7-2006

Comparing Infinite Sets

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Comparing Infinite Sets
Expository Paper

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In partial fulfillment of the requirements for the Master of Arts in Teaching with a
Specialization in the Teaching of Middle Level Mathematics
in the Department of Mathematics.
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July 2006

I have been assigned to explore the theorem stating that there is no largest (infinite) set as established and proven by Georg Cantor. To do this I need to start by defining what it means to say that a set is infinite. This can be quite difficult because the tendency might be to say that a set is infinite if it is not finite, and I don't believe that grants us the clarity of definition we are looking for. When trying to understand the size of a given set, the number of objects (elements) in the set, we may not be able to count them as the total might be quite large. So we look to pair them evenly with objects of other sets or proper subsets of themselves: this is known as finding a one-to-one correspondence.

A set *A* is finite if it is impossible to have a one-to-one correspondence between the set *A* and a proper subset of the set *A*. (This is essentially the Pigeonhole Principle.) For example, the set {A, B, C, …Y, Z} is finite because we cannot pair every element in the alphabet with the proper subset consisting of the alphabet not including Z, the set = {A, B, C,…Y}. We run out of elements to pair with the final letter.

Building on this definition, a set *A* is infinite if it IS possible to have a one-to-one correspondence between the set *A* and a proper subset of *A*. For example, the natural numbers are infinite because they can be paired in a one-to-one correspondence with a proper subset of themselves (i.e. the natural numbers not including 1). There is a function $f(x)=x+1$ that defines such a function from the natural numbers to a proper subset of the natural numbers. The moment we mention a particular number from the set of natural numbers, we also know which number it corresponds with in the proper subset.

In order to address the question of whether there is a largest infinite set, we must establish a way to compare sizes of infinite sets. We don't state the size of a set by telling the total number of elements in the set, instead we give the *cardinality of the set*. The cardinality of a set means the "number" of things in the set, with the understanding that the set may contain infinitely many things. To compare sets we concern ourselves more with the question of whether or not two sets have the *same* cardinality.

With finite (small) sets it can be easily demonstrated that two sets have the same cardinality. For example, when comparing set X={*A*, *B*, *C*, *D*, *E*,} with the set Y={*1*, *2*, *3*, *4*,*5*} we can show the one-to-one correspondence by pairing elements from each set.

Set X		***Set Y***
A	↔	*1*
B	↔	*2*
C	↔	*3*
D	↔	*4*
E	↔	*5*

What about infinite sets? The natural numbers and the integers are two sets that are each infinite sets. We can show that they have the same cardinality by using this idea of one-to-one correspondence, but we may have to be creative about organizing our pairings.

Natural numbers		***Integers***
1	↔	0
2	↔	1
3	↔	-1
4	↔	2
5	↔	-2
6	↔	3
7	↔	-3
…		…
2*n* (even numbers)	↔	*n*
2*n*+1 (odd numbers)	↔	-*n*
…		…

This might seem uncomfortable since many of us learn about our number system using more of a concept of containment (i.e. we define the integers in terms of the natural numbers). We often use a Venn Diagram to illustrate these relationships. The “containment” idea does not address size of sets, only relationships of sets. We also define the rational numbers in terms of the integers, yet this is another infinitely large number set. But recall that by definition, a set *A* (like the rationals) is infinite if it IS possible to have a one-to-one correspondence between the set *A* (the rationals) and a proper subset of *A* (like the naturals). We can show that they have the same cardinality by using this idea of one-to-one correspondence, but again we have to be creative about organizing our pairings. Recall that rational numbers are numbers that can be expressed as a quotient of integers, so we can build a list of all the rational numbers in a way that we are certain we have included all of the rational numbers. One way to write down the rational numbers is this way:

							…	**…**	**…**	**…**	**…**	
							1/5	**2/5**	**3/5**	**4/5**	**5/5**	**…**
							1/4	**2/4**	**3/4**	**4/4**	**5/4**	**…**
							1/3	**2/3**	**3/3**	**4/3**	**5/3**	**…**
							1/2	**2/2**	**3/2**	**4/2**	**5/2**	**…**
							1/1	**2/1**	**3/1**	**4/1**	**5/1**	**…**
						0						
…	**-5/1**	**-4/1**	**-3/1**	**-2/1**	**-1/1**							
…	**-5/2**	**-4/2**	**-3/2**	**-2/2**	**-1/2**							
…	**-5/3**	**-4/3**	**-3/3**	**-2/3**	**-1/3**							
…	**-5/4**	**-4/4**	**-3/4**	**-2/4**	**-1/4**							
…	**-5/5**	**-4/5**	**-3/5**	**-2/5**	**-1/5**							
	…	**…**	**…**	**…**	**…**							

We can use this listing to help build our table of one-to-one correspondences. If we can write a set out as an ordered/organized list that includes every element of the set, then we can make a one-to-one correspondence with the natural numbers. Therefore, the rational numbers have the same cardinality (hence same size) as the natural numbers.

