Summer 2012

The Weak Discrepancy and Linear Extension Diameter of Grids and Other Posets

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THE WEAK DISCREPANCY AND LINEAR EXTENSION DIAMETER OF GRIDS AND OTHER POSETS

by

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A DISSERTATION

Presented to the Faculty of

The Graduate College at the University of Nebraska

In Partial Fulfilment of Requirements

For the Degree of Doctor of Philosophy

Major: Mathematics

Under the Supervision of Professor Jamie Radcliffe

Lincoln, Nebraska

August, 2012

THE WEAK DISCREPANCY AND LINEAR EXTENSION DIAMETER OF GRIDS AND OTHER POSETS

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University of Nebraska, 2012

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A linear extension of a partially ordered set is simply a total ordering of the poset that is consistent with the original ordering. The linear extension diameter is a measure of how different two linear extensions could be, that is, the number of pairs of elements that are ordered differently by the two extensions. In this dissertation, we calculate the linear extension diameter of grids. This also gives us a nice characterization of the linear extensions that are the farthest from each other, and allows us to conclude that grids are diametrally reversing.

A linear extension of a poset might be considered "good" if incomparable elements appear near to one another. The linear discrepancy of a poset is a natural way of measuring just how good the best linear extension of that poset can be, i.e.

$$\operatorname{ld}(P) = \min_{L} \max_{x \,||\, y} |L(x) - L(y)|,$$

where L ranges over all linear extensions of P mapping P to $\{1, 2, \ldots, |P|\}$. In certain situations, it makes sense to weaken the definition of a linear extension by allowing elements of the poset to be sent to the same integer, while still requiring that $x < y$ implies $L(x) < L(y)$. This is known as a weak labeling. Similar to linear discrepancy, the weak discrepancy measures how nicely we can weakly label the elements of the poset. In this dissertation, we calculate the weak discrepancy of grids, the permutohedron, the partition lattice, and the two-dimensional Young's lattice.

ACKNOWLEDGMENTS

I would like to thank Jamie Radcliffe, who has been my advisor since my first day of graduate school and whose Combinatorics course confirmed for me that I wanted to study discrete math. I have never enjoyed someone's exams so much. Thank you also to Judy Walker who has always been approachable and a wonderful source of guidance ever since I attended the Nebraska Conference for Undergraduate Women in Math as a senior in college. I also owe a great deal of thanks to Stephen Hartke for his interesting topics courses, encouragement in giving seminar talks, and pointing me towards the area of linear and weak discrepancy. Finally, I would like to thank the other members of my committee, Srikanth Iyengar and Vinod Variyam, for taking the time to learn about my small, little corner of discrete math.

In addition to my committee, I would like to thank everyone else who helped me through the complicated job search process, notably Liz Youroukos, John Orr, Cindy Farthing, Susan Hermiller, Allan Donsig, Becky Swanson, and Katie Morrison. I also would like to thank all the graduate students who listened with open ears to our struggles during this process. I hope I did not scare you too much for your future job searches; my experience is living proof that dreams can come true.

On that note, I would also like to thank all my Lincoln friends for making graduate school an unforgettable, and actually fun(!), experience. This especially applies to Michael Uhrig, one of my best friends and officemate, and Derek Boeckner, the official Johnson travel companion. Stay warm and visit often. ☺

Without my family and my teachers at Lee Academy, I would never have developed, nor been able to pursue, my love of mathematics at such a young age. Thanks Mom and Dad for always encouraging me to keep going and believing in me without hesitation. Thank you to my older sister, Jennifer, for always being someone I could depend on, and for playing school with me when I was too little to go to school myself. (I think you graded my first math test!)

To Brian, I could thank you for making my morning coffee and lunch every day, or for driving/accompanying me almost anywhere I got it into my head to go. Or for helping me with Algebra or covering my classes when I was sick or reminding me of induction when I was too tired to think straight. But this wouldn't even begin to describe how you've changed my life. In the words of Goethe, thank you for making this life a garden.

GRANT INFORMATION

This work was supported in part by the University of Nebraska Othmer Fellowship and a Nebraska Mentoring through Critical Transition Points Traineeship (NSF Grant DMS-0838463).

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Chapter 1

Introduction

This dissertation focuses on the study of partially ordered sets, specifically their linear extensions and weak labelings. Special consideration is given to the grid poset. We are interested in how different two linear extensions can be, and in how good a weak labeling can be.

A *linear extension* L of a finite poset P is a total ordering of the elements that respects the initial partial order. So $x <_P y$ in the poset implies $x <_L y$ in the linear extension. How different can two linear extensions be? In other words, what is the greatest number of pairs of elements that could be ordered differently in two linear extensions? This parameter is known as the *linear extension diameter*, $\mathrm{led}(P)$, because it is the diameter of the *linear extension graph*, $G(P)$, of the poset. The vertices of this graph are all linear extensions of P. If we think of a linear extension as a listing of the elements of P from smallest to largest, then two linear extensions are adjacent in $G(P)$ if and only if they swap two adjacent elements. For example, when P is an antichain (where any two elements are incomparable), any permutation is a linear extension and the distance between π and σ is the number of inversions of $\pi\sigma^{-1}$, i.e. the number of pairs (i, j) with $1 \leq i < j \leq |P|$ and $\pi\sigma^{-1}(i) > \pi\sigma^{-1}(j)$. As an additional example, Figure 2.17 shows the grid poset 2×3 and its linear extension graph $G(2 \times 3)$.

The graph $G(P)$ was introduced in a 1991 paper by Pruesse and Ruskey [18], and its diameter was first studied in 1999 by Felsner and Reuter [8]. The linear extension diameter of the Boolean lattice, 2^n, was not computed until 2011 though, when Felsner and Massow [7] developed a new method that involved looking at sub-cubes of the lattice. In this thesis, I generalize their method to extend their results to grids, i.e. products of chains.

The linear extension graph of a grid is a natural and interesting graph to study. While this graph contains a Hamiltonian path [22], and its minimum and maximum degrees are known [6], there are still some mysteries to solve. For example, we do not know the number of vertices. Brightwell and Tetali [4] gave an asymptotic count for the case of the Boolean lattice, namely

$$\frac{\log_2 |V(G(2^n))|}{2^n} = \log_2 \binom{n}{\lfloor n/2 \rfloor} - \frac{3}{2} \log_2 e + O\left(\frac{\log_2 n}{n}\right),$$

using the entropy method of Kahn [15].

Although it is a very difficult problem to generate all of the linear extensions of a poset, and thereby count how many exist, it is easy to find a single (or small number) of linear extensions. For this reason, one may ask whether certain linear extensions could be deemed "better" than other ones. Perhaps I need a linear extension for some application, but I would like to find a good one (or a fair one).

This concept was first introduced in a 2001 paper by Tanenbaum, Trenk, and Fishburn [21], and is known as linear discrepancy. For this definition, we need to think of a linear extension as a listing of the elements of the poset, so that $L(X) = 1$ if X is the smallest element in the linear extension and $L(Y) = |P|$ if Y is the largest element.

Definition 1.1. The *linear discrepancy* is the minimum over all linear extensions of the poset of the distance between incomparable elements, i.e.

$$\mathrm{ld}(P) := \min_L \max_{X \,||\, Y} |L(X) - L(Y)|,$$

where L ranges over all linear extensions.

Tanenbaum, Trenk, and Fishburn proved that computing the linear discrepancy of a poset is an NP-complete problem. In their 2001 paper, they also show some basic relationships between linear and weak discrepancy, which had been somewhat studied previously in a 1998 paper by Gimbel and Trenk [9]. Weak discrepancy is similar to linear discrepancy, except that now elements are allowed to be assigned the same labels (while preserving the requirement that $X < Y$ in P implies $L(X) < L(Y)$). In this latter paper, Gimbel and Trenk show that there is a polynomial time algorithm to compute the weak discrepancy of a poset, and that weak discrepancy is a comparability invariant.

More recently, a 2010 paper of Howard and Young [14] characterizes the posets that have equal linear and weak discrepancy, and a 2008 paper of Choi and West [5] makes progress in the direction of computing the linear discrepancy of grids by focusing on asymptotics for three and four-dimensional grid posets where the dimensions have the same size. In this thesis, we compute the weak discrepancy of arbitrary-dimensional grid posets, where the dimensions may have different sizes. We also compute the weak discrepancy of the permutohedron, the partition lattice, and the two-dimensional Young's lattice.

To begin, we will introduce the standard notation to be used throughout this thesis, along with definitions and other background material. We will first discuss the linear extension diameter problem in Chapter 3. In Section 3.1, we will introduce the proof methods used for general grids by focusing on the ternary lattice. Then, in Section 3.2, we will prove the main result for this chapter, calculating the linear extension diameter of grids. Building on this foundation, we will also characterize all diametral pairs of $G(P)$ in this section, and ultimately, we will show that all grids have the property of being diametrally reversing.

Chapter 4 focuses on calculating the weak discrepancy of various classes of posets. Section 4.1 presents our results on grid posets, and Section 4.2 uses these results to show how we can calculate the weak discrepancy of the permutohedron. In Section 4.3, we calculate the weak discrepancy of the partition lattice by proving a more general result about graded posets with long incomparable chains. Finally, in Section 4.4, we compute the weak discrepancy of a two-dimensional Young's lattice and present ideas of how to generalize this result to higher-dimensional Young's lattices.

Both chapters conclude with ideas for future work.

Chapter 2

Background, Definitions, and Notation

2.1 Partially Ordered Sets

Definition 2.1. A *partially ordered set*, or *poset*, P is a set of elements with an ordering $\leq$ such that for all $x, y, z \in P$, we have

1. $x \leq x$ (reflexive),
2. $x \leq y$ and $y \leq x$ imply $x = y$ (antisymmetric),
3. $x \leq y$ and $y \leq z$ imply $x \leq z$ (transitive).

Specifically, note that there are usually elements $x, y \in P$ with $x \not\leq y$ and $y \not\leq x$. We call such pairs of elements incomparable, and denote this by $x \,||\, y$.

Whether we are aware of it or not, we encounter partially ordered sets on a daily basis. A set of tasks to be completed is partially ordered by the fact that some tasks must be done before others. The teams in a little league could be partially ordered by record if we allow ties. Employees in a company are partially ordered by their upward progress along the corporate ladder. As a final example, patients waiting in an emergency room can be partially ordered by the severity of their conditions.

Keeping in mind the previous examples, there are some natural properties we would like a poset to have. If the Huskers are a better team than the Eagles, who are a better team than the Tribe, then we ought to consider the Huskers a better team than the Tribe. This motivates the transitivity requirement. In addition, we would like to avoid any catch-22's. It doesn't do us a lot of good to consider scheduling tasks when task A must be done before task B, but task B must be done before task A. I need to put on clean clothes before I go to the laundromat, but I need to do laundry before I can put on clean clothes. So, we require the poset to be antisymmetric. Similarly, when combined with transitivity, the antisymmetry property helps us avoid longer cyclic catch-22's. The Huskers are a better team than the Eagles, who are a better team than the Tribe, who are a better team than the Huskers.

If the poset does not have any incomparable elements, then it is *totally ordered* and we call it a *chain*.

Definition 2.2. A poset is a *chain* if it is totally ordered.

On the other end of the spectrum, we have *antichains*, where all of the elements are incomparable.

Definition 2.3. A poset A is an antichain if $x \,||\, y$ for all $x, y \in A$.

Before proceeding, we should clarify that whenever we use the word *comparable*, it is with respect to the poset, where one element is less than the other. This is different from the standard English usage where comparable means similar (and is usually pronounced by stressing the first syllable). For example, in standard English, one might say that a broken toe is comparable to a dislocated finger. But in our ER poset (see 2.6), it is clear that these are incomparable, because one is not more severe than the other. On the other hand, a broken toe and a heart attack are comparable ailments.

We often represent a poset with a *Hasse diagram*, which is a visual representation of the relations in the poset. In theory, we could draw a directed edge from x to y whenever $x \leq y$, but this would be wasteful. We can take advantage of the transitivity property in Definition 2.1, and only draw an edge from x to y if y directly *covers* x, i.e. $x \leq y$ and there are no elements z with $x \leq z \leq y$. Furthermore, we can consider drawing these edges upward, instead of orienting them, because antisymmetry together with transitivity guarantees we will never get stuck having to draw an upward edge down towards an element.

Definition 2.4. For $x \neq y$ in a poset P, we say $x < y$ if there is an upward path from x to y in the poset's Hasse diagram.

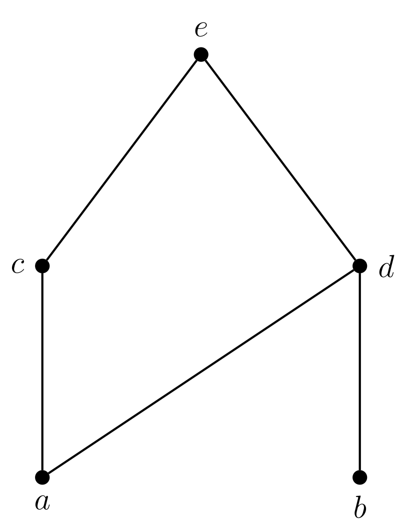

Figure 2.1: The Hasse diagram of a poset on five elements.

In Figure 2.1, we see that $a \leq c$ because there is an upward edge from a to c, but we also have that $a \leq e$ because we can follow an upward path from a to c to e. In all, this diagram gives us the following relations:

$$a \leq c, \quad a \leq e, \quad a \leq d, \quad b \leq d, \quad b \leq e, \quad c \leq e, \quad \text{and} \quad d \leq e.$$

For clarity, Figures 2.2 and 2.3 show the Hasse diagrams of a chain on five elements and an antichain on five elements.

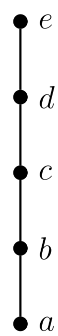

Figure 2.2: A chain with 5 elements.

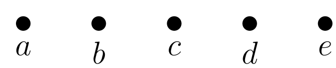

Figure 2.3: An antichain with 5 elements.

Figures 2.4, 2.5, and 2.6 show possible Hasse diagrams for three of the real-life examples I described, to which I will refer back throughout this thesis.

Definition 2.5. Two elements x, y are said to have a *meet*, called $x \wedge y$, if there is a unique maximal element z such that $z \leq x$ and $z \leq y$. Two elements x, y are said to have a *join*, called $x \vee y$, if there is a unique minimal element w such that $x \leq w$ and $y \leq w$.

Definition 2.6. A poset P is called a *lattice* if it has the additional properties that for all $x, y \in P$, x and y have both a meet and a join. It is a *distributive lattice* if the joins and meets distribute over one another, i.e. if for all $x, y, z \in P$, we have $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$, or equivalently $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$.

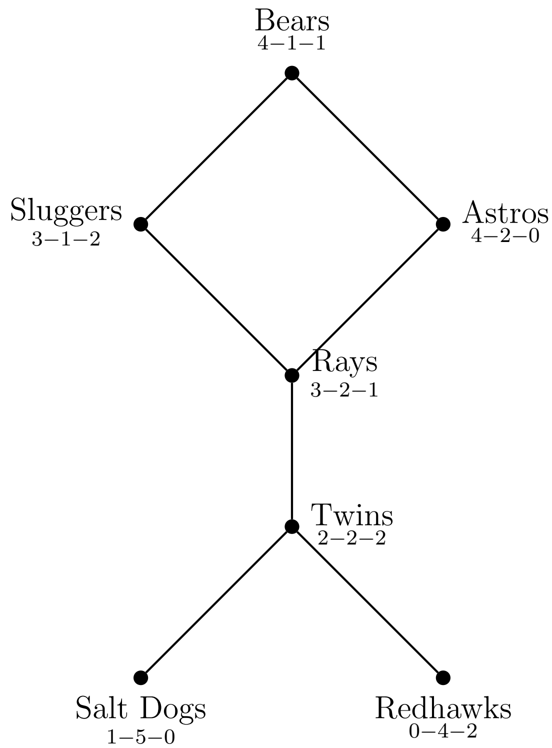

Figure 2.4: Little League Rankings given by wins–losses–ties, where $A \geq B$ if A has at least as many wins as B and at most as many losses.

The poset in Figure 2.7 is not a lattice for two reasons. First, a and b do not have a join—there are no elements in the poset greater than or equal to both a and b. Second, they do not have a meet because there are two maximal elements that are less than or equal to them both, namely c and d. It is easy to see that a finite lattice must have unique minimal and maximal elements; hence, the posets in Figures 2.4, 2.5, and 2.6 are not lattices. However, this is not a sufficient condition: the poset in Figure 2.8 takes the previous poset and adds an element that is the join of a and b, yet even now it is not a lattice because the elements a and b still do not have a meet. (Similarly, c and d do not have a join.)

Definition 2.7. If a poset has a unique minimal element, then the *atoms* are the elements that cover the minimal element.

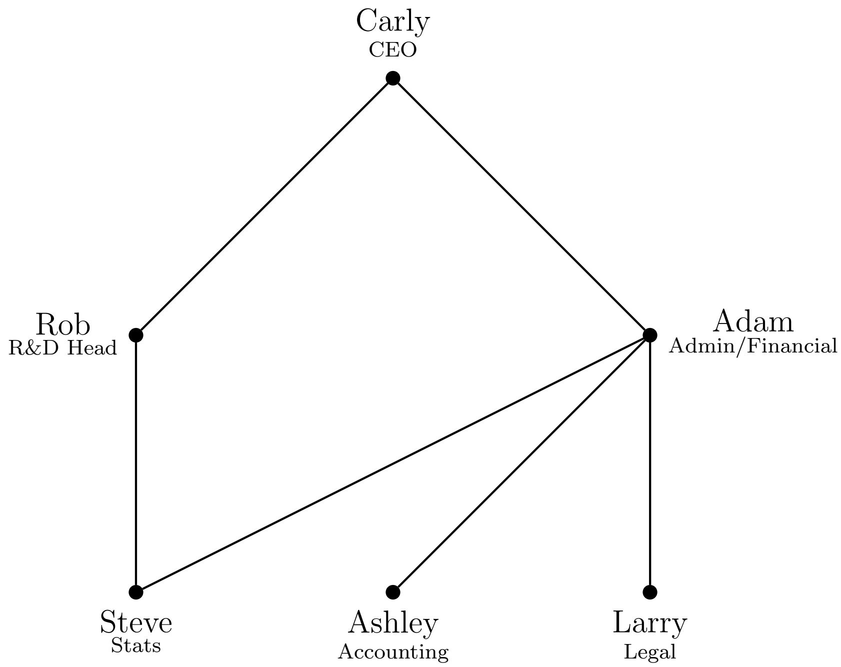

Figure 2.5: Employees in a company, ordered by supervisors.

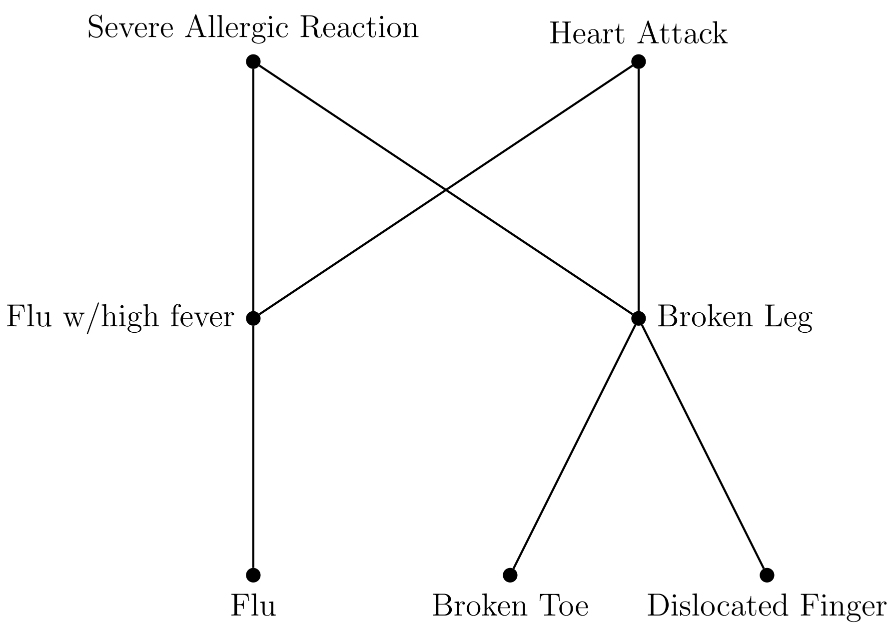

Figure 2.6: Patients in an ER waiting room, ordered by severity of their conditions.

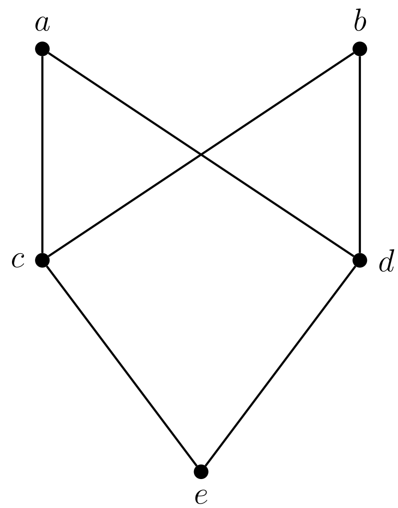

Figure 2.7: A poset that is not a lattice.

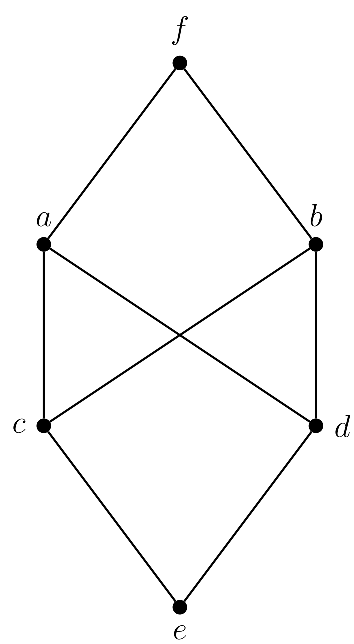

Figure 2.8: A poset with unique minimal and maximal elements that is not a lattice.

Because every finite lattice has a unique minimal element, it also has a set of atoms. If every element is the join of a set of atoms, the lattice is known as a *finite atomic lattice*. Nevertheless, posets that are not lattices may still have atoms, such as the posets in Figures 2.7

and 2.8.

Definition 2.8. A poset is *graded* if its Hasse diagram can be drawn so that each element appears on a level, and the edges only transverse consecutive levels. The *rank* of a graded poset is the minimum number of levels needed to grade the poset.

