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## Decompositions of Modules and Matrices

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## DECOMPOSITIONS OF MODULES AND MATRICES

BY THOMAS S. SHORES AND ROGER WIEGAND<sup>1</sup>

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**ABSTRACT** A canonical form for a module  $M$  over a commutative ring  $R$  is a decomposition  $M \cong R/I_1 \oplus \cdots \oplus R/I_n$ , where the  $I_j$  are ideals of  $R$  and  $I_1 \subseteq \cdots \subseteq I_n$ . A complete structure theory is developed for those rings for which every finitely generated module has a canonical form. The (possibly larger) class of rings, for which every finitely generated module is a direct sum of cyclics, is also considered, and partial results are obtained for rings with fewer than  $2^c$  prime ideals. For example, if  $R$  is countable and every finitely generated  $R$ -module is a direct sum of cyclics, then  $R$  is a principal ideal ring. Finally, some topological criteria are given for Hermite rings and elementary divisor rings.

All rings in this announcement are commutative with 1, and all modules are unital. A *canonical form* for an  $R$ -module  $M$  is a decomposition  $M \cong R/I_1 \oplus \cdots \oplus R/I_n$ , where  $I_1 \subseteq \cdots \subseteq I_n \neq R$ . If  $M$  has a canonical form, the ideals  $I_j$  are uniquely determined [K]. A CF-ring is a ring for which every finitely generated direct sum of cyclics has a canonical form. It can be shown that  $R$  is CF if and only if

$$R/I \oplus R/J \cong R/(I \cap J) \oplus R/(I + J)$$

for every pair of ideals  $I, J$ .

By a valuation ring we shall mean a ring, possibly with zero-divisors, whose lattice of ideals is totally ordered. A ring  $R$  is arithmetical, provided the local ring  $R_{\mathfrak{m}}$  is a valuation ring for each maximal ideal  $\mathfrak{m}$ . Finally, an  $h$ -local domain [M1] is an integral domain such that (1) every nonzero ideal is contained in only finitely many maximal ideals, and (2) every nonzero prime ideal is contained in a unique maximal ideal.

**THEOREM 1.** *Every CF-ring is a finite direct product of indecomposable CF-rings. The indecomposable CF-rings are precisely the rings  $R$  such that (i)  $R$  is arithmetical, (ii)  $R$  has a unique minimal prime  $P$ , (iii)  $R/P$  is an  $h$ -local domain, and (iv) every ideal contained in  $P$  is comparable with every ideal of  $R$ .*

Thus valuation rings and arithmetical  $h$ -local domains are CF-rings.

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The next theorem gives a fairly complete picture of the remaining indecomposable CF-rings.

**THEOREM 2.** *Let  $R$  be an indecomposable CF-ring with minimal prime  $P$ . Suppose  $R$  is neither a valuation ring nor an  $h$ -local domain. Then  $P^2 = 0$ ,  $P$  is an indecomposable, torsion, divisible  $R/P$ -module, and  $P = P_{\mathfrak{m}}$  for a unique maximal ideal  $\mathfrak{m}$ .*

**EXAMPLE.** Let  $A$  be an arithmetical  $h$ -local domain (for example a Dedekind domain) with more than one maximal ideal, and let  $K$  be the quotient field of  $A$ . Let  $\mathfrak{m}$  be any maximal ideal of  $A$ , let  $P = K/A_{\mathfrak{m}}$ , and make  $R = A \oplus P$  into a ring by setting  $(a, p)(a', p') = (aa', ap' + a'p)$ . Then  $R$  is a CF-ring which is neither a domain nor a valuation ring.

Let us define an FGC-ring to be a ring  $R$  such that every finitely generated  $R$ -module is a direct sum of cyclic modules. If, in addition,  $R$  is CF, we say  $R$  is an FGCF-ring. The local FGC-rings have been characterized as the almost maximal valuation rings [G]. Thus, if  $R$  is FGC, it follows that  $R_{\mathfrak{m}}$  is almost maximal for every  $\mathfrak{m}$ , and it is not hard to see that  $R$  is Bezout, that is, every finitely generated ideal of  $R$  is principal. In general, these two conditions are far from sufficient. Surprisingly, CF-rings have the requisite “finiteness” conditions:

**THEOREM 3.** *A CF-ring  $R$  is FGCF if and only if  $R$  is Bezout and  $R_{\mathfrak{m}}$  is almost maximal for each maximal ideal  $\mathfrak{m}$ .*

If, in the example above,  $A_{\mathfrak{m}}$  is maximal and all the other localizations of  $A$  are almost maximal, the ring  $R = A \oplus P$  can be shown to have almost maximal localizations. If, in addition,  $A$  is Bezout (for example, a ring of type I [M2]) then  $R$  is an FGCF-ring.

A direct proof of Theorem 3 would appear to be difficult. Theorems 1 and 2, however, reduce the task to consideration of the three types of indecomposable CF-rings. Only the third type presents any difficulty, and it is handled by techniques similar to those in [G].

