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Periodic modules over Gorenstein local rings

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PERIODIC MODULES OVER GORENSTEIN LOCAL RINGS

by

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It is proved that the minimal free resolution of a module M over a Gorenstein local ring R is eventually periodic if, and only if, the class of M is torsion in a certain $\mathbb{Z}[t^{\pm 1}]$-module associated to R. This module, denoted $J(R)$, is the free $\mathbb{Z}[t^{\pm 1}]$-module on the isomorphism classes of finitely generated R-modules modulo relations reminiscent of those defining the Grothendieck group of R. The main result is a structure theorem for $J(R)$ when R is a complete Gorenstein local ring; the link between periodicity and torsion stated above is a corollary.

DEDICATION

To Nicholas, for everything

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Contents

Chapter 1

Introduction

This dissertation makes a contribution to commutative algebra, particularly the study of resolutions of modules over local rings. The generators of a module over a ring, unlike the basis elements of a vector space over a field, may satisfy nontrivial relations. Given a module M over a commutative ring R and a set of generators $m_1, \ldots, m_s$, a *syzygy of* M is an element $(r_1, \ldots, r_s) \in R^s$ such that $r_1 m_1 + \cdots + r_s m_s = 0$. The set of all syzygies of M with respect to this generating set is a submodule of R^s. When R is a (commutative) *local ring*, that is, a ring with a unique maximal ideal, selecting a minimal set of generators makes this submodule unique; it is called the *first syzygy of* M and denoted ΩM. For example, consider the ring $R = \mathbb{C}[[x, y]]$ of formal power series in x and y over $\mathbb{C}$. The field $\mathbb{C}$ is an R-module with R-action given by mapping x and y to zero, and the module $\Omega\mathbb{C}$ is generated by x and y. Note that $-y \cdot x + x \cdot y = 0$, and hence $(-y, x)$ is an element of the first syzygy of $\Omega\mathbb{C}$, which is the *second syzygy of* $\mathbb{C}$. For $n \geq 2$, the *nth syzygy* of a module M is $\Omega^n M = \Omega\left(\Omega^{n-1} M\right)$.

Hilbert [9] proved that for a polynomial ring over a field the process of calculating syzygies of a module is finite; more precisely, for a polynomial ring in n variables, the

$(n+1)$st syzygy of any module is zero. Modules with syzygies that are eventually zero can be studied using techniques inspired by linear algebra. For general rings, however, there exist modules such that every syzygy is nonzero.

Among modules for which all syzygies are nonzero, the simplest are those whose syzygies repeat periodically or eventually do so; these modules are said to be *periodic* or *eventually periodic* , respectively. For instance, consider $R = \mathbb{C}[x]/(x^3)$, the ring of polynomials in x modulo the relation $x^3 = 0$. The first syzygy of the R-module $\mathbb{C}$ is generated by x, and $\Omega^2\mathbb{C}$ is generated by x^2. In fact, $\Omega^{2\ell-1}\mathbb{C} = (x)$ and $\Omega^{2\ell}\mathbb{C} = (x^2)$ for all $\ell \in \mathbb{N}$.

For complete intersection rings, which are the nicest class of rings besides polynomial rings, Eisenbud [5, Thm 6.1] proved that for any module M which is eventually periodic one has $\Omega^{2+\ell}M \cong \Omega^{\ell}M$ for $\ell \gg 0$. However, little is known about such modules over the much broader class of Gorenstein rings, which play an important role in commutative algebra and algebraic geometry. Gorenstein rings arise naturally in many places, including as the coordinate ring of a Grassmannian variety and the ring of invariants given by a finite subgroup of the special linear group $\mathrm{SL}(n, \mathbb{C})$ acting on $\mathbb{C}[x_1, \ldots, x_n]$, the polynomial ring in n variables over $\mathbb{C}$. Examples of Gorenstein rings include $\mathbb{C}[x_1, \ldots, x_d]$, $\mathbb{C}[x_1, \ldots, x_d]/(x_1^{a_1}, \ldots, x_d^{a_d})$, $\mathbb{C}[x, y, z, w]/(xy - zw)$, and $\mathbb{C}[x, y, z]/(x^2 - y^2, x^2 - z^2, xy, xz, yz)$.

Given any $n \in \mathbb{N}$, Gasharov and Peeva [7] give an example of a Gorenstein ring with a periodic module M such that $M \cong \Omega^n M$ and $M \ncong \Omega^m M$ for $m < n$. Using [2] one can also construct Gorenstein rings over which no nonfree module is eventually periodic. In 1990 Avramov [1] posed the problem of characterizing the rings which have a periodic module. One of the main results of this dissertation is a characterization of complete Gorenstein local rings with periodic modules in terms of a certain $\mathbb{Z}[t^{\pm 1}]$-module associated to the ring, where $\mathbb{Z}[t^{\pm 1}]$ denotes the ring of Laurent polynomials.

In order to state this result, we define the module $J(R)$, which is the main object of study in this dissertation. For an R-module M, we write $[M]$ for the isomorphism class of M. The module $J(R)$ is the free $\mathbb{Z}[t^{\pm 1}]$-module with basis given by the set of isomorphism classes of finitely generated R-modules modulo the relations $[P] = 0$, $[M] = t[\Omega M]$, and $[M \oplus N] = [M] + [N]$ for all finitely generated R-modules M, N, and P with P projective (that is, a direct summand of a free module); see Definition 3.1.1 and Proposition 3.2.4. The structure of the module $J(R)$ is described in Theorem 5.1.3:

Let R be a Gorenstein local ring with unique maximal ideal $\mathfrak{m}$ such that R is complete with respect to the $\mathfrak{m}$-adic topology. Then $J(R)$ is a free $\mathbb{Z}$-module with basis given by the set of isomorphism classes of maximal Cohen-Macaulay, nonfree, indecomposable modules.

The maximal Cohen-Macaulay modules are the most important part of the module category of a ring. Over a Gorenstein local ring, the high syzygies of every module are maximal Cohen-Macaulay; in other words, when taking successive syzygies of a module, there will always be a point after which every syzygy is maximal Cohen-Macaulay.

The structure of $J(R)$ given by Theorem 5.1.3 allows for a better understanding of torsion in $J(R)$. If M is eventually periodic, the relation between the class of M and the classes of its syzygies shows that $[M]$ is *torsion* in $J(R)$, that is, there exists $f(t) \in \mathbb{Z}[t^{\pm 1}]$ such that $f(t)[M] = 0$ in $J(R)$. In Theorem 5.2.6, it is shown that over a Gorenstein local ring, the class of a finitely generated module is torsion in $J(R)$ if and only if the module is eventually periodic. Corollary 5.2.8 gives an answer to Avramov's question concerning existence of periodic modules in the case of complete

Gorenstein local rings: the ring R has a periodic module if and only if $J(R)$ has a nonzero torsion submodule.

This research is motivated by work of D.R. Jordan [13], who defined the module $J(R)$ and proved that if the class of a module in $J(R)$ is torsion then the module has a rational Poincaré series. In addition, Jordan proved that the converse does not hold, even for the residue field of a complete intersection ring with codimension at least two.

Chapters 2 and 4 of this dissertation contain background material. Chapter 2 discusses syzygies and cosyzygies, periodic modules, and the Krull-Remak-Schmidt property. Chapter 4 covers maximal Cohen-Macaulay modules over Gorenstein rings. We collect several well-known results on maximal Cohen-Macaulay modules over Gorenstein local rings; these results are hard to find in the literature in the form we need. In Chapter 3, the module $J(R)$ is defined and some of its basic properties are stated. The main results of this dissertation are contained in Chapter 5, which is a study of the module $J(R)$ for Gorenstein local rings.

Chapter 2

Periodic modules

This chapter covers background material on syzygies, cosyzygies, and the Krull-Remak-Schmidt property. We discuss periodic modules and some well-known results concerning such modules. For a local ring R with maximal ideal $\mathfrak{m}$, periodicity of an R-module M is equivalent to periodicity of the $\widehat{R}$-module $\widehat{M}$, where $\widehat{R}$ and $\widehat{M}$ denote the $\mathfrak{m}$-adic completions of R and M, respectively; see Lemma 2.3.4.

2.1 Syzygies

All rings considered in this dissertation are commutative and Noetherian. In order to discuss syzygies, we recall Schanuel's Lemma; a proof of this lemma can be found in [14, Thm 4.1.A].

Schanuel's Lemma. *Given exact sequences of R-modules*

$$0 \to K \to P \to M \to 0 \quad \text{and} \quad 0 \to K' \to P' \to M \to 0$$

with P and P' projective, there is an isomorphism $K \oplus P' \cong K' \oplus P$ of R-modules. $\square$

Let R be a ring and M an R-module. Denote by $\Omega_R M$ any R-module that is the kernel of an R-module homomorphism $P \twoheadrightarrow M$ with P a finitely generated projective. While $\Omega_R M$ depends on the choice of P, Schanuel's Lemma shows that M determines $\Omega_R M$ up to a projective summand. Any module isomorphic to a module $\Omega_R M$ is called a *syzygy* of M. For any $d > 1$, a *dth syzygy* of M is a module $\Omega_R^d M$ such that $\Omega_R^d M = \Omega_R(\Omega_R^{d-1} M)$ for some $(d-1)$st syzygy of M. By Schanuel's Lemma, $\Omega_R^d M$ is also determined by M up to a projective summand. For any $n \geq 0$, we write $\Omega^n M$ when the ring is clear from context.

Definition 2.1.1. An R-module M is said to have *finite projective dimension* if for some $i \geq 0$, an ith syzygy module $\Omega^i M$ is projective; in this case, we write $\operatorname{pd}_R M < \infty$. If no such i exists, M is said to have *infinite projective dimension*.

By Schanuel's Lemma, an ith syzygy module is projective if and only if every ith syzygy module is projective. Observe that if $\Omega^i M$ is projective, then $\Omega^j M$ is projective for all $j \geq i$.

Definition 2.1.2. Let $(R, \mathfrak{m}, k)$ be a local ring with residue field k and maximal ideal $\mathfrak{m}$, and let M be a finitely generated R-module. Set

$$\beta_i(M) = \operatorname{rank}_k \operatorname{Tor}_i^R(M, k);$$

this is the *ith Betti number* of M. The *Poincaré series* of M is given by

$$P_M^R(t) = \sum_{i=0}^{\infty} \beta_i(M)\, t^i$$

viewed as an element in the formal power series ring $\mathbb{Z}[t^{\pm 1}]$.

