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Jean Baptiste Joseph Fourier

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Jean Baptiste Joseph Fourier was born in Auxerre, France on March 21, 1768. He was the ninth of twelve children from his father's second marriage. When he was nine, his mother died. The following year, his father, a tailor, also passed.

His first schooling was a Pallais's school, run by the music master from a cathedral. There Joseph studied Latin and French and showed great promise. He proceeded in 1780 to the Ecole Royale Militaire of Auxerre where at first he showed talents for literature but very soon, by the age of thirteen, mathematics became his real interest. He completed a study of the six volumes of Bezout's *Cours de mathematiques* by the age of 14. In 1873 he received the first prize for his study of Bossut's *Mecanique en general* (O'Connor and Robertson, 1997).

After entering the Benedictine abbey of St. Benoit-sur-Loire in 1787, Fourier's interest for mathematics continued. He corresponded with C. L. Bonard, the professor of mathematics at Auxerre. Joseph Fourier's letters to Bonard suggest that he really wanted to make a major impact in mathematics. In one letter, Fourier wrote

> *"Yesterday was my 21st birthday, at that age Newton and Pascal had already acquired many claims to immortality."*

Joseph Fourier left St. Benoit in 1789 and visited Paris. His first memoir on the numerical solution of algebraic equations was read before the French Academy of Sciences. The commissions in the scientific corps of the army were, as is still the case in Russia, reserved for those of good stature, and being thus ineligible he accepted a military lectureship on mathematics at the Ecole Royale Militaire of Auxerre (Ball, 1908). Up until this time there had been a conflict inside Fourier about whether he should study religion or mathematics. Then, in 1793, a third element to his professional conflict was added when he became involved in politics and joined the Revolutionary Committee. According to O'Connor and Robertson, Fourier wrote:

> *As the natural ideas of equality developed it was possible to conceive the sublime hope of establishing among us a free government exempt from kings and priests, and to free from this double yoke the long-usurped soil of Europe. I readily became enamoured of this cause, in my opinion the greatest and most beautiful which any nation has ever undertaken.*

Fourier took a prominent part in his own district in promoting the revolution, and was rewarded by an appointment in 1795 in the Normal school, an institution set up for training teachers. He was taught by Lagrange, Laplace, and Monge.

Fourier began teaching at the College of France and, having excellent relations with Lagrange, Laplace, and Monge, began further mathematical research. He was appointed to a position at the Ecole Centrale des Travaux Publiques or Ecole Polytechnique, the school being under the direction of Lazare Carnot and Gaspard Monge. In 1797, he succeeded Lagrange in being appointed to the chair of analysis and mechanics. He was renowned as an outstanding lecturer but he did not undertake any original research at this time (O'Conner and Robertson, 1997).

Fourier was 30 when Napoleon requested his participation as scientific advisor on an expedition to Egypt. Fourier served from 1798 to 1802 as secretary of the Institute of Egypt, established by Napoleon to explore systematically the archeological riches of that ancient land. Cut off from France by the English fleet, he organized the workshops on which the French army had to rely for their munitions of war. He also contributed several mathematical papers to the Egyptian Institute which Napoleon founded at Cairo which ranged from the general solution of algebraic equations to irrigation projects. After the British victories and the capitulation of the French under General Menouin 1801, Fourier returned to France, and was made Prefect of Grenoble (Ball, 1908).

It was during his time in Grenoble that Fourier did his important mathematical work on the theory of heat. His work on the topic began around 1804 and by 1807 he had completed his important memoir *On the Propagation of Heat in Solid Bodies.* The memoir, which told of heat conduction and its dependency on the essential idea of analyzing the temperature distribution into spatially sinusoidal components, was read to the Paris Institut on December 21, 1807. A committee consisting of Lagrange, Laplace, Monge, and Lacroix was set up to report on the work (O'Connor and Robertson, 1997). This memoir is very highly regarded today, but at the time it caused much controversy.

There were two reasons for the committee to feel unhappy with the work. The first objection, made by Lagrange and Laplace in 1808, was to Fourier's expansions of functions as trigonometrical series. We now call the expansions the Fourier series. Further clarification by Fourier still failed to convince them.

The second objection was made by Biot against Fourier's derivation of the equations of transfer of heat. Fourier had not made reference to Biot's 1804 paper on this topic. Laplace, and later Poisson, had similar objections. Even though there were objections, the Institut set the propagation of heat in solid bodies as the topic for the prize of mathematics for 1811. The prize was granted to Fourier but with a citation mentioning lack of generality and rigor (Bracewell, 1986). The fact that publication was further delayed until 1815 can be seen as an indication of the deep uneasiness about Fourier analysis that was felt by the great mathematicians of the day.

Fourier considered heat flow in a ring instead of a straight bar distribution. The bar was bent into a circle. In this way, the temperature distribution is forced to be spatially periodic. There is essentially no loss of generality because the circumference of the ring can be supposedly larger than the greatest distance that could be of physical interest on a straight bar conducting heat.

Fourier concluded that the flow of heat between two adjacent molecules is proportional to the extremely small difference of their temperatures. In his work he claims that any function of a variable, whether continuous or discontinuous, can be expanded in a series of sines of multiples of the variable (Answers.com). Fourier's observation that some discontinuous functions are the sum of infinite series was a breakthrough. The question of determining when a function is the sum of its Fourier series has been fundamental for centuries. His observations and work contained important mathematical techniques which later were developed into a special branch of mathematics – Fourier analysis and Fourier integrals.

The Fourier series is a mathematical tool used for analyzing periodic functions by decomposing such a function into a weighted sum of much simpler sinusoidal component functions. These functions are sometimes referred to a normal Fourier modes, or simply modes for short. The weight, or coefficients, together with the modes, are uniquely determined by the original function.

