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The Square Root of i

Tiffany Lothrop
University of Nebraska-Lincoln

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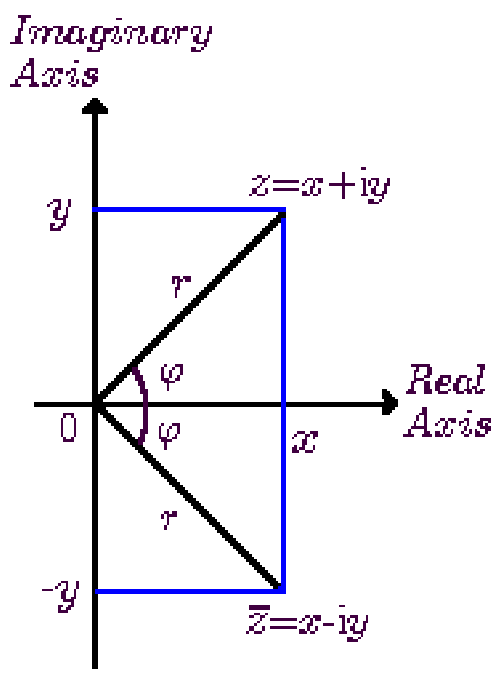
While Girolamo Cardano was working on solving cubic and quadratic equations in 1539, he encountered some formulas that involved square roots of negative numbers. In 1545 Cardano published *Ars Magna*, where he presents the first recorded calculations that involve complex numbers. Then in 1572, Rafael Bombelli published the first three parts of his *Algebra*. He is known as the inventor of complex numbers, because he identifies some rules for working with them. Bombelli also shows how complex numbers are very important and useful. From Bombelli's list of rules for adding, subtracting and multiplying the complex numbers, he was able to analyze the cubic equations that Cardano was trying to solve in his paper *Ars Magna*. Bombelli was able to use his rules for operations with complex numbers to solve the cubic equations that produced an expression that contained a square root of a negative number.

The next big discovery in complex numbers was made by Abraham de Moivre. De Moivre published papers in 1707 and in 1722 where he used trigonometric functions to represent complex numbers. He developed the representation [r (cos x + i sin x)]. This was a very important in the development and theory of complex numbers. However, it was Leonhard Euler's publications that really brought complex numbers to the forefront. In 1748, Euler published *Analysis of the Infinite*. Mathematical analysis is said to have started with Euler where he used Bernoulli's ideas of functions and refines them. From this mathematical analysis Euler based his study on functions and introduced the formulas $e^{\pi i} = -1$ and $e^{ix} = \cos x + i \sin x$. Then in 1751, Euler published his theory of logarithms of complex numbers and introduced the symbol i to represent $\sqrt{-1}$. Still, most mathematicians of that day rejected the notion of complex numbers, despite Euler's publications.

In 1799, Caspar Wessel published a paper giving a geometrical representation of complex numbers. This was not a well know paper and was not even published in English until 1999, 200 years after its first publication. Jean Robert Argand rediscovered Wessel's work in 1806 with his publication of the Argand diagram. In this geometrical representation of complex numbers, Argand interpreted i as a rotation of 90°.

Complex numbers are of the form $x + iy$ where x and y are real numbers and i is the imaginary part. The diagram below is the Argand diagram that shows a graphical representation of a complex number. The x-axis is the real number line and an ordered pair that represents a point on this line has coordinates (x, 0) or the complex number $x+0i$. The y-axis is the imaginary axis and an ordered pair that represents a point on this line has coordinates (0, y) or the complex number $0+iy$, a pure imaginary number.

This graph gives us a two-dimensional view of a complex number. Using the complex



plane, or Argand diagram, de Moivre's formula and Euler's formula we now have

$z = x + iy = r(\cos\alpha + i\sin\alpha) = re^{i\alpha}$, where $i = 0 + iy = e^{\frac{i\pi}{2}}$.

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In this equation, multiplication by i results in a counter-clockwise rotation of 90° about the origin, or $\frac{\pi}{2}$ radians. So, when we look at the geometric representation of $i^2 = -1$, it is shown as two 90° turns (180°) or π radians.

