1-1-1975

Simple Harmonic Motion

STUDY GUIDE

SIMPLE HARMONIC MOTION

INTRODUCTION

Have you ever felt you were the slave of a clock? Clocks are mechanisims that include a pendulum or balance wheel whose repeated patterns of movement define equal time intervals, one after another. Such repeated movements are called periodic motion. Periodic motion may occur when a particle or body is confined to a limited region of space by the forces acting on it and does not have sufficient energy to escape.

In this module you will study the special kind of periodic motion that results when the net force acting on a particle, often called the restoring force, is directly proportional to the particle's displacement from its equilibrium position; this is known as simple harmonic motion. Actually, simple harmonic motion is an idealization that applies only when friction, finite size, and other small effects in real physical systems are neglected. But it is a good enough approximation that it ranks in importance with other special kinds of motion (free fall, circular and rotational motion) that you have already studied. Examples of simple harmonic motion include cars without shock absorbers, a child's swing, violin strings, and, more importantly, certain electrical circuits and vibrations of a tuning fork that you may study in later modules.

PREREQUISITES

Before you begin this module, you should be able to:	Location of Prerequisite Content
*Define kinetic energy (needed for Objective 3 of this module)	Work and Energy Module
*Define potential energy, and use the conservation of energy to solve simple problems (needed for Objectives 2 and 4 of this module)	Rotational Motion Module
*Define angular velocity, acceleration, displacement, and torque (needed for Objectives 2 and 4 of this module)	Rotational Motion Module
*Apply Newton's second law for rotation to solve simple problems (needed for Objectives 2 and 4 of this module)	Rotational Dynamics Module

LEARNING OBJECTIVES

After you have mastered the content of this module, you will be able to:

1. Definitions - Define the following terms or relate them to the solution of Newton's second law for simple harmonic motion, $x = A \cos(\omega t + \theta)$ (instead of θ you may see ϕ or δ):

   simple harmonic motion, amplitude,
   frequency, phase constant (or phase angle),
   angular frequency, period,
   spring constant, restoring force.

2. Identify simple harmonic motion - Analyze the motion of a particle to determine whether simple harmonic motion occurs, and if so, determine its angular frequency.

3. Linear simple harmonic motion - Organize the necessary data about a particle undergoing linear simple harmonic motion to find any or all of the following quantities: the particle's position as a function of time, angular frequency, period, amplitude, phase, frequency, velocity, acceleration, mass, and the restoring force, kinetic energy, or potential energy of the system.

4. Rotational or approximate simple harmonic motion - Apply Newton's second law or conservation of energy to simple physical systems carrying out rotational or approximately linear simple harmonic motion to determine any or all of the quantities listed in Objective 3.

TEXT: Frederick J. Bueche, Introduction to Physics for Scientists and Engineers (McGraw-Hill, New York, 1975), second edition

SUGGESTED STUDY PROCEDURE

Read Chapter 13, Sections 13.1 through 13.5, 13.7, 13.8, and work at least Problems A through H, and 1, 4, 16, 9, 20 in Chapter 13 before attempting the Practice Test.

The general solution of a differential equation is discussed in General Comment 1. Study that carefully. Read the discussion on the small-angle approximation, $\sin\theta \simeq \theta$, in General Comment 3.

Conservation of Energy

If you forgot how to obtain the potential energy of a simple harmonic oscillator, read Section 9.6 in Chapter 9. Since the forces within a spring that make it resist compression and extension are conservative, the sum of kinetic and potential energy in any harmonic oscillator is a constant. This observation can often be used to solve for the amplitude of vibration. For instance, if we know the velocity of the oscillator as it passes equilibrium (when the potential energy is zero), we can find its maximum displacement (when kinetic energy is zero) from

$$0 + (1/2)mv_0^2 = (1/2)kx_0^2 + 0, \qquad x_0 = v_0\sqrt{m/k}.$$

BUECHE

Objective Number	Readings	Problems with Solutions: Study Guide	Assigned Problems: Study Guide	Assigned Problems: Text (Chap. 13)	Additional Problems (Chap. 13)
1	Secs. 13.1 to 13.4, General Comment 1	A	E	1	2
2	Sec. 13.3, General Comments 2, 3	B	F		
3	Secs. 13.5, 13.7	C	G	4, 16	11
4	Secs. 13.7, 13.8, General Comment 3	D	H	9, 20	6, 15, 18

Similarly, if we know x_0, we can solve for the maximum velocity of the oscillator, $v_0 = x_0(k/m)^{1/2}$ without even thinking about derivatives [Note, however, that this <u>is</u> $v(t)_{max}$.]:

$$v(t)_{max} = (\frac{dx}{dt})_{max} = [\frac{d[x_0 \cos(\omega t + \theta)]}{dt}]_{max} = [-x_0\omega \sin(\omega t + \theta)]_{max}.$$

The maximum value of $[-\sin(\omega t + \theta)]$ is +1. Thus

$$v(t)_{max} = x_0\omega = x_0\sqrt{k/m}.$$

TEXT: David Halliday and Robert Resnick, Fundamentals of Physics (Wiley, New York, 1970; revised printing, 1974)

SUGGESTED STUDY PROCEDURE

Read Chapter 13, Sections 13-1 through 13-6, and work at least Problems A through H, and 1, 9, 17, 27, 37 in Chapter 13 before attempting the Practice Test.

The general solution of a differential equation is discussed in General Comment 1. Study that carefully. Read the discussion on the small-angle approximation, $\sin\theta \simeq \theta$, in General Comment 3. There are no problems on Section 13-6, but reading it should help to clarify angular frequency.

Example: Physical Pendulum

A rather simple example of simple harmonic motion is the physical pendulum or compound pendulum as shown in Figure 1. A rigid body of mass m is suspended from an axis O. The center of mass is a distance h from the axis. The torque about the axis O on the body is equal to the moment of inertia about the axis, I, times the angular acceleration:

$$\tau = I\alpha = I(d^2\theta/dt^2).$$

The restoring torque is provided by the weight $m\vec{g}$:

$$\tau = -mgh \sin\theta.$$

HALLIDAY AND RESNICK

Objective Number	Readings	Problems with Solutions	Assigned Problems		Additional Problems (Chap. 13)
		Study Guide	Study Guide	Text (Chap. 13)	
1	Secs. 13-1 to 13-3, General Comment 1	A	E	1, 17	16, 18
2	Sec. 13-3, General Comments 2, 3	B	F		
3	Secs. 13-3 to 13-4	C	G	9, 27	5, 14, 32
4	Sec. 13-5, General Comment 3	D	H	37	35, 40, 42

Therefore,

$$-mgh \sin \theta = I(d^2\theta/dt^2).$$

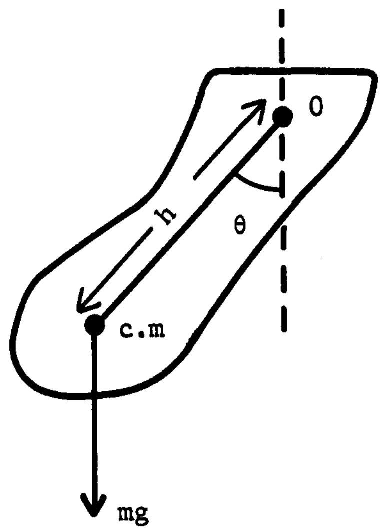
If θ is small, $\sin \theta \simeq \theta$, thus

$$-mgh\theta = I(d^2\theta/dt^2).$$

This is the equation for simple harmonic motion. Compare this with Eq. (13-21) in the text, setting k equivalent to mgh, and thus

$$\omega^2 = mgh/I.$$

Figure 1



TEXT: Francis Weston Sears and Mark W. Zemansky, University Physics (Addison-Wesley, Reading, Mass., 1970), fourth edition

SUGGESTED STUDY PROCEDURE

Read Chapter 11, Sections 11-1 through 11-6, 11-8 and 11-9, and work at least Problems A through H, and 11-1, 11-5, 11-13, and 11-29 before attempting the Practice Test.

