1920

A Set of Completely Independent Postulates for the Linear Order η^*

M. G. Gaba
University of Nebraska - Lincoln

Published in Bull. Amer. Math. Soc. 26 (1920) 158-159.

A SET OF COMPLETELY INDEPENDENT POSTULATES FOR THE LINEAR ORDER η*.

BY PROFESSOR M. G. GABA.

(Read before the American Mathematical Society September 4, 1919.)

PROFESSOR E. V. HUNTINGTON has published† three sets of completely independent postulates for serial order. His set A involves four postulates, which is as high a number of postulates as had been proved completely independent. In the present paper are given seven postulates which form a categorical and completely independent set for the linear order.

Our basis is a class of elements $[p]$ and an undefined dyadic relation (called 'less than') among the elements. If we are given two elements p_1p_2 and if the relation p_1 less than p_2 holds, we will symbolize it by $p_1 < p_2$. If the relation p_1 less than p_2 does not hold, we will symbolize it by $p_1 \nless p_2$.

Our postulates are:

I. If $p_1 < p_2$, then $p_2 \nless p_1$.

II. If $p_1 \nless p_2$, then $p_2 < p_1$; p_1, p_2 distinct.

III. If $p_1 < p_2$ and $p_2 < p_3$, then $p_1 < p_3$.

IV. If $p_1 < p_2$, then there exists a p_3 such that $p_1 < p_3$ and $p_3 < p_2$.

V. For every p_1 there exists a p_2 such that $p_2 < p_1$.

VI. For every p_1 there exists a p_2 such that $p_1 < p_2$.

VII. The class of elements $[p]$ form a denumerable set.

That the set is categorical follows from the fact that the seven postulates stated are the necessary and sufficient conditions for the linear order η. To show complete independence it will be necessary to cite 128 (2^7) examples showing all possible combinations ($\pm \pm \pm \pm \pm \pm \pm$) of our postulates holding and not holding. This is done by giving eight definitions of $<$, and sixteen sets of points such that each definition is applicable to every one of the sets, and every combination

* The linear order η is an ordered set equivalent to that of all the rational numbers.

† "Sets of completely independent postulates for serial order." This BULLETIN, March, 1917. This paper contains a bibliography of complete independence.

of definition of $<$ and set yields a different example. The eight definitions give the eight $(\pm\pm\pm)$ groups of cases for the implicational postulates I, II and III, whereas each of the sixteen sets gives all the eight cases where any particular set $(\pm\pm\pm\pm)$ of the existential postulates IV, V, VI and VII hold or do not hold.

For the independence examples, the set $[p]$ consists of points on a line such that

	IV	V	VI	VII				
1)	−	−	−	−	$p = -3$,	$-2 \leqq p \leqq 2$,	$p = 3$	and p real.
2)	−	−	−	+	$p = -3$,	$-2 \leqq p \leqq 2$,	$p = 3$	and p rational.
3)	−	−	+	−	$p = -3$,	$-2 \leqq p < 3$,		and p real.
4)	−	−	+	+	$p = -3$,	$-2 \leqq p < 3$,		and p rational.
5)	−	+	−	−		$-3 < p \leqq 2$,	$p = 3$	and p real.
6)	−	+	−	+		$-3 < p \leqq 2$,	$p = 3$	and p rational.
7)	−	+	+	−	$-3 < p \leqq 3/2$,	$2 \leqq p < 3$,		and p real.
8)	−	+	+	+	$-3 < p \leqq 3/2$,	$2 \leqq p < 3$,		and p rational.
9)	+	−	−	−		$-3 \leqq p \leqq 3$,		and p real.
10)	+	−	−	+		$-3 \leqq p \leqq 3$,		and p rational.
11)	+	−	+	−		$-3 \leqq p < 3$,		and p real.
12)	+	−	+	+		$-3 \leqq p < 3$,		and p rational.
13)	+	+	−	−		$-3 < p \leqq 3$,		and p real.
14)	+	+	−	+		$-3 < p \leqq 3$,		and p rational.
15)	+	+	+	−		$-3 < p < 3$,		and p real.
16)	+	+	+	+		$-3 < p < 3$,		and p rational.

A definition of $<$ requires that whenever we are given two numbers of our set p_1p_2 we have a criterion whereby we can tell whether the relation $p_1 < p_2$ holds or does not hold. In all the eight definitions of $<$ the relation holds for any pair of numbers p_1p_2 if it holds in the case of ordinary linear order,

	I	II	III	
1′)	−	−	−	except $0 \nless 1$, $-1 < -2$, $0 < -1$ and $0 < -2$.
2′)	−	−	+	except $1 < -1$, $1 < 0$, $0 < -1$, $p_1 < -1$, $p_1 < 0$, $p_1 < 1$, $-1 \nless p_2$, $0 \nless p_2$ and $1 \nless p_2$; $p_1 \neq -1, 0, 1$; $p_2 \neq -1, 0, 1$.
3′)	−	+	−	except $0 < -m/2^n$, n positive integer and m odd positive integer.
4′)	−	+	+	except $p_1 < -1$, $p_1 < 0$, $p_1 < 1$, $-1 \nless p_2$, $0 \nless p_2$, and $1 \nless p_2$; $p_1 \neq 3$; $p_2 \neq -1, 0, 1$.
5′)	+	−	−	and $p_2 - p_1 < 1/3$.
6′)	+	−	+	and $p_2 - p_1 = m/2^n$, n positive integer and m odd integer.
7′)	+	+	−	except $0 < -m/2^n$ and $-m/2^n \nless 0$, n positive integer and m odd positive integer.
8′)	+	+	+	with no exceptions.

To illustrate: The independence example where postulates II, III, V, and VII hold and postulates I, IV and VI do not hold $(-++-+-+)$ is definition 4′ used on set 6.

UNIVERSITY OF NEBRASKA.