According to sosmath.com, a Fourier series can be defined by letting $f(x)$ be a 2π -periodic function which is integrable on $[-\pi, \pi]$. Then set

$$\begin{cases} a_0 & = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx, \\ a_n = & = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)dx, \quad 1 \le n \\ b_n & = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)dx, \quad 1 \le n. \end{cases}$$

(where n is an integer)

The trigonometric series

$$a_0 + \sum \left(a_n \cos(nx) + b_n \sin(nx)\right)$$

is called the **Fourier series** associated to the function *f*(*x*). We will use the notation

$$f(x) \sim a_0 + \sum_{n=1}^{\infty} \left(a_n \cos(nx) + b_n \sin(nx)\right).$$

Fourier series serve many useful purposes, as manipulation and conceptualization of the modal coefficients are often easier than with the original function. Areas of application include electrical engineering vibration analysis, acoustics, optics, signal and image processing, and data compression. Using the tools and techniques of spectroscopy for example, astronomers can deduce the chemical composition of a star by analyzing the frequency components (i.e. the n's and the modal weights, a_n and b_n in the series), or spectrum of the star's emitted light. Similarly, engineers can optimize the design of a telecommunications system using information about the spectral components of the data that the system will carry.

According to hyperphysics.phy-astr.gsu.edu, the Fourier series has far-reaching implications for the reproduction and synthesis of sound. A pure sine wave can be converted into sound by a loudspeaker and will be perceived to be a steady, pure tone of a single pitch. The sounds from orchestral instruments usually consist of a fundamental and a complement of harmonics, which can be considered to be a superposition of sine waves of a fundamental frequency f and integer multiples of that frequency.

The process of decomposing a musical instrument sound or any other periodic function into its constituent sine or cosine waves is called Fourier analysis. The Fourier analysis can characterize the sound wave in terms of the amplitudes of the constituent sine waves which make it up. This set of numbers tells the harmonic content of the sound and is sometimes referred to as the harmonic spectrum of the sound. The harmonic content is the most important determiner of the quality or timbre of a sustained musical note.

Once the harmonic content of a sustained musical sound from Fourier analysis is known, then a person is capable of synthesizing that sound from a series of pure tone generators by properly adjusting their amplitudes and phases and adding them together. This is called Fourier synthesis.

The Fourier analysis was originally developed to solve a particular heat equation, about 200 years ago. However, over the years, the Fourier analysis has been shown to be an indispensable tool not only for mathematics but also for many different fields of science and technology, and generalized to various different forms. Its philosophy is still the same: analyze a function by decomposing it into a linear combination of elementary building blocks (e.g., sines and cosines). Other topics can be chosen from the following list:

- Gibbs's phenomena
- Trigonometric approximations
- Chebychev polynomial and the best approximations
- Random walks and Brownian motions
- The central limit theorem
- Shannon's sampling theorem and discretization of signals and functions
- Heisenberg's uncertainty principle
- Discrete Fourier Transforms
- Fast Fourier Transforms

- Gaussian quadratures for numerical integrations
- Spherical harmonics

The Fourier series can easily be adapted for functions that do not have a period of 2π.

Suppose there is a functions *f(x)* that is periodic with a period interval of $a \le x < b$ or of *b-a*.

Define $x = \frac{(b-a)t}{2\pi} + \frac{(b+a)}{2}$ then *x* varies over the interval [*a*, *b*] when *t* varies over the interval $[-\pi, \pi]$. Thus, if $x = \frac{(b-a)t}{2\pi} + \frac{(b+a)}{2}$ is substituted into *f* and define a new function *g* by $g(t) = f\left(\frac{(b-a)t}{2\pi} + \frac{(b+a)}{2}\right) = f(x)$, then *g* has a period of 2π. The Fourier series for *g* can be found: $g(t) = a_o + a_1 \cos t + a_2 \cos(2t) + a_3 \cos(3t) + \cdots + b_1 \sin t + b_2 \sin(2t) + b_3 \sin(3t) + \cdots$.

Substituting $t = (x - (b+a)/2)(2\pi/(b-a)) = \frac{(2x-b-a)\pi}{(b-a)}$ allows the conversion back:

$$f(x) = g\left(\frac{(x-b-a)\pi}{b-a}\right) = a_0 + a_1 \cos\left(\frac{(2x-b-a)\pi}{b-a}\right) + a_2 \cos\left(\frac{(2x-b-a)2\pi}{b-a}\right) + a_3 \cos\left(\frac{(2x-b-a)3\pi}{b-a}\right) + \cdots$$
$$+ b_1 \sin\left(\frac{(2x-b-a)\pi}{b-a}\right) + b_2 \sin\left(\frac{(2x-b-a)2\pi}{b-a}\right) + b_3 \sin\left(\frac{(2x-b-a)3\pi}{b-a}\right) + \cdots$$

Note that the terms in the Fourier series of a periodic function of period *b-a* are not cos(*kx*) and sin(*kx*), but instead are $\cos(\pi k(2x-b-a)/(b-a))$ and $\sin(\pi k(2x-b-a)/(b-a))$.

Given the function f(*x*) = -1 for $-1 \le x < 0$ and f(*x*) = 1 for $0 \le x < 1$ with a period interval of [-1, 1], the Fourier series can be found by solving for the coefficients of a_k and b_k.