Complex Numbers

a + bi where $a, b \in \boldsymbol{R}$

Ex. 3 + 2i, 7i, 5 + 0i,

$-i\sqrt{2}$, 5, $\sqrt[3]{5}$, π, $\frac{-2}{7}$, $5.\overline{67}$

Real Numbers

a + bi, b = 0

Ex. 5, $\sqrt[3]{5}$, π, $\frac{-2}{7}$,

$5.\overline{67}$, $\sqrt{2}$

Imaginary Numbers

a + bi, b $\neq$ 0

Ex. 3 + 2i, $\frac{2}{3} - \frac{1}{2}i$, $-i\sqrt{2}$

Pure Imaginary Numbers

a + bi, a = 0, b $\neq$ 0

Ex. 7i, $-i\sqrt{2}$

Rational Numbers

$\frac{a}{b}$, $a, b \in R, b \neq 0$

Ex. 5, $\frac{-2}{7}$, $5.\overline{67}$

Irrational Numbers

Infinite & non-repeating decimals

Ex. $\sqrt[3]{5}$, π, $\sqrt{2}$

Now using our knowledge of complex and imaginary numbers where

$i = \sqrt{-1}, i^2 = -1$ we can look at basic operations with them.

Finding square roots of some negative numbers:

a) $\sqrt{-5}$

$= \sqrt{-1}\sqrt{5}$

$= i\sqrt{5}$

b) $\sqrt{-25}$

$= \sqrt{-1}\sqrt{25}$

= 5i

c) $\sqrt{-50}$

$= \sqrt{-1}\sqrt{25}\sqrt{2}$

$= 5i\sqrt{2}$

d) $\sqrt{-16} - \sqrt{-49}$

$= \sqrt{-1}\sqrt{16} - \sqrt{-1}\sqrt{49}$

= 4i – 7i

= -3i

e) $\sqrt{-2} + \sqrt{-18}$

$= \sqrt{-1}\cdot\sqrt{2} + \sqrt{-1}\cdot\sqrt{9}\cdot\sqrt{2}$

$= i\sqrt{2} + 3i\sqrt{2}$

$= (1+3)i\sqrt{2}$

$= 4i\sqrt{2}$

f) $\sqrt{-4}\cdot\sqrt{-25}$

$= \sqrt{-1}\cdot\sqrt{4}\cdot\sqrt{-1}\cdot\sqrt{25}$

$= (\sqrt{-1})^2\cdot 2\cdot 5$

= -10

g) $i\sqrt{2}\cdot i\sqrt{3}$

$= i^2\cdot\sqrt{2\cdot 3}$

$= -1\cdot\sqrt{6}$

$= -\sqrt{6}$

h) $\dfrac{2}{3i}$

$= \dfrac{2\cdot i}{3i\cdot i}$

$= \dfrac{2i}{3i^2}$

$= \dfrac{2i}{3(-1)}$

$= \dfrac{-2}{3}i$

i) $\dfrac{6}{\sqrt{-2}}$

$= \dfrac{6}{i\sqrt{2}}$

$= \dfrac{6}{i\sqrt{2}}\cdot\dfrac{i\sqrt{2}}{i\sqrt{2}}$

$= \dfrac{6i\sqrt{2}}{2i^2}$

$= \dfrac{6i\sqrt{2}}{2(-1)}$

$= -3i\sqrt{2}$

j) $2x^2 + 19 = 3$

$2x^2 + 19 - 19 = 3 - 19$

$\dfrac{2x^2}{2} = \dfrac{-16}{2}$

$\sqrt{x^2} = \pm\sqrt{-8}$

$x = \pm\sqrt{-1}\sqrt{4}\sqrt{2}$

$x = \pm 2i\sqrt{2}$

Some basic operations with complex numbers:

Sum of Complex Numbers

$(a + b\,i) + (c + d\,i)$

$= (a + c) + i(b + d)$

Ex. $(3 + 6i) + (4 + 2i)$

$= (3 + 4) + i(6 + 2)$

$= 7 + 8i$

Ex. $(3 + 6i) + (4 - 2i)$

$(3 + 6i) + (4 + -2i)$

$= (3 + 4) + [6 + (-2)]\,i$

$= 7 + 4i$

Differences of Complex Numbers

$(a + bi) - (c + d\,i)$

$= (a - c) + i(b - d)$

Ex. $(3 + 6i) - (4 + 2i)$

$= (3 - 4) + i(6 - 2)$

$= -1 + 4i$

Ex. $(3 + 6i) - (4 - 2i)$

$(3 + 6i) - (4 + -2i)$

$= (3 - 4) + [6 - (-2)]\,i$

$= -1 + 8i$

Product of Complex Numbers

$(a + bi)(c + d\,i)$

$= ac + bd\,i^2 + ad\,i + cbi$

$= ac + (-1)bd + i(ad + bc)$

$= ac - bd + i(ad + bc)$

Quotient of Complex Numbers

$$\frac{a+bi}{c+di}$$

$$= \frac{(a+bi)(c-di)}{(c+di)(c-di)}$$

$$= \frac{ac - bdi^2 - adi + cbi}{c^2 - d^2i^2}$$

$$= \frac{ac - (-1)bd + i(cb - ad)}{c^2 - (-1)d^2}$$

$$= \frac{ac + bd + i(cb - ad)}{c^2 + d^2}$$

Product of Complex Numbers

Ex. $(6 + 4i)(2 - 5i)$

$=(6)(2) + (4i)(-5i) + (6)(-5i) +(2)(4i)$

$= 12 - 20i^2 - 30i + 8i$

$= 12 - 20(-1) - 22i$

$= 12 + 20 - 22i$

$= 32 - 22i$

Quotient of Complex Numbers

Ex. $\dfrac{6+4i}{2-5i}$

$$= \frac{(6+4i)(2+5i)}{(2-5i)(2+5i)}$$

$$= \frac{6\cdot 2 + 4\cdot 5i^2 + 6\cdot 5i + 4i\cdot 2}{2\cdot 2 - (-5)5i^2}$$

$$= \frac{12 + 20(-1) + 30i + 8i}{4 + 25(-1)}$$

$$= \frac{-8}{-21} + \frac{38}{-21}i$$

$$= \frac{8}{21} + -1\frac{17}{21}i$$

The algebra of complex numbers involves treating i as a number and using the basic number and operation properties (such as the distributive, associative, and commutative properties) to rewrite the expression in the form $a + b\,i$. We can use the information about complex numbers and operations, along with formulas such as de Moivre's formula and Euler's formula, to study $\sqrt{i}$.

Evaluating $\sqrt{i}$.

We begin by reiterating that any complex number, z, can be written as $z = a + b\,i$. We also note that the complex numbers we are looking for will satisfy the equation $z^4 + 1 = 0$, which, by the Fundamental Theorem of Algebra, has four solutions (two of which satisfy $z^2 - i = 0$, two which satisfy $z^2 + i = 0$). Then,