In Section 11-4, note the integral on the left-hand side of Eq. (11-7). You will not be expected to do integrals like this by yourself; instead, you can look them up in a book of tables. The general solution of a differential equation is discussed in General Comment 1. Study that carefully. Read the discussion on the small-angle approximation, $\sin\theta \simeq \theta$, in General Comment 3.

Reference Circle

First, let us define "simple harmonic motion." If a particle, displaced a distance x from a position of rest and released, experiences a force toward that position of rest of magnitude proportional to the magnitude of its displacement, the particle will move with simple harmonic motion. Expressed algebraically, this is $F = -kx$, where x is the displacement from the position of rest, F the restoring force, and k a positive constant.

Now, leave this for a moment, and consider the projection of the motion of a particle, moving at constant angular speed ω in a circular path of radius A,

SEARS AND ZEMANSKY

Objective Number	Readings	Problems with Solutions: Study Guide	Assigned Problems: Study Guide	Assigned Problems: Text	Additional Problems
1	Secs. 11-1 to 11-4, General Comment 1	A	E	11-1	
2	Sec. 11-2, General Comments 2, 3	B	F		
3	Secs. 11-4 to 11-5	C	G	11-5, 11-13	11-2, 11-3, 11-4, 11-18
4	Secs. 11-6, 11-8, 11-9, General Comment 3	D	H	11-29	11-22, 11-26, 11-30

upon a diameter of the circle, as in Figure 2. Arbitrarily, let the projection be at the center of the diameter at $t = 0$, and call x the displacement of the projection from the center. Then, at some later time t, the particle will have turned through an angle θ, equal to ωt, and the projection will then have moved a distance $x = A \sin \omega t$.

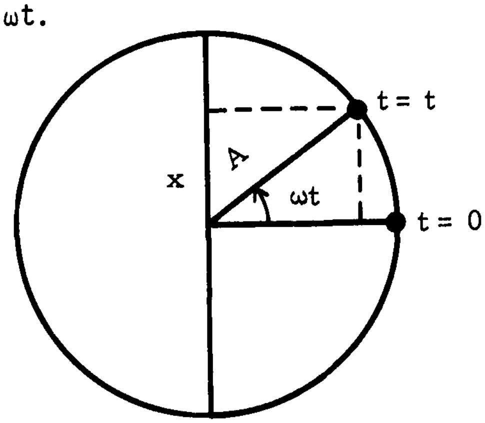


Figure 2

Now, differentiate this expression twice with respect to time, obtaining

$$dx/dt = \omega A \cos \omega t \quad \text{and} \quad d^2x/dt^2 = -\omega^2 A \sin \omega t.$$

By definition, d^2x/dt^2 is acceleration, ω^2 is a positive constant, and $A \sin \omega t$ is x, the displacement of the projection from the center of the circle. From Newton's second law:

$$F = md^2x/dt^2, \qquad F = -m\omega^2 A \sin \omega t.$$

Therefore our final equation is $F = -m\omega^2 x = -kx$, and the projection moves with simple harmonic motion, with the center of the diameter as the position of rest. The circle used here is referred to as the circle of reference.

The amplitude of the simple harmonic motion is defined in Section 11-3 as the maximum value of x. Since $\sin \omega t$ cannot be greater than one, the maximum value of x is A in our equation $x = A \sin \omega t$, and A, the coefficient of the trigonometric term, is the amplitude of the simple harmonic motion. The period of the motion is the time required for one complete vibration. In this time, then, the projection must move from the center of the diameter, up to a maximum positive displacement, down to a maximum negative displacement, and back to the central point. In the same time, then, the particle moving with angular speed ω in a circular path will go just once around the circle. The angle turned through by this particle is 2π rad, and we see, from the definition of angular velocity $\omega = \theta/t$, that $\omega = 2\pi/T$, where T is the period of the simple harmonic motion. Also, since the frequency $f = 1/T$, $\omega = 2\pi f$. Thus, in our equation $x = A \sin \omega t$, the coefficient of t is $2\pi f$ or $2\pi/T$.

By definition, ds/dt is the velocity, and d^2s/dt^2 is the acceleration. By the use of just our equation $x = A \sin \omega t$ and its derivatives, we may now find the velocity and acceleration in our particular simple harmonic motion for any position x or time t.

The constant θ_0 in Eq. (11-11) is of use only when the position of the projection is specified when $t = 0$ at some point other than the position of rest, and in the absence of such specification may arbitrarily be set equal to zero. Suppose, though, that x has some value x_0 at $t = 0$. Then Eq. (11-11) becomes

$$x_0 = A \sin \theta_0,$$

and θ_0 may be evaluated.

Another use of the circle of reference is to simplify the kind of problem we encounter in part (c) of Problems 11-3 and 11-4, where we are asked for the minimum time necessary to move from point x_1 to point x_2 in the simple harmonic motion. See Figure 3. We can determine the angles ϕ_1 and ϕ_2, since $\sin \phi_1 = x_1/A$ and $\sin \phi_2 = x_2/A$. Their sum is the angle θ turned through by the particle in the circle of reference while the projection moves from x_1 to x_2. As the angular velocity ω is a constant, we may say that $\omega = \theta/t = 2\pi/T$ or $t = (\theta/2\pi)T$, and, knowing θ and the period T, the time is immediately determined.

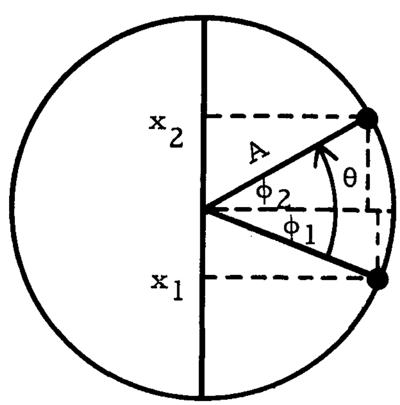


Figure 3

TEXT: Richard T. Weidner and Robert L. Sells, Elementary Classical Physics (Allyn and Bacon, Boston, 1973), second edition, Vol. 1

SUGGESTED STUDY PROCEDURE

Read Chapter 14, Sections 14-1 to 14-4, and work at least Problems A through H, and 14-1, 14-5, 14-12, 14-15, and 14-29 before attempting the Practice Test.

The general solution of a differential equation is discussed in General Comment 1. Study that carefully. Example 14-3 should be studied before working Problems B and F for Objective 2. Read the discussion on the small-angle approximation, $\sin\theta \approx \theta$, in the General Comment 3.

WEIDNER AND SELLS

Objective Number	Readings	Problems with Solutions		Assigned Problems		Additional Problems
		Study Guide	Text	Study Guide	Text	
1	Sec. 14-1, General Comment 1	A		E	14-1	14-4
2	Sec. 14-3, General Comments 2, 3	B	Ex.* 14-3	F	14-12	
3	Secs. 14-1, 14-2	C		G	14-5, 14-15	14-9, 14-10, 14-11
4	Sec. 14-4, General Comment 3	D		H	14-29	14-22, 14-25, 14-28

*Ex. = Example(s).

GENERAL COMMENTS

1. Differential Equations

Your present calculus course may not have acquainted you with differential equations. Hence we shall discuss them briefly without getting too fancy or formal. The equation

$$d^2x/dt^2 = -\omega^2 x \tag{1}$$

is called a second-order differential equation because it contains a second derivative. It is not like an algebraic equation for which certain constant values of x satisfy the equality. As is shown in your text the solution of Eq. (1) is a function of the time. Although the function

$$x = A\cos(\omega t + \theta) = A\cos\theta\cos\omega t - A\sin\theta\sin\omega t \tag{2}$$

can be thought of simply as being arrived at by a very clever guess, it can be shown (by advanced mathematical techniques) to be the most general possible solution of Eq. (1).