$$a_0 = \frac{1}{2}\int_{-1}^{0} -1dx + \frac{1}{2}\int_{0}^{1} 1dx$$

$$a_0 = \frac{1}{2}(-x)\Big|_{-1}^{0} + \frac{1}{2}x\Big|_{0}^{1}$$

$$a_0 = \left(0 - \frac{1}{2}\right) + \left(\frac{1}{2} - 0\right)$$

$$a_0 = 0$$

$$a_1 = \int_{-1}^{0} -\cos \pi x dx + \int_{0}^{1} \cos \pi x dx$$

$$a_1 = -\frac{1}{\pi}\sin \pi x\Big|_{-1}^{0} + \frac{1}{\pi}\sin \pi x\Big|_{0}^{1}$$

$$a_1 = \frac{1}{\pi}(-\sin 0 + \sin(-\pi)) + \frac{1}{\pi}(\sin \pi - \sin 0)$$

$$a_1 = \frac{1}{\pi}(0-0) + \frac{1}{\pi}(0-0)$$

$$a_1 = 0$$

$$a_2 = \int_{-1}^{0} -\cos 2\pi x dx + \int_{0}^{1} \cos 2\pi x dx$$

$$a_2 = -\frac{1}{2\pi}\sin 2\pi x\Big|_{-1}^{0} + \frac{1}{2\pi}\sin 2\pi x\Big|_{0}^{1}$$

$$a_2 = \frac{1}{2\pi}(-\sin 0 + \sin(-2\pi)) + \frac{1}{2\pi}(\sin 2\pi - \sin 0)$$

$$a_2 = \frac{1}{2\pi}(0-0) + \frac{1}{2\pi}(0-0)$$

$$a_2 = 0$$

$$a_3 = \int_{-1}^{0} -\cos 3\pi x dx + \int_{0}^{1} \cos 3\pi x dx$$

$$a_3 = -\frac{1}{3\pi}\sin 3\pi x\Big|_{-1}^{0} + \frac{1}{3\pi}\sin 3\pi x\Big|_{0}^{1}$$

$$a_3 = \frac{1}{3\pi}(-\sin 0 + \sin(-3\pi)) + \frac{1}{3\pi}(\sin 3\pi - \sin 0)$$

$$a_3 = \frac{1}{3\pi}(0-0) + \frac{1}{3\pi}(0-0)$$

$$a_3 = 0$$

$$b_1 = \int_{-1}^{0} -\sin \pi x dx + \int_{0}^{1} \sin \pi x dx$$

$$b_1 = \frac{1}{\pi}\cos \pi x\Big|_{-1}^{0} - \frac{1}{\pi}\sin \pi x\Big|_{0}^{1}$$

$$b_1 = \frac{1}{\pi}(\cos 0 - \cos(-\pi)) - \frac{1}{\pi}(\cos \pi - \cos 0)$$

$$b_1 = \frac{1}{\pi}(1-(-1)) - \frac{1}{\pi}(-1-1)$$

$$b_1 = \frac{4}{\pi}$$

$$b_2 = \int_{-1}^{0} -\sin 2\pi x dx + \int_{0}^{1} \sin 2\pi x dx$$

$$b_2 = \frac{1}{2\pi}\cos 2\pi x\Big|_{-1}^{0} - \frac{1}{2\pi}\sin 2\pi x\Big|_{0}^{1}$$

$$b_2 = \frac{1}{2\pi}(\cos 0 - \cos(-2\pi)) - \frac{1}{2\pi}(\cos 2\pi - \cos 0)$$

$$b_2 = \frac{1}{2\pi}(1-1) - \frac{1}{2\pi}(1-1)$$

$$b_2 = 0$$

$$b_3 = \int_{-1}^{0} -\sin 3\pi x dx + \int_{0}^{1} \sin 3\pi x dx$$

$$b_3 = \frac{1}{3\pi}\cos 3\pi x\Big|_{-1}^{0} - \frac{1}{3\pi}\sin 3\pi x\Big|_{0}^{1}$$

$$b_3 = \frac{1}{3\pi}\left(\cos 0 - \cos(-3\pi)\right) - \frac{1}{3\pi}\left(\cos 3\pi - \cos 0\right)$$

$$b_3 = \frac{1}{3\pi}\left(1-(-1)\right) - \frac{1}{3\pi}\left(-1-1\right)$$

$$b_3 = \frac{4}{3\pi}$$

$$b_4 = \int_{-1}^{0} -\sin 4\pi x dx + \int_{0}^{1} \sin 4\pi x dx$$

$$b_4 = \frac{1}{4\pi}\cos 4\pi x\Big|_{-1}^{0} - \frac{1}{4\pi}\sin 4\pi x\Big|_{0}^{1}$$

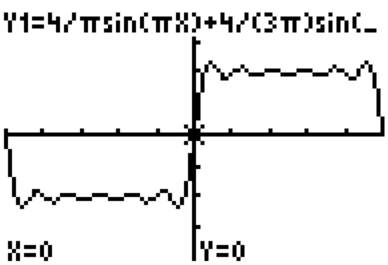
$$b_4 = \frac{1}{4\pi}\left(\cos 0 - \cos(-4\pi)\right) - \frac{1}{4\pi}\left(\cos 4\pi - \cos 0\right)$$

$$b_4 = \frac{1}{4\pi}\left(1-1\right) - \frac{1}{4\pi}\left(1-1\right)$$

$$b_4 = 0$$

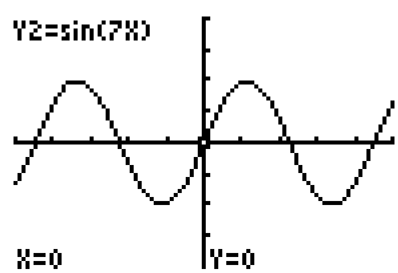
The Fourier series for the function f(x) is given by the generalized function:

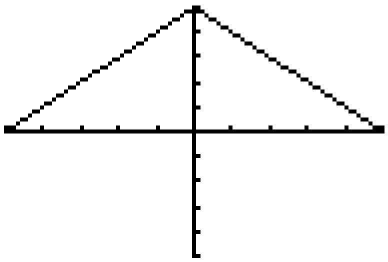
$$F(x) = \frac{4}{\pi}\sin \pi x + \frac{4}{3\pi}\sin 3\pi x + \frac{4}{5\pi}\sin 5\pi x + \cdots \text{ or } F(x) = \sum_{k=1,3,5\ldots}^{n} \frac{4}{k\pi}\sin(k\pi x)$$



The graph of the 11th degree Fourier polynomial of the square wave f(x) is graphed here or in Appendix A. The graph is beginning to resemble a line at y = -1 for x < 0 and y = 1 for x > 0.

