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Covariant Representations of C*-dynamical systems Involving Compact Groups

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COVARIANT REPRESENTATIONS OF C^*-DYNAMICAL SYSTEMS
INVOLVING COMPACT GROUPS

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Given a C^*-dynamical system (A, G, σ) the crossed product C^*-algebra $A \times_\sigma G$ encodes the action of G on A. By the universal property of $A \times_\sigma G$ there exists a one to one correspondence between the set all covariant representations of the system (A, G, σ) and the set of all *-representations of $A \times_\sigma G$. Therefore, the study of representations of $A \times_\sigma G$ is equivalent to that of covariant representations of (A, G, σ).

We study induced covariant representations of systems involving compact groups. We prove that every irreducible (resp. factor) covariant representation of (A, G, σ) is induced from an irreducible (resp. factor) representation of a subsystem (A, G_0, σ) where π_0 is a factor representation. This extends a result obtained in [3] for finite groups. It was shown in [10] that if G is an amenable group then every primitive ideal of $A \times_\sigma G$ is induced from a stability group. If G is compact then we obtain a stronger result, that is, every irreducible representation of (A, G, σ) is induced from a stability group. In addition, we show that (A, G, σ) satisfies the strong-EHI property introduced by Echterhoff and Williams in [5].

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0.1 Introduction

Let G be a locally compact group, A a C^*-algebra, and σ a point-norm continuous homomorphism of G into the automorphism group of A. The triple (A, G, σ) is a C^*-dynamical system. Given a C^*-dynamical system the crossed product C^*-algebra $A \times_\sigma G$ encodes the action of G on A. Crossed product C^*-algebras originate from the study of group action on a topological space. Given an action of G on a locally compact space X there is a natural way of defining an action of G on $C_0(X)$. One can study the action of G on X by studying the crossed product C^*-algebra $C_0(X) \times_\sigma G$. Crossed products have become a source of many interesting examples of C^*-algebras such as the "rotation" algebras and the Bunce-Deddens algebras.

It is well known that there exists a one to one correspondence between the set of all covariant representations of the system (A, G, σ) and the set of all *-representations of $A \times_\sigma G$. Therefore, the study of representations of $A \times_\sigma G$ is equivalent to that of covariant representations of (A, G, σ). Our goal is to study induced covariant representations of systems involving compact groups. In the context of unitary representations of locally compact groups, the study of induced representations was initiated by Mackey in [13, 14]. Using Mackey's approach Takesaki extended the theory to crossed products in [17]. Subsequently, Rieffel recast that theory in terms of Hilbert modules and Morita equivalence with [16]. It follows from Proposition 5.4 in [18] that the construction of induced representations for crossed products by Rieffel is equivalent to that of Takesaki.

The importance of induced representations arises from the fact that the fundamental structure of a crossed product $A \times_\sigma G$ is reflected in the structure of the orbit space for the G-action on Prim A together with the subsystems (A, G_P, σ) where G_P is the stability group at $P \in$ Prim A. In particular, one gets a complete description of the primitive ideal space and its topology for transformation group C^*-algebra $C_0(X) \times_\sigma G$ when G is abelian. In many important cases we also get a characterization of when $A \times_\sigma G$ is GCR or CCR. Williams presents all these results and more in his book [18].

Although induced representations have been studied extensively there remains a considerable gap in the theory. We outline below two questions for which answers are not known. Using structure theorems obtained in this paper we give a positive answer to both questions in the case of separable C^*-dynamical systems with compact groups.

1. Is every irreducible representation of $A \times_\sigma G$ induced from a stability group?

2. Suppose that (π, U) is an irreducible representation of (A, G_P, σ) such that $\ker(\pi) = P$. Is the corresponding induced representation of $A \times_\sigma G$ irreducible?

The first question is closely related to a classical result in the theory of crossed products known as the GRS theorem. One of the key ingredients in building the connection between Prim $A \times_\sigma G$ and the G-action on Prim A is establishing that every primitive ideal of $A \times_\sigma G$ is induced from a stability group ([18]; p. 235). The latter result was conjectured by Effros and Hahn, and systems for which the conjecture holds are called EH-regular. The proof that the Effros-Hahn conjecture holds for separable crossed products where G is amenable is due Gootman, Rosenberg and Sauvageot (see Chapters 8 and 9 in [18] for the proof of the GRS theorem and its applications). There exists a stronger notion of EH-regularity namely the requirement that every irreducible representation of $A \times_\sigma G$ is induced from a stability group. The latter requirement is known to hold for many dynamical systems ([18]; Theorem 8.16) but the general case remains open.

The second question was raised by Echterhoff and Williams in [5]. Following their nomenclature we say that (A, G, σ) satisfies strong Effros-Hahn Induction Property (strong-EHI) if, for each primitive ideal P of A and a covariant irreducible representation (π, U) of (A, G_P, σ) such that $\ker(\pi) = P$ the corresponding induced representation of (A, G, σ) is irreducible. A very nice summary of the results regarding the (strong)-EHI property can be found in [5].

We give positive answers to above questions in the case of compact groups. To answer the above questions we first prove a theorem which, in part, states that every irreducible representation (π, U) of (A, G, σ) is equivalent to the representation induced from a representation (π_0, U_0) of (A, G_0, σ), for an appropriate subgroup G_0 of G, and where π_0 is a factor representation. A very similar result was obtained in [3] for the case of finite groups. We show that the results in [3] follow directly from our results.

In this paper we use Takesaki's approach to the theory of induced representations for crossed products. As in [17] we will often assume basic countability conditions. These assumptions seem to be natural in the context of other results in the theory of induced representations. Chapter 1 is devoted to presenting the necessary background and machinery used in the proof of our results. Chapter 2 contains our original results.

Chapter 1

Preliminaries

1.1 Von Neumann Algebras

In this section we will present basic results in the theory of von Neumann algebras that we will need. The subject of von Neumann algebras is important in of itself, but we are mostly interested in its applications in the study of representations of C^*-algebras. Let $\mathcal{H}$ be a Hilbert space and $\mathcal{B}(\mathcal{H})$ be the set of all bounded operators on $\mathcal{H}$. The norm-topology on $\mathcal{B}(\mathcal{H})$ is the most widely used topology because it has very nice properties such as continuity of joint multiplication and involution. However, there are several other useful topologies on $\mathcal{B}(\mathcal{H})$ in addition to the norm topology.

Definition 1. The *strong operator topology* on $\mathcal{B}(\mathcal{H})$ is the topology of pointwise norm convergence, i.e. $T_i \to T$ strongly if $T_i\zeta \to T\zeta$ for all $\zeta \in \mathcal{H}$.

The *weak operator topology* on $\mathcal{B}(\mathcal{H})$ is the topology of pointwise weak convergence, i.e. $T_i \to T$ weakly if $\langle T_i\zeta, \eta\rangle \to \langle T\zeta, \eta\rangle$ for all $\zeta, \eta \in \mathcal{H}$.

Addition and scalar multiplication is jointly continuous in all topologies. Multiplication is separately continuous in all topologies and is jointly strongly continuous on (norm) bounded sets. In general, the multiplication is not jointly continuous.

The *commutant* of a subset $\mathbf{S}$ of $\mathcal{B}(\mathcal{H})$ is defined by

$$\mathbf{S}' = \{T \in \mathcal{B}(\mathcal{H}) : TS = ST \text{ for all } S \in \mathbf{S}\}$$

It is a unital subalgebra of $\mathcal{B}(\mathcal{H})$, closed in the weak operator topology, hence in the strong operator topology. The double commutant of $\mathbf{S}$ is $\mathbf{S}'' = (\mathbf{S}')'$. We say that a subset $\mathbf{S}$ of $\mathcal{B}(\mathcal{H})$ acts nondegenerately on $\mathcal{H}$ if $\overline{\mathbf{S}\mathcal{H}} = \mathcal{H}$.

Theorem 2 ([4]; Theorem I.9.1.1)**.** *Let* $\mathbf{A}$ *be a nondegenerate self adjoint subalgebra of* $\mathcal{B}(\mathcal{H})$*. Then* $\mathbf{A}''$ *coincides with the weak and strong operator closures of* A*.*

The above theorem, known as the Double Commutant Theorem, allows us to characterize the elements of $\mathbf{A}''$ as limit points of $\mathbf{A}$ in the strong (weak) operator topologies. However, it is possible that the converging net is not norm bounded. The advantage of dealing with norm bounded sets in the strong operator topology, as mentioned above, is that multiplication is jointly continuous. Fortunately, we have the Kaplansky Density Theorem which addresses our problem.

Theorem 3 ([4]; I.9.1.3)**.** *Let* $\mathbf{A}$ *be a nondegenerate self adjoint subalgebra of* $\mathcal{B}(\mathcal{H})$*. Then the unit ball of* $\mathbf{A}$ *is strong operator topology dense in the unit ball of* $\mathbf{A}''$*.*

Definition 4. A *von Neumann algebra* is a unital, self adjoint algebra $\mathbf{A} \subseteq \mathcal{B}(\mathcal{H})$ that is closed in the strong operator topology.

Remark 5. If $\mathbf{S}$ is a self adjoint subset of $\mathcal{B}(\mathcal{H})$ then a simple argument shows that $\mathbf{S}' = \mathbf{S}'''$; since $\mathbf{S}'$ clearly contains the identity it follows that the commutant of any self adjoint set of operators is a von Neumann algebra.

Remark 6. Let $\mathbf{A}$ be a von Neumann algebra in $\mathcal{B}(\mathcal{H})$. We define the *center* of $\mathbf{A}$ to be $\mathbf{Z}(\mathbf{A}) = \mathbf{A} \cap \mathbf{A}'$. It follows that $\mathbf{Z}(\mathbf{A}) = \mathbf{Z}(\mathbf{A}')$.

Definition 7. A von Neumann algebra $\mathbf{A}$ in $\mathcal{B}(\mathcal{H})$ is a called a *factor* if it has a trivial center, i.e. $\mathbf{Z}(\mathbf{A}) = \mathbf{C}1_{\mathcal{H}}$

Example 8. Let (X, μ) be a σ-finite measure space and denote $L^\infty(X, \mu)$ to be the set of essentially bounded Borel functions on X. For $f \in L^\infty(X, \mu)$ define a multiplication operator M_f on $\mathcal{H} = L^2(X, \mu)$ by $M_f\zeta = f\zeta$ for $\zeta \in L^2(X, \mu)$. The map $f \mapsto M_f$ carries $L^\infty(X, \mu)$ isometrically to a subalgebra of $\mathcal{B}(\mathcal{H})$, and it is customary to identify $L^\infty(X, \mu)$ with its image. Viewed in this way $L^\infty(X, \mu)$ is an abelian von Neumann algebra. In fact, $L^\infty(X, \mu)$ is equal to its commutant [7].

Definition 9. Let $\mathbf{A}$ be an abelian von Neumann algebra. We say that $\mathbf{A}$ is a maximal self adjoint algebra (*masa*) if,

$$\mathbf{A} = \mathbf{A}'.$$

Let $L^\infty(X, \mu)$ be as in Example 8. Then $L^\infty(X, \mu) = L^\infty(X, \mu)'$. Thus $L^\infty(X, \mu)$ is a masa in $\mathcal{B}(L^2(X, \mu))$.

Recall that if $\mathbf{S}$ is any subset of $\mathcal{B}(\mathcal{H})$. Then a vector $\zeta \in \mathcal{H}$ is called a *cyclic* vector for $\mathbf{S}$ if $\mathbf{S}\zeta$ is dense in $\mathcal{H}$. A vector $\zeta \in \mathcal{H}$ is called a *separating* vector for $\mathbf{S}$ if $S\zeta \neq 0$ for all $S \in \mathbf{S}$. We have the following useful alternative characterization of a masa in terms of such vectors.

Proposition 10 ([4]; III.1.5.19)**.** *Let $\mathcal{H}$ be a separable Hilbert space and let $\mathbf{A} \subseteq \mathcal{B}(\mathcal{H})$ be a von Neumann algebra. Then $\mathbf{A}$ is a masa if and only if $\mathbf{A}$ has a cyclic, separating vector.*

Masas are well studied objects, especially in the case of a separable Hilbert space where a complete classification of masas is available. Moreover, the study of abelian von Neumann algebras reduces to the study of masas:

Theorem 11 ([7]; 4.3.8 and 4.6.2)**.** *Let* $\mathbf{A}$ *be an abelian von Neumann algebra on a separable Hilbert space* $\mathcal{H}$*. Then there exists a unique, up to equivalence, sequence of projections* $p_n \in \mathbf{A}$ *with* $n = 1, 2, ...dim(\mathcal{H})$ *such that* $\sum p_n = I_{\mathcal{H}}$ *and each* $\mathbf{A}p_n$ *is spatially isomorphic to* $\mathrm{R}_n \otimes I_n$ *where* R_n *is a masa.*

Remark 12. Let $\mathbf{A}$ be an abelian von Neumann algebra on a separable Hilbert space $\mathcal{H}$. Then by the above theorem there exist a unique sequence of projections $p_n \in \mathbf{A}$ so that $\mathbf{A}p_n$ is spatially isomorphic to $R_n \otimes I_n$. Let U be a unitary acting on the same space $\mathcal{H}$ such that $U\mathbf{A}U^* = \mathbf{A}$. Then $Up_nU^* \in \mathbf{A}$ for each $n = 1, 2, ...dim(\mathcal{H})$ and $\sum Up_nU^* = I_{\mathcal{H}}$. Also $\mathbf{A} = U\mathbf{A}U^* = \sum U\mathbf{A}p_nU^* = \sum(U\mathbf{A}U^*)(Up_nU^*) = \sum \mathbf{A}Up_nU^*$. It follows from uniqueness of p_n's that $Up_nU^* = p_n$ for each n.

1.2 C^*-algebras

In this section we will present the necessary facts and results related to the theory of C^*-algebras that will be used later in the dissertation. In particular, we will discuss the representation theory for abelian C^*-algebras. This theory is important in of itself but it is also useful in studying representations of arbitrary C^*-algebras. We will limit ourselves to discussing the representation theory for separable C^*-algebras as our results in the next chapter are mainly concerned with separable spaces.

Definition 13. Let A be C^*-algebra and π be a representation of A on a Hilbert space $\mathcal{H}$. We say that π is a *nondegenrate* representation if $\pi(A)\mathcal{H}$ is dense in $\mathcal{H}$. Two representations π and ρ are *equivalent* if there is a unitary operator $U : \mathcal{H}_\pi \to \mathcal{H}_\rho$

such that $\rho(a) = U\pi(a)U^*$ for all $a \in A$. We say that π is *irreducible* if there are no closed invariant subspaces. We say that π is a *factor* representation if $\pi(A)''$ is a factor as von Neumann algebra.

Example 14. Every irreducible representation of a C^*-algebra A is a factor representation, and in fact any multiple of an irreducible representation is a factor representation.

For the remainder of the section (as throughout most of this paper) we shall assume that all representations are nondegenerate unless otherwise specified.

Definition 15. A representation π of a C^*-algebra A is called *multiplicity free* provided there is no nonzero subrepresentation σ of A such that $\sigma \oplus \sigma$ is equivalent to a subrepresentation of π. Equivalently, π is multiplicity free if $\pi(A)'$ is abelian. We say π is a multiplicity n representation if it is equivalent to n copies of a multiplicity free representation. Two representations π and σ of A are called *disjoint* if no nonzero subrepresentation of π is equivalent to any subrepresentation of σ. A subrepresentation of π is called a *central* subrepresentation if the corresponding orthogonal projection belongs to the center of $\pi(A)''$. We call π a *type I* representation if every central subrepresentation of π has a multiplicity free subrepresentation. We call the C^*-algebra A of type I if every representation of A is of type I. Equivalently, C^*-algebra A of type I if $\pi(A)''$ is type I as a von Neumann algebra for all representations π of A.

Example 16. If A is an abelian C^* algebra then every representation of A is trivially of type I. Hence abelian C^*-algebras are of type I. We note that a type I von Neumann algebra is not necessarily type I as a C^*-algebra (for instance $\mathcal{B}(\mathcal{H})$ [4];IV.1.1.5).

Example 17. Let X be a second countable locally compact Hausdorff space. Let μ be a finite measure on X and define a representation π_μ of $C_0(X)$ on the Hilbert space $L^2(X,\mu)$ to be pointwise multiplication. Then $\pi_\mu(A)' = L^\infty(X,\mu)$ and π_μ is a multiplicity free representation. Conversely, if π is a multiplicity free representation of $C_0(X)$ then there exist a finite measure μ such that π is equivalent to π_μ. We also note that if ν is another finite measure on X then π_μ is equivalent to π_ν if and only if μ is equivalent, in the sense of absolute continuity, to ν ([18]; p.401-402).

Type I representations have been well studied partly because there exists a nice decomposition theory for such representations:

Theorem 18 ([18]; Theorem E.12)**.** *Suppose that π is a type I representation of a C^*-algebra on a separable Hilbert space. Then there is a unique orthogonal family $\{\pi_n\}$ of central subrepresentations of π such that*

(a) each π_n has multiplicity n or is the zero representation, and

(b) $\pi = \bigoplus \pi_n$.

Since abelian C^*-algebras are type I the above theorem applies to $C_0(X)$. We know from Example 17 that every multiplicity n representation of $C_0(X)$ is of the form $n \cdot \pi_\mu$. Thus we obtain the following characterization of representations of $C_0(X)$.

Corollary 19 ([18]; Theorem E.14)**.** *Suppose that $A = C_0(X)$ is a separable commutative C^*-algebra and that π is a separable representation of A. Then π is equivalent to a representation of the form*

$$(\pi_{\mu_\infty} \otimes 1_{\mathcal{H}_\infty}) \oplus \pi_{\mu_1} \oplus (\pi_{\mu_2} \otimes 1_{\mathcal{H}_2}) \oplus \cdots$$

where each μ_n is a finite Borel measure on X with $\mu_n \perp \mu_m$ whenever $n \neq m$. If

$$\sigma = (\pi_{\nu_\infty} \otimes 1_{\mathcal{H}_\infty}) \oplus \pi_{\nu_1} \oplus (\pi_{\nu_2} \otimes 1_{\mathcal{H}_2}) \oplus \cdots$$

is another such representation, then σ is equivalent to π if and only if μ_n and ν_n are equivalent measures for all n.

Thus the study of representations of $C_0(X)$ is reduced to the study of its multiplicity free representations, which are well studied. One can read Arveson's book [1] for a good treatment of the topic. We will now briefly sketch the main ideas needed for our purposes. A good summary of the main results related to these representations is given in Section E.2 of [18] and p. 316-318 of [4]. Our discussion is borrowed from Section E.2 of [18].

We assume that X is a second countable locally compact Hausdorff space, A is a separable C^*-algebra, and $\mathcal{H}$ is a separable Hilbert space. Let $\mathcal{B}^b(X,\mathcal{H})$ be the set of all bounded functions $F : X \to \mathcal{B}(\mathcal{H})$ such that $x \mapsto \langle F(x)h, k\rangle$ is Borel for all $h, k \in \mathcal{H}$. The usual pointwise operations make $\mathcal{B}^b(X,\mathcal{H})$ into a *-algebra with norm

$$\|F\| = \sup_{x\in X}\|F(x)\|.$$

For each $F \in \mathcal{B}^b(X,\mathcal{H})$, we define an operator L_F on $L^2(X,\mu,\mathcal{H})$ by $L_F\zeta(x) = F(x)\zeta(x)$. The subalgebra

$$\mathcal{L}\otimes 1_{\mathcal{H}} = \{L_f \otimes 1_{\mathcal{H}} : f \in \mathcal{B}^b(X)\}$$

is called the diagonal operators. We call a bounded operator T on $L^2(X,\mu,\mathcal{H})$ *decomposable* if there exists $F \in \mathcal{B}^b(X,\mathcal{H})$ such that $T = L_F$.