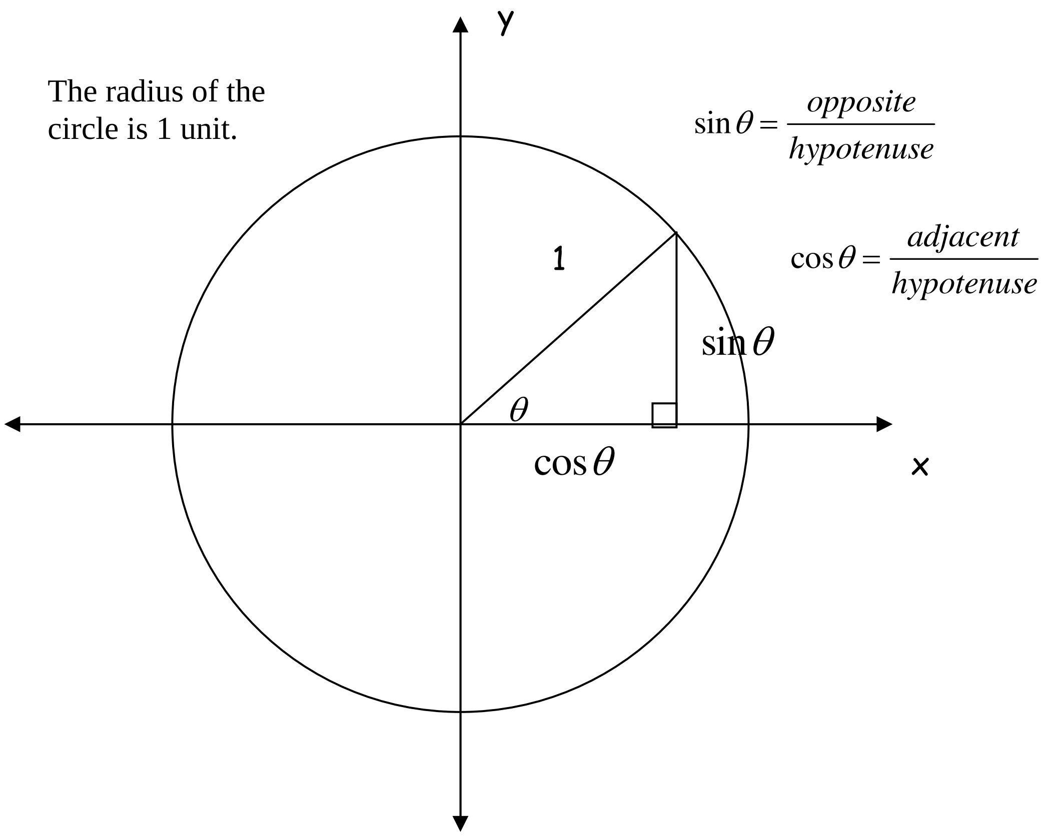
applying Euler's formula for writing complex numbers, we can write z as:

$$z = re^{i(n\theta)} = (e^{i\theta})^n = \cos(n\theta) + i\sin(n\theta) = (\cos\theta + i\sin\theta)^n$$

From the first part of Euler's formula we write $i = e^{\frac{i\pi}{2}}$. Then, proceeding formally

(and admittedly abusing some notation) we take the square root of both sides:

$$\begin{aligned}\sqrt{i} &= \sqrt{e^{\frac{i\pi}{2}}} \\ &= e^{\frac{i\pi}{2}\bullet\frac{1}{2}} \\ &= e^{\frac{i\pi}{4}} \\ &= \cos(\frac{1}{4}\pi) + i\sin(\frac{1}{4}\pi)\end{aligned}$$



Thus we need to evaluate the sine and cosine of π/4. Since $\frac{\pi}{4} = 45^{\circ}$, the reference triangle for our calculations is an isosceles, 45°-45°-90° triangle. To find the length of each side of the triangle we use Pythagorean's Theorem $a^2 + b^2 = c^2$; the hypotenuse is 1 unit and we let the legs each be of length x units.

Then solving for x we have: $x^2 + x^2 = 1^2$

$$\frac{2x^2}{2} = \frac{1}{2}$$

$$x^2 = \frac{1}{2}$$

$$\sqrt{x^2} = \sqrt{\frac{1}{2}}$$

$$x = \frac{\sqrt{1}}{\sqrt{2}}$$

$$x = \frac{1}{\sqrt{2}}$$

Thus, $\cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$ and $\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$, so we write $\sqrt{i} = \cos\left(\frac{1}{4}\pi\right) + i\sin\left(\frac{1}{4}\pi\right)$.

However, we stated previously that we were actually looking for four complex numbers of the form $z = a + bi$ which satisfy $z^4 + 1 = 0$. Based on the calculation above, it makes sense to consider four possibilities for $z = \sqrt{i}$; specifically

$$\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i, \quad \frac{-1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i, \quad \frac{-1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i, \quad \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i.$$

To determine which are actual answers to $\sqrt{i}$, I will square each and solve for i which should equal $\sqrt{-1}$.