Equation (2) can also be written in terms of two new constants B and C as

$$x(t) = B\cos\omega t + C\sin\omega t. \tag{3}$$

(What are the relations among B, C, A, and θ?) The velocity is

$$v(t) = dx(t)/dt = -\omega B\sin\omega t + \omega C\cos\omega t. \tag{4}$$

These last two equations are especially helpful. For instance, if you are told that the particle begins its simple harmonic motion from rest at the point x_0, you know that $x(0) = x_0$ and $v(0) = 0$, hence since $\cos(0) = 1$ and $\sin(0) = 0$ you immediately have $B = x_0$ and $C = 0$. If the particle starts at the origin with velocity v_0, then you can conclude that $B = 0$ and $C = v_0$. Look at the equations and check these results for yourself. If you have a more complicated case in which the particle starts at x_0 with velocity v_0, then you can find B and C yourself, using the same method. Try it. Once you have found B and C, you can find then A and θ.

Note in the above discussion that a change in phase of $\pi/2$ does not change the solution. That is, let $\theta = \theta' + \pi/2$:

$$x = A\cos(\omega t + \theta), \qquad x = A\cos(\omega t + \theta' + \pi/2), \qquad x = A\sin(\omega t + \theta').$$

This last equation is just as valid a solution of the differential equation as the cosine function. Try substituting it in Eq. (1) and see for yourself.

2. Outline of Method for Investigating a System for Simple Harmonic Motion

$$F(x) = -kx, \tag{5}$$

$$m(d^2x/dt^2) = -kx. \tag{6}$$

I. Determine the net force acting on the particle.
(a) Identify forces acting on the particle by drawing a free-body diagram. Choose a convenient coordinate system.
(b) Find the net force acting on the particle as a function of its position in the chosen coordinate system.

II. Describe the particle's displacement from the equilibrium position.
(a) Find the position where the net force is equal to zero. That is the equilibrium position of the particle.
(b) If necessary, introduce a new coordinate system with origin at the equilibrium position.

III. Use the coordinate system introduced in II(b) to state Eqs. (5) and (6).
(a) Express the net force as a function of the new coordinates. Compare this with Eq. (5).
(b) Express the acceleration in terms of the second time derivative of the new coordinates.
(c) Use the expressions derived in steps (a) and (b) to state Newton's second law $\vec{F} = m\vec{a}$ in terms of the new coordinates. Compare its form with Eq. (6).

3. Approximation: $\sin\theta \simeq \theta$ for Small Angles

The first thing to note is that this is true only if θ is in radians. Obviously $\sin 1.00° \neq 1$. But $1° = 0.0174$ rad, and $\sin(0.0174 \text{ rad}) = 0.0174$. This approximation is good up to about 15.0° or 0.262 rad. $\sin(0.262 \text{ rad}) = 0.258$. This is only an error of about 1% so the approximation is pretty good. The error in the period of a pendulum when the amplitude = 15.0° is only 0.50%. Thus, even though a system actually does not execute simple harmonic motion, if the angular displacement is kept small enough its motion will be essentially simple harmonic.

A few of the many examples of simple harmonic oscillators are listed below, along with the expressions for F_x or τ. You should verify these expressions for yourself. The determination of ω, f, and T for each is left as an exercise.

Examples of Simple Harmonic Motion

a. Object on a spring (Fig. 4). Equilibrium occurs at the height for which the spring force equals $-m\vec{g}$. When the object is displaced, the spring force changes, but mg remains the same. Restoring force is $F_x = -kx$.

b. Object fastened to 2 identical stretched springs. (Fig. 5). When the object is displaced, one spring pulls more, and the other pulls less. Restoring force is $F_x = -(2k)x$.

c. Object fastened to two stretched springs, but displaced sideways (Fig. 6). If the displacement is small, the forces exerted by the springs change in direction, but hardly at all in magnitude. Restoring force is $F_x = -2F_0(x/\ell)$.

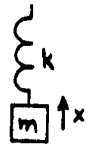


Figure 4

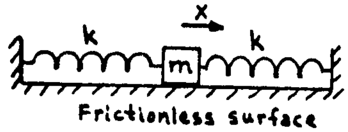


Figure 5

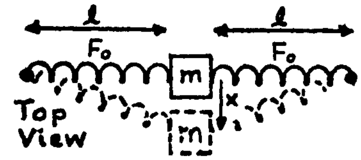


Figure 6

d. Object fastened to two stretched elastic strings (Fig. 7). Essentially the same as above. For small displacements, the tension F_0 in the strings does not change appreciably. Restoring force is $F_x = -2F_0(x/\ell)$.

e. Massive object on a "massless flagpole" (Fig. 8). For small displacements, the motion is almost linear. Restoring force is $F_x = -Kx$.

f. Object on a string (pendulum, Fig. 9). The restoring force is the component of $m\vec{g}$ perpendicular to the string, $-mg \sin\theta$. For small displacements, the motion is almost linear, and $\sin\theta \simeq \theta$. Restoring force is $F_x = -mg\theta = -mg(x/\ell)$.

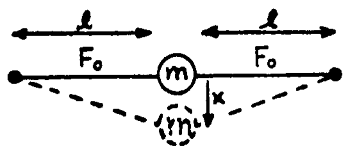


Figure 7

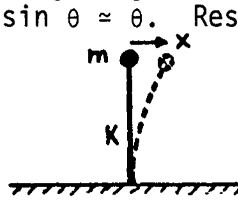


Figure 8

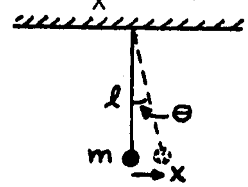


Figure 9

g. Small object sliding in a frictionless spherical bowl (Fig. 10). Same as above. Restoring force is $F_x = -mg\theta$. Or, use the restoring torque $\tau = -mg\ell\theta$ with $I = \ell^2 m$ and $\tau = I\alpha = I\ d^2\theta/dt^2$.

h. Pivoted plank on spring (Fig. 11). Same as a car with good shocks in front, very bad shocks in back. As the object bounces up and down, the force of gravity is constant, but the spring force changes. Restoring torque is $\tau = -\ell k(\ell\theta) = -k\ell^2\theta$.

i. Object hung from wire and rotating about a vertical axis (torsion pendulum, Fig. 12). Some mantlepiece clocks use a pendulum of this kind. Generally, the wire provides a restoring torque $\tau = -\kappa\theta$.

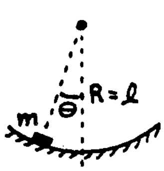


Figure 10

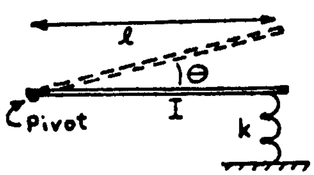


Figure 11

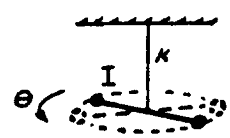


Figure 12

ADDITIONAL LEARNING MATERIALS

Film loops: "Simple Harmonic Motion"

"Velocity and Acceleration in Simple Harmonic Motion"

Available from Ealing Corporation.

PROBLEM SET WITH SOLUTIONS*

A(1). Define the following terms: amplitude, angular frequency, phase constant, period.

Solution

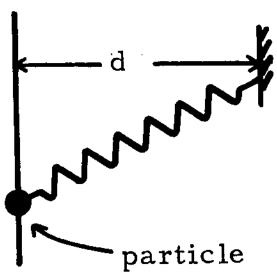
Amplitude: the maximum displacement from equilibrium of an oscillating particle.

Angular frequency: 2π divided by the time required to complete one cycle of the motion (or $2\pi f$). Another definition is the number of radians completed per second, knowing that 2π rad equal one cycle.