Given the function f(x) = sin(7x) on the period interval of $[-2\pi, 2\pi]$, the Fourier series can be found. The function will have one term at b_7 with a coefficient of 1. The other terms will have a coefficient of zero, thus leaving a Fourier series of F(x) = sin(7x). It's graph is shown at the right or in Appendix B.





The function defined as f(x) = 1 + x for -1 $\le$ x < 0 and 1 – x for 0 $\le$ x < 1 with a period interval of [-1, 1] is called the saw tooth function. If the function is extended periodically, the result with look similar to the teeth of a saw. The function can be approximated by finding a Fourier polynomial of degree n. The construction of the Fourier series is first began by finding the Fourier coefficients.

$$a_0 = \frac{1}{2}\int_{-1}^{0}(1+x)dx + \frac{1}{2}\int_{0}^{1}(1-x)dx$$

$$a_0 = \frac{1}{2}\left(x+\frac{1}{2}x^2\right)\Bigg|_{-1}^{0} + \frac{1}{2}\left(x-\frac{1}{2}x^2\right)\Bigg|_{0}^{1}$$

$$a_0 = \frac{1}{2}\left[0-\left(-1+\frac{1}{2}\right)\right] + \frac{1}{2}\left[\left(1-\frac{1}{2}\right)-0\right]$$

$$a_0 = \frac{1}{2}\left(\frac{1}{2}\right) + \frac{1}{2}\left(\frac{1}{2}\right)$$

$$a_0 = \frac{1}{2}$$

$$a_k = \int_{-1}^{0}(1+x)(\cos k\pi x)dx + \int_{0}^{1}(1-x)(\cos k\pi x)dx$$

$$a_k = \frac{1}{k\pi}(1+x)(\sin k\pi x)\Bigg|_{-1}^{0} - \frac{1}{k\pi}\int_{-1}^{0}(\sin k\pi x)dx + \frac{1}{k\pi}(1-x)(\sin k\pi x)\Bigg|_{0}^{1} + \frac{1}{k\pi}\int_{0}^{1}(\sin k\pi x)dx$$

$$a_k = \frac{1}{k\pi}(1+x)(\sin k\pi x)\Bigg|_{-1}^{0} + \frac{1}{(k\pi)^2}(\cos k\pi x)\Bigg|_{-1}^{0} + \frac{1}{k\pi}(1-x)(\sin k\pi x)\Bigg|_{0}^{1} - \frac{1}{(k\pi)^2}(\cos k\pi x)\Bigg|_{0}^{1}$$

$$a_1 = \int_{-1}^{0}(1+x)(\cos \pi x)dx + \int_{0}^{1}(1-x)(\cos \pi x)dx$$

$$a_1 = \frac{1}{\pi}(1+x)(\sin \pi x)\Bigg|_{-1}^{0} - \frac{1}{\pi}\int_{-1}^{0}(\sin \pi x)dx + \frac{1}{\pi}(1-x)(\sin \pi x)\Bigg|_{0}^{1} + \frac{1}{\pi}\int_{0}^{1}(\sin \pi x)dx$$

$$a_1 = \frac{1}{\pi}(1+x)(\sin \pi x)\Bigg|_{-1}^{0} + \frac{1}{(\pi)^2}(\cos \pi x)\Bigg|_{-1}^{0} + \frac{1}{\pi}(1-x)(\sin \pi x)\Bigg|_{0}^{1} - \frac{1}{(\pi)^2}(\cos \pi x)\Bigg|_{0}^{1}$$

$$a_1 = \frac{1}{\pi}[\sin 0 - (0)(\sin(-\pi))] + \frac{1}{\pi^2}[(\cos 0) - (\cos(-\pi)] + \frac{1}{\pi}[(0)(\sin\pi) - (\sin 0)] - \frac{1}{\pi^2}[\cos\pi - \cos 0]$$

$$a_1 = 0 + \frac{2}{\pi^2} + 0 + \frac{2}{\pi^2}$$

$$a_1 = \frac{4}{\pi^2}$$

$$a_2 = \int_{-1}^{0} (1+x)(\cos 2\pi x)dx + \int_{0}^{1} (1-x)(\cos 2\pi x)dx$$

$$a_2 = \frac{1}{2\pi}(1+x)(\sin 2\pi x)\Big|_{-1}^{0} - \frac{1}{2\pi}\int_{-1}^{0}(\sin 2\pi x)dx + \frac{1}{2\pi}(1-x)(\sin 2\pi x)\Big|_{0}^{1} + \frac{1}{2\pi}\int_{0}^{1}(\sin 2\pi x)dx$$

$$a_2 = \frac{1}{2\pi}(1+x)(\sin 2\pi x)\Big|_{-1}^{0} + \frac{1}{(2\pi)^2}(\cos 2\pi x)\Big|_{-1}^{0} + \frac{1}{2\pi}(1-x)(\sin 2\pi x)\Big|_{0}^{1} - \frac{1}{(2\pi)^2}(\cos 2\pi x)\Big|_{0}^{1}$$

$$a_2 = \frac{1}{2\pi}[\sin 0 - (0)(\sin(-2\pi))] + \frac{1}{(2\pi)^2}[(\cos 0) - (\cos(-2\pi)] + \frac{1}{2\pi}[(0)(\sin 2\pi) - (\sin 0)] - \frac{1}{(2\pi)^2}[\cos 2\pi - \cos 0]$$