Every example given so far has been a graded poset, and each has been drawn in a way so that the levels are easy to see. The poset in Figure 2.9 is not graded; we must have at least two levels between a and e so that the edges from a to b, b to c, and c to e transverse different levels, but then we are stuck placing d on one of these two levels, too many levels away from either a or e. This is also an example of a non-distributive lattice. (For instance, $b \wedge (c \vee d) = b$ and $(b \wedge c) \vee (b \wedge d) = c$.)

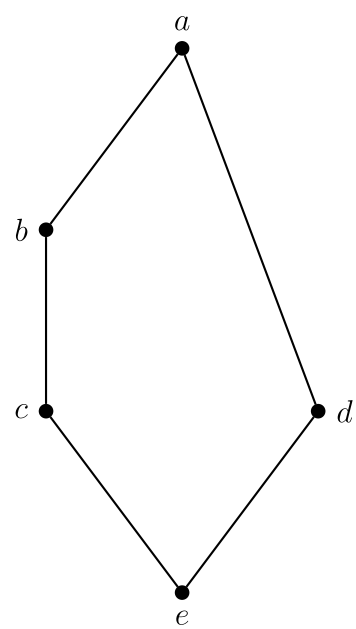

Figure 2.9: A poset that is neither graded nor distributive.

2.2 Divisibility Poset and Grids

The real-life posets described in the previous section are convenient to think about since we already have some intuition for how to work with them. Unfortunately, though, they do not have a lot of structure. We have no idea what the ER poset will look like on a given night at the hospital. There may be many maximal or minimal elements; in fact, on occasion it may be disconnected. So we will need to focus instead on mathematical posets which provide lots of structure, and the goal is that one day we will be able to use all we have learned from these posets to be able to reason about an arbitrary poset, akin to the real-life ones we encounter.

Often in mathematics, the first natural example to consider is the set of aptly named natural numbers. If we consider this set with the usual meaning of $\leq$, then we have a totally ordered set, which is not a terribly interesting poset on its own. So instead we will consider the poset where $x \leq y$ if and only if x divides y; this is an infinite poset called $\mathrm{Div}(\mathbb{N})$. One can check that this poset is a graded, distributive lattice with a unique minimal element and infinitely many atoms. For each natural number, we get a finite poset corresponding to that number which we will call $\mathrm{Div}(n)$, the *divisibility poset of* n; more precisely, $\mathrm{Div}(n)$ is the restriction of $\mathrm{Div}(\mathbb{N})$ to all numbers which divide n. For some numbers, specifically prime powers, we still get a chain, but for many numbers, we reap a more complex poset. For examples, see Figures 2.10 and 2.11.

Consider the divisibility poset of $N = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_n^{\alpha_n}$, where the p_i are distinct primes and the α_i are natural numbers. You may notice that the specific primes $\{p_1, \dots, p_n\}$ that appear in the factorization do not affect the structure of the poset. The divisibility poset of $18 = 2 \cdot 3^2$ is isomorphic to the poset in Figure 2.10. The structure of the poset is completely determined by the exponents $\{\alpha_1, \dots, \alpha_n\}$.

Specifically, $\mathrm{Div}(N)$ is the set of elements $\{p_1^{x_1} p_2^{x_2} \dots p_n^{x_n} : 0 \leq x_i \leq \alpha_i \text{ for } 1 \leq i \leq n\}$

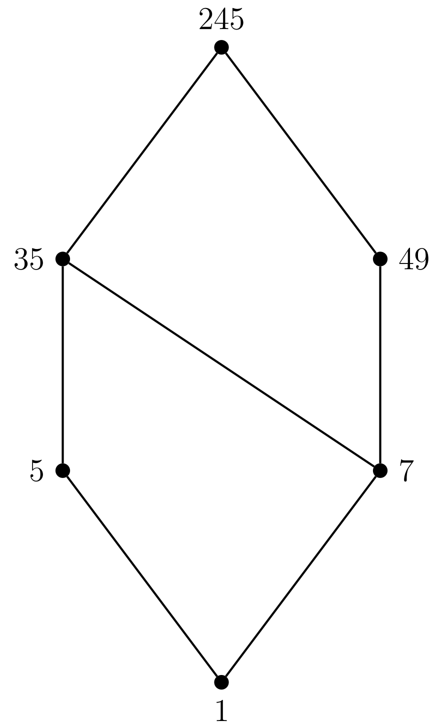

Figure 2.10: Div(245)

where

$$p_1^{x_1} p_2^{x_2} \dots p_n^{x_n} \leq p_1^{y_1} p_2^{y_2} \dots p_n^{y_n}$$

if and only if $x_i \leq y_i$ for all $i \in [n]$. Figures 2.10 and 2.12 show the divisibility poset of 245; in the latter, each element has been factored.

Furthermore, because the specific primes do not affect the structure of the poset, we can consider each element $p_1^{x_1} p_2^{x_2} \dots p_n^{x_n}$ as simply an ordered n-tuple of its exponents, $(x_1, x_2, \dots, x_n)$. We will often shorten this n-tuple as $x_1 x_2 \dots x_n$ when it makes sense to do so. Figures 2.11 and 2.13 show the divisibility poset of 30; in the latter, each element is represented by its exponent tuple.

As might be expected, we switch to studying this family of posets where we are no longer concerned with the specific primes. The set of posets with n-tuple elements where each entry

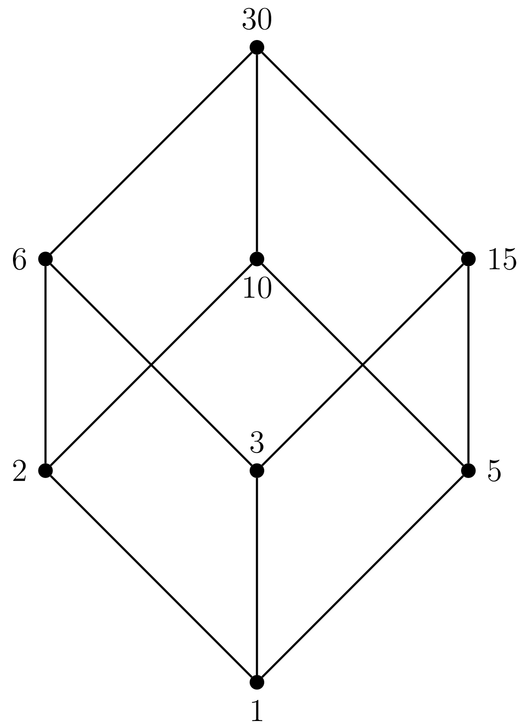

Figure 2.11: Div(30)

x_i in the tuple is bounded between 0 and $m_i - 1$ is precisely the set of n-dimensional grids.

Definition 2.9. The *grid poset* $m_1 \times \cdots \times m_n$ is the product of n chains of lengths $m_1, m_2, \ldots, m_n$, respectively. We set $m_1 \times \cdots \times m_n := \{(x_i)_1^n : 0 \leq x_i < m_i\}$ where $x_1 x_2 \ldots x_n \leq y_1 y_2 \ldots y_n$ if $x_i \leq y_i$ for all $i \in [n]$. For the duration of this thesis, we will refer to this grid as $P(m_1, \ldots, m_n)$, or simply P.

Once again, we may occasionally omit the parentheses and commas when writing elements of P if the context is clear. Writing $X = (x_i)_1^n$ and $Y = (y_i)_1^n$, we have that the grid is a distributive lattice with

$$X \wedge Y = (\min(x_i, y_i))_1^n$$

and

$$X \vee Y = (\max(x_i, y_i))_1^n.$$

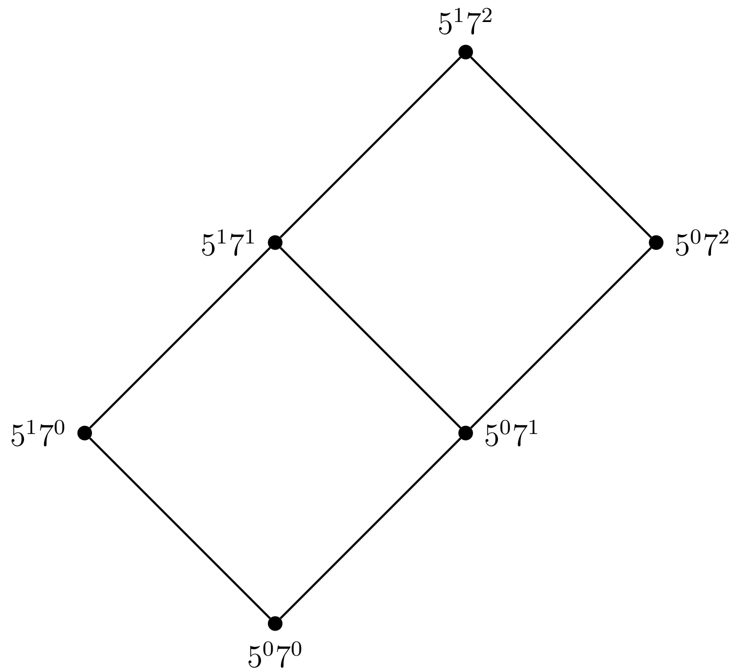

Figure 2.12: Div(245), redux.

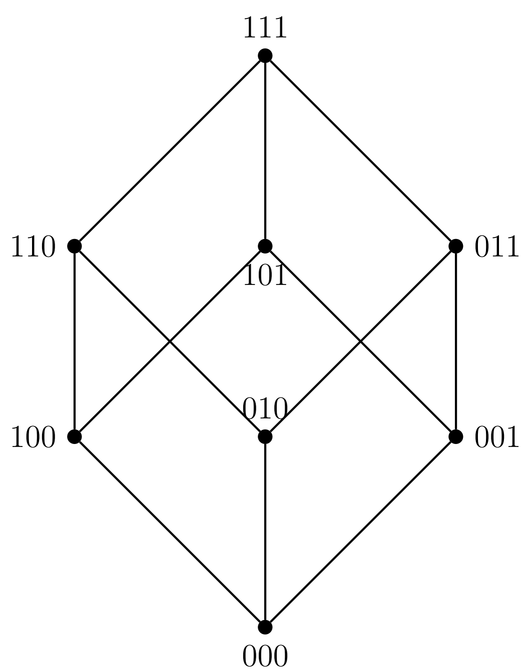

Figure 2.13: Div(30), redux.

We write $\bar{0}$ for $(0,\ldots,0)$ and let $d(X,Y) := |\{i : x_i \neq y_i\}|$ be the Hamming metric.

An important and often studied subset of grids is the set of *Boolean lattices*, 2^n. This is sometimes referred to as the subset lattice because of the canonical bijection between binary vectors of length n and subsets of $[n]$. Figure 2.13 is the Boolean lattice 2^3 which is isomorphic to the subset lattice in Figure 2.14.

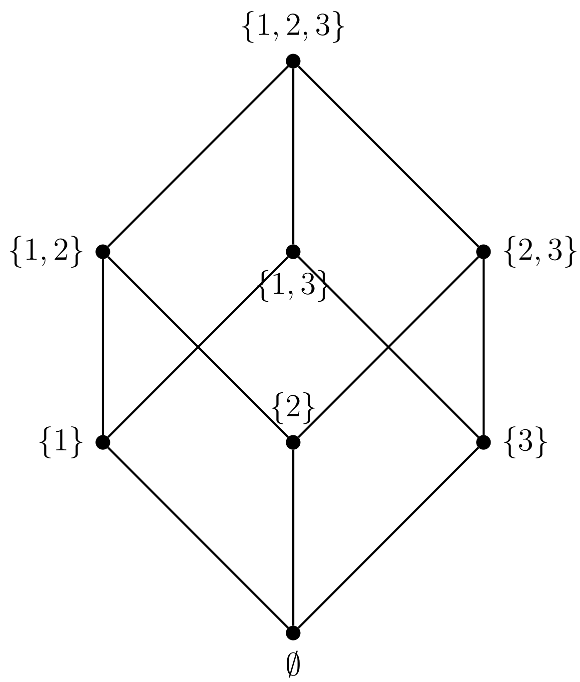

Figure 2.14: The subset lattice of [3].

2.3 Linear Extensions and Labelings

Let us return to our three real-life examples of posets and consider what properties they might motivate us to study. In the little league example in Figure 2.4, we could think about how to rank the teams; perhaps we are deciding how to seed them for a tournament. Of course, we would want our ranking to respect our partial ordering, and so this is exactly a question of totally ordering our poset. When we expand our poset by adding relations (that respect the antisymmetry condition) until we arrive at a chain, we are creating a linear extension of a poset.

Definition 2.10. A *linear extension* of a poset P is a total ordering of the poset that respects the initial partial ordering.

Remark 2.11. There is a natural correspondence between total orderings of a poset P and bijective maps ϕ from P to $\{1, 2, \ldots, |P|\}$ such that $X < Y$ in P implies $\phi(X) < \phi(Y)$. We will often think about linear extensions in this way, as a *linear labeling* of the poset where each element receives a unique integer label and larger elements in P have larger labels.

For the little league teams in Figure 2.4, this means that the best team, the Bears, will be assigned the largest label, 7. So in fact, we are ranking them from worst to best. One possible linear extension of this poset is shown in Figure 2.15.

In this example, we are fixed in our choices of labels for the Bears, Rays, and Twins. We have a choice whether the Sluggers or the Astros should be ranked higher, and whether the Salt Dogs or the Redhawks should be ranked higher. In all, this will give us four different possible linear extensions.

In general, how many linear extensions of a poset are there? This is actually a very difficult question. In a larger poset, we may have many pairs of incomparable elements, but these pairs may be related in intricate ways. For example, the number of linear extensions of the Boolean lattice is unknown, although good asymptotics are known. (See Chapter 1.)

A second interesting question asks how different two linear extensions can be. If I rank the little league teams and you rank the little league teams, how much could we possibly disagree? In other words, how many pairs of teams (A, B) could we have where I thought team A was better than team B, but you thought team B was better than team A? This is

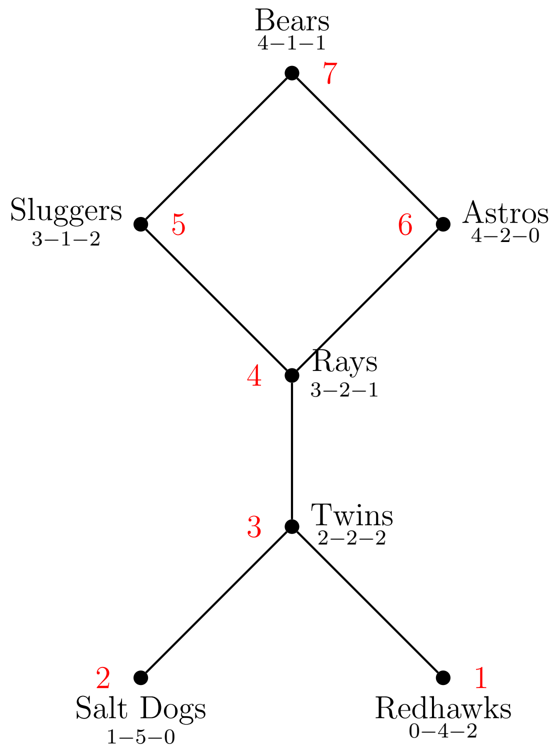

Figure 2.15: A linear extension of seven little league teams, represented by a linear labeling.

precisely what the linear extension diameter measures, which will be introduced in the next section and studied in more detail in Chapter 3.

Finally, which of these linear extensions is the best? How do we determine what best even means, or how to measure the "fairness" of a linear extension? Once we decide what this could mean, can we find the best linear extension in a nice way? What does it look like? These are questions that can be studied by considering the linear discrepancy of a poset, which I will define and discuss in Section 2.5.

My work in this area has consisted of studying not the linear discrepancy, but the weak discrepancy of a poset. For this, we need to consider a weak labeling.

Definition 2.12. A *weak labeling*, L, of a poset P is a mapping from P to $\mathbb{Z}$ so that if $x < y$ in P, then $L(x) < L(y)$.

Notice that this is a relaxation of the requirements of a linear labeling, as now our labels may be used multiple times. Hence, every linear labeling is also a weak labeling.

Suppose that we are working with the employee poset in Figure 2.5 and we are asked to assign pay grades to each worker in the company in such a way that each person's supervisor has a higher pay grade than himself or herself. For instance, we want Carly to have a higher pay grade than Rob, but we don't much care if Steve makes more money than Larry or Larry makes more money than Steve. In fact, we don't mind if they earn the same salary. For this situation, it would make more sense to use a weak labeling than a linear labeling, and we would actually be interested in finding a "good" weak labeling. An example of a weak labeling for the employee poset is given in Figure 2.16.

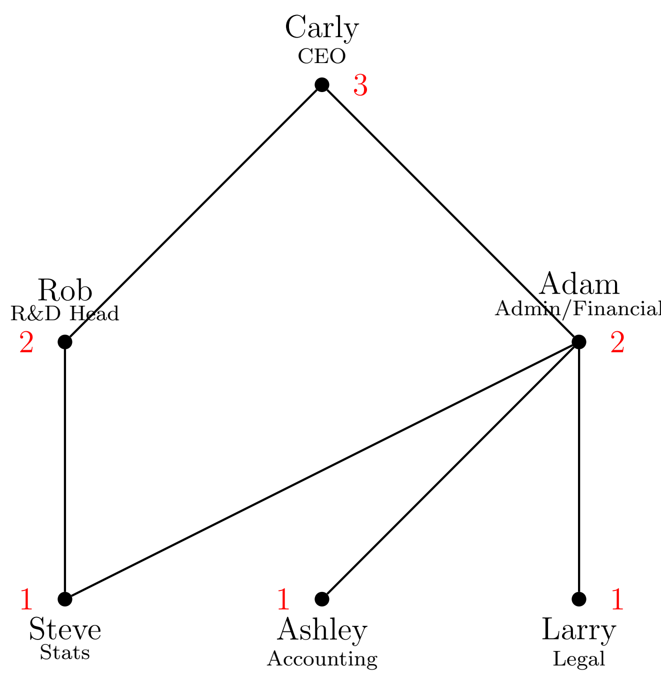

Figure 2.16: Assigning pay grades to employees in a company using a weak labeling.

Is this a "fair" weak labeling? How do we measure its fairness and determine if there are other weak labelings that are more fair? These questions are considered by studying the weak discrepancy of the poset. An introduction to weak discrepancy is in Section 2.5 and my results appear in Chapter 4.

2.4 Linear Extension Graph

When examining how different two linear extensions of a poset P can be, it will be helpful to visualize their similarities and differences using the linear extension graph of P. An example of a linear extension graph is given in Figure 2.17.

Definition 2.13. The *linear extension graph of* P, denoted $G(P)$, is the graph that has as its vertices all linear extensions of P, and two vertices are adjacent if and only if the linear extensions differ in an adjacent transposition. The *linear extension diameter of* P, denoted $\mathrm{led}(P)$, is the diameter of $G(P)$. A *diametral pair* of vertices (u, v) of a graph G is a pair of vertices such that $\mathrm{dist}_G(u, v) = \mathrm{diam}(G)$.

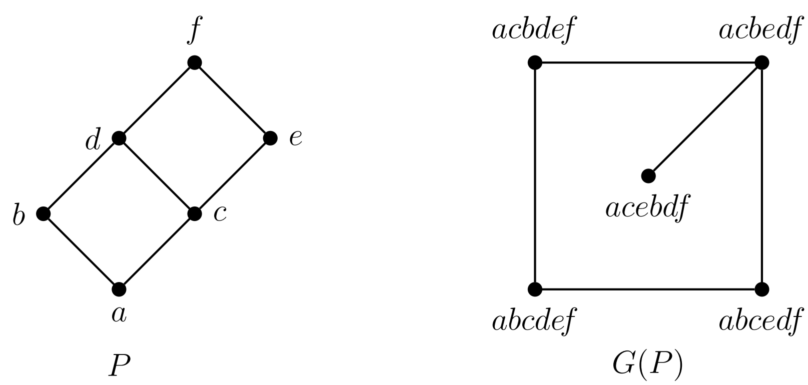

Figure 2.17: The grid 2×3 and its linear extension graph.

A diametral pair (L_1, L_2) of vertices in $G(P)$ would be two linear extensions that are the most different. By definition, their distance in the graph is equal to the linear extension diameter of P.

In Figure 2.17, the linear extensions are written as words, so that *abcdef* represents the linear extension $a < b < c < d < e < f$. We see that $\mathrm{led}(2 \times 3) = 3$ and there is a single diametral pair, namely the linear extensions *abcdef* and *acebdf*. For the little league poset in Figure 2.4, the linear extension diameter is 2, as discussed previously, and a diametral pair of linear extensions is shown in Figure 2.18.

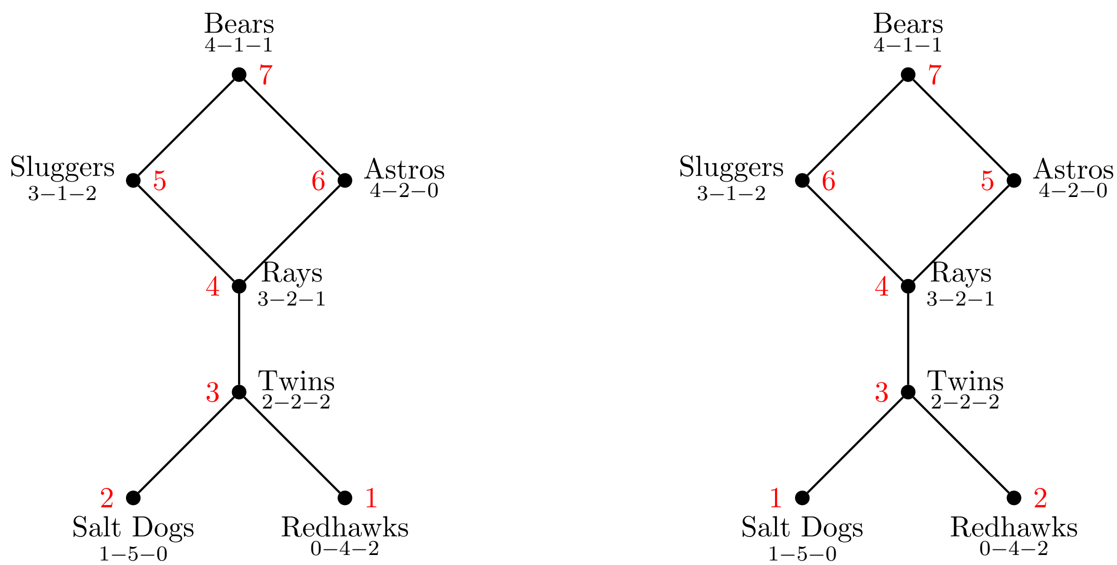

Figure 2.18: A diametral pair of linear extensions for the Little League poset.