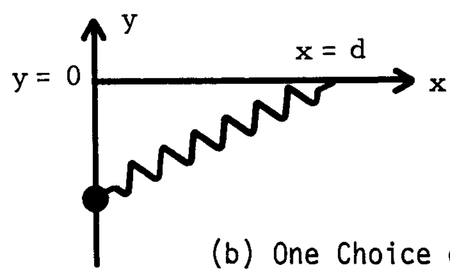
Phase constant: if $x = A\cos(\omega t + \theta)$ then $(\omega t + \theta)$ is called the phase of the motion. θ is called the phase constant (phase angle). The amplitude and the phase of the motion determine the initial velocity and position of the particle (or vice versa if you like). For example, if $\phi = \pi/2$ at $t = 0$, then $x = A\cos(\pi/2) = 0$ and the particle starts at $x = 0$. The unit of the phase will be radians.

Period: the time required to complete one cycle.

B(2). A particle of mass m is restricted to move on a vertical frictionless track. It is attached to one end of the massless spring with spring constant k and unextended length $\ell_0 = 0$ m (small compared to other lengths in the problem). The other end of the spring is hooked to a peg at the distance d from the track (see Fig. 13). (Use $g = 9.8\ m/s^2$.)
(a) Show that the particle carries out simple harmonic motion when displaced from its equilibrium position.
(b) Find the period of oscillation of the particle.



(a) The Physical System



(b) One Choice of a Coordinate System

Figure 13

*The key to Problems B and F is to find the net force acting on the particle, find the particle's equilibrium position, and then express the force as a function of the displacement from equilibrium. If and only if the force can be written as $F(x) = -kx$, then the motion is simple harmonic. The three-step procedure is described in General Comment 2.

Solution

(a) I. In Figure 14, $\vec{S}$ is the force of the spring, $\vec{N}$ is the constraining force of the track, $\vec{W}$ is the gravitational force, y is the vertical position of the particle,

$\vec{W} = -mg\hat{j}$, $\vec{N} = -N\hat{i}$, $\vec{S}$ = -(spring constant) × (extended length) = $-k(-d\hat{i} + y\hat{j})$.

The total force is then $\vec{F}(y) = \vec{S} + \vec{W} + \vec{N} = (kd - N)\hat{i} + (-ky - mg)\hat{j}$.

II. Where is the net force = 0? Set $\vec{F} = 0$ and solve for y.

$$\vec{F} = 0 = (kd - N)\hat{i} + (-ky - mg)\hat{j}.$$

Thus the net force is zero when N = dk and y = -mg/k. Introduce a new coordinate system at the equilibrium position:

$$x' = x, \quad y' = y + mg/k.$$

III. In this new system, the total force is

$$\vec{F}(y') = -ky'\hat{j}.$$

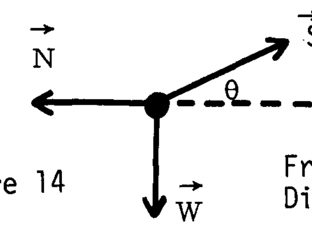


Figure 14 Free-Body Diagram

This is the same form of the force that gives rise to simple harmonic motion, F(x) = -kx, thus the motion of the mass on the vertical track is simple harmonic.

(b) Newton's second law is $\vec{F} = m\vec{a}$, so

$$-ky'\hat{j} = m(d^2y'/dt^2)\hat{j}, \quad -ky' = m(d^2y'/dt^2), \quad \omega^2 = k/m, \quad T = 2\pi/\omega = 2\pi\sqrt{m/k}.$$

C(3). One day you visit a friend who has a chair suspended on springs. When you sit down on the chair, it oscillates vertically at 0.50 Hz. After the oscillations have died down, you stand up slowly, and the chair rises 0.50 m. Next, your friend sits in the chair, and you find that the oscillations have a period of 2.10 s. Assume that your mass is 60 kg:
(a) What is the spring constant for the two springs together?
(b) What is the mass of the chair?
(c) What is the mass of your friend?
(d) While you are sitting in the chair, at a certain instant (t = 0) the chair is 0.300 m above its equilibrium position, and momentarily at rest. Find the expression for y(t), its displacement from equilibrium as a function of time.
(e) Under these conditions, what is the maximum kinetic energy of you and the chair? What is your maximum speed?

Solution

(a) $F = -kx$, $-mg = -kx$, $k = mg/x = 60(9.8)/0.50 = 1180 \approx 1200$ N/m.

(b) $\omega^2 = k/m$, $m = m_{chair} + m_{you} = k/\omega^2 = k/(2\pi f)^2$,

$$m_{chair} = k/(2\pi f)^2 - m_{you} = \frac{1200 \text{ N/m}}{[(2\pi)(0.50)]^2(1/s^2)} - 60 \text{ kg}$$

$$= (120 - 60) \text{ kg} = 60 \text{ kg}.$$

(c) $m_{friend} = k/(2\pi f)^2 - m_{chair} = 72$ kg.

(d) $A = 0.300$ m $= 2\pi f = 2\pi(0.50) = \pi$ rad/s. $y = A\cos(\omega t + \theta)$.

At $t = 0$, $y = A$, $A = A\cos(0 + \theta)$. Therefore $\cos\theta = 1$, $\theta = 0$.

$$y(t) = 0.300 \cos(\pi t) \text{ m}.$$

(e) Maximum kinetic energy = maximum potential energy

$$= (1/2)kx^2 = (1/2)(1200)(0.300)^2 = 54 \text{ J}.$$

Maximum speed $= (dy/dt)_{max} = (-A\omega \sin \omega t)_{max} = A\omega = 0.94$ m/s.

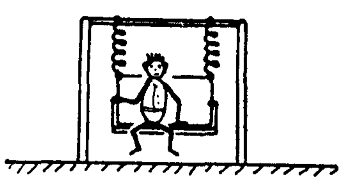


Figure 15

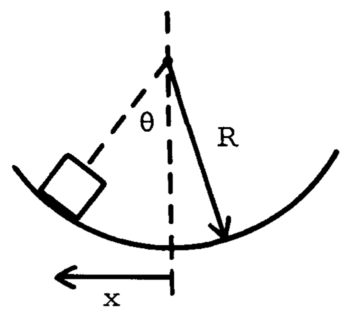


Figure 16

D(4). "A challenging problem." A particle with mass M slides freely in a hemispherical bowl of radius R, as shown in Figure 16.
(a) Find the potential energy U(x), making the approximations

$$1 - \cos\theta = 1 - [1 - (1/2)\theta^2 + \cdots] \simeq (1/2)\theta^2 \quad \text{and} \quad \theta \simeq x/R.$$

(b) Suppose the particle starts from rest with a displacement x_0; what is its kinetic energy K(x)?
(c) What is its velocity v_x as a function of x?
(d) What is its acceleration a_x as a function of x? (Hint: Differentiate!)
(e) Does this particle undergo simple harmonic motion? If not, explain why not; if so, find the angular frequency ω for this motion.

Solution

(a) As the particle slides up the slope to an angle θ it has increased in height a distance $R - R\cos\theta = R(1 - \cos\theta)$. If θ is small then $(1 - \cos\theta) \simeq (1/2)\theta^2$ and $\theta \simeq x/R$, so that

$$R(1 - \cos\theta) \simeq R[(1/2)\theta^2] = R[(1/2)(x^2/R^2)] = (1/2)(x^2/R).$$

Potential energy = mgh = $(mg/2R)x^2$.

(b) Kinetic energy = loss in potential energy

$$= (mg/2R)(x_0^2) - (mg/2R)(x^2) = (mg/2R)(x_0^2 - x^2).$$

(c) Total kinetic energy is $(1/2)mv^2$. Gain in kinetic energy = loss in potential energy:

$$(1/2)mv^2 = (mg/2R)(x_0^2 - x^2), \qquad v_x = \pm[(g/R)(x_0^2 - x^2)]^{1/2}.$$

Is this reasonable? First check the units. The resulting units should be the same on both the left- and right-hand sides of the equation:

$$[\frac{(g)}{(R)}(x)^2]^{1/2} = ? \qquad [\frac{(g)}{(R)}(x^2)]^{1/2} = [\frac{(m/s^2)m^2}{m}]^{1/2} = (m^2/s^2)^{1/2} = m/s = \text{unit of velocity}.$$

Thus the units are correct. Now what would happen if $x = x_0$? $(x_0 - x) = 0$ and $v = 0$. As we expect, the velocity is zero at the starting point. Now let's take x approaching zero. As $x \to 0$, $(x_0^2 - x^2) \to x_0^2$, a maximum value. Thus the velocity becomes maximum at $x = 0$, which is where we have the maximum kinetic energy.