Theorem 20 ([18]; Theorem E.17)**.** *Suppose $\mathcal{H}$ is a separable Hilbert space, X is a second countable locally compact Hausdorff space, and μ is a finite Borel measure on X. Then $T \in \mathcal{B}(L^2(X,\mu,\mathcal{H}))$ is decomposable if and only if $T \in (L\otimes 1_{\mathcal{H}})'$. Furthermore, $L\otimes 1_{\mathcal{H}}$ is an abelian von Neumann algebra, and $\pi_\mu \otimes 1_{\mathcal{H}}(C_0(X))$ is* *dense in $L \otimes 1_{\mathcal{H}}$ in the strong operator topology.*

Let $L^2(X,\mu,\mathcal{H})$ be as in Theorem 20 and f_1, f_2 be a μ-almost everywhere bounded Borel functions on X. Then for each $i = 1,2$, we can define an operator L_{f_i} on $L^2(X,\mu,\mathcal{H})$ by $(L_{f_i}\zeta)(x) = f_i(x)\zeta(x)$. If $f_1(x) = f_2(x)$ for almost all $x \in X$ then $L_{f_1} = L_{f_2}$. Therefore, there is a well defined injective map from $L^\infty(X,\mu)$ into $\mathcal{B}(L^2(X,\mu,\mathcal{H}))$ whose image we denote by $L^\infty(X,\mu) \otimes 1_{\mathcal{H}}$. It not hard to see that $L^\infty(X,\mu)\otimes 1_{\mathcal{H}} = \mathcal{L}\otimes 1_{\mathcal{H}}$. Therefore, we use $L^\infty(X,\mu)\otimes 1_{\mathcal{H}}$ and $\mathcal{L}\otimes 1_{\mathcal{H}}$ interchangeably.

Let B be a separable C^*-subalgebra of $(L^\infty(X,\mu) \otimes 1_{\mathcal{H}})'$. By the above theorem for each $T \in B$ there exists an $F \in \mathcal{B}^b(X,\mathcal{H})$ such that $T = L_F$. Suppose that $F_1, F_2 \in \mathcal{B}^b(X,\mathcal{H})$ such that $F_1(x) = F_2(x)$ for almost all $x \in X$ then $L_{F_1} = L_{F_2}$. We can make these choices so that the map $\pi : B \to \mathcal{B}^b(X,\mathcal{H})$ with $T = L_{\pi(T)}$ is an isometric *-isomorphism ([18]; Theorem E.18). In this case, we can define for each $x \in X$ a representation $\pi_x : B \to \mathcal{B}(\mathcal{H})$ by $\pi_x(T) = \pi(T)(x)$. Now suppose we have a representation ρ of a separable C^*-algebra A on $L^2(X,\mu,\mathcal{H})$ such that $\rho(A) \subseteq (L^\infty(X,\mu) \otimes 1_{\mathcal{H}})'$. By letting $B = \rho(A)$ we obtain a decomposition of ρ into $\{\rho_x = \pi_x \circ \rho\}_{x\in X}$. If ρ is a nondegenerate representation of A then ρ_x is a nondegenerate representation for almost all $x \in X$.

Proposition 21 ([18]; Proposition E.20, p.316-317; [4])**.** *Let $\mathcal{H}$ be a separable Hilbert space, X a second countable locally compact Hausdorff space, and μ a finite measure on X. Suppose that B is a separable C^*-algebra of $(L^\infty(X,\mu) \otimes 1_{\mathcal{H}})'$ with $\pi : B \to \mathcal{B}^b(X,\mathcal{H})$ as above.*

(a) If $L_F \in B''$, then $F(x) \in \pi_x(B)''$ for almost all x.

(b) If $L_F \in B'$, then $F(x) \in \pi_x(B)'$ for almost all x.

(c) If $L^\infty(X,\mu) \otimes 1_{\mathcal{H}} = B' \cap B''$, then π_x is a factor representation for almost all x.

1.3 Locally Compact Groups

In this section we present some basic results regarding locally compact groups and their representations. Although our ultimate goal is to study actions of compact groups on C^*-algebras, we want to discuss the theory of compact groups in the context of locally compact groups. The main point of this section is to discuss the construction of induced representations for groups developed by Mackey which we will later generalize to covariant representations of C^*-dynamical systems. We also address some of the topological considerations and talk a bit about the Haar measure. Most of the material below appears in Mackey's papers ([15], [13], [14]) and Chapter 1 of Williams' book [18].

Definition 22. A *topological group* is a group $(G, \cdot)$ together with a topology such that

1. points are closed in G, and
2. the map $(s, r) \mapsto sr^{-1}$ is continuous from $G \times G$ to G.

Example 23. Any group endowed with the discrete topology is a topological group. If $n \in \mathbf{N}$ then $\mathbf{R}^n$, $\mathbf{T}^n$, and $\mathbf{Z}^n$ are all topological groups with their usual topologies. An example of a non-abelian topological group would be the set of all unitary operators, $U(\mathcal{H})$, on a Hilbert space $\mathcal{H}$ together with the strong operator topology; that is, $U_i \to U$ if and only $U_i(\zeta) \to U(\zeta)$ for all $\zeta \in \mathcal{H}$. If A is a C^*-algebra then the collection $\mathrm{Aut}(A)$ of automorphims of A is a group under composition. We give

Aut(A) the point norm topology; that is, $\sigma_i \to \sigma$ if and only if $\sigma_i(a) \to \sigma(a)$ for all $a \in A$.

Lemma 24 ([18]; Lemma 1.13). *If G is topological group, then G is Hausdorff and regular.*

Definition 25. A *locally compact group* is a topological group for which the underlying topology is locally compact.

Any discrete group G is locally compact as are any of the basic abelian groups – $\mathbf{R}^n$, $\mathbf{T}^n$, and $\mathbf{Z}^n$. The unitary group $U(\mathcal{H})$ is locally compact if and only if $\dim(\mathcal{H}) < \infty$. The group Aut$(A)$ is not locally compact in general.

We would like to talk a bit more about Aut(A) in the case when A is abelian, which is a very important example in the theory of C^*-dynamical systems. If X is a locally compact Hausdorff space then the collection Homeo(X) of homeomorphisms of X is a group under composition. We give Homeo(X) the following topology: $h_n \to h$ in Homeo(X) if and only if both $h_n(x_n) \to h(x)$ and $h_n^{-1}(x_n) \to h^{-1}(x)$ whenever $x_n \to x$.

Lemma 26 ([18]; Lemma 1.33). *If $\sigma \in Aut\, C_0(X)$, then there is $h \in Homeo(X)$ such that $\sigma(f)(x) = f(h(x))$ for all $f \in C_0(X)$. The map $\sigma \mapsto h$ is a homeomorphism of $Aut(C_0(X))$ with $Homeo(X)$.*

We now present basic facts about the quotient space of a topological group by its subgroup. Suppose H is a subgroup of a topological group G. The set of *right* cosets, H/G, inherits a topology, called the quotient topology, from G which is the smallest topology making the quotient map $q : G \to H/G$ continuous. In particular, $W \subseteq H/G$ is open in the quotient topology if and only if $q^{-1}(W)$ is open in G.

Remark 27. It is customary, in the modern literature, to consider the left coset space G/H. However, in order to adhere to the original works of Mackey we would like to consider the right coset space. All the results regarding G/H can be easily seen to be true for H/G so no harm is done.

Lemma 28 ([18]; Lemma 1.44)**.** *If H is a subgroup of a topological group G, then the quotient map $q : G \to H/G$ is open and continuous.*

Proposition 29 ([18]; Proposition 1.48)**.** *Let H be a subgroup of a topological group G. The right coset space H/G equipped with the quotient topology is Hausdorff if and only if H is closed in G. If G is locally compact, then H/G is locally compact. If G is second countable, then H/G is second countable .*

1.3.1 Invariant Measures on G and H/G

The reason we concentrate on locally compact groups is because they have a uniquely defined measure class that respects the group action. Given a topological group G we endow it with the Borel structure generated by the open sets in G. Similarly, if H is a closed subgroup of G then we endow H/G with the Borel structure generated by the open sets coming from the quotient topology.

Definition 30. A (right) *Haar* measure on G is a nonzero Radon measure μ on G that satisfies $\mu(Es) = \mu(E)$ for every Borel set $E \subseteq G$ and every $s \in G$.

Remark 31. It is conventional to define the Haar measure to be a left invariant Radon measure on G. Our preference for right over left Haar measures is due to Mackey's original works.

Theorem 32 ([8]; Theorem 2.10 and 2.20)**.** *Every locally compact group G has a Haar measure which is unique up to a positive scalar.*

Regularity and invariance imply that every non empty open set of G has positive Haar measure. If G is compact then the Haar measure must be finite. In fact, the Haar measure μ on a compact group G is bivariant; that is, $\mu(sE) = \mu(Es) = \mu(E)$ for every Borel set $E \subseteq G$ and every $s \in G$.

Example 33. If G is a discrete group the Haar measure is simply the counting measure and if G is $\mathbf{R}^n$ or $\mathbf{T}^n$ then the Haar measure is the Lebesgue measure.

If G is a compact group then it is automatically normal as a topological space. The classical version of the Urysohn Lemma shows that $C_c(G)$ is dense in $L^2(G, \mu)$, where μ is the Haar measure. If G is a locally compact group the result is still true but one needs to employ a "generalized" version of the Urysohn Lemma.

Theorem 34 ([18]; Lemma 1.41)**.** *Suppose that X is a locally compact Hausdorff space and F is an open neighborhood of a compact set K in X. Then there is a $f \in C_c(X)$ such that $0 \leq f(x) \leq 1$ for all $x \in X$, $f(x) = 1$ for all $x \in K$, and $f(x) = 0$ for all $x \notin F$.*

Corollary 35. *Suppose X is a locally compact Hausdorff space and μ is a Radon measure on X. Then $C_c(X)$ is dense in $L^2(X, \mu)$.*

Although there always exists an invariant measure on a locally compact group G the same is not true for the quotient space H/G. Fortunately, if G is a compact group the situation is salvageable.

Proposition 36 ([8]; Corollary 2.50)**.** *Let G be a compact group and H be a closed subgroup of G. Then there exists a unique, up to scalar multiple, finite left G-invariant Radon measure on G/H.*

Corollary 37. *Let G be a compact group and H be a closed subgroup of G. Then there exists a unique, up to scalar multiple, finite (right) G-invariant Radon measure on H/G.*

Proof. Let $q_1 : G \to G/H$ and $q_2 : G \to H/G$ be the quotient maps. Define a function $\phi : H/G \to G/H$ by $\phi(Hs) = s^{-1}H$. Then ϕ is clearly a bijection. In fact, ϕ is a homeomorphism. Let $E \subseteq H/G$ be an open set. Then $\phi(E) = q_1((q_2^{-1}(E))^{-1})$. Since q_1, q_2 are open, continuous maps and $s \mapsto s^{-1}$ is a homeomorphism of G then $\phi(E)$ is open in G/H. Similarly, if $F \subseteq G/H$ is an open set then $\phi^{-1}(F) = q_2((q_1^{-1}(F))^{-1})$ is open in H/G.

Let μ be a measure on G/H as in Proposition 36. Define a function ν on H/G by $\nu(E) = \mu(\phi(E))$ where E is a Borel set in H/G. Since ϕ is a homeomorphism then ν is a Radon measure. Finally, to see that ν is (right) G-invariant let $s \in G$ and $E \subseteq H/G$ then

$$\begin{aligned}
\nu(Es) &= \mu(q_1((q_2^{-1}(Es))^{-1})) \\
&= \mu(s^{-1}q_1((q_2^{-1}(E))^{-1})) \\
&= \mu(q_1((q_2^{-1}(E))^{-1})) \\
&= \nu(E)
\end{aligned}$$

□

Remark 38. Let ϕ be the function defined in Corollary 37. Since ϕ is a Borel isomorphism we can define a bijective map Φ from the space of Borel functions on G/H onto the space of Borel functions on H/G by $\Phi(f) = f \circ \phi$. Let μ a finite (left) invariant Radon measure on G/H and ν the corresponding measure on H/G as constructed in Corollary 37. Then Φ is an isometry from the set of all characteristic functions in $L^2(G/H,\mu)$ onto the set of all characteristic functions in $L^2(H/G,\nu)$. By linearity Φ is an isometry from the set of simple functions in $L^2(G/H,\mu)$ onto the set of simple functions in $L^2(H/G,\nu)$. Since simple functions are dense in $L^2(G/H,\mu)$ and $L^2(H/G,\nu)$ we get that Φ is an isometry from $L^2(G/H,\mu)$ onto $L^2(H/G,\nu)$.

If G is not compact then a weaker but still useful result exists. Suppose G is second countable locally compact group and H is a closed subgroup of G. Let ν be a Borel measure on H/G which we assume to be finite on compact sets. For each $s \in G$ define the translate ν_s of ν by $\nu_s(E) = \nu(Es)$. The measure ν is said to be *quasi-invariant* if the measures ν_s are all equivalent (i.e. mutually absolutely continuous). To see that such measures always exist choose any finite measure υ on G from the measure class containing the Haar measure and let $\nu(E) = \upsilon(q^{-1}(E))$ for each Borel set $E \subseteq H/G$. It is not hard to check that the measure ν thus defined is quasi-invariant. It turns out that every quasi-invariant measure on H/G is equivalent (i.e. mutually absolutely continuous) to the one described above ([13]; Lemma 1.3). Moreover, each quasi-invariant measure arises from a unique member of a class of Borel functions on G in the following manner. Let $\triangle_G$ and $\triangle_H$ be the modular functions of the right Haar measure for G and H respectively. Define a *rho-function* to be a positive Borel function ρ on G which is bounded on compact sets and such that $\rho(st) = \frac{\triangle_H(s)}{\triangle_G(s)}\rho(t)$ for all $t \in G$ and $s \in H$. Note that if ρ is a rho-function then $\frac{\rho(rt)}{\rho(r)}$ is a Borel function of (r,t) which is constant on the right cosets of $H \times G$ in $G \times G$. There exists a quasi-invariant measure μ on H/G such that for all $t \in G$, $\frac{\rho(rt)}{\rho(r)}$ is the Radon-Nikodym derivative of the measure $E \to \mu(Et)$ with respect to the measure μ. Conversely, given a quasi-invariant measure μ on H/G there exists a corresponding rho-function ([13]; Theorem 1.1).

Remark 39. There have been notable advances in the theory of quasi-invariant measures on quotient spaces since the original results obtained by Mackey. In particular, given *any* locally compact group G and a closed subgroup H it has been shown ([8]; Theorem 2.56 and 2.59) that there exists a strongly quasi-invariant measure on H/G; that is, a quasi-invariant Radon measure whose corresponding rho-function is continuous.

Remark 40. If G is a compact group and H is a closed subgroup of G. We can obtain a G-invariant Borel measure ν on H/G by letting $\nu(E) = \mu(q^{-1}(E))$ for each Borel set $E \subseteq H/G$ where μ is the Haar measure on G. However, it is not clear that G-invariant Borel measures on H/G are unique up to a scalar multiple (it is also unclear that the measure ν constructed above is outer regular).

1.3.2 Unitary Representations of Locally Compact Groups

Representations of a group on a Hilbert space play a fundamental role in the study of non abelian locally compact groups. A major tool in the study of group representations is the Mackey machine that allows one to study the relationship between the representations of a group and the representations of its subgroups. We would like to outline some of the key ideas of the Mackey's constructions.

Remark 41. Mackey developed his theory for locally compact, second countable groups. Therefore, we will also work with these groups. It is important to point out that Mackey used the word "separable" to mean second countable which initially was a source of my personal confusion. One of the main reasons Mackey chose to work with second countable groups is because the theory of decomposable operators over standard Borel spaces is well understood (see Theorem 20). Mackey's ideas were later extended to arbitrary locally compact groups using a somewhat different approach ([8]; Chapter 6).

Definition 42. A (unitary) *representation* of a locally compact group G is a continuous homomorphism $U : G \to U(\mathcal{H})$ where $U(\mathcal{H})$ is given the strong operator topology, i.e. $U(s_i) \to U(s)$ if and only if $U(s_i)\zeta \to U(s)\zeta$ for all $\zeta \in \mathcal{H}$. A pair of representations U and V of G are *equivalent* if there exists a unitary map $W : \mathcal{H}_U \to \mathcal{H}_V$ such that $V = WUW^*$. We say U is irreducible if $\mathcal{H}_U$ has no closed invariant subspaces.

Note that a homomorphism $U : G \to U(\mathcal{H})$ is continuous in the strong operator topology if and only if the map $s \mapsto \langle U(s)\zeta, \eta\rangle$ is Borel for all $\zeta, \eta \in \mathcal{H}$ ([18]; D.11, D.42).

Example 43. For each $z \in \mathbf{T}$ we can define a homomorphism $\theta_z : \mathbf{Z} \to \mathbf{T}$ by $\theta_z(n) = z^n$. It is not hard to see that every irreducible representation of $\mathbf{Z}$ is of the form θ_z for some $z \in \mathbf{T}$. In general, if G is an abelian locally compact group then every irreducible representation of G is a homomorphism into the unit circle.

Example 44. Let G be a locally compact group together with a Haar measure μ. Define a representation λ of G on $L^2(G)$ by $\lambda(t)\zeta(s) = \zeta(st)$. Since each $\zeta \in C_c(G)$ is uniformly continuous ([18]; Lemma 1.62) then λ is continuous homomorphism on $C_c(G)$. It follows from Corollary 35 that λ is continuous on $L^2(G)$. This representation is called the (right) regular representation of G.

Let G be a locally compact second countable group and H be a closed subgroup of G. Suppose $U_0 : H \to U(\mathcal{H}_0)$ is a unitary representation then there is a process of "inducing" U_0 to a unitary representation of G developed originally by Mackey [13],[12]. Let μ be any fixed quasi-invariant measure on H/G. Let $\mathcal{H}$ denote the induced representation space which is the space of all $\mathcal{H}_0$ valued functions ξ on G satisfying the following conditions:

1. $\langle \xi(s), h_0\rangle$ is Borel function of s for all $h_0 \in \mathcal{H}_0$.

2. $\xi(ts) = U_0(t)\xi(s)$ for all $t \in H$ and all $s \in G$.

3. $\int_{H/G} \langle \xi(s), \xi(s) \rangle d\mu(\overline{s}) < \infty$

where the integrand is constant on the right cosets of H, by condition 2, and hence defines a function on H/G. The inner product on $\mathcal{H}$ is given by

$$\langle \xi_1, \xi_2 \rangle = \int_{H/G} \langle \xi_1(s), \xi_2(s) \rangle d\mu(\overline{s})$$

for all $\xi_1, \xi_2 \in \mathcal{H}$. After identifying functions that are equal almost everywhere, $\mathcal{H}$ becomes a Hilbert space.

Remark 45. We can observe that the induced representation space $\mathcal{H}$ is complete by identifying it with the complete space $L^2(H/G, \mathcal{H}_0, \mu)$. Let E be the Borel subset of G that intersects each right coset of H in exactly one point ([13]; Lemma 1.1). Then $H \times E$ is a Borel subset of $G \times G$ and hence it is a standard Borel space when equipped with the relative Borel structure in $G \times G$. The multiplication map from $H \times E$ to G is a Borel bijection. Since $H \times E$ and G are both standard Borel spaces then the multiplication map is a Borel isomorphism ([1]; Theorem 3.3.2). Define a $U(\mathcal{H}_0)$-valued function V on G by $V(rt) = U_0(r)$ for all $r \in H$ and $t \in E$. Then V is a Borel function.

Define a map from $L^2(H/G, \mathcal{H}_0, \mu)$ into $\mathcal{H}$ by $\xi \mapsto \overline{\xi}$ where $\overline{\xi}(s) = V(s)\xi(q(s))$. This map is an isometry from $L^2(H/G, \mathcal{H}_0, \mu)$ onto $\mathcal{H}$. Since $L^2(H/G, \mathcal{H}_0, \mu)$ is complete then $\mathcal{H}$ is also complete.

Let ρ be a rho-function corresponding to the measure μ. Define U to be the homomorphism of G into the unitary group of $\mathcal{B}(\mathcal{H})$ given by:

$$(U(t)\xi)(s) = \sqrt{\frac{\rho(st)}{\rho(s)}}\xi(st)$$

for all $\xi \in \mathcal{H}$ and $s, t \in G$. Then $U : G \to U(\mathcal{H})$ is a unitary representation ([14]; Theorem 4.1).

Remark 46. Induced representations generalize the right regular representation. Let G be a locally compact, second countable group and $H = \{e\}$ be the trivial subgroup. Then the representation of G induced from the trivial representation of $\{e\}$ produces the right regular representation of G.

