1-1-1975

Planar Motion

PLANAR MOTION

INTRODUCTION

Enough of this physics where things move along straight lines only! We know that most interesting real-life motions involve curves of many and varied shapes. This module extends your understanding of kinematics from one dimension to two dimensions. To accomplish this, you will combine your knowledge of calculus and vectors with concepts like position, displacement, velocity, speed, and acceleration.

Two important applications that will be utilized many times in later modules are covered here. First is the motion of a particle experiencing constant acceleration, e.g., a baseball in flight. Second is the motion of a particle in a circular path with a constant speed, e.g., an earth satellite in circular orbit.

PREREQUISITES

Before you begin this module, you should be able to:	Location of Prerequisite Content
*Use vector algebra in the following operations (needed for Objectives 1 through 5 of this module): Addition Multiplication by a scalar Unit vectors Magnitude of a vector Scalar product	Dimensions and Vector Addition Module
*Differentiate polynomial, sine, and cosine functions (needed for Objectives 2, 4, and 5 of this module)	Calculus Review
*Use the chain rule for derivatives (needed for Objectives 2, 4, and 5 of this module)	Calculus Review
*Solve kinematics problems in one dimension (needed for Objectives 1 through 5 of this module)	Rectilinear Motion Module
*Compute angles in radians (needed for Objectives 1, 2, 4, and 5 of this module)	Trigonometry Review

LEARNING OBJECTIVES

After you have mastered the content of this module, you will be able to:

1. Graphing the trajectory - Given a particle's time-dependent position vector $\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j}$, draw its path in the plane.

2. Velocity, speed, and acceleration - Given $\vec{r}(t)$, calculate velocity $\vec{v}(t)$, speed $v(t)$, and acceleration $\vec{a}(t)$.

3. Interpreting velocity and acceleration - Given a particle's position $\vec{r}$, velocity $\vec{v}$, and acceleration $\vec{a}$ at a specified time, determine whether at this instant:
   (a) its distance from the origin is increasing, decreasing, or not changing;
   (b) $\vec{r}$ is turning clockwise, counterclockwise, or not turning;
   (c) its speed is increasing, decreasing, or not changing;
   (d) $\vec{v}$ is turning clockwise, counterclockwise, or not turning.

4. Projectiles - Given that a particle moves with constant acceleration in two dimensions, solve problems involving position, velocity, acceleration, and time.

5. Uniform circular motion - Given that a particle moves in a circular path at a constant speed, solve problems involving position, velocity, acceleration, and time.

GENERAL COMMENTS

This study guide may be different from others in that a large portion of your studying will be done in the Problem Set. Each of the objectives is discussed in some detail in the 14 problems. Seven of these problems develop the basic ideas and present detailed solutions to typical problems. The remaining seven represent challenges for your attention.

Your text will be used to provide supplementary readings and problems. If you have a calculus text, you will also find it helpful.

TEXT: Frederick J. Bueche, Introduction to Physics for Scientists and Engineers (McGraw-Hill, New York, 1975), second edition

SUGGESTED STUDY PROCEDURE

Your primary reading for this module will be Sections 4.1, 4.2, and 4.8 of Chapter 4 and Section 10.6 of Chapter 10. For Objective 1, first study Problem A and the material preceding it, then work Problem H. Next review the definitions of velocity and acceleration in Sections 4.1 and 4.2 of the text (for Objective 2: calculating velocity, speed, and acceleration in two dimensions). Study Problems B and C, with their explanatory material, before working Problems I and J. Then study the section Interpreting Velocity and Acceleration and Problem D. Read Section 4.8 of the text before studying Problems E and F and working Problem K. Next study Problem G, with its preliminary material, before working Problems L, M, and N.

Study the text, Chapter 10, Section 10.6 and the first two paragraphs of Section 10.7. Don't let angular speed ω scare you. It's just the rate (in radians per second) at which the angle θ of Figure 10.7 is changing. You have already seen ω in Problem G. Do the Practice Test before attempting the Mastery Test.

BUECHE

Objective Number	Readings	Problems with Solutions: Study Guide	Assigned Problems: Study Guide	Assigned Problems: Text	Additional Problems
1		A	H		
2	Sec. 4.1 Sec. 4.2	B, C	I, J		
3		D	D		
4	Sec. 4.8	E, F	K	Chap. 4: Problem 17	Chap. 4: Problems 19, 21, 25
5	Sec. 10.6	G	L, M, N		

TEXT: David Halliday and Robert Resnick, Fundamentals of Physics (Wiley, New York, 1970; revised printing, 1974)

SUGGESTED STUDY PROCEDURES

Study Trajectories and Problem A before working Problem H. Then study the text, Chapter 3, Sections 3-3, 3-4, 3-6 and Chapter 4, Section 4-1, along with the sections Velocity and Speed and Acceleration of this module. Study Problems B and C and work Problems I and J for Objective 2. Read Interpreting Velocity and Acceleration before working through Problem D.

For Objectives 4 and 5, study Sections 4-2 to 4-4 of the text, and Problems E, F, and G of the module along with their preliminary material. Then work Problems K through N along with Problems 37 and 39 in Chapter 4 of the text. Do the Practice Test before attempting the Mastery Test.

HALLIDAY AND RESNICK

		Problems with Solutions	Assigned Problems		
Objective Number	Readings	Study Guide	Study Guide	Text	Additional Problems
1		A	H		
2	Secs. 3-3, 3-4, 3-6, 4-1	B, C	I, J		Chap. 4: Problems 3, 5
3		D	D		
4	Secs. 4-2, 4-3	E, F	K		Chap. 4: Problems 9, 11, 13
5	Sec. 4-4	G	L, M, N	Chap. 4: Problems 37, 39	

TEXT: Francis Weston Sears and Mark W. Zemansky, University Physics (Addison-Wesley, Reading, Massachusetts, 1970), fourth edition

SUGGESTED STUDY PROCEDURE

Study Trajectories and Problem A and work Problem H. Then study the text, Chapter 6, Sections 6-1 through 6-3, and Problems B and C along with their accompanying material, before working Problems I and J. Study Problem D. Next study the text, Section 6-5, and Problems E and F (Objective 4) before working Problems 6-1 and 6-3 of the text, Problem K of this module. Read Section 6-6 and study Uniform Circular Motion and Problem G before working Problems 6-27 and L to N.

Try the Practice Test before doing the Mastery Test.

SEARS AND ZEMANSKY

		Problems with Solutions	Assigned Problems	
Objective Number	Readings	Study Guide	Study Guide	Text
1		A	H	
2	Secs. 6-1, 6-2, 6-3	B, C	I, J	
3		D	D	
4	Sec. 6-5	E, F	K	6-1, 6-3
5	Sec. 6-6	G	L, M, N	6-27

TEXT: Richard T. Weidner and Robert L. Sells, Elementary Classical Physics (Allyn and Bacon, Boston, 1973), second edition, Vol 1

SUGGESTED STUDY PROCEDURE

Study Trajectories and Problem A and work Problem H. Then read the text, Chapter 4, Section 4-1, and study Problems B and C before working Problems I and J. Next study Interpreting Velocity and Acceleration and work through Problem D. Then read Sections 4-2 and 4-3 and study Motion with Constant Acceleration: Projectiles with Problems E and F before working Problems 4-7, 4-9 (text) and K on your own. Read Section 4-4 (text) and study Problem G with its discussion before working Problems L, M, N, and 4-23 of the text.

Try the Practice Test before working the Mastery Test.

WEIDNER AND SELLS

Objective Number	Readings	Problems with Solutions: Study Guide	Assigned Problems: Study Guide	Assigned Problems: Text
1		A	H	
2	Sec. 4-1	B, C	I, J	
3		D	D	
4	Secs. 4-2, 4-3	E, F	K	4-7, 4-9
5	Sec. 4-4	G	L, M, N	4-23

PROBLEM SET WITH SOLUTIONS

Trajectories

As a particle moves in a plane it traces out a curve, or path. A knowledge of this path and of the time at which the particle passed through each point constitutes a knowledge of the particle's trajectory.