(d) Before differentiating by brute force, remember that $d(y^2) = 2y\,dy$, therefore differentiate

$$v_x^2 = (g/R)(x_0^2 - x^2).$$

Thus $\quad 2v_x\,dv_x = (g/R)(-2dx\,dx).$

Divide by dt: $\quad v_x(dv_x/dt) = -x(g/R)(dx/dt).$

Now since $dx/dt = v_x, \quad dv_x/dt = a_x,$

divide each side by v_x to get

$$a_x = (-g/R)x.$$

(e) From Newton's second law $F = ma = m(dx^2/dt^2)$, thus

$$m(d^2x/dt^2) = -(g/R)x \quad \text{or} \quad d^2x/dt^2 + (g/Rm)x = 0.$$

This is a form of the equation for simple harmonic motion:

$$d^2x/dt^2 + (k/m)x = 0 \quad \text{or} \quad F = -kx.$$

So yes, simple harmonic motion does occur. The angular frequency $\omega = \sqrt{k/m} = \sqrt{g/mR}$. Part (e) could also be answered by finding the restoring force.

Problems

E(1). The displacement of an object undergoing simple harmonic motion is given by the equation

$$x(t) = \underset{\text{m}}{3.00}\,\sin(\underset{\frac{\text{rad}}{\text{s}}}{8\pi}\,\underset{\text{s}}{t} + \underset{\text{rad}}{\pi/4}).$$

Note that the mks units of each term are shown underneath.
(a) What is the amplitude of motion?
(b) What is the frequency of the motion?
(c) Sketch the position of the particle as a function of time, starting at $t = 0$.

F(2). In the book Tik-Tok of Oz, Queen Anne, Hank the mule, the Rose Princess, Betsy, Tik-Tok, Polychrome, the Shaggy Man, and the entire Army of Oogaboo all fall through the straight Hollow Tube to the opposite side of the earth. The retarding force of the air is evidently negligible during this trip, since they all pop out neatly at the other end. For an object at a distance r from the center of such a spherical mass distribution the gravitational force has the magnitude (you will not have to derive something like this):

$$F_g(r) = (r/R_e)^3 GM_e(m/r^2) = mgr/R_e,$$

and is directed toward the center of the earth. Use $R_e = 6.4 \times 10^6$ m for the radius of the earth:
(a) Do they undergo simple harmonic motion? How do you know?
(b) How long does their trip last?

G(3). An automobile with very bad shock absorbers behaves as though it were simply mounted on a spring, as far as vertical oscillations are concerned. When empty, the car's mass is 1000 kg, and the frequency of oscillation is 2.00 Hz.
(a) What is the spring constant?
(b) How much energy does it take to set this car into oscillation with an amplitude of 5.0 cm (assuming all damping can be neglected)?
(c) What is the maximum speed of the vertical motion in (b)?
(d) Suppose that four passengers with an average weight of 60 kg now enter the car. What is the new frequency of oscillation?

H(4). The rotor of the electric generator in Figure 17 is to be driven by a long shaft. Since any rotational oscillations about axis AA′ of the rotor would cause fluctuations in the electrical output, an engineer decides to investigate this possibility, starting with the case of completely undamped motion (i.e., no friction).
(a) It takes a torque $\tau = \kappa\theta$ to twist the shaft an angle θ. When one end is clamped as in the figure, will the rotor undergo simple harmonic motion? How do you know?

(b) With the parameters indicated in the figure, what will she find for the frequency?
(c) What are the maximum potential and kinetic energies of the oscillation if the amplitude is 0.00100 rad? What is the total mechanical energy?
(d) What change in the stiffness of the shaft (κ) would be necessary to double the frequency of oscillation?

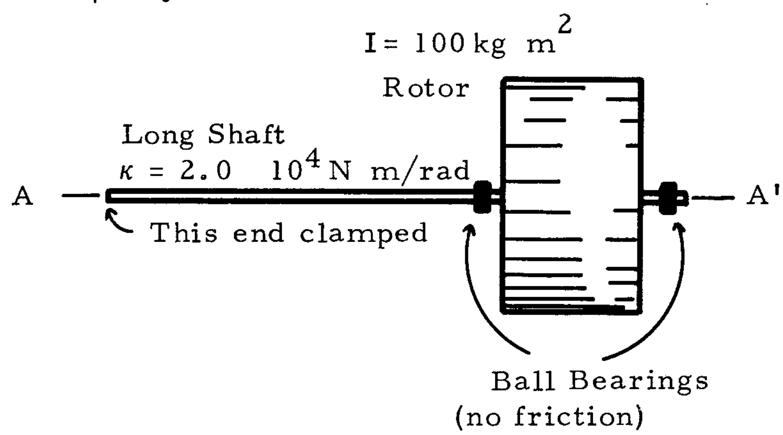


Figure 17

Solutions

E(1). (a) 3.00 m.

(b) $\omega = 8\pi$ rad/s,

if $x = 3.00 \sin(8\pi t + \theta)$ m.

$f = \omega/2\pi = 4.0$ Hz.

(c)

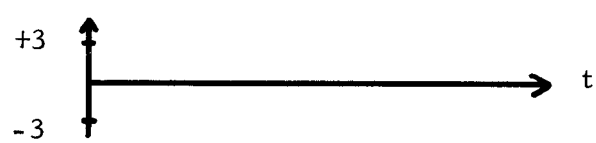


Figure 18

F(2). (a) Yes; the gravitational force they experience has a magnitude proportional to the distance from the center of the earth, and is directed toward the center.
(b) If you take the x axis to lie along the tube, then $m(d^2x/dt^2) = F_x = -mgx/R_e$, or $d^2x/dt^2 = -(g/R_e)x$. But from (a) above we know that their motion is given by an expression of the form $x = A\cos(\omega t + \theta)$, for which $d^2x/dt^2 = -\omega^2 x$. Therefore, $\omega = \sqrt{g/R_e}$; and the duration of their trip is $(1/2)T = \pi\sqrt{R_e/g} = 800\pi$ s $= 0.70$ h.

G(3). (a) $k = m\omega^2 = m(2\pi f)^2 = 1.60 \times 10^5$ N/m. Does this seem reasonable?
(b) Total energy = maximum potential energy = 2.00×10^2 J.
(c) From conservation of energy,

$$v_{max} = \sqrt{k/m}A = 0.63 \text{ m/s}.$$

(d) $f = (1/2\pi)\sqrt{k/m} = 1.80$ Hz. (Which mass should you use?)

H(4). (a) Yes, τ is proportional to θ.
(b) $I\alpha = \tau = -\kappa\theta$, or $d^2\theta/dt^2 = -(\kappa/I)\theta$. But $\theta = \theta_0 \cos(\omega t + \theta)$, from part (a), so that $d^2\theta/dt^2 = -\omega^2\theta$. Therefore, $\omega = \sqrt{\kappa/I}$, and $f = \omega/2\pi = 2.25$ Hz.
(c) 0.0100 J each.
(d) Since κ varies as the square of ω, if ω is doubled, κ becomes four times as stiff.

PRACTICE TEST

1. Seesaws at parks often go unused because two small children seldom decide to play on them simultaneously. As Technical Consultant to the Park Board, your first assignment is to provide specifications for a One-Tot-Teeter. The design is partly determined by the existing equipment. (See Figure 19.) A child of mass m is to receive a ride with a period of T seconds. Without child or counterweight, the teeter-totter has a rotational inertia I_1; the child and counterweight, of course, have I_2. Note: the spring is attached to the teeter. Start with Newton's second law for rotational motion to find the answer.
   (a) Does simple harmonic motion occur? Why?
   (b) What spring constant is needed?