In the linear extension on the left, we have chosen to rank the Astros higher than the Sluggers and the Salt Dogs higher than the Redhawks. Perhaps we decided that the number of wins should be a tie breaker; the Astros have won more games than the Sluggers so they should be considered a better team. On the right, we have decided that the number of losses is more important. The Sluggers have only lost one game while the Astros have lost two. In each case, our ranking of the Salt Dogs and Redhawks is consistent with our philosophy for how to rank the Sluggers and Astros.

With grids, we can develop a similar philosophy by prioritizing the dimensions. We must always start with $\bar{0}$ as the lowest element. The next element will be an atom, but which atom? Conversely, which atom will appear greatest in our linear extension? Once we have an ordering on the atoms, an *atomic ordering* if you will, we can generate a linear extension that is consistent with our priorities in the atomic ordering. In Chapter 3, we will see that these linear extensions are exactly what we are looking for.

Formalizing the previous paragraph, let $P = P(m_1, \ldots, m_n)$. Suppose $\sigma = \sigma_1 \ldots \sigma_n$ is a linear ordering of $[n]$, the index set of P. We define L_σ to be the *σ-lexicographic order* of P (a linear extension), namely $(x_i)_1^n < (y_i)_1^n$ in L_σ if and only if there exists $k \in [n]$ with $x_{\sigma_i} = y_{\sigma_i}$ for $i > k$ and $x_{\sigma_k} < y_{\sigma_k}$. Hence, we call the σ_n index the *σ-most important index*, and the σ_1 the *σ-least*. Let $\operatorname{rev}(\sigma) := \sigma_n \ldots \sigma_1$ be the *reversed priority ordering*.

Example 2.14. Consider the grid $2 \times 2 \times 3$, and let $\sigma = 321$. Then L_σ is the ordering

$$000 < 001 < 002 < 010 < 011 < 012 < 100 < 101 < 102 < 110 < 111 < 112$$

and $L_{\operatorname{rev}(\sigma)} = L_{123}(= L_{\mathrm{id}})$ is given by

$$000 < 100 < 010 < 110 < 001 < 101 < 011 < 111 < 002 < 102 < 012 < 112.$$

Example 2.15. Consider the grid $2 \times 2 \times 3$, and let $\sigma = 213$. Then L_σ is the ordering

$$000 < 010 < 100 < 110 < 001 < 011 < 101 < 111 < 002 < 012 < 102 < 112$$

and $L_{\operatorname{rev}(\sigma)} = L_{312}$ is given by

$$000 < 001 < 002 < 100 < 101 < 102 < 010 < 011 < 012 < 110 < 111 < 112.$$

Figure 2.19 shows the grid $P(2, 2, 3)$. The linear extension L_{213} begins by moving in the second dimension to 010, then the first to 100. The next smallest element is the join of these two atoms, namely 110. From there, L_{213} moves to the next layer in the third dimension

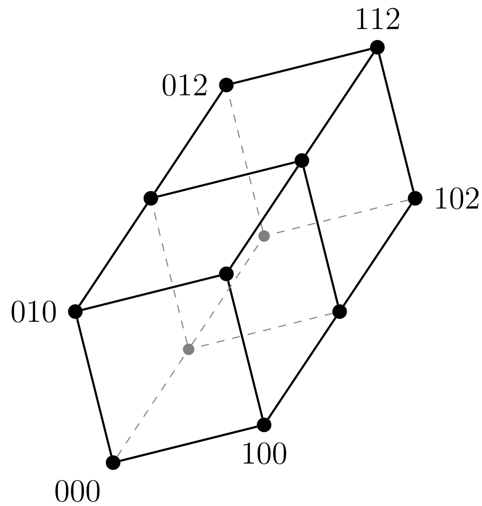

Figure 2.19: The grid $2 \times 2 \times 3$.

and builds it up in the same way as the first. We can see that for this σ, we begin with the front-most square and work our way to the back.

For contrast, the linear extension L_{312} begins by moving in the third dimension to 001 and then to 002. Next, it jumps to the first dimension to 100, then 101 and 102. Only then do we move to the second dimension and repeat the process. For this ordering, we first build up the bottom layer and then move upwards.

A *reversal* between two linear extensions L and L' is a pair of elements (x, y) such that $L(x) < L(y)$ but $L'(x) > L'(y)$. We let $R(L, L')$ be the set of reversals between L and L'. The following is a standard result.

Lemma 2.16. *The number of reversals between L and L' equals the distance between L and L' in $G(P)$, i.e. $|R(L, L')| = \mathrm{dist}_{G(P)}(L, L')$.*

Proof. It is easy to see that $|R(L, L')| \leq \mathrm{dist}_{G(P)}(L, L') =: d$. Take a path of length d between L and L'. Each edge in this path can account for at most one reversal. By the end of the path, every reversal between L and L' must have been accounted for by at least one edge. Hence, the inequality follows.

To prove equality, we need to show that an efficient path exists; in other words, we need to show that given two distinct vertices, L and L', there is always an edge that is incident to L that leads to a vertex $\widehat{L}$ such that $\text{dist}_{G(P)}(\widehat{L}, L') < d$.

Suppose not. Say that $L : x_1 < x_2 < \cdots < x_N$ where the x_i are elements of the poset, and suppose that none of the pairs of adjacent elements are reversals. So, $L'(x_i) < L'(x_{i+1})$ for all $i \in [N-1]$. By transitivity, this means $L = L'$, which contradicts that these were distinct vertices. $\square$

To wrap up this section, we will define a grid restriction that will be useful in the proof of Theorem 3.18.

Definition 2.17. Let $f : [n] \to \mathcal{P}(\mathbb{N}_0)$ be a function such that $f(k) \subseteq \{0, 1, 2, \ldots, m_k - 1\}$. In other words, f is a function that specifies a restriction of each index k to a subset of $\{0, 1, \ldots, m_k - 1\}$. We define the sub-grid $P[f]$ to be $\{(x_k)_1^n \in P : x_k \in f(k) \text{ for all } k\}$. Similarly, we write $L[f]$ for the restriction of a linear extension L of P to $P[f]$.

Note that $P[f]$ is isomorphic to the lattice $|f(1)| \times \cdots \times |f(n)|$. Figure 3.2 shows an example where $P = P(3, 3, 3)$, $f(1) = f(3) = \{0, 1, 2\}$, and $f(2) = \{1\}$. If we let $L = L_{321}$, then $L[f]$ would be the linear extension

$$010 < 011 < 012 < 110 < 111 < 112 < 210 < 211 < 212.$$

2.5 Linear, Weak, and t-Discrepancy

Recall the definitions of linear and weak labelings from Section 2.3. We can also define a weak labeling in the following way.

Definition 2.18. A *weak labeling* L on a poset P is a strictly increasing function from P to $\mathbb{Z}$, i.e. it satisfies the following properties:

1. $L(x) \in \mathbb{Z}$ for all $x \in P$, and

2. $x < y \Rightarrow L(x) < L(y)$.

Occasionally, it makes sense to give a poset a labeling that is not quite a linear labeling, but is stronger than a weak labeling. Consider the ER poset in Figure 2.6. A linear labeling would be akin to ranking the patients by who should be seen first by a doctor, who second, etc. But if our hospital employs more than one doctor, and hopefully it does, then we should be able to reuse labels so that multiple patients are seen simultaneously. This is good news for our heart attack patient and the one with the severe allergic reaction!

However, we certainly cannot treat arbitrarily large numbers of patients simultaneously, so a weak labeling does not accurately model this situation. Instead, if we have t doctors on call, we would like to be able to reuse a label at most t times. See Figure 2.20 for an example of a labeling where we let $t = 2$.

Definition 2.19. A *t-labeling* of a poset P assigns each element $x \in P$ an integer label, $L(x)$, so that

1. if $x < y$ in P then $L(x) < L(y)$, and

2. each label is used at most t times.

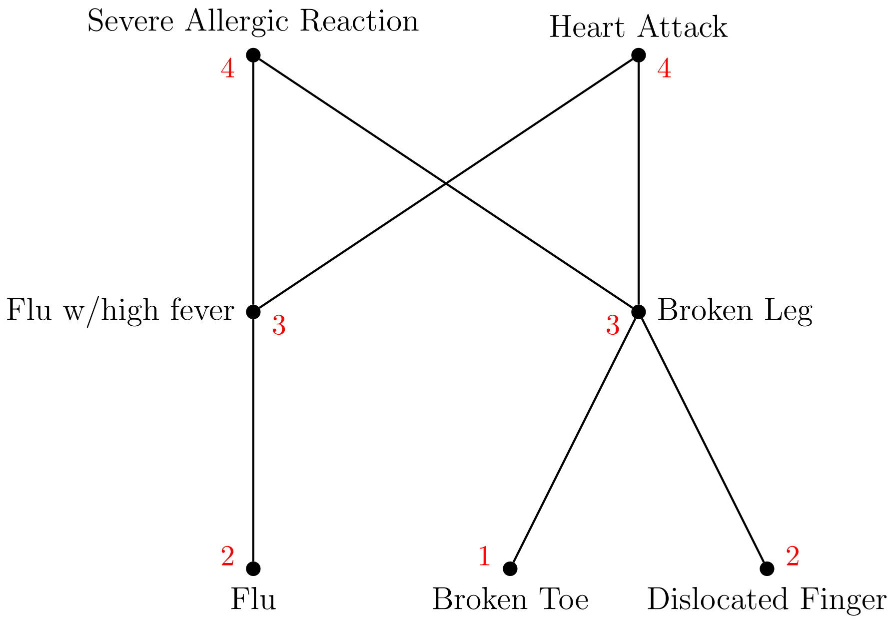

Figure 2.20: Patients in an ER waiting room, given a 2-labeling.

Notice that this is a generalization of both linear and weak labelings, as a linear labeling is simply a t-labeling where $t = 1$, and a weak labeling is a $|P|$-labeling (or an ∞-labeling). For the remainder of this thesis, the word labeling without any qualifiers could refer to any of these three types.

Now, what makes a linear, weak, or t-labeling a good labeling? We could spend the rest of this thesis entering into a complex philosophical argument about fairness, but we'll leave this to the BCS system debaters. Instead, we will consider a labeling "fair" if it gives incomparable elements labels that aren't too far apart. If I were a patient in the ER waiting room, I would be understanding that those with more severe conditions ought to be treated before myself. But I might be less understanding if I have to wait for a long time when no one in the room has a more severe problem than my own.

This measure is called the *discrepancy* of a labeling, and we can consider this measure for our three types of labelings.

Definition 2.20. The *discrepancy* of a labeling L, called $\Delta(L)$, is the maximum difference between the labels of two incomparable elements, i.e.

$$\Delta(L) = \max_{x \,||\, y} |L(x) - L(y)|.$$

If there are no incomparable elements, then $\Delta(L) = 0$. We say that x and y are a *discrepant pair* of L if and only if $x \,||\, y$ and $|L(x) - L(y)| = \Delta(L)$.

Figure 2.21 shows the grid $P(2,4)$ with a weak labeling. The discrepancy of this labeling is 2, which is achieved by the discrepant pair 10 and 03.

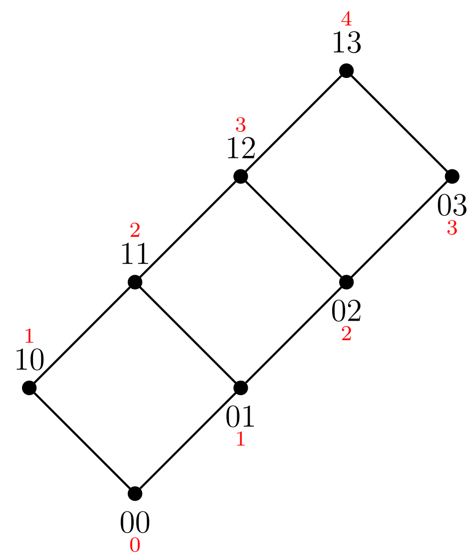

Figure 2.21: The grid 2×4 with a weak labeling.

Definition 2.21. The *weak discrepancy* of a poset, $\mathrm{wd}(P)$, is the minimum over all weak labelings of the discrepancy of the labeling, i.e.

$$\mathrm{wd}(P) := \min_{\substack{L \\ \text{weak labeling}}} \Delta(L) = \min_{\substack{L \\ \text{weak labeling}}} \max_{x \,||\, y} |L(x) - L(y)|.$$

The *linear discrepancy* is the minimum over all linear labelings of the discrepancy of the labeling, i.e.

$$\mathrm{ld}(P) := \min_{\substack{L \\ \text{linear labeling}}} \Delta(L).$$

The t-*discrepancy* is the minimum over all t-labelings of the discrepancy of the labeling, i.e.

$$d_t(P) := \min_{\substack{L \\ t\text{-labeling}}} \Delta(L).$$

For the grid in Figure 2.21, have we achieved an optimal labeling? In other words, is the weak discrepancy of this grid equal to 2 or can we do better with a different labeling? This labeling is a natural first labeling to try, as it simply labels by level, summing the entries of each element. However, it is not optimal.

We can bring the labels of our discrepant pair, 10 and 03, closer together by shifting all the labels of the upper level (where $x_1 = 1$) up by 1. This does not cause any other incomparable pair of elements to move more than 1 apart. So this new labeling has discrepancy 1, an improvement. See Figure 2.22 for this new labeling.

This is an optimal labeling, because it is not difficult to show that $\mathrm{wd}(2\times 4) \geq 1$. Consider the element 10 which is incomparable to the chain $01 < 02 < 03$. The three elements in this chain must be assigned different labels by the definition of a weak labeling. So, the best we can do, with respect to having a low discrepancy, is to assign the element 10 the same label as the element 02, and ensure that the elements 01 and 03 each have labels that are one away.

In general, how are these three discrepancies related? For a chain C_n, $\mathrm{wd}(C_n) = \mathrm{ld}(C_n) = d_t(C_n) = 0$ as there are no incomparable elements. In Figure 2.23, we have a poset where the three values differ when $t = 2$.

An easy result from [12] is the following.

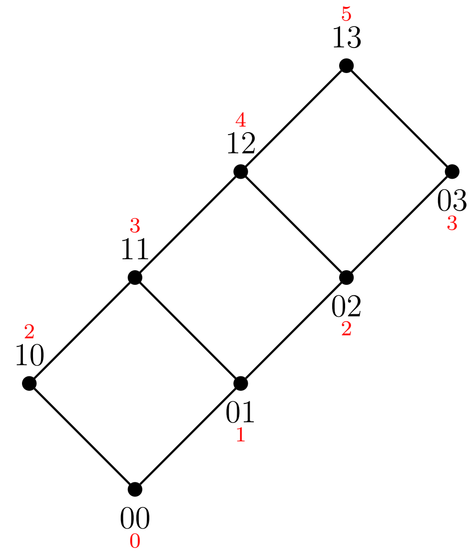

Figure 2.22: The grid 2×4 with an optimal weak labeling.

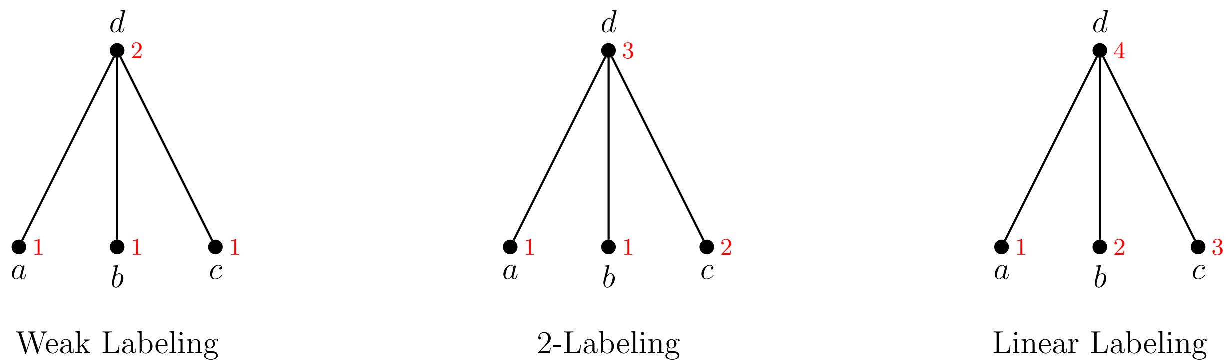

Figure 2.23: A poset where $\mathrm{wd}(P) = 0$, $d_2(P) = 1$, and $\mathrm{ld}(P) = 2$.

Proposition 2.22. *For any poset P and any $t \in \mathbb{N}$, we have* $\mathrm{ld}(P) \geq d_t(P) \geq \mathrm{wd}(P)$.

Proof. For the first inequality, consider a linear labeling L with $\Delta(L) = \mathrm{ld}(P)$. Because $t \geq 1$, L is necessarily a t-labeling. Hence, $d_t(P) \geq \Delta(L) = \mathrm{ld}(P)$. Similarly, any t-labeling is necessarily a weak labeling, so $d_t(P) \geq \mathrm{wd}(P)$. □

For grids, it is straightforward to determine what the discrepant pairs of elements look like, so we will include this result of ours here.

Proposition 2.23. *Let* $P = P(m_1, \ldots, m_n)$ *for some* $m_i \in \mathbb{N}$. *Given any labeling* L, *if* x *and* y *are a discrepant pair for* L *with* $L(y) > L(x)$, *then*

$$x = 0 \ldots 0\alpha 0 \ldots 0$$

and

$$y = (m_1 - 1) \ldots (m_{i-1} - 1)(\alpha - 1)(m_{i+1} - 1) \ldots (m_n - 1)$$

for some index i *and value* α. *We refer to an element like* x *as* $\Delta_{i,\alpha}$ *where* i *is the non-zero index, and to its discrepant partner* y *as* $\overline{\Delta}_{i,\alpha}$.

Proof. Let x, y be a discrepant pair for L with $L(y) > L(x)$. Because $x \,||\, y$, there exists an i_1 such that $x_{i_1} > y_{i_1}$ and an i_2 such that $x_{i_2} < y_{i_2}$. Suppose there exists another $j \neq i_1$ with $x_j > 0$. Define

$$\widehat{x}_k = \begin{cases} x_k & \text{if } k \neq j \\ x_k - 1 & \text{if } k = j \end{cases}.$$

Consider $L(y) - L(\widehat{x})$. (Note that $\widehat{x} \,||\, y$ because $\widehat{x}_{i_1} = x_{i_1} > y_{i_1}$ and $\widehat{x}_{i_2} \in \{x_{i_2}, x_{i_2} - 1\}$ so $\widehat{x}_{i_2} < y_{i_2}$.) Since $\widehat{x} < x$, we have

$$L(y) - L(\widehat{x}) \geq L(y) - L(x) + 1.$$

This is a contradiction. Hence, for only $k = i_1$ do we have that $x_k > 0$. (In particular, only $x_{i_1} > y_{i_1}$.)

At this point, we have shown that $x = \Delta_{i,\alpha}$ for some i, α. By similar methods, it is seen that $y = \overline{\Delta}_{i,\alpha}$. □

Chapter 3

Linear Extension Diameter

In this chapter, we investigate the linear extension diameter of grids. We use methods that were developed in [7] to calculate the linear extension diameter of the Boolean lattice, which was a problem that originated in [8]. In Section, 3.1, we look at the specific case of the ternary lattice, which will help illuminate the methods used in Section 3.2 to prove results about arbitrary grids.

3.1 Ternary Lattice

The *ternary lattice* 3^n is the n-dimensional grid $P(3, \ldots, 3)$, i.e. each element is a vector $\{0, 1, 2\}^n$ where $x_1 \ldots x_n \leq y_1 \ldots y_n$ if and only if $x_i \leq y_i$ for all $i \in [n]$.

Recall from Section 2.4 that L_σ is the σ-lexicographic ordering of P, and that (x, y) is a *reversal* between L and L' if $L(x) < L(y)$ and $L'(x) > L'(y)$. Reversals are ordered pairs of elements from the poset; hence, to study the linear extension diameter, we will first define a partition of $3^n \times 3^n$ that will make it easier to count the number of pairs that are reversals. We will exhibit a simple way of determining the number of reversals that occur in each part, and will use this to calculate the distance between L_σ and $L_{\text{rev}(\sigma)}$. Then we will show that the pairs of linear extensions $\{(L_\sigma, L_{\text{rev}(\sigma)})\}$ are exactly the diametral pairs of the linear extension graph of the ternary lattice.

Fix $S, T \in 3^n$. We define a difference vector $S \triangle T := D = (d_1, \ldots, d_n)$ where $d_k := |s_k - t_k|$, and an intersection vector $S \wedge T := I = (i_1, \ldots, i_n)$ where $i_k := \min\{s_k, t_k\}$. Also, it will be helpful to specify the indices where S and T differ, so let $S \blacktriangle T := \{k : s_k \neq t_k\}$.

For this section, it may be clearer to think of σ as a permutation of $[n]$ where $\sigma_i = \sigma(i)$, instead of an atomic ordering. Then, we say $i <_\sigma j$ if $\sigma^{-1}(i) < \sigma^{-1}(j)$. For $S, T \in 3^n$, we write $S <_\sigma T$ if and only if $s_m < t_m$ where $m = \max_\sigma\{S \blacktriangle T\}$.

Example 3.1. If $n = 5$, $\sigma = 41325$, $S = (2, 1, 2, 0, 1)$, and $T = (1, 0, 2, 1, 1)$, then $S \triangle T = (1, 1, 0, 1, 0)$, $S \blacktriangle T = \{1, 2, 4\}$, $S \wedge T = (1, 0, 2, 0, 1)$, $m = 2$, and $T <_\sigma S$.

This process of comparing elements creates a linear extension of 3^n by putting everything in σ-lexicographic order. This linear extension is identical to L_σ.

The following definition will give our partition of $3^n \times 3^n$. It is a partition because two elements of our poset will have a unique intersection and symmetric difference, as 3^n is in fact a lattice.

Definition 3.2. For $I, D \in 3^n$ with $i_k + d_k \leq 2$ for all $k \in [n]$, define

$$C_{D,I} := \{(S, T) : S, T \in 3^n,\ S \triangle T = D,\ S \wedge T = I\}.$$

Notice that the set $C_{D,I}$ is in bijection with the "antipodal corners" of the interval $[I, D + I]$, namely the set

$$\{A \in 3^n : a_k = i_k \text{ or } a_k = i_k + d_k\},$$

because each of these elements A appear as an S in an ordered pair of $C_{D,I}$. In Figure 3.1, we see the "sub-cube" $C_{D,I}$ where $D = (1,1,0,1,0)$ and $I = (1,0,2,0,1)$ as in Example 3.1. Here, $S = (2,1,2,0,1)$ and $T = (1,0,2,1,1)$ are antipodal corners.