$$a_2 = 0 + 0 + 0 - 0$$

$$a_2 = 0$$

$$a_3 = \int_{-1}^{0} (1+x)(\cos 3\pi x)dx + \int_{0}^{1} (1-x)(\cos 3\pi x)dx$$

$$a_3 = \frac{1}{3\pi}(1+x)(\sin 3\pi x)\Big|_{-1}^{0} - \frac{1}{3\pi}\int_{-1}^{0}(\sin 3\pi x)dx + \frac{1}{3\pi}(1-x)(\sin 3\pi x)\Big|_{0}^{1} + \frac{1}{3\pi}\int_{0}^{1}(\sin 3\pi x)dx$$

$$a_3 = \frac{1}{3\pi}(1+x)(\sin 3\pi x)\Big|_{-1}^{0} + \frac{1}{(3\pi)^2}(\cos 3\pi x)\Big|_{-1}^{0} + \frac{1}{3\pi}(1-x)(\sin 3\pi x)\Big|_{0}^{1} - \frac{1}{(3\pi)^2}(\cos 3\pi x)\Big|_{0}^{1}$$

$$a_3 = \frac{1}{3\pi}[\sin 0 - (0)(\sin(-3\pi))] + \frac{1}{(3\pi)^2}[(\cos 0) - (\cos(-3\pi)] + \frac{1}{3\pi}[(0)(\sin 3\pi) - (\sin 0)] - \frac{1}{(3\pi)^2}[\cos 3\pi - \cos 0]$$

$$a_3 = 0 + \frac{2}{(3\pi)^2} + 0 + \frac{2}{(3\pi)^2}$$

$$a_3 = \frac{4}{(3\pi)^2}$$

$$b_k = \int_{-1}^{0} (1+x)(\sin k\pi x)dx + \int_{0}^{1} (1-x)(\sin k\pi x)dx$$

$$b_k = -\frac{1}{k\pi}(1+x)(\cos k\pi x)\Big|_{-1}^{0} + \frac{1}{k\pi}\int_{-1}^{0}(\cos k\pi x)dx - \frac{1}{k\pi}(1-x)(\cos k\pi x)\Big|_{0}^{1} + \frac{1}{k\pi}\int_{0}^{1}(\cos k\pi x)dx$$

$$b_k = -\frac{1}{k\pi}(1+x)(\cos k\pi x)\Big|_{-1}^{0} + \frac{1}{(k\pi)^2}(\sin k\pi x)\Big|_{-1}^{0} - \frac{1}{k\pi}(1-x)(\cos k\pi x)\Big|_{0}^{1} - \frac{1}{(k\pi)^2}(\sin k\pi x)\Big|_{0}^{1}$$

$$b_1 = \int_{-1}^{0}(1+x)(\sin \pi x)dx + \int_{0}^{1}(1-x)(\sin \pi x)dx$$

$$b_1 = -\frac{1}{\pi}(1+x)(\cos \pi x)\Big|_{-1}^{0} + \frac{1}{\pi}\int_{-1}^{0}(\cos \pi x)dx - \frac{1}{\pi}(1-x)(\cos \pi x)\Big|_{0}^{1} + \frac{1}{\pi}\int_{0}^{1}(\cos \pi x)dx$$

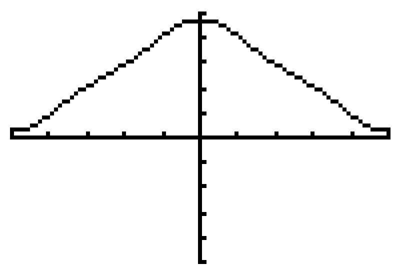
$$b_1 = -\frac{1}{\pi}(1+x)(\cos \pi x)\Big|_{-1}^{0} + \frac{1}{(\pi)^2}(\sin \pi x)\Big|_{-1}^{0} - \frac{1}{\pi}(1-x)(\cos \pi x)\Big|_{0}^{1} - \frac{1}{(\pi)^2}(\sin \pi x)\Big|_{0}^{1}$$

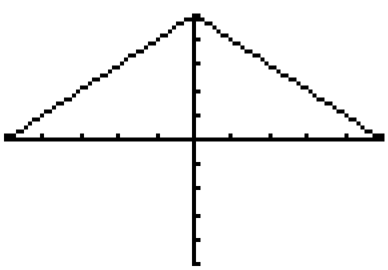
$$b_1 = -\frac{1}{\pi}[(\cos 0)-(0)(\cos(-\pi))] + \frac{1}{(\pi)^2}[\sin 0 - \sin(-\pi)] - \frac{1}{\pi}[(0)(\cos \pi)-(\cos 0)] - \frac{1}{(\pi)^2}[\sin \pi - \sin 0]$$

$$b_1 = -\frac{1}{\pi} + 0 + \frac{1}{\pi} - 0$$

$$b_1 = 0$$

The coefficients b_2, b_3, and so on will continue to have a value of zero. The function for the Fourier series can be written as $F(x) = \frac{1}{2} + \frac{4}{\pi^2}\cos \pi x + \frac{4}{(3\pi)^2}\cos 3\pi x + \frac{4}{(5\pi)^2}\cos 5\pi x + \cdots$ or $F(x) = \frac{1}{2} + \sum_{k=1,3,5,\ldots}^{n} \frac{4}{(k\pi)^2}\cos k\pi x$. The graph of the 3rd degree Fourier polynomial at the right shows a close approximation of the original saw tooth function.





However, the 7th degree Fourier polynomial at the left or in Appendix C shows a better approximation of the original saw tooth function.

The final function given is f(x) = 0 for -1 $\leq x <$ -1/10, = 10 + 100x for -1/10 $\leq x <$ 0, = 10 – 100x for 0 $\leq x <$ 1/10, = 0 for 1/10 $\leq x <$ 1. The function is defined as periodic on the interval [-1, 1]. Similarly as the previous function, the Fourier coefficients can be found.