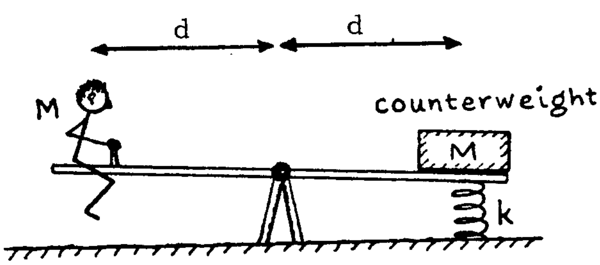


Figure 19

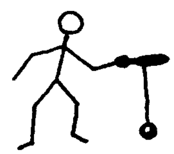


Figure 20

2. A child is bouncing a 50-g rubber ball on the end of a rubber string, in such a way as to give the ball and string a total energy of 0.050 J (counting the potential energy as zero at the equilibrium position). If the ball were just hanging at rest, it would stretch the string 20.0 cm. For the motion of the bouncing ball, find
   (a) the angular frequency ω,
   (b) the amplitude,
   (c) the maximum kinetic energy,
   (d) the maximum speed of the ball, and
   (e) the expression for its acceleration as a function of time, if the position = +0.200 m at t = 0 s.

Practice Test Answers

1. (a) Total moment of inertia equals $I_1 + I_2 = I_{total} = I$.

Total torque = $\tau_{child} + \tau_{counterweight} + \tau_{spring}$, $\tau_{child} = -\tau_{counterweight}$.

Total torque = τ_{spring} = (force) × (distance) = $-kyd$,
$y = (\sin\theta)d \approx \theta d$ approximately.

Thus $\tau = -k\theta d^2$, which is a restoring torque proportional to displacement. Therefore motion will be simple harmonic.

(b) $\tau = I\alpha = I(d^2\theta/dt^2) = -k\theta d^2$, $d^2\theta/dt^2 = -(kd^2/I)\theta = -\omega^2\theta$

$\omega^2 = kd^2/I$, $\omega^2 = (2\pi f)^2 = (2\pi/T)^2$,

thus

$kd^2/I = (2\pi/T)^2$, $k = I(2\pi/dT)^2$.

2. (a) $F = -kx$, $-mg = -kx$, $k = mg/x$,

$\omega = \sqrt{k/m} = \sqrt{mg/xm} = \sqrt{g/x} = \sqrt{9.8/0.200} = 7.0$ rad/s.

(b) $A = ?$ $k = mg/x = (0.50\text{ kg})(9.8\text{ m/s}^2/0.200\text{ m}) = 2.45$ N/m.

$U = (1/2)kx^2$, $U_{max} = (1/2)kA^2$, $A = \sqrt{2U_{max}/k} = \sqrt{2(0.050)/2.45}$
$= \sqrt{0.0408} = 0.200$ m.

(c) $K_{max} = U_{max} = E_{total} = 0.050$ J.

(d) $K_{max} = (1/2)mv^2_{max}$, $v_{max} = \sqrt{2K_{max}/m} = \sqrt{2(.05/0.050} = 1.40$ m/s.

(e) $a = -A\omega^2\cos(\omega t + \theta) = -(0.200)(7^2)\cos(7t + \theta) = -9.8\cos(7t + \theta)$ m/s^2.

At $t = 0$, $x = A\cos\theta = 0.200\cos\theta = 0.200$. Thus $\cos\theta = 1$, $\theta = 0$ rad.

SIMPLE HARMONIC MOTION

Mastery Test Form A

Name________________________________

Date________________________

pass recycle

1 2 3 4

Tutor________________________

1. A block sliding on a frictionless surface is held in place by two springs attached to opposite sides and <u>stretched</u> to clamps at the edge of the surface as in Figure 1. The springs have an unextended length = ℓ and a force constant k.

   (a) Show that the block will carry out simple harmonic motion on the surface if it is pushed closer to one of the clamps and then released.

   (b) Find the angular frequency ω for this motion.

2. You have been retained as a consultant to a traveling circus to advise on problems of a trapeze act. See Figure 2. The artists each have a mass of 80 kg and will use a trapeze hung from ropes 30.0 m long to travel between platforms 15.0 m apart. The approximation, $\sin\theta \simeq \theta$ is valid here.

   (a) The musical director wants to know how long it will take them to swing back and forth with one of the two persons on the trapeze.

   (b) What is the magnitude of the maximum velocity?

   (c) The property manager wants to know what the maximum tension on the ropes will be with <u>two</u> people swinging.

   (d) Write an expression for the displacement as a function of time, assuming the artists start at the right-hand side at t = 0.

Furthermore, the circus owner insists that you <u>start from Newton's second law</u> to find the answer to (a). We await your answers.

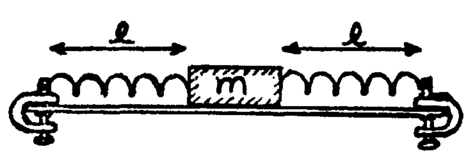


Figure 1

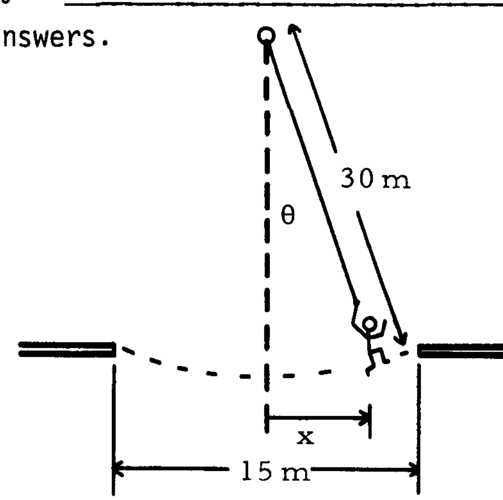


Figure 2

SIMPLE HARMONIC MOTION

Mastery Test Form B

Name________________________________

Date________________________

pass recycle

1 2 3 4

Tutor________________________

1. A car with good shocks in front and very bad shocks in the back can be modeled by a pivoted plank on a spring as in Figure 1. The spring constant is k, the length is ℓ, and the moment of inertia I about the pivot is $m\ell^2/3$.

   (a) Show that the plank will carry out simple harmonic motion if it is pushed up at a small angle θ and then released.

   (b) Find the angular frequency for this motion.

2. You have just been hired by Tinseltown Movie Studios to design a "jungle elevator" for Tarzan. We want 90-kg Tarzan to grab the end of a hanging elastic vine, step off his tree branch, and be brought to rest at the ground 15.0 m below.

   (a) Where should the equilibrium point be for the system of Tarzan-plus-vine?

   (b) What must be the force constant of the vine?

   (c) How long does the trip take?

   (d) What is the maximum pull on Tarzan's hands?

   (e) What is Tarzan's maximum velocity.

   (f) Write Tarzan's velocity as a function of time if he starts from the limb at t = 0.

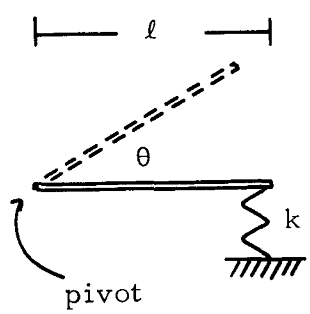


Figure 1

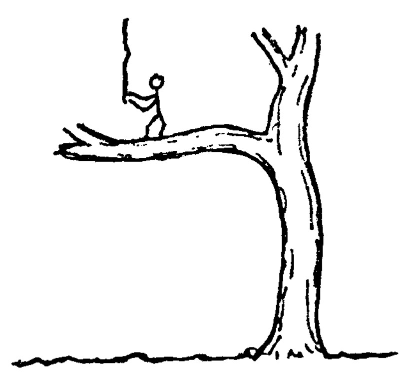


Figure 2

SIMPLE HARMONIC MOTION

Mastery Test   Form C

Name________________________________

Date__________________________

pass   recycle

1   2   3   4

Tutor__________________________

1. A long thin rod of length ℓ and mass m is pivoted about a point on the rod that is a distance h above the center of the rod. The moment of inertia about the pivot point is $m(\ell^2/12 + h^2)$. See Figure 1.