In the construction of an induced representation for locally compact groups one has various types of measures from which to choose. In the classical construction Mackey used quasi-invariant measures [13] and in the more modern approach one usually uses a strongly quasi-invariant measure ([8]; Chapter 6). The induced representation does not depend on the choice of the quasi-invariant measure ([13]; Theorem 2.1).

1.3.3 Unitary Representations of Compact Groups

For the remainder of this section we will assume, unless otherwise specified, that G is a second countable compact group, H a closed subgroup of G, and H/G the right coset space endowed with the quotient topology. Since we are mostly interested in working with compact groups, it is advantageous to use a G-invariant Radon measure on H/G because the independence of the induced representation on the choice of such a measure will be immediately clear. In addition, if using a G-invariant measure in the construction of induced representation, one can do away with the rho-function, which makes the calculations less cumbersome. Therefore, our preference for using G-invariant Radon measures, when G is compact, is essentially to avoid the extra work involved with quasi-invariant measures. We shall make a separate definition for induced representations of compact groups.

Definition 47. Let G be a compact second countable group and let H be a closed subgroup of G. Suppose $U_0 : H \to U(\mathcal{H}_0)$ is a unitary representation. Let μ be any G-invariant measure on H/G as in Corollary 37. Define $\mathcal{H}$ to be the space of all $\mathcal{H}_0$ valued functions ξ on G satisfying the following conditions:

1. $\langle \xi(s), h_0 \rangle$ is Borel function of s for all $h_0 \in \mathcal{H}_0$.

2. $\xi(ts) = U_0(t)\xi(s)$ for all $t \in H$ and all $s \in G$.

3. $\int_{H/G} \langle \xi(s), \xi(s) \rangle d\mu(\overline{s}) < \infty$.

Define U to be the homomorphism of G into the unitary group of $\mathcal{B}(\mathcal{H})$ given by $(U(t)\xi)(s) = \xi(st)$ for all $\xi \in \mathcal{H}$ and $s, t \in G$. We call U the *induced* representation of G by U_0.

We now investigate a certain type of representation of G that will be of particular use later in this paper. Our discussion closely follows Sections 5 and 6 of [14] where the same material is discussed in the context of second countable, locally compact groups.

Let μ be a G-invariant Radon measure on H/G and let $\mathcal{H}_k$ be a separable Hilbert space of dimension $k = 1, 2, .., \infty$. Define $\mathcal{H} = L^2(H/G, \mu, \mathcal{H}_k)$ to be the space of all square integrable functions from H/G to $\mathcal{H}_k$. For each Borel subset E of H/G we can define an operator P_E on $\mathcal{H}$ to be multiplication by the characteristic function of E. Suppose $U : G \to U(\mathcal{H})$ is a representation such that $U(s)P_E U(s)^* = P_{Es^{-1}}$ for all $s \in G$ and all Borel subsets E of H/G. Then H/G together with the map $E \mapsto P_E$ is called the canonical system of imprimitivity for the representation U ([14]; p. 279). As we will see below, this representation is induced from a certain representation of H. Before we proceed it is useful to discuss the case when the group is discrete because one can see the construction of the induced representation without the measure-theoretic technicalities.

Example 48. Let G be a countable discrete group, H a subgroup of G, and μ the counting measure. Let $\mathcal{H}_k$ be a separable Hilbert space and define $\mathcal{H} = l^2(H/G, \mathcal{H}_k)$ to be space of $\mathcal{H}_k$-valued square-summable sequences indexed by H/G. For each subset E of H/G define an operator P_E on $\mathcal{H}$ to be multiplication by the characteristic function of E. Suppose $U : G \to U(\mathcal{H})$ is a representation such that $U(s)P_E U(s)^* = P_{Es^{-1}}$ for all $s \in G$ and all subsets E of H/G. *Our goal is show that there exists a representation U_0 of H such that U is equivalent to the representation induced by U_0.*

We first show that U can be viewed as a generalized permutation matrix acting on $\mathcal{H}$. For each $s \in G$ define an operator $V(s)$ on $\mathcal{H}$ by $(V(s)\zeta)(Ht) = \zeta(Hts)$. It is easily observed that the map $s \mapsto V(s)$ is a homomorphism of G. Since G is discrete, V is automatically continuous. A simple calculation shows that $V(s)P_E V(s)^* = P_{Es^{-1}}$ for all $s \in G$ and all $E \subseteq H/G$. It follows that

$$U(s)V(s)^* P_E V(s)U(s)^* = U(s)P_{Es}U(s)^* = P_E.$$

In particular, $U(s)V(s)^*$ commutes with P_E for all $s \in G$ and all $E \subseteq H/G$. If we view $U(s)V(s)^*$ and P_E as operator valued matrices then it follows that $U(s)V(s)^*$ is a diagonal matrix; that is $U(s)V(s)^*$ is a decomposable operator. Let $U(s)V(s)^*(Ht) \in U(\mathcal{H}_k)$ denote the corresponding diagonal entry of the operator valued matrix $U(s)V(s)^*$ and define a function $W : G \times H/G \to U(\mathcal{H}_k)$ by $W(s, Ht) = U(s)V(s)^*(Ht)$. Then

$$\begin{aligned}
(U(s)\zeta)(Ht) &= (U(s)V^*(s)V(s)\zeta)(Ht) \\
&= W(s, Ht)(V(s)\zeta)(Ht) \\
&= W(s, Ht)\zeta(Hts)
\end{aligned}$$

for each $s \in G$, $\zeta \in \mathcal{H}$, and $Ht \in H/G$. Moreover, it is not hard to verify that W

satisfies the following properties

1. $W(s_1s_2, Ht) = W(s_1, Ht)W(s_2, Hts_1)$ for all $s_1, s_2 \in G$ and all $Ht \in H/G$.

2. $W(e, Ht)$ is the identity for all $Ht \in H/G$.

We extend W to the function $\overline{W} : G \times G \to U(\mathcal{H}_k)$ by defining $\overline{W}(s,t) = W(s, Ht)$. Then the properties of W can be translated to the properties of $\overline{W}$ in the following way:

1. $\overline{W}(s_1s_2, t) = \overline{W}(s_1, t)\overline{W}(s_2, ts_1)$ for all $s_1, s_2 \in G$ and all $t \in G$.

2. $\overline{W}(e, t)$ is the identity for all $t \in G$.

3. $\overline{W}(s, t) = \overline{W}(s, rt)$ for all $s, t \in G$ and all $r \in H$.

We next show that the function $\overline{W}$ can be expressed via a $U(\mathcal{H}_k)$-valued function on G. Let t_0 be a fixed element of G and define a function $B : G \to U(\mathcal{H}_k)$ by $B(s) = \overline{W}(t_0^{-1}s, t_0)$, then

$$
\begin{aligned}
B^{-1}(s)B(st) &= \overline{W}(t_0^{-1}s, t_0)^{-1}\overline{W}(t_0^{-1}st, t_0) \\
&= \overline{W}(t_0^{-1}s, t_0)^{-1}\overline{W}(t_0^{-1}s, t_0)\overline{W}(t, t_0t_0^{-1}s) \\
&= \overline{W}(t, s).
\end{aligned}
$$

Furthermore, we claim there exists a representation $U_0 : H \to U(\mathcal{H}_k)$ such that $B(rs) = U_0(r)B(s)$ for all $s \in G$ and all $r \in H$. To see this, let $r \in H$ then by property (3) of $\overline{W}$ we have

$$
B^{-1}(s)B(st) = B^{-1}(rs)B(rst)
$$

for all $s, t \in G$. After rearranging terms we get

$$B(rs)B^{-1}(s) = B(rst)B^{-1}(st) \tag{1.1}$$

for all $s, t \in G$. Let s_1, s_2 be any two elements in G then by setting $s = s_1$ and $t = s_1^{-1}s_2$ in the Equation (1.1) above we get that $B(rs_1)B^{-1}(s_1) = B(rs_2)B^{-1}(s_2)$. For each $r \in H$ define $U_0(r) = B(rs)B^{-1}(s)$. Then it is not hard to check that U_0 defines the necessary representation of H.

We are now in position to show that the representation $U : G \to U(l^2(H/G, \mathcal{H}_k))$ is equivalent to the one induced from the representation $U_0 : H \to U(\mathcal{H}_k)$. Indeed, let $\overline{U}$ denote the representation of G induced from U_0 and $\overline{\mathcal{H}}$ denote the induced representation space. Define a map $L : l^2(H/G, \mathcal{H}_k) \to \overline{\mathcal{H}}$ by $\zeta \mapsto \overline{\zeta}$ where $\overline{\zeta}(t) = B(t)\zeta(Ht)$. We claim that L is the desired unitary equivalence. To this end, first note that $\overline{\zeta}(rt) = B(rt)\zeta(Hrt) = U_0(r)B(t)\zeta(Ht) = U_0(r)\overline{\zeta}(t)$ for all $r \in H$. Moreover, $\|\overline{\zeta}\|^2 = \sum_{H/G} \|\overline{\zeta}(t)\|^2 = \sum_{H/G} \|B(t)\zeta(Ht)\|^2 = \|\zeta\|^2$. So L is an isometry. For each $\overline{\zeta} \in \overline{\mathcal{H}}$ define a function $\zeta \in \mathcal{H}$ by $\zeta(Ht) = B(t)^{-1}\overline{\zeta}(t)$ then $L(\zeta) = \overline{\zeta}$. So L is surjective and it is a unitary operator. Finally, we check that

$$\begin{aligned}
(LU(s)L^*\overline{\zeta})(t) &= B(t)(U(s)L^*\overline{\zeta})(Ht) \\
&= B(t)\overline{W}(s,t)(L^*\overline{\zeta})(Hts) \\
&= B(t)\overline{W}(s,t)B^{-1}(ts)\overline{\zeta}(ts) \\
&= B(t)(B^{-1}(t)B(ts))B^{-1}(ts)\overline{\zeta}(ts) \\
&= \overline{\zeta}(ts) \\
&= (\overline{U}(s)\overline{\zeta})(t).
\end{aligned}$$

for all $\overline{\zeta} \in \overline{\mathcal{H}}$ and $s \in G$.

We now return to our discussion of the case where G is a compact, second countable group. Let $U : G \to U(\mathcal{H})$ be the representation described in the start of the section where $\mathcal{H} = L^2(H/G, \mu, \mathcal{H}_k)$. One can follow essentially the same steps as outlined in the preceding example to show that U is equivalent to a representation of G induced from a representation $U_0 : H \to U(\mathcal{H}_k)$. However, if G is not discrete, one needs to address various measure-theoretic considerations.

For each $s \in G$ define an operator $V(s)$ on $\mathcal{H}$ by $(V(s)\zeta)(Ht) = \zeta(Hts)$. Observe that the map $s \mapsto V(s)$ is a homomorphism of G. Using the uniform continuity arguments similar to Proposition 83 we can show that $V(s_i)\zeta \to V(s)\zeta$ whenever $s_i \to s$ for all $\zeta \in C(H/G, \mathcal{H}_k)$. Since $C(H/G, \mathcal{H}_k)$ is dense in $L^2(H/G, \mu, \mathcal{H}_k)$ it follows that V is continuous in the strong operator topology (see also [14]; Theorem 5.3). For each $s \in G$ let $W(s) = U(s)V(s)^{-1}$ then it is not hard to check that $W(s)P_E = P_E W(s)$ for all $s \in G$ and for all Borel subsets E of H/G. Hence $W(s)$ is decomposable (Theorem 20); that is, for each $s \in G$ there exists a $U(\mathcal{H}_k)$-valued Borel function $W(s, Ht)$ on $G \times H/G$ such that $(W(s)\zeta)(Ht) = W(s, Ht)\zeta(Ht)$ for every $\zeta \in \mathcal{H}$ and almost all $Ht \in H/G$.

The $U(\mathcal{H}_k)$-valued function $W(s, Ht)$ on $G \times H/G$ may be chosen to be a Borel function.

Theorem 49 ([14]; Theorem 5.6)**.** *Let $\mathcal{H} = L^2(H/G, \mu, \mathcal{H}_k)$. Suppose that $U : G \to U(\mathcal{H})$ is a representation such that $U(s)P_E U(s)^* = P_{Es^{-1}}$ for all $s \in G$ and all Borel subsets E of H/G. Then there exists an essentially unique function $W : G \times H/G \to U(\mathcal{H}_k)$ such that:*

1. *For each $s_1, s_2 \in G$ we have $W(s_1 s_2, Ht) = W(s_1, Ht)W(s_2, Hts_1)$ for almost all $Ht \in H/G$.*

2. *$W(e, Ht)$ is the identity for almost all $Ht \in H/G$.*

3. *For all $v_1, v_2 \in \mathcal{H}_k$, $\langle W(s, Ht)v_1, v_2\rangle$ is measurable as a function on $G \times H/G$, and for each $s \in G$ is measurable as a function on H/G.*

Remark 50. Since $W(s) = U(s)V(s)^{-1}$ we have

$$(U(s)\zeta)(Ht) = W(s, Ht)\zeta(Hts)$$

for all $s \in G$, $\zeta \in \mathcal{H}$, and almost all $Ht \in H/G$.

Recall that the quotient map $q : G \to H/G$ is Borel. Thus we can extend the function W from the above theorem to a function on $G \times G$ by $\overline{W}(s, t) = W(s, Ht)$. The conditions of the above theorem can be restated for the function $\overline{W}$ as follows:

1. For each $s_1, s_2 \in G$, $\overline{W}(s_1s_2, t) = \overline{W}(s_1, t)\overline{W}(s_2, ts_1)$ for almost all $t \in G$.

2. $\overline{W}(e, t)$ is the identity for almost all $t \in G$.

3. For all $v_1, v_2 \in \mathcal{H}_k$, $\langle\overline{W}(s, t)v_1, v_2\rangle$ is measurable as a function on $G \times G$, and for each $s \in G$, is measurable as a function on G.

Remark 51. The above equalities hold almost everywhere because $\mu(E) = 0$ if and only if $\nu(q^{-1}(E)) = 0$ where ν is the Haar measure on G.

It turns out that the function $\overline{W}$ can be expressed via a $U(\mathcal{H}_k)$-valued Borel function on G ([14], Lemma 6.1 and 6.2):

Lemma 52. *Let $\overline{W}$ be a function on $G \times G$ as described above, then there exists an essentially unique $U(\mathcal{H}_k)$-valued Borel function on G such that $\overline{W}(s, t) = B^{-1}(t)B(ts)$ for almost all pairs s, t. Moreover the function B may be chosen so that $B(rs) = U_0(r)B(s)$ for all $r \in H$ and all $s \in G$ where U_0 is a unitary representation of H on $\mathcal{H}_k$ which is uniquely determined by $\overline{W}$ up to unitary equivalence.*

We are now in position to state the main theorem of this section.

Theorem 53 ([14]; Theorem 6.5)**.** *Let $\mathcal{H} = L^2(H/G, \mu, \mathcal{H}_k)$. Suppose $U : G \to U(\mathcal{H})$ is a representation such that $U(s)P_E U(s)^* = P_{Es^{-1}}$ for all $s \in G$ and all Borel subsets E of H/G. Let $B : G \to U(\mathcal{H}_k)$ and $U_0 : H \to U(\mathcal{H}_k)$ be as in the preceding lemma. Then U is unitarily equivalent to the representation of G induced from U_0.*

Example 54. Let G be a compact, second countable group, H be a closed subgroup of G, and $\mathcal{H} = L^2(H/G, \mu, \mathcal{H}_k)$ where μ is a G-invariant Radon measure on H/G. Define a representation $U : G \to U(\mathcal{H})$ by $(U(s)\zeta)(Ht) = \zeta(Hts)$. Then using the above notation $\overline{W}(s,t) = 1_{\mathcal{H}_k}$ and $U(r) = 1_{\mathcal{H}_k}$ for all $s, t \in G$ and all $r \in H$. In other words, U is induced from the trivial representation of H.

Example 55. Let D_3 be the dihedrial group of order 6 with discrete topology and $\mathbf{T}$ be the group of elements of the circle under multiplication with the Eucledian topology; define $G = D_3 \times \mathbf{T}$ with the product topology. Let λ be the usual representation of D_3 on $\mathbf{C}^3$ via permutation matrices. Let μ be the Lesbegue measure and define U to be the representation of G on $L^2(\mathbf{T}, \mu, \mathbf{C}^3)$ given by

$$U(s, e^{i\theta})\zeta(z) = \lambda(s)\zeta(ze^{i\theta})$$

for all $(s, e^{i\theta}) \in G$ and all $\zeta \in L^2(\mathbf{T}, \mu, \mathbf{C}^3)$. Then U is equivalent to the representation of G induced from λ.

1.4 C^*-dynamical systems

In this section we will discuss actions of locally compact groups on C^*-algebras. In particular, we will study covariant representations of C^*-dynamical systems. We will show that many of the constructions related to unitary representations of locally compact groups can be carried over to covariant representations of C^*-dynamical systems.

We say that G *acts* on the right on a set X if there is a map such that

$$(x, s) \mapsto x \cdot s \tag{1.2}$$

from $X \times G \to X$ such that for all $s, r \in G$ and all $x \in X$

$$x \cdot e = x \text{ and } x \cdot rs = (x \cdot s) \cdot r$$

If G and X are both topological spaces we say that the action is continuous if the map in (1.2) is (jointly) continuous. In this case, X is called a *topological G-space.* If both X and G are locally compact spaces then we say that X is a locally compact G-space.

Example 56. Let X be any topological space and $h \in$Homeo(X) then $\mathbf{Z}$ acts on X by $x \cdot n = h^n(x)$. An important example of a $\mathbf{Z}$-space is the unit circle together with rotation by an angle θ; that is, $X = \mathbf{T}$ and $h(z) = ze^{i\theta}$.

Example 57. Let G be any locally compact group and H be a subgroup of G then H/G is a topological G-space and G is a topological H-space where in each case the action is given by right multiplication together with the usual topologies. Note that if H is a closed subgroup then H/G is Hausdorff.

Let X be a locally compact G-space. Since every homeomorphism of X defines an automorphism of $C_0(X)$ we obtain a group homomorphism $\sigma : G \to$Aut$(C_0(X))$ defined by

$$\sigma_s(f)(x) = f(x \cdot s).$$

It is not hard to check that σ is continuous with respect to the point-norm topology on Aut$(C_0(X))$. The triple $(C_0(X), G, \sigma)$ is a classical example of a C^*-dynamical system.

Definition 58. A C^**-dynamical system* is a triple (A, G, σ) consisting of a C^*-algebra A, a locally compact group G, and a continuous homomorphism $\sigma : G \to \mathrm{Aut}(A)$. We say that (A, G, σ) is *separable* if A is separable and G is second countable. If H is a closed subgroup of G then by restricting σ to H we obtain C^*-dynamical system (A, H, σ_H) which we often simply denote by (A, H, σ) and refer to it as a subsystem of (A, G, σ).

We will often refer to (A, G, σ) simply as a dynamical system.

Remark 59. Recall that $\mathrm{Aut}(A)$ is endowed with the point-norm topology, i.e. $\sigma_{s_i} \to \sigma_s$ if and only if $\sigma_{s_i}(a) \to \sigma_s(a)$ for all $a \in A$.

As we have mentioned above every locally compact G-space X gives rise to a C^*-dynamical system $(C_0(X), G, \sigma)$ where σ is defined by the action of G on X. Conversely, we know from Lemma 26 that every automorphism of $C_0(X)$ is implemented via an action of G on X.

Proposition 60 ([18]; Proposition 2.7)**.** *Suppose that* $(C_0(X), G, \sigma)$ *is a* C^**-dynamical system. Then there is an action of* G *on* X *such that* X *is a* G*-space and*

$$\sigma_s(f)(x) = f(x \cdot s). \tag{1.3}$$

Given a C^*-dynamical system (A, G, σ) one can study its properties by looking at representations of the system on a Hilbert space, i.e. pairs of representations of A and G that suitably implement the group action on the C^*-algebra.

Definition 61. Let (A, G, σ) be a C^*-dynamical system. A *covariant representation* of (A, G, σ) is a pair (π, U) consisting of a representation $\pi : A \to \mathcal{B}(\mathcal{H})$ and a unitary representation $U : G \to U(\mathcal{H})$ such that

$$\pi(\sigma_s(a)) = U(s)\pi(a)U^*(s). \tag{1.4}$$

We say (π, U) is a *nondegenerate* covariant representation if π is a nondegenerate representation. We say that (π, U) is an *irreducible* covariant representation if there is no nontrivial closed subspace $\mathcal{H}_1$ of $\mathcal{H}$ such that $\pi(A)(\mathcal{H}_1) \subseteq \mathcal{H}_1$ and $U(G)\mathcal{H}_1 \subseteq \mathcal{H}_1$. We say that (π, U) is a *factor* covariant representation if the von Neumann algebra generated by $\pi(A)$ and $U(G)$ is a factor.