When R is local, an R-module M has finite projective dimension if and only if $\beta_i(M) = 0$ for $i \gg 0$; see [4, Cor 1.3.2].

2.2 Cosyzygies

For the discussion in this section, let $(R, \mathfrak{m}, k)$ be a local ring with maximal ideal $\mathfrak{m}$ and residue field k, and let M be a finitely generated R-module. Set $(-)^* = \mathrm{Hom}_R(-, R)$. A *free cover* of M [6, Def 5.1.1] is a homomorphism $\varphi : G \to M$ with G a free R-module such that

1. for any homomorphism $g : G' \to M$ with G' free there exists a homomorphism $f : G' \to G$ such that $g = \varphi f$, and
2. any endomorphism f of G with $\varphi = \varphi f$ is an automorphism.

A free cover is unique up to isomorphism.

Let $\nu_R(M)$ denote the minimal number of generators of M, that is,

$$\nu_R(M) = \mathrm{rank}_k(k \otimes_R M).$$

Remark 2.2.1. Every R-module M admits a free cover. Indeed, the homomorphism $\varphi : R^n \to M$ is a free cover of M if and only if φ is surjective and $n = \nu_R(M)$.

A *free envelope* of M [6, Def 6.1.1] is a homomorphism $\varphi : M \to G$ with G a free R-module such that

1. for any homomorphism $g : M \to G'$ with G' free there exists a homomorphism $f : G \to G'$ such that $g = f\varphi$, and
2. any endomorphism f of G with $\varphi = f\varphi$ is an automorphism.

A free envelope is unique up to isomorphism.

Remark 2.2.2. Every finitely generated R-module M admits a free envelope. Indeed, the homomorphism $f = (f_1, \ldots, f_n) : M \to R^n$, where $f_1, \ldots, f_n$ is a minimal system of generators of M^*, is a free envelope of M.

The free envelope of M can also be constructed as follows. Let $R^n \twoheadrightarrow M^*$ be the free cover of M^*. Applying $(-)^*$ to this map, one has an injection $M^{**} \to R^n$. The composite map $M \to M^{**} \to R^n$ is the free envelope of M, where $M \to M^{**}$ is the natural biduality map.

Remark 2.2.3. For a local ring R, one may choose $\Omega M = \mathrm{Ker}(\varphi)$ for a free cover φ of M. This choice of ΩM is unique up to isomorphism.

Definition 2.2.4. The *cosyzygy module* of M is $\Omega_R^{-1} M = \mathrm{Coker}(\varphi)$, where φ is the free envelope of M. For $n > 1$, the *nth cosyzygy module* of M is

$$\Omega_R^{-n} M = \Omega_R^{-1}(\Omega_R^{-(n-1)} M).$$

In [6, Sect 8.1], the authors refer to the cosyzygy module as the *free cosyzygy module*; since this is the only cosyzygy module studied in this article, we simply call it the cosyzygy module. We note that, when the module M is torsion-free, the cosyzygy module is also called the *pushforward*; see [11].

2.3 Periodic modules

Hilbert [9] proved that for a polynomial ring over a field every module has finite projective dimension; more precisely, for a polynomial ring in n variables, the $(n+1)$st syzygy of any module is zero. Modules with finite projective dimension can be studied using techniques inspired by linear algebra. For general rings, however, there exist modules with infinite projective dimension. Among modules with infinite projective dimension, the simplest are those whose syzygies repeat periodically or eventually repeat periodically.

Definition 2.3.1. Let $(R, \mathfrak{m})$ be a local ring. An R-module M is said to be *periodic* if there exists an $n \in \mathbb{N}$ such that $M \cong \Omega^n M$. The module M is said to be *eventually periodic* if there exists an $n \in \mathbb{N}$ and $\ell \in \mathbb{Z}_{\geq 0}$ such that $\Omega^\ell M \cong \Omega^{n+\ell} M$. In either case, the minimal such integer n is called the *period* of M.

For complete intersection rings, Eisenbud [5] proved that a periodic module always exists and necessarily has period at most two. However, little is known about periodic modules over the much broader class of Gorenstein rings. Examples of Gasharov and Peeva [7] show that the behavior over Gorenstein rings is more varied: for each $n \in \mathbb{N}$, they give an example of a Gorenstein ring with a periodic module M of period n. In 1990, Avramov [1] posed the following problem which we study in this dissertation.

Problem 2.3.2. Characterize the rings which have a periodic module.

Rings with the Krull-Remak-Schmidt property are particularly important for our approach to this characterization problem.

Definition 2.3.3. The ring R has the *Krull-Remak-Schmidt property* if the following condition holds: given an isomorphism of finitely generated R-modules

$$\bigoplus_{i=1}^{m} M_i \cong \bigoplus_{j=1}^{n} N_j$$

where M_i and N_j are indecomposable and nonzero, then $m = n$ and, after renumbering if necessary, $M_i \cong N_i$ for each i.

A local ring R has the Krull-Remak-Schmidt property if R is Henselian [16, Thm 1.8]; in particular, a complete ring has this property [16, Cor 1.9].

One of the main results of this dissertation is a characterization of Gorenstein local rings with the Krull-Remak-Schmidt property that have a periodic module in terms of a certain $\mathbb{Z}[t^{\pm 1}]$-module associated to the ring; see Corollary 5.2.8.

For the remainder of this chapter, let $(R, \mathfrak{m})$ be a local ring. We write $\widehat{M}$ for the $\mathfrak{m}$-adic completion of the R-module M. The following lemma, which shows that periodicity of an R-module M is equivalent to periodicity of the $\widehat{R}$-module $\widehat{M}$, will be used to obtain a solution to Problem 2.3.2 for Gorenstein local rings with the Krull-Remak-Schmidt property.

Lemma 2.3.4. *Let M be a finitely generated R-module, and let $i, j \in \mathbb{Z}_{\geq 0}$. Then $\Omega^i_R M \cong \Omega^j_R M$ if and only if $\Omega^i_{\widehat{R}} \widehat{M} \cong \Omega^j_{\widehat{R}} \widehat{M}$. In particular, M is eventually periodic if and only if the $\widehat{R}$-module $\widehat{M}$ is eventually periodic as an $\widehat{R}$-module.*

Proof. Given that $\Omega^i_R M \cong \Omega^j_R M$, one has

$$\Omega^i_{\widehat{R}}(\widehat{M}) \cong \widehat{\Omega^i_R M} \cong \widehat{\Omega^j_R M} \cong \Omega^j_{\widehat{R}}(\widehat{M}).$$

Suppose that $\Omega^i_{\widehat{R}} \widehat{M} \cong \Omega^j_{\widehat{R}} \widehat{M}$. Then we have the following isomorphisms:

$$\widehat{\Omega^i_R M} \cong \Omega^i_{\widehat{R}}(\widehat{M}) \cong \Omega^j_{\widehat{R}}(\widehat{M}) \cong \widehat{\Omega^j_R M}.$$

Hence $\Omega^i_R M \cong \Omega^j_R M$ by [16, Cor 1.15]. $\square$

Chapter 3

The module $J(R)$

In this chapter, we introduce the module $J(R)$, which was defined by D.R. Jordan in [13]. The module $J(R)$ is the main object of study in this dissertation. In what follows, we discuss some basic properties of this module and give an alternate definition for $J(R)$; see Proposition 3.2.4.

3.1 Definition and basic properties of $J(R)$

Let R be a ring, and let $\mathcal{C}(R)$ be the set of isomorphism classes of finitely generated R-modules; write $[M]$ for the class of an R-module M in $\mathcal{C}(R)$. When the ring is clear from context, we write $\mathcal{C}$ instead of $\mathcal{C}(R)$.

Definition 3.1.1. Let F be the free $\mathbb{Z}[t^{\pm 1}]$-module $\mathbb{Z}[t^{\pm 1}]^{(\mathcal{C})}$, that is,

$$F = \bigoplus_{[M]\in\mathcal{C}} \mathbb{Z}[t^{\pm 1}][M],$$

and let I be the $\mathbb{Z}[t^{\pm 1}]$-submodule generated by the following elements:

(R1) $[M] - [M']$ for every exact sequence of finitely generated R-modules
$0 \to P \to M \to M' \to 0$ with P projective;

(R2) $[M] - t[M']$ for every exact sequence of finitely generated R-modules
$0 \to M' \to P \to M \to 0$ with P projective;

(R3) $[M \oplus M'] - [M] - [M']$ for all finitely generated R-modules M and M'.

The main object of study in this work is the $\mathbb{Z}[t^{\pm 1}]$-module:

$$J(R) = F/I.$$

In the following remark, we make a few observations about the module $J(R)$.

Remark 3.1.2. Let M, M', and P be finitely generated R-modules with P projective.

1. $[P] = 0$ in $J(R)$.

2. If $0 \to M \to M' \to P \to 0$ is an exact sequence, then $[M] - [M'] = 0$ in $J(R)$.

Indeed, for 1, note that there is an exact sequence $0 \to P \to P \to 0 \to 0$, so the desired result follows from (R1) .

To prove 2, notice that $M' \cong M \oplus P$ since P is projective. Then in $J(R)$, $[M'] = [M] + [P]$ by (R3). Since $[P] = 0$ in $J(R)$, it follows that $[M'] = [M]$.

The module $J(R)$ was defined by D.R. Jordan in [13] and called the *Grothendieck module*. In Jordan's definition, the submodule I is generated by four types of elements: the three given in Definition 3.1.1 as well as elements of the form $[M] - [M']$ where M and M' are modules as in Remark 3.1.2.2.

Remark 3.1.3. The *Grothendieck group* $\mathcal{G}$ of a ring R is the free $\mathbb{Z}$-module $\mathbb{Z}^{(\mathcal{C})}$ modulo the subgroup generated by the Euler relations, that is, elements of the form

$[M'] - [M] + [M'']$ for each exact sequence $0 \to M' \to M \to M'' \to 0$ of finitely generated R-modules. The *reduced Grothendieck group* $\overline{\mathcal{G}}$ of R is the group $\mathcal{G}$ modulo the subgroup generated by classes of modules of finite projective dimension. We note that $\overline{\mathcal{G}} = J(R)/L$, where L is the submodule generated by the Euler relations.

3.2 Syzygies

The syzygy gives a well-defined functor on $J(R)$, as shown in Lemma 3.2.2. The following remark will aid in this discussion.