At each instant the particle's position is specified by its position vector:

$$\vec{r} = x\hat{i} + y\hat{j}. \tag{1}$$

As the particle moves x, y, and $\vec{r}$ change. This time dependence is emphasized by writing Eq. (1) as

$$\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j}. \tag{2}$$

A vector equation such as (2) is a compact method for representing a trajectory in two dimensions.

A(1). A particle's position is given by

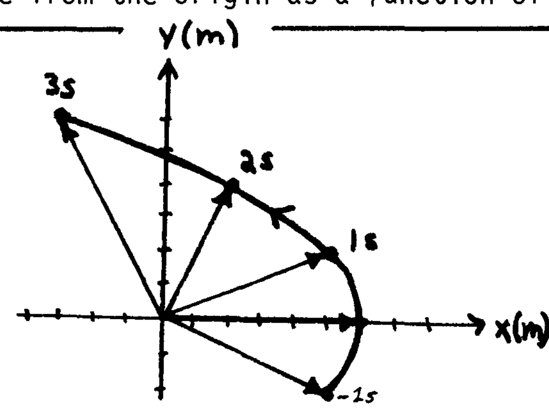
$$\vec{r}(t) = [(6-t^2)\hat{i} + (2t)\hat{j}]\ \text{m},$$

where t is measured in seconds.
(a) Draw the particle's trajectory for $-1 \le t \le 3$ s.
(b) Calculate the particle's distance from the origin at t = 2 s.
(c) Determine the particle's distance from the origin as a function of time.

Solution

(a) First evaluate $\vec{r}$ for

$t = -1$ s, $\vec{r} = (5\hat{i}-2\hat{j})$ m,
$t = 0$, $\vec{r} = (6\hat{i})$ m,
$t = 1$ s, $\vec{r} = (5\hat{i}+2\hat{j})$ m,
$t = 2$ s, $\vec{r} = (2\hat{i}+4\hat{j})$ m,
$t = 3$ s, $\vec{r} = (-3\hat{i}+6\hat{j})$ m;



Plot these vectors and connect their tips by a smooth curve. The arrow along the path between t = 1 s and t = 2 s is used to indicate the direction of motion along the path.

(b) The distance from the origin to the particle at t = 2 s is just the magnitude of $\vec{r}$ at this time. Thus,

$$r = \sqrt{2^2 + 4^2} = 2\sqrt{5}\ \text{m}.$$

(c) At any time t the distance from the origin to the particle is

$$r(t) = \sqrt{(6-t^2)^2 + (2t)^2} = \sqrt{36-8t^2+t^4}\ \text{m}.$$

Velocity and Speed

The velocity $\vec{v}$ for a particle is defined as the time derivative of the particle's position vector, that is,

$$\vec{v} = d\vec{r}/dt. \tag{3}$$

If $\vec{r}$ is written in component form, i.e.,

$$\vec{r} = x\hat{i} + y\hat{j},$$

then

$$\vec{v} = (dx/dt)\hat{i} + (dy/dt)\hat{j}, \tag{4}$$

so that

$$v_x = dx/dt, \qquad v_y = dy/dt. \tag{5}$$

The particle's speed v at any instant is defined as the magnitude of its velocity at that time,

$$v = \sqrt{v_x^2 + v_y^2}\ . \tag{6}$$

Speed, a scalar quantity, measures the time rate of change of distance traveled by the particle.

At each instant the velocity vector (if drawn with its initial point on the particle) is tangential to the particle's path. Thus, the magnitude of the velocity indicates "how fast" the particle is moving, whereas its direction points in the direction of motion of the particle.

B(2). For

$$\vec{r}(t) = [(6-t^2)\hat{i} + (2t)\hat{j}]\ \text{m},$$

as in Problem A,
(a) determine the velocity and speed as functions of time;
(b) draw the trajectory for $-1 \le t \le 3$ s and show $\vec{v}$ at t = 0 and t = 2 s.

Solution

(a) From the definition of velocity,

$$\vec{v}(t) = d\vec{r}/dt = (-2t\hat{i} + 2\hat{j}) \text{ m/s};$$

and thus the speed is

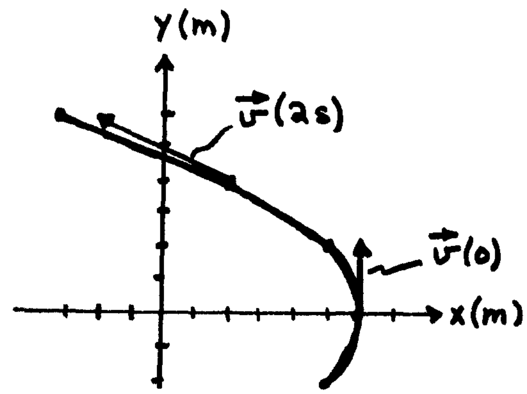
$$v(t) = \sqrt{(-2t)^2 + 2^2} \text{ m/s} = 2\sqrt{t^2 + 1} \text{ m/s}.$$

(b) The trajectory was drawn in Problem A. Now

$$\vec{v}(0) = 2\hat{j} \text{ m/s},$$

$$\vec{v}(2\text{ s}) = (-4\hat{i} + 2\hat{j}) \text{ m/s}.$$

(Notice that the velocity vectors are tangential to the path.)



Acceleration

The acceleration $\vec{a}$ for a particle is defined as the time derivative of the particle's velocity, i.e.,

$$\vec{a} = d\vec{v}/dt. \tag{7}$$

If $\vec{v}$ is written in component form:

$$\vec{v} = v_x\hat{i} + v_y\hat{j},$$

then

$$\vec{a} = (dv_x/dt)\hat{i} + (dv_y/dt)\hat{j}, \tag{8}$$

so that

$$a_x = dv_x/dt, \qquad a_y = dv_y/dt. \tag{9}$$

C(2). For

$$\vec{r}(t) = [(6-t^2)\hat{i} + (2t)\hat{j}] \text{ m},$$

as in Problems A and B, determine $\vec{a}(t)$, $a_x(t)$, $a_y(t)$.

Solution

From Problem B,

$$\vec{v}(t) = (-2t\hat{i} + 2\hat{j})\ \text{m/s},$$

so that

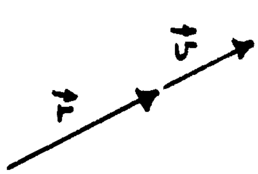
$$\vec{a}(t) = d\vec{v}/dt = (-2\hat{i})\ \text{m/s}^2.$$

Hence

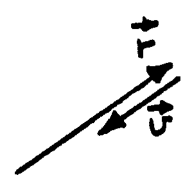
$$a_x(t) = -2\ \text{m/s}^2 \qquad \text{and} \qquad a_y(t) = 0.$$

Interpreting Velocity and Acceleration

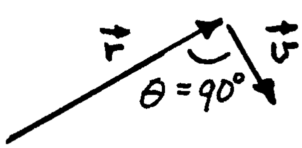
Remember that the velocity is defined as the time derivative of position. Since position is a vector, it may change by changing magnitude only, by changing direction only, or by changing both magnitude and direction. If the position vector changes its magnitude only, then the velocity vector is parallel to $\vec{r}$. If $\vec{r}$ changes direction only, then $\vec{v}$ is perpendicular to $\vec{r}$. Examples are shown below.



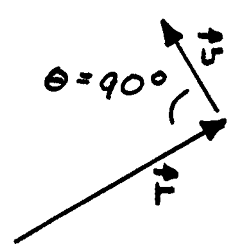
$\vec{r}$ increasing in length but not changing direction.



$\vec{r}$ decreasing in length but not changing direction.