Natural Numbers	**Rational Numbers**
1	**0**
2	**1/1**
3	**-1/1**
4	**2/1**
5	**1/2**
6	**-2/1**
7	**-1/2**
8	**3/1**
9	**3/2**
10	**2/3**
11	**1/3**
12	**-3/1**
…	**…**

Next it is logical to address whether the real numbers have the same cardinality as the natural numbers. We might want to ask ourselves if the Real numbers are "countable" – could they be written in an ordered list? Would we know that we had included all the real numbers in our list? This is where Georg Cantor can show an argument that the real numbers are larger than the natural numbers, i.e. they do not have the same cardinality, as they cannot be paired in a one-to-one correspondence.

Georg Cantor, born March 3, 1845, put forth the modern theory on infinite sets that revolutionized almost every mathematics field. However, his new ideas also created many dissenters and made him one of the most assailed mathematicians in history. Georg's father saw his mathematical giftedness when he was a child and pushed him in that direction, however he encouraged a more profitable field of engineering. Georg tried this path, but eventually convinced his father to let him study mathematics. After receiving his doctorate in 1867, he was unable to find good employment and was forced to accept a position as an unpaid lecturer and later as an assistant professor at the Backwater University of Halle. In 1874 Cantor published his first paper on the theory of sets. In a series of papers from 1874 to 1897, he was able to prove among other things that the set of integers had an equal number of members as the set of even numbers, squares, cubes, and roots to equations; that the number of points in a line segment is equal to the number of points in an infinite line, a plane and all mathematical space; and that the number of transcendental numbers, values such as pi and e that can never be the solution to any algebraic equation, were much larger than the number of integers. Before in mathematics, infinity had been a taboo subject. Previously, Gauss had stated that infinity should only be used as "a way of speaking" and not as a mathematical value. Most mathematicians followed his advice and stayed away. However, Cantor would not leave it alone. He considered infinite sets not as merely going on forever but as completed entities, that is having an actual though infinite number of members. He called these actual infinite numbers transfinite numbers. By considering the infinite sets with a transfinite number of members, Cantor was able to come up with his amazing discoveries. For his work, he was promoted to full professorship in 1879. However, his new ideas also gained him numerous enemies. Many mathematicians just would not accept his groundbreaking ideas that shattered their safe world of mathematics. Leopold Kronecker was a firm believer that the only numbers were integers and that negatives, fractions, imaginary and especially irrational numbers had no business in mathematics. He simply could not handle "actual infinity." Using his prestige as a professor at the University of Berlin, he did all he could to suppress Cantor's ideas and ruin his life. Among other things, he delayed or suppressed completely Cantor's and his followers' publications, raged both written and verbal personal attacks against him, belittled his ideas in front of his students and blocked Cantor's life ambition of gaining a position at the University of Berlin. Stuck in a third-rate institution, stripped of well-deserved recognition for his work and under constant attack by Kronecker, he suffered the first of many nervous breakdowns in 1884. The rest of his life was spent in and out of mental institutions and his work nearly ceased completely. Much too late for him to really enjoy it, his theory finally began to gain recognition by the turn of the century. In 1904, he was awarded a medal by the Royal Society of London and was made a member of both the London Mathematical Society and the Society of Sciences in Gottingen. He died in a mental institution on January 6, 1918.

To understand Cantor's argument let's first consider the Dodge Ball game played in an Experiment, Conjecture and Reason (ECR) course. The game involved two players and the goal of player two (Dodger) was to create a list of six O's and X's different from the six lists generated by player one (Matcher).

Dodge Ball

Player 1(Matcher)

Row 1	X	X	X	O	O	O
Row 2	O	O	X	X	O	O
Row 3	O	X	O	X	O	X
Row 4	O	X	X	O	X	X
Row 5	O	X	X	X	X	X
Row 6	O	X	X	X	O	???

Player 2 (Dodger)

Play 1	*Play 2*	*Play 3*	*Play 4*	*Play 5*	*Play 6*
O	X	X	X	O	???