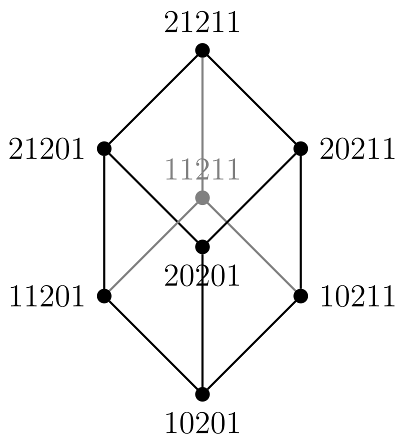

Figure 3.1: The "sub-cube" $C_{11010,10201}$ in 3^5.

Also, there is a natural bijection between $\{X \in 3^n : x_k \in \{0, d_k\}\}$ and $(I+X, D+I-X) \in C_{D,I}$, so for many purposes we can assume that $I = \bar{0}$. A simple result that follows from this fact is the following lemma.

Lemma 3.3. *Each class $C_{D,I}$ contains exactly $2^{|D|}$ ordered pairs, where $|D| = |\{k : d_k > 0\}|$.*

Proof. Because of the previous bijection, for each positive entry of D, we can choose whether to include it in X or not, i.e. we have $2^{|D|}$ choices. □

Furthermore, we can determine the exact number of reversals that appear as an ordered pair in a particular $C_{D,I}$ simply by considering the size of D.

Lemma 3.4. *Let $D, I \in 3^n$ with $D + I \in 3^n$, and let σ be a linear ordering of $[n]$. Then,*

$$|C_{D,I} \cap R(L_\sigma, L_{\text{rev}(\sigma)})| = \begin{cases} 0 & \textit{if } |D| \leq 1 \\ 2^{|D|-2} & \textit{otherwise} \end{cases}.$$

Proof. Note that $S <_{\mathrm{rev}(\sigma)} T$ if and only if $s_{\widehat{m}} < t_{\widehat{m}}$ where $\widehat{m} = \min_\sigma(S\blacktriangle T)$.

If $|D| = 0$, then $C_{D,I} = \{(I, I)\}$, and so there are no reversals. If $|D| = 1$, then $C_{D,I} = \{(I, D+I), (D+I, I)\}$ and we have that $I < D+I$ in 3^n, so there can be no reversals here either.

Now suppose that $|D| \geq 2$. Then $m' := \min_\sigma\{k : d_k > 0\} \neq \max_\sigma\{k : d_k > 0\} =: m$. A pair (X, Y) corresponds to a reversal if and only if $y_{m'} = i_{m'}$ and $y_m = i_m + d_m$ (and hence $x_{m'} = i_{m'} + d_{m'}$ and $x_m = i_m$). Each other non-zero index in D can be included in X or Y so there are $2^{|D|-2}$ such sets. □

Lemma 3.5. *There are a total of* $\frac{1}{4} \cdot 3^n \cdot (3^n - 2n - 1)$ *reversals between* L_σ *and* $L_{\mathrm{rev}(\sigma)}$.

Proof. Lemma 3.4 tells us the number of reversals in a given corner set $C_{D,I}$, given the number of positive entries in D. So, how many corner sets are there with a certain number of positive entries? We will show there are $3^n\binom{n}{j}$ legal pairs (D, I) with $|D| = j$.

We first pick the j indices to be positive in D. For the positive entries of D, we have three choices for how these indices differ in our ordered pairs:

1. $d_k = 1, i_k = 0$,
2. $d_k = 1 = i_k$, or
3. $d_k = 2, i_k = 0$.

For the indices not chosen to be positive in D, we still have three choices, $i_k = 0, 1, 2$. Overall, there are $\binom{n}{j}$ ways to pick the positive elements in D, and then 3^n ways to assign the d_k, i_k.

From Lemma 3.4, we sum over all $j \geq 2$ to get

$$3^n \sum_{j=2}^{n} \binom{n}{j} 2^{j-2} = \frac{1}{2^2} \cdot 3^n \cdot (3^n - 2n - 1)$$

total reversals between L_σ and $L_{\mathrm{rev}(\sigma)}$. □

Now that we have counted the distance between L_σ and $L_{\mathrm{rev}(\sigma)}$, we will show that this is the greatest number of reversals that can occur between two linear extensions of 3^n. To do so, we will show that each corner set $C_{D,I}$ contributes the greatest number of reversals possible for these particular linear extensions.

Theorem 3.6. *The linear extension diameter of the ternary lattice is* $\mathrm{led}(3^n) = \frac{1}{4} \cdot 3^n \cdot (3^n - 2n - 1)$.

Proof. We have $\mathrm{led}(3^n) \geq \frac{1}{4} \cdot 3^n \cdot (3^n - 2n - 1)$ by Lemma 3.5.

Let L_1 and L_2 be two linear extensions of 3^n. We will show each $C_{D,I}$ can contribute at most $2^{|D|-2}$ reversals between L_1 and L_2.

Fix $C_{D,I}$. In fact, without loss of generality, let $I = \bar{0}$. So now, for $X \in 3^n$ such that $x_i = 0$ or d_i for all i, we have $(X, D - X) \in C_{D,I}$. Say X is "down" in a linear extension if $X < D - X$. Let $\mathcal{F}_1$ be the family of such X which are down in L_1, and $\mathcal{F}_2$ those down in L_2. Then $(X, D - X) \in C_{D,I}$ yields a reversal between L_1 and L_2 exactly if X is down in L_1 but not in L_2. Since X down in L_i implies that $D - X$ is not down in L_i, we can find an upper bound for $|\mathcal{F}_1 \triangle \mathcal{F}_2|$, and the number of reversals will be half this quantity.

Because the elements X and $D - X$ in this $C_{D,I}$ all have indices that are 0 or d_i, there is a natural bijection between these elements and subsets of D. Note also that these sets are downwards closed, because of transitivity. So, we can use Kleitman's Lemma (see Theorem 3.12) to deduce that $|\mathcal{F}_1| \cdot |\mathcal{F}_2| \leq 2^d |\mathcal{F}_1 \cap \mathcal{F}_2|$.

Now, for all L_i and X, either X or $D - X$ is down in L_i. Hence, $|\mathcal{F}_1| = |\mathcal{F}_2| = 2^{d-1}$. So, $|\mathcal{F}_1 \cap \mathcal{F}_2| \geq 2^{d-2}$ (again by Kleitman's Lemma).

Also, if X is down in both L_1 and L_2, then $D - X$ is down in neither, i.e. $X \in \mathcal{F}_1 \cap \mathcal{F}_2$ if and only if $D - X \notin \mathcal{F}_1 \cup \mathcal{F}_2$. Therefore, $|\mathcal{F}_1 \cap \mathcal{F}_2| = |\{D - X : X \in \mathcal{F}_1 \cup \mathcal{F}_2\}|$.

Because there are 2^d choices for X, we have

$$|\mathcal{F}_1 \triangle \mathcal{F}_2| = 2^d - |\mathcal{F}_1 \cap \mathcal{F}_2| - |\{D - X : X \in \mathcal{F}_1 \cup \mathcal{F}_2\}| \leq 2^d - 2^{d-2} - 2^{d-2} = 2^{d-1}.$$

As discussed above, every reversal is counted twice in $\mathcal{F}_1 \triangle \mathcal{F}_2$: if $(X, D - X)$ is a reversal, then both X and $D - X$ are in the set $\mathcal{F}_1 \triangle \mathcal{F}_2$. Therefore, there are at most 2^{d-2} reversals per $C_{D,I}$. □

For a fixed permutation σ of $[n]$, L_σ and $L_{\mathrm{rev}(\sigma)}$ form a diametral pair of 3^n. This gives us $\frac{n!}{2}$ diametral pairs, and we will show this is all of them.

Definition 3.7. Let $J \subseteq [n]$ be a set of *free* indices and $f : [n] \setminus J \to \{0, 1, 2\}$ be any function that fixes all non-free indices. Define the sub-grid $\mathcal{G}[J, f]$ to be the sub-poset on the elements $\{X : x_i = f(i) \text{ for all } i \notin J\}$. Similarly, if L is any linear extension of 3^n, define $L[J, f]$ to be the restriction of L to $\mathcal{G}[J, f]$. Finally, if $C_{D,I}$ is a corner set of 3^n, let $C_{D,I}[J, f]$ be the restriction of $C_{D,I}$ to $\mathcal{G}[J, f]$.

This is a more specific version of Definition 2.17 that has been customized to fit this section. Figure 3.2 shows the two-dimensional sub-grid of 3^3 where $J = \{1, 3\}$ and $f(2) = 1$. The following lemma shows that diametral pairs of linear extensions restrict to diametral pairs when certain indices are fixed.

Lemma 3.8. *Let $J \subseteq [n]$ and $f : [n] \setminus J \to \{0, 1, 2\}$. If $L, \bar{L}$ is a diametral pair of 3^n, then $L[J, f]$ and $\bar{L}[J, f]$ are a diametral pair of $\mathcal{G}[J, f]$.*

Proof. We need to show each $C_{D,I}[J, f]$ contributes exactly $2^{|D|-2}$ reversals between $L[J, f]$ and $\bar{L}[J, f]$.

Fix $D, I \in 3^n$ such that $D|_{[n]\setminus J} = 0$, $I|_{[n]\setminus J} = f$, and $d_j + i_j < 3$ for $j \in J$.

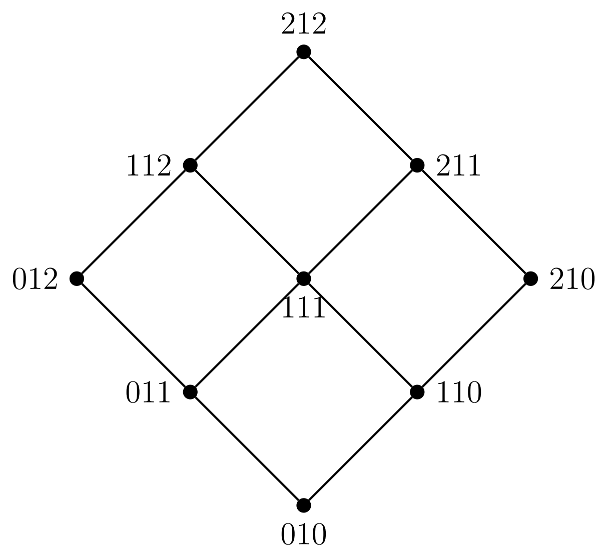

Figure 3.2: The sub-grid of 3^3 where the second index is fixed as 1.

Consider $C_{D,I}$ of the original, unrestricted grid. This $C_{D,I}$ contains $2^{|D|-2}$ reversals, but since the restriction to this sub-grid keeps the linear extensions in the same order and the elements of the pairs in $C_{D,I}$ are chosen specifically to all be in $\mathcal{G}[J, f]$, these are also all reversals in $C_{D,I}[J, f]$. □

Using the previous lemma, we can now show that if two linear extensions are a diametral pair for 3^n, then they must be generated by the σ-lex and rev(σ)-lex ordering for some σ. As a basic introduction to the structure of the proof, let us first look at the specific case of the three-dimensional ternary lattice 3^3, and assume we have already proved the result for the two-dimensional ternary lattice.

Suppose L_1 and L_2 are two linear extensions of 3^3 where there are exactly $\frac{1}{4} \cdot 3^3 \cdot (3^3 - 2 \cdot 3 - 1) = 135$ reversals that occur between L_1 and L_2. Define σ to be the order of the atoms in L_1; let's use the example that $\sigma = 123$ so that in L_1 we have $100 < 010 < 001$.

To begin, consider restricting to the sub-grids where $J = \{1, 2\}$ and $f(3) = i$ for

$i \in \{0, 1, 2\}$. The previous lemma and induction tell us that in L_1 we have

$$000 < 100 < 200 < 010 < 110 < 210 < 020 < 120 < 220,$$

$$001 < 101 < 201 < 011 < 111 < 211 < 021 < 121 < 221,$$

and

$$002 < 102 < 202 < 012 < 112 < 212 < 022 < 122 < 222.$$

Consequentially, to show that $L_1 = L_\sigma$ is simply a matter of showing that $220 < 001$ and $221 < 002$ in L_1. We'll come back to this, but we need to work with L_2 a little first. We can show that the atoms of L_2 are in rev(σ)-lex order by considering the corner sets where $I = 000$ and $D = d_1 d_2 d_3$ where two of the d_k are 1 and one is 0. Because L_1 and L_2 achieve the linear extension diameter of 3^3, we know from Theorem 3.6 that each of these corner sets must contribute exactly 1 reversal. But this means that the two atoms must appear in a different order in L_2. Hence, the atoms are all reversed in L_2.

As a detailed example, consider when $D = 101$. Then,

$$C_{101,000} = \{(000, 101), (100, 001), (001, 100), (101, 000)\}.$$

We know that $(000, 101)$ and $(101, 000)$ cannot be reversals because their order is fixed by the poset structure. We also know that $(001, 100)$ can't be a reversal because $001 > 100$ in L_1. So $(100, 001)$ is a reversal and that means $001 < 100$ in L_2. Continuing this process with the other possibilities for D shows that we must have $001 < 010 < 100$ in L_2.

Then, similar to what we did for L_1, we can consider the sub-grid where $f(1) = j \in \{0, 1, 2\}$

to see that in L_2 we have

$$000 < 001 < 002 < 010 < 011 < 012 < 020 < 021 < 022,$$

$$100 < 101 < 102 < 110 < 111 < 112 < 120 < 121 < 122,$$

and

$$200 < 201 < 202 < 210 < 211 < 212 < 220 < 221 < 222.$$

Finally, consider the corner set where $D = 221$ and $I = 000$. For reference,

$$C_{221,000} = \{(000, 221), (200, 021), (020, 201), (220, 001),$$

$$(001, 220), (201, 020), (021, 200), (221, 000)\}.$$

This $C_{D,I}$ must contribute 2 reversals. Because of the structure of the poset, $(000, 221)$ and $(221, 000)$ cannot be reversals. We also know that $020 < 001$ in L_1 by restricting the first entry to 0, and so $020 < 001 < 201$ in L_1. Similarly, $020 < 200$ in L_2 so $020 < 200 < 201$ in L_2. Hence, $(020, 201)$ and $(201, 020)$ cannot be reversals.

Because $001 < 010 < 220$ in L_2, we have $(220, 001)$ is not a reversal. Because $200 < 020 < 021$ in L_1, we have $(021, 200)$ is not a reversal. Hence, both $(200, 021)$ and $(220, 001)$ must be reversals. So in particular, we have $220 < 001$ in L_1. We can use similar methods to show $221 < 002$ in L_1, and $022 < 100$ and $122 < 200$ in L_2.

The proof of the following theorem for the n-dimensional ternary lattice is similar to this example.

Theorem 3.9. *If L_1, L_2 is a diametral pair of linear extensions of 3^n and σ is the order of the atoms in L, then $L_1 = L_\sigma$ and $L_2 = L_{\text{rev}(\sigma)}$.*

Proof. We'll induct on n. For $n = 1$, we have only one linear extension, to which L_1 and L_2 must both be equal.

For $n > 1$, first consider $C_{D,I}$ where $I = \bar{0}$ and D has two nonzero entries, say $d_a = d_b = 1$. Call the atom with a 1 in index a v_a and the atom with a 1 in index b v_b. Without loss of generality, suppose $a <_\sigma b$. Because $|D| = 2$, this corner set must contribute 1 reversal, but $(\bar{0}, D)$ and $(D, \bar{0})$ aren't reversals, and the ordering of the atoms in L_1 tells us that (v_b, v_a) can't be a reversal. So (v_a, v_b) is a reversal, and we have that $v_b < v_a$ in L_2. Hence, all of the atoms appear in reversed order in L_2.

Fix $f(\sigma_n) = i$ where $0 \leq i \leq 2$. By the lemma, the other entries of L_1 must be in σ-lex order. Similarly, fixing $f(\sigma_1) = j \in \{0, 1, 2\}$ shows that the other entries of L_2 are in rev(σ)-lex order. To conclude that $L_1 = L_\sigma$ and $L_2 = L_{\mathrm{rev}(\sigma)}$, we need to show that for all $0 < i \leq 2$, we have

$$(\overline{2})(i-1)(\overline{2}) < \bar{0}i\bar{0}$$

in L_1, where the i is in position σ_n, and

$$(\overline{2})(j-1)(\overline{2}) < \bar{0}j\bar{0}$$

in L_2, where the j is in position σ_1.

Now, consider C_{D,I_i} where $D = \overline{2}(1)\overline{2}$ and $I_i = \bar{0}(i-1)\bar{0}$ (the 1 of D and the nonzero entry of I_i are in position σ_n). For any $X \in 3^n$ with $x_l \in \{0, 2\}$ for $l \neq \sigma_1$ or σ_n, and $x_{\sigma_1} = 0 = x_{\sigma_n}$, we can consider fixing $f(\sigma_1) = 0$. Then by induction we have $X + I_i < \bar{0}i\bar{0}$ in L_1 where i is in position σ_n, and so $X + I_i < X^c + I_i$ in L_1, where the complement is with respect to D, i.e. $X^c = D - X$.

For this same X, we can also show that $X + I_i < X^c + I_i$ in L_2 by the following. By induction and fixing $f(\sigma_n) = 0$, we have that $X + I_i < \bar{0}1\bar{0}$ in L_2 where the 1 is in position

σ_1. Therefore, $X + I_i < X^c + I_i$ in L_2 and $(X + I_i, X^c + I_i)$ cannot be a reversal between L_1 and L_2.

This also shows that $(X^c + I_i, X + I_i)$ cannot be a reversal for any such X. Our corner set $C_{D,I}$ contains 2^n ordered pairs and we have just shown that 2^{n-2} cannot be reversals.

For each remaining X with $x_i \in \{0, d_i\}$, the pairs $(X + I_i, X^c + I_i)$ and $(X^c + I_i, X + I_i)$ cannot both be reversals as either $X + I_i < X^c + I_i$ in L_1 or vice-versa. In fact, exactly half of these must be reversals, since we need 2^{n-2} reversals to appear in this $C_{D,I}$. In particular, since $\bar{0}i\bar{0} < \bar{2}(i-1)\bar{2}$ in L_2 (with i in position σ_n), we must have that $(\overline{2}(i-1)\overline{2}, \bar{0}i\bar{0})$ is a reversal. Hence, $\overline{2}(i-1)\overline{2} < \bar{0}i\bar{0}$ in L_1 for each i.

Using similar methods, we can show that we must have $\overline{2}(j-1)\overline{2} < \bar{0}j\bar{0}$ in L_2 (where j is in position σ_1). □

3.2 General Grids

In this section, we calculate the linear extension diameter of general grids, where each dimension can have size larger than 3, and the dimensions may all differ in sizes. Many of the proofs will be similar to those of the ternary lattice, except for Theorem 3.18 which was much harder to generalize. Surprisingly, the generalization we eventually developed gives a much easier proof of Theorem 3.9 and only becomes complicated when m_{σ_1} or m_{σ_n} equals 2.

In this section, we also change from using the vectors D and I to describe the symmetric difference and intersection between vectors in the corner sets. Instead, it will be more straightforward to consider these sets as intervals $[B, T]$ of the poset. The transition between these notations is easy: $B = I$ and $T = I + D$.

Further, in Section 3.2.3, we prove that grids have the additional property of being diametrally reversing by characterizing the critical pairs.

3.2.1 Calculating the Diameter

We will first define a partition of P^2. We will use this partition to show that the pairs of linear extensions $\{(L_\sigma, L_{\text{rev}(\sigma)})\}$ are exactly the diametral pairs of the linear extension graph of the grid. Then, we will calculate the distance between them.

Definition 3.10. Let $B, T \in P$ with $B \leq T$. The *corner set* with bottom B and top T is the set of pairs $C_{B,T} = \{(X, Y) \in P^2 : X \wedge Y = B \text{ and } X \vee Y = T\}$. In other words, $C_{B,T}$ is the set of ordered pairs of opposite corners of the "sub-cube" with minimum element B and maximum element T. This sub-cube is $P[f]$ where $f(k) = \{b_k, t_k\}$ (see Definition 2.17).

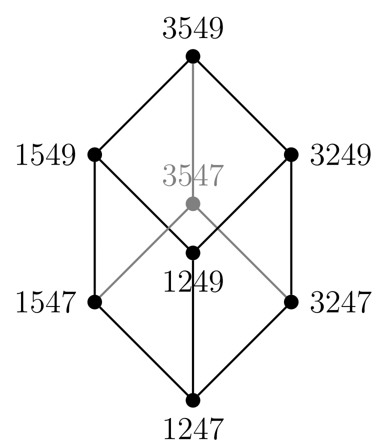

Figure 3.3: The "sub-cube" $C_{1247,3549}$ in 10^4.

Note that $|C_{B,T}| = 2^{d(B,T)}$. (Recall that $d(B,T) := |\{k : b_k \neq t_k\}|$.) Also,

$$\{C_{B,T} : B, T \in P, B \leq T\}$$

is a partition of P^2 since $(X, Y) \in C_{B,T}$ if and only if $B = X \wedge Y$ and $T = X \vee Y$.

For any two linear extensions L_1 and L_2, the set $\{C_{B,T} \cap R(L_1, L_2) : B, T \in P, B \leq T\}$ is a partition of the reversals between them. We will now show that when $L_1 = L_\sigma$ and $L_2 = L_{\text{rev}(\sigma)}$, the sets $C_{B,T} \cap R(L_1, L_2)$ have a predictable cardinality.

Lemma 3.11. *Let $B, T \in P$ with $B \leq T$, and let σ be a linear ordering of $[n]$. Then,*

$$|C_{B,T} \cap R(L_\sigma, L_{\mathrm{rev}(\sigma)})| = \begin{cases} 0 & \text{if } d(B,T) \leq 1 \\ 2^{d(B,T)-2} & \text{otherwise} \end{cases} .$$

Proof. Fix $d = d(B,T)$. If $d = 0$, then $B = T$ and our sub-cube is simply a point, so there can be no reversals. If $d = 1$, then our sub-cube is a chain with two elements, and the ordering is forced for all linear extensions, so once again there can be no reversals.