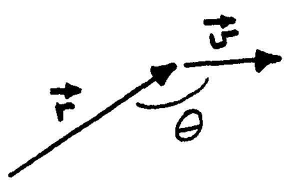


$\vec{r}$ turning clockwise but not changing length.

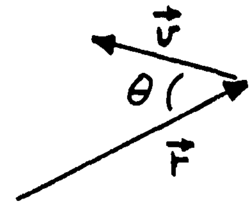


$\vec{r}$ turning counterclockwise but not changing length.

If $\vec{r}$ is both changing in magnitude and turning, then $\vec{v}$ will have components parallel to $\vec{r}$ and perpendicular to $\vec{r}$. Examples are shown below.

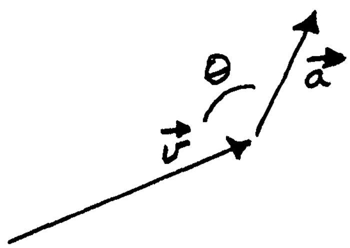


$\vec{r}$ increasing in length and turning clockwise.

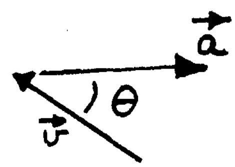


$\vec{r}$ decreasing in length and turning counterclockwise.

Since acceleration is obtained from velocity in precisely the same manner that velocity is determined from position, similar conclusions can be deduced about $\vec{v}$ from a knowledge of $\vec{a}$. Examples are shown below.



Speed increasing, $\vec{v}$ turning counterclockwise.



Speed decreasing, $\vec{v}$ turning clockwise.

D(3). At a certain instant a particle's position, velocity, and acceleration are

$$\vec{r} = 4\hat{i}\ \text{m}, \qquad \vec{v} = (-2\hat{i} + 3\hat{j})\ \text{m/s}, \qquad \vec{a} = (6\hat{i} + 4\hat{j})\ \text{m/s}^2.$$

(a) Is the distance from the origin to the particle increasing, decreasing, or not changing at this time?
(b) How is $\vec{r}$ turning at this instant?
(c) Is the particle's speed increasing, decreasing, or not changing at this instant?
(d) How is $\vec{v}$ turning at this instant?

Solution

(a) Decreasing;
(b) counterclockwise;
(c) not changing (since $\vec{v}$ is perpendicular to $\vec{a}$);
(d) clockwise.

Motion with Constant Acceleration: Projectiles

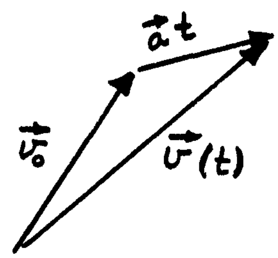
Suppose an object is known to move with a constant acceleration $\vec{a}$. Then

$$d\vec{v}/dt = \vec{a} = \text{const}; \qquad (10)$$

and, just as with rectilinear motion,

$$\vec{v}(t) = \vec{v}_0 + \vec{a}t \qquad (\vec{a} = \text{const}), \qquad (11)$$

where $\vec{v}$ is the velocity at time t and $\vec{v}_0$ is the velocity at t = 0. This equation for velocity is given graphical display below.



As t increases the $\vec{a}t$ term increases proportionately in magnitude without changing direction.

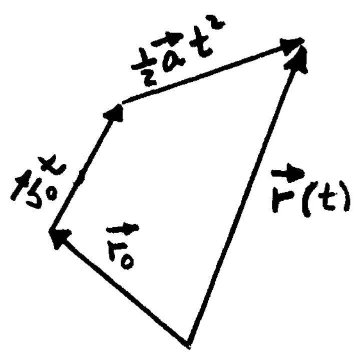
The position vector is obtained from

$$d\vec{r}/dt = \vec{v} = \vec{v}_0 + \vec{a}t. \qquad (12)$$

Again, just as in rectilinear motion,

$$\vec{r}(t) = \vec{r}_0 + \vec{v}_0 t + (1/2)\vec{a}t^2 \qquad (\vec{a} = \text{const}), \qquad (13)$$

where $\vec{r}$ is the position vector at time t and $\vec{r}_0$ is the position at t = 0. This equation is also graphically displayed below.



A word of caution is in order here. Equations (11) and (13) for $\vec{v}(t)$ and $\vec{r}(t)$ were obtained from an assumption of <u>constant acceleration</u> and <u>should be used only in such a circumstance</u>.

Perhaps the most familiar constant-acceleration phenomenon is that of a body in free fall near the earth's surface. If air resistance and other small effects are neglected, such an object moves with the downward acceleration

$$\vec{a} = -g\,\hat{j},$$

where $g = 9.8\ m/s^2$ and $\hat{j}$ is a unit vector pointing vertically upward.

E(4). A ball is projected from ground level above flat land with an initial velocity $\vec{v}_0 = v_{0x}\hat{i} + v_{0y}\hat{j}$ where $v_{0x} = 30.0$ m/s and $v_{0y} = 50$ m/s. Assuming that $\vec{r} = 0$ at $t = 0$, determine

(a) its velocity $\vec{v}(t)$,

(b) its position $\vec{r}(t)$.

Solution

(a) Since $\vec{a}$ ($a_x = 0$, $a_y = -g$) is constant,

$$\vec{v} = \vec{v}_0 + \vec{a}t = v_{0x}\hat{i} + v_{0y}\hat{j} + (-g\hat{j})t$$
$$= v_{0x}\hat{i} + (v_{0y} - gt)\hat{j}.$$

Substituting numerical values:

$$\vec{v} = [30.0\hat{i} + (50 - 9.8t)\hat{j}]\ m/s.$$

(b) Again, since $\vec{a}$ is constant,

$$\vec{r} = \vec{r}_0 + \vec{v}_0 t + (1/2)\,\vec{a}t^2$$
$$= (x_0\hat{i} + y_0\hat{j}) + (v_{0x}\hat{i} + v_{0y}\hat{j})t + (1/2)(-g\hat{j})t^2$$
$$= (x_0 + v_{0x}t)\hat{i} + [y_0 + v_{0y}t - (1/2)gt^2]\hat{j}.$$

Substituting numerical values, we find

$$\vec{r} = [(30.0t)\hat{i} + (50t - 4.9t^2)\hat{j}]\ m.$$

Comment. From the equations for $\vec{v}$ and $\vec{r}$,

$$v_x = v_{0x} = 30.0\ m/s, \qquad x = x_0 + v_{0x} = (30.0t)\ m.$$

Since the acceleration has a zero x component, the x coordinate is identical to that of a particle moving along the x axis with a constant speed of 30 m/s. Now consider the y components of $\vec{v}$ and $\vec{r}$:

$$v_y = v_{0y} - gt = (50 - 9.8t)\ \text{m/s},$$

$$y = y_0 + v_{0y}t - (1/2)gt^2 = (50t - 4.9t^2)\ \text{m}.$$

Not surprisingly the y coordinate is identical to that of a particle projected vertically with an initial speed of 50 m/s.

F(4). For the ball of Problem E,
(a) draw its flight path;
(b) determine its maximum height;
(c) determine its horizontal range (the distance between the origin and the point where the ball returns to its original height).