Suppose that (π, U) is a (possibly degenerate) representation of (A, G, σ) on a Hilbert space $\mathcal{H}$. Let $\mathcal{H}_1 = \overline{\pi(A)\mathcal{H}}$. Then the restriction of (π, U) to $\mathcal{H}_1$ is a nondegenerate covariant representation of (A, G, σ). Since π acts trivially on the orthogonal complement of $\mathcal{H}_1$ we can without loss of generality work with nondegenerate covariant representations.

Example 62. Suppose that $(C_0(X), G, \sigma)$ is a dynamical system and μ is a G-invariant measure on X. Then define a covariant representation (π_μ, λ) of $(C_0(X), G, \sigma)$ on $L^2(X, \mu)$ by $(\pi_\mu(f)\zeta)(x) = f(x)\zeta(x)$ and $(\lambda_s\zeta)(x) = \zeta(x \cdot s)$ for all $f \in C_0(X)$, $s \in G$, and $\zeta \in L^2(X, \mu)$.

(a) Let $0 \leq \theta < 2\pi$ and define an action of $\mathbf{Z}$ on $C(\mathbf{T})$ by $(\theta^n f)(z) = f(ze^{in\theta})$. Then $(C(\mathbf{T}), \mathbf{Z}, \theta)$ is a dynamical system. We can define a covariant representation (π_μ, λ) of $(C(\mathbf{T}), \mathbf{Z}, \theta)$ on $L^2(\mathbf{T}, \mu)$ as above where μ is the Lebesgue measure.

(b) Let G be a compact group and H be a closed subgroup of G. Let G act on H/G by right multiplication and $(C_0(H/G), G, \sigma)$ be the corresponding dynamical system. We can define a covariant representation (π_μ, λ) of $(C_0(H/G), G, \sigma)$ on $L^2(H/G, \mu)$ as above where μ is a G-invariant measure on H/G.

Example 63. Let (A, G, σ) be C^*-dynamical system and π_0 be a representation of A on a Hilbert space $\mathcal{H}$. Let U be the right regular representation of G on $L^2(G, \mu, \mathcal{H})$ that is $(U(t)\zeta)(s) = \zeta(st)$ for all $s \in G$ and $\zeta \in L^2(G, \mu, \mathcal{H})$. Define a representation of A on $L^2(G, \mu, \mathcal{H})$ by $(\pi(a)\zeta)(s) = \pi_0(\sigma_s a)\zeta(s)$ for all $a \in A$ and all $\zeta \in L^2(G, \mu, \mathcal{H})$. Then (π, U) is a covariant representation of (A, G, σ) called the right regular representation based on π_0. This shows that covariant representations always exist.

1.4.1 Induced Covariant Representations

We have already discussed the concept of induced representations in the context of locally compact groups. It turns out that one can naturally extend the construction of induced representations for unitary representations of groups to the context of covariant representations of dynamical systems. We will describe induced covariant representations following the construction given in Section 3 of [17]. Note that since we are mostly interested in C^*-dynamical systems involving compact groups our definition of the induced representation is slightly different from the one given in [17].

Let H be a closed subgroup of a compact second countable group G. Let (π_0, U_0) be a covariant representation of (A, H, σ) on a separable Hilbert space $\mathcal{H}_0$. Let U be the induced representation of G from U_0 as described in the previous section. Let $\mathcal{H}$ denote the induced representation space. Recall that $\mathcal{H}$ is the space of all $\mathcal{H}_0$ valued functions ζ on G satisfying the following conditions:

1. $\langle \zeta(s), h_0 \rangle$ is Borel function of s for all $h_0 \in \mathcal{H}_0$.

2. $\zeta(ts) = U_0(t)\zeta(s)$ for all $t \in H$ and all $s \in G$.

3. $\int_{H/G} \langle \zeta(s), \zeta(s) \rangle d\mu(\overline{s}) < \infty$.

We require $\mathcal{H}_0$ to be separable to avoid technicalities involved with non separable spaces ([5]; Definition I.14). Also recall that U is the homomorphism of G into the unitary group of $\mathcal{B}(\mathcal{H})$ given by:

$$(U(t)\zeta)(s) = \zeta(st)$$

for all $\zeta \in \mathcal{H}$ and $s, t \in G$. We define a representation π of A on $\mathcal{H}$ by:

$$(\pi(a)\zeta)(s) = \pi_0(\sigma_s a)\zeta(s)$$

for all $\zeta \in \mathcal{H}$ and $s \in G$. Then (π, U) is easily checked to be a covariant representation of (A, G, σ):

$$\begin{aligned} U(t)\pi(a)U(t^{-1})\zeta(s) &= (\pi(a)U(t^{-1})\zeta)(st) \\ &= \pi_0(\sigma_{st}a)(U(t^{-1})\zeta)(st) \\ &= \pi_0(\sigma_{st}a)\zeta(s) = \pi(\sigma_t a)\zeta(s) \end{aligned}$$

for all $s, t \in G$ and $a \in A$. Since the G-invariant measure μ is unique up to a scalar multiple the induced representation is independent of the choice of the G-invariant measure. The covariant representation (π, U) is called the induced representation from (π_0, U_0).

Remark 64. Induced covariant representations generalize induced unitary representations. Induced covariant representations also generalize the right regular representations in Example 63.

Remark 65. Let (A, G, σ) be a C^*-dynamical system where G is a locally compact group. Let H be a closed subgroup of G and let (π_0, U_0) be a covariant representation of (A, H, σ) on a separable Hilbert space $\mathcal{H}_0$. Then we can construct the induced covariant representation of (A, G, σ) exactly as above by taking U to be the induced representation of G from the representation U_0 of H and defining the representation π of A exactly as above ([17]; Section 3).

We will now discuss a special type of covariant representations of separable dynamical systems which are closely related to the representations presented in the end of the previous section. Our discussion follows closely Section 4 of [17].

Let (A, G, σ) be a separable dynamical system where G is compact. Let H be a closed subgroup of G and μ be a G-invariant Radon measure on H/G and let $\mathcal{H}_k$ be a separable Hilbert space of dimension $k = 1, 2, .., \infty$. Define $\mathcal{H} = L^2(H/G, \mu, \mathcal{H}_k)$ to be the space of all square integrable function from H/G to $\mathcal{H}_k$. For each Borel subset E of H/G we define an operator P_E on $\mathcal{H}$ to be multiplication by the characteristic function of E. Let (π, U) be a covariant representation of (A, G, σ) on $\mathcal{H}$ such that

1. $\pi(A) \subseteq (L^\infty(H/G, \mu) \otimes 1_{\mathcal{H}_k})'$
2. $U(s)P_E U(s)^* = P_{Es^{-1}}$ for all $s \in G$ and all Borel subsets E of H/G.

Our goal is to show that there exists a covariant representation (π_0, U_0) of (A, H, σ) such that (π, U) is equivalent to the representation induced from (π_0, U_0). In fact, we have already found the representation U_0 in Section 1.3.3. It remains to show that there exists an appropriate representation π_0. We first show that π can be expressed as a $\mathrm{Rep}(A : \mathcal{H}_k)$-valued function on G given by $s \mapsto \overline{\pi_s}$. Then we show that there exists a representation $\pi_0 : A \to \mathcal{B}(\mathcal{H}_k)$ such that $\overline{\pi_t}$ is equivalent to $\pi_0 \circ \sigma_t$ for almost all $t \in G$. Finally, we show that (π_0, U_0) is the covariant representation of (A, H, σ) that induces to a representation equivalent to (π, U).

It follows from condition 1 that π is decomposable (see discussion following Theorem 20); that is, there exists a set of representations $\{\pi\}_{Ht}$ of A on $\mathcal{H}_k$ such that $(\pi(a)\zeta)(Ht) = \pi_{Ht}(a)\zeta(Ht)$ for each $a \in A$, $\zeta \in \mathcal{H}$ and almost all $Ht \in H/G$. Recall from Theorem 49 that there exists a $U(\mathcal{H}_k)$-valued Borel function W on $G \times H/G$ such that $(U(s)\zeta)(Ht) = W(s, Ht)\zeta(Hts)$ for each $s \in G$, $\zeta \in \mathcal{H}$ and almost all $Ht \in H/G$. It follows that

$$\begin{aligned}
(\pi(\sigma_s a)\zeta)(Ht) &= (U(s)\pi(a)U(s)^*\zeta)(Ht) \\
&= W(s, Ht)(\pi(a)U(s)^*\zeta)(Hts) \\
&= W(s, Ht)\pi_{Hts}(a)(U(s)^*\zeta)(Hts) \\
&= W(s, Ht)\pi_{Hts}(a)W(s^{-1}, Hts)\zeta(Ht) \\
&= W(s, Ht)\pi_{Hts}(a)W(s, Ht)^{-1}\zeta(Ht).
\end{aligned}$$

for each $a \in A$, $s \in G$, $\zeta \in \mathcal{H}$ and almost all $Ht \in H/G$. The above calculation implies that for each $s \in G$ and $a \in A$ the operator given by $Ht \mapsto \pi_{Ht}(\sigma_s a)$ is equal to the operator given by $Ht \mapsto W(s, Ht)\pi_{Hts}(a)W(s, Ht)^{-1}$. Since two decomposable operators are equal if and only if they are equal almost everywhere we get $\pi_{Ht}(\sigma_s a) = W(s, Ht)\pi_{Hts}(a)W(s, Ht)^{-1}$ for each $s \in G$, $a \in A$ and almost all $Ht \in H/G$.

Fix an $s \in G$ and let $\{a_i\}$ be a dense subset of A. Define $E_i = \{Ht \in H/G : \pi_{Ht}(\sigma_s a_i) = W(s, Ht)\pi_{Hts}(a_i)W(s, Ht)^{-1}\}$ then $\mu(H/G - E_i) = 0$ for all i. Let $E = \bigcap_i E_i$ then $\pi_{Ht}(\sigma_s a) = W(s, Ht)\pi_{Hts}(a)W(s, Ht)^{-1}$ for all $Ht \in E, a \in A$ and $\mu(H/G - E) = 0$. It follows that for each $s \in G$

$$\pi_{Hts} = W(s, Ht)\pi_{Hts}W(s, Ht)^{-1}$$

for almost all $Ht \in H/G$.

Since the quotient map $q : G \to H/G$ is Borel we can extend the function $Ht \mapsto \pi_{Ht}$ to a function on G by $\overline{\pi}_t = \pi_{Ht}$. Notice we have $\overline{\pi}_{rt} = \overline{\pi}_t$ for all $t \in G$ and all $r \in H$. Let $\overline{W}$ be the extension of W to $G \times G$ as in the previous section. Then $\overline{W}$ and $\overline{\pi}_t$ satisfy

$$\overline{W}(s,t)\overline{\pi}_{ts}\overline{W}(s,t)^{-1} = \overline{\pi}_t \circ \sigma_s \tag{1.5}$$

for each $s \in G$ and almost every $t \in G$. By Lemma 52 there exists a $U(\mathcal{H}_k)$-valued Borel function on G such that $\overline{W}(s,t) = B^{-1}(t)B(ts)$ for almost all pairs s, t. Moreover, the function B may be chosen so that $B(rs) = U_0(r)B(s)$ for all $r \in H$ and all $s \in G$ where U_0 is a unitary representation of H on $\mathcal{H}_k$ determined by $\overline{W}$. Then Equation (1.5) can be restated in terms of $B(t)$ as

$$B(t)^{-1}B(ts)\overline{\pi}_{ts}B(ts)^{-1}B(t) = \overline{\pi}_t \circ \sigma_s$$

for almost all $s, t \in G$. A simple rearrangement of terms yields

$$B(ts)(\overline{\pi}_{ts} \circ \sigma_{(ts)^{-1}})B(ts)^{-1} = B(t)(\overline{\pi}_t \circ \sigma_{t^{-1}})B(t)^{-1}$$

for almost all $s, t \in G$. Hence, there is $t_0 \in G$ such that

$$B(t_0s)(\overline{\pi}_{t_0s} \circ \sigma_{(t_0s)^{-1}})B(t_0s)^{-1} = B(t_0)(\overline{\pi}_{t_0} \circ \sigma_{t_0^{-1}})B(t_0)^{-1}$$

for almost all $s \in G$. We set $\pi_0 = B(t_0)(\overline{\pi}_{t_0} \circ \sigma_{t_0^{-1}})B(t_0)^{-1}$.

Next we show that (π_0, U_0) is a covariant representation of (A, H, σ). Let ν be the Haar measure on G. Then for every pair of vectors $h_1, h_2 \in \mathcal{H}_k$ and every $a \in A$ we have,

$$\langle \pi_0(a)h_1, h_2\rangle = \int_G \langle B(t)\overline{\pi}_t(\sigma_{t^{-1}}(a))B(t)^{-1}h_1, h_2\rangle d\nu(t)$$

And for every $r \in H$ we have,

$$\begin{aligned}
\langle \pi_0(\sigma_r(a))h_1, h_2\rangle &= \int_G \langle B(t)\overline{\pi}_t(\sigma_{t^{-1}r}(a))B(t)^{-1}h_1, h_2\rangle d\nu(t) \\
&= \int_G \langle B(rt)\overline{\pi}_{rt}(\sigma_{t^{-1}}(a))B(rt)^{-1}h_1, h_2\rangle d\nu(t) \\
&= \int_G \langle U_0(r)B(t)\overline{\pi}_t(\sigma_{t^{-1}}(a))B(t)^{-1}U_0(r)^{-1}h_1, h_2\rangle d\nu(t) \\
&= \int_G \langle B(t)\overline{\pi}_t(\sigma_{t^{-1}}(a))B(t)^{-1}U_0(r)^{-1}h_1, U_0(r)^{-1}h_2\rangle d\nu(t).
\end{aligned}$$

Since $B(t)(\overline{\pi}_t \circ \sigma_{t^{-1}})B(t)^{-1} = \pi_0$ for almost all $t \in G$ it follows that the last expression above is equal to $\langle \pi_0(a)U_0(r)^{-1}h_1, U_0(r)^{-1}h_2\rangle$. Hence,

$$\pi_0 \circ \sigma_r = U_0(r)\pi_0 U_0(r)^{-1} \tag{1.6}$$

for every $r \in H$. In other words, (π_0, U_0) is a covariant representation of (A, H, σ). We are now ready to show that the covariant representation (π, U) of (A, G, σ) is induced from the covariant representation (π_0, U_0) of (A, H, σ).

Theorem 66 ([17]; Theorem 4.2)**.** *Let (A, G, σ) be a separable dynamical system where G is compact and $k \in \{1, 2, .., \infty\}$. Let μ be a G-invariant Radon measure on H/G and let $\mathcal{H}_k$ be a separable Hilbert space of dimension k. Define $\mathcal{H} = L^2(H/G, \mu, \mathcal{H}_k)$ to be the space of all square integrable function from H/G to $\mathcal{H}_k$. Let (π, U) be a covariant representation on $\mathcal{H}$ such that*

1. *$\pi(A) \subseteq (L^\infty(H/G, \mu) \otimes 1_{\mathcal{H}_k})'$*

2. *$U(s)P_E U(s)^* = P_{Es^{-1}}$ for all $s \in G$ and all Borel subsets E of H/G.*

Then the covariant representation (π, U) *of* (A, G, σ) *is unitarily equivalent to the covariant representation* $(\overline{\pi}, \overline{U})$ *induced from the covariant representation* (π_0, U_0) *of* (A, H, σ).

Furthermore, if

$$L^\infty(H/G, \mu) \otimes 1_{\mathcal{H}_k} \subseteq \pi(A)' \cap \pi(A)''$$

then π_0 *is a factor representation.*

Proof. Let $\overline{\mathcal{H}}$ denote the representation space for the induced covariant representation $(\overline{\pi}, \overline{U})$. It follows from Theorem 53 that the map $\zeta \mapsto \overline{\zeta}$ where $\overline{\zeta}(s) = B(s)\zeta(Hs)$ defines a unitary operator V from $\mathcal{H}$ onto $\overline{\mathcal{H}}$ such that $VU(s)V^* = \overline{U}(s)$ for all $s \in G$. Hence, we only need to show that $V\pi V^* = \overline{\pi}$. To this end, let $a \in A$ and $\zeta \in \mathcal{H}$ then

$$\begin{aligned}
(V\pi(a)\zeta)(s) &= B(s)(\pi(a)\zeta)(Hs) \\
&= B(s)\overline{\pi_s}(a)\zeta(Hs) \\
&= \pi_0(\sigma_s(a))B(s)\zeta(Hs) \\
&= \pi_0(\sigma_s(a))(V\zeta)(s) \\
&= (\overline{\pi}(a)V\zeta)(s)
\end{aligned}$$

for almost all $s \in G$. It follows that (π, U) is unitarily equivalent to $(\overline{\pi}, \overline{U})$ via V.

The last statement of the theorem follows directly from part "c" of Proposition 21.

□

Let (π, U) be a covariant representation of (A, G, σ) on $\mathcal{H}$. Define $(\pi, U)' = \{T \in \mathcal{B}(\mathcal{H}) : T\pi(a) = \pi(a)T, TU(s) = U(s)T, \forall a \in A, s \in G\}$ to be the commutant of the covariant representation. Then by Remark 5 $(\pi, U)'$ is a von Neumann algebra.

Theorem 67 ([17]; Theorem 4.3)**.** *In the same situation as in Theorem 66, the von*

Neumann algebra $(\pi, U)' \cap (L^\infty(H/G, \mu) \otimes 1_{\mathcal{H}_k})'$ *is isomorphic to* $(\pi_0, U_0)'$.

The next two propositions can be deduced from the above theorem albeit each has a rather easy independent proof.

Proposition 68. *Let* (π_0, U_0) *be a covariant representation of* (A, H, σ) *and* (π, U) *be the corresponding induced representation of* (A, G, σ). *If* (π, U) *is irreducible then* (π_0, U_0) *is also irreducible.*

Proof. Let $\mathcal{H}$ denote the representation space for the induced covariant representation (π, U). Suppose that $T_0 \in (\pi_0, U_0)'$. Define an operator T on $\mathcal{H}$ by $(T\zeta)(s) = T_0\zeta(s)$ for all $\zeta \in \mathcal{H}$ and $s \in G$. Since $T_0U_0(s) = U_0(s)T_0$ for all $s \in H$ then $T \in \mathcal{B}(\mathcal{H})$. Clearly, $U(s)T = TU(s)$ for all $s \in G$. For each $a \in A$,

$$\begin{aligned}
(\pi(a)T\zeta)(s) &= \pi_0(\sigma_s(a))(T\zeta)(s) \\
&= \pi_0(\sigma_s(a))T_0\zeta(s) \\
&= T_0\pi_0(\sigma_s(a))\zeta(s) \\
&= (T\pi(a)\zeta)(s)
\end{aligned}$$

for all $\zeta \in \mathcal{H}$ and almost all $s \in G$. It follows that $\pi(a)T = T\pi(a)$ for all $a \in A$. Since (π, U) is an irreducible representation then T is a scalar operator. Hence T_0 is also a scalar operator. □

The proof of the next proposition is similar to the proposition above.

Proposition 69. *Let* (π_0, U_0) *be a covariant representation of* (A, H, σ) *and* (π, U) *be the corresponding induced representation of* (A, G, σ). *If* (π, U) *is a factor representation then* (π_0, U_0) *is also a factor representation.*

Example 70. Let H be a closed subgroup of a compact second countable group G and let μ be a G invariant measure on H/G. Define a covariant representation of the dynamical system $(C(H/G), G, \sigma)$ on $L^2(H/G, \mu)$ by $(\pi_\mu(f)\zeta)(Ht) = f(Ht)\zeta(Ht)$ and $(\lambda(s)\zeta)(Ht) = \zeta(Hts)$ for all $f \in C(H/G)$, $s \in G$, and $\zeta \in L^2(H/G, \mu)$. Then (π_μ, λ) is equivalent to the representation induced from the representation (π_0, U_0) of $(C(H/G), H, \sigma)$ on $\mathbf{C}$ where π_0 is the evaluation at He and U_0 is the trivial representation.