Remark 3.2.1. If $0 \to P \to M \to M' \to 0$ is an exact sequence of R-modules with P projective, then there is a module that is a syzygy of both M and M'.

Indeed, pick a surjective map $G' \twoheadrightarrow M'$, with G' a projective R-module. Let X be the pullback of $M \to M'$ and $G' \to M'$. Since $G' \to M'$ is surjective, $X \to M$ is also surjective. Since G' and P are projective, X is projective. Hence the kernel of $X \twoheadrightarrow M$ is a syzygy of M; let N be this kernel. Let N' denote the kernel of $G \twoheadrightarrow M'$. Then there is a commutative diagram with exact rows as follows.

$$\begin{tikzcd}
0 \arrow[r] & P \arrow[r] & M \arrow[r] & M' \arrow[r] & 0 \\
0 \arrow[r] & P \arrow[r] \arrow[u, equal] & X \arrow[r] \arrow[u, two heads] & G' \arrow[r] \arrow[u, two heads] & 0 \\
& & N \arrow[r, "\cong"] \arrow[u, hook] & N' \arrow[u, hook] &
\end{tikzcd}$$

Note that $N \cong N'$ by the Snake Lemma, and this justifies the claim.

Lemma 3.2.2. *Assigning $[M]$ to $[\Omega M]$ induces a $\mathbb{Z}[t^{\pm 1}]$-linear map*

$$\Omega \colon J(R) \to J(R).$$

Proof. By Schanuel's Lemma, the assignment of $[M] \mapsto [\Omega M]$ gives a homomorphism

$$\tilde{\Omega} : \bigoplus_{[M]\in\mathcal{C}} \mathbb{Z}[t^{\pm 1}][M] \to J(R)$$

of $\mathbb{Z}[t^{\pm 1}]$-modules. It is enough to check that (R1), (R2), and (R3) from Definition 3.1.1 are in $\mathrm{Ker}(\tilde{\Omega})$ and so $\tilde{\Omega}$ factors through $J(R)$; the induced map is Ω.

For (R3), note that for any syzygies ΩM of M and $\Omega M'$ of M', the R-module $\Omega M \oplus \Omega M'$ is a syzygy of $M \oplus M'$. Since $[\Omega M \oplus \Omega M'] = [\Omega M] + [\Omega M']$ in $J(R)$, one finds that $\tilde{\Omega}([M \oplus M']) = \tilde{\Omega}([M]) \oplus \tilde{\Omega}([M'])$.

Next, consider (R2). Given an exact sequence of finitely generated R-modules $0 \to M' \to P \to M \to 0$ with P projective, we show that $[M'] - t^{-1}[M]$ is in $\mathrm{Ker}(\tilde{\Omega})$. In $J(R)$, one has $\tilde{\Omega}([M']) = t^{-1}[M']$. Since M' is a syzygy of M, we have $[M'] = [\Omega M]$ in $J(R)$. But $[\Omega M] = \tilde{\Omega}([M])$, so the $\mathbb{Z}[t^{\pm 1}]$-linearity of $\tilde{\Omega}$ implies that

$$\tilde{\Omega}([M']) = t^{-1}\tilde{\Omega}([M]) = \tilde{\Omega}(t^{-1}[M]).$$

Therefore (R2) is in $\mathrm{Ker}(\tilde{\Omega})$.

Finally, we verify that (R1) is in $\mathrm{Ker}(\tilde{\Omega})$. Given an exact sequence of finitely generated R-modules $0 \to P \to M \to M' \to 0$ with P projective, Remark 3.2.1 implies there exists a module L which is a syzygy of both M and M'. Hence $\tilde{\Omega}([M]) = [L] = \tilde{\Omega}([M'])$. □

For $i > 1$, define $\Omega^i : J(R) \to J(R)$ by $\Omega^i = \Omega \circ \Omega^{i-1}$. The next lemma demonstrates a relationship in $J(R)$ between the class of a module and the classes of its syzygies.

Lemma 3.2.3. *Let M be a finitely generated R-module. Then $[M] = t^n[\Omega^n M]$ in $J(R)$ for any $n \in \mathbb{N}$.*

Proof. By (R2), one gets $t[\Omega M] = [M]$ in $J(R)$. Iterating this, $[M] = t^n[\Omega^n M]$ for all $n \in \mathbb{N}$. $\square$

In the next proposition, we give an alternate description of $J(R)$ which makes the relations in this module more transparent.

Proposition 3.2.4. *Let F be the free $\mathbb{Z}[t^{\pm 1}]$-module given by $\mathbb{Z}[t^{\pm 1}]^{(\mathcal{C})}$, and let L be the $\mathbb{Z}[t^{\pm 1}]$-submodule generated by the following elements:*

(R1′) *$[P]$ for every finitely generated projective R-module P;*

(R2′) *$[M] - t[\Omega M]$ for every finitely generated R-module M;*

(R3) *$[M \oplus M'] - [M] - [M']$ for all finitely generated R-modules M and M'.*

There is an isomorphism of $\mathbb{Z}[t^{\pm 1}]$-modules

$$J(R) \cong F/L.$$

Before proving Proposition 3.2.4, we show that for an R-module M, $[\Omega M]$ is well defined in F/L.

Lemma 3.2.5. *Let F and L be as defined in the previous proposition. Assigning $[M]$ to $[\Omega M]$ induces a $\mathbb{Z}[t^{\pm 1}]$-linear map $\Omega : F/L \to F/L$.*

Proof. By Schanuel's Lemma, the assignment $[M] \mapsto [\Omega M]$ gives a homomorphism

$$\tilde{\Omega} : F \to F/L$$

of $\mathbb{Z}[t^{\pm 1}]$-modules. It is enough to check that (R1′), (R2′), and (R3) from Proposition 3.2.4 are in $\mathrm{Ker}(\tilde{\Omega})$ and so $\tilde{\Omega}$ factors through F/L; the induced map is Ω.

Note that (R1$'$) is in $\operatorname{Ker}(\tilde{\Omega})$ since the zero module is a syzygy of every projective module.

Next, we consider (R2$'$). Let ΩM be a syzygy of M, and let $\Omega^2 M$ be a syzygy of ΩM. The following equalities hold in F/L:

$$\tilde{\Omega}([M] - t[\Omega M]) = [\Omega M] - t[\Omega^2 M] = 0,$$

and hence (R2$'$) is in $\operatorname{Ker}(\tilde{\Omega})$.

For (R3), note that for any syzygies ΩM of M and $\Omega M'$ of M', the module $\Omega M \oplus \Omega M'$ is a syzygy of $M \oplus M'$. Since $[\Omega M \oplus \Omega M'] = [\Omega M] + [\Omega M']$ in F/L, one finds that $\tilde{\Omega}([M \oplus M']) = \tilde{\Omega}([M]) \oplus \tilde{\Omega}([M'])$. □

Proof of Proposition 3.2.4. Let $\tilde{q} : F \to J(R)$ be the quotient map. We show that (R1$'$), (R2$'$), and (R3) are in $\operatorname{Ker}(\tilde{q})$, and hence $\tilde{q}$ factors through the quotient F/L via a map $q : F/L \to J(R)$.

The elements given by (R1$'$) are in $\operatorname{Ker}(\tilde{q})$ by Remark 3.1.2.1, and those from (R2$'$) are in $\operatorname{Ker}(\tilde{q})$ by Lemma 3.2.3. The elements given by (R3) are in $\operatorname{Ker}(\tilde{q})$ by the definition of $J(R)$.

Let $\tilde{p} : F \to F/L$ be the quotient map. We show that (R1), (R2), and (R3) are in $\operatorname{Ker}(\tilde{p})$, and hence $\tilde{p}$ factors through the quotient $J(R)$ by a map $p : J(R) \to F/L$. Note that the elements given by (R3) are in $\operatorname{Ker}(\tilde{p})$ by the definition of L. It remains to verify that (R1) and (R2) are in $\operatorname{Ker}(\tilde{p})$.

First, consider (R1). Let $0 \to P \to M \to M' \to 0$ be an exact sequence of finitely generated R-modules with P projective. By (R2$'$), one has $[M] - [M'] = [M] - t[\Omega M']$ in F/L for any syzygy $\Omega M'$ of M'. Given a syzygy ΩM of M, Remark 3.2.1 shows that there is a projective R-module G such that $\Omega M \oplus G$ is a syzygy of M'. Hence

$[\Omega M'] = [\Omega M \oplus G] = [\Omega M]$ in F/L, and thus

$$[M] - [M'] = [M] - t[\Omega M] = 0$$

in F/L. Therefore (R1) is in $\operatorname{Ker}(\tilde{p})$.

Finally, we show that (R2) is in $\operatorname{Ker}(\tilde{p})$. Let $0 \to M' \to P \to M \to 0$ be an exact sequence of finitely generated R-modules. Then M' is a syzygy of M, so $t[M'] - [M] = t[\Omega M] - [M] = 0$ by (R2′). Hence (R2) is in $\operatorname{Ker}(\tilde{p})$.

Note that $p \circ q$ is the identity map on F/L; thus p is injective. Since p is a quotient map and hence also surjective, p is an isomorphism. $\square$

Recall that a homomorphism of rings $\varphi : R \to S$ is said to be *flat* if S is flat as an R-module via φ.

Lemma 3.2.6. *Let $\varphi : R \to S$ be a homomorphism of rings. When φ is flat, the assignment $[M] \mapsto [S \otimes_R M]$ induces a homomorphism of $\mathbb{Z}[t^{\pm 1}]$-modules*

$$J_R(\varphi) : J(R) \to J_S(t).$$

Proof. Consider the unique $\mathbb{Z}[t^{\pm 1}]$-linear map

$$\gamma : \bigoplus_{[M] \in \mathcal{C}(R)} \mathbb{Z}[t^{\pm 1}][M] \to J_S(t)$$

with $\gamma([M]) = [S \otimes_R M]$. It suffices to show that the elements (R1), (R2), and (R3) are in $\operatorname{Ker}(\gamma)$.

Let $0 \to P \to M \to M' \to 0$ be an exact sequence of R-modules with P projective.

Since S is flat over R, the following sequence of S-modules is exact:

$$0 \to S \otimes_R P \to S \otimes_R M \to S \otimes_R M' \to 0.$$

Since $S \otimes_R P$ is projective as an S-module, $[S \otimes_R M] = [S \otimes_R M']$ in $J_S(t)$. Thus relation (R1) is in the kernel of γ. The proof for (R2) is similar.