However, the integrals for the function from -1 $\le$ x < -1/10 and 1/10 $\le$ x < 1 do not need to be found since the graph is on the horizontal axis. The integral will then be zero at all times.

$$a_0 = \frac{1}{2}\int_{-.1}^{0}(10+100x)dx + \frac{1}{2}\int_{0}^{.1}(10-100x)dx$$

$$a_0 = \frac{1}{2}\left(10x+50x^2\right)\Big|_{-.1}^{0} + \frac{1}{2}\left(10x-50x^2\right)\Big|_{0}^{.1}$$

$$a_0 = \frac{1}{2}\left[0-\left(-1+\frac{1}{2}\right)\right] + \frac{1}{2}\left[\left(1-\frac{1}{2}\right)-0\right]$$

$$a_0 = \frac{1}{2}\left(\frac{1}{2}\right)+\frac{1}{2}\left(\frac{1}{2}\right)$$

$$a_0 = \frac{1}{2}$$

$$a_k = \int_{-.1}^{0}(10+100x)(\cos k\pi x)dx + \int_{0}^{.1}(10-100x)(\cos k\pi x)dx$$

$$a_k = \frac{1}{k\pi}(10+100x)(\sin k\pi x)\Big|_{-.1}^{0} - \frac{100}{k\pi}\int_{-.1}^{0}(\sin k\pi x)dx + \frac{1}{k\pi}(10-100x)(\sin k\pi x)\Big|_{0}^{.1} - \frac{100}{k\pi}\int_{0}^{.1}(\sin k\pi x)dx$$

$$a_k = \frac{1}{k\pi}(10+100x)(\sin k\pi x)\Big|_{-.1}^{0} + \frac{100}{(k\pi)^2}(\cos k\pi x)\Big|_{-.1}^{0} + \frac{1}{k\pi}(10-100x)(\sin k\pi x)\Big|_{0}^{.1} - \frac{100}{(k\pi)^2}(\cos k\pi x)\Big|_{0}^{.1}$$

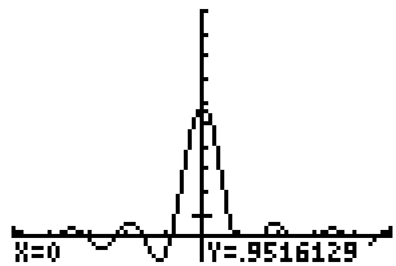
$$b_k = \int_{-.1}^{0}(10+100x)(\sin k\pi x)dx + \int_{0}^{.1}(10-100x)(\sin k\pi x)dx$$

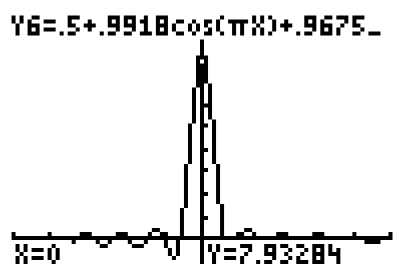
$$b_k = -\frac{1}{k\pi}(10+100x)(\cos k\pi x)\Big|_{-.1}^{0} + \frac{100}{k\pi}\int_{-.1}^{0}(\cos k\pi x)dx - \frac{1}{k\pi}(10-100x)(\cos k\pi x)\Big|_{0}^{.1} + \frac{100}{k\pi}\int_{0}^{.1}(\cos k\pi x)dx$$

$$b_k = -\frac{1}{k\pi}(10+100x)(\cos k\pi x)\Big|_{-.1}^{0} + \frac{100}{(k\pi)^2}(\sin k\pi x)\Big|_{-.1}^{0} - \frac{1}{k\pi}(10-100x)(\cos k\pi x)\Big|_{0}^{.1} - \frac{100}{(k\pi)^2}(\sin k\pi x)\Big|_{0}^{.1}$$

Using the mathematic program of *Geogebra* the coefficients can be calculated. The values for b_k all equal zero. The values for a_k can be expressed as $\dfrac{200\left(1-\cos\left(\frac{k\pi}{10}\right)\right)}{(k\pi)^2}$. The first three coefficients are $a_1 = .99180$, $a_2 = .96753$, and $a_3 = .92814$. The Fourier series can then be written as $F(x) = \frac{1}{2} + 0.99180\cos \pi x + 0.96753\cos 2\pi x + 0.92814\cos 3\pi x + \cdots$ or

$$F(x) = \frac{1}{2} + \sum_{k=1,2,3,\ldots}^{n} \frac{200\left(1-\cos\left(\frac{k\pi}{10}\right)\right)}{(k\pi)^2}\cos(k\pi x).$$





The graph of the 6th degree Fourier polynomial shows an approximation, however it has a maximum of about six instead of ten. A better approximation is the 10th degree Fourier polynomial seen at the right or in Appendix D. It has a maximum of about eight. The farther the polynomial is carried out, the closer and closer the approximation is to the actual function.

Fourier series were introduced by Joseph Fourier in the early 19th century as a way to solve an idealized heat transfer problem. Fourier seems to have discovered this series in much the same way an archeologist might discover an ancient city, one step at a time without knowing what lies ahead until logic suggests the next step. Evidently he was passionately excited about his discoveries, but most of the rest of the mathematicians of the time were skeptical if not outright hostile toward Fourier and his claims. Fourier persisted and with numerous examples was able to convince the mathematical community that his ideas had substantial merit and were worthy of investigation. These investigations produced fundamentally important theorems and properties and eventually some of the greatest mathematicians of the middle 19th century were able to give Fourier's ideas a solid proof-based foundation while developing calculus as we know it today. The 20th century saw an amazing array of applications of Fourier series from engineering applications to filtering transmission signals to the analysis of harmonics to data compression to number theory and more.