1.5 Crossed Product C^*-algebras

In this section we will assume that G is a locally compact group and A is a (possibly non separable) C^*-algebra. Given a C^*-dynamical system (A, G, σ) there is a natural way of building an associated C^*-algebra that encodes the action of the G on A; this algebra is called the *crossed product* C^*-algebra and it is denoted by $A \times_\sigma G$. Crossed product C^*-algebras are a source of many interesting examples of C^*-algebras. The construction of crossed product C^*-algebras is similar to the construction of group C^*-algebras and has similar universal properties. The universal property of $A \times_\sigma G$ is a one to one correspondence between the representations of $A \times_\sigma G$ and the covariant representations of (A, G, σ). Thus, the study of covariant representations of (A, G, σ) is equivalent to that of representations of $A \times_\sigma G$.

Let (A, G, σ) be a C^*-dynamical system. Then the set $C_c(G, A)$ of continuous functions with compact support from G to A becomes a *-algebra under the following operations. For each pair $f, g \in C_c(G, A)$, define the multiplication to be

$$(f * g)(s) = \int_G f(t)\sigma_t(g(t^{-1}s))d\mu(s)$$

where μ is the left Haar measure. For each $f \in C_c(G, A)$, define the involution to be

$$f^*(s) = \triangle(s^{-1})\sigma_s(f(s^{-1})^*)$$

where $\triangle$ is the modular function on G corresponding to the left Haar measure.

Remark 71. At first glance it may seem unnatural to define multiplication and involution on $C_c(G, A)$ using the left Haar measure since so far we have been working mostly with the right Haar measure. However, the resulting crossed product, and the universal property, do not depend on the choice of measure. For instance, the left and right regular representations are different (and are constructed using different Haar measures) but they both satisfy *the same covariance condition*. It is tempting to define the multiplication and involution on $C_c(G, A)$ in terms of the right Haar measure and we are confident one can do so (and we will get the same crossed product as above). However, to remain consistent with the presentation given in [18] we stick to the left Haar measure. In fact, Takesaki uses the same definitions for multiplication and involution as given above ([17]; p. 1).

Lemma 72 ([18]; Proposition 2.23)**.** *Suppose (π, U) is a covariant representation of (A, G, σ) on $\mathcal{H}$. Then*

$$\pi \times U(f) = \int_G \pi(f(s))U(s)d\mu(s)$$

*defines an L^1-norm decreasing *-representation of $C_c(G, A)$ on $\mathcal{H}$ called the integrated form of (π, U). Furthermore, $\pi \times U$ is nondegenerate if π is nondegenerate.*

Definition 73. Let (A, G, σ) be a C^*-dynamical system. For each $f \in C_c(G, A)$ we define

$$\|f\| = \sup\{\|\pi \times U(f)\| : (\pi, U) \text{ is a covariant representation of } (A, G, \sigma)\}.$$

Then $\|\cdot\|$ is a norm on $C_c(G, A)$ called the universal norm. The completion of $C_c(G, A)$ with respect to the universal norm is called the crossed product C^*-algebra and is denoted by $A \times_\sigma G$ ([18]; Lemma 2.27).

It follows from Lemma 72 that every covariant representation (π, U) of (A, G, σ) defines a representation $\pi \times U$ of $A \times_\sigma G$. We want show the converse, i.e. every representation L of $A \times_\sigma G$ is of the form $\pi \times U$. If G is a discrete group and A is a unital C^*-algebra we can define $\pi(a) = L(a \times e)$ and $U(s) = L(1_A \times s)$ then it is easy to check that $L = \pi \times U$. In general, $a \times e$ and $1_A \times s$ are not elements of $A \times_\sigma G$ so $L(a \times e)$ and $L(1_A \times s)$ are not defined. We use multiplier algebras to salvage the situation.

Let A be a C^*-algebra. Recall that the *multiplier* algebra, $M(A)$, of A is the largest unital C^*-algebra containing A as an essential ideal. There are several different (equivalent) concrete realizations of $M(A)$ in the literature ([4]; II.7.3.2). The most natural way of realizing $M(A)$ is via identification with the set of adjointable operators on the right Hilbert C^*-module A_A ([4]; II.7.3.1). By the universal property of multiplier algebras, every nondegenerate representation of A extends uniquely to a representation of $M(A)$.

Proposition 74 ([18]; Proposition 2.34)**.** *Suppose (A, G, σ) is a C^*-dynamical system. Then there is a nondegenerate faithful homomorphism*

$$i_A : A \to M(A \times_\sigma G)$$

and an injective, strictly continuous unitary valued homomorphism

$$i_G : G \to UM(A \times_\sigma G)$$

such that for $f \in C_c(G, A)$, $t, s \in G$ *and* $a \in A$ *we have*

$$i_G(t)f(s) = \sigma_t(f(t^{-1}s)) \text{ and } i_A(a)f(s) = af(s).$$

If (π, U) *is nondegenerate, then*

$$(\pi \times U)(i_A(a)) = \pi(a) \text{ and } (\pi \times U)(i_G(s)) = U(s).$$

Given a nondegenerate representation L of $A \times_\sigma G$ we define a representation of A by $\pi = L \circ i_A$ and a representation of G by $U = L \circ i_G$. Then (π, U) is a covariant representation of (A, G, σ) and $\pi \times U = L$. It follows that there is a natural one to one correspondence between nondegenerate covariant representations of (A, G, σ) and nondegenerate representations of $A \times_\sigma G$.

Example 75 ([7]; 8.4.2). Let $(C(\mathbf{T}), \mathbf{Z}, \theta)$ be as in Example 62 (a).

(a) If θ is a rational angle, i.e. $\theta = \frac{2\pi}{m}$ where m is a positive integer then $C(\mathbf{T}) \times_\theta \mathbf{Z}$ is isomorphic to $M_m(C(S))$.

(b) If θ is an irrational angle, then $C(\mathbf{T}) \times_\theta \mathbf{Z}$ is isomorphic to the universal C^*-algebra generated by a pair of unitaries satisfying the covariance condition $UV = e^{i\theta}VU$. This algebra is called the irrational rotation algebra ([7]; 3.9).

Another interesting example of crossed products is the Bunce-Deddens algebra ([7]; 8.4.4). See Section 2.5 in [18] for more examples.

The next theorem illustrates another deep connection between (A, G, σ) and $A \times_\sigma G$. Recall that if (X, G) is a topological G-space then the action is called free if the stabilzer subgroup is trivial for each element of X; and the action is called minimal if the orbit of each element of X is dense in X.

Theorem 76. *Let (X, G) be a topological G-space and $(C(X), G, \sigma)$ be the corresponding C^*-dynamical system. If G is amenable and acts freely, then $C_0(X) \times_\sigma G$ is simple if and only if the action is minimal ([7]; 8.4.1).*

It follows from the above theorem that the irrational rotation algebras are simple.

1.6 Induced Representations via Hilbert Modules

We have so far followed Takesaki's construction of induced representations. Takesaki's construction follows closely Mackey's original construction of induced representations for groups. There exists an alternative way of constructing induced representations via Hilbert modules. This approach is based on the work of Green and Rieffel [16]. We would like to show that Takesaki's construction of induced representations is equivalent to the one developed by Green/Rieffel. In order to present Green/Rieffel approach we need to know some basic facts about Hilbert modules. We borrow our discussion from Section II.7 of [4].

Let A and B be C^*-algebras. If $\mathcal{E}$ is a (right) Hilbert A-module, $\mathcal{F}$ is a (right) Hilbert B-module, and ϕ is a *-homomorphism of A into $\mathcal{L}(\mathcal{F})$, the space of adjointable operators on $\mathcal{F}$, then $\mathcal{F}$ can be regarded as a left Hilbert A-module. We can form an algebraic tensor product $\mathcal{E} \odot_A \mathcal{F}$ of $\mathcal{E}$ and $\mathcal{F}$ over A as the quotient space of the regular algebraic tensor product $\mathcal{E} \odot_{\mathbf{C}} \mathcal{F}$ by the subspace spanned by

$$\{\xi a \otimes \eta - \xi \otimes \phi(a)\eta : \xi \in \mathcal{E}, \eta \in \mathcal{F}, a \in A\}.$$

Then $\mathcal{E} \odot_A \mathcal{F}$ becomes a (right) Hilbert B-module in the following way: $(\xi \otimes \eta)b = \xi \otimes \eta b$. We define a B-valued pre-inner product on $\mathcal{E} \odot_A \mathcal{F}$ by

$$\langle \xi_1 \otimes \eta_1, \xi_2 \otimes \eta_2 \rangle = \langle \eta_1, \phi(\langle \xi_1, \xi_2 \rangle_A)\eta_2 \rangle_B.$$

The completion of $\mathcal{E} \odot_A \mathcal{F}$ with respect to this pre-inner product is a Hilbert B-module called the (internal) tensor product of $\mathcal{E}$ and $\mathcal{F}$, denoted $\mathcal{E} \otimes_A \mathcal{F}$ (assuming there is no ambiguity about the map ϕ).

If $T \in \mathcal{L}(\mathcal{E})$, then there is a a natural operator $T \otimes I \in \mathcal{L}(\mathcal{E} \otimes_A \mathcal{F})$ defined by

$$(T \otimes I)(\xi \otimes \eta) = T\xi \otimes \eta. \tag{1.7}$$

Thus there is a *-homomorphism from $\mathcal{L}(\mathcal{E})$ into $\mathcal{L}(\mathcal{E} \otimes_A \mathcal{F})$.

Let (A, G, σ) be a separable dynamical system where G is compact and $A \times_\sigma G$ be the corresponding crossed product C^*-algebra. Let H be a closed subgroup of G and L be a representation of $A \times_\sigma H$ on $\mathcal{H}_L$. We would like to build a corresponding representation of $A \times_\sigma G$. To this end, we need a (right) Hilbert $A \times_\sigma H$-module $\mathcal{X}$ and a *-homomorphism of $\psi : A \times_\sigma G \to \mathcal{L}(\mathcal{X})$. Then, as described above, we can form a Hilbert space $\mathcal{X} \otimes_{A \times_\sigma H} \mathcal{H}_L$ where the map $\phi : A \times_\sigma H \to \mathcal{L}(\mathcal{H}_L)$ is the representation L. The *-homomorphism ψ defines a representation of $A \times_\sigma G$ on $\mathcal{X} \otimes_{A \times_\sigma H} \mathcal{H}_{\mathcal{L}}$ via Equation (1.7).

A natural choice for the space $\mathcal{X}$ is Green's imprimitivity (bi)module(see [18]; Section 4.3 for details). Let $\mathcal{X}_0 = C(G, A)$ and $B_0 = C(H, A)$ where the latter is viewed as a dense subset of $A \times_\sigma H$. For each $f, g \in C(G, A)$ and $b \in B_0$, define

1. $f \cdot b(s) = \int_H f(sr)\sigma_{sr}(b(r^{-1}))d\mu_H(r)$

2. $\langle f, g \rangle_{B_0}(r) = \int_G \sigma_s^{-1}(f(s)^* g(sr))d\mu_G(s)$

Then $\mathcal{X}_0$ is a right B_0-pre-Hilbert module. Let $\mathcal{X}$ be the completion of $\mathcal{X}_0$. Then $\mathcal{X}$ is a right Hilbert $A \times_\sigma H$-module ([18]; Theorem 4.22). Furthermore, there exists a nondegenerate covariant homomorphism (N, υ) of (A, G, σ) into $\mathcal{L}(\mathcal{X})$ defined by

$$N(a)f(s) = af(s) \text{ and } \upsilon_t(f)(s) = \sigma_t(f(t^{-1}s))$$

for all $f \in \mathcal{X}_0$. The corresponding crossed product homomorphism is given by $N \times \upsilon(g)(f) = g * f$ using the convolution product, $g \in C(G, A) \subseteq (A, G, \sigma)$ and $f \in \mathcal{X}_0$.

Remark 77. The space $\mathcal{X}$ constructed above is in fact a bimodule viewed as a left $C(G/H, A)$-module. The bimodule structure of $\mathcal{X}$ is important in its own right, but we will not need it for the purposes of building the induced representation. A more detailed analysis of Green's imprimitivity (bi)module can be found in Section 4.3 of [18].

Definition 78. Suppose $A \times_\sigma G$ is a crossed product C^*-algebra and L is a representation of $A \times_\sigma H$ where $H \leq G$. Then $\mathrm{Ind}L$ will denote the representation of $A \times_\sigma G$ induced from L via Green's imprimitivity (bi)module $\mathcal{X}$ and the homomorphism $N \times \upsilon : A \times_\sigma G \to \mathcal{L}(\mathcal{X})$.

Note that if $g \in C(G, A) \subseteq A \times_\sigma G$ and $f \in C(G, A) \subseteq \mathcal{X}_0$ then on $\mathcal{X}_0 \odot \mathcal{H}_L$

$$\mathrm{Ind}L(g)(f \otimes h) = (g * f) \otimes h.$$

Moreover, $\mathrm{Ind}\, L = (N \otimes I) \times (\upsilon \otimes I)$ where

$$(N \otimes I)(a)(f \otimes h) = N(a)(f) \otimes h$$

and

$$(\upsilon \otimes I)(f \otimes h) = \upsilon_s(f) \otimes h.$$

Our goal is to show that the construction of induced representations given in Section 1.4.1 is equivalent to the construction given in Definition 78. We will take advantage of Proposition 5.4 in [18] where is it shown that the induced representation Ind$L$ is equivalent to a representation much in the same spirit as the induced representation constructed in Section 1.4.1.

Suppose that $L = \pi_0 \times U_0$ is a representation of $A \times_\sigma H$ on $\mathcal{H}_L$. Let $\overline{\mathcal{H}}$ be the space of all $\mathcal{H}_L$ valued functions ζ on G satisfying the following conditions:

1. $\langle \zeta(s), h \rangle$ is Borel function of s for all $h \in \mathcal{H}_L$.

2. $\zeta(rt) = U_0(t^{-1})\zeta(r)$ for all $t \in H$ and all $r \in G$.

3. $\int_{G/H} \langle \zeta(r), \zeta(r) \rangle d\mu_{G/H}(\overline{r}) < \infty$, where $\mu_{G/H}$ is a left invariant measure on G/H.

Define a new representation $\overline{\pi} \times \overline{U} \to \mathcal{B}(\overline{\mathcal{H}})$ by

$$\overline{\pi}(a)\zeta(r) = \pi_0(\sigma_{r^{-1}}(a))\zeta(r)$$

$$\overline{U}(s)\zeta(r) = \zeta(s^{-1}r)$$

for $s, r \in G$, $a \in A$, $\zeta \in \overline{\mathcal{H}}$

Proposition 79 ([18]; Proposition 5.4)**.** *Suppose that (A, G, σ) is a separable dynamical system and H is a closed subgroup of a compact group G, and $L = \pi_0 \times U_0$ is a representation of $A \times_\sigma H$ on $\mathcal{H}_L$. Let $\overline{\mathcal{H}}$ be the space of $\mathcal{H}_L$ valued functions on G described above. Then IndL is unitarily equivalent to the representation $\overline{\pi} \times \overline{U}$ on $\overline{\mathcal{H}}$.*

Let W be an operator from $\mathcal{X}_0 \odot \mathcal{H}_L$ into $\overline{\mathcal{H}}$ given by

$$W(f \otimes h)(r) = \int_H \pi_0(\sigma_{r^{-1}}(f(rt)))U_0(t)h \, \mathrm{d}\mu_H(t)$$

for $f \in \mathcal{X}_0 \odot \mathcal{H}_L$, $h \in \mathcal{H}_L$, and $r \in G$. Then W is an isometry that extends to a unitary operator from $\mathcal{X} \otimes_{A \times_\sigma H} \mathcal{H}_L$ onto $\overline{\mathcal{H}}$. In fact, W implements the unitary equivalence between Ind$L$ and $\overline{\pi} \times \overline{U}$.

It is not hard to see that the difference between the induced covariant representation $(\overline{\pi}, \overline{U})$ and the induced covariant representation (π, U) as defined in Section 1.4.1 is essentially the same as the difference between the left regular representation and the right regular representation. To make this more precise suppose (π_0, U_0) is a covariant representation of the subsystem (A, H, σ) on $\mathcal{H}_0$. Let $(\overline{\pi}, \overline{U})$ be the induced covariant representation on $\overline{\mathcal{H}}$ as in Proposition 79 and (π, U) be the induced representation on $\mathcal{H}$ as defined in Section 1.4.1. Define a map V on $\overline{\mathcal{H}}$ by $(V\zeta)(s) = \zeta(s^{-1})$ for $\zeta \in \overline{\mathcal{H}}$, $s \in G$. Then

$$\begin{aligned}
(V\zeta)(rs) &= \zeta((rs)^{-1}) \\
&= \zeta(s^{-1}r^{-1}) \\
&= U_0(r)\zeta(s^{-1}) \\
&= U_0(r)(V\zeta)(s)
\end{aligned}$$

for all $\zeta \in \overline{\mathcal{H}}$, $r \in H$, $s \in G$. It follows from Remark 38 that V is in fact a unitary from $\overline{\mathcal{H}}$ onto $\mathcal{H}$. Moreover, for each $\eta \in \mathcal{H}$, $a \in A$

$$\begin{aligned}(V\overline{\pi}(a)V^*\eta)(s) &= (\overline{\pi}(a)V^*\eta)(s^{-1}) \\ &= \pi_0(\sigma_s(a))(V^*\eta)(s^{-1}) \\ &= \pi_0(\sigma_s(a))\eta(s) \\ &= (\pi(a)\eta)(s)\end{aligned}$$

for almost every $s \in G$, and

$$\begin{aligned}(V\overline{U}(t)V^*\eta)(s) &= (\overline{U}(t)V^*\eta)(s^{-1}) \\ &= (V^*\eta)(t^{-1}s^{-1}) \\ &= \eta(st) \\ &= (U(t)\eta)(s)\end{aligned}$$

for each $\eta \in \mathcal{H}$, $t \in G$ and almost every $s \in G$. It follows that V is the desired intertwining operator between $(\overline{\pi}, \overline{U})$ and (π, U).

Proposition 80. *Suppose (π_0, U_0) is a covariant representation of the subsystem (A, H, σ) on $\mathcal{H}_0$. Let $(\overline{\pi}, \overline{U})$ be the induced covariant representation on $\overline{\mathcal{H}}$ as in Proposition 79 and let (π, U) be the induced representation on $\mathcal{H}$ as defined in Section 1.4.1. Then $(\overline{\pi}, \overline{U})$ is unitarily equivalent to (π, U).*

Chapter 2

Covariant Representations of C^*-dynamical systems with Compact Groups

In this chapter we will investigate covariant representations of (A, G, σ) under the assumption that G is a compact group. The main results of this chapter are proved in the context of separable dynamical systems, i.e. where A is separable and G is second countable. This should not come as a surprise as our results are based on the theory developed by Mackey and Takesaki who worked under the same assumptions. Moreover, many of the fundamental results in the theory such as the GRS Theorem [10] also assume the above countability conditions. Nevertheless, some of the tools we developed for proving our main results are valid without the countability assumptions.

In Section 1, we consider dynamical systems of the form $(C(X), G, \sigma)$. In particular, we show that if the action of G on $C(X)$ is ergodic then X is homeomorphic to the right coset space G_0/G where G_0 is a closed subgroup of G.

In Section 2, we consider irreducible and factor covariant representations of (A, G, σ).

Our main result in this section states that every such representation is induced from a representation (π_0, U_0) of (A, G_0, σ), for an appropriate subgroup $G_0 \leq G$, with a key additional property that π_0 is a factor representation. As a corollary, we show that every irreducible representation of (A, G, σ) is induced from a stability group. The latter result is a stronger version of the GRS Theorem [10]. We also consider covariant representations of (A, G, σ) in a pair of special cases: the case when A is an abelian C^*-algebra and the case when G is a finite group. If A is abelian we show explicitly that π_0 must be a multiple of an irreducible representation. If G is a finite group we show that our findings generalize the results in [3]. In particular, we show that π_0 must be a multiple of an irreducible representation with multiplicity less than the order of G.

In Section 3, we consider dynamical systems of the form (A, G_P, σ) where P is a primitive ideal of A and G_P is the subgroup of elements of G that stabilize P. In particular, we show that every irreducible representation (π, U) of (A, G_P, σ) with ker $\pi = P$ induces to an irreducible representation of (A, G, σ). This result is known the strong-EHI property.