Solution

(a) From the previous problem,

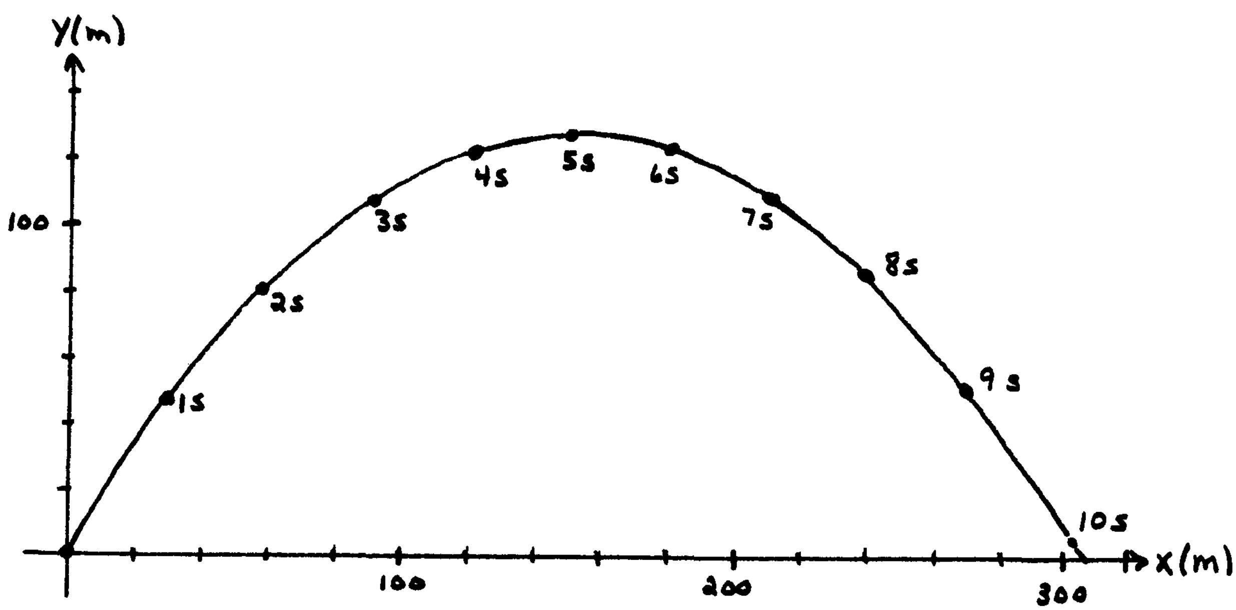
$$\vec{r}(t) = [(30.0t)\hat{i} + (50t - 4.9t^2)\hat{j}]\ \text{m}.$$

Evaluate $\vec{r}$ for integral values of $t \geq 0$, e.g.,

$$\vec{r}(1.00\ \text{s}) = 30.0\hat{i} + (50 - 4.9)\hat{j} = (30.0\hat{i} + 45.1\hat{j})\text{m}$$

$$\vec{r}(10.0\ \text{s}) = 300\hat{i} + (500 - 490)\hat{j} = (300\hat{i} + 10\hat{j})\ \text{m}.$$

Graphing these gives



(b) From the graph the maximum height is apparently near 130 m. To determine this value accurately, use the fact that at the point of maximum height, the y component of the velocity is necessarily zero:

$$v_y = v_{0y} - gt = 0.$$

Thus the time t_{max} for this maximum height is

$$t_{max} = \frac{v_{0y}}{g} = \frac{50\ m/s}{9.8\ m/s^2} = 5.1\ s.$$

The maximum height is obtained by substituting this value for t into the expression for y, i.e.,

$$y_{max} = v_{0y}t_{max} - \frac{1}{2}gt_{max}^2 = \frac{v_{0y}^2}{g} - \frac{1}{2}g\left(\frac{v_{0y}}{g}\right)^2 = \frac{v_{0y}^2}{2g}.$$

Substituting numerical values gives

$$y_{max} = \frac{50^2\ m^2/s^2}{2(9.8)\ m/s^2} = 128\ m.$$

(c) The horizontal range is just the x coordinate when the ball returns to its initial height. From the graph this distance is apparently slightly greater than 300 m. To determine this value more accurately, first determine the time t_R when the ball strikes the ground:

$$y = 0 = v_{0y}t_R - (1/2)gt_R^2 = t_R[v_{0y} - (1/2)gt_R].$$

Of the two solutions to this quadratic equation, the solution $t_R = 0$ is discarded (why?), and thus

$$t_R = 2v_{0y}/g.$$

(Notice that this time of flight is exactly twice the time to achieve the maximum height.) The horizontal range R is obtained by substituting t_R into the x equation:

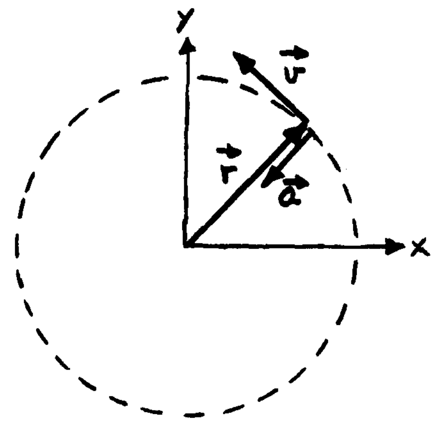
$$R = v_{0x}t_R = \frac{2v_{0x}v_{0y}}{g} = \frac{2(30.0\ m/s)(50\ m/s)}{9.8\ m/s^2} = 310\ m.$$

Uniform Circular Motion

First look back over your solutions of Problems H, I, and J. In these problems you considered the trajectory given by

$$\vec{r}(t) = 2[\cos(\pi t/4)\hat{i} + \sin(\pi t/4)\hat{j}]\ m.$$

In Problem H you might have noted that this trajectory is a circle of radius 2 m centered on the origin. Problem I revealed that the speed is constant and equal to $(\pi/2)$ m/s. Finally, in Problem J you found that the acceleration is constant in magnitude $[(\pi^2/8)\ m/s^2]$ and always opposite to $\vec{r}$, i.e., $\vec{a}$ always points toward the center of the circle for uniform circular motion. This situation is depicted in the diagram at the right.



As you might guess, circular motion at a constant speed can be described by

$$\vec{r}(t) = R[(\cos \omega t)\hat{i} + (\sin \omega t)\hat{j}],$$

where R and ω are constants.

G(5). Using this expression for $\vec{r}(t)$, show that

(a) $r(t) = R$ (constant);

(b) $\vec{v}(t) = \omega R[-(\sin \omega t)\hat{i} + (\cos \omega t)\hat{j}]$;

(c) $v(t) = \omega R$ (constant);

(d) $\vec{a}(t) = -\omega^2 R[(\cos \omega t)\hat{i} + (\sin \omega t)\hat{j}] = -\omega^2\vec{r}$;

(e) $a(t) = +\omega^2 R = v^2/R$ (constant).

Solution

(a) Since

$$\vec{r}(t) = \sqrt{(R \cos \omega t)^2 + (R \sin \omega t)^2} = R,$$

R (the magnitude of $\vec{r}$) = radius of circle.

(b)
$$\vec{v}(t) = d\vec{r}/dt$$
$$= R[(d/dt)(\cos \omega t)\hat{i} + (d/dt)(\sin \omega t)\hat{j}]$$
$$= \omega R[(-\sin \omega t)\hat{i} + (\cos \omega t)\hat{j}].$$

[Don't forget the chain rule, i.e., $df(u)/dx = (df/du)(du/dx)$.]

(c)
$$v(t) = \sqrt{(\omega R \sin \omega t)^2 + (\omega R \cos \omega t)^2} = \omega R.$$

Thus, the particle's speed is constant and $\omega = v/R$.

(d) $\vec{a}(t) = d\vec{v}/dt$

$$= \omega R[-(d/dt)(\sin \omega t)\hat{i} + (d/dt)(\cos \omega t)\hat{j}]$$

$$= -\omega^2 R[(\cos \omega t)\hat{i} + (\sin \omega t)\hat{j}]$$

$$= -\omega^2 \vec{r}(t).$$

Thus $\vec{a}$ is at all times opposite to $\vec{r}$; and therefore, if drawn starting on the particle, it always points toward the center of the circle.

(e) $a(t) = \sqrt{(\omega^2 R \cos \omega t)^2 + (\omega^2 R \sin \omega t)^2} = \omega^2 R = v^2/R.$

In summary, if a particle moves in a circular path of radius R at a constant speed v, its acceleration is radially inward and of magnitude v^2/R. This acceleration is usually called centripetal acceleration.