Now suppose $d \geq 2$. Let i be the σ-most important index where B and T differ, and similarly j the σ-least important. Because B and T differ in at least 2 indices, we have that $i \neq j$. Also, we see that $X <_\sigma Y$ if and only if $x_i < y_i$, and $X <_{\mathrm{rev}(\sigma)} Y$ if and only if $x_j < y_j$. So a pair (X, Y) is a reversal exactly if $x_i < y_i$ and $x_j > y_j$. Therefore, two of our choices are fixed but the other $d-2$ are free, and we have 2^{d-2} reversals in each $C_{B,T}$ with $d \geq 2$. □

In fact, we will show that for any linear extensions L and L' and any $B \leq T$ with $B, T \in P$, the corner set $C_{B,T}$ cannot intersect $R(L, L')$ in a larger set than this. Therefore, L_σ and $L_{\mathrm{rev}(\sigma)}$ must be a diametral pair. To do so, we will need the following theorem.

Theorem 3.12 (Daykin's Extension of Kleitman's Lemma [2])**.** *If $\mathcal{P}$ is a distributive lattice and $A, B \subset \mathcal{P}$, then*

$$|A||B| \leq |A \vee B||A \wedge B|,$$

where $A \vee B = \{a \vee b : a \in A, b \in B\}$ and $A \wedge B = \{a \wedge b : a \in A, b \in B\}$. In particular, if $A, B \subseteq P = P(m_1, \ldots, m_n)$ are downsets, then

$$|A||B| \leq |P||A \cap B|$$

since $A \wedge B = A \cap B$.

Lemma 3.13. *Let L_1 and L_2 be two linear extensions of P. For every $B, T \in P$ with $B < T$ and $d(B,T) \geq 2$,*

$$|C_{B,T} \cap R(L_1, L_2)| \leq 2^{d(B,T)-2}.$$

In addition, $C_{B,T}$ cannot contain any reversals when $d(B,T) < 2$.

Proof. As in the proof of Lemma 3.11, the claim is immediate when $d(B,T) < 2$.

Suppose $d(B,T) = d \geq 2$, and fix T. Since $C_{B,T} = B + C_{\bar{0},T-B}$, we may assume that $B = \bar{0}$ without loss of generality. So now, $C_{B,T} = \{(X, T-X) : x_k = 0 \text{ or } x_k = t_k\}$. Following [7], we say X is "T-down" in a linear extension if $X < T - X$ in the extension. Let $\mathcal{F}_1$ be the family of such X which are T-down in L_1, and $\mathcal{F}_2$ those that are T-down in L_2. Then $(X, T-X) \in C_{B,T}$ is a reversal between L_1 and L_2 exactly if X is T-down in L_1 but not in L_2. Since X T-down implies that $T - X$ is not, we have that $(X, T-X)$ is a reversal if and only if $X \in \mathcal{F}_1 \setminus \mathcal{F}_2$ and $T - X \in \mathcal{F}_2 \setminus \mathcal{F}_1$. So, we want to find an upper bound for $|\mathcal{F}_1 \triangle \mathcal{F}_2|$, which will be twice the number of reversals.

Note that these are downsets, because of transitivity. So, Theorem 3.12 shows that $|\mathcal{F}_1||\mathcal{F}_2| \leq 2^d |\mathcal{F}_1 \cap \mathcal{F}_2|$. Now, for all L_i and X, either X or $T - X$ is T-down in L_i. Hence, $|\mathcal{F}_1| = |\mathcal{F}_2| = 2^{d-1}$. So, $|\mathcal{F}_1 \cap \mathcal{F}_2| \geq 2^{d-2}$ by the previous inequality. Also, if X is T-down in L_1 and L_2, then $T - X$ is T-down in neither, i.e. $X \in \mathcal{F}_1 \cap \mathcal{F}_2$ if and only if $T - X \notin \mathcal{F}_1 \cup \mathcal{F}_2$. Therefore, $|\mathcal{F}_1 \cap \mathcal{F}_2| = |\{T - X : X \in \mathcal{F}_1 \cup \mathcal{F}_2\}|$.

Because there are 2^d choices for X, we have

$$|\mathcal{F}_1 \triangle \mathcal{F}_2| = 2^d - |\mathcal{F}_1 \cap \mathcal{F}_2| - |\{T - X : X \in \mathcal{F}_1 \cup \mathcal{F}_2\}| \leq 2^d - 2^{d-2} - 2^{d-2} = 2^{d-1}.$$

As noted previously, each reversal $(X, T-X)$ counts doubly towards the cardinality of

$|\mathcal{F}_1 \cap \mathcal{F}_2|$. Hence,

$$|C_{B,T} \cap R(L_1, L_2)| = \frac{1}{2} |\mathcal{F}_1 \cap \mathcal{F}_2| \leq 2^{d(B,T)-2}. \qquad \square$$

Corollary 3.14. *L_1 and L_2 are diametral pairs of $G(P)$ if and only if $|C_{B,T} \cap R(L_1, L_2)| = 2^{d(B,T)-2}$ for all $B, T \in P$ with $B \leq T$ and $d(B,T) \geq 2$.*

Proof. We have just shown that this is the most number of reversals a corner set could contain, and from Lemma 3.11 we know that this number is achievable. $\square$

Corollary 3.15. *Let σ be a linear ordering of $[n]$. Then, L_σ and $L_{\mathrm{rev}(\sigma)}$ are a diametral pair for P.*

Proof. From Lemma 3.11, we know $|C_{B,T} \cap R(L_\sigma, L_{\mathrm{rev}(\sigma)})| = 2^{d(B,T)-2}$ for all $B < T$ with $d(B,T) \geq 2$. Applying Corollary 3.14, we see that L_σ and $L_{\mathrm{rev}(\sigma)}$ must be as far apart as possible. $\square$

We have just shown that for each σ, L_σ and $L_{\mathrm{rev}(\sigma)}$ are a diametral pair of $G(P)$, and so we can now count the number of reversals between these to calculate the linear extension diameter of P.

Theorem 3.16. *The linear extension diameter of the grid $P(m_1, \ldots, m_n)$ is*

$$\frac{1}{4} \left(\prod_k m_k \right) \left(\prod_k m_k - \sum_k m_k + n - 1 \right).$$

Proof. From Corollary 3.14, we know that

$$|C_{B,T} \cap R(L_\sigma, L_{\mathrm{rev}(\sigma)})| = 2^{d(B,T)-2}$$

for all $B < T$ with $d(B,T) \geq 2$. Let's count these pairs (B,T) with $\{i : b_i \neq t_i\} = D$. We need to pick a pair of elements from $\{0, 1, \ldots, m_i - 1\}$ for each $i \in D$, and a single element from $\{0, 1, \ldots, m_i - 1\}$ for each $i \notin D$. Therefore, there are

$$\sum_{\substack{D \subseteq [n] \\ |D| = d}} \left[\prod_{k \notin D} m_k \cdot \prod_{k \in D} \binom{m_k}{2} \right]$$

possible pairs for each fixed $d \geq 2$.

Then, for each of these pairs with difference d, we have 2^{d-2} reversals, from Lemma 3.11. Summing over all $d \geq 2$, we get

$$\begin{aligned}
\operatorname{led}(P) &= \sum_{d=2}^{n} 2^{d-2} \sum_{\substack{D \subseteq [n] \\ |D| = d}} \left[\prod_{k \notin D} m_k \cdot \prod_{k \in D} \binom{m_k}{2} \right] \\
&= \sum_{d=2}^{n} 2^{d-2} \frac{1}{2^d} \sum_{\substack{D \subseteq [n] \\ |D| = d}} \left[\prod_{k \in [n]} m_k \cdot \prod_{k \in D} (m_k - 1) \right] \\
&= \frac{1}{4} \left(\prod_{k \in [n]} m_k \right) \sum_{d=2}^{n} \sum_{\substack{D \subseteq [n] \\ |D| = d}} \left[\prod_{k \in D} (m_k - 1) \right] \\
&= \frac{1}{4} \left(\prod_{k \in [n]} m_k \right) \left(\prod_{k \in [n]} (1 + (m_k - 1)) - \sum_{k \in [n]} (m_k - 1) - 1 \right) \\
&= \frac{1}{4} \left(\prod_{k \in [n]} m_k \right) \left(\prod_{k \in [n]} m_k - \sum_{k \in [n]} m_k + n - 1 \right) \qquad \square
\end{aligned}$$

3.2.2 Diametral Pairs of $G(P)$

For a fixed permutation σ, we have shown that L_σ and $L_{\mathrm{rev}(\sigma)}$ form a diametral pair of P. This gives us $n!/2$ diametral pairs, and we will show this is all of them. To do so, we must expand to looking at all sub-grids of P, rather than simply the sub-cubes. Recall from the notation section, that $P[f]$ and $L[f]$ are the sub-grid and linear extension, respectively, generated by the restriction function f.

Lemma 3.17. *Let $f : [n] \to \mathcal{P}(\mathbb{N}_0)$ such that $f(k) \subseteq \{0, 1, 2, \dots, m_k - 1\}$. If L_1 and L_2 are a diametral pair of $G(P)$, then $L_1[f]$ and $L_2[f]$ are a diametral pair of the linear extension graph of $P[f]$.*

Proof. We need to show

$$|C_{B,T}[f] \cap R(L_1[f], L_2[f])| = \begin{cases} 0 & \text{if } d(B,T) \leq 1 \\ 2^{d(B,T)-2} & \text{otherwise} \end{cases}$$

for each $B, T \in P[f]$ with $B \leq T$.

Fix such a B and T, and let $d := d(B,T)$. Notice that $C_{B,T} = C_{B,T}[f]$ because $B, T \in P[f]$ implies $b_k, t_k \in f(k)$ for all k, and so any vector in $C_{B,T}$ has b_k or t_k as each of its entries. Therefore, each vector that appears as part of a pair in $C_{B,T} \cap R(L_1, L_2)$ is still in our sub-grid $P[f]$. We also know that $|C_{B,T} \cap R(L_1, L_2)| = 2^{d-2}$ if $d \geq 2$ and 0 otherwise, because L_1 and L_2 are a diametral pair. Moreover, the linear extension restriction keeps the restricted elements in the same order, so any pair in $C_{B,T} \cap R(L_1, L_2)$ is also a reversal between $L_1[f]$ and $L_2[f]$. Hence, $|C_{B,T}[f] \cap R(L_1[f], L_2[f])| = |C_{B,T} \cap R(L_1, L_2)|$. □

The previous lemma allows us to use induction to show that as each dimension of a grid is built up, the diametral linear extensions maintain the same structure. In other words, for a diametral pair (L_1, L_2) to remain a diametral pair under any restriction, as the lemma requires, we must have that L_1 is the σ-lex ordering for some σ, and L_2 is its counterpart $L_{\text{rev}(\sigma)}$.

Theorem 3.18. *If L_1 and L_2 are a diametral pair of linear extensions of P and σ is the order of the atoms in L_1, then $L_1 = L_\sigma$ and $L_2 = L_{\text{rev}(\sigma)}$.*

Proof. From Corollary 3.14, we know $|C_{B,T} \cap R(L_1, L_2)| = 2^{d(B,T)-2}$ for $d(B,T) \geq 2$. We can also assume $m_k > 1$ for all k, because the cases where some $m_k = 1$ are isomorphic to lattices with a smaller dimension that have no $m_k = 1$.

We proceed by induction on $M := \sum_{k=1}^{n}(m_k - 1)$. When $M = 1$, there is only the trivial linear extension, to which both L_1 and L_2 must be equal. When $M > 1$, let σ_n be the σ-most important index (the position where the 1 occurs in the atom that is greatest in σ). If $m_{\sigma_n} > 2$, then we can piece together two grid restrictions. Let

$$f_1(k) := \begin{cases} \{0,1\} & \text{if } k = \sigma_n \\ \{0,1,\ldots,m_k - 1\} & \text{otherwise} \end{cases}$$

and

$$g_1(k) := \begin{cases} \{1,2,\ldots m_k - 1\} & \text{if } k = \sigma_n \\ \{0,1,\ldots,m_k - 1\} & \text{otherwise} \end{cases}.$$

By induction we have that $L_1[f_1]$ and $L_1[g_1]$ are in σ-lex order. Because the atom with a 1 in position σ_n is in $P[f_1] \cap P[g_1]$, the result that $L_1 = L_\sigma$ follows from transitivity.

Similarly, let σ_1 be the σ-least important index (which is also the rev(σ)-most important). If $m_{\sigma_1} > 2$, then we can piece together two grid restrictions once again to show that $L_2 = L_{\text{rev}(\sigma)}$. Let

$$f_2(k) := \begin{cases} \{0,1\} & \text{if } k = \sigma_1 \\ \{0,1,\ldots,m_k - 1\} & \text{otherwise} \end{cases}$$

and

$$g_2(k) := \begin{cases} \{1,2,\ldots m_k - 1\} & \text{if } k = \sigma_1 \\ \{0,1,\ldots,m_k - 1\} & \text{otherwise} \end{cases}.$$

By induction we have that $L_2[f_2]$ and $L_2[g_2]$ are in rev(σ)-lex order. Because the atom with a 1 in position σ_1 is in $P[f_2] \cap P[g_2]$, the result that L_2 is in rev(σ)-lex order follows from transitivity.

We must do a little more work when $m_{\sigma_1} = 2$ or $m_{\sigma_n} = 2$. Let us first tackle the case where $m_{\sigma_n} = 2$. By considering the sub-grids with the restriction functions

$$f_3(k) := \begin{cases} \{0\} & \text{if } k = \sigma_n \\ \{0, 1, \ldots, m_k - 1\} & \text{otherwise} \end{cases}$$

and

$$g_3(k) := \begin{cases} \{1\} & \text{if } k = \sigma_n \\ \{0, 1, \ldots, m_k - 1\} & \text{otherwise} \end{cases},$$

in essence decreasing the dimension by 1, we know that elements with the σ_n index fixed are in σ-lex order in L_1 and rev(σ)-lex order in L_2. We only need to show that the maximum element of $L_1[f_3]$ is less than the minimum element of $L_1[g_3]$, and again we will be done by transitivity. For simplicity, we can assume $\sigma_1 = 1$ and $\sigma_n = n$. So we would like to show that

$$(m_1 - 1) \ldots (m_{n-1} - 1)0 < 0 \ldots 01$$

in L_1. We will follow a similar proof to that of Felsner and Massow from [7].

First, we will show that the atoms are in rev(σ)-lex order in L_2. Note that they are in the right order in L_1 by definition. Let X and Y both be atoms and assume $X < Y$ in L_1. Consider $C_{B,T}$ where $T = X + Y$ and $B = \bar{0}$. Since $d(B, T) = 2$, this particular $C_{B,T}$ must contain exactly one reversal between L_1 and L_2. But

$$C_{B,T} = \{(0, T), (X, Y), (Y, X), (T, 0)\}.$$

Clearly, $(0, T)$ and $(T, 0)$ cannot be reversals, and $Y < X$ in L_1 implies that (Y, X) cannot be a reversal either. Hence, (X, Y) must be a reversal. Therefore, $Y < X$ in L_2, and it follows that all the atoms in L_2 are in rev(σ)-lex order.

Now consider $C_{B,T}$ where $T = (m_k - 1)_{k=1}^{n}$ and $B = \bar{0}$. We will show that $(X,Y) \in C_{B,T}$ cannot be a reversal if $x_1 = x_n = 0$, or similarly if $y_1 = y_n = 0$. Suppose the first were true, so σ-lexicographically we have $X < Y$. In P, we have $X < X + 10\dots0$, and by the function restriction

$$\widehat{f}(k) := \begin{cases} \{1, 2, \dots m_k - 1\} & \text{if } k = 1 \\ \{0, 1, \dots, m_k - 1\} & \text{otherwise} \end{cases}$$

and induction, we have $X + 10\dots0 < Y$ in L_1. Similarly, in L_2, we have $X < X + 0\dots01 < Y$. Therefore neither (X,Y) nor (Y,X) can be a reversal between L_1 and L_2.

We give a counting argument to show that all the rest must be reversals or "flipped" reversals, i.e. (Y,X) where (X,Y) is a reversal. We know $C_{B,T}$ must contain $2^{d(B,T)-2}$ reversals, and that there are $2^{d(B,T)}$ pairs in $C_{B,T}$. We have just shown that $2^{d(B,T)-1}$ pairs cannot be reversals. So the other $2^{d(B,T)} - 2^{d(B,T)-1} = 2^{d(B,T)-1}$ pairs must all be reversals or flipped reversals. In particular, $(m_1 - 1)\dots(m_{n-1} - 1)0$ and $0\dots01$ must be a reversal or flipped reversal. But in L_2 we have already shown that $0\dots01 < 10\dots0$, so

$$0\dots01 < (m_1 - 1)\dots(m_{n-1} - 1)0$$

in L_2 by transitivity. Hence, we must have

$$(m_1 - 1)\dots(m_{n-1} - 1)0 < 0\dots01$$

in L_1.

Similarly, suppose that $m_{\sigma_1} = 2$. This time, consider the sub-grids with the restriction functions

$$f_4(k) := \begin{cases} \{0\} & \text{if } k = \sigma_1 \\ \{0, 1, \dots, m_k - 1\} & \text{otherwise} \end{cases}$$

and

$$g_4(k) := \begin{cases} \{1\} & \text{if } k = \sigma_1 \\ \{0, 1, \ldots, m_k - 1\} & \text{otherwise} \end{cases}.$$

We know that elements with the σ_1 index fixed are in rev(σ)-lex order in L_2 and σ-lex order in L_1. We only need to show that the maximum element of $L_2[f_4]$ is less than the minimum element of $L_2[g_4]$, and once more we will be done by transitivity. Again assuming $\sigma_1 = 1$ and $\sigma_n = n$, we need to show that

$$0(m_2 - 1) \ldots (m_n - 1) < 10 \ldots 0$$

in L_2.

However, we still know that for $T = (m_k - 1)_{k=1}^n$ and $B = \bar{0}$, any $(X, Y) \in C_{B,T}$ with $x_1 = x_n = 0$ or $y_1 = y_n = 0$ cannot be a reversal between L_1 and L_2. Therefore, by the counting argument given above, we must have that $0(m_2 - 1) \ldots (m_n - 1)$ and $10 \ldots 0$ are a reversal. But by assumption we have that $10 \ldots 0 < 0 \ldots 01$ in L_1 and so by transitivity,

$$10 \ldots 0 < 0(m_2 - 1) \ldots (m_n - 1).$$

Therefore, we must have

$$0(m_2 - 1) \ldots (m_n - 1) < 10 \ldots 0$$

in L_2 and by transitivity, L_2 is in rev(σ)-lex order. □

3.2.3 Critical Pairs

In the study of linear extensions of partially ordered sets, an important concept that appears frequently is critical pairs. Two elements form a critical pair if they have a special relationship, described precisely below; critical pairs are important in dimension theory because to check if a set of linear extensions forms a realizer (i.e. generates the poset), we only need to check that every critical pair is reversed in some linear extension in the set.

Definition 3.19. A *critical pair* of a poset $\mathcal{P}$ is an ordered pair (x, y) of incomparable elements of $\mathcal{P}$ such that

1. $a < x$ implies $a < y$, and

2. $b > y$ implies $b > x$.

Equivalently, (x, y) is a critical pair if the addition of $x < y$ to the relations of $\mathcal{P}$ does not transitively force any other additional relation. A third way to characterize critical pairs is that $y < x$ cannot be transitively forced by adding any other relation (besides $y < x$).

If (x, y) is a critical pair of $\mathcal{P}$, then we say that a linear extension L of $\mathcal{P}$ *reverses* (x, y) if $y < x$ in L. Note that then nothing can appear between y and x in L. A poset is *diametrally reversing* if every linear extension contained in a diametral pair reverses a critical pair. In [7], Felsner and Massow showed that Boolean lattices are diametrally reversing, and we will now extend their result to grids.

Corollary 3.20. *Grids are diametrally reversing.*

Proof. Let L be a linear extension of P that is contained in a diametral pair of $G(P)$. By Theorem 3.18, we know that $L = L_\sigma$ for some ordering σ of $[n]$.

Let σ_n be the σ-most important index. Then consider $X := (x_k)_1^n$ where $x_k = 0$ for all $k \neq \sigma_n$ and $x_{\sigma_n} = 1$, and $Y := (y_k)_1^n$ where $y_k = m_k - 1$ for all $k \neq \sigma_n$ and $y_{\sigma_n} = 0$. It's clear that (X, Y) is a critical pair of P. In L_σ, we have $X > Y$, and so this is actually a reversed critical pair. □

Remark 3.21. We can easily extend this proof to show there are at least $m_{\sigma_n} - 1$ reversed critical pairs between L_σ and $L_{\mathrm{rev}(\sigma)}$. For each $i \in [m_{\sigma_n} - 1]$, we can let $x_{\sigma_n} = i$ and $y_{\sigma_n} = i - 1$, and keep $x_k = 0$ and $y_k = m_k - 1$ for every $k \neq \sigma_n$. Then $(x_k)_1^n$ and $(y_k)_1^n$ are also reversed.

In the following proposition, we characterize all of the critical pairs of the grid. From the previous Remark, we can then conclude that there are exactly $m_{\sigma_n} - 1$ reversed critical pairs between L_σ and $L_{\mathrm{rev}(\sigma)}$.

Proposition 3.22. *Let $X = (x_k)_1^n$ and $Y = (y_k)_1^n$. If (X, Y) is a critical pair of P, then there exists k_0 such that $x_{k_0} - y_{k_0} = 1$, and $x_k = 0$ and $y_k = m_k - 1$ for all $k \neq k_0$.*

Proof. Suppose (X, Y) is a critical pair. Because X and Y are incomparable by definition, there must be an index, say k_0, where $x_{k_0} > y_{k_0}$. If $x_{k_0} - y_{k_0} > 1$, then the element $X - 0 \ldots 1 \ldots 0$, where we subtract the atom with a 1 in index k_0, would be less than X but not less than Y, which contradicts the definition of a critical pair. So we must have $x_{k_0} - y_{k_0} = 1$.

Similarly, suppose $x_{k_1} \neq 0$ for some $k_1 \neq k_0$. Then we could decrease index k_1 in X by 1, and we would have an element that is less than X, but this element is not less than Y, because index k_0 is still larger than in Y. So we must have $x_k = 0$ for all $k \neq k_0$.

Finally, suppose $y_{k_1} < m_{k_1} - 1$ for some $k_1 \neq k_0$. Then the element $Y + 0 \ldots 1 \ldots 0$ where we add the atom with a 1 in index k_1 is greater than Y but not greater than X. Hence, $y_k = m_k - 1$ for all $k \neq k_0$. □

3.3 Poset of Partially Defined Functions

In this section, we consider a different generalization of the Boolean lattice, called the Poset of Partially Defined Functions. Although we don't quite have results for this poset yet, we have several promising conjectures. They seem relevant to include here, as they build off our results for the grid.