2.1 Ergodic Actions on C(X)

In this section we show that ergodicity and transitivity are equivalent notions for an action of a compact group on $C(X)$.

Definition 81. Let (A, G, σ) be a dynamical system where A is unital and G is locally compact. We say that the action of G on A is *ergodic* if the only G invariant elements of A are the scalars; that is, $\sigma_s(a) = a$ for all $s \in G$ implies that $a \in \mathbf{C}1_A$.

Definition 82. Let X be a topological G-space. The action of G on X is called *transitive* if for each pair of elements $x_1, x_2 \in X$ there is $s \in G$ such that $x_1 \cdot s = x_2$.

Let X be a topological G-space where X is a compact Hausdorff space and G is a locally compact group. Suppose the action of G on X is transitive. Then the corresponding action of G on $C(X)$ is certainly ergodic. However, the converse is not true in general. For instance, the action of $\mathbf{Z}$ on $C(\mathbf{T})$ by an irrational angle rotation is ergodic, but the corresponding action of G on $\mathbf{T}$ is not transitive. The following proposition shows that if G is a compact group the two notions are equivalent. The first part of Lemma 83 is similar, with a different proof, to a result by Albeverio and Hegh-Krohn ([2]; Lemma 2.1).

Proposition 83. *Let G be a compact group. Let X be a compact, Hausdorff topological G-space. Suppose the action of G on $C(X)$ given by $(\sigma_s f)(x) = f(x \cdot s)$ is ergodic, i.e. the only G invariant functions are the constant functions. Then the action of G on X is transitive.*

Moreover, there exists a closed subgroup G_0 of G such that the right coset space G_0/G with the quotient topology is homeomorphic to X.

Proof. For each $x \in X$ define the orbit of x to be $O_x = \{x \cdot s : s \in G\}$. Since the map $s \mapsto x \cdot s$ is continuous from $G \to X$ and G is compact then O_x is compact for each $x \in X$. In particular, O_x is closed for each $x \in X$.

Fix $x_0 \in X$. Suppose there is $x_1 \in X - O_{x_0}$ then O_{x_0} and O_{x_1} are disjoint closed subsets of X. By Urysohn's Lemma there exists a continuous function $f : X \to [0,1]$ such that $f(x_0 \cdot s) = 0$ and $f(x_1 \cdot s) = 1$ for all $s \in G$. Define a function $g : X \to [0,1]$ by $g(x) = \int_G f(x \cdot s) dm(s)$. We want to show that g is continuous. To this end, let $\epsilon > 0$ be given; extend f to $\overline{f} : G \times X \to [0,1]$ by defining $\overline{f}(x,s) = f(x \cdot s)$. Then $\overline{f}$ is continuous function with compact support so we can find a finite open cover $\{G_i \times F_i\}_{i=1}^n$ of $G \times X$ such that $|f(x \cdot s) - f(y \cdot t)| < \epsilon$ whenever (s,x) and (t,y) are both in $G_i \times F_i$ for some $i = 1, .., n$. Given any $x \in X$ define $F_x = \bigcap\{F_i : x \in F_i\}$. Let $y \in F_x$ and $s \in G$. Choose j such that $(s, x) \in G_j \times F_j$. Then $(s, y) \in G_j \times F_j$. It follows that $|f(x \cdot s) - f(y \cdot s)| < \epsilon$ for all $y \in F_x$ and $s \in G$. Then $|g(x) - g(y)| \leq \int_G |f(x \cdot s) - f(y \cdot s)| dm(s) \leq \epsilon$ for all $y \in F_x$. It follows that g is continuous.

It is routine to check that g is a G-invariant function and hence must be constant on X. But $g(x_0) = 0$ and $g(x_1) = 1$, contradiction. It follows that $O_{x_0} = X$.

To prove the second part of the statement let $G_{x_0} = \{s \in G : x_0 \cdot s = x_0\}$. Then G_{x_0} is a closed subgroup of G and the right coset space G_{x_0}/G is compact in the quotient topology. Moreover, it is easy to see that the map $G_{x_0} \cdot s \mapsto x_0 \cdot s$ is a continuous bijection from G_{x_0}/G onto X. Since G_{x_0}/G is compact and X is Hausdorff it follows that G_{x_0}/G is in fact homeomorphic to X. $\square$

Remark 84. Let G be a compact group. Let X be a *locally compact*, Hausdorff topological G-space. Since $C_0(X)$ is not unital our definition of ergodicity does not apply. However, some of the key arguments in Proposition 83 carry over to the action of G on $C_0(X)$.

By the same argument as above O_x is compact for every $x \in X$. Suppose $x_1, x_2 \in X$ have disjoint orbits. Since both O_{x_1} and O_{x_2} are compact then by a straightforward compactness argument we can find an open set F containing O_{x_1} and disjoint from O_{x_2}. By the "generalized" Urysohn Lemma (Theorem 34) there is $f \in C_c(X)$ such that $0 \leq f(x) \leq 1$ for all $x \in X$, $f(x) = 1$ for all $x \in O_{x_1}$, and $f(x) = 0$ for all $x \notin F$. Define a function $g : X \to [0, 1]$ by $g(x) = \int_G f(x \cdot s) dm(s)$ as above. Then $g \in C_c(X)$ and $g(x_1) = 1$ and $g(x_2) = 0$.

Corollary 85. *Let G be a second countable compact group. Let X be a compact, Hausdorff topological G-space. Suppose the action of G on $C(X)$ given by $(\sigma_s f)(x) = f(x \cdot s)$ is ergodic. Then X is a second countable topological space.*

Remark 86. Let X be a topological G-space where X is a compact Hausdorff space and G is a locally compact group. In the view of the above discussion it is natural to ask the following questions:

1. What is a necessary and sufficient condition on the action of G on X to obtain ergodic action of G on $C(X)$?
2. What is a necessary and sufficient condition on the action of G on $C(X)$ to obtain transitive action of G on X?

2.2 Covariant Representations of (A, G, σ)

Our goal is to show that every irreducible (resp factor) representation (π, U) of (A, G, σ) is induced from an irreducible (resp factor) representation (π_0, U_0) of a subsystem (A, G_0, σ) with the key additional property that π_0 is a factor representation of A. As a corollary, we get a strengthening of the GRS theorem for compact groups.

The key step is to show that every irreducible (factor) representation of (A, G, σ) can turned into a representation of the form described in Theorem 66. First, we need to introduce W^*-dynamical systems and systems of imprimitivity.

Definition 87. Let G be a locally compact group, $\mathbf{A}$ a von Neumann algebra, and τ a homomorphism of G into the automorphism group of $\mathbf{A}$ such that $\tau_{s_i} \to \tau_s$ in the strong operator topology whenever $s_i \to s$ in G. We call the triple $(\mathbf{A}, G, \tau)$ a *W^*-dynamical system.*

Let $(\mathbf{A}, G, \tau)$ be a W^*-dynamical system. Define

$$\mathbf{A}^c = \{x \in \mathbf{A} : s \mapsto \tau_s(x) \text{ is norm continuous}\}.$$

Then $\mathbf{A}^c$ is a G-invariant C^*-subalgebra and it is σ-weakly dense in $\mathbf{A}$ ([4]; Proposition III.3.2.4). Since $\mathbf{A}^c$ is unital it follows from the Double Commutant Theorem that $\mathbf{A}^c$ is strong operator topology-dense in $\mathbf{A}$.

For the rest of the section we will assume that (A, G, σ) is a separable C^*-dynamical system, G is a compact group, and (π, U) is a nondegenerate covariant representation of (A, G, σ) on a separable Hilbert space $\mathcal{H}$. Following [17] we define a *system of imprimitivity* for (π, U) to be a commutative von Neumann algebra $\mathbf{A}$ acting on $\mathcal{H}$ such that:

1. $\mathbf{A} \subseteq \pi(A)'$.

2. $U(s)\mathbf{A}U(s)^* = \mathbf{A}$ for all $s \in G$.

Note that condition 2 implies that G acts by automorphisms on $\mathbf{A}$. Moreover, since U is assumed to be strongly continuous, then for each $x \in \mathbf{A}$ the map $s \mapsto U(s)xU(s)^*$ is continuous in the strong operator topology. Thus we obtain a W^*-dynamical system $(\mathbf{A}, G, \tau)$ where $\tau_s(x) = U(s)xU(s)^*$ for each $s \in G$ and $x \in \mathbf{A}$. If the only G invariant elements of $\mathbf{A}$ are scalars then $\mathbf{A}$ is called an *ergodic* system of imprimitivity. In particular, if (π, U) is an irreducible covariant representation then *every* system of imprimitivity is ergodic.

Covariant representations with ergodic systems of imprimitivity are equivalent to representations described in Theorem 66. Suppose $\mathbf{A}$ is an ergodic system of imprimitivity for (π, U). Let $(\mathbf{A}, G, \tau)$ be the corresponding W^*-dynamical system. Then $(\mathbf{A}^c, G, \tau)$ is a C^*-dynamical system. Since $\mathbf{A}^c$ is a unital, abelian C^*-algebra then $\mathbf{A}^c \cong C(X)$ where X is a compact Hausdorff space. Note that the action of G on $C(X)$ is also ergodic. It follows from Proposition 83 that X is homeomorphic to G_0/G. Since G is second countable then G_0/G is also second countable. Let ϱ be the isomorphism from $C(G_0/G)$ onto $\mathbf{A}^c$. Then ϱ is a representation of $C(G_0/G)$ on $\mathcal{H}$.

It follows from Corollary 19 that ϱ is unitarily equivalent to a representation of the form

$$\rho = (\rho_{\mu_\infty} \otimes 1_{\mathcal{H}_\infty}) \oplus \rho_{\mu_1} \oplus (\rho_{\mu_2} \otimes 1_{\mathcal{H}_2}) \oplus \cdots \tag{2.1}$$

where each $\mathcal{H}_n$ is a Hilbert space of dimension n and each μ_n is a finite Borel measure on G_0/G with μ_n disjoint from μ_m. We want to show that the strong operator topology closure of $\rho(C(G_0/G))$ is equal to

$$\mathbf{B} = (L^\infty(G_0/G, \mu_\infty) \otimes 1_{\mathcal{H}_\infty}) \oplus L^\infty(G_0/G, \mu_1) \oplus (L^\infty(G_0/G, \mu_2) \otimes 1_{\mathcal{H}_2}) \oplus \cdots$$

By Theorem 20 the strong operator closure of $\rho_{\mu_k} \otimes 1_{\mathcal{H}_k}(C(G_0/G))$ is equal to $L^\infty(G_0/G, \mu_k) \otimes 1_{\mathcal{H}_k}$ for all k. Let p_k be projection onto $L^2(G_0/G, \mu_k, \mathcal{H}_k)$. Suppose that $T \in (\rho(C(G_0/G)))'$. Then $p_m T p_n$ is an intertwining operator of the pair of disjoint representations $\rho_{\mu_m} \otimes 1_{\mathcal{H}_m}$ and $\rho_{\mu_n} \otimes 1_{\mathcal{H}_m}$, $m \neq n$. Thus $p_m T p_n = 0$ and $T = \sum p_k T p_k$ where $p_k T p_k \in (\rho_{\mu_m} \otimes 1_{\mathcal{H}_m}(C(G_0/G)))'$ for all k. It follows that $\mathbf{B} \subseteq (\rho(C(G_0/G)))''$. Conversely, if $S \in (\rho(C(G_0/G)))''$ then $Sp_k = p_k S$ for all k. Hence, $S = \sum p_k S p_k$ where $p_k S p_k \in (\rho_{\mu_m} \otimes 1_{\mathcal{H}_m}(C(G_0/G)))''$ for all k. It follows that $(\rho(C(G_0/G)))'' \subseteq \mathbf{B}$. Thus the unitary equivalence intertwining ϱ with ρ carries $\mathbf{A}$ onto $\mathbf{B}$.

Let U' denote the image of U under the unitary equivalence intertwining ϱ with ρ. Then $U'(s)\mathbf{B}U'(s)^* = \mathbf{B}$ for all $s \in G$. It follows from Remark 12 that $U'(s)p_k U'(s)^* = p_k$ for all $s \in G$ and all k. Since the action of G on $\mathbf{B}$ is ergodic then $p_k = 0$ for all but a single k. We obtain the following theorem from our discussion.

Theorem 88. *Let G be a compact group. Suppose that (A, G, σ) is a separable C^*-dynamical system and (π, U) is a nondegenerate covariant representation of (A, G, σ) on a separable Hilbert space $\mathcal{H}$. Suppose $\mathbf{A}$ is an ergodic system of imprimitivity for*

(π, U). Then $\mathcal{H}$ is unitarily equivalent to $L^2(G_0/G, \mu, \mathcal{H}_k)$ for some $k \in \{1, 2, ...\infty\}$. Furthermore, the unitary equivalence carries $\mathbf{A}^c$ onto $\pi_\mu(C(G_0/G)) \otimes 1_{\mathcal{H}_k}$, where π_μ is faithful, and carries $\mathbf{A}$ onto $L^\infty(G_0/G, \mu) \otimes 1_{\mathcal{H}_k}$.

We want to show that the measure μ in the above theorem can be chosen to be G-invariant. By Corollary 19 μ can be replaced with any other finite Borel measure ν which is absolutely continuous with respect to μ. Recall that all quasi-invariant measures on G_0/G are absolutely continuous with respect to one another. Hence, if we can show that μ is a quasi-invariant measure on G_0/G then we can replace it with a G-invariant Radon measure. The action of G on $L^\infty(G_0/G, \mu) \otimes 1_{\mathcal{H}_k}$ induces an action, which we will call τ, of G on $L^\infty(G_0/G, \mu)$ such that $U(s)'(g \otimes 1_{\mathcal{H}_k})U(s)'^* = \tau_s(g) \otimes 1_{\mathcal{H}_k}$ for all $g \in L^\infty(G_0/G, \mu)$ and $s \in G$. By construction, $(\tau_s f)(G_0 t) = f(G_0 ts)$ for all $t, s \in G$ and $f \in C(G_0/G)$. We would like to extend the last equality to $L^\infty(G_0/G, \mu)$ functions.

Proposition 89. *Let X be a topological G-space where X is a compact Hausdorff, second countable space and G is a compact, second countable group. Denote τ to be the corresponding action of G on $C(X)$. Let π_μ be a faithful representation of $C(X)$ on $L^2(X, \mu)$ where μ is a finite Borel measure. Suppose the action of G extends from $C(X)$ to $L^\infty(X, \mu)$. Then for each $s \in G$ and $g \in L^\infty(X, \mu)$,*

$$(\tau_s g)(x) = g(x \cdot s)$$

for almost all $x \in X$.

Proof. Let $g \in L^\infty(X, \mu)$ then there is a (norm)-bounded sequence $\{f_i\}$ in $C(X)$ such that $f_i(x) \to g(x)$ for almost every x. It follows from the dominated convergence theorem that $f_i \to g$, as multiplication operators, in the strong operator topology. Since automorphisms of von Neumann algebras are strong operator topology continuous on bounded sets ([4]; Proposition III.2.2.2) then $\tau_s f_i \to \tau_s g$ strong operator topology. In particular, $\tau_s f_i \to \tau_s g$ in $L^1(X,\mu)$. Therefore, there exists a subsequence such that $\tau_s f_{i_j}(x) \to \tau_s g(x)$ for almost every x. By replacing the original sequence with the subsequence we can assume, without the loss of generality, that $\tau_s f_i \to \tau_s g$ almost everywhere. Since $f_i \subseteq C(X)$ then $(\tau_s f_i)(x) = f_i(x \cdot s)$ for all $x \in X$ and i. It follows that $(\tau_s g)(x) = g(x \cdot s)$ for almost all $x \in X$. □

Corollary 90. *Let (X,μ) be as in Proposition 89. Then μ is a quasi-invariant measure.*

Proof. Let Y be a Borel subset of X. Then $\mu(Y) = 0 \iff \chi_Y = 0 \iff \tau_s(\chi_Y) = 0 \iff \chi_{(Y\cdot s)} = 0 \iff \mu(Y \cdot s) = 0$ □

Applying Proposition 89 and Corollary 90 to the situation in Theorem 88 we can assume that the measure μ in the statement of Theorem 88 is a G-invariant Radon measure.

The most natural system of imprimitivity for (π, U) is the center of $\pi(A)''$ which we denote $\mathbf{Z}(\pi(A)'')$. If (π, U) is a factor representation, then $\mathbf{Z}(\pi(A)'')$ is an ergodic system of imprimitivity for (π, U) ([17]; Theorem 5.1).

Theorem 91. *Let (π, U) be a factor (resp. irreducible) representation of a separable system (A, G, σ) on a separable Hilbert space $\mathcal{H}$ where G is compact. Then there exists a closed subgroup G_0 of G and a unique covariant representation (π_0, U_0) of the subsystem (A, G_0, σ) such that (π, U) is induced by (π_0, U_0), where the uniqueness is up to equivalence. Moreover,*

1. *(π_0, U_0) is a factor (resp. irreducible) representation.*

2. *π_0 is a factor representation.*

Proof. Suppose (π, U) is a factor representation. Then $\mathbf{Z}(\pi(A)'')$ is an ergodic system of imprimitivity for (π, U). Using Theorem 88 we can assume, without the loss of generality, that $\mathcal{H} = L^2(G_0/G, \mu, \mathcal{H}_k)$ and $\mathbf{Z}(\pi(A)'') = L^\infty(G_0/G, \mu) \otimes 1_{\mathcal{H}_k}$ where μ is a G-invariant Radon measure. It follows from Theorem 66 that there exists a covariant representation (π_0, U_0) of (A, G_0, σ) such that the corresponding induced representation is equivalent to (π, U). Moreover, since $L^\infty(G_0/G, \mu) \otimes 1_{\mathcal{H}_k} = \mathbf{Z}(\pi(A)'') = \pi(A)' \cap \pi(A)''$ then π_0 is a factor representation. By Proposition 69, (π_0, U_0) is a factor representation. Similarly, if (π, U) is an irreducible representation, then (π_0, U_0) is irreducible by Proposition 68. It is not hard to check that if (π_1, U_1) is a another representation of (A, G_0, σ) and (π_1, U_1) is unitarily equivalent to (π_0, U_0) then the corresponding induced representations are equivalent. $\square$

Remark 92. Theorem 91 does not hold for dynamical systems with discrete groups. Let (π_μ, λ) be the canonical covariant representation of $(C(\mathbf{T}), \mathbf{Z}, \theta)$ on $L^2(\mathbf{T}, \mu)$ where θ is an irrational angle. Then (π_μ, λ) is an irreducible covariant representation. Suppose (π_μ, λ) is induced from a covariant representation (π_0, U_0) of $(C(\mathbf{T}), \mathbf{Z}_n, \theta)$ with π_0 a factor representation. Then π_0 must be equivalent to a subrepresentation of π_μ, but π_μ has no factor subrepresentations, a contradiction.

The above theorem has very interesting applications one of which we will discuss next. Let P be a primitive ideal of A and define

$$G_P = \{s \in G : \sigma_s P = P\}.$$

Note that G_P is a closed subgroup of G. Applying Theorem 91 we get the following corollary.

Corollary 93. *Let (A, G, σ) be a separable dynamical system where G is compact.*

Suppose that (π, U) *is an irreducible representation of* (A, G, σ)*. Then there exists a primitive ideal* P *of* A *and a covariant representation* (π_P, U_P) *of* (A, G_P, σ) *such that* (π, U) *is induced by* (π_P, U_P)*. Moreover, ker* $\pi_P = P$*.*

Proof. By Theorem 91, there exists a closed subgroup G_0 of G and a covariant representation (π_0, U_0) of the subsystem (A, G_0, σ) such that (π, U) is induced by (π_0, U_0). Since A is separable and π_0 is a factor representation, ker $\pi_0 \in$ Prim A. Let $P :=$ ker π_0. Then $G_0 \subseteq G_P$. We take (π_P, U_P) to be the representation of (A, G_P, σ) induced by the representation (π_0, U_0) of the subsystem (A, G_0, σ).

In addition, it follows from Lemma 102 in the next section that ker $\pi_P = \bigcap_{r \in G_P} \sigma_r P = P$. □

We note that the above corollary generalizes the GRS Theorem in the case of compact groups. Our next proposition illustrates the use of the theory of induced representations.