For relation (R3), note that $S \otimes_R (M \oplus N) \cong (S \otimes_R M) \oplus (S \otimes_R N)$. $\square$

3.3 Finite projective dimension

In [13, Prop 3], Jordan proves the following: if R is a commutative local Noetherian ring and M a finitely generated R-module, then the projective dimension of M is finite if and only if $[M] = 0$ in $J(R)$. In Proposition 3.3.2, we extend this result to all commutative Noetherian rings.

Let $\mathbb{Z}((t))$ denote the *ring of formal Laurent series*, $\mathbb{Z}[[t]]\left[\frac{1}{t}\right]$; we view it as a module over $\mathbb{Z}[t^{\pm 1}]$. Notice that $\mathbb{Z}[t^{\pm 1}]$ is a $\mathbb{Z}[t^{\pm 1}]$-submodule of $\mathbb{Z}((t))$.

The following proposition is [13, Lem 1]. We include the statement here along with a more detailed proof for ease of reference.

Proposition 3.3.1. *Let R be a commutative local Noetherian ring with residue field k. The assignment $[M] \mapsto P_M^R(t)$ induces a $\mathbb{Z}[t^{\pm 1}]$-linear map*

$$\pi : J(R) \to \mathbb{Z}((t))/\mathbb{Z}[t^{\pm 1}].$$

Proof. Consider the unique $\mathbb{Z}[t^{\pm 1}]$-module homomorphism

$$\theta : \bigoplus_{[M] \in \mathcal{C}} \mathbb{Z}[t^{\pm 1}][M] \to \mathbb{Z}((t))/\mathbb{Z}[t^{\pm 1}]$$

with $\theta([M]) = P_M^R(t)$. We show that the elements (R1), (R2), and (R3) from Definition 3.1.1 are in $\mathrm{Ker}(\theta)$.

For (R3), note that $\mathrm{Tor}_i^R(M \oplus N, k) \cong \mathrm{Tor}_i^R(M, k) \oplus \mathrm{Tor}_i^R(N, k)$, and hence $\beta_i(M \oplus N) = \beta_i(M) + \beta_i(N)$. So one has $\theta([M \oplus N]) = \theta([M]) + \theta([N])$, which implies that (R3) is in $\mathrm{Ker}(\theta)$.

Next, we consider the element (R2). Let $0 \to M' \to P \to M \to 0$ be an exact sequence of R-modules with P projective. This gives a long exact sequence in Tor:

$$\cdots \to \mathrm{Tor}_{i+1}^R(P, k) \to \mathrm{Tor}_{i+1}^R(M, k) \to \mathrm{Tor}_i^R(M', k) \to \mathrm{Tor}_i(P, k) \to \cdots .$$

For $i \geq 1$, one has $\mathrm{Tor}_i^R(P, k) = 0$ and so $\mathrm{Tor}_{i+1}^R(M, k) \cong \mathrm{Tor}_i^R(M', k)$. Hence $\beta_i(M') = \beta_{i+1}(M)$ for all $i \geq 1$, and so, in $\mathbb{Z}((t))/\mathbb{Z}[t^{\pm 1}]$,

$$\begin{aligned} \theta([M] - t[M']) &= \theta([M]) - t\theta([M']) \\ &= \sum_{i=0}^{\infty} \beta_i(M) t^i - \sum_{i=0}^{\infty} \beta_i(M') t^{i+1} \\ &= \beta_0(M) + \beta_1(M) t - \beta_0(M') t \\ &= 0. \end{aligned}$$

The argument for the element (R1) is similar to the argument for (R2). In this case, the long exact sequence in Tor implies that $\mathrm{Tor}_i^R(M, k) \cong \mathrm{Tor}_i^R(M', k)$ for all $i \geq 2$, and therefore $\theta([M] - [M']) = 0$ in $\mathbb{Z}((t))/\mathbb{Z}[t^{\pm 1}]$. $\square$

The following proposition was proved in [13, Prop 3] for local rings.

Proposition 3.3.2. *Let R be a commutative Noetherian ring and M a finitely generated R-module. Then $[M] = 0$ in $J(R)$ if and only if M has finite projective dimension.*

Proof. If the projective dimension of M is finite, then $[\Omega^n M] = 0$ for some $n \in \mathbb{N}$. Hence $[M] = 0$ by Lemma 3.2.3.

Suppose $[M] = 0$ in $J(R)$. First, we consider the case when R is local. Using the homomorphism π from Proposition 3.3.1, one finds that $P_R(M) \in \mathbb{Z}[t^{\pm 1}]$. Hence $P_R(M)$ is a polynomial, and it follows that $\beta_i(M) = 0$ for $i \gg 0$. Thus the projective dimension of M is finite.

For a general ring R, the map $R \to R_{\mathfrak{m}}$ is flat for each maximal ideal $\mathfrak{m}$. Lemma 3.2.6 gives a homomorphism $J(R) \to J_{R_{\mathfrak{m}}}(t)$ with $[M] \mapsto [M_{\mathfrak{m}}]$. Thus $[M_{\mathfrak{m}}] = 0$ in $J_{R_{\mathfrak{m}}}(t)$, and hence $\operatorname{pd}_{R_{\mathfrak{m}}} M_{\mathfrak{m}} < \infty$. Hence the projective dimension of M over R is finite by [3, Thm 4.5]. $\square$

Chapter 4

Maximal-Cohen Macaulay modules

This chapter covers background material on depth and maximal Cohen-Macaulay modules. We collect several well-known results on maximal Cohen-Macaulay modules over Gorenstein local rings; these results are difficult to find in the literature in the form that we need.

Throughout this chapter, let $(R, \mathfrak{m}, k)$ be a (commutative, Noetherian) local ring. For a finitely generated R-module M, we denote by ΩM the syzygy of M given by Remark 2.2.3; this syzygy is unique up to isomorphism.

4.1 Injective modules

An R-module E is said to be *injective* if the functor $\operatorname{Hom}_R(-, E)$ is exact. A complex of injective R-modules

$$\mathbf{I}: \quad 0 \xrightarrow{\partial^0} I^0 \xrightarrow{\partial^1} I^1 \xrightarrow{\partial^2} I^2 \longrightarrow \cdots$$

is an *injective resolution* of an R-module M if $H^n(\mathbf{I}) = 0$ for all $n \geq 1$ and $H^0(\mathbf{I}) = M$, where $H^n(\mathbf{I}) = \operatorname{Ker}\partial^{n+1} / \operatorname{im} \partial^n$. While it is clear that every module has a projective resolution, it is not immediately obvious that every module has an injective resolution. However, the fact that every R-module injects into an injective R-module allows for the construction of an injective resolution for any R-module; see [4, Thm 3.1.8].

The *length* of an injective resolution $\mathbf{I}$ is the infimum over $n \in \mathbb{N}$ such that $I^m = 0$ for all $m > n$. The *injective dimension* of M, denoted $\operatorname{id}_R M$, is the infimum over $n \in \mathbb{N}$ such that M has an injective resolution of length n. In [4, Prop 3.1.14], it is shown that

$$\operatorname{id}_R M = \sup\{i \mid \operatorname{Ext}_R^i(k, M) \neq 0\}.$$

Recall that a local ring R is *Gorenstein* if $\operatorname{id}_R R$ is finite.

4.2 Maximal-Cohen Macaulay modules over Gorenstein rings

For the remainder of this chapter, let M be a finitely generated nonzero R-module. The *depth* of the module M is given by

$$\operatorname{depth}_R M = \min\{i \mid \operatorname{Ext}_R^i(k, M) \neq 0\}.$$

When R is clear from context, we write $\operatorname{depth} M$. The following well-known result, which can be found in [4, Prop 1.2.9], allows for a comparison of the depths of R-modules in a short exact sequence.

Depth Lemma. *Let R be a local ring and $0 \to U \to M \to N \to 0$ an exact sequence of R-modules. Then*

$$\operatorname{depth} U \geq \min\{\operatorname{depth} M, \operatorname{depth} N + 1\}. \qquad \square$$

Definition 4.2.1. A nonzero R-module M is said to be *maximal Cohen-Macaulay* (abbreviated to MCM) if $\operatorname{depth} M = \dim R$. A local ring R is *Cohen-Macaulay* if R is MCM as an R-module.

The maximal Cohen-Macaulay modules are the most important part of the module category of a ring. Indeed, the Depth Lemma implies that the high syzygies of every module over a Cohen-Macaulay local ring are either zero or maximal Cohen-Macaulay.

Remark 4.2.2. Let R be a Cohen-Macaulay ring with $d = \dim R$ and M a nonzero R-module. Then either $\Omega^d M$ is MCM or zero.

If a ring R has a nonzero module of finite injective dimension, the following result of Ischebeck [12, Satz 2.6] allows for a characterization of Gorenstein rings in terms of maximal Cohen-Macaulay modules.

Theorem 4.2.3. *Let R be a local ring and M and N nonzero R-modules. If N has finite injective dimension, then*

$$\operatorname{depth} R - \operatorname{depth} M = \sup\{i \mid \operatorname{Ext}^i_R(M, N) \neq 0\}. \qquad \square$$

The following corollary shows that a ring R is Gorenstein if R is Cohen-Macaulay and R is an injective object in the category of MCM R-modules.

Corollary 4.2.4. *Let R be a local ring. Then R is Gorenstein if and only if R is Cohen-Macaulay and $\operatorname{Ext}^i_R(M, R) = 0$ for all MCM modules M and all $i \geq 1$.*

Proof. Suppose R is Gorenstein. Then R is Cohen-Macaulay [4, Prop 3.1.20]. Let M be an MCM R-module. By Theorem 4.2.3, $\operatorname{Ext}^i_R(M, R) = 0$ for all $i \geq 1$.