If you were player one (the matcher), what would you place in that last spot? If you were player two, how will you play your last spot to guarantee that you defeat(dodge) player one? If you recall, the strategy for Dodger (player two) was to consider only the letter in the box for Matcher's (player one) row that corresponded with the play number. So, if you wanted to guarantee that your list would be different than Matcher's row four list, than you look at his fourth letter in that fourth row, and on play 4 "dodge" with the opposite letter. Notice the connection between Dodger's list and the letters in the diagonal (they are highlighted). The Dodger can always generate a list distinctly different from any of the Matcher's lists.

Cantor used this same kind of strategy when making what we call *his diagonalization argument*. For every pairing of the natural numbers with the real numbers, Cantor showed how he could describe a real number that was not a part of the pairing. Recall that real numbers cannot all be written as a quotient of integers, and they include numbers like pi and the square root of two. All reals can be expressed as decimals carried out infinitely. Starting with the assumption that a pairing exists, Cantor could find a number that was missing from the list of reals. Consider the beginnings of a pairing of natural numbers to real numbers.

Natural numbers	Real numbers
1 ⟷	0.7893294337…
2 ⟷	0.98765432111…
3 ⟷	0.0345543345…
4 ⟷	0.3333333333…
5 ⟷	0.85271969922…
6 ⟷	0.7500000000…
…	…

For each element of the natural numbers there corresponds a real number in decimal form, and we assume the correspondence is one-to-one. Will this also mean that for every possible real number, the real number corresponds back to a natural number? Cantor used the same kind of strategy that we did for the Dodge Ball game to show that there was always an extra real number that didn't have a natural number paired with it. Consider the numbers highlighted. They form sort of a diagonal of place values and I could write down the decimal formed by them: 0.784310… . Cantor's argument follows that we can write a real number that we know for certain does not exist in the list of reals corresponding to the list of naturals by changing the digit at each place value and making it different than those diagonal digits. To determine this "missing" real number, we establish some convenient guidelines. At each digit we are going to change the digit to one of two choices. If the digit is a 3, then change it to 8, but if the digit is not a 3, then change it to a three. This "choosing" idea makes some other ideas that I will discuss later a little easier to understand and relate to the diagonalization argument. This is somewhat like our combinatorics proofs in Discrete math when we had to consider whether a number was "in" or "out" of a chosen set. So, finally, our missing number would be: 0.333833… . How do I really know that this number is not on the list? Well, if you were Dodger (player 2) in Dodge Ball how did you know the series of O's and X's you generated was not the same as any of the series that Matcher (player 1) listed? In this case the decimal corresponding to the natural number 1 has a 7 in the tens place, so our missing number can't be the one corresponding to that natural number. The decimal corresponding to the natural number 2 has a 9 in the second digits (or hundredths) place, so the missing number can't be the same. Go a little further down to the natural number 6 and notice that the digit in the 6^{th} place after the decimal point is a zero, not a three like our missing number, so again no match. This continues proving that this "missing" number truly is missing from the list of reals said to be in correspondence with the naturals. Since this number is not on the list, but we know it exists as a real number, then we have leftover real numbers that are not paired up in a one-to-one correspondence with the naturals. Therefore there must be more real numbers than natural numbers. A more general proof of Cantor's Theorem follows.

> ***Cantor's Diagonalization Argument***
>
> Suppose that the infinity of decimal numbers between zero and one is the same as the infinity of counting numbers. Then all the decimal numbers can be denumerated in a list.

1 $d_1 = 0.d_{11}d_{12}d_{13}d_{14}$ ....

2 $d_2 = 0.d_{21}d_{22}d_{23}d_{24}$ ....

3 $d_3 = 0.d_{31}d_{32}d_{33}d_{34}$ ....

4 $d_4 = 0.d_{41}d_{42}d_{43}d_{44}$ ....

.
.
.

n $d_n = 0.d_{n1}d_{n2}d_{n3}d_{n4}$ ....

.
.
.

Consider the decimal number $x = 0.\ x_1x_2x_3x_4x_5$ .... , where x_1 is any digit other than d_{11}; x_2 is different from d_{22}; x_3 is not equal to d_{33}; x_4 is not d_{44}; and so on. Now, x is a decimal number, and x is less than one, so it must be in our list. But where? x can't be first, since x's first digit differs from d_1's first digit. x can't be second in the list, because x and d_2 have different hundredths place digits. In general, x is not equal to d_n, since their nth digits are not the same.

x is nowhere to be found in the list. In other words, we have exhibited a decimal number that ought to be in the list but isn't. No matter how we try to list the decimal numbers, at least one will be left out. Therefore, "listing" the decimal numbers is impossible, so the infinity of decimal numbers is greater than the infinity of counting numbers.