Does $\sqrt{i} = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} i$?

$$\sqrt{i}^2 = \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} i \right)^2$$

$$i = \frac{1}{2} - i + \frac{1}{2} i^2$$

$$i + \left(i - \frac{1}{2} \right) = \frac{1}{2} - i + \frac{1}{2} i^2 + \left(i - \frac{1}{2} \right)$$

$$-\frac{1}{2} = \frac{1}{2} i^2$$

$$\left(-\frac{1}{2} \right) \cdot 2 = \left(\frac{1}{2} i^2 \right) \cdot 2$$

$$-1 = i^2$$

$$\sqrt{i} = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} i$$

Does $\sqrt{i} = \frac{-1}{\sqrt{2}} + \frac{1}{\sqrt{2}} i$?

$$\sqrt{i}^2 = \left(\frac{-1}{\sqrt{2}} + \frac{1}{\sqrt{2}} i \right)^2$$

$$i = \frac{1}{2} - i + \frac{1}{2} i^2$$

$$i + \left(i - \frac{1}{2} \right) = \frac{1}{2} - i + \frac{1}{2} i^2 + \left(i - \frac{1}{2} \right)$$

$$2i - \frac{1}{2} = \frac{1}{2} i^2$$

$$\left(2i - \frac{1}{2} \right) \cdot 2 = \left(\frac{1}{2} i^2 \right) \cdot 2$$

$$4i - 1 = i^2$$

$$\sqrt{i} \neq \frac{-1}{\sqrt{2}} + \frac{1}{\sqrt{2}} i$$

Does $\sqrt{i} = \frac{-1}{\sqrt{2}} + \frac{-1}{\sqrt{2}} i$?

$$\sqrt{i}^2 = (\frac{-1}{\sqrt{2}} + \frac{-1}{\sqrt{2}} i)^2$$

$$i = \frac{1}{2} + i + \frac{1}{2} i^2$$

$$i + (\frac{-1}{2} - i) = \frac{1}{2} + i + \frac{1}{2} i^2 + (\frac{-1}{2} - i)$$

$$\frac{-1}{2} = \frac{1}{2} i^2$$

$$(\frac{-1}{2}) \cdot 2 = (\frac{1}{2} i^2) \cdot 2$$

$$-1 = i^2$$

$$\sqrt{-1} = i$$

$$\sqrt{i} = \frac{-1}{\sqrt{2}} + \frac{-1}{\sqrt{2}} i$$

Does $\sqrt{i} = \frac{1}{\sqrt{2}} + \frac{-1}{\sqrt{2}} i$?

$$\sqrt{i}^2 = (\frac{1}{\sqrt{2}} + \frac{-1}{\sqrt{2} i})^2$$

$$i = \frac{1}{2} - i + \frac{1}{2} i^2$$

$$i + (\frac{-1}{2} + i) = \frac{1}{2} - i + \frac{1}{2} i^2 + (\frac{-1}{2} + i)$$

$$2i - \frac{1}{2} = \frac{1}{2} i^2$$

$$(2i - \frac{1}{2}) \cdot 2 = (\frac{1}{2} i^2) \cdot 2$$

$$4i - 1 = i^2$$

$$\sqrt{4i - 1} = i$$

$$\sqrt{i} \neq \frac{1}{\sqrt{2}} + \frac{-1}{\sqrt{2}} i$$

Therefore there are two solutions to $\sqrt{i}$; they are $\sqrt{i} = \frac{-1}{\sqrt{2}} + \frac{-1}{\sqrt{2}}i$ and

$\sqrt{i} = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i$.

The two complex numbers which are solutions to $\sqrt{i}$ lead us to consider the question: Is there a pattern for finding the nth roots of i? To look for a pattern I will return to the case where n = 2, the square root of i:

$$\sqrt{i} = \left(e^{i\theta}\right)^{\frac{1}{2}} = e^{i\frac{\theta}{2}}$$

I can rotate any point in the complex plane about the origin by 360° or 2π radians and return to the same location on the Argand diagram. Likewise I can rotate the point again by 360° for a total of 4π radians. Each time I perform this rotation I need to take 2π times k, where k is the number of times I have gone around the circle. Thus, I can write $\theta = \frac{\pi}{2} + 2k\pi$. When calculating a root, I need to check where $\frac{\theta}{2}$ is less than 2π because the sine and cosine functions are periodic with a period of 2π. Thus, to determine all values for $\frac{\theta}{2}$ (since we are considering the square root) we consider $\frac{\theta}{2} = \frac{1}{2}\left(\frac{\pi}{2} + 2k\pi\right)$, where k = 0, 1, 2, … .