Definition 3.23. $F_{n,k}$ is the poset of partial functions from $[n] \to \Sigma$ where Σ is a set of (unrelated) elements of cardinality k. (We use $*$ to represent a place where the function is not defined.) We say $f \leq g$ iff $f(i) = g(i)$ for each $i \in [n]$ where f is defined.

This is a generalization of the Boolean lattice because $F_{n,1} \cong 2^n$. Figure 3.4 shows $F_{2,2}$ where $\Sigma = \{a, b\}$.

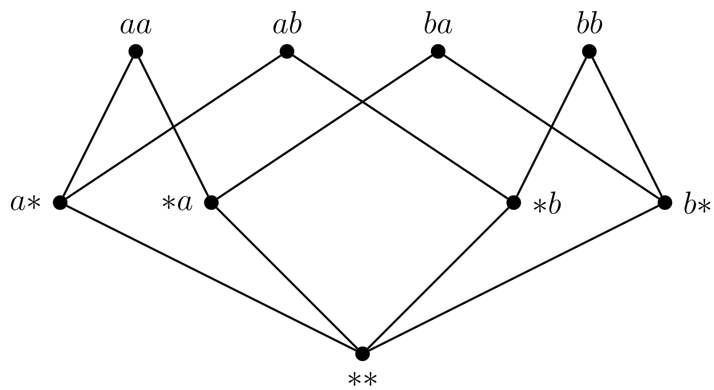

Figure 3.4: $F_{2,2}$ where $\Sigma = \{a, b\}$.

What do the diametral pairs of linear extensions of this poset look like? For small n and k, preliminary results using SAGE suggest that the diametral pairs are also built from atomic orderings. The poset $F_{n,k}$ has $n \cdot k$ atoms, but not all of these atoms can be combined together. For instance, in $F_{2,2}$ we cannot combine the atoms $a*$ and $b*$, although we can combine $a*$ with $*a$ to get aa, or $a*$ with $*b$ to get ab. We denote a linear extension that is the lexicographic extension of the atomic ordering α by L_α.

Example 3.24. For $F_{2,2}$, let α be the atomic ordering $a* < *a < *b < b*$. Then L_α is the linear extension

$$** < a* < *a < aa < *b < ab < b* < ba < bb.$$

Due to the restrictions of how we can combine atoms, not all atomic orderings lead to diametral pairs. The atomic ordering must satisfy a particular pattern to ensure that the atoms are intertwined in as complex a manner as possible. For $k = 2$, we have a conjecture as to what this pattern is; for larger k, it is unclear at this point what makes the atomic orderings that work different from those that don't.

Conjecture 3.25. *Let $\alpha = \alpha_1 < \alpha_2 < \cdots < \alpha_{2n}$ be an atomic ordering of $F_{n,2}$. Then L_α and $L_{\mathrm{rev}(\alpha)}$ are a diametral pair of linear extensions of $F_{n,2}$ if α satisfies the following pattern:*

1. *α_1 and α_{2n} have the same defined element (i.e. the same non-$*$ index),*
2. *α_2 and α_{2n-1} have the same defined element,*

   $\vdots$

n. *α_n and α_{n+1} have the same defined element.*

We say α is a diametral atomic ordering *if L_α and $L_{\mathrm{rev}(\alpha)}$ form a diametral pair of linear extensions.*

Example 3.26. One diametral atomic ordering for $F_{2,2}$ is $a* < *a < *b < b*$. One diametral atomic ordering for $F_{3,2}$ is

$$a{*}{*} < {*}a{*} < {*}{*}a < {*}{*}b < {*}b{*} < b{*}{*}.$$

Another is the atomic ordering

$${*}b{*} < {*}{*}a < a{*}{*} < b{*}{*} < {*}{*}b < {*}a{*}\,.$$

For $k > 2$, the pattern is unclear. Results from SAGE show that diametral atomic orderings for $F_{2,3}$ range from

$$a* < b* < *a < *b < *c < c*$$

to

$$*c < b* < *b < a* < c* < *a.$$

Chapter 4

Weak Discrepancy

Weak discrepancy is the usual mechanism for studying the question, "what is a good weak labeling?" We consider labelings with low discrepancies to be more fair, and we wish to calculate the discrepancy of the most fair labeling that exists. Recall Definition 2.21:

Definition 4.1. The *weak discrepancy* of a poset, $\mathrm{wd}(P)$, is the minimum over all weak labelings of the discrepancy of the labeling, i.e.

$$\mathrm{wd}(P) := \min_{\substack{L \\ \text{weak labeling}}} \Delta(L) = \min_{\substack{L \\ \text{weak labeling}}} \max_{X \,||\, Y} |L(X) - L(Y)|.$$

In this Chapter, we calculate the weak discrepancy of grids, the permutohedron, the partition lattice, and the two-dimensional Young's lattice.

4.1 Grids

For this Section, let $P := P(m_1, m_2, \ldots, m_n)$ where the m_i are nondecreasing, i.e. $m_1 \leq m_2 \leq \ldots \leq m_n$. Also, without loss of generality, assume $m_i \geq 2$ for all i. Set $T := \sum_{i=1}^{n}(m_i - 1)$

and $\widehat{T} := \sum_{i=2}^{n}(m_i - 1) = T - m_1 + 1$.

Lemma 4.2. *If L is a weak labeling of P, and $x_1 \ldots x_n \leq y_1 \ldots y_n$ in P, then $L(y_1 \ldots y_n) \geq L(x_1 \ldots x_n) + \sum_{i=1}^{n}(y_i - x_i)$.*

Proof. Consider the chain

$$x_1 \ldots x_n < (x_1 + 1) \ldots x_n < \cdots < y_1 x_2 \ldots x_n < y_1(x_2 + 1) \ldots x_n < \cdots < y_1 \ldots y_n,$$

where each index increases by one until reaching the value of y_i.

The proof is then immediate from the fact that this chain has length $\sum_{i=1}^{n}(y_i - x_i)$, and each element must be assigned a different integer in the labeling. □

Using this simple lemma, we can now prove that the weak discrepancy of P is $\widehat{T} - \left\lfloor \frac{m_2}{m_1} \right\rfloor$. The optimal labeling that is produced coincides with the labeling in Figure 2.22; it is always the case that we want to "shift" (perhaps "stretch" is better) the first dimension to match the second.

In the proof, we begin by describing a weak labeling and calculating the discrepancy of this labeling. The labeling we describe is inspired by the example of labeling the grid 2×4 in Section 2.5. It is easy to see that this labeling has a lower discrepancy than the canonical labeling where $L(X) = \sum_{k=1}^{n} x_k$. However, it is unexpected that this labeling is the optimal one, as it is not intuitive at all.

To achieve the lower bound requires examining just the right constraints at just the right time. There are many relations, and non-relations, between elements of P, but many of them do not help us. For instance, you will see in the proof of the lower bound that we only need consider upper bounds on the labels, although it would be easy to also construct similar lower bounds.

Theorem 4.3. *The weak discrepancy of the grid $P(m_1, \ldots, m_n)$ is $\widehat{T} - \left\lfloor \frac{m_2}{m_1} \right\rfloor$, and this is achieved by the labeling $L(X) = \left\lfloor \frac{m_2}{m_1} \right\rfloor \cdot x_1 + \sum_{k=2}^{n} x_k$.*

Proof. To begin, we will show

$$\mathrm{wd}(m_1 \times \cdots \times m_n) \leq \widehat{T} - \left\lfloor \frac{m_2}{m_1} \right\rfloor$$

by showing that the given labeling achieves this discrepancy.

Consider the labeling $L(X) = \left\lfloor \frac{m_2}{m_1} \right\rfloor \cdot x_1 + \sum_{k=2}^{n} x_k$. It is clear that this is a weak labeling of the grid. To calculate $\Delta(L)$, the discrepancy of this labeling, we only need to consider $L(\overline{\Delta}_{i,\alpha}) - L(\Delta_{i,\alpha})$ for all possible choices of i and α, by Proposition 2.23.

Suppose first that $i = 1$. Then

$$L(\overline{\Delta}_{i,\alpha}) - L(\Delta_{i,\alpha}) = \left(\left\lfloor \frac{m_2}{m_1} \right\rfloor \cdot (\alpha - 1) + \sum_{k=2}^{n} (m_k - 1) \right) - \left\lfloor \frac{m_2}{m_1} \right\rfloor \cdot \alpha = \widehat{T} - \left\lfloor \frac{m_2}{m_1} \right\rfloor .$$

Now suppose $i \neq 1$. Then

$$L(\overline{\Delta}_{i,\alpha}) - L(\Delta_{i,\alpha}) = \left(\left\lfloor \frac{m_2}{m_1} \right\rfloor \cdot (m_1 - 1) + \sum_{k=2}^{n} (m_k - 1) - (m_i - 1) + (\alpha - 1) \right) - \alpha$$

$$= \left\lfloor \frac{m_2}{m_1} \right\rfloor \cdot (m_1 - 1) + \widehat{T} - m_i < \widehat{T} - \left\lfloor \frac{m_2}{m_1} \right\rfloor ,$$

because $\left\lfloor \frac{m_2}{m_1} \right\rfloor \cdot (m_1 - 1) < \left\lfloor \frac{m_2}{m_1} \right\rfloor \cdot m_1 \leq m_2 \leq m_i$.

Therefore, $\Delta(L) = \widehat{T} - \left\lfloor \frac{m_2}{m_1} \right\rfloor$.

Now, let L be any weak labeling of P with $\omega := \Delta(L)$, and without loss of generality, say

$L(010\ldots0) = 1$. By Lemma 4.2, we see that

$$L(0(m_2-1)\ldots(m_n-1)) \geq \sum_{i=2}^{n}(m_i-1) = \widehat{T}.$$

Because $10\ldots0 \,||\, 0(m_2-1)\ldots(m_n-1)$, we must then have

$$L(0(m_2-1)\ldots(m_n-1)) - L(10\ldots0) \leq \omega$$

and hence

$$L(10\ldots0) \geq \widehat{T} - \omega.$$

We can continue this process as follows:

The base case, $i = 1$, is given above. During stage i, we showed $L(i0\ldots0) \geq i(\widehat{T} - \omega)$.

Now apply Lemma 4.2 to $i0\ldots0$ and $i(m_2-1)\ldots(m_n-1)$ to see that

$$L(i(m_2-1)\ldots(m_n-1)) \geq (i+1)\widehat{T} - i\omega.$$

Because $(i+1)0\ldots0 \,||\, i(m_2-1)\ldots(m_n-1)$, we must then have

$$L((i+1)0\ldots0) \geq (i+1)(\widehat{T} - \omega).$$

We continue this process until $i = m_1 - 1$. During this last stage, we show that

$$L((m_1-1)0\ldots0) \geq (m_1-1)(\widehat{T} - \omega).$$

Notice that then

$$L((m_1-1)0(m_3-1)\dots(m_n-1)) \geq (m_1-1)(\widehat{T}-\omega)+\widehat{T}-m_2+1$$

and therefore

$$L((m_1-1)0(m_3-1)\dots(m_n-1)) - L(010\dots0) \geq (m_1-1)(\widehat{T}-\omega)+\widehat{T}-m_2.$$

But these elements are incomparable, so

$$\begin{aligned} L((m_1-1)0(m_3-1)\dots(m_n-1)) - L(010\dots0) &\leq \omega \\ (m_1-1)(\widehat{T}-\omega)+\widehat{T}-m_2 &\leq \omega \\ m_1\widehat{T}-m_2 &\leq m_1\omega \\ \widehat{T}-\left\lfloor\frac{m_2}{m_1}\right\rfloor &\leq \omega \end{aligned}$$

Hence, our initial labeling attains the best weak discrepancy that it is possible for us to achieve. □

Remark 4.4. The discrepant pairs for this optimal labeling form the set

$$\{(\Delta_{1,\alpha},\overline{\Delta}_{1,\alpha}) : 1 \leq \alpha \leq m_1-1\}.$$

Corollary 4.5. *Theorem 4.3 gives us the following, more specific results:*

1. $\mathrm{wd}(m \times m \times m_3 \times \cdots \times m_n) = T - m = \widehat{T}-1$

2. $\mathrm{wd}(m \times (m+1) \times m_3 \times \cdots \times m_n) = T - m = \widehat{T}-1$

3. $\mathrm{wd}(m^n) = mn - m - n$

4. $\mathrm{wd}(2^n) = n - 2$

4.2 Permutohedron

The permutohedron is a beautiful class of posets where the ground set is permutations of $[n]$. These structures are somewhat grid-like; we will see there is a nice bijection between the set of permutations of $[n]$ and elements of the grid $2 \times 3 \times \cdots \times n$. However, the permutohedron has far fewer edges than the grid, and so there is less structure with which to work. Surprisingly, we will see that enough chains remain to show that the permutohedron must have the same weak discrepancy as the grid.

Note that we will always write permutations in one-line notation, so that $\pi = 3124$ means $\pi(1) = 3$, $\pi(2) = 1$, $\pi(3) = 2$, and $\pi(4) = 4$. We will also often denote $\pi(i)$ as π_i.

Definition 4.6. Define a covering relation on permutations of $[n]$ by $\pi < \sigma$ if there exists an $i \in [n]$ such that

1. for all $j \neq i, i+1$, $\pi_j = \sigma_j$,

2. $\pi_i = \sigma_{i+1}$ and $\pi_{i+1} = \sigma_i$,

3. $\pi_{i+1} > \pi_i$.

The weak Bruhat order $<_B$ on permutations is the transitive closure of this covering relation.

For example, when $n = 3$, the above covering relation gives $213 < 231$, with $i = 2$. Figure 4.1 shows the Hasse diagram of the weak Bruhat order on permutations of [3]. Figure 4.2

shows the permutohedron of [4]; this particular Hasse diagram is why this poset is known as the permutohedron instead of simply the permutation lattice, as it is a truncated octahedron.

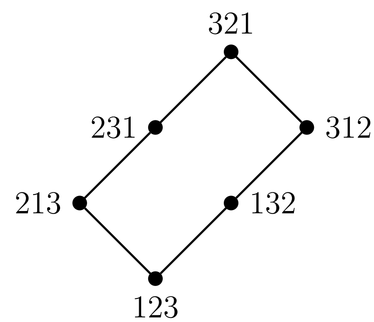

Figure 4.1: The permutohedron of [3].

A well-known fact about permutations is the following proposition.

Proposition 4.7. *The set of permutations of* $[n]$ *is in one-to-one correspondence, by an inversion count, with the set* $2 \times 3 \times \cdots \times n$*, which we will call the vector representation.*

Proof. Given a permutation π in one-line notation, we generate the vector representation, $x_1 \dots x_{n-1}$ of π by:

$$x_i = |\{j < i+1 : \pi^{-1}(j) > \pi^{-1}(i+1)\}|,$$

i.e. x_i is the number of elements less than $i+1$ that appear after $i+1$ in the one-line notation.

Each x_i is valid, i.e. $0 \leq x_i \leq i$, because there are precisely i elements less than $i+1$, each of which occurs once, so clearly no more than this could appear after $i+1$. Therefore, $x_1 \dots x_{n-1} \in 2 \times \cdots \times n$.

To go the other way, we begin with an element $x_1 \dots x_{n-1} \in 2 \times \cdots \times n$. We will show this is the vector representation of a permutation π by building up π in stages.

First, x_1 tells you the relative ordering of 1 and 2. If $x_1 = 1$, we know 1 must appear after 2, while if $x_1 = 0$, then 1 must appear first.

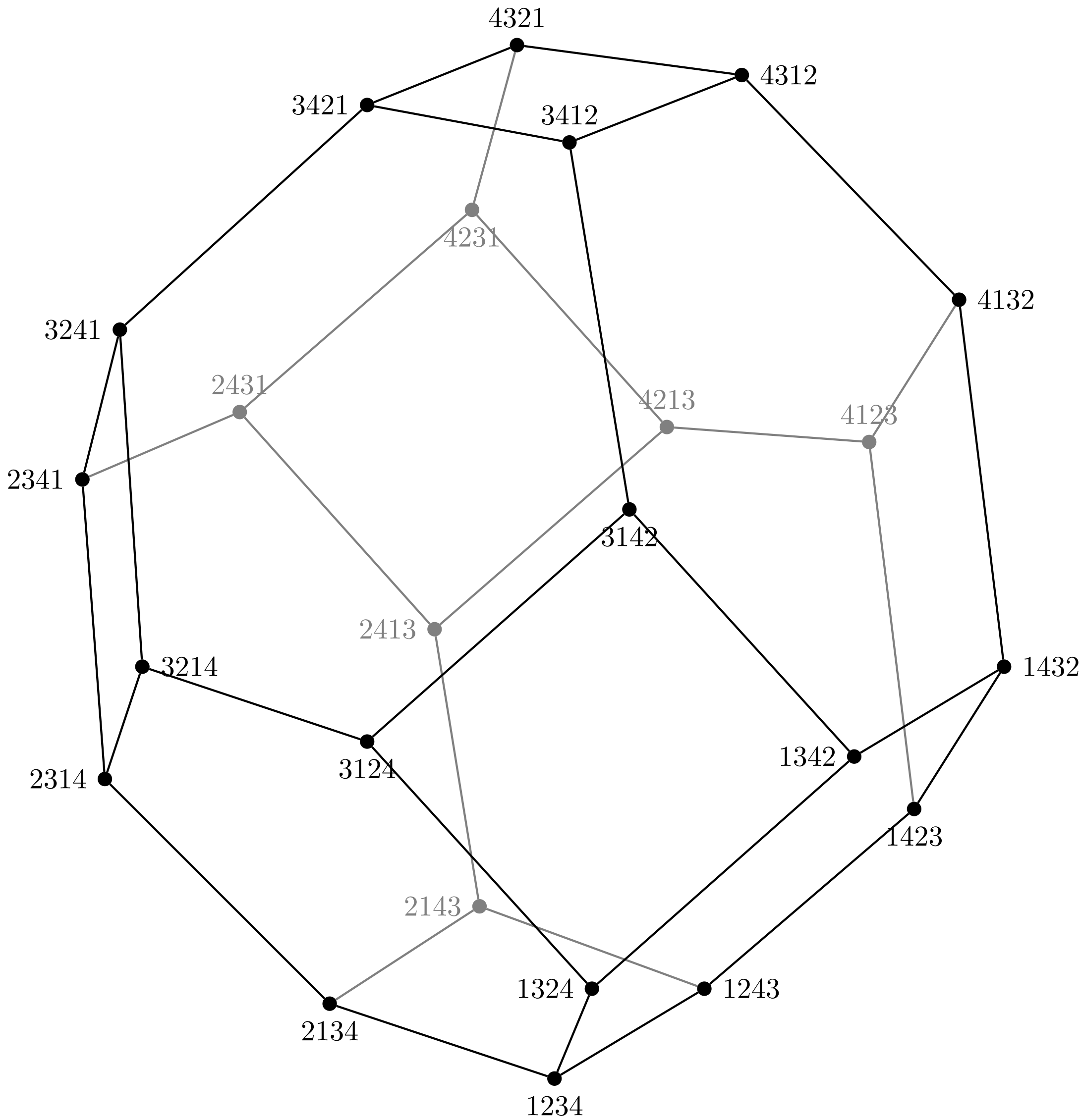

Figure 4.2: The permutohedron of [4].

At the start of stage i, we have ordered (relatively) the numbers $1, \ldots, i$. Now, from the perspective of the element $(i + 1)$, these elements are all the same because they're all smaller than itself, and $(i + 1)$ can then position itself so that there are exactly x_i of these lesser elements following it. Of course this is possible because $0 \leq x_i \leq i$ and there are i elements listed already. Because $(i + 1)$ is larger than every element listed, the position of $(i + 1)$

doesn't affect the values of x_j for $j < i$.

Continue this process to the end of stage $n-1$, and π will be written in one-line notation. □

Example 4.8. The permutation 32514 corresponds to the vector $(1, 2, 0, 2)$. From the vector, we construct the one-line notation in the following stages:

$$21 \to 321 \to 3214 \to 32514.$$

There is a second ordering on permutations of $[n]$ that is often studied called the strong Bruhat order, $<_S$. For this ordering, we have $\pi <_S \sigma$ if and only if $x_1 \dots x_n < y_1 \dots y_n$ as vectors, where $x_1 \dots x_n$ represents π in vector notation, and $y_1 \dots y_n$ represents σ.

The strong Bruhat order on permutations of $[n]$ is isomorphic to the grid $2 \times \cdots \times n$, so we have already discussed the weak discrepancy of this poset. Also, we can think of this poset as the transitive closure of the covering relations: $\pi <_S \sigma$ if there exists $i, j \in [n]$ such that

1. for all $k \neq i, j$, $\pi_k = \sigma_k$,
2. $\pi_i = \sigma_j$ and $\pi_j = \sigma_i$,
3. $j > i$ and $\pi_j > \pi_i$,
4. for all $i < k < j$, π_k has the same relationship with π_i and π_j, i.e. $\pi_k < \pi_i$ if and only if $\pi_k < \pi_j$.

Note that this last requirement guarantees that the total number of inversions in σ is exactly one more than the total number in π.

It is now easy to see that the Hasse diagram of the weak Bruhat order on permutations of $[n]$, call it P_n, is a subgraph of the grid $2 \times 3 \times \cdots \times n$. In Figure 4.1, the only difference is the missing edge $132 \to 231$, which is shown in Figure 4.3.

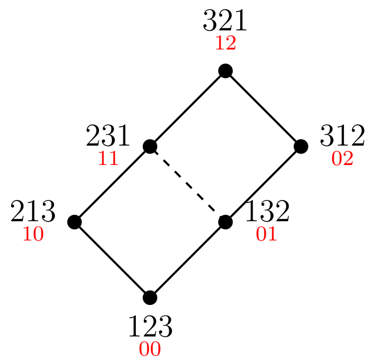

Figure 4.3: The weak Bruhat order on permutations of $[n]$ as a subgraph of $P(2, 3, \ldots, n)$.

Although P_n is a sub-grid of $P(2, 3, \ldots, n)$, this does not immediately imply that the weak discrepancy of P_n is bounded above by the weak discrepancy of $2 \times \cdots \times n$, since the missing edges means there are more incomparable elements. The following example illustrates this phenomenon.

Example 4.9. In the Hasse diagrams below, the poset P' on the right is a sub-poset of the one on the left, P. However, P' has larger weak discrepancy than P.