PROBLEMS

H(1). Draw the paths for each of the following trajectories:

(a) $\vec{r} = (t^2\hat{i} + 2t\hat{j})$ m, $-1 \leq t \leq 2$ s;

(b) $\vec{r} = [2 \cos (\pi t/4)\hat{i} + 2 \sin (\pi t/4)\hat{j}]$ m, $0 \leq t \leq 4$ s.

I(2). (a) Determine $\vec{v}(t)$ and $v(t)$ for

$$\vec{r}(t) = [2 \cos (\pi t/4)\hat{i} + 2 \sin (\pi t/4)\hat{j}] \text{ m}.$$

Hint: Don't forget that $d(\cos \omega t)/dt = -\omega \sin \omega t$ if ω is constant. Similarly $d(\sin \omega t)/dt = \omega \cos \omega t$.

(b) What is the significance of the result for $v(t)$?

J(2). Using the position function

$$\vec{r}(t) = [2 \cos (\pi t/4)\hat{i} + 2 \sin (\pi t/4)\hat{j}] \text{ m},$$

determine $\vec{a}(t)$, $a(t)$, $a_x(t)$, $a_y(t)$. Show that for this motion the acceleration vector always points in the opposite direction to the position vector.

K(4). A cannon with a muzzle speed of 800 m/s fires at a target at a horizontal distance of 30 000 m. What initial angles of elevation will ensure success? Sketch the two paths. Hint: $2 \sin \theta \cos \theta = \sin 2\theta$.

L(5). Write an equation for a counterclockwise circular trajectory such that the radius is 1.5 m and the speed is 6 m/s. Let (x,y) = (1.5 m,0) at t = 0. Determine the magnitude of the centripetal acceleration.

M(5). Consider the trajectory

$$\vec{r}(t) = 1.5[(-\sin 4t)\hat{i} + (\cos 4t)\hat{j}] \text{ m}.$$

Show that this is circular motion at a constant speed. Determine r, v, and a. How is this trajectory different from that of Problem L?

N(5). Consider the trajectory

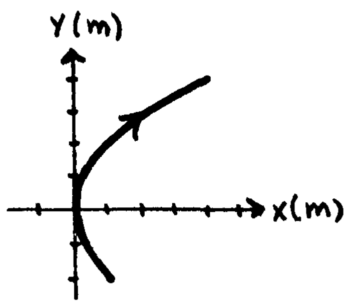
$$\vec{r}(t) = 1.5[(\cos 4t)\hat{i} - (\sin 4t)\hat{j}] \text{ m}.$$

How is it different from those of Problems L and M?

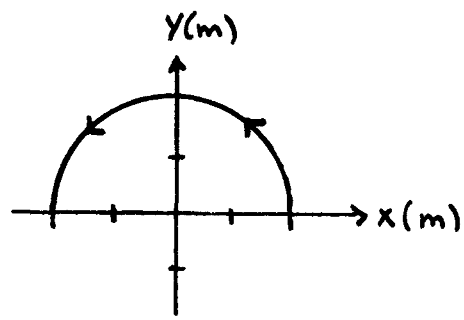
Solutions

H(1).

(a)



(b)



I(2). $$\vec{v}(t) = (\pi/2)[-\sin(\pi t/4)\hat{i} + \cos(\pi t/4)\hat{j}] \text{ m/s},$$

$$v(t) = (\pi/2) \text{ m/s},$$

i.e., the speed is constant, but the velocity is not (why?).

J(2). $\vec{a} = -(\pi^2/8)[\cos(\pi t/4)\hat{i} + \sin(\pi t/4)\hat{j}]$ m/s^2,

$a = (\pi^2/8)$ m/s^2,

$a_x = -(\pi^2/8)\cos(\pi t/4)$ m/s^2,

$a_y = -(\pi^2/8)\sin(\pi t/4)$ m/s^2.

Since $\vec{a} = -(\pi^2/16)\vec{r}$, $\vec{a}$ always points in the opposite direction to $\vec{r}$.

K(4). $\theta_0 = 13.7°$; $\theta_0 = 76.3°$

L(5). $\vec{r}(t) = 1.5[(\cos 4t)\hat{i} + (\sin 4t)\hat{j}]$ m,

$a = 24$ m/s^2.

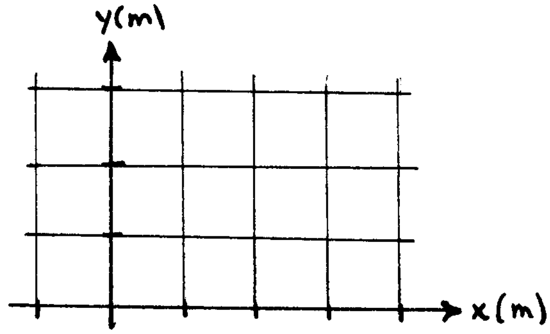
M(5). $r = 1.5$ m, $v = 6$ m/s, $a = 24$ m/s^2.

Problem L's trajectory has t = 0 displacement of $1.5\hat{i}$ m, but for this trajectory $r = 1.5j$ m at t = 0. Thus the M trajectory is "one-quarter of a lap" ahead of the L trajectory.

N(5). The radius, speed, and acceleration (magnitude) are the same, but the particle starts at t = 0 at $\vec{r} = (1.5\hat{i})$ m and moves <u>clockwise</u> rather than counterclockwise as in L and M.

PRACTICE TEST

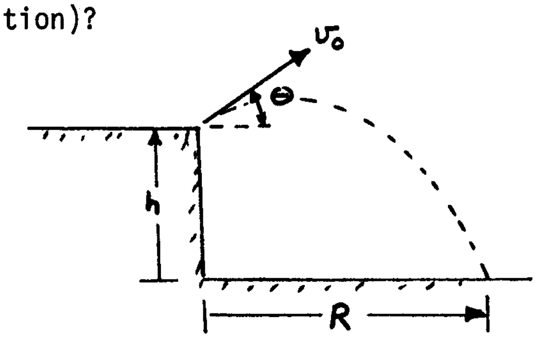
1. Draw the trajectory
   $\vec{r}(t) = [t\hat{i} + 2 \sin (\pi t/2)\hat{j}]$m
   for $0 \leq t \leq 2$ s.



2. Determine the velocity, acceleration, and speed at t = 1 s for the trajectory of Problem 1.

3. At an instant when $\vec{r} = (2\hat{i} + 3\hat{j})$ m, $\vec{v} = (6\hat{i} - 4\hat{j})$ m/s, $\vec{a} = 6\hat{j}$ m/s^2:
   (a) How is $\vec{r}$ changing (magnitude and direction)?
   (b) How is $\vec{v}$ changing (magnitude and direction)?

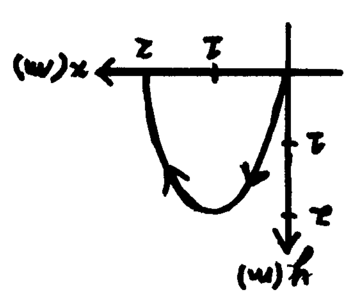
4. A ball is thrown from the cliff as shown. If v_0 = 20.0 m/s, θ = 30.0°, and h = 40 m,
   (a) When does the ball strike the ground?
   (b) Calculate R.



5. Write an equation for a constant-speed circular trajectory such that
   (a) the centripetal acceleration has a magnitude of 32.0 m/s^2;
   (b) the motion is clockwise; and
   (c) at t = 0, $\vec{v} = 4.0\hat{j}$ m/s.

Practice Test Answers

1.