Now suppose R is Cohen-Macaulay and $\operatorname{Ext}^i_R(M, R) = 0$ for all MCM modules M and all $i \geq 1$. Let N be an R-module, and let $d = \operatorname{depth} R$. By Remark 4.2.2, $\Omega^d N$ is either MCM or zero. Let $\mathbf{P}$ be a projective resolution of N. Breaking this resolution into short exact sequences, we have the following:

$$
\begin{gathered}
0 \to \Omega^d N \to P_{d-1} \to \Omega^{d-1} N \to 0 \\
\vdots \\
0 \to \Omega^2 N \to P_1 \to \Omega N \to 0 \\
0 \to \Omega N \to P_0 \to N \to 0
\end{gathered}
\tag{4.1}
$$

Applying $(-)^*$ to sequence (4.1) yields a long exact sequence in Ext, and we conclude that $\operatorname{Ext}^i_R(\Omega^{d-1}N, R) = 0$ for all $i \geq 2$. Iterating this process on each sequence above, $\operatorname{Ext}^i_R(N, R) = 0$ for all $i \geq d+1$. Then by [4, Prop 3.1.10], the injective dimension of R is less than d and hence finite. Thus R is Gorenstein. □

For the remainder of this chapter, we focus on Gorenstein rings. The following are well-known results on MCM modules that will be used throughout this dissertation; for lack of adequate references, some of the proofs are given here.

Remark 4.2.5. Let R be a Gorenstein local ring and M an MCM R-module.

1. The natural homomorphism $M \to M^{**}$ is an isomorphism.
2. The free envelope of M is an injective homomorphism.
3. The modules ΩM and $\Omega^{-1} M$ are MCM.
4. $(\Omega^{-1}M)^* \cong \Omega(M^*)$.
5. If M is indecomposable, then ΩM and $\Omega^{-1}M$ are also indecomposable.
6. If M has no free summands, then ΩM and $\Omega^{-1}M$ also have no free summands.
7. If M has no free summands, then $\Omega^{-n}\Omega^n M \cong M$ for all $n \in \mathbb{Z}$.

Property 1 is proved in [17, Cor 2.3]. Before we give proofs for the remaining properties, we make an observation which will be useful in the proof of property 6 as well as later in the dissertation.

Remark 4.2.6. Let R be a ring. An R-module M has a free summand if and only if the evaluation map $ev : M^* \otimes_R M \to R$, where $\varphi \otimes m \mapsto \varphi(m)$, is surjective.

If M has a free summand, it is clear that ev is surjective. If ev is surjective, there is an $f \in M^*$ and $n \in M$ such that $f(n) = 1$. Then $f : M \to R$ is surjective, which implies that f splits. Hence R is a direct summand of M.

We return to the proof of Remark 4.2.5.

Proof of 2. As seen in Remark 2.2.2, the free envelope of M is the composition

$$M \to M^{**} \hookrightarrow F^*,$$

where $F \twoheadrightarrow M^*$ is the free cover of M^*. So (2) follows from (1).

Proof of 3. By [10, Lem 1.3], ΩM is an MCM module.

One proof that $\Omega^{-1}M$ is MCM is given in [11, Prop 1.6.(2)]; we provide a different proof here. Let $i : M \to G$ be the free envelope of M. By Remark 4.2.5.2, the following sequence is exact:

$$0 \to M \xrightarrow{i} G \to \Omega^{-1}M \to 0.$$

Apply $(-)^*$ to this sequence. By Corollary 4.2.4, $\operatorname{Ext}^i_R(M, R) = 0$ for each $i \geq 1$. Hence $\operatorname{Ext}^n_R(\Omega^{-1}M, R) = 0$ for all $n \geq 2$. We show that the map $i^* : G^* \to M^*$ is surjective. By Remark 2.2.2, we have $i = (f_1, \ldots, f_n)$ where $f_1, \ldots, f_n$ is a minimal system of generators of M^*. Let $\pi_j : R^n \to R$ be the jth projection map. Then

$i^*(\pi_j) = f_j$, and hence i^* is surjective. Therefore $\mathrm{Ext}^1_R(\Omega^{-1}M, R) = 0$, and hence $\Omega^{-1}M$ is MCM by Corollary 4.2.4.

Proof of 4. Let $\pi : F \to M^*$ be the free cover of M^*. Then $\pi^* : M \to F^*$ is the free envelope of M by Remark 2.2.2. Thus $\Omega^{-1}M$ is defined by an exact sequence

$$0 \longrightarrow M \xrightarrow{\pi^*} F^* \longrightarrow \Omega^{-1}M \longrightarrow 0.$$

Since M is MCM, applying $(-)^*$ to this sequence yields the exact sequence

$$0 \longrightarrow (\Omega^{-1}M)^* \longrightarrow F \xrightarrow{\pi} M^* \longrightarrow 0.$$

As π is the free cover of M^*, one gets $\Omega(M^*) \cong (\Omega^{-1}M)^*$.

Proof of 5. It is shown that ΩM is indecomposable in [10, Lem 1.3]. We prove that $\Omega^{-1}M$ is indecomposable. Let G be the free envelope of M. By property 2, the following sequence is exact:

$$0 \to M \to G \to \Omega^{-1}M \to 0.$$

Since M is MCM, applying $(-)^*$ to this sequence yields the exact sequence

$$0 \to (\Omega^{-1}M)^* \to G^* \to M^* \to 0.$$

By property 4, $(\Omega^{-1}M)^* \cong \Omega(M^*)$. Since M is indecomposable and isomorphic to M^{**}, it follows that M^* is indecomposable. As M^* is an indecomposable MCM module, $(\Omega^{-1}M)^*$ is indecomposable by the result for syzygies. Hence $\Omega^{-1}M$ is also indecomposable.

Proof of 6. Suppose $\Omega M \cong N \oplus R$. Let G be the free cover of M, and let X be the pushout of $N \oplus R \twoheadrightarrow R$ and $N \oplus R \to G$. Then we have the following commutative diagram with exact rows.

$$\begin{tikzcd}
0 \arrow[r] & N \oplus R \arrow[r] \arrow[d] & G \arrow[r] \arrow[d] & M \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & R \arrow[r] & X \arrow[r] & M \arrow[r] & 0
\end{tikzcd}$$

Since M is MCM, Corollary 4.2.4 implies that $\mathrm{Ext}^1_R(M, R) = 0$. Note that there is a bijective correspondence between elements of $\mathrm{Ext}^1_R(M, R)$ and equivalence classes of extensions $0 \to R \to E \to M \to 0$ of M by R. Since M is MCM, Corollary 4.2.4 implies that $X \cong M \oplus R$. Then $\nu_R(G) \geq \nu_R(M) + 1$ as G maps onto $M \oplus R$. However, this is a contradiction since G is the free cover of M. Therefore ΩM has no free summand.

Since M is MCM with no free summand, M^* is MCM with no free summand. By property 4 and the result for syzygies, $(\Omega^{-1}M)^*$ has no free summand. Hence $\Omega^{-1}M$ has no free summand.

Proof of 7. First, note that $M \cong \Omega(\Omega^{-1}M) \oplus F'$ for some free module F' by Schanuel's Lemma. Since M has no free summands, $M \cong \Omega(\Omega^{-1}M)$. Next, we show that $\Omega^{-1}(\Omega M) \cong M$. The result then follows by induction on n since $\Omega^{-i}\Omega^{i}M$ is MCM for all $i \in \mathbb{Z}$ by property 3.

Let $i : \Omega M \to G$ be the free envelope of ΩM, and let $G' \to M$ be the free cover of M. We have the following commutative diagram.

$$\begin{tikzcd}
0 \arrow[r] & \Omega M \arrow[r, "i"] \arrow[d, equal] & G \arrow[r] & \Omega^{-1}(\Omega M) \arrow[r] & 0 \\
0 \arrow[r] & \Omega M \arrow[r, "j"] & G' \arrow[r] \arrow[u, "f"'] & M \arrow[r] \arrow[u] & 0 \\
 & & K \arrow[r, "\cong"] \arrow[u, hook] & K \arrow[u, hook] &
\end{tikzcd}$$

Indeed, since i is the free envelope of ΩM, there exists a homomorphism $g : G \to G'$ such that $g \circ i = j$. As M is MCM, the map $j^* : (G')^* \to (\Omega M)^*$ is surjective. Hence there is a homomorphism $f : G' \to G$ such that $f \circ j = i$. Then since $i = i \circ (f \circ g)$ and i is the free envelope of ΩM, the composition $f \circ g$ is an isomorphism. Hence $f : G' \to G$ is surjective, which implies that the map $M \to \Omega^{-1}(\Omega M)$ is also surjective. Note that the kernels of $G' \twoheadrightarrow G$ and $M \twoheadrightarrow \Omega^{-1}(\Omega M)$ are isomorphic by the Snake Lemma. As f is a surjection and both G and G' are free, $K = \mathrm{Ker}(f)$ is also free. Since $\Omega^{-1}(\Omega M)$ is MCM, we have $\mathrm{Ext}^1_R(\Omega^{-1}\Omega M, R) = 0$. Then since K is free, the sequence $0 \to K \to M \to \Omega^{-1}(\Omega M) \to 0$ splits. Hence $M \cong \Omega^{-1}(\Omega M) \oplus K$. Since M has no free summands, $M \cong \Omega^{-1}(\Omega M)$. $\square$

Chapter 5

Gorenstein local rings

This chapter is a study of the module $J(R)$ for Gorenstein local rings, and it contains the main results of this dissertation. Theorem 5.1.3 describes the structure of $J(R)$ for Gorenstein local rings with the Krull-Remak-Schmidt property. Theorem 5.2.6, shows that for a Gorenstein local ring the class of a module is torsion in $J(R)$ if and only if the module is eventually periodic. As a consequence of the previous theorem, we give an answer to Avramov's characterization problem for Gorenstein local rings with the Krull-Remak-Schmidt property. In particular, such a ring has an eventually periodic module if and only if $J(R)$ has nonzero torsion; see Corollary 5.2.8.

Throughout this chapter, let $(R, \mathfrak{m}, k)$ be a local ring and M a finitely generated R-module. We denote by ΩM the syzygy of M given by Remark 2.2.3; this syzygy is unique up to isomorphism. Set $(-)^* = \mathrm{Hom}_R(-, R)$.

5.1 $\mathbb{Z}[t^{\pm 1}]$-module structure of $J(R)$

In this section, we prove a structure theorem for $J(R)$ when R is a Gorenstein local ring. Define, for any local ring R,

$$\mathcal{M}(R) = \left\{ [M] \in \mathcal{C}(R) \;\middle|\; \begin{array}{c} M \text{ is MCM, nonfree,} \\ \text{and indecomposable} \end{array} \right\}.$$

When R is clear from context, we write $\mathcal{M}$ for $\mathcal{M}(R)$. If R is Cohen-Macaulay, the set $\mathcal{M}$ generates $J(R)$ as a module over $\mathbb{Z}[t^{\pm 1}]$ since each R-module has a syzygy that is either MCM or zero as noted in Remark 4.2.2. If R is Gorenstein and has the Krull-Remak-Schmidt property, one can do better: $\mathcal{M}$ generates $J(R)$ over $\mathbb{Z}$; see Theorem 5.1.3.

