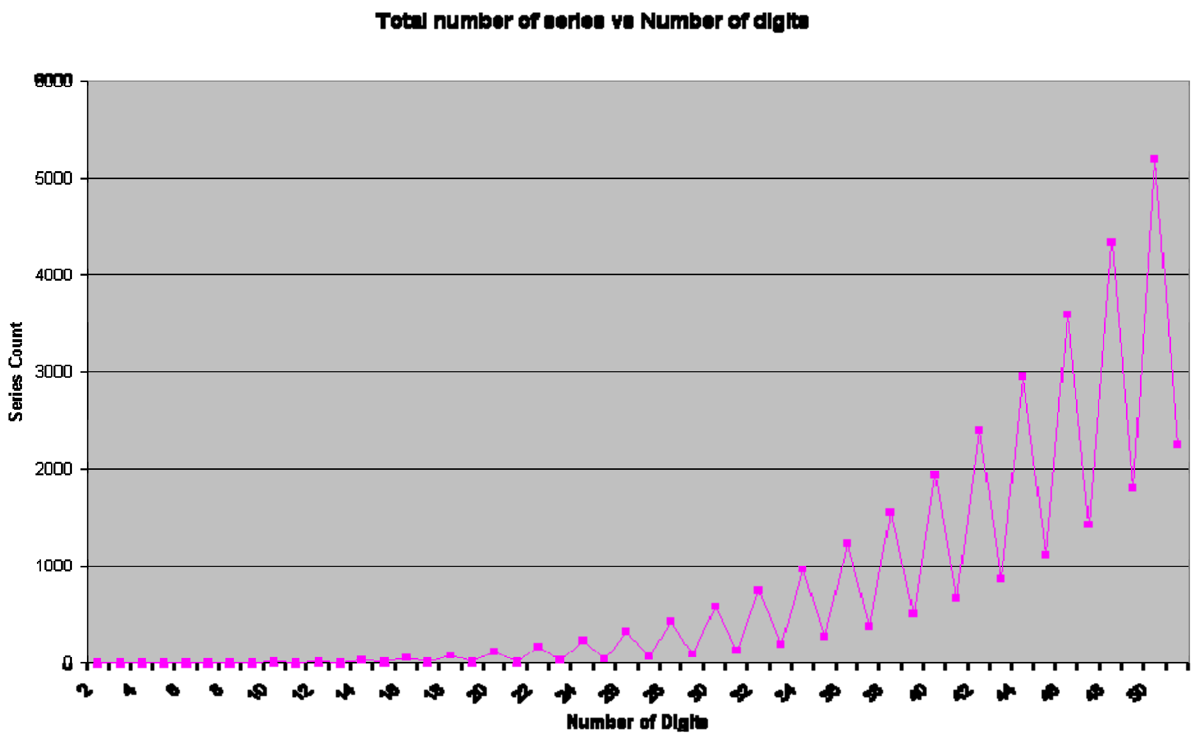
59994 > 53955 > 59994

This is a cycle of length two.

Five-digit numbers actually have three different possible endings:

- one cycle of length two, or
- two cycles of length four

When graphing the number of series plotted against the number of digits, some interesting observations can be made. The following graph was obtained at http:Kaprekar.sourceforge.net/output/numplay/numplay.php.



We can note from the above graph that there is a definite pattern between even and odd numbers of digits in the number to be worked with. Each odd number has a greater number of series that it can be cycled down to than the following even number. If you separate the “odd” and “even” graphs, the “odd” line is consistently greater than the “even” line.

References

faculty.sfasu.edu/robersonpamel/txcmj/Black-Holes-Loops.pdf "Subtractive Holes and Black Loops"

http:Kaprekar.sourceforge.net/output/numplay/numplay.php

http://mathpoint.blogspot.com/2006/12/mysterious-6174-revisited

http://mathworld.wolfram.com/KaprekarRoutine.html

http://www-groups.dcs.st-and.ac.uk/~history/Biographies/Kaprekar.html

Mathworld.com

"Subtractive Black Holes and Black Loops": faculty.sfasu.edu/robersonpamel/txcmj/Black-Holes-Loops.pdf

Wolfram.com