   (a) Show that the rod will carry out simple harmonic motion if it is pushed to one side and then released.

   (b) Find the angular frequency for this motion.

2. A body of mass 100 g hangs on a long spiral spring. When pulled down 10.0 cm below its equilibrium position and released, it vibrates with a period of 2.00 s.

   (a) What is its velocity as it first passes through the equilibrium position?

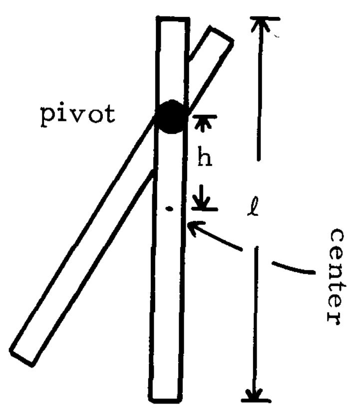
   (b) Write the position of the body as a function of time assuming that $x = -10.0$ cm at $t = 0$..

   (c) What is its acceleration when it is 5.0 cm above the equilibrium position?

   (d) When it is moving upward, how long a time is required for it to move from a point 10.0 cm below its equilibrium position to its equilibrium position.

   (e) How much will the spring shorten if the body is removed?

Figure 1



SIMPLE HARMONIC MOTION

Date____________________________

Mastery Test   Form D

pass   recycle

Name____________________________

1 2 3 4

Tutor____________________________

1. A simple pendulum may be used to determine the acceleration due to gravity. In doing an experiment a student reports that a simple pendulum with a length of 1 m has a period of 2 seconds.

   a) what is his calculated value of g?

   b) If this pendulum has an amplitude of 0.2 meters, what is its velocity and acceleration for a displacement of 0.1 m?

2. A spring has a natural length of 20 cm and a force of 1N produces a stretch of 2 cm. An unknown mass is hung upon this spring in a vertical position and it is set into vibration with an amplitude of 10 cm and it is observed that the period is $2\pi/25$ seconds.

   a) What is the mass of the body?

   b) If the body is at the lower end of its path at $t = 0$, write the equation of motion.

   c) What is the acceleration when the mass has a displacement of 5 cm?

   d) What is the maximum kinetic energy of the body?

   e) What is the minimum distance travelled during any one-quarter period?

SIMPLE HARMONIC MOTION

Mastery Test Form E

Name______________________

Date______________________

pass recycle

1 2 3 4 5

Tutor______________________

1. A thin metal ring is hung on a horizontal knife efge and displaced through a small angle θ and released. See Figure 1. Given: mass of ring = 0.5 kgm and radius r = 0.1 and the moment of inertia about knife edge is given by $I = 2mr^2$.

   a) Show that the ring will carry out simple harmonic motion.

   b) What is the period of motion?

2. A 0.1 kgm body is moving along a straight line so that its displacement is given by

   $x = 8 \cos 4\pi t$ cm.

   Find: a) amplitude of motion
   b) velocity when x = 0
   c) acceleration when x = 8
   d) restoring force when x = 6
   e) period of motion
   f) minimum time to go from +4 to -4 cm
   g) maximum P.E. of body
   h) kinetic energy when x = 2.

SIMPLE HARMONIC MOTION DATE__________

Mastery Test Form F pass recycle

Name__________ 1 2 3 4 5

Tutor__________

1. A particle oscillating in simple harmonic motion travels a distance of 40 cm during the time of one complete cycle. The maximum acceleration is 8 cm/sec^2.

   a) What is the frequency of oscillation?

   b) Write the equation of motion if the displacement of the particle is + 10 cm at t = 0.

2. When a man of mass 70 kgm stands on the end of a diving board, the end is lowered 30 cm from its original position. Assume the displacement is proportional to the force applied to it and that the mass of the board is small compared with that of the man. A force is applied to the end of the diving board and lowers it another 50 cm and it is then released.

   a) Describe the motion which will follow.
   b) what is the period of the man on the diving board?
   c) What is the maximum velocity that the man will experience? Where will this be?
   d) What will be his maximum acceleration? Where?
   e) What is the maximum distance the man travels during any 1/4 period?

MASTERY TEST GRADING KEY - Form A

1. What To Look For: If student shows that $F \propto x$, that's enough for part (a).

Solution: (a) See Figure 21. Let origin be at midpoint. Since at that point the springs are not extended, the horizontal forces are equal and opposite. Thus, that is the equilibrium position as well. If it is displaced a distance x from equilibrium then

$$\vec{F}_1 = -kx\hat{i}, \quad \vec{F}_2 = -kx\hat{i}, \quad \vec{F}_{total} = -2kx\hat{i} = m(d^2x/dt^2)\hat{i}.$$

Thus $m(d^2x/dt^2) = -2kx$: same form for simple harmonic motion. Thus simple harmonic motion does occur.

(b) $\dfrac{d^2x}{dt^2} + \dfrac{2k}{m}x = 0$, $\omega = \sqrt{2k/m}$.

2. What To Look For: (a) See if student uses Newton's second law to get equation of motion. (c) Could also use

$$F = -mg \sin\theta \simeq -mg\,\theta \simeq -mg(x/L) = (d^2x/dt^2),$$

so that $d^2x/dt^2 + (g/L)x = 0$.

(d) Why is it not $7.5 \cos(0.57t + \theta)$, where θ is some angle other than zero?

Solution: (a) $\tau = -mgL \sin\theta$. For small θ, $\sin\theta \simeq \theta$,

$\tau = mgL\theta = I(d^2\theta/dt^2)$, $\quad I = mL^2$ for a particle around an axis.

$-mgL\theta = mL^2(d^2\theta/dt)$, $\quad d^2\theta/dt^2 + (g/L)\theta = 0.$ $\quad \omega = \sqrt{g/L}$.

$T = 1/f = 2\pi/\omega = 2\pi\sqrt{L/g} = 2\pi\sqrt{30.0/9.8} = 11$ s, the time for one cycle, back and forth. T is independent of mass, thus time is same for one or two persons.

(b) From conservation of energy:

$(1/2)mv_{max}^2 = mgh$,

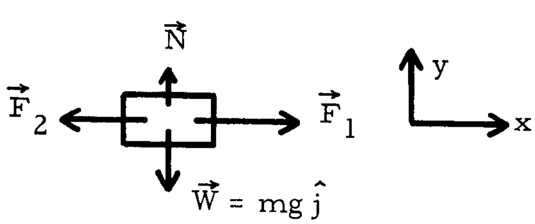


Figure 21

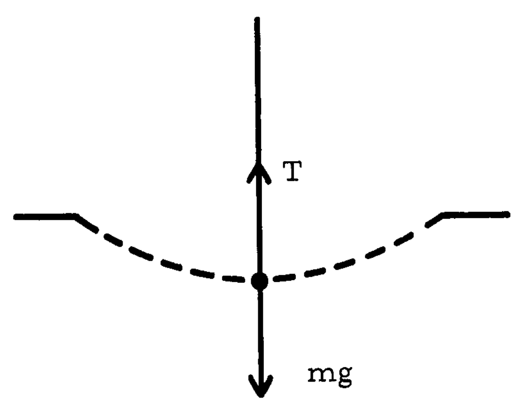


Figure 22

where h is the distance from ledge to bottom of swing: $h = L(1 - \cos\theta)$ $= L\{1 - \cos[\sin^{-1}(7.5/30)]\}$, $v_{max} = \sqrt{2gh} = 4.32$ m/s.