The problem of classifying FGC-rings seems to be much harder. It would be a great help to know that an FGC-ring has only finitely many minimal primes. (Indeed, this was a major step in the characterization of CF-rings.) We have some partial results in this direction:

**THEOREM 4.** *Let  $R$  be an FGC-ring. Then every compact set of minimal primes is finite. If  $R$  has fewer than  $2^c$  prime ideals, then  $R$  has only finitely many minimal primes.*

**COROLLARY.** *If  $R$  is an FGC-ring with fewer than  $2^c$  prime ideals then  $R$  has noetherian maximal ideal space.*

**COROLLARY.** *Every countable FGC-ring is a principal ideal ring.*

To prove Theorem 4, one uses methods similar to those in [P] to show that a fairly benign condition on  $\text{spec}(R)$ , the prime spectrum of  $R$ , prevents FGC. The statement of this condition involves two topologies on  $\text{spec}(R)$ —the usual (Zariski) topology, and the patch topology, which is generated by declaring that the quasicompact open sets in the usual topology shall be both open and closed.

**THEOREM 5.** *Suppose  $\text{spec}(R)$  has a point in the patch closure of each of three pairwise disjoint Zariski open sets. Then  $R$  is not an FGC-ring.*

We know of no examples of FGC-rings which are not already FGCF. Nonetheless it seems plausible that a semilocal, locally almost maximal domain might have FGC without being  $h$ -local.

When we restrict our attention to decompositions of finitely presented modules, we are led inevitably to questions about diagonalization of matrices. A module  $M$  with  $m$  generators and  $n$  relations is isomorphic to  $R^m/K$ , where  $K$  is generated by the columns of an  $m \times n$  matrix  $A$ . In this case we say  $M$  is *presented* by  $A$ . It is well known that if  $A$  and  $B$  are equivalent then they present isomorphic modules, but that the converse fails, even if  $A$  and  $B$  have the same size. It has recently been shown, however, that if every finitely presented  $R$ -module is a direct sum of cyclics then every matrix over  $R$  is equivalent to a diagonal matrix [LLS].

The situation is somewhat different for canonical forms. We say a matrix  $A$  has a canonical form provided  $A$  is equivalent to a diagonal matrix  $[d_{ij}]$  in which  $d_{i+1,i+1}$  divides  $d_{i,i}$  for all  $i$ . In [LLS] it was proved that  $R$  is Bezout if and only if every diagonal matrix over  $R$  has a canonical form. On the other hand,  $R$  is arithmetical if and only if every finitely presented direct sum of cyclics has a canonical form.

An elementary divisor ring is a ring for which every matrix has a canonical form. (By [LLS] it is enough to check that every finitely presented module is a direct sum of cyclics.) In the standard examples of elementary divisor rings (for example, the adequate rings [LLS]), the dimension of the maximal ideal space is at most one. The following theorem, and a construction due to Heinzer [H], show that higher-dimensional examples exist.

**THEOREM 6.** *Every Bezout ring with noetherian maximal ideal space is an elementary divisor ring.*

Finally, we turn our attention to an intermediate class of rings, between Bezout rings and elementary divisor rings. A ring  $R$  is *Hermite* provided every matrix is equivalent to a triangular matrix. (Equivalently, every 1 by 2 matrix is equivalent to a diagonal matrix [K].) M. Henriksen has raised the following question: If  $R$  is a Bezout ring with compact minimal prime spectrum, is  $R$  Hermite? While we suspect that the answer is “no”,

the following results indicate that a counterexample would have to have a rather perverse nilradical.

**THEOREM 7.** *Let  $R$  be a Bezout ring with compact minimal prime spectrum. Then  $R$  is Hermite if and only if every 1 by 2 matrix with nilpotent entries is equivalent to a diagonal matrix.*

**COROLLARY.** *Let  $R$  be a Bezout ring with compact minimal prime spectrum. If the nilradical of  $R$  is  $T$ -nilpotent, then  $R$  is Hermite.*

#### REFERENCES

- [G] D. T. Gill, *Almost maximal valuation rings*, J. London Math. Soc. (2) **4** (1971), 140–146.
- [H] W. Heinzer,  *$J$ -noetherian integral domains with 1 in the stable range*, Proc. Amer. Math. Soc. **19** (1968), 1369–1372. MR **38** #145.
- [K] I. Kaplansky, *Elementary divisors and modules*, Trans. Amer. Math. Soc. **66** (1949), 464–491. MR **11**, 155.
- [LLS] M. D. Larsen, W. J. Lewis and T. S. Shores, *Elementary divisor rings and finitely presented modules*, Trans. Amer. Math. Soc., (to appear).
- [M1] E. Matlis, *Decomposable modules*, Trans. Amer. Math. Soc. **125** (1966), 147–179. MR **34** #1349.
- [M2] ———, *Rings of type I*, J. Algebra **23** (1972), 76–87.
- [P] R. S. Pierce, *Modules over commutative regular rings*, Mem. Amer. Math. Soc. No. 70 (1967). MR **36** #151.

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