Lemma 5.1.1. *Let R be a local Gorenstein ring and M a finitely generated R-module.*

1. *If M is an MCM module, then $t^{-n}[\Omega^{-n} M] = [M]$ in $J(R)$ for each $n \in \mathbb{Z}$.*

2. *There is an MCM R-module N with $[M] = [N]$ in $J(R)$.*

Proof. Lemma 3.2.3 proved 1 for $n \leq 0$. Using Remark 4.2.5.2, a proof similar to the proof of Lemma 3.2.3 yields the desired result.

For 2, let $d = \dim R$. Lemma 3.2.3 implies that $t^d[\Omega^d M] = [M]$. By part 1, $t^{-d}[\Omega^{-d}\Omega^d M] = [\Omega^d M]$. Hence $[M] = [\Omega^{-d}\Omega^d M]$. □

Remark 5.1.2. In order to set up notation for the next theorem, we first discuss a special type of $\mathbb{Z}[t^{\pm 1}]$-module. For this, we view $\mathbb{Z}[t^{\pm 1}]$ as the group algebra over $\mathbb{Z}$ of the free group $G = \langle t \rangle$ on a single generator t; that is, $G \cong (\mathbb{Z}, +)$. Let X be a set with a G-action. Let $\mathbb{Z}X = \mathbb{Z}^{(X)}$, the free $\mathbb{Z}$-module with basis given by the elements of X, and let $\mathbb{Z}G$ be the group algebra over $\mathbb{Z}$ of G. Then $\mathbb{Z}X$ is naturally a $\mathbb{Z}G$-module [15, Ch.III, §1].

Remark 4.2.5 properties 3, 5, and 6 imply that $[\Omega M]$ and $[\Omega^{-1}M]$ are in $\mathcal{M}$ if $[M] \in \mathcal{M}$. Thus there is an action of G on $\mathcal{M}$ with $t[M] = [\Omega^{-1}M]$ and $t^{-1}[M] = [\Omega M]$. Since $\Omega(-)$ and $\Omega^{-1}(-)$ are well-defined up to isomorphism, this action is well-defined. Let $\mathcal{A} = \mathbb{Z}^{(\mathcal{M})}$ be the $\mathbb{Z}[t^{\pm 1}]$-module induced by this t-action, and set

$$\Phi : \mathcal{A} \to J(R)$$

to be the unique $\mathbb{Z}$-module homomorphism with $\Phi([M]) = [M]$. It is clear that Φ is $\mathbb{Z}[t^{\pm 1}]$-linear.

Assume R has the Krull-Remak-Schmidt property. We define a $\mathbb{Z}[t^{\pm 1}]$-linear map

$$\psi : \bigoplus_{[M] \in \mathcal{C}} \mathbb{Z}[t^{\pm 1}][M] \quad \longrightarrow \mathcal{A} \tag{5.1}$$

by setting $\psi([M]) = \sum_{i=1}^{n} [M_i]$, where each M_i is indecomposable and

$$\Omega^{-(d+1)}\Omega^{d+1}M \cong \bigoplus_{i=1}^{n} M_i$$

with $d = \dim R$. Since $\Omega^{-(d+1)}\Omega^{d+1}M$ is either zero or MCM with no free summands, $\sum_{i=1}^{n}[M_i]$ is in $\mathcal{A}$. Since R has the Krull-Remak-Schmidt property, ψ is well-defined.

Theorem 5.1.3. *Let R be a Gorenstein local ring that has the Krull-Remak-Schmidt property. Then the $\mathbb{Z}[t^{\pm 1}]$-linear map*

$$\Phi : \mathcal{A} \to J(R)$$

is an isomorphism with inverse Ψ *induced by* ψ *defined in* (5.1).

Proof. To show that ψ induces a $\mathbb{Z}[t^{\pm 1}]$-linear map $\Psi : J(R) \to \mathcal{A}$, it suffices to show that the elements described in (R1), (R2), and (R3) from Definition 3.1.1 are in the kernel of ψ.

For elements given by (R3), note that

$$\Omega^{-(d+1)}(\Omega^{d+1}(M \oplus N)) \cong \Omega^{-(d+1)}\Omega^{d+1}M \oplus \Omega^{-(d+1)}\Omega^{d+1}N.$$

Then $\psi([M \oplus N]) = \psi([M]) + \psi([N])$, and hence (R3) is in $\mathrm{Ker}(\psi)$.

Next, we consider (R2): given an exact sequence $0 \to M' \to P \to M \to 0$ with P projective, we show that $\psi([M']) = \psi(t^{-1}[M])$. By Schanuel's Lemma, $M' \cong \Omega M \oplus G$ for some free R-module G. Since $\Omega G = 0$, one has

$$\Omega^{-(d+1)}\Omega^{d+1}M' \cong \Omega^{-(d+1)}\Omega^{d+1}(\Omega M).$$

Then $\psi([M']) = \psi([\Omega M]) = \psi([t^{-1}\Omega M])$, and therefore (R2) is in $\mathrm{Ker}(\psi)$.

It remains to verify that (R1) is in $\mathrm{Ker}(\psi)$. Let $0 \to P \to M \to M' \to 0$ be an exact sequence of R-modules with P projective. By Remark 3.2.1, there are free R-modules G and G' such that $\Omega M \oplus G \cong \Omega M' \oplus G'$. Since $\psi([G]) = \psi([G']) = 0$, $\psi([\Omega M] = \psi([\Omega M'])$. As (R2) is in $\mathrm{Ker}(\psi)$, one finds that $\psi(t^{-1}[M]) = \psi(t^{-1}[M'])$ and thus $\psi([M]) = \psi([M'])$. Hence (R1) is in $\mathrm{Ker}(\psi)$.

Thus ψ factors through the quotient $J(R)$ via a homomorphism $\Psi : J(R) \to \mathcal{A}$. Notice that $\Psi \circ \Phi$ is the identity. Indeed, if M is an MCM module with no free summands, then $\Omega^{-(d+1)}\Omega^{d+1}M \cong M$ by Remark 4.2.5.(7). Hence Φ is injective. For each R-module N, Lemma 5.1.1.(2) shows that there is an MCM R-module M such that $[M] = [N]$. Thus Φ is also surjective and hence an isomorphism. $\square$

Remark 5.1.4. If R is Gorenstein, Theorem 5.1.3 implies that $J(R)$ is torsion-free as an abelian group. We do not know whether this holds also when R is Cohen-Macaulay.

The following result is proved in [13, Lem 8] for Artinian complete intersection rings.

Corollary 5.1.5. *Let R be a Gorenstein local ring, and let M and N be finitely generated MCM R-modules. Then $[M]=[N]$ in $J(R)$ if and only if*

$$M \oplus R^m \cong N \oplus R^n$$

for some $m,n \in \mathbb{Z}_{\geq 0}$. Thus if neither M nor N has a free summand, $M \cong N$.

Proof. If $M \oplus R^m \cong N \oplus R^n$ for some $m,n \in \mathbb{Z}_{\geq 0}$, then in $J(R)$ one has

$$[M] = [M \oplus R^m] = [N \oplus R^n] = [N].$$

Suppose that $[M]=[N]$ in $J(R)$. We may assume M and N have no free summands. We first prove the result under the assumption that R is complete with respect to the maximal ideal. As shown in [16, Cor 1.10], complete rings have the Krull-Remak-Schmidt property for finitely generated modules; so Theorem 5.1.3 applies. Let $d = \dim R$, and let $\Psi : J(R) \to \mathcal{A}$ be the isomorphism given in Theorem 5.1.3. Suppose

$$M = \bigoplus_{[M_\lambda]\in\mathcal{M}} M_\lambda^{e_\lambda} \quad \text{and} \quad N = \bigoplus_{[M_\lambda]\in\mathcal{M}} M_\lambda^{f_\lambda}$$

where $e_\lambda, f_\lambda \geq 0$. From Remark 4.2.5.(6) and the definition of ψ given in (5.1), one gets an equality

$$\sum_{[M_\lambda]\in\mathcal{M}} e_\lambda [M_\lambda] = \Psi([M]) = \Psi([N]) = \sum_{[M_\lambda]\in\mathcal{M}} f_\lambda [M_\lambda].$$

Since $\mathcal{A}$ is free on $\mathcal{M}$, we have $e_\lambda = f_\lambda$ for all λ. Therefore $M \cong N$ as R-modules.

Now suppose that R is any local ring with maximal ideal $\mathfrak{m}$. Write $\widehat{R}$ for the $\mathfrak{m}$-adic completion of R. If $[M] = [N]$ in $J(R)$, then $[M \otimes_R \widehat{R}] = [N \otimes_R \widehat{R}]$ in $J_{\widehat{R}}(t)$ by Lemma 3.2.6.

By Remark 4.2.6, M has a free summand if and only if the evaluation map $ev : M^* \otimes_R M \to R$, where $\varphi \otimes m \mapsto \varphi(m)$, is surjective. The map ev is surjective if and only if the map $ev \otimes_R \widehat{R}$ is surjective. Note that the map $ev \otimes_R \widehat{R}$ can be identified with the map $\left(\widehat{M}\right)^* \otimes_{\widehat{R}} \widehat{M} \to \widehat{R}$. Then since M and N have no free summands, $M \otimes_R \widehat{R}$ and $N \otimes_R \widehat{R}$ also have no free summands.

The result for complete rings then shows that $M \otimes_R \widehat{R} \cong N \otimes_R \widehat{R}$ as $\widehat{R}$-modules, and [16, Cor 1.15] implies that $M \cong N$.

Note that cancellation of direct summands is valid over local rings [16, Cor 1.16]. Then $M \oplus R^m \cong N \oplus R^n$ implies that $M \oplus R^{m'} \cong N$ or $M \cong N \oplus R^{n'}$. Thus if neither M nor N has a free summand, $M \cong N$. $\square$

Example 5.1.6. Let $R = k[x]/(x^n)$ where $n \geq 2$. The nonfree, indecomposable, MCM R modules are given by $M_i = k[x]/(x^i)$ for $i = 1, \dots, n-1$. By Theorem 5.1.3,

$$J(R) \cong \bigoplus_{i=1}^{n-1} \mathbb{Z}[M_i]$$

as $\mathbb{Z}$-modules. Note that $\Omega(M_i) = M_{n-i}$ for each i, and so $[M_i] = t[M_{n-i}]$. Then $(t^2 - 1)[M_i] = 0$ for all i, and hence

$$J(R) \cong \begin{cases} \left(\dfrac{\mathbb{Z}[t]}{(t^2-1)}\right)^{\frac{n-1}{2}} & n \text{ odd} \\ \mathbb{Z} \bigoplus \left(\dfrac{\mathbb{Z}[t]}{(t^2-1)}\right)^{\frac{n-2}{2}} & n \text{ even} \end{cases}$$

as $\mathbb{Z}[t^{\pm 1}]$-modules.