Corollary 94. *Let* (A, G, σ) *be a separable dynamical system where* G *compact. Suppose that the action of* G *on Prim* A *is free, i.e.* $G_P = \{e\}$ *for all* $P \in$ *Prim* A*. Then every irreducible covariant representation of* (A, G, σ) *is equivalent to the right regular representation based on an irreducible representation of* A*.*

Remark 95. It is tempting to suggest that the converse of Corollary 94 is also true. It is easy to show that the converse is true if G is a finite group and A is type I C^*-algebra. However, the situation is not clear if A is not type I, even if G is a finite group.

Example 96. Let $\mathbf{D}$ denote the closed unit disc in $\mathbf{R}^2$ and let $A = C(\mathbf{D})$. Define the action of $\mathbf{T}$ on A by $(\sigma_s f)(z) = f(zs)$ for all $f \in A$, $z \in \mathbf{D}$ and $s \in \mathbf{T}$. We would like to investigate irreducible representations of the dynamical system $(A, \mathbf{T}, \sigma)$ following the ideas outlined above. Note that we will make minimal use of the fact $\mathbf{T}$ is an abelian group. Let (π, U) be an irreducible representation of $(A, \mathbf{T}, \sigma)$ on a Hilbert space $\mathcal{H}$. Since $\pi(A)$ is commutative then $\pi(A)'' = \mathbf{Z}(\pi(A)'')$. Therefore, we can use $\pi(A)$ as our dense, point-norm continuous C^*-subalgebra of $\mathbf{Z}(\pi(A)'')$. Let X be a compact, Hausdorff space such that $\pi(A) \cong C(X)$ and let φ denote the isomorphism map from $C(X)$ onto $\pi(A)$. We translate the action of $\mathbf{T}$ on $\pi(A)$ to an action on $C(X)$ by $\sigma_s(g) = U(s)\varphi(g)U(s^{-1})$ for $g \in C(X)$. Since the action of $\mathbf{T}$ on $C(X)$ is ergodic we know that X is homeomorphic to a right coset space of $\mathbf{T}$. We will consider the case $X = \mathbf{T}$ and the case $X = \{e\}$. It will turn out that these are the only cases we need to consider.

Suppose first that $X = \mathbf{T}$. We can view φ as a representation of $C(\mathbf{T})$ on $\mathcal{H}$. By the discussion preceding Theorem 88 φ is equivalent to the representation of $C(\mathbf{T})$ on $L^2(\mathbf{T}, \mathcal{H}_n, \mu)$ given by $(\pi_\mu \otimes 1_{\mathcal{H}_n}(f)\zeta)(z) = f(z)\zeta(z)$ for all $f \in C(\mathbf{T})$, $\zeta \in L^2(\mathbf{T}, \mathcal{H}_n, \mu)$, and $z \in \mathbf{T}$ where μ is a quasi-invariant measure on $\mathbf{T}$. We can assume, without the loss of generality, that μ is the Lebesgue measure. Let $V : \mathcal{H} \to L^2(\mathbf{T}, \mathcal{H}_n)$ be the unitary implementing the equivalence between φ and $\pi_\mu \otimes 1_{\mathcal{H}_n}$. Define $\phi = \varphi^{-1} \circ \pi : A \to C(\mathbf{T})$, then π is equivalent to the representation given by the map $a \mapsto (\pi_\mu \otimes 1_{\mathcal{H}_n})(\phi(a))$. The last statement is essentially the content of Theorem 88. So, without the loss of generality, we can assume that $\pi = (\pi_\mu \otimes 1_{\mathcal{H}_n}) \circ \phi$. For each $z \in \mathbf{T}$ define $\pi_z(a) = \phi(a)(z)$ to be the representation of A on $\mathcal{H}_n$. Then $(\pi(a)\zeta)(z) = \pi_z(a)\zeta(z)$ for all $a \in A$, $z \in \mathbf{T}$, and $\zeta \in L^2(\mathbf{T}, \mathcal{H}_n)$. We know by Theorem 49 that for each $s \in \mathbf{T}$

$$(U(s)\zeta)(z) = W(s, z)\zeta(zs)$$

for all $\zeta \in L^2(\mathbf{T}, \mathcal{H}_n)$ and almost every $z \in \mathbf{T}$, where $W(s, z) \in U\mathcal{B}(\mathcal{H}_n)$. By the same theorem we can choose $W(s,z)$ to be a Borel function on $\mathbf{T}\times\mathbf{T}$. Recall that for every $s,t\in\mathbf{T}$,

$$W(st,z)=W(s,z)W(t,zs) \tag{2.2}$$

for almost every $z\in\mathbf{T}$. Since W is a Borel function (in both variables) and $\mathcal{H}_n$ is a separable Hilbert space, then the characteristic function of the set for which equation (2.2) holds is Borel. It follows from the Fubini theorem ([9]; Theorem 2.36) that there exists $z_0\in\mathbf{T}$ such that

$$W(st,z_0)=W(s,z_0)W(t,z_0s)$$

for almost every $s,t\in\mathbf{T}$. We want to show that $\mathcal{H}_n$ is a one dimensional vector space. To this end, let T_0 be any operator in $\mathcal{B}(\mathcal{H}_n)$. Since the map $s\mapsto W(s,z_0)$ is Borel, we can define an operator T on $L^2(\mathbf{T},\mathcal{H}_n)$ by $z_0s\mapsto T_{z_0s}=W(s,z_0)^{-1}T_0W(s,z_0)$; that is $(T\zeta)(z_0s)=T_{z_0s}\zeta(z_0s)$ for all $\zeta\in L^2(\mathbf{T},\mathcal{H}_n)$ and $s\in\mathbf{T}$. Note that T is a decomposable operator by construction so $T \in \pi(A)'$. For each $s \in \mathbf{T}$,

$$\begin{aligned}
(U(s)TU(s^{-1})\zeta)(z_0t) &= W(s,z_0t)(TU(s^{-1})\zeta)(z_0ts) \\
&= W(s,z_0t)T_{z_0ts}(U(s^{-1})\zeta)(z_0ts) \\
&= W(s,z_0t)T_{z_0ts}W(s^{-1},z_0ts)\zeta(z_0t) \\
&= W(s,z_0t)T_{z_0ts}W(s,z_0t)^{-1}\zeta(z_0t) \\
&= W(s,z_0t)(W(ts,z_0)^{-1}T_0W(ts,z_0))W(s,z_0t)^{-1}\zeta(z_0t) \\
&= W(t,z_0)^{-1}T_0W(t,z_0)\zeta(z_0t) \\
&= T_{z_0t}\zeta(z_0t) \\
&= (T\zeta)(z_0t)
\end{aligned}$$

for every $\zeta \in L^2(\mathbf{T},\mathcal{H}_n)$ and almost every $t \in \mathbf{T}$. Thus $U(s)T = TU(s)$ for all $s \in \mathbf{T}$. It follows, from irreducibility of (π, U), that T_0 is a scalar operator and $\mathcal{H}_n = \mathbf{C}$.

Next we show that U is equivalent to the right regular representation. Define a unitary operator Q on $L^2(\mathbf{T})$ by $(Q\zeta)(z_0t) = W(t,z_0)\zeta(z_0t)$ for all $\zeta \in L^2(\mathbf{T})$ and $t \in \mathbf{T}$. For every $s \in \mathbf{T}$

$$\begin{aligned}
(QU(s)Q^*\zeta)(z_0t) &= W(t,z_0)(U(s)Q^*\zeta)(z_0t) \\
&= W(t,z_0)W(s,z_0t)(Q^*\zeta)(z_0ts) \\
&= W(t,z_0)W(s,z_0t)W(ts,z_0)^{-1}\zeta(z_0ts) \\
&= \zeta(z_0ts)
\end{aligned}$$

for every $\zeta \in L^2(\mathbf{T})$ and almost every $t \in \mathbf{T}$. Note that by construction Q is a decomposable operator so $Q \in \pi(A)'$. Hence, after conjugating by Q, one can assume that $(\pi(a)\zeta)(z) = \pi_z(a)\zeta(z)$, $(U(s)\zeta)(z) = \zeta(zs)$ for every $z, s \in \mathbf{T}$, $\zeta \in L^2(\mathbf{T})$, and

$a \in A$. It follows that (π, U) is equivalent to the right regular representation induced from the representation π_z of A, where z can be taken to be any point in $\mathbf{T}$. Recall that every irreducible representation of A is given by evaluation on $\mathbf{D}$ so there exists $w \in \mathbf{D}$ such that $\pi_z(f) = f(w)$ for all $f \in C(\mathbf{D})$. Hence, (π, U) is equivalent to the right regular representation induced from an irreducible representation π_0 of A given by $\pi_0(f) = f(w)$ where $w \in \mathbf{D}$. Moreover, if $w = 0$ then the right regular representation induced from $\pi_0(f) = f(0)$ is not irreducible, so (π, U) is induced from a representation $\pi_0(f) = f(w)$ where $w \in \mathbf{D}$ and $w \neq 0$.

Next we want to consider the case when $\pi(A)$ is isomorphic to $C(\{e\})$. Then $\mathcal{H} = \mathcal{H}_n$ and $\pi(A) = \mathbf{C}1_{\mathcal{H}_n}$. Since $\mathbf{T}$ is abelian and $\pi(A) = \mathbf{C}1_{\mathcal{H}_n}$ then for each $s \in \mathbf{T}$

$$U(s)U(t) = U(t)U(s) \text{ and } U(s)\pi(a) = \pi(a)U(s)$$

for every $t \in \mathbf{T}$ and $a \in A$. Thus $U(s) \in \mathbf{C}1_{\mathcal{H}_n}$. Since (π, U) is irreducible then $\mathcal{H}$ is a one dimensional vector space and π is equivalent to evaluation at a point in $\mathbf{D}$. Suppose that $\pi(f) = f(z_0)$ for some $z_0 \in \mathbf{D}$ then $f(z_0) = \pi(f) = U(s)\pi(f)U(s^{-1}) = \pi(\sigma_s f) = f(z_0 s)$ for all $f \in A$ and $s \in \mathbf{T}$. It follows that $z_0 = 0$. Since (π, U) is a representation on $\mathbf{C}$ and $\pi(f) = f(0)$ then U can be taken to be any representation of $\mathbf{T}$ on $\mathbf{C}$.

Using Theorem 91 one can show that the two cases outlined above are in fact the only irreducible representations of $(A, \mathbf{T}, \sigma)$. We will return to this example in Section 2.2.1 where we will use a slightly different approach to explicitly determine all the irreducible representations of $(A, \mathbf{T}, \sigma)$.

2.2.1 Covariant Representations of $(C_0(X), G, \sigma)$

Let $(C_0(X), G, \sigma)$ be a separable dynamical system where G is a compact group. Let (π, U) be an *irreducible* representation of $(C_0(X), G, \sigma)$ on $\mathcal{H}$. We can assume without a loss of generality that π is of the form

$$\pi = (\pi_{\mu_\infty} \otimes 1_{\mathcal{H}_\infty}) \oplus \pi_{\mu_1} \oplus (\pi_{\mu_2} \otimes 1_{\mathcal{H}_2}) \oplus \cdots \tag{2.3}$$

where each μ_n is a finite Borel measure on X with μ_n disjoint μ_m. Then

$$\pi(A)'' = (L^\infty(X, \mu_\infty) \otimes 1_{\mathcal{H}_\infty}) \oplus L^\infty(X, \mu_1) \oplus (L^\infty(X, \mu_2) \otimes 1_{\mathcal{H}_2}) \oplus \cdots$$

It is routine to check that $\pi(A)''$ is an ergodic system of imprimitivity for (π, U). Therefore, we can assume that $\mathcal{H} = L^2(X, \mu, \mathcal{H}_k)$ and $\pi = \pi_\mu \otimes 1_{\mathcal{H}_k}$. Consider the dynamical system $(C_0(X), G, \sigma)$ where $C_0(X)$ is viewed as a subalgebra of $\mathcal{B}(L^2(X, \mu))$. We want to show that the action of G on X is transitive. Unfortunately, we cannot use Proposition 83 directly in this case because a non constant continuous function on X does not necessarily produce a non constant multiplication operator on $L^2(X, \mu)$ unless π is a faithful representation. Also note that the action of G on X is given by the original dynamical system therefore the transitivity property must be inherent to $(C_0(X), G, \sigma)$ independent of a particular representation. Nevertheless, we can use the ideas developed earlier in this section to show that the action of G on X is *essentially* transitive.

Lemma 97. *Let N be a masa on a Hilbert space $\mathcal{K}$ and $\zeta \in \mathcal{K}$ be a cyclic separating vector for N. Suppose M is a unital C^*-subalgebra of N such that $\overline{M\zeta} = \mathcal{K}$. Then there exists a compact Hausdorff space Y and a finite Borel measure ν and a unitary $V : \mathcal{K} \to L^2(Y, \nu)$ such that $VNV^* = L^\infty(Y, \nu)$ and $VMV^* = C(Y)$.*

Proof. Let $\rho : M \to C(Y)$ be the Gelfand isomorphism. Define a positive linear functional ϕ on M by $\phi(x) = \langle x\zeta, \zeta\rangle$. Then there is a finite positive Borel measure ν on Y such that

$$\phi(x) = \int_Y \rho(x) d\nu$$

for all $x \in M$.

Let $\pi_\phi : M \to \mathcal{B}(L^2(Y,\nu))$ be the corresponding GNS representation with 1_Y as the cyclic vector. Since ζ is a separating vector then the map $V : M\zeta \to \pi_\phi(M)1_Y$ given by $V(x\zeta) = \pi_\phi(x)1_Y$ is well defined. Clearly, V is an isometry. Hence we can extend V to a unitary from $\mathcal{K}$ onto $L^2(Y,\nu)$. Moreover, $\pi_\phi(x) = VxV^*$ for all $x \in M$ so that $VMV^* = \pi_\phi(M) = C(Y)$. To see that $VNV^* = L^\infty(Y,\nu)$ let $x_1 \in M$ and $x_2 \in N$ then

$$(Vx_1V^*)(Vx_2V^*) = (Vx_2V^*)(Vx_1V^*).$$

So $(Vx_2V^*) \subseteq (VMV^*)' = (C(Y))' = L^\infty(Y,\nu)$. Conversely, if $T \in L^\infty(Y,\nu) \subseteq (VNV^*)'$ then $T(VxV^*) = (VxV^*)T$, for all $x \in N$. So $x(V^*TV) = (V^*TV)x$, for all $x \in N$. Thus $V^*TV \in N' = N$ and $T = V(V^*TV)V^* \in VNV^*$.

$\square$

We apply Lemma 97 to the W^*-dynamical system $L^\infty(X,\mu)$ and the subalgebra $L^\infty(X,\mu)^c$. Then $L^\infty(X,\mu)$ is equivalent to $L^\infty(Y,\nu)$ and $L^\infty(X,\mu)^c$ is equivalent to $C(Y)$. The group action also translates via the unitary equivalence. In particular, the action of G on $C(Y)$ is ergodic. By Proposition 83 the action of G on Y must be transitive. Moreover, by Corollary 90 the measure ν on Y is quasi-invariant. Similarly, the measure μ on X is also quasi-invariant. Since $L^\infty(X,\mu)$ is equivalent to $L^\infty(Y,\nu)$, it follows from Mackey's Theorem 2 in [15] that there are invariant Borel subsets $Y' \subseteq Y$ and $X' \subseteq X$ and a Borel isomorphism $\theta : Y' \to X'$ such that

1. $\mu(X - X') = \nu(Y - Y') = 0$.

2. $\theta(y \cdot s) = \theta(y) \cdot s$ for all $y \in Y', s \in G$.

We want to show that X' is an orbit of G. To this end, let $x_1, x_2 \in X'$. Let $y_1, y_2 \in Y'$ such that $\theta(y_i) = x_i$. We know that G acts transitively on Y' so there is $s \in G$ such that $y_1 \cdot s = y_2$. It follows $x_2 = \theta(y_2) = \theta(y_1 \cdot s) = \theta(y_1) \cdot s = x_1 \cdot s$ as claimed. Suppose that $x \in X'$ then $L^2(X, \mu) = L^2(O_x, \mu)$ and $\pi_\mu = \pi_\mu|_{O_x}$. We know that $O_x = G_0/G$. Then it follows from Theorem 66 that (π, U) is induced from $(\pi_0 \otimes 1_{\mathcal{H}_k}, U_0)$ where $\pi_0(f) = f(x)$ for all $f \in C_0(X)$.

Example 98. We would like to return to Example 96 and consider it from the point of view outlined in this section. Let (π, U) be an irreducible representation of $(A, \mathbf{T}, \sigma)$ on $\mathcal{H}$. Then we can assume that $\mathcal{H} = L^2(\mathbf{D}, \mu, \mathcal{H}_k)$ and $\pi = \pi_\mu \otimes 1_{\mathcal{H}_k}$ where μ is a quasi-invariant measure on $\mathbf{D}$. We know that there exists $z_0 \in \mathbf{D}$ such that the measure μ is supported on the orbit of z_0, i.e. $\mu(\mathbf{D} - O_{z_0}) = 0$.

Suppose that $z_0 \neq 0$ then $L^2(\mathbf{D}, \mu, \mathcal{H}_k) = L^2(O_{z_0}, \mu, \mathcal{H}_k) = L^2(\mathbf{T}, \mu, \mathcal{H}_k)$. Since μ is a quasi-invariant measure we can assume that μ is the Lebesgue measure. It is now not hard to see that (π, U) is equivalent to the right regular representation induced from evaluation at z_0. Suppose next that $z_0 = 0$ then π is equivalent to evaluation at 0 and U is any representation of $\mathbf{T}$ on $\mathbf{C}$.

Thus there are two classes of irreducible representations of $(A, \mathbf{T}, \sigma)$,

1. (π, U) is equivalent to the right regular representation induced from an irreducible representation of $C(\mathbf{D})$ given by $f \mapsto f(z)$ where $z \in \mathbf{D}, z \neq 0$. Moreover, if $z_1, z_2 \in \mathbf{D}$ then the corresponding induced representations are equivalent if and only if z_1 and z_2 are in the same orbit.

2. (π, U) is equivalent to (π_0, U_0) where $\pi_0(f) = f(0)$ and U_0 is any one dimensional

representation of $\mathbf{T}$. In this case, the equivalence class of (π, U) is determined by U.

2.2.2 Covariant Representations Involving Finite Groups

In this section we will consider irreducible representations of (A, G, σ) when G is a finite group. Therefore, we will assume throughout this section that G is a *finite* group endowed with a discrete topology. We will show that if (π, U) is an irreducible representation of (A, G, σ) then π is a direct sum of finitely many irreducible representations. A similar result, with a different proof, is obtained in [3]. Using Theorem 91 together with Proposition 99 we can deduce the main results in [3].

Let M be a von Neumann algebra in $\mathcal{B}(\mathcal{H})$. Let p and q be a pair of projections in M. We denote $p \wedge q$ to be the orthogonal projection onto the space $p\mathcal{H} \cap q\mathcal{H}$. Then $\lim\, (pq)^n = p \wedge q$ where the limit is taken in the strong operator topology. Hence, $p \wedge q \in M$. We denote $p \vee q$ to be the orthogonal projection onto the space $\overline{p\mathcal{H} + q\mathcal{H}}$. Then $p \vee q = (p^\perp \wedge q^\perp)^\perp \in M$. We say that $q \leq p$ if $q\mathcal{H} \subseteq p\mathcal{H}$. We say that p is a minimal projection if $q \leq p$ implies that either $q = 0$ or $q = p$.

Proposition 99. *Let (π, U) be an irreducible representation of (A, G, σ) on $\mathcal{H}$. Then π is a direct sum of n irreducible representations with $n \leq |G|$.*

Proof. We will show that there exists a minimal projection $p \in \pi(A)'$ together with a subset $S \subseteq G$ such that $\oplus_S p_s = 1_{\mathcal{H}}$ where $p_s = U(s)pU(s)^*$. If p is a minimal projection together with $S \subseteq G$ as above then $\pi = \oplus_S \pi p_s$. Since p is a minimal projection then each πp_s is an irreducible subrepresentation of π.