2. $\vec{v}(1\text{ s}) = \hat{i}$ m/s. $\vec{a}(1\text{ s}) = -(\pi^2/2)\hat{j}$ m/s^2.
   v(1 s) = 1 m/s.
3. (a) r not changing, $\vec{r}$ turning clockwise.
   (b) v decreasing, $\vec{v}$ turning counterclockwise.
4. (a) 4.1 s. (b) 70 m.
5. $\vec{r}(t) = (1/2)[-(\cos 8.0t)\hat{i} + (\sin 8.0t)\hat{j}]$ m.

pass recycle

1 2 3 4 5

PLANAR MOTION

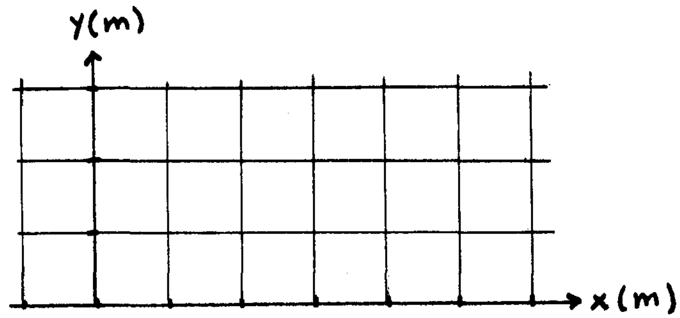
Mastery Test Form A

Name________________________ Tutor________________________

1. Draw the path

$\vec{r}(t) = [(4t/\pi)\hat{i} + (2 \sin t)\hat{j}]$ m

for $0 \leq t \leq \pi$ s.



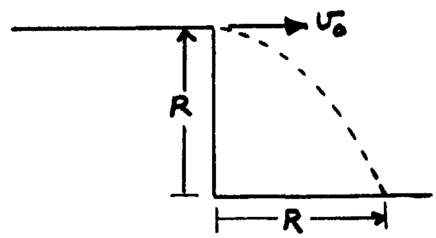
2. Determine the particle's acceleration at $t = (\pi/2)$ s (for Problem 1).

3. At an instant when $\vec{r} = 4\hat{j}$ m, $\vec{v} = (2\hat{i} - 3\hat{j})$ m/s, $\vec{a} = (-4\hat{i} + 6\hat{j})$ m/s^2:

(a) How is $\vec{r}$ changing (magnitude and direction)?

(b) How is $\vec{v}$ changing (magnitude and direction)?

4. A thrown rock follows the trajectory shown. Determine the initial speed v_0 if R = 50 m.



5. Consider the circular trajectory

$\vec{r}(t) = 2.00[(\sin 6.0t)\hat{i} + (\cos 6.0t)\hat{j}]$ m.

Determine
(a) the radius,
(b) the speed,
(c) the magnitude of the centripetal acceleration,
(d) the direction of motion (clockwise or counterclockwise).

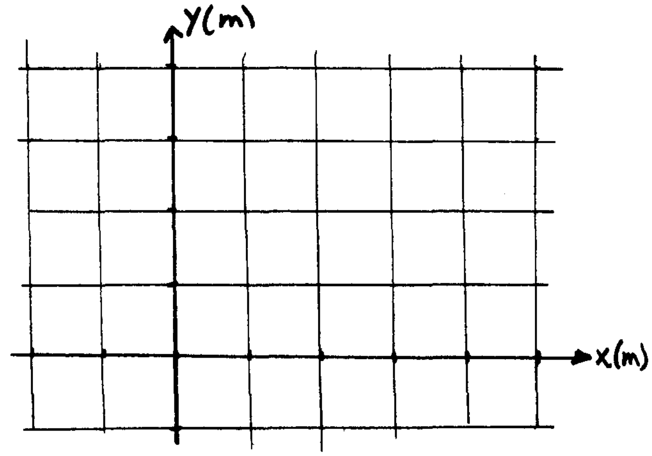
Date ______________

Pass 1 2 3 Recycle 4 5

PLANAR MOTION

Mastery Test Form B

Name ______________ Tutor ______________



1. Draw the path

   $\vec{r}(t) = [(1/2)t^3\hat{i} + t^2\hat{j}]$ m

   for $-1 \leq t \leq 2$ s.

2. For the trajectory of Problem 1, determine the speed at t = 1 s.

3. At an instant when $\vec{r} = (-2\hat{i} + 4\hat{j})$ m, $\vec{v} = 2\hat{i}$ m/s, $\vec{a} = 6\hat{j}$ m/s^2;

   (a) How is $\vec{r}$ changing (magnitude and direction)?

   (b) How is $\vec{v}$ changing (magnitude and direction)?

4. A ball is thrown with an initial speed of 30.0 m/s at an angle of 60° above the horizontal.
   (a) What is its acceleration when it reaches its maximum height?
   (b) How far (distance) is the ball from the release point 3.00 s after release?

5. Consider the circular trajectory

   $$\vec{r}(t) = 0.240[(\sin 5.0t)\hat{i} - (\cos 5.0t)\hat{j}] \text{ m}.$$

   Determine
   (a) the radius,
   (b) the speed,
   (c) the magnitude of the centripetal acceleration,
   (d) the direction of motion (clockwise or counterclockwise).

PLANAR MOTION

Mastery Test Form C

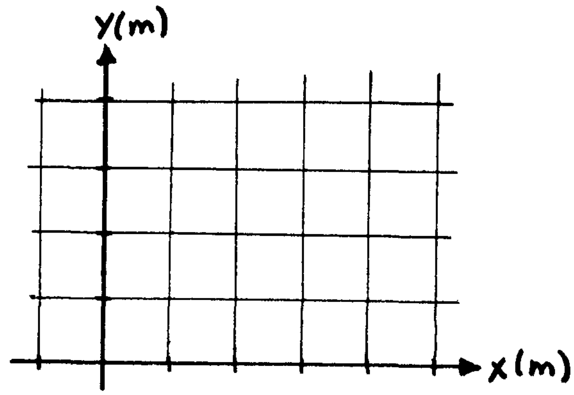
Pass			Recycle	
1	2	3	4	5

Name______________________________ Tutor______________________________

1. Draw the path

$\vec{r}(t) = [4 \cos (\pi t/4)\hat{i} + 2t\hat{j}]$ m

for $0 \leq t \leq 2$ s.



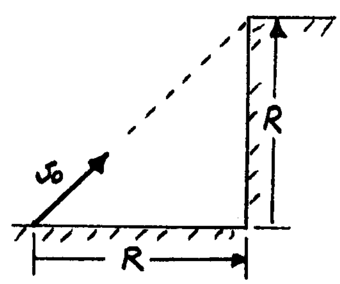
2. For the trajectory of Problem 1, determine the velocity at t = 1 s.

3. At an instant when $\vec{r} = -3\hat{i}$ m, $\vec{v} = 2\hat{j}$ m/s, $\vec{a} = (2\hat{i} - \hat{j})$ m/s^2;

(a) How is $\vec{r}$ changing (magnitude and direction)?

(b) How is $\vec{v}$ changing (magnitude and direction)?

4. A ball is thrown toward a wall as shown. Where will it hit if v_0 = 20.0 m/s and R = 20.0 m?



5. Consider the circular trajectory

$\vec{r}(t) = 1.20[(\cos 15.0t)\hat{i} - (\sin 15.0t)\hat{j}]$ m.

Determine

(a) the speed,

(b) the magnitude of the centripetal acceleration,

(c) the direction of motion (clockwise or counterclockwise).

Date_______________

pass recycle
1 2 3 4 5

PLANAR MOTION

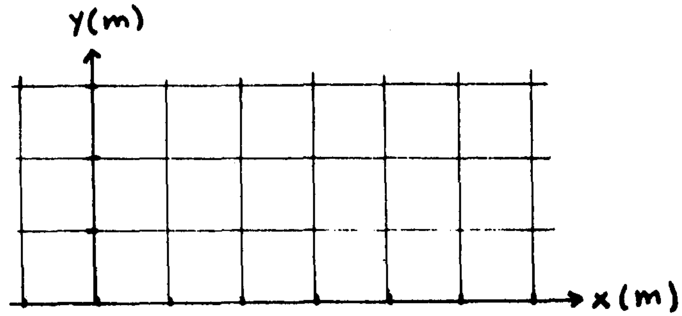
Mastery Test Form D

Name_______________ Tutor_______________

1. Draw the path

$\vec{r}(t) = [(2t/\pi)^2\hat{i} + (2\sin 2t)\hat{j}]$ m

for $0 \leq t \leq \pi/2$ s.