5.2 Torsion in $J(R)$

The main result of this section, Theorem 5.2.6, is that when R is Gorenstein the class of a module is torsion in $J(R)$ if and only if the module is eventually periodic. This result does not extend verbatim to Cohen-Macaulay local rings; see Example 5.2.14.

In the next lemma, we give a decomposition for the special type of $\mathbb{Z}[t^{\pm 1}]$-modules discussed in Remark 5.1.2.

Lemma 5.2.1. *Let $G = \langle t \rangle$, and let X be a set with a G-action. Then there is an isomorphism of $\mathbb{Z}G$-modules*

$$\mathbb{Z}X \cong \bigoplus_{n=1}^{\infty} \left(\frac{\mathbb{Z}[t]}{(t^n - 1)} \right)^{b_n} \bigoplus (\mathbb{Z}G)^{b_\infty}$$

where $b_\infty, b_n \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$ for all n. $\square$

Proof. Note that for any $x \in X$, either $t^n x \neq x$ for all $n \neq 0$ and the orbit of x is $Gx = \{t^i x : i \in \mathbb{Z}\}$, or there is an $n \neq 0$ with $t^n x = x$ and $Gx = \{x, tx, t^2 x, \ldots, t^{n-1}x\}$. Then for each $x \in X$ either $\mathbb{Z}Gx \cong \mathbb{Z}G$ or $\mathbb{Z}Gx \cong \mathbb{Z}[t]/(t^n - 1)$ as $\mathbb{Z}G$-modules; in either case, the map assigning x to 1 induces an isomorphism. Thus the decomposition of X into orbits gives the desired isomorphism. $\square$

Definition 5.2.2. The *torsion submodule* of a $\mathbb{Z}G$-module L is

$$T_{\mathbb{Z}G}(L) = \{u \in L : ru = 0 \text{ for some } r \in \mathbb{Z}G \setminus \{0\}\}.$$

An element $u \in T_{\mathbb{Z}G}(L)$ is said to be a *torsion element* of L.

Proposition 5.2.3. *An element $u \in \mathbb{Z}X$ is torsion if and only if there exists an $n \in \mathbb{N}$ such that $(t^n - 1)u = 0$.*

Proof. Suppose u is torsion in $\mathbb{Z}X$. Identifying $\mathbb{Z}X$ with the right hand side of the isomorphism in Lemma 5.2.1, one finds that u belongs to the submodule $\oplus_{n=1}^{\infty}(L_n)^{b_n}$ of $\mathbb{Z}X$ where

$$L_n = \frac{\mathbb{Z}[t]}{(t^n - 1)}.$$

Consider the case when $u = v + w$ where $v \in L_\ell$ and $w \in L_m$ for some $\ell, m \in \mathbb{N}$. Then $(t^\ell - 1)v = 0$ and $(t^m - 1)w = w$, and hence $(t^{m\ell} - 1)(v + w) = 0$. Indeed, if y is an indeterminate, we have $(1 - y^n) = (1 - y)(1 + y + \cdots + y^{n-1})$. Set $y = t^\ell$, and one finds that $(t^\ell - 1)$ is a factor of $(t^{n\ell} - 1)$. Similarly, $(t^n - 1)$ is a factor of $(t^{n\ell} - 1)$. By induction on the number of terms in u, there exists an $n \in \mathbb{N}$ such that $(t^n - 1)u = 0$.

The reverse implication is immediate. □

In light of Lemma 5.2.1 and Proposition 5.2.3, we have the following corollaries to Theorem 5.1.3.

Corollary 5.2.4. *Let R be a Gorenstein local ring that has the Krull-Remak-Schmidt property.*

1. *Then an element $u \in J(R)$ is torsion if and only if there exists an $n \in \mathbb{Z}$ such that $(t^n - 1)u = 0$.*

2. *The $\mathbb{Z}[t^{\pm 1}]$-module $J(R)$ has nonzero torsion if and only if there is a finitely generated R-module M such that $[M]$ is torsion.*

Proof. For 1, note that $G = \langle t \rangle$ acts on $\mathcal{M}$. By Theorem 5.1.3, $J(R) \cong \mathcal{A} = \mathbb{Z}\mathcal{M}$ as $\mathbb{Z}[t^{\pm 1}]$-modules. The result then follows from Proposition 5.2.3.

To prove 2, suppose u is a nonzero torsion element of $J(R)$. In the notation of Lemma 5.2.1, there is an $n \in \mathbb{N}$ such that $b_n \neq 0$. By Theorem 5.1.3, there are some $[M_\alpha] \in \mathcal{M}$ that generate L_n, and thus $[M_\alpha]$ is torsion for each α.

The reverse implication is immediate. □

We make some observations about torsion in $J(R)$ and eventually periodic modules. In Corollary 5.2.8, we show that torsion of the class of a module in $J(R)$ is equivalent to periodicity of a module under certain assumptions on R.

Remark 5.2.5. Let M be a finitely generated R-module. It is easy to see that the following statements hold.

1. $[M]$ is torsion in $J(R)$ if and only if $[\Omega^n M]$ is torsion for some (equivalently, all) $n \in \mathbb{N}$.

2. M is eventually periodic if and only if $\Omega^n M$ is eventually periodic for some (equivalently, all) $n \in \mathbb{N}$.

Theorem 5.2.6. *Let R be a Gorenstein local ring, and let M be a finitely generated R-module. Then $[M]$ is torsion in $J(R)$ with respect to the $\mathbb{Z}[t^{\pm 1}]$-action if and only if M is eventually periodic. Moreover, for any $n \in \mathbb{N}$, the following conditions are equivalent:*

1. $(t^n - 1)[M] = 0$ *in* $J(R)$.

2. $\Omega^\ell M \cong \Omega^{n+\ell} M$ *for* $\ell \gg 0$.

Proof. Suppose M is eventually periodic. Then there are $i, j \in \mathbb{Z}_{\geq 0}$ with $i \neq j$ such that $\Omega^i M \cong \Omega^j M$. In $J(R)$, $t^{-i}[M] = [\Omega^i M] = [\Omega^j M] = t^{-j}[M]$, and hence $(t^{-i} - t^{-j})[M] = 0$.

Assume $[M]$ is torsion in $J(R)$. We first show that we can reduce to the case when R is complete with respect to the maximal ideal $\mathfrak{m}$. Let $\widehat{M}$ denote the $\mathfrak{m}$-adic completion of M. Since the canonical homomorphism $\varphi : R \to \widehat{R}$ is flat, Lemma 3.2.6 implies that there is a homomorphism of $\mathbb{Z}[t^{\pm 1}]$-modules $J_R(\varphi) : J(R) \to J_{\widehat{R}}(t)$ with

$J_R(\varphi)([M]) = [\,\widehat{M}\,]$. Hence $[M]$ torsion implies that $[\widehat{M}]$ is torsion. If the result holds for complete rings, then $\widehat{M}$ is eventually periodic as an $\widehat{R}$-module. Hence Lemma 2.3.4 implies that M is eventually periodic as an R-module.

Assume R is complete. To show that M is eventually periodic, it is enough to show that some syzygy of M is eventually periodic. We may assume M is MCM with no free summands.

By Remark 4.2.2, $\Omega^d M$ is zero or MCM for $d \gg 0$. If $\Omega^d M = 0$, the proof is complete. If not, then replacing M by $\Omega^d M$ we may assume that M is MCM. If $M = N \oplus R$, then $[M] = [N]$ and hence $[M]$ is torsion in $J(R)$ if and only if $[N]$ is torsion. Note that M is eventually periodic if and only if N is eventually periodic, since $\Omega M \cong \Omega N$. Thus we may assume M has no free summands.

As $[M]$ is torsion, Corollary 5.2.4.(1) implies that there is an $n \in \mathbb{N}$ such that $(t^n - 1)[M] = 0$ in $J(R)$. Lemma 5.1.1.(1) shows that $[\Omega^{-n}M] = t^n[M] = [M]$. By Remark 4.2.5.(3), the R-module $\Omega^{-n}M$ is MCM, and thus Corollary 5.1.5 implies that $\Omega^{-n}M \oplus F \cong M \oplus G$ for some free R-modules F and G. Then, as R-modules, $\Omega^n\left(\Omega^{-n}M \oplus F\right) \cong \Omega^n\left(M \oplus G\right)$, and thus $\Omega^n\Omega^{-n}M \cong \Omega^n M$. Since M is MCM with no free summands, $M \cong \Omega^n\Omega^{-n}M$ by Remark 4.2.5.(7). Hence $M \cong \Omega^n M$, and therefore M is eventually periodic.

It is clear that *2* implies *1*. The argument in the previous paragraph along with Lemma 2.3.4 shows that *1* implies *2*. $\square$

In what follows, we write $\widehat{M}$ for the $\mathfrak{m}$-adic completion of the R-module M.

Corollary 5.2.7. *Let R be a Gorenstein local ring and M a finitely generated R-module. Then $[M]$ is torsion in $J(R)$ with respect to the $\mathbb{Z}[t^{\pm 1}]$-action if and only if $[\,\widehat{M}\,]$ is torsion in $J_{\widehat{R}}(t)$ with respect to the $\mathbb{Z}[t^{\pm 1}]$-action.*

Proof. In the proof of Theorem 5.2.6, it is shown that $[M]$ torsion in $J(R)$ implies $[\widehat{M}]$ is torsion in $J_{\widehat{R}}(t)$.

The reverse implication is immediate from Theorem 5.2.6 and Lemma 2.3.4. $\square$

The following corollary uses $J(R)$ to characterize Gorenstein local rings with the Krull-Remak-Schmidt property that have an eventually periodic module.