(c) See Figure 22. For circular motion: Net force = centripetal force,

$T - mg = mv^2/L$, $T = mg + mv^2/L$.

T_{max} occurs at v_{max} which occurs at the bottom of the swing. From Part (b) $v^2 = 2gL(1 - \cos\theta)$.

Tension $= 2mgh/L + mg = 2mg\{1 - \cos[\sin^{-1}(7.5/30)]\} + mg = 1.66 \times 10^3$ N.

(d) $x = 7.5\cos(0.57t)$.

MASTERY TEST GRADING KEY - Form B

1. What To Look For: (a) If they show that $\tau \propto \theta$, that is enough for part (a). Note: $y'/\ell \simeq \sin\theta \simeq \theta$.

Solution: (a) See Figure 23.

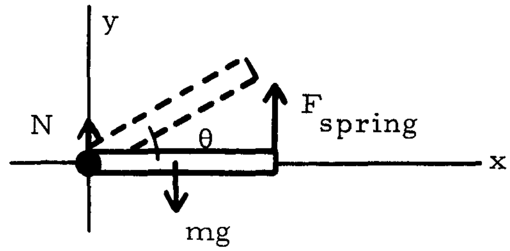


Figure 23

Considering rotational motion only, the plank is in equilibrium when

$\text{torque}_{mg} = \text{torque}_{spring}$, $\quad -mg\ell/2 = ky\ell$, $\quad y = -mg/2k$ = equilibrium position.

If it is displaced a distance y' from equilibrium then the net restoring torque is given by

$$\tau_{restoring} = -ky'\ell = I(d^2\theta/dt^2) = (m\ell^2/3)(d^2\theta/dt^2).$$

Thus

$$d^2\theta/dt^2 + \frac{3k}{m}\frac{y'}{\ell} = 0, \quad \frac{y'}{\ell} \simeq \theta, \quad \frac{d^2\theta}{dt^2} + \frac{3k}{m}\theta = 0.$$

Thus it is simple harmonic motion: $\omega = \sqrt{3k/m}$.

2. What To Look For: (c) How long would it take to go from the tree limb to 7.5 m above ground? (d) If Tarzan were oscillating up and down, is there a point where the pull would be zero? If so, where? (f) Check for minus sign if they used the sine function.

Solution: (a) Since the points of zero kinetic energy are the tree limb and ground, the point of maximum kinetic energy or equilibrium is halfway or 7.5 m.

(b) If Tarzan were to oscillate slowly and finally come out to rest, he would be at equilibrium and

$$F_{vine} = -kx_{eq} = -mg,$$

Thus

$$k = mg/x_{eq} = 90(9.8)/7.5 = 118 \text{ N/m}.$$

(c) Trip would take one-half cycle or (1/2)T.

$T = 1/f = 2\pi/\omega = 2\pi\sqrt{m/k}$,

$$(1/2)T = \pi\sqrt{m/k} = \pi\sqrt{\frac{m}{mg/x_{eq}}} = \pi\sqrt{x_{eq}/y} = \pi\sqrt{7.5/9.8} = 2.75 \text{ s.}$$

(d) Maximum pull = maximum restoring force of vine

$$= F_{vine\ (max)} = -kx_{max} = (mg/x_{eq})x_{max} = 90(9.8)(15/7.5) = 1760 \text{ N.}$$

(e) $(1/2)kx^2 = (1/2)mv^2$ from conservation of energy.

$$v_{max} = \sqrt{k/m}\, x_{max} = \sqrt{118/90}\ 7.5 = 8.58 \text{ m/s.}$$

(f) $v = v_{max} \cos(\omega t + \theta)$.

At $t = 0$, $v = 0$, thus $\theta = \pi/2$.

$v(t) = 8.58 \cos(1.14t + \pi/2) = -8.58 \sin(1.14t)$ m/s.

MASTERY TEST GRADING KEY - Form C

1. What To Look For: (a) For how large an angle does $\sin\theta \simeq \theta$ hold?

Solution: This is the same thing as a physical pendulum. See Figure 24.

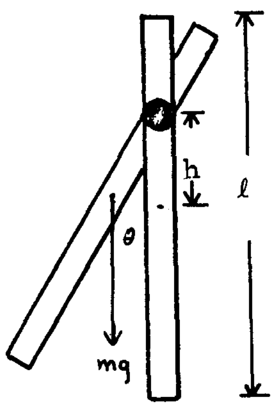


Figure 24

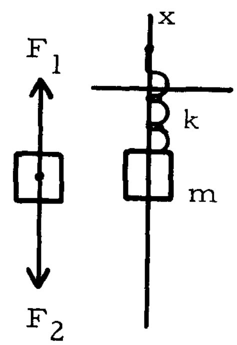


Figure 25

restoring torque = $-mgh \sin\theta$, torque = $I\alpha = I(d^2\theta/dt^2)$,

$-mgh \sin\theta = I(d^2\theta/dt^2)$.

$\sin\theta \simeq \theta$ for small θ, $-mgh\theta = I(d^2\theta/dt^2)$, $d^2\theta/dt^2 + mgh\theta/I = 0$.

This is the same as for simple harmonic motion. Thus simple harmonic motion occurs.

(b) $\omega = \sqrt{\dfrac{mgh}{I}} = \sqrt{\dfrac{mgh}{m(\ell^2/12 + h^2)}} = \sqrt{\dfrac{gh}{(\ell^2/12 + h^2)}}$.

2. What To Look For: (b) It is more confusing if we say $A = +10.0$ cm, $\theta = \pi$, although it can be done.

(c) Alternate solution for (c):

$F = -kx$, $-kx = -ma$, $a = (-k/m)x$, $a = -\omega^2 x = -0.49\ \mathrm{m/s^2}$.

(d) How long would it take to go from $x = -10.0$ cm to $x = 10.0$ cm?

Solution: (a) From conservation of energy:

$(1/2)mv^2_{max} = (1/2)kx^2_{max}$, $\omega^2 = k/m$ or $k = m\omega^2$, $(1/2)mv^2 = (1/2)m\omega^2 x^2$.

$v_{max}^2 = \omega^2 x_{max}^2$, $v_{max} = \omega x_{max} = 2\pi(1/T)x_{max} = 2\pi(1/2)(0.100) = 0.310$ m/s.

($x_{max} = A$, thus $v_{max} = A\omega$.)

(b) $x = A\cos(\omega t + \theta)$, $\omega = 2\pi/T = \pi$.

At $t = 0$, $x = -10.0$ cm, thus $\theta = 0$, $A = -10.0$.

$x = -10.0\cos(\omega t)$.

(c) at $x = +5.0$ cm, $t = t_1$,

5.0 cm $= -10.0\cos(\omega t_1)$ cm or $0.50 = \cos(\omega t_1)$ at $x = +5.0$.

$$\text{Acc. (at } x = +5.0 \text{ cm)} = \left.\frac{d^2x}{dt^2}\right|_{(x\,=\,+5.0)} = A\omega^2\cos(\omega t_1)\Big|_{(x\,=\,+5.0)} = A\omega^2(0.50)$$
$$= 0.100[2\pi(1/2)]^2 0.50 \text{ m/s}^2 = 0.49 \text{ m/s}^2.$$

(d) As in part (b), $A = -10.0$ and $\theta = 0$. Let time $= t_2$:

$x = A\cos(\omega t_2)$, $0 = -10.0\cos(\omega t_2)$, $\cos(\omega t_2) = 0$,

$\omega t_2 = \pi/2$, $t_2 = 0.50$ s.

(e) See Figure 25.

$k = m\omega^2 = m[2\pi(1/T)]^2$, $F_1 = -kx\tau$, $F_2 = -mg\tau$,

$-kx = -mg$, $m[2\pi(1/T)]^2 x = -mg$, $x = gT^2/(2\pi)^2 = -(9.8)(2)^2/(2\pi)^2 = -1.00$ m.

Thus, spring will shorten by one meter.