Proposition 4.10. *The weak discrepancy of P_n is equal to the weak discrepancy of the grid $P(2,3,\ldots,n)$, i.e.* $\mathrm{wd}(P_n) = \frac{n(n-1)}{2} - 2$*, and is achieved by the same labeling, namely $L(x) = \sum_{i=1}^{n} x_i$ (where this is the vector representation of the permutation).*

Proof. To begin, we will show $\Delta(L) = \frac{n(n-1)}{2} - 2$, where L is as given above. We no longer know what all the discrepant pairs of the poset are, but notice that every element is comparable to the unique maximal element and unique minimal element of the permutohedron.

Between these two labels of 0 and $T := \sum_{i=2}^{n}(i-1) = \frac{n(n-1)}{2}$, there are only $\frac{n(n-1)}{2} - 1$ unique labels. Therefore, any two incomparable elements must have labels with a difference of at most $\frac{n(n-1)}{2} - 2$. In particular, the elements $10\ldots0$ and $023\ldots(n-1)$ are incomparable and achieve this distance.

Now, we will show that the previous method for lower bounding the weak discrepancy can still be used, by exhibiting two particular unbroken chains.

We would like to use the chain, in vector representation:

$$010\ldots0 < 020\ldots0 < 0210\ldots0 < \cdots < 023\ldots(n-1).$$

In one-line notation, this corresponds to the chain:

$$1324\ldots n <_B 3124\ldots n <_B 3142\ldots n <_B \cdots <_B n\ldots4312.$$

As you can see, each step in this chain is actually a covering relationship (transposition as defined in Definition 4.6). First, we move element 3 forward, a step at a time, and then we move element 4 towards the front, and so on. So, this chain is unbroken in P_n, and $L(023\ldots(n-1)) \geq L(010\ldots0) + T - 2$.

We also need to consider the chain:

$$10\dots0 < 1010\dots0 < 1020\dots0 < \cdots < 103\dots(n-1).$$

In one-line notation, this corresponds to the chain:

$$2134\dots n <_B 2143\dots n <_B 2413\dots n <_B \cdots <_B n\dots4213.$$

Each step here is also a covering relationship, as elements 4 through n successively move toward the front. So $L(103\dots(n-1)) \geq L(10\dots0) + T - 3$.

Similar to the proof for grids, let L be a weak labeling with $L(010\dots0) = 1$ without loss of generality, and $\omega := \Delta(L)$. Then, $L(023\dots(n-1)) \geq T-2$ and because $10\dots0 \,||\, 023\dots(n-1)$, we must have $L(10\dots0) \geq T-2-\omega$. Then, $L(103\dots(n-1)) \geq T-2-\omega+T-3$, from above. But $1034\dots(n-1) \,||\, 010\dots0$, so $2T-\omega-5 \leq \omega$.

Finally, since ω is an integer, this implies $\omega \geq T-2 = \frac{n(n-1)}{2} - 2$. □

4.3 Partition Lattice

We call $(X_1|X_2|\dots|X_k)$ a partition of a set $\mathcal{S}$ if $X_i \cap X_j = \emptyset$ for all $i \neq j$ and $\bigcup_{i=1}^k X_i = \mathcal{S}$. The partition lattice of $[n]$ is the set of all partitions of $[n]$ with the following order relation.

Definition 4.11. Define a relation $<_P$ on partitions of $[n]$ by $(X_1|X_2|\dots|X_k) <_P (Y_1|Y_2|\dots|Y_\ell)$ if and only if there's a partition $\pi_1,\dots,\pi_k$ of $[\ell]$ with $X_i = \bigcup_{\alpha\in\pi_i} Y_\alpha$ for each $i \in [k]$. (Note that $k < \ell$ necessarily.) We can think of the covering relation as breaking a single set of the partition into two sets; in this case, $k = \ell - 1$.

For $n = 5$, we have $(124|35) <_P (12|3|4|5)$ because we can let $\pi_1 = \{1,3\}$ and $\pi_2 = \{2,4\}$.

However, (12|3|4|5) does not *cover* (124|35) because two new sets are created. Figure 4.4 shows the partition lattice for $n = 4$.

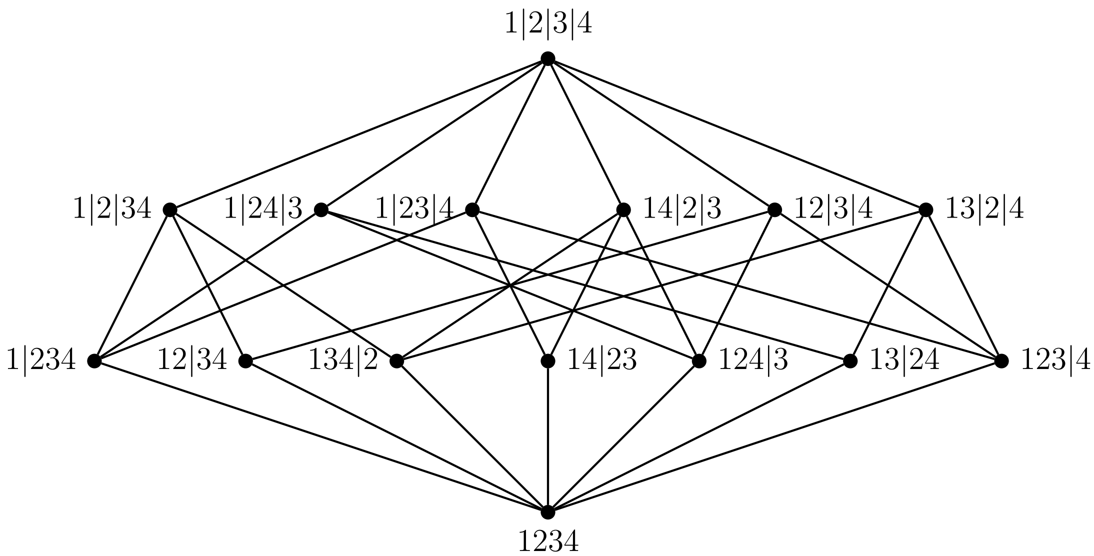

Figure 4.4: The partition lattice of [4].

Lemma 4.12. *If there exist two incomparable chains C_1 and C_2 in a poset P with $|C_1| = |C_2| = c$, then* $\mathrm{wd}(P) \geq c - 1$*. Moreover, if the poset is graded with rank $c + 2$, and has unique maximal and minimal elements, then* $\mathrm{wd}(P) = c - 1$*.*

Proof. Let $C_1 = x_1 < x_2 < \cdots < x_c$ and $C_2 = y_1 < y_2 < \cdots < y_c$ where $x_i \,||\, y_j$ for all $i, j \in [c]$. For any weak labeling L, we have

$$(L(y_c) - L(x_1)) + (L(x_c) - L(y_1)) = (L(y_c) - L(y_1)) + (L(x_c) - L(x_1)) \geq 2c - 2.$$

Therefore, $L(y_c) - L(x_1) \geq c - 1$ or $L(x_c) - L(y_1) \geq c - 1$, and so $\mathrm{wd}(P) \geq c - 1$.

Note that if we want to minimize the discrepancy, we can assume $L(x_1) \leq L(y_c)$ or else $L(x_c) - L(y_1) \geq 2c - 1$. Similarly, we can assume $L(y_1) \leq L(x_c)$.

Now, if we label the elements of the poset by grade level, we will achieve weak discrepancy $c-1$. Because the maximal and minimal elements are unique, and so are comparable to every other element, we only need to be concerned with elements in the middle levels. These have minimal label 1 and maximal label c, so the difference of their labels is at most $c-1$. □

Corollary 4.13. *If $\mathcal{P}_n$ is the partition lattice of $[n]$, then $\mathrm{wd}(\mathcal{P}_n) = n-3$. This is achieved by labeling by grade level, i.e. $L(X_1|\ldots|X_j) = j$.*

Proof. Let

$$C_1 := (1|23\ldots n) < (1|2|3\ldots n) < \cdots < (1|2|3|\ldots|(n-1)n),$$

where $(n-1)$ and n are always in the same part, but every other element is detached one at a time, starting at the beginning.

Let

$$C_2 := (12\ldots(n-1)|n) < (12\ldots(n-2)|(n-1)|n) < \cdots < (12|3|4|\ldots|n),$$

where 1 and 2 are always in the same part, but every other element is detached one at a time, starting at the end.

These chains each contain $n-2$ elements, and they are incomparable, as $(n-1)$ and n are together in every element in the first chain but never together in the second, and 1 and 2 are always together in the second but never in the first.

We can also easily see that $\mathcal{P}_n$ is graded by considering number of parts, and that the poset has rank n since there could be anywhere from 1 to n parts. Finally, $123\ldots n$ is the unique minimal element of grade level 1, and $1|2|3|\ldots|n$ is the unique maximal element of grade level n.

Therefore, by Lemma 4.12, we have $\mathrm{wd}(\mathcal{P}_n) = n-3$. □

4.4 Young's Lattice

The final poset for which we have results is the two-dimensional Young's lattice. We are currently working on extending these results to higher-dimensional Young's lattices.

For this section, we will write a partition $(\lambda_1|\lambda_2|\dots|\lambda_n)$ as $\lambda_1\lambda_2\dots\lambda_n$ when it is convenient and clear to do so.

4.4.1 Two-Dimensional

Definition 4.14. Let $\lambda = \lambda_1 \dots \lambda_n$ be a partition of some integer m such that $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$. Define the *Young's lattice*, Y_λ, to be the poset of all partitions (of integers $\leq m$) whose Ferrers diagrams fit inside the Ferrers diagram of λ, i.e. all elements $x_1 \dots x_n$ with $x_1 \geq \dots \geq x_n$ and $0 \leq x_i \leq \lambda_i$, where the partial ordering is also by inclusion.

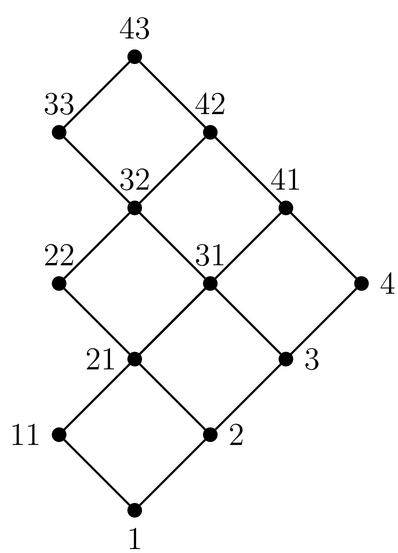

Figure 4.5: The Young's lattice with $\lambda = 43$.

To avoid any confusion, note that the single digit elements of Figure 4.5 have their second index equal to 0. For example, the element 2 is really the element x_1x_2 where $x_1 = 2$ and $x_2 = 0$.

We begin by characterizing the discrepant pairs of a weak labeling of Y_λ; we don't necessarily know that two elements will be a discrepant pair, but if two elements are a discrepant pair then we know what they must look like. This result holds for Young's lattices of all dimensions.

Proposition 4.15. *If L is a weak labeling of Y_λ and $x := x_1 \dots x_n$ and $y := y_1 \dots y_n$ are a discrepant pair of L with $L(x) > L(y)$, then there exists $j \in [n]$ such that*

1. $x_i = \lambda_i$ *for all* $i < j$,
2. $x_j = y_j - 1$,
3. $x_i = \min\{y_j - 1, \lambda_i\}$ *for all* $i > j$,
4. $y_1 = y_2 = \cdots = y_j$, *and*
5. $y_i = 0$ *for all* $i > j$.

Proof. Suppose $x_1 \dots x_n$ and $y_1 \dots y_n$ are a discrepant pair for L, and assume $L(x) > L(y)$. Let $j = \min\{i : y_i > x_i\}$; this is well-defined because $x \,||\, y$.

First suppose there exists $i < j$ with $x_i < \lambda_i$. Then define $x' = x'_1 \dots x'_n$ as follows.

$$x'_k = \begin{cases} x_k & \text{if } k \neq i \\ \lambda_i & \text{if } k = i \end{cases}$$

Then, $L(x') - L(y) > L(x) - L(y)$ which is a contradiction. Hence, $x_i = \lambda_i$ for $i < j$.

The proof is similar for the other conditions. Condition (2) is necessary so that $x \not> y$ but x_j is as large as possible. Condition (3) is a minimum to ensure that the entries of x are non-increasing so that we have $x \in Y_\lambda$. Conditions (4) and (5) similarly construct y as small as possible while requiring the entries to be non-increasing. □

In Figure 4.5, the possible discrepant pairs (x, y) for any weak labeling are: $(11, 2)$, $(22, 3)$, $(33, 4)$, $(4, 11)$, $(41, 22)$, and $(42, 33)$.

Note that $\mathrm{wd}(Y_{kk}) = \mathrm{wd}(Y_{k(k-1)})$ because the difference between the Hasse diagrams is a single edge going from $k(k-1)$ to kk. So, we will assume $\lambda_1 > \lambda_2$ for the remainder of this section.

We can now calculate the weak discrepancy of a two-dimensional Young's lattice. This class of posets is somewhat grid-like; $Y_{\lambda_1\lambda_2}$ is an induced sub-poset of the grid $P(\lambda_2 + 1, \lambda_1)$. However, unlike grids, Young's lattices have a staircase structure. Consider labeling Y_{43} in Figure 4.5 by level, a natural starting place. For this example, we want the labels of 11 and 4 to be "close" and the labels of 4 and 33 to be "close." (These are possible discrepant pairs that are dependent/related through the element 4.) So we cannot stretch either dimension to bring one pair of these elements closer together without pushing the other pair apart. In fact, for Y_{43}, labeling by level is the optimal labeling.

But now consider $Y_{10,4}$ in Figure 4.6. The possible discrepant pairs are the long diagonals: $(10, 1|1), (10|1, 2|2), (10|2, 3|3)$, and $(10|3, 4|4)$, and the shorter diagonals: $(1|1, 2), (2|2, 3)$, $(3|3, 4), (4|4, 5), (5|4, 6), (6|4, 7), (7|4, 8), (8|4, 9)$, and $(9|4, 10)$. For this poset, if we begin with the canonical weak labeling, then $L(1|1) = 2$ and $L(10) = 10$ which are the single discrepant pair. Therefore, it seems natural to scale the second dimension in order to bring these labels closer together. However, notice that because of the staircase structure of the poset, we do not need to shift the label of $2|2$ as much as the label of $1|1$ (one less in fact, proportionally speaking), and similarly for $3|3$ and $4|4$. So, instead of shifting the canonical labeling by scaling the second dimension, we will shift the labels by different amounts, which will still be dependent on the entry in the second dimension. For instance, $\mathrm{wd}(Y_{10,4}) = 6$ and here is one optimal labeling:

$$L(x_1x_2) = \begin{cases} x_1 + x_2 + 0 & \text{if } x_2 = 0 \\ x_1 + x_2 + 2 & \text{if } x_2 = 1 \\ x_1 + x_2 + 3 & \text{if } x_2 \geq 2 \end{cases}$$

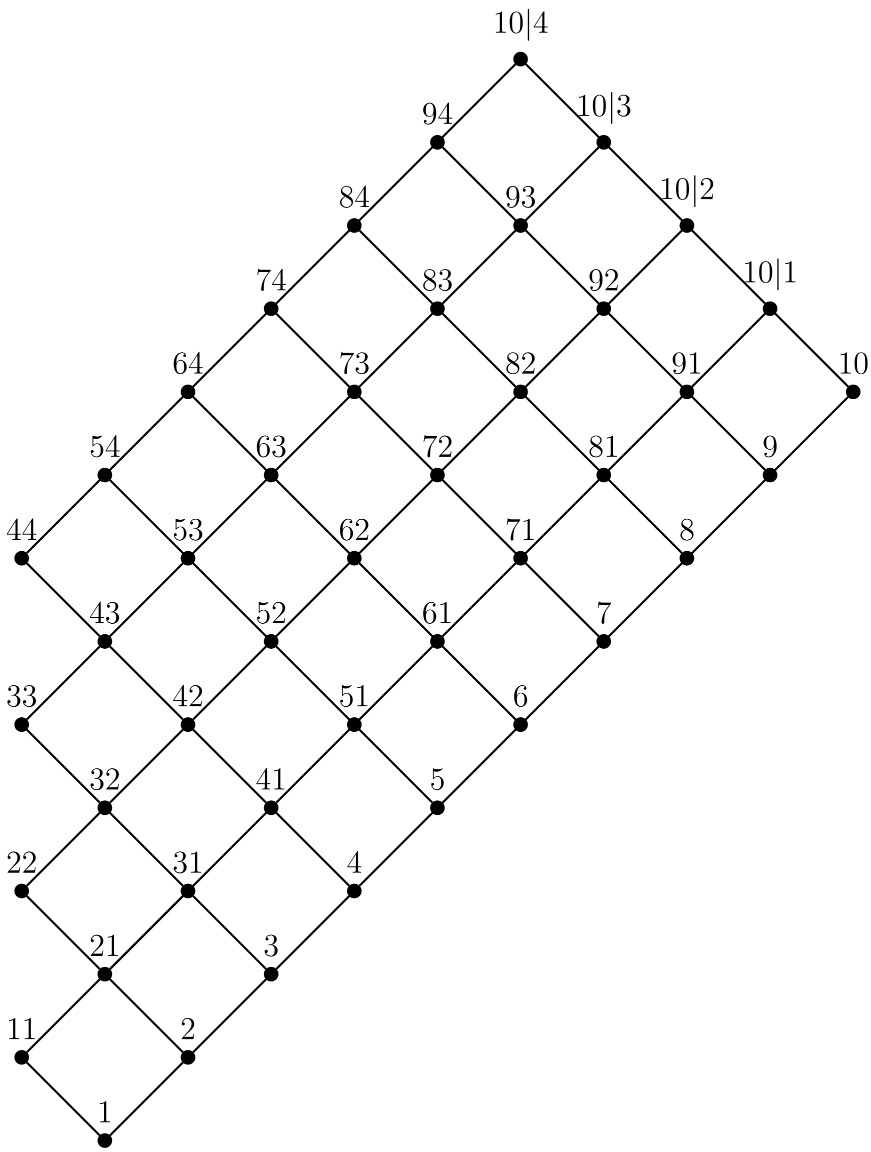

Figure 4.6: The Young's lattice with $\lambda = 10|4$.

The following result formalizes this idea.

Theorem 4.16. *Let i be the largest integer such that $\binom{i+1}{2} \leq \lambda_1 - \lambda_2$.*

1. *If $i \leq \lambda_2$, then $\mathrm{wd}(Y_{\lambda_1\lambda_2}) = \lambda_1 - 1 - i$, achieved by the labeling $L(x_1x_2) = x_1 + x_2 + \binom{i}{2} - \binom{i-x_2}{2}$, where we take $\binom{i-x_2}{2} = 0$ for $x_2 \geq i-1$.*

2. *If $i > \lambda_2$, then $\mathrm{wd}(Y_{\lambda_1\lambda_2}) = \lambda_1 - 2 - \ell$, achieved by the labeling $L(x_1x_2) = x_1 + x_2 + x_2\ell - \binom{x_2}{2}$, where $\ell := \left\lceil \frac{\lambda_1}{\lambda_2+1} + \frac{\lambda_2}{2} \right\rceil - 2$.*

Proof. First, taking i as defined in the statement of the Theorem, we will consider the labeling $L(x_1x_2) = x_1 + x_2 + \binom{i}{2} - \binom{i-x_2}{2}$. We will show that this labeling achieves a discrepancy of $\lambda_1 - 1 - i$ for any λ. Then, we will prove that this is the best you can do when $i \leq \lambda_2$ but that for large i, it is possible to do better. Finally, we will calculate the discrepancy of the second labeling for this case, when $i > \lambda_2$, and show that this is the best possible for this class of posets.

From Proposition 4.15, we know the possible discrepant pairs are $\lambda_1(k-1)$ and kk, and kk and $(k+1)$ for $k \in [\lambda_2]$, along with $k\lambda_1$ and $(k+1)$ for $\lambda_2 < k \leq \lambda_1 - 1$. We will check these three types within two cases: when $k \leq i-1$ and when $k > i-1$. This is to simplify the arithmetic when determining whether the term $\binom{i-x_2}{2}$ is zero.

First, suppose $k \leq i-1$. The first type of discrepant pair is $(\lambda_1(k-1), kk)$ for $1 \leq k \leq \lambda_2$. The difference between the labels of these elements is

$$L(\lambda_1(k-1)) - L(kk) = \lambda_1 + k - 1 + \binom{i}{2} - \binom{i-k+1}{2} - \left[2k + \binom{i}{2} - \binom{i-k}{2}\right]$$

$$= \lambda_1 - k - 1 - \left[\binom{i-k+1}{2} - \binom{i-k}{2}\right] = \lambda_1 - k - 1 - i + k = \lambda_1 - 1 - i.$$

The second type of discrepant pair is $(kk, k+1)$ where $1 \leq k \leq \lambda_2$, which has a labeling difference of

$$L(kk) - L(k+1) = 2k + \binom{i}{2} - \binom{i-k}{2} - k - 1 = k - 1 + \binom{i}{2} - \binom{i-k}{2} \leq \lambda_2 - 1 + \binom{i}{2}.$$

The last inequality follows from $k - \binom{i-k}{2} \leq k \leq \lambda_2$.

Finally, the third type of discrepant pair is $(k\lambda_2, k+1)$ where $\lambda_2 < k \leq \lambda_1 - 1$. The difference between the labels of these elements is

$$L(k\lambda_2) - L(k+1) = k + \lambda_2 + \binom{i}{2} - \binom{i-\lambda_2}{2} - k - 1 = \lambda_2 - 1 + \binom{i}{2} - \binom{i-\lambda_2}{2} \leq \lambda_2 - 1 + \binom{i}{2}.$$

Now we will consider the possible discrepant pairs for $k > i - 1$. Note that for many λ, several of these cases will overlap, but the most straightforward proof simply calculates all of these labeling differences to be thorough, since they are fairly straightforward to compute.