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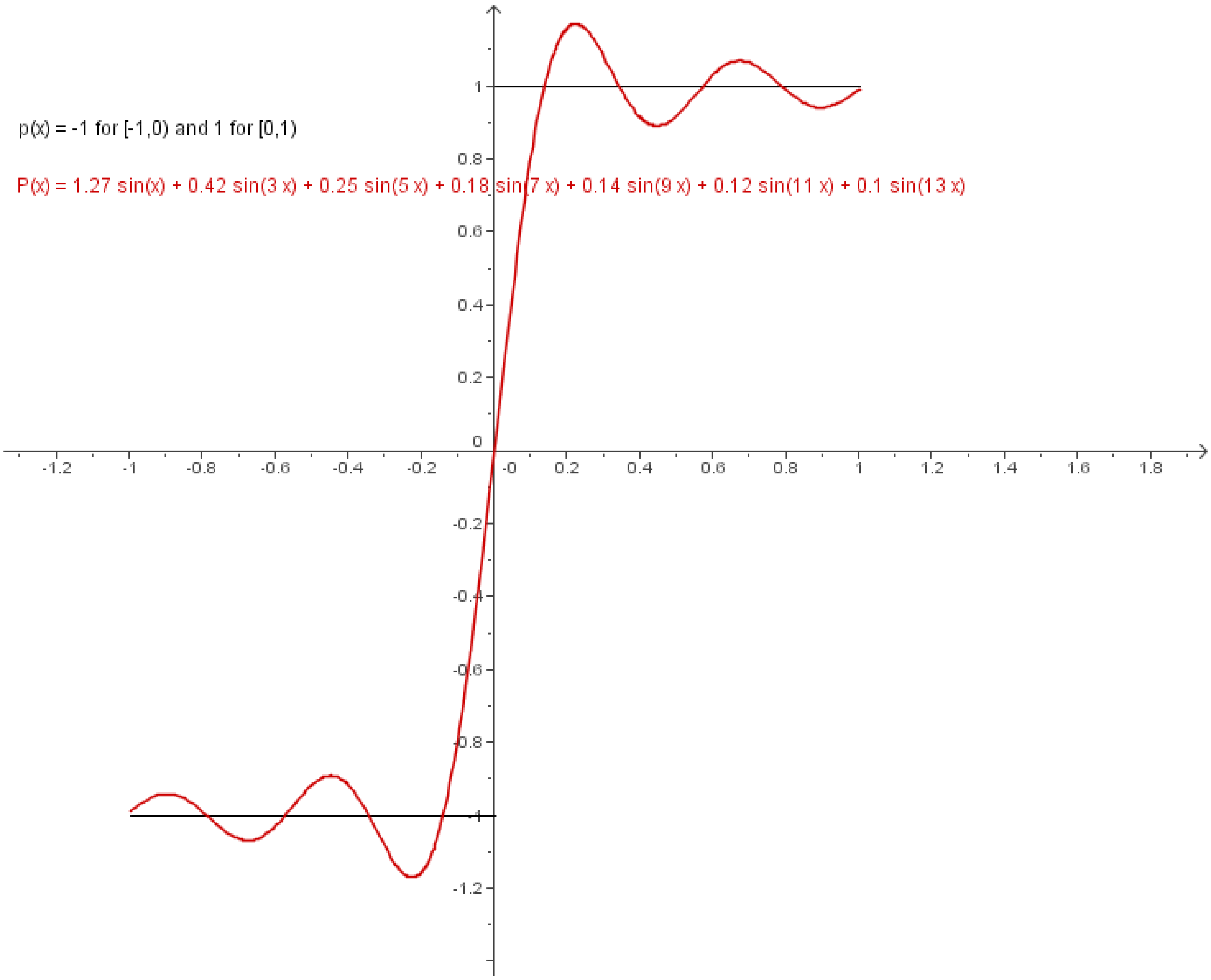
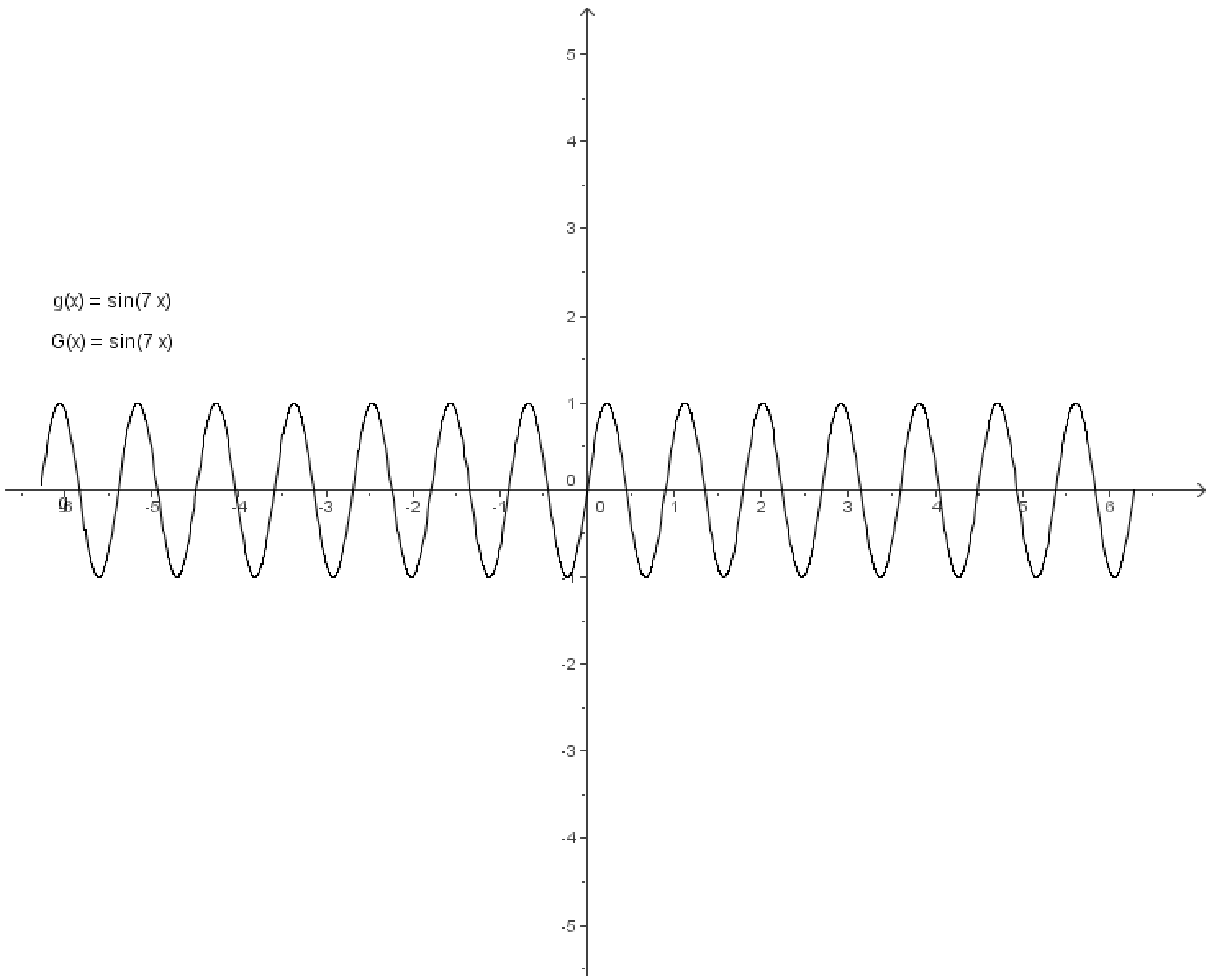
Appendix A