Cantor's Theorem states: There are more real numbers than natural numbers.

Now that we have established that the infinity of the real numbers is greater than the infinity of the natural numbers, we are led to consider whether or not there is a largest infinity. Is there an infinity between the cardinality of the natural numbers and the cardinality of the real numbers? We will come back to this after laying some more groundwork about sets.

First we will describe subsets. These are sets that include some, all, or none of the elements from the original set. Also recall that the empty set is the set containing no elements – nothing. Here are few examples of sets and their subsets:

Set: {apples, oranges}

Subsets: { }, {apples}, {oranges}, {apples, oranges}

Set: {cohort1, cohort2, cohort3}

Subsets: { }, {cohort1}, {cohort2}, {cohort3}, {cohort1, cohort2}, {cohort1, cohort3}, {cohort2, cohort3}, {cohort1, cohort2, cohort3}

Set: {M, A, T, H}

Subsets: { }, {M}, {A}, {T}, {H}, {M, A}, {M, T}, {M, H}, {A, T}, {A, H}, {H, T}, {M, A, T}, {M, A, H}, {A, H, T}, {H, T, M}, {M, A, T, H}

We begin to notice a pattern when looking at sizes of sets and numbers of subsets for that set. The numbers of subsets are powers of two.

Number of *Elements* in the Set	**Number of *Subsets* for the Set**
{apples, oranges} = 2	$2^2 = 4$
{cohort1, cohort2, cohort3}= 3	$2^3 = 8$
{M, A, T, H}= 4	$2^4 = 16$

This leads us to conjecture that the number of subsets for any given set *S* containing *seven* elements is 2^7=*128 subsets*. We can prove this by using that idea of an element either being in the set or not in the set. Consider our set of {apples, oranges}. If we add one more element to that set and create the set{apples, oranges, bananas}, then we would of course expect more subsets. We should be able to build on the subsets we have already listed for the set {apples, oranges}. Each subset we already listed consists of subsets without the bananas. We must list the subsets with the bananas. There is a “doubling” of subsets occurring. Adding one more element to a set doubles its number of subsets. Therefore we have proven our conjecture.

Previous subsets (bananas not in)	New subsets (bananas in)	Total subsets
{ }	{bananas}	2
{apples}	{apples, bananas}	2
{oranges}	{oranges, bananas}	2
{apples, oranges}	{apples, oranges, bananas}	2

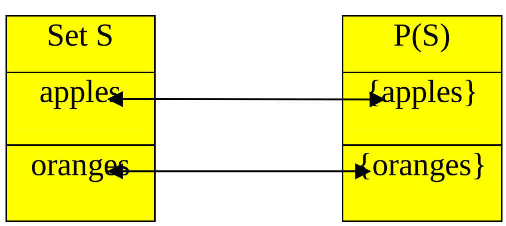
The Subset Count: A set containing *n* elements has 2^n subsets.

When we refer to the elements of a given set, we must realize that an element is an object in the set. There is a set of all possible subsets for a given set *S* that is called the *Power Set of S*, and is written *P*(*S*) and we use big brackets to notate them. The elements of the Power Set are all subsets generated for set *S*.

Set S = {apples, oranges}

P(S) **=** **{**{ }, {apples}, {oranges}, {apples, oranges}**}**

Notice how the size (cardinality) of set S compares to the cardinality of P(S). In this example it is obvious that the Power Set is larger than the Set. We could try to establish a one-to-one correspondence between these two sets, but we would of course have some subsets left over. It may help to see this.



Set S is out of elements, but *P(S)* still has subset elements left over (i.e. { }, {apples, oranges}). So, it seems to follow that for any finite set its power set will have a greater cardinality. Okay, but what about infinite sets and their power sets? What about power sets of power sets, denoted *P(P((S)).* These power sets get REALLY BIG. It should seem rather intuitive that a set's power set cardinality would be greater than the set's cardinality, but Cantor proved it. He again used the concept of the diagonalization argument and the idea that when creating a one-to-one correspondence there will be leftovers. Cantor showed that he could construct an element of P(S) that is not paired with any element of S. We're kind of back to Dodge Ball strategy. Suppose there is a correspondence between a set S and its power set, P(S). For each $x \in S$, we ask whether *x* is in the set it is matched with or not. If *x* is not in the corresponding subset, then it gets sort of put aside for consideration later. Then if the next element is *y* and corresponds with another subset of S, we will look to see if *y* was in the subset. If y is not in the subset, then again we set it aside for later. At each stage we're asking ourselves whether the element is "in" the subset or "not in" the subset that it is being matched with. Cantor set these elements aside creating a leftover subset that did not correspond with an element in the original set at any step in the pairing. Therefore, the size of a sets power set will always be greater than the cardinality of the set.