For the first type of discrepant pair, $(\lambda_1(k-1), kk)$ for $1 \leq k \leq \lambda_2$, we have a difference of

$$L(\lambda_1(k-1)) - L(kk) = \lambda_1 + k - 1 + \binom{i}{2} - \left[2k + \binom{i}{2}\right] = \lambda_1 - k - 1 < \lambda_1 - 1 - i.$$

For the second type, $(kk, k+1)$ where $1 \leq k \leq \lambda_2$, we have

$$L(kk) - L(k+1) = 2k + \binom{i}{2} - k - 1 = k - 1 + \binom{i}{2} \leq \lambda_2 - 1 + \binom{i}{2}.$$

Finally, for the third type, $(k\lambda_2, k+1)$ where $\lambda_2 < k \leq \lambda_1 - 1$, we have

$$L(k\lambda_2) - L(k+1) = k + \lambda_2 + \binom{i}{2} - \binom{i-\lambda_2}{2} - k - 1 \leq \lambda_2 - 1 + \binom{i}{2}.$$

Notice that all of these labeling differences are either bounded by $\lambda_1 - 1 - i$ or $\lambda_2 - 1 + \binom{i}{2}$;

therefore, $\Delta(L) \leq \max\{\lambda_1 - 1 - i, \lambda_2 - 1 + \binom{i}{2}\}$. For our choice of i,

$$\lambda_1 - \lambda_2 \geq \binom{i+1}{2} \quad \Rightarrow \quad \lambda_1 - \lambda_2 \geq \binom{i}{2} + i \quad \Rightarrow \quad \lambda_1 - 1 - i \geq \lambda_2 - 1 + \binom{i}{2}.$$

Moreover, the labeling difference between λ_1 and 11 is tight, so $\Delta(L) = \lambda_1 - 1 - i$.

Next, we will show that if $i \leq \lambda_2$, this is the best we can do.

Suppose that $i \leq \lambda_2$, and that L is a weak labeling with $\Delta(L) \leq \lambda_1 - i - 2$ and $L(\lambda_1) = \lambda_1$ without loss of generality.

Then since $\lambda_1 \,||\, 11$, we must have $L(11) \geq \lambda_1 - (\lambda_1 - i - 2) = i + 2$. This implies $L(\lambda_1 1) \geq (i+2) + (\lambda_1 - 1)$, and so

$$\lambda_1 - i - 2 \geq L(\lambda_1 1) - L(22) \geq (i+2) + (\lambda_1 - 1) - L(22)$$

$$\Rightarrow L(22) \geq (i+2) + (i+2) - 1.$$

Continuing this process, for $i \leq \lambda_2$, we get $L(ii) \geq i(i+2) - \binom{i}{2} = \frac{1}{2}i(i+5)$. (Note that this is the same i we defined previously.)

Then, by reasoning similar to Lemma 4.2,

$$L((\lambda_1 - 1)\lambda_2) \geq (\lambda_1 - 1 - i) + (\lambda_2 - i) + L(ii)$$

$$= (\lambda_1 - 1 - i) + (\lambda_2 - i) + \frac{1}{2}i(i+5) = \lambda_1 + \lambda_2 - 1 + \binom{i+1}{2}.$$

Finally, this implies $L((\lambda_1 - 1)\lambda_2) - L(\lambda_1) \geq \lambda_2 - 1 + \binom{i+1}{2}$. Because $\lambda_1 - \lambda_2 < \binom{i+2}{2}$, we have $\lambda_1 - \lambda_2 - i - 1 < \binom{i+1}{2}$ and $\lambda_2 - 1 + \binom{i+1}{2} > \lambda_1 - i - 2$. This is strict, so we have a contradiction. L cannot possibly have a smaller discrepancy than $\lambda_1 - i - 1$.

Therefore, for $i \leq \lambda_2$, $\mathrm{wd}(Y_\lambda) = \lambda_1 - i - 1$.

Now suppose that $i > \lambda_2$. Consider the labeling $L(x_1x_2) = x_1 + x_2 + x_2\ell - \binom{x_2}{2}$, where $\ell := \left\lceil \frac{\lambda_1}{\lambda_2+1} + \frac{\lambda_2}{2} \right\rceil - 2$. We will show that this labeling has a lower discrepancy than the previous one for these particular λ, and that this labeling is optimal.

Once more, we'll consider the labelings of the possible discrepant pairs.

For $(\lambda_1(k-1), kk)$ with $1 \le k \le \lambda_2$, we have

$$L(\lambda_1(k-1)) - L(kk) = \lambda_1 + k - 1 + (k-1)\ell - \binom{k-1}{2} - \left[2k + k\ell - \binom{k}{2}\right]$$

$$= \lambda_1 - k - 1 - \ell + k - 1 = \lambda_1 - 2 - \ell.$$

For $(kk, k+1)$ with $1 \le k \le \lambda_2$, we have

$$L(kk) - L(k+1) = 2k + k\ell - \binom{k}{2} - (k+1) = k(\ell+1) - \binom{k}{2} - 1.$$

This is a quadratic, concave-down function with a vertex at $k = \ell + \frac{3}{2}$. But notice that since $i > \lambda_2$ and $\binom{i+1}{2} \le \lambda_1 - \lambda_2$, we have

$$\lambda_1 - \lambda_2 \ge \binom{\lambda_2+1}{2} \ge \frac{1}{2}(\lambda_2^2+1) \quad \Rightarrow \quad \lambda_1 \ge \frac{1}{2}(\lambda_2+1)^2 \quad \Rightarrow \quad \frac{\lambda_1}{\lambda_2+1} \ge \frac{1}{2}(\lambda_2+1)$$

$$\Rightarrow \quad \frac{\lambda_1}{\lambda_2+1} + \frac{\lambda_2}{2} \ge \lambda_2 + \frac{1}{2} \quad \Rightarrow \quad \frac{\lambda_1}{\lambda_2+1} + \frac{\lambda_2}{2} - \frac{1}{2} \ge \lambda_2 \quad \Rightarrow \quad \ell + \frac{3}{2} \ge \lambda_2.$$

Hence, when $k \le \lambda_2$, this function is increasing with respect to k, and therefore maximized when $k = \lambda_2$ giving $\lambda_2(\ell+1) - \binom{\lambda_2}{2} - 1$.

Now we must show

$$\begin{aligned}
& \lambda_2(\ell+1) - \binom{\lambda_2}{2} - 1 \le \lambda_1 - 2 - \ell \\
\Leftrightarrow \quad & \ell(\lambda_2+1) \le \lambda_1 - \lambda_2 - 1 + \binom{\lambda_2}{2} \\
\Leftrightarrow \quad & \ell \le \left\lfloor \frac{\lambda_1 - \lambda_2 - 1 + \binom{\lambda_2}{2}}{\lambda_2+1} \right\rfloor \\
\Leftrightarrow \quad & \frac{\lambda_1}{\lambda_2+1} + \frac{\lambda_2}{2} - 2 \le \frac{\lambda_1 - \lambda_2 - 1 + \binom{\lambda_2}{2}}{\lambda_2+1} \\
\Leftrightarrow \quad & \lambda_1 + \binom{\lambda_2+1}{2} - 2(\lambda_2+1) \le \lambda_1 - \lambda_2 - 1 + \binom{\lambda_2}{2} \\
\Leftrightarrow \quad & 0 \le 1.
\end{aligned}$$

Finally, consider the third type of discrepant pair, namely $(k\lambda_2, k+1)$ when $\lambda_2 < k < \lambda_1$:

$$L(k\lambda_2) - L(k+1) = k + \lambda_2 + \lambda_2\ell - \binom{\lambda_2}{2} - k - 1 = \lambda_2(\ell+1) - \binom{\lambda_2}{2} - 1,$$

which we just showed was at most $\lambda_1 - 2 - \ell$. Hence, the discrepancy of this labeling is at most $\lambda_1 - 2 - \ell$.

To conclude, suppose we could do better than this, i.e. let L be another weak labeling with $\Delta(L) \le \lambda_1 - 3 - \ell$ and $L(\lambda_1) = \lambda_1$ without loss of generality.

Then since $\lambda_1 \,||\, 11$, $L(11) \ge L(\lambda_1) - (\lambda_1 - 3 - \ell) = \ell + 3$. This implies $L(\lambda_1 1) \ge (\ell+3) + (\lambda_1 - 1)$, and so

$$\lambda_1 - 3 - \ell \ge L(\lambda_1 1) - L(22) \ge (\ell+3) + (\lambda_1 - 1) - L(22)$$

$$\Rightarrow L(22) \ge (\ell+3) + (\ell+3) - 1.$$

Continuing this process, we get $L(\lambda_2\lambda_2) \geq \lambda_2(\ell+3) - \binom{\lambda_2}{2}$. Then,

$$L((\lambda_1 - 1)\lambda_2) \geq (\lambda_1 - 1 - \lambda_2) + \lambda_2(\ell+3) - \binom{\lambda_2}{2} = \lambda_1 - 1 + \lambda_2(\ell+2) - \binom{\lambda_2}{2}.$$

Finally, this implies $L((\lambda_1 - 1)\lambda_2) - L(\lambda_1) \geq \lambda_2(\ell+2) - 1 - \binom{\lambda_2}{2}$. The only thing left to show to arrive at a contradiction is that

$$\begin{aligned}
& & \lambda_2(\ell+2) - 1 - \binom{\lambda_2}{2} &> \lambda_1 - 3 - \ell \\
&\Leftarrow & \lambda_2\left(\frac{\lambda_1}{\lambda_2+1} + \frac{\lambda_2}{2}\right) - \binom{\lambda_2}{2} &> \lambda_1 - 2 - \ell \\
&\Leftrightarrow & \frac{\lambda_2\lambda_1}{\lambda_2+1} + \frac{\lambda_2^2}{2} - \frac{\lambda_2^2 - \lambda_2}{2} &> \lambda_1 - 2 - \ell \\
&\Leftrightarrow & \frac{\lambda_2\lambda_1}{\lambda_2+1} + \frac{\lambda_2}{2} &> \lambda_1 - 2 - \ell \\
&\Leftrightarrow & \ell &> \frac{\lambda_1}{\lambda_2+1} - \frac{\lambda_2}{2} - 2,
\end{aligned}$$

which is true by definition of ℓ.

Therefore, for $i > \lambda_2$, $\mathrm{wd}(Y_\lambda) = \lambda_1 - 2 - \ell$. □

4.4.2 Higher Dimensional

It would be interesting to generalize this result to higher-dimensional Young's lattices. As a step in this direction, Figure 4.7 shows a three-dimensional Young's lattice, namely Y_{941}. The following is a proof that the weak discrepancy of this poset is 9, whereas the canonical labeling has a discrepancy of 10.

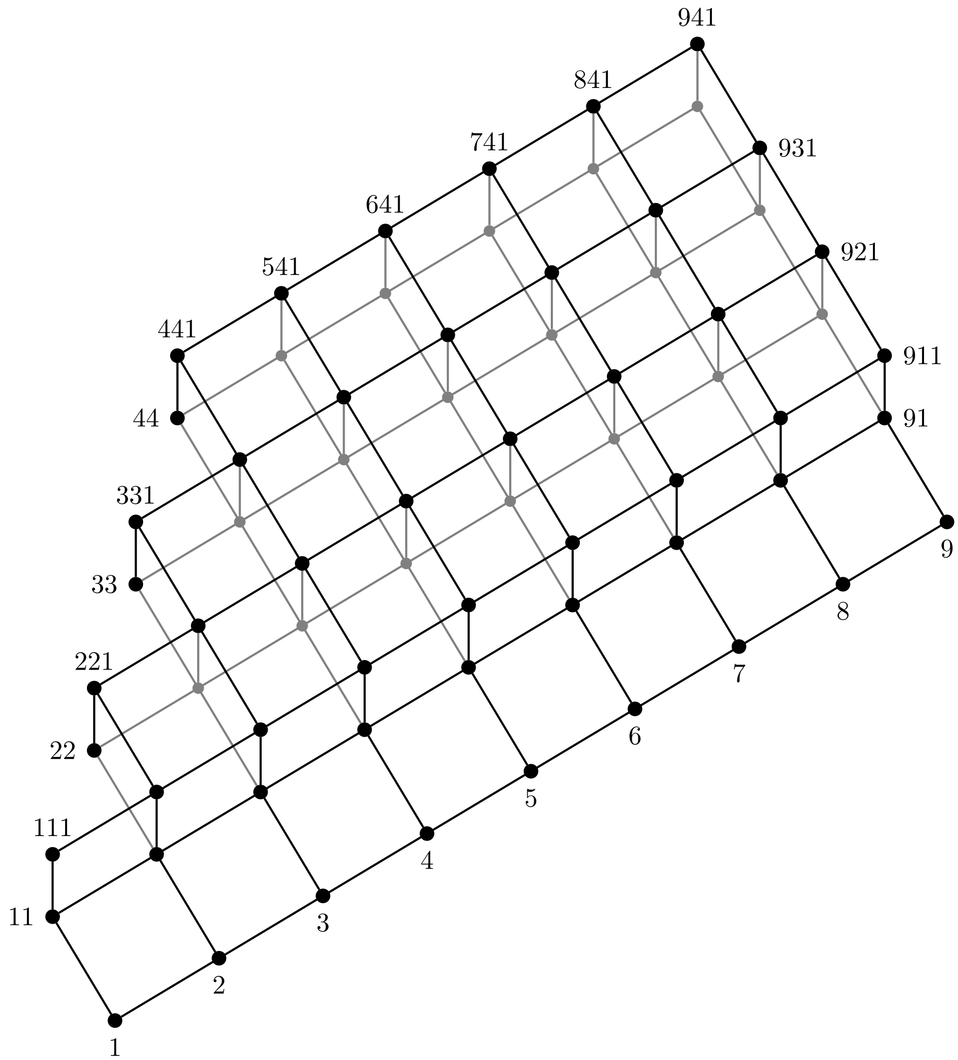

Figure 4.7: The Young's lattice with $\lambda = 941$.

First consider the canonical labeling:

$$L_c(x_1x_2x_3) = x_1 + x_2 + x_3.$$

From Proposition 4.15, we can list the possible discrepant pairs and calculate their discrepancies, which are included in Table 4.1. This shows that $\Delta(L_c) = 10$; however, we can do better.

$j = 1$	**Discrepancy**	$j = 2$	**Discrepancy**	$j = 3$	**Discrepancy**
$(111, 2)$	1	$(9, 11)$	7	$(94, 111)$	10
$(221, 3)$	2	$(911, 22)$	7		
$(331, 4)$	3	$(921, 33)$	6		
$(441, 5)$	4	$(931, 44)$	5		
$(541, 6)$	4				
$(641, 7)$	4				
$(741, 8)$	4				
$(841, 9)$	4				

Table 4.1: Determining the discrepancy of L_c for Y_{941}.

Proposition 4.17. *The weak discrepancy of the Young's lattice* Y_{941} *equals 9, and is achieved by the labeling*

$$L(x_1x_2x_3) = \begin{cases} x_1 + x_2 + x_3 + 0 & \textit{if } x_3 = 0 \\ x_1 + x_2 + x_3 + 1 & \textit{if } x_3 = 1 \end{cases}.$$

Proof. Table 4.2 shows that the discrepancy of this labeling is 9.

Now suppose L' is any weak labeling of Y_{941} and that $\Delta(L') =: \omega$. Without loss of generality, suppose $L'(111) = 3$. Then $L'(911) \geq 11$ and since $911 \,||\, 22$, we have $L'(22) \geq 11 - \omega$. This implies that $L'(94) \geq 20 - \omega$. But $94 \,||\, 111$, and hence,

$$\omega \geq L(94) - L(111) \geq 17 - \omega.$$

Therefore, $\omega \geq 9$ necessarily. $\square$

$j=1$	**Discrepancy**	$j=2$	**Discrepancy**	$j=3$	**Discrepancy**
$(111,2)$	2	$(9,11)$	7	$(94,111)$	9
$(221,3)$	3	$(911,22)$	8		
$(331,4)$	4	$(921,33)$	7		
$(441,5)$	5	$(931,44)$	6		
$(541,6)$	5				
$(641,7)$	5				
$(741,8)$	5				
$(841,9)$	5				

Table 4.2: Determining the discrepancy of L for Y_{941}.

4.5 Future Directions

As discrepancy is a fairly new area of study with respect to partially ordered sets, there are many open questions left to tackle. A few variations such as t-discrepancy, fractional weak discrepancy, or total linear discrepancy have only been looked at in the last two to three years. There are also many interesting and natural variations one can consider that have not been studied at all. We present several of these here.

4.5.1 t-Discrepancy

Recall from Section 2.5 that a *t-labeling* of a poset P is a function L from P to $\mathbb{Z}$ such that $X < Y$ in P implies $L(X) < L(Y)$ and $|L^{-1}(a)| \leq t$ for each $a \in \mathbb{Z}$, i.e. each label can be used up to t times. Then the *t-discrepancy*, denoted $d_t(P)$, is the minimum over all t-labelings of the discrepancy of the labeling; in other words,

$$d_t(P) := \min_{L} \max_{X \,||\, Y} |L(X) - L(Y)|,$$

where L ranges over all t-labelings of P.

In [12], Howard and Trenk calculate the t-discrepancy of the union of disjoint chains, represented by the sum $m_1 + m_2 + \cdots + m_n$.

Theorem 4.18. *(Howard, Trenk, 2010) Let $P = m_1 + m_2 + \cdots + m_n$ where $m_1 \geq m_2 \geq \cdots \geq m_n$ and let $t \geq 2$ be an integer. Furthermore, let $q = \left\lceil \frac{m_1+\cdots+m_n}{t} \right\rceil$ and $M = \max\left\{m_2, \left\lceil \frac{m_2+\cdots+m_n}{t-1} \right\rceil\right\}$. Then*

$$d_t(P) = \begin{cases} q-1 & \text{if } q > m_1 \\ \lceil (m_1+M)/2 \rceil - 1 & \text{if } q \leq m_1 \end{cases}$$

Can this result be used to get insight into calculating the t-discrepancy of grids which are products of chains as opposed to disjoint sums? Can it be extended to give bounds on the t-discrepancy of the permutohedron or the partition lattice?

4.5.2 Fractional Weak Discrepancy

A further generalization of weak discrepancy allows labels to be real numbers, while requiring that labels of comparable elements be at least 1 apart.

Definition 4.19. A *fractional weak labeling* of a poset P is a function L from P to $\mathbb{R}$ such that $X < Y$ in P implies $L(X) + 1 \leq L(Y)$. The *fractional weak discrepancy*, $\mathrm{wd}_F(P)$, is the minimum over all fractional weak labelings of the maximum difference between the labels of incomparable elements, i.e.

$$\mathrm{wd}_F(P) = \min_{L} \max_{X \,||\, Y} |L(X) - L(Y)|$$

where L ranges over all fractional weak labelings.

This variation of weak discrepancy was introduced in a 2007 paper by Shuchat, Shull, and Trenk [19], where they showed that for any poset P, $\mathrm{wd}_F(P)$ is in fact rational, and all the labels may be taken to be rational numbers. Hence, it makes sense to call it the *fractional* weak discrepancy.

In their paper, Shuchat, Shull, and Trenk also proved that an optimal fractional weak labeling will produce an optimal weak labeling if one takes the ceiling of each label. Our results for the weak discrepancy of grids and Young's lattice appear that they would translate over nicely to fractional weak discrepancy. If we simply remove the ceilings in each of these results, will an optimal fractional weak labeling result?

4.5.3 Other Variations

The ER waiting room example given in Figure 2.6 motivates several new variations of labelings and discrepancy. Suppose first that we have t doctors on call, so labels may be used up to t times as in a standard t-labeling. However, we don't ever want a doctor to be waiting around without a patient. So we want to loosen the requirement that comparable elements are given distinct labels, yet maintain that labels of larger elements are *at least* as large as smaller ones.

Definition 4.20. Let $t \geq 2$ be an integer. A *t-loose labeling* of a poset P is a map from P to $\mathbb{Z}$ such that $|L^{-1}(a)| \leq t$ for all $a \in \mathbb{Z}$, and if $X < Y$ in P, then $L(X) \leq L(Y)$. The *t-loose discrepancy*, $\ell_t(P)$, is the minimum over all t-loose labelings of the maximum difference between the labels of incomparable elements, i.e.

$$\ell_t(P) = \min_{L} \max_{X \,||\, Y} |L(X) - L(Y)|,$$

where L ranges over all t-loose labelings of P.

Proposition 4.21. *For any poset P and $t \in \mathbb{N}$, we have $\ell_t(P) \leq d_t(P)$.*

Proof. Let L be a t-labeling of P such that $\Delta(L) = d_t(P)$. L is necessarily a t-loose labeling, as each label is used at most t times, and if $X < Y$ in P, then $L(X) < L(Y)$. Therefore, $\ell_t(P) \leq \Delta(L) = d_t(P)$. □

Corollary 4.22. *For any poset P and $t \in \mathbb{N}$, we have $\ell_t(P) \leq \mathrm{ld}(P)$.*

Proof. See Proposition 2.22. □

However, there is no clear relationship between the t-loose discrepancy of a poset and its weak discrepancy. Figure 4.8 shows a poset where the weak discrepancy is smaller than the 2-loose discrepancy, and Figure 4.9 exhibits a poset where the weak discrepancy is larger than the 3-loose discrepancy. Is there a poset for which $\ell_2(P) < \mathrm{wd}(P)$? Are there posets with $d_t(P) < \ell_{t-1}(P)$?

It would also be interesting to study weighted posets. Suppose each element x in a poset P is given a weight $w(x)$. Then, we could construct a t-weighted labeling by requiring

$$\sum_{\substack{x \in P \\ L(x)=a}} w(x) \leq t$$

for each $a \in \mathbb{Z}$. This is analogous to saying that a patient in our ER may require multiple doctors simultaneously. Alternately, we could define a t-weighted labeling by requiring

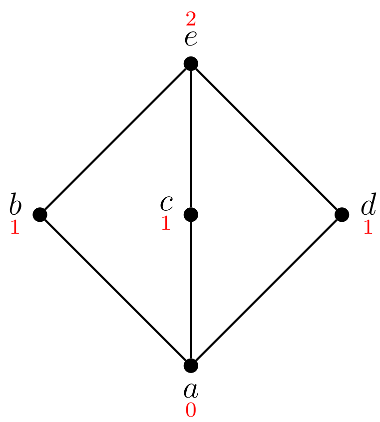

Figure 4.8: A poset with smaller weak discrepancy than 2-loose discrepancy. On the left is an optimal weak labeling and on the right is an optimal 2-loose labeling.

Figure 4.9: A poset with larger weak discrepancy than 3-loose discrepancy. On the left is an optimal weak labeling and on the right is an optimal 3-loose labeling.

an element x with weight $w(x)$ to be assigned $w(x)$ consecutive labels. Then, t-weighted discrepancy would need to specify whether we are comparing two elements' smallest, largest, or average labels. This is like saying that one patient may take longer to treat than another.

Finally, what if we allow our labels to come from someplace other than $\mathbb{N}$ or $\mathbb{R}$? We could consider labeling by ordered pairs, or perhaps by another poset entirely. Suppose we labeled a poset P by another poset Q; then we could require that $X < Y$ in P implies $L(X) < L(Y)$ in Q. What might result from this projection?

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