Let p be a projection in $\pi(A)'$. Then

$$\begin{aligned} U(s)pU(s)^*\pi(a) &= U(s)p\pi(\sigma_{s^{-1}}(a))U(s)^* \\ &= U(s)\pi(\sigma_{s^{-1}}(a))pU(s)^* \\ &= \pi(a)U(s)pU(s)^* \end{aligned}$$

for all $s \in G$. In other words, $U(s)pU(s)^* \in \pi(A)'$ for all $p \in \pi(A)'$ and all $s \in G$. In particular, $\sum_G U(s)pU(s)^* \in \pi(A)'$ for all $p \in \pi(A)'$. Note that $U(t)(\sum_G U(s)pU(s)^*)U(t)^* = \sum_G U(s)pU(s)^*$ for all $t \in G$. Since (π, U) is irreducible then $\sum_G U(s)pU(s)^* = c1_{\mathcal{H}}$ for some complex number c. It follows that

$$\vee_G U(s)pU(s)^* = 1_{\mathcal{H}} \text{ for all } p \in \pi(A)'. \tag{2.4}$$

Suppose that $1_{\mathcal{H}}$ is not a minimal projection. Choose $p \in \pi(A)'$ such that $0 < p < 1_{\mathcal{H}}$. By Equation 2.4, we know that there is $s \in G$ such that $U(s)pU(s)^* \wedge (1_{\mathcal{H}} - p) \neq 0$. Let $q = U(s)pU(s)^* \wedge (1_{\mathcal{H}} - p)$. Then $0 < U(s)^*qU(s) \leq p$ and $U(s)^*qU(s) \perp q$. We reset $p = U(s)^*qU(s)$. If p is not a minimal projection, we can choose $q \in \pi(A)'$ such that $0 < q < p$. We reset $p = q$. Then $p \oplus U(s)pU(s)^* < 1_{\mathcal{H}}$. By Equation 2.4, we know that there is $t \in G - \{e, s\}$ such that $U(t)pU(t)^* \wedge (1_{\mathcal{H}} - p - U(s)pU(s)^*) \neq 0$. Let $q = U(t)pU(t)^* \wedge (1_{\mathcal{H}} - p - U(s)pU(s)^*)$ and reset $p = U(s)^*qU(s)$. Then $p \oplus U(s)pU(s)^* \oplus U(t)pU(t)^* \leq 1_{\mathcal{H}}$. We iterate this process until p is a minimal projection. $\square$

The following examples are borrowed from Example 4.3 and 5.2 in [3].

Example 100. Let $A = M_2(\mathbf{C}) \oplus M_2(\mathbf{C})$ and define $\sigma(M \oplus N) = WNW^* \oplus M$ where

$$W = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Then σ defines a dynamical system $(A, \mathbf{Z}_4, \sigma)$. Let $\pi_i : M_1 \oplus M_2 \in A \mapsto M_i$ for $i = 1, 2$. Observe that each π_i is an irreducible representation and $\pi_2 = \pi_1 \circ \sigma$. Consider the following covariant representation of $(A, \mathbf{Z}_4, \sigma)$ on $\mathbf{C}^4$

$$\pi = \begin{bmatrix} \pi_1 & 0 \\ 0 & \pi_1 \circ \sigma \end{bmatrix}$$

and

$$U = \begin{bmatrix} 0 & 1 \\ W & 0 \end{bmatrix}.$$

It is not hard to check that (π, U) is an irreducible representation. Define a covariant representation of $(A, \mathbf{Z}_2, \sigma^2)$ on $\mathbf{C}^2$ by

$$\pi_0 = \pi_1 \text{ and } U_0 = W.$$

Then (π, U) is equivalent to the representation induced from (π_0, U_0).

Example 101. Let $G = D_3$ be the dihedrial group of order 6 acting on the set $\{1, 2, 3\}$. Recall that D_3 has two generators s and t satisfying $s^2 = t^3 = e$ and $sts = t^2$. Let $\mathbf{F}_3$ be the free group on three generators and $A = C^*(\mathbf{F}_3)$ be the corresponding group C^*-algebra. Let $\phi : D_3 \to \text{Aut}\, \mathbf{F}_3$ be the homomorphism corresponding to the action of D_3 on the set $\{1, 2, 3\}$. Given a finitely supported function $f : \mathbf{F}_3 \to \mathbf{C}$ we define the action of D_3 to be $(\sigma_s f)(t) = f(\phi(s)^{-1}(t)$ for all $s \in D_3$ and $t \in \mathbf{F}_n$. By extending

σ_s to $C^*(\mathbf{F}_n)$ we obtain the dynamical system $(C^*(\mathbf{F}_n), D_3, \sigma)$.

Define the action of D_3 on A by $\sigma_r(U_i) = U_{r(i)}$ where U_i are the canonical unitary generators of A. Consider the following covariant representation of (A, G, σ) on $\mathbf{C}^2 \oplus \mathbf{C}^2 \oplus \mathbf{C}^2$

$$\pi = \begin{bmatrix} \pi_1 & 0 & 0 \\ 0 & \pi_1 \circ \sigma_t & 0 \\ 0 & 0 & \pi_1 \circ \sigma_{t^2} \end{bmatrix}$$

where $\pi_1 : A \to M_2(C)$ given by

$$\pi_1(U_1) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \pi_1(U_2) = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}, \pi_1(U_1) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

And

$$U_t = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, U_s = \begin{bmatrix} W & 0 & 0 \\ 0 & 0 & W \\ 0 & W & 0 \end{bmatrix}$$

where

$$W = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Then (π, U) is an irreducible representation. Let $H = \{e, s\}$ be a subgroup of G. Define a representation of H on $\mathbf{C}^2$ by $U_0(s) = W$ and $\pi_0 = \pi_1$. We want to show explicitly that (π, U) is equivalent to the representation induced from (π_0, U_0). To this end, note that (π, U) can be viewed as a representation on $L^2(H/G, \mathbf{C}^2)$ where

$$(U(t)\zeta)(Ht^i) = \zeta(Ht^i t), (U(s)\zeta)(Ht^i) = W\zeta(Ht^i s)$$

$$(\pi(a)\zeta(Ht^i) = \pi_1(\sigma_{t^i}(a))\zeta(Ht^i)$$

for all $\zeta \in L^2(H/G, \mathbf{C}^2)$, $a \in A$, and $i = 0, 1, 2$. Define a map $V : L^2(H/G, \mathbf{C}^2) \to L^2(G, \mathbf{C}^2)$ by $(V\zeta)(t^i) = \zeta(t^i)$ and $(V\zeta)(st^i) = W\zeta(t^i)$ for all $\zeta \in L^2(H/G, \mathbf{C}^2)$ and $i = 0, 1, 2$. Then it is easy to see that V defines a unitary onto the induced representation space of (π_0, U_0). For each $\zeta \in L^2(H/G, \mathbf{C}^2)$ and $i = 0, 1, 2$ we have

$$\begin{aligned}
(VU(t)\zeta)(t^i) &= (U(t)\zeta)(Ht^i) \\
&= \zeta(Ht^i t) \\
&= (V\zeta)(t^i t) \\
&= (V\overline{U}(t)\zeta)(t^i)
\end{aligned}$$

$$\begin{aligned}
(VU(t)\zeta)(st^i) &= W(U(t)\zeta)(Ht^i) \\
&= W\zeta(Ht^i t) \\
&= (V\zeta)(st^i t) \\
&= (V\overline{U}(t)\zeta)(st^i)
\end{aligned}$$

$$\begin{aligned}
(VU(s)\zeta)(t^i) &= (U(s)\zeta)(Ht^i) \\
&= W\zeta(Ht^i s) \\
&= W\zeta(Ht^{2i}) \\
&= (V\zeta)(st^{2i}) \\
&= (V\zeta)(t^i s) \\
&= (V\overline{U}(s)\zeta)(t^i)
\end{aligned}$$

$$\begin{aligned}
(VU(s)\zeta)(st^i) &= W(U(s)\zeta)(Ht^i) \\
&= \zeta(Ht^i s) \\
&= \zeta(Ht^{2i}) \\
&= (V\zeta)(t^{2i}) \\
&= (V\zeta)(st^i s) \\
&= (V\overline{U}(s)\zeta)(st^i)
\end{aligned}$$

$$\begin{aligned}
(V\pi(a)\zeta)(t^i) &= (\pi(a)\zeta)(Ht^i) \\
&= \pi_1(\sigma_{t^i}(a))\zeta(Ht^i) \\
&= \pi_1(\sigma_{t^i}(a))(V\zeta)(t^i) \\
&= (\overline{\pi}(a)V\zeta)(t^i)
\end{aligned}$$

$$
\begin{aligned}
(V\pi(a)\zeta)(st^i) &= W(\pi(a)\zeta)(Ht^i)\\
&= W\pi_1(\sigma_{t^i}(a))\zeta(Ht^i)\\
&= \pi_1(\sigma_{st^i}(a))W\zeta(t^i)\\
&= \pi_1(\sigma_{st^i}(a))(V\zeta)(st^i)\\
&= (\overline{\pi}(a)V\zeta)(st^i)
\end{aligned}
$$

2.3 Strong EHI

In this section we continue working with separable dynamical systems (A, G, σ) where G is a compact group. Our goal is to show that such systems satisfy the strong-EHI property. The property of strong-EHI was introduced by Echterhoff and Williams in an attempt to establish a connection between Prim $A \times_\sigma G$ and the G-action on Prim A. They showed that the strong-EHI holds under various conditions including if A is a type I C^*-algebra or if G is an abelian group.

Recall that for each $P \in$ Prim A, we define $G_P = \{s \in G : \sigma_s(P) = P\}$. Our key result is to show that if (π, U) is an irreducible representation of (A, G_P, σ) with ker $\pi = P$, then π is a homogeneous representation. It will follow by a result of Echterhoff and Williams that in the above situation the induced representation is always irreducible.

Let π be a representation of A on a separable Hilbert space $\mathcal{H}$. If E is a projection in $\pi(A)'$ then we denote π^E to be the subrepresentation of π acting on $E\mathcal{H}$. We call π a *homogeneous* representation if ker $\pi^E =$ ker π for every nonzero projection $E \in \pi(A)'$.

It follows from Lemma G.3 in [18] that π is a homogeneous representation if ker π^E = ker π for every nonzero projection $E \in \pi(A)' \cap \pi(A)''$.

Let G_0 be a closed subgroup of G and (π_0, U_0) be a covariant representation of (A, G_0, σ) on $\mathcal{H}_0$. Let (π, U) be the covariant representation of (A, G, σ) on $\mathcal{H}$ induced by (π_0, U_0). There is a natural family of projections in $\pi(A)'$ associated with Borel subsets of G_0/G. Consider the map $i : L^\infty(G_0/G, \mu) \to \pi(A)'$ given by $(i(f)\xi)(s) = f(\overline{s})\xi(s)$. For each nonzero Borel subset E of G_0/G, we denote π^E to be the subrepresentation of π acting on $i(\chi_E)\mathcal{H}$.

Lemma 102. *In the above situation, let $Q := ker\, \pi_0$. If F is an open subset of G_0/G then*

$$ker\ \pi^F = \bigcap_{s \in q^{-1}(F)} \sigma_{s^{-1}} Q.$$

Proof. Clearly, $\bigcap_{s \in q^{-1}(F)} \sigma_{s^{-1}} Q \subseteq \pi^F$. For the reverse inclusion, recall that the quotient map $q : G \to G_0/G$ is continuous and open. Let F be an open subset of G_0/G and suppose there is an $a \in A$ such that $a \notin \bigcap_{s \in q^{-1}(F)} \sigma_{s^{-1}} Q$. We will show that $\pi^F(a) \neq 0$. Let $s \in q^{-1}(F)$ such that $\pi_0(\sigma_s a) \neq 0$. Choose a unit vector $h \in \mathcal{H}_0$ and $\epsilon > 0$ so that

$$\|\pi_0(\sigma_s a)h\| \geq 2\epsilon$$

Then as in the proof of Lemma 6.19 in [18] we will to construct a function $\xi \in C(G, \mathcal{H}_0) \subseteq \mathcal{H}$ such that

$$\|\xi(s) - h\| \leq \frac{\epsilon}{\|a\|}.$$

Using the strong continuity of U_0, we can find an open neighborhood $N \subseteq G_0$ of e such that $\|U_0(t)h - h\| < \frac{\epsilon}{\|a\|}$ for all $t \in N$. We can assume without of loss of generality that $N = N^{-1}$ (replace $N \cap N^{-1}$). Using Urysohn's Lemma we can find a function $g \in C(G)$ such that $g(e) = 1$ and $g(t) = 0$ for all t in the complement of N in G. Note that $g^{-1}((\frac{1}{2},\infty))$ is an open neighborhood of e in G therefore its intersection with G_0 is open in the relative topology of G_0. Recall that every open set has a positive measure with respect to the Haar measure. In particular, $\mu_{G_0}(g^{-1}((\frac{1}{2},\infty))\cap G_0) > 0$ so, after dividing by $\mu_{G_0}(g^{-1}((\frac{1}{2},\infty))\cap G_0)$, we can assume that $\int_{G_0} g(t)d\mu_{G_0}(t) = 1$. Let $f(r) = g(rs^{-1})$ and define $\zeta \in \mathcal{H}$ by

$$\zeta(r) = \int_{G_0} f(tr)U_0(t^{-1})(h)d\mu_{G_0}(t).$$

It follows from the Dominated Convergence Theorem that $\zeta \in C(G, \mathcal{H}_0)$ and ζ satisfies all the conditions of an element of $\mathcal{H}$. Then

$$\begin{aligned}
\|\zeta(s) - h\| &= \| \int_{G_0} f(ts)(U_0(t^{-1})h - h)d\mu_{G_0}(t)\| \\
&= \| \int_{G_0} g(t)(U_0(t^{-1})h - h)d\mu_{G_0}(t)\| \\
&= \| \int_{N} g(t)(U_0(t^{-1})h - h)d\mu_{G_0}(t)\| \\
&\leq \frac{\epsilon}{\|a\|}.
\end{aligned}$$

It follows that $\|\pi_0(\sigma_s a)\xi(s) - \pi_0(\sigma_s a)h\| \leq \|\pi_0(\sigma_s a)\| \cdot \|\xi(s) - h\| \leq \|a\| \cdot (\frac{\epsilon}{\|a\|}) = \epsilon$. By the reverse triangle inequality,

$$\|\pi_0(\sigma_s a)\xi(s)\| \geq \epsilon.$$

Since $\pi_0(\sigma_{s_j} a) \to \pi_0(\sigma_s a)$ whenever $s_j \to s$ and $\xi \in C(G, \mathcal{H}_0)$ there exists an open neighborhood $F_s \subseteq G_0/G$ of $G_0 s$ such that

$$\|\pi_0(ta)\xi(t)\| > \epsilon/2$$

for all $t \in q^{-1}(F_s)$. Then $\pi^F(a)(\chi_{q^{-1}(F_s \cap F)}\xi) \neq 0$. $\square$

A structure theory developed by Effros in [6] allows us to decompose arbitrary representations into a direct integral of homogeneous representations that has very useful properties. Using this decomposition theory Echterhoff and Williams established a criterion for irreducibility of induced representation.

Theorem 103 ([5]; Theorem 1.7)**.** *Let (A, G, σ) be a separable system. Suppose that ρ is a homogeneous representation of A with ker $\rho = P$, and that $\rho \times_\sigma V$ is an irreducible representation of $A \times_\sigma G_P$. Then the representation of $A \times_\sigma G$ induced by $\rho \times_\sigma V$ is irreducible.*

In light of Theorem 103 we make the following definition.

Definition 104. We say that (A, G, σ) satisfies the strong Effros-Hahn Induction Property (strong-EHI) if given any $P \in \mathrm{Prim}A$ and an irreducible covariant representation (π_P, U_P) of (A, G_P, σ) such that $\ker\pi_P = P$ then the corresponding induced representation of (A, G, σ) is irreducible.

We would like to use Theorem 103 to prove the strong-EHI property for separable systems involving compact groups.

Theorem 105. *Let (A, G, σ) be a separable system where G is a compact group. Suppose P is a primitive ideal of A and (π, U) is an irreducible covariant representation of (A, G_P, σ) on $\mathcal{H}$ such that ker $\pi = P$. Then π is a homogeneous representation of A.*

Proof. Note that G_P is a closed subgroup of G so G_P is compact. We know by Theorem 91 that there exists a closed subgroup G_0 of G_P and an irreducible covariant representation (π_0, U_0) of the subsystem (A, G_0) such that (π, U) is equivalent to the representation induced by (π_0, U_0). Moreover, there is an isomorphism

$i : L^\infty(G_0/G_P, \mu) \to \mathbf{Z}(\pi(A)')$ given by $(i(f)\xi)(s) = f(\overline{s})\xi(s)$. Let E be a Borel subset of G_0/G_P of nonzero measure and denote π^E to be the subrepresentation of π acting on $i(\chi_E)\mathcal{H}$. It is enough to show that ker $\pi^E =$ ker π .

Let $Q =$ker π_0. If F is an open subset of G_0/G_P, let $F' := \{s^{-1} : s \in q^{-1}(F)\}$. By Lemma 102, ker $\pi^F = \bigcap_{s\in F'} \sigma_s Q$. Since G_0/G_P is compact, there is $\{t_j\}_{1\leq j\leq n} \subseteq G_P$ such that $G_P = \bigcup t_j F'$. Then by Lemma 102 $P =$ ker $\pi = \bigcap_{r\in G_P} \sigma_r Q = \bigcap \sigma_{t_j}(\bigcap_{s\in F'} \sigma_s Q) = \bigcap \sigma_{t_j}$ (ker π^F). Since P is a prime ideal and P is G_P-invariant, it follows that $P =$ ker π^F. In particular, $\|\pi^F(a)\| = \|\pi(a)\|$ for all $a \in A$.

Now let K be a compact subset of G_0/G_P of nonzero measure. By a simple compactness argument we can find $G_0 s \in K$ such that every open neighborhood of $G_0 s$ intersects with K in a set of positive measure. We claim that ker $\pi^K \subseteq$ ker $\pi_0 \circ s$. To this end, suppose that $\pi_0(\sigma_s a) \neq 0$ for some $a \in A$. Then as in Lemma 102 we can construct a function $\zeta \in C(G, \mathcal{H}_0) \subseteq \mathcal{H}$ such that

$$\|\pi_0(\sigma_s a)\xi(s)\| \geq \epsilon.$$

Since $\pi_0(\sigma_{s_j} a) \to \pi_0(\sigma_s a)$ whenever $s_j \to s$ and $\xi \in C(G, \mathcal{H}_0)$ there exists an open neighborhood $F_s \subseteq G_0/G$ of $G_0 s$ such that

$$\|\pi_0(ta)\xi(t)\| > \epsilon/2$$

for all $t \in q^{-1}(F_s)$. Then $\pi^K(a)(\chi_{q^{-1}(F_s\cap K)}\xi) \neq 0$. We want to show that ker $\pi_0 \circ s \subseteq$ ker π. To this end, suppose $\pi_0(\sigma_s a) = 0$ and let $\epsilon > 0$ be given. Since $\pi_0(\sigma_{s_j} a) \to 0$ whenever $s_j \to s$ we can find an open neighborhood F' of s in G_P such that $\|\pi_0(\sigma_t a)\| < \epsilon$ for all $t \in F'$. Then $\|\pi(a)\| = \|\pi^{q(F')}(a)\| < \epsilon$. Thus $\pi(a) = 0$ as claimed. It follows ker $\pi^K = P$.

Finally, if E a nonzero Borel subset of G_0/G_P then we can choose a compact subset $K \subseteq E$ such that $\mu(K) > 0$. Suppose $\pi^E(a) = 0$ then $\pi^K(a) = 0$. It follows $\|\pi(a)\| = \|\pi^K(a)\| = 0$. So ker $\pi^E = P$. □

Combining Theorem 103 and Theorem 105 we obtain the following corollary.

Corollary 106. *Let (A, G, σ) be a separable C^*-dynamical system where G is compact. Then (A, G, σ) satisfies the strong-EHI property.*

As mentioned in the introduction it remains unknown whether the strong-EHI property holds for an arbitrary C^*-dynamical system. We can inquire about a weaker property of C^*-dynamical systems, called simply the EHI property, where we ask every primitive ideal of $A \times_\sigma G$ to be induced from a stability group ([5]). However, even with an additional assumption that G is amenable it is not known whether all separable C^*-dynamical systems satisfy the EHI property.

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