2. Determine the particle's acceleration at $t = (\pi/4)$ s (for Problem 1).

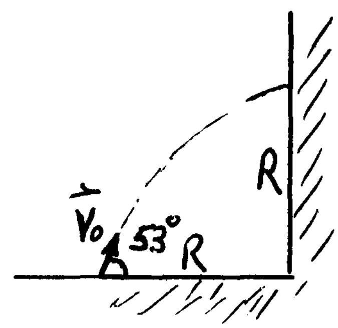
3. At an instant when $\vec{r} = (3\hat{i} + 2\hat{j})$ m

$\vec{v} = (2\hat{i} - 3\hat{j})$ m/s, $\vec{a} = (4\hat{i} + 6\hat{j})$ m/s^2:

(a) How is $\vec{r}$ changing (magnitude and direction)?

(b) How is $\vec{v}$ changing (magnitude and direction)?

4. A thrown rock follows the trajectory shown. Determine the initial speed v_0 if R = 50 m.



5. Consider the circular trajectory

$\vec{r}(t) = 5.00[(\sin 6.0t)\hat{i} - (\cos 6.0t)\hat{j}]$ m.

Determine
(a) the radius,
(b) the speed,
(c) the magnitude of the centripetal acceleration,
(d) the direction of motion (clockwise or counterclockwise).

Date_______________

Pass 1 2 3 Recycle 4 5

PLANAR MOTION

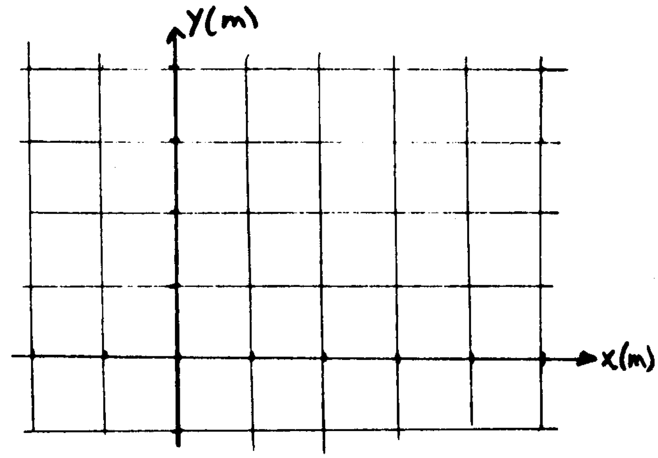
Mastery Test Form *E*

Name_______________ Tutor_______________

1. Draw the path

$\vec{r}(t) = [(1/2)t^3\hat{i} + 2t\hat{j}$

for $-1 \leq t \leq 2$ s.



2. For the trajectory of Problem 1, determine the speed at t = 1 s.

3. At an instant when $\vec{r} = 4\hat{j}$ m
$\vec{v} = 2\hat{i}$ m/s, $\vec{a} = -3\hat{j}$ m/s^2;

(a) How is $\vec{r}$ changing (magnitude and direction)?

(b) How is $\vec{v}$ changing (magnitude and direction)?

4. A ball is thrown with an initial speed of 30.0 m/s at an angle of 53° above the horizontal.
(a) What is its distance(x) when it reaches its maximum height?
(b) How far (distance) is the ball from the release point 3.00 s after release?

5. Consider the circular trajectory

$$\vec{r}(t) = 3.0[(\cos 5.0t)\hat{i} - (\sin 5.0t)\hat{j}] \text{ m}.$$

Determine
(a) the radius,
(b) the speed,
(c) the magnitude of the centripetal acceleration,
(d) the direction of motion (clockwise or counterclockwise).

Date____________________

PLANAR MOTION

Mastery Test Form F

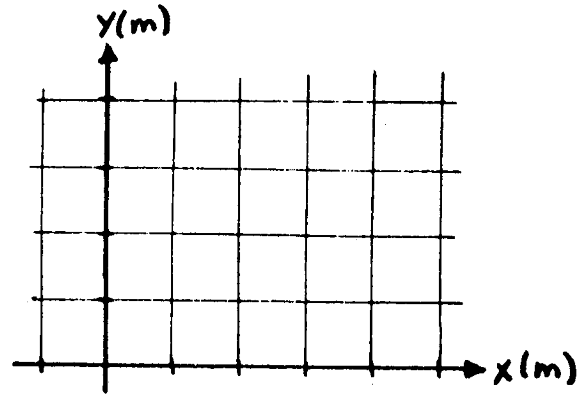
Pass 1 2 3 Recycle 4 5

Name____________________ Tutor____________________

1. Draw the path

$\vec{r}(t) = [4 \sin(\pi t/4)\hat{i} + t^2\hat{j}]$ m

for $0 \leq t \leq 2$ s.



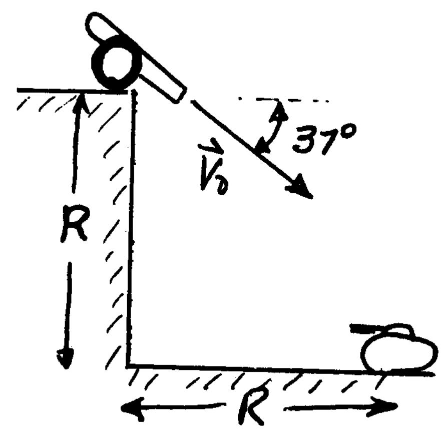
2. For the trajectory of Problem 1, determine the velocity at t = 1 s.

3. At an instant when $\vec{r} = +3\hat{i}$ m, $\vec{v} = -2\hat{j}$ m/s, $\vec{a} = (2\hat{i} - 3\hat{j})$ m/s^2;

(a) How is $\vec{r}$ changing (magnitude and direction)?

(b) How is $\vec{v}$ changing (magnitude and direction)?

4. A cannon located on a cliff is fired at a tank as shown in the sketch. If R = 200 m calculate the velocity of the shell leaving the cannon if it is to hit the tank.



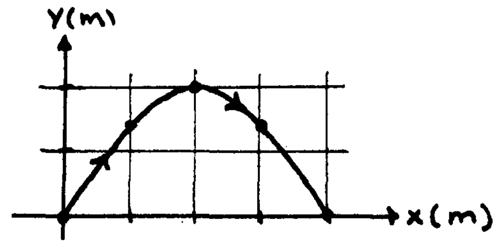
5. Consider the circular trajectory

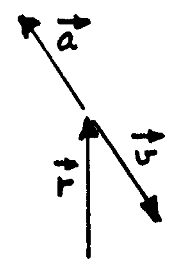
$\vec{r}(t) = 7.0\,[(\cos 15.0t)\hat{i} + (\sin 15.0t)\hat{j}]$ m.

Determine
(a) the speed,
(b) the magnitude of the centripetal acceleration,
(c) the direction of motion (clockwise or counterclockwise).