CANTOR'S POWER SET THEOREM:
Let S be a set (finite or infinite). Then the cardinality of the power set of S, P(S), is strictly greater than the cardinality of S.

A complete yet brief proof of this theorem might amaze you – it certainly amazed me.

Consider: $f: S \to P(S)$. *Then* $\{x \in S | x \notin f(x)\} \notin f(S)$. *Q.E.D.*

Since we know that the natural numbers are an infinite set, and it's power set is larger, then we are logically led to question whether there is a largest infinity? How big is P(P(P(P(S))))? Is there an order of infinities from smallest infinity to largest? Is there a largest infinite set? Are there infinitely many infinities? When you ponder such things, is it a wonder that Cantor spent the last part of his lifetime in an insane asylum?
In order to answer these questions we would need to have a sense of the size of these infinities that would allow us to order them from least to greatest. This created a new problem. We know that the natural numbers are an infinite set. We also know that the

power set of the natural numbers is an infinite set and that it has greater cardinality than the set of naturals. Is there a set of infinite size somewhere in between these two? In 1874 Cantor discovered that there is more than one level of infinity. The lowest level is called "countable infinity" and higher levels are called "uncountable infinities." The natural numbers are an example of a countably infinite set and the real numbers are an example of an uncountably infinite set. In 1877 Cantor hypothesized that the number of real numbers is the next level of infinity above countable infinity. Since the real numbers are used to represent a linear continuum, this hypothesis is called "the Continuum Hypothesis".

> **The Continuum Hypothesis:** *There is no cardinality between the cardinality of the set of natural numbers and the cardinality of the set of real numbers (sometimes referred to as the continuum).*

Some might be curious enough to want to prove this or have it proven to them. We would need to prove the hypothesis by demonstrating that there truly is no set with a cardinality between these two, or we would have to disprove the hypothesis by finding a set whose cardinality is clearly in between them. Sometimes we come across hypotheses that can neither be proven nor disproven. In 1940 Kurt Godel proved that it is impossible to disprove the Continuum Hypothesis; then in 1963 Paul Cohen proved that it is impossible to prove the Continuum Hypothesis. It is a statement that is neither true or false!
An example problem helped me see this kind of challenge.

> ***Russell's barber's puzzle****: In a certain village there is one male barber who shaves all those men, and only those men, who do not shave themselves. Does the barber shave himself?*
> Well suppose that all men cut their own hair are named with capital letters. All capital lettered names are thereby in the set of all men who cut their own hair {A, B, C, D…}. Now let's suppose that all men who do not cut their own hair are jammed with lowercase letters. The lowercase lettered names are thereby in the set of all men who have their hair cut by the barber {a, b, c, d…}.
> If the barber is a member of the first set (capital letters), he cuts his own hair. But, wait! If he cuts his own hair, it follows that his hair is cut by the barber. That makes him a member of the second set (lowercase letters). Alas! Another conundrum. The barber explicitly states that he does not cut the hair of anyone who cuts his own hair. So if he cuts his own hair (has his hair cut by the barber – himself), He, the barber, will not cut his own hair. Are you dizzy yet?

This example is called “Russell’s” because of an insight by Bertrand Russell in 1901 that became known as *Russell’s Paradox.* The paradox arises within naive set theory by considering the set of all sets that are not members of themselves. Such a set appears to be a member of itself if and only if it is not a member of itself, hence the paradox. Russell wanted to know if there was a “set of all sets”. Russell wrote to Gottlob Frege with news of his paradox on June 16, 1902. The paradox was of significance to Frege's logical work since, in effect, it showed that the axioms Frege was using to formalize his logic were inconsistent. Specifically, Frege's Rule V, which states that two sets are equal if and only if their corresponding functions coincide in values for all possible arguments, requires that an expression such as $f(x)$ be considered both a function of the argument x and a function of the argument f. In effect, it was this ambiguity that allowed Russell to construct R in such a way that it could both be and not be a member of itself. Russell's paradox is the most famous of the logical or set-theoretical paradoxes.

This paradox shows us that some ideas have limitations, and in the case of infinite set cardinality we approach power sets that are so enormous that they are just too hard to define within those limitations. Cantor’s Power set theorem convinced Russell that there was no such thing as a “Set of all sets” within our limited idea of sets.

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