An approximation of the square wave function with a 13th degree Fourier polynomial, using Geogebra to plot the graph.



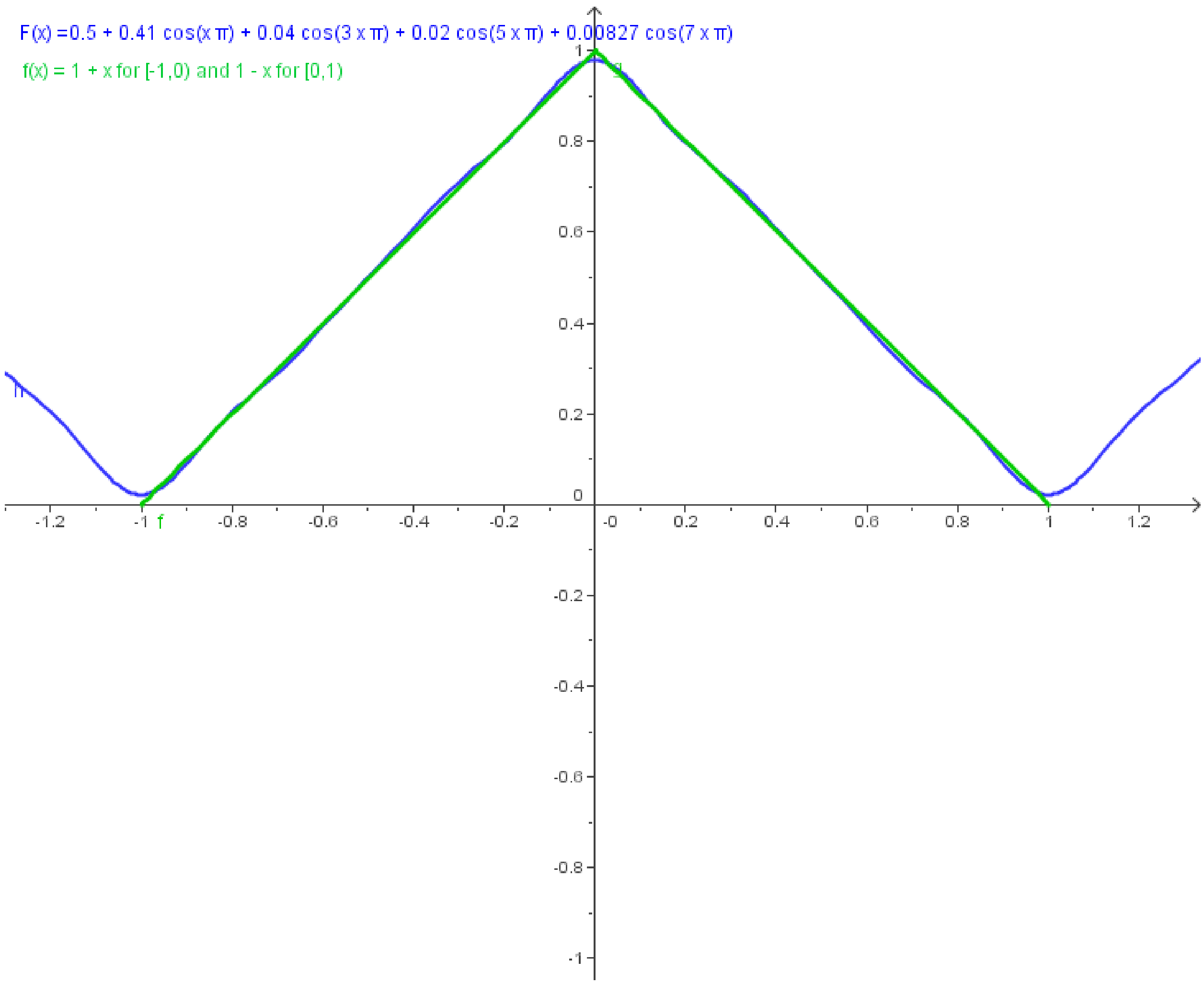
Appendix B

A graph of a 7th degree Fourier polynomial plotted using Geogebra.



Appendix C

An approximation for the saw tooth function with a 7^{th} degree Fourier polynomial, using Geogebra to plot the graph.



Appendix D

An approximation of another saw tooth function with a 10th degree Fourier polynomial, using Geogebra to plot the graph.