MASTERY TEST GRADING KEY - Form A

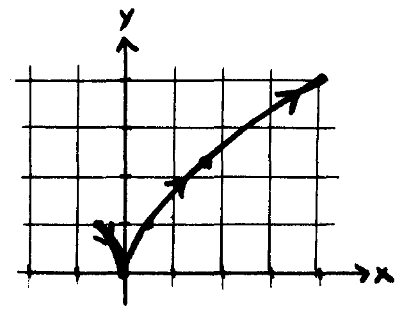
What To Look For	Solutions
1. Correct end points; correct shape.	1. $\vec{r}(0) = 0$, $\vec{r}(\pi/4) = (\hat{i} + \sqrt{2}\hat{j})$ m, $\vec{r}(\pi/2) = (2\hat{i} + 2\hat{j})$ m, $\vec{r}(3\pi/4) = (3\hat{i} + \sqrt{2}\hat{j})$ m, $\vec{r}(\pi) = 4\hat{i}$ m.
2. Correct answer.	2. $\vec{v}(t) = d\vec{r}/dt = [(4/\pi)\hat{i} + (2 \cos t)\hat{j}]$ m/s, $\vec{a}(t) = d\vec{v}/dt = -(2 \sin t)\hat{j}$ m/s^2, $\vec{a}(\pi/2) = -2 \sin (\pi/2)\hat{j} = -2\hat{j}$ m/s^2.
3. Correct answers. Ask for verbal explanation of at least one answer.	3. (a) r (magnitude) decreasing, $\vec{r}$ turning clockwise. (b) v (speed) decreasing, $\vec{v}$ not turning.
4. Correct answer. v_0 in x equation, not in y equation.	4. Since $\vec{a} = -g\hat{j}$ is constant, $\vec{r} = \vec{r}_0 + \vec{v}_0 t + (1/2)\vec{a}t^2$, where $\vec{r}_0 = R\hat{j}$, $\vec{v}_0 = v_0\hat{i}$, so $x = v_0 t$ and $y = R - (1/2)gt^2$. When $x = R$, $y = 0$, thus $R = v_0 t$. $0 = R - (1/2)gt^2$. Eliminating t gives $t = R/v_0$. $R = (1/2)g(R/v_0)^2$. Thus $v_0^2 = (gR/2)$, or $v_0 = \sqrt{gR/2} = 16.0$ m/s.
5. Correct answers. Ask for verbal explanation of part (d).	5. (a) radius = r $= \sqrt{(2.00 \sin 6.0t)^2 + (2.00 \cos 6.0t)^2}$ = 2.00 m. (b) $\vec{v} = d\vec{r}/dt = 12.0[(\cos 6.0t)\hat{i} - (\sin 6.0t)\hat{j}]$ m/s. speed $= v = \sqrt{(12.0 \cos 6.0t)^2 + (-12.0 \sin 6.0t)^2}$ = 12.0 m/s. (c) $a = v^2/r = (12.0\text{ m/s})^2/(2.00\text{ m}) = 72$ m/s^2 (d) at t = 0, $\vec{r}(0) = 2.00\hat{j}$ m, $\vec{v}(0) = 12.0\hat{i}$ m/s, so motion is clockwise.

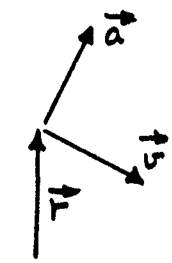




MASTERY TEST GRADING KEY - Form B

What to Look For	Solutions
1. Correct end points. Correct shape.	1. $\vec{r}(-1) = [-(1/2)\hat{i} + \hat{j}]$ m. $\vec{r}(0) = 0$. $\vec{r}(1) = [(1/2)\hat{i} + \hat{j}]$ m. $\vec{r}(2) = (4\hat{i} + 4\hat{j})$ m. $\vec{r}(3/2) = [(27/16)\hat{i} + (9/4)\hat{j}]$ m.
2. Correct answer.	2. $\vec{v}(t) = [(3/2)t^2\hat{i} + 2t\hat{j}]$ m/s. speed = $v(1\text{ s}) = \sqrt{(9/4) + 4} = \sqrt{25/4} = (5/2)$ m/s.
3. Correct answers. Ask for verbal explanation of part (b).	3. (a) r (magnitude) decreasing, $\vec{r}$ turning clockwise. (b) v not changing, $\vec{v}$ turning counterclockwise.
4. Correct answers. You might ask student to explain why $a_y \neq 0$ when $v_y = 0$ in part (a).	4. (a) $\vec{a} = -g\hat{j} = -9.8\hat{j}$ m/s^2 at all times. (b) Since $\vec{a}$ constant, $\vec{r} = \vec{v}_0 t + (1/2)\vec{a}t^2$ $(\vec{r}_0 = 0)$ or $\vec{r}(t) = (v_0 \cos\theta t)\hat{i} + [v_0 \sin\theta t - (1/2)gt^2]\hat{j}$. Thus $r(3.00\text{ s}) = \{[(30.0)(0.50)(3.00)]\hat{i}$ $+ [(30.0)(1.73)(3.00) - (1/2)(9.8)(9.0)]\hat{j}\}$ m $= [45\hat{i} + (78 - 44)\hat{j}]$ m $= (45\hat{i} + 34\hat{j})$ m. Distance $= \sqrt{(45)^2 + (34)^2} =$ 56 m.
5. Correct answers. Ask for verbal explanation of part (d).	5. (a) radius = r $= \sqrt{(0.240 \sin 5.0t)^2 + (0.240 \cos 5.0t)^2}$ = 0.240 m. (b) $\vec{v}(t) = 1.20\,[(\cos 5.0t)\hat{i} + (\sin 5.0t)\hat{j}]$ m/s. speed = v $= \sqrt{(1.20 \cos 5.0t)^2 + (1.20 \sin 5.0t)^2}$ = 1.20 m/s. (c) $a = v^2/r = [(1.44\ \text{m}^2/\text{s}^2)/(0.240\text{ m})] = 6.0$ m/s^2. (d) $\vec{r}_0 = -0.240\hat{j}$, $\vec{v}_0 = 1.20\hat{i}$ m/s. Thus the motion is counterclockwise.





MASTERY TEST GRADING KEY - Form C

What To Look For	Solutions
1. Correct end points. Correct shape.	1. $\vec{r}(0) = 4\hat{i}$ m. $\vec{r}(1) = (2\sqrt{2}\hat{i} + 2\hat{j})$ m. $\vec{r}(2) = 4\hat{j}$ m.
2. Correct answer.	2. $\vec{v} = d\vec{r}/dt = [\,-\pi \sin(\pi t/4)\hat{i} + 2\hat{j}\,]$ m/s. $\vec{v}(1\text{ s}) = [-\pi \sin(\pi/4)\hat{i} + 2\hat{j}] = (-\sqrt{2}\pi/2\hat{i} + 2.0\hat{j})$ m/s.
3. Correct answers. Ask for verbal explanation of at least one part.	3. (a) r not changing, $\vec{v} \perp \vec{r}$. $\vec{r}$ turning clockwise. (b) v decreasing, $\vec{v}$ turning clockwise.
4. Correct answer. See that initial velocity is resolved correctly into x and y components.	4. $v_{0y} = v_{0x} = v_0 \cos 45° = v_0/\sqrt{2}$. When will x = R? $R = v_{0x}t_R = (v_0/\sqrt{2})t_R$; $t_R = \sqrt{2}R/v_0$. What is y at this time? $y = v_{0y}t - \frac{1}{2}gt^2 = \frac{v_0}{\sqrt{2}}\frac{\sqrt{2}R}{v_0} - \frac{1}{2}g\frac{2R^2}{v_0^2} = R - \frac{gR^2}{v_0^2}$ $= 20.0[1 - \frac{(9.8)(20.0)}{400}] = 20.0(0.51) = 10.2$ m. Ball hits wall 10.2 m above ground.
5. Correct answers. Ask for verbal explanation of part (c).	5. (a) $\vec{v}(t) = 18.0[(-\sin 15.0t)\hat{i} - (\cos 15.0t)\hat{j}]$ m/s. speed = v $= \sqrt{(18.0 \sin 15.0t)^2 + (18.0 \cos 15.0t)^2}$ = 18.0 m/s. (b) $a = v^2/r = (324\text{ m}^2/\text{s}^2)/1.20\text{ m} = 270\text{ m/s}^2$. (c) $\vec{r}_0 = 1.20\hat{i}$ m, $\vec{v}_0 = -18.0\hat{j}$ m/s. Motion is clockwise.