So, for the specific cases where k = 0, 1, and 2 I calculated the following:

$$\frac{\theta}{2} = \frac{1}{2}\left(\frac{\pi}{2} + 2k\pi\right) \text{ for k = 0}$$
$$\frac{\theta}{2} = \frac{1}{2}\left(\frac{\pi}{2} + 2(0)\pi\right)$$
$$\frac{\theta}{2} = \frac{1}{2}\left(\frac{\pi}{2}\right)$$
$$\frac{\theta}{2} = \frac{\pi}{4} < 2\pi$$

$$\frac{\theta}{2} = \frac{1}{2}\left(\frac{\pi}{2} + 2k\pi\right) \text{ for k = 1}$$
$$\frac{\theta}{2} = \frac{1}{2}\left(\frac{\pi}{2} + 2(1)\pi\right)$$
$$\frac{\theta}{2} = \frac{1}{2}\left(\frac{5\pi}{2}\right)$$
$$\frac{\theta}{2} = \frac{5\pi}{4} < 2\pi$$

$$\frac{\theta}{2} = \frac{1}{2}\left(\frac{\pi}{2} + 2k\pi\right) \text{ for k = 2}$$
$$\frac{\theta}{2} = \frac{1}{2}\left(\frac{\pi}{2} + 2(2)\pi\right)$$
$$\frac{\theta}{2} = \frac{1}{2}\left(\frac{9\pi}{2}\right)$$
$$\frac{\theta}{2} = \frac{9\pi}{4} > 2\pi$$

Thus I need only consider the cases where k = 0 and k = 1, since for larger values of k I have $\frac{\theta}{2}$ > 2π. Now, replacing $\frac{\theta}{2}$ with $\frac{\pi}{4}$ *and* $\frac{5\pi}{4}$, I can return to the formula

$$\sqrt{i} = \left(e^{i\theta}\right)^{\frac{1}{2}} = e^{i\frac{\theta}{2}} = \cos\frac{\theta}{2} + i\sin\frac{\theta}{2}$$

and calculate the following:

$$\sqrt{i} = \cos\left(\frac{1}{4}\pi\right) + i\sin\left(\frac{1}{4}\pi\right)$$

$$\sqrt{i} = \cos 45^o + i\sin 45^o$$

$$\sqrt{i} = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i$$

$$\sqrt{i} = \cos\left(\frac{5}{4}\pi\right) + i\sin\left(\frac{5}{4}\pi\right)$$

$$\sqrt{i} = \cos 135^o + i\sin 135^o$$

$$\sqrt{i} = \frac{-1}{\sqrt{2}} + \frac{-1}{\sqrt{2}}i$$

These are the same values for $\sqrt{i}$ that I found before.

I can extend this idea to calculate the cube root of i. I begin by writing $\sqrt[3]{i} = \left(e^{i\theta}\right)^{\frac{1}{3}} = e^{i\frac{\theta}{3}}$ and rotating $\frac{\theta}{3}$ around the unit circle. Then I determine the number of rotations, k, for which $1/3\ (\theta + 2k\pi) < 2\pi$:

$$\frac{\theta}{3} = \frac{1}{3}\left(\frac{\pi}{2} + 2k\pi\right) \quad \text{for k = 0}$$

$$\frac{\theta}{3} = \frac{1}{3}\left(\frac{\pi}{2} + 2(0)\pi\right)$$

$$\frac{\theta}{3} = \frac{1}{3}\left(\frac{\pi}{2}\right)$$

$$\frac{\theta}{3} = \frac{\pi}{6} < 2\pi$$

$$\frac{\theta}{3} = \frac{1}{3}\left(\frac{\pi}{2} + 2k\pi\right) \quad \text{for k = 1}$$

$$\frac{\theta}{3} = \frac{1}{3}\left(\frac{\pi}{2} + 2(1)\pi\right)$$

$$\frac{\theta}{3} = \frac{1}{3}\left(\frac{5\pi}{2}\right)$$

$$\frac{\theta}{3} = \frac{5\pi}{6} < 2\pi$$

$$\frac{\theta}{3}=\frac{1}{3}\left(\frac{\pi}{2}+2k\pi\right) \text{ for k = 2}$$

$$\frac{\theta}{3}=\frac{1}{3}\left(\frac{\pi}{2}+2(2)\pi\right)$$

$$\frac{\theta}{3}=\frac{1}{3}\left(\frac{9\pi}{2}\right)$$

$$\frac{\theta}{3}=\frac{9\pi}{6}=\frac{3\pi}{2}<2\pi$$

$$\frac{\theta}{3}=\frac{1}{3}\left(\frac{\pi}{2}+2k\pi\right) \text{ for k = 3}$$

$$\frac{\theta}{3}=\frac{1}{3}\left(\frac{\pi}{2}+2(3)\pi\right)$$

$$\frac{\theta}{3}=\frac{1}{3}\left(\frac{13\pi}{2}\right)$$

$$\frac{\theta}{3}=\frac{13\pi}{6}>2\pi$$

Thus, I only need to consider the cases where $\frac{\theta}{3}=\frac{1}{3}\left(\frac{\pi}{2}+2k\pi\right)$ for k = 0, 1, 2; specifically when $\frac{\theta}{3}$ is $\frac{\pi}{6},\frac{5\pi}{6} and \frac{3\pi}{2}$. Then, substituting these values into the formula

$$\sqrt{i}=\left(e^{i\theta}\right)^{\frac{1}{3}}=e^{i\frac{\theta}{3}}=\cos\frac{\theta}{3}+i\sin\frac{\theta}{3}$$

leads to the following calculations:

$$\sqrt[3]{i}=\cos\left(\frac{1}{6}\pi\right)+i\sin\left(\frac{1}{6}\pi\right)$$

$$\sqrt[3]{i}=\cos 30^o+i\sin 30^o$$

$$\sqrt[3]{i}=\frac{\sqrt{3}}{2}+\frac{1}{2}i$$

$$\sqrt[3]{i}=\cos\left(\frac{5}{4}\pi\right)+i\sin\left(\frac{5}{4}\pi\right)$$

$$\sqrt[3]{i}=\cos 150^o+i\sin 150^o$$

$$\sqrt[3]{i}=\frac{-\sqrt{3}}{2}+\frac{1}{2}i$$

$$\sqrt[3]{i}=\cos\left(\frac{3}{2}\pi\right)+i\sin\left(\frac{3}{2}\pi\right)$$

$$\sqrt[3]{i}=\cos 270^o+i\sin 270^o$$

$$\sqrt[3]{i}=0+(-1)i = -i$$

Therefore, $\sqrt[3]{i} = -i, \dfrac{-\sqrt{3}}{2} + \dfrac{1}{2}i, and \dfrac{\sqrt{3}}{2} + \dfrac{1}{2}i$.

Now I can generalize this idea to find the solution set for any root of i . From the previous two examples, I noticed that the number of rotations, k, is always one less than the root that I am trying to find. This will allow me to list a complete set of the values of θ:

$$\frac{\theta}{n} = \frac{1}{n}\left(\frac{\pi}{2} + 2k\pi\right) \text{ for k = 0, 1, 2, …, (n-1).}$$

Then the complex solutions to the equation $z^n + 1 = 0$ (i.e. the n^{th} roots of i) are given by

$$\boxed{\sqrt[n]{i} = \cos\left(\frac{\theta}{n}\right) + i\sin\left(\frac{\theta}{n}\right)}$$

where

$$\boxed{\frac{\theta}{n} = \frac{\pi}{2n}(1 + 4k) \text{ for k = 0, 1, 2, …, (n-1).}}$$

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