Corollary 5.2.8. *Let R be a local Gorenstein ring with the Krull-Remak-Schmidt property. The ring R has a periodic module if and only if $J(R)$ has nonzero torsion.*

Proof. Combining Corollary 5.2.4.2 and Theorem 5.2.6, one arrives at the desired result. $\square$

Corollary 5.2.9. *Suppose $M = \oplus_{i=1}^{m} M_i$ for some R-modules M_i. Then $[M]$ is torsion in $J(R)$ if and only if $[M_i]$ is torsion in $J(R)$ for all i.*

Proof. Assume $[M_i]$ is torsion in $J(R)$ for all i. For each $i \in \{1, \ldots, m\}$, there is an $f_i(t) \in \mathbb{Z}[t^{\pm 1}]$ such that $f_i(t)[M_i] = 0$ in $J(R)$. Then $f_1(t) \cdots f_m(t)[M] = 0$.

Suppose $[M]$ is torsion in $J(R)$. We first prove the result under the assumption that R is complete. It suffices to consider the case when each M_i is indecomposable. Since $[M]$ is torsion in $J(R)$, Theorem 5.2.6 implies that M is eventually periodic. So there is an $n \in \mathbb{N}$ and an $\ell \in \mathbb{Z}_{\geq 0}$ such that $\Omega^{n+\ell} M \cong \Omega^{\ell} M$, and therefore

$$\bigoplus_{i=1}^{m} \Omega^{n+\ell}(M_i) \cong \bigoplus_{i=1}^{m} \Omega^{\ell}(M_i).$$

We prove that each $[M_i]$ is torsion by using induction on m, the number of indecomposable summands of M. Suppose $M = M_1 \oplus M_2$. Then by the Krull-Remak-Schmidt property, either $\Omega^{n+\ell}(M_i) \cong \Omega^{\ell}(M_i)$ for $i = 1, 2$ or $\Omega^{n+\ell}(M_1) \cong \Omega^{\ell}(M_2)$ and $\Omega^{n+\ell}(M_2) \cong \Omega^{\ell}(M_1)$. In the first case, it is clear that M_1 and M_2 are eventually periodic. In the second case, note that

$$\Omega^{2n+\ell}(M_1) \cong \Omega^{n+\ell}(M_2) \cong \Omega^{\ell}(M_1),$$

and hence M_1 is eventually periodic. Similarly, M_2 is eventually periodic. Then by Theorem 5.2.6, $[M_i]$ is torsion in $J(R)$ for $i = 1, 2$.

Suppose $M = \oplus_{i=1}^m M_i$ and that the conclusion holds for $s < m$. By the Krull-Remak-Schmidt property, for each i there exists j such that $\Omega^{n+\ell}(M_i) \cong \Omega^{\ell}(M_j)$. If there is an i such that $\Omega^{n+\ell}(M_i) \cong \Omega^{\ell}(M_i)$, then the result follows from the inductive hypothesis. Without loss of generality, suppose $\Omega^{n+\ell}(M_i) \cong \Omega^{\ell}(M_{i+1})$ for $1 \leq i \leq m-1$ and $\Omega^{n+\ell}(M_m) \cong \Omega^{\ell}(M_1)$. The following isomorphisms of R-modules show that M_1 is eventually periodic:

$$\Omega^{mn+\ell}(M_1) \cong \Omega^{(m-1)n+\ell}(M_2) \cong \cdots \cong \Omega^{n+\ell}(M_m) \cong \Omega^{\ell}(M_1).$$

Similarly M_i is eventually periodic for $2 \leq i \leq m$, and consequently Theorem 5.2.6 implies that $[M_i]$ is torsion in $J(R)$ for all i.

Now suppose that R is any local ring and $[M]$ is torsion in $J(R)$. By Corollary 5.2.7, $[\widehat{M}]$ is torsion in $J_{\widehat{R}}(t)$. Write

$$\widehat{M} = \bigoplus_{i=1}^{m} \left(\bigoplus_{j=1}^{a_i} M_{ij} \right) \quad \text{with} \quad \widehat{M_i} = \bigoplus_{j=1}^{a_i} M_{ij}$$

and each M_{ij} an indecomposable $\widehat{R}$-module. The result for complete rings implies that $[M_{ij}]$ is torsion in $J_{\widehat{R}}(t)$ for each i and j. Then $[\widehat{M_i}]$ is torsion in $J_{\widehat{R}}(t)$ for all i, and so Corollary 5.2.7 implies that $[M_i]$ is torsion in $J(R)$ for all i. $\square$

Definition 5.2.10. A *hypersurface* is a local ring R such that $\widehat{R} \cong S/(f)$ where S is a regular local ring and f is an S-regular element.

Theorem 5.2.6 also gives a characterization of hypersurface rings in terms of $J(R)$. We note that [13, Thm 7] shows that *4* implies *1* for an Artinian complete intersection ring.

Corollary 5.2.11. *Let $(R, \mathfrak{m}, k)$ be a Gorenstein local ring. Then the following conditions are equivalent:*

1. *R is a hypersurface;*
2. *$(1 - t^2) \cdot J(R) = 0$;*
3. *$J(R)$ is a torsion module;*
4. *$[k]$ is torsion in $J(R)$ with respect to the $\mathbb{Z}[t^{\pm 1}]$-action.*

Proof. *1* $\Rightarrow$ *2*. For any module M over a hypersurface one has $\Omega^{2+\ell}M \cong \Omega^{\ell}M$ for $\ell \gg 0$; see [5, Thm 6.1]. Hence $(1-t^2)[M] = 0$ for each module M over a hypersurface.

2 $\Rightarrow$ *3* and *3* $\Rightarrow$ *4*. These implications are immediate.

4 $\Rightarrow$ *1*. Observe that M eventually periodic implies the Betti numbers of M are bounded. Then since the Betti numbers of k are bounded, [8, Cor 1] implies that R is a hypersurface. $\square$

The statement of Theorem 5.2.6 can fail for non-Gorenstein rings. In order to give a class of examples where the statement does not hold, we recall the following definitions.

Definition 5.2.12. Let $(R, \mathfrak{m}, k)$ be a local ring. The *embedding dimension* of R is

$$e = \operatorname{edim}(R) = \operatorname{rank}_k \mathfrak{m}/\mathfrak{m}^2.$$

Definition 5.2.13. Let $(R, \mathfrak{m}, k)$ be a local ring. The *socle* of R is

$$\operatorname{Soc} R = (0 : \mathfrak{m}) \cong \operatorname{Hom}_R(k, R).$$

Example 5.2.14. Let $(R, \mathfrak{m}, k)$ be a local ring with $\mathfrak{m}^2 = 0$ and $e \geq 2$. Note that R is Cohen-Macaulay but not Gorenstein because $\operatorname{rank}_k \operatorname{Soc} R = e$. Then $(1-et)J(R) = 0$, but R has no nonzero nonfree eventually periodic module.

First, we note that k is not eventually periodic but $[k]$ is torsion in $J(R)$. Indeed, the sequence

$$0 \to \mathfrak{m} \to R \to k \to 0$$

is exact, and $\Omega k \cong \mathfrak{m}$ as R-modules. Therefore $\Omega k \cong k^e$, which implies that k is not eventually periodic. On the other hand, $t^{-1}[k] = e[k]$ in $J(R)$, and therefore $(1 - et)[k] = 0$.

Let M be a nonzero, nonfree R-module. Since $\mathfrak{m}^2 = 0$, we have $\Omega M \cong k^{\beta_1(M)}$. Since k is not eventually periodic, the module M is not eventually periodic. However,

$$t^{-1}[M] = [\Omega M] = \beta_1(M)[k]$$

in $J(R)$, and therefore

$$(1 - et)[M] = t(1 - et)\beta_1(M)[k] = 0.$$

Remark 5.2.15. Using Corollary 5.2.4.1, one can determine the torsion submodule of $J(R)$ for a Gorenstein local ring that has the Krull-Remak-Schmidt property:

$$T_{\mathbb{Z}[t^{\pm 1}]}(J(R)) = \bigcup_{n=1}^{\infty} \operatorname{Ann}_{J(R)}(1 - t^n).$$

If R is a complete intersection, then $T_{\mathbb{Z}[t^{\pm 1}]}(J(R)) = \mathrm{Ann}_{J(R)}(1-t^2)$ by Theorem 5.2.6, since [5, Thm 5.2] shows that a module M over a complete intersection is periodic if and only if $\Omega^\ell M \cong \Omega^{2+\ell} M$ for $\ell \gg 0$. For a Gorenstein ring R, however, $[M]$ torsion in $J(R)$ for an R-module M need not imply that $(1-t^2)[M] = 0$. Indeed, for each $n \in \mathbb{N}$, there exists an Artinian Gorenstein local ring with a periodic module of period n; see [7, Ex 3.6].

We note that there also exist Gorenstein local rings with no nonfree eventually periodic modules. In order to give such an example, we recall the following definition from [2].

Definition 5.2.16. Let $(R, \mathfrak{m}, k)$ be a local ring and M a finitely generated R-module. Note that the property $\partial(\mathbf{F}) \subseteq \mathfrak{m}F$ of a minimal free resolution $\mathbf{F}$ of M allows one to form for each $j \geq 0$ a complex

$$\mathrm{lin}_j(\mathbf{F}) = 0 \to \frac{F_j}{\mathfrak{m}F_j} \to \cdots \to \frac{\mathfrak{m}^{j-n}F_n}{\mathfrak{m}^{j+1-n}F_n} \to \cdots \to \frac{\mathfrak{m}^j F_0}{\mathfrak{m}^{j+1}F_0} \to \frac{\mathfrak{m}^j M}{\mathfrak{m}^{j+1}M} \to 0$$

of k-vector spaces. M is *Koszul* if every complex $\mathrm{lin}_j(\mathbf{F})$ is acyclic.

When R is a graded k-algebra generated in degree 1 and M is a graded R-module generated in a single degree, say d, then M is Koszul if and only if M has a d-linear free resolution.

The following proposition shows that, using [2], one can construct Gorenstein rings over which no nonzero nonfree module is eventually periodic.

Proposition 5.2.17. *Let $(R, \mathfrak{m}, k)$ be a Gorenstein local ring with $\mathfrak{m}^3 = 0$. If $e \geq 3$, then every nonfree R-module has Betti numbers which are eventually strictly increasing. In particular, R has no nonzero nonfree eventually periodic modules.*

Proof. It suffices to consider indecomposable modules. In [2, Thm 4.6], it is proved that any indecomposable R-module is either Koszul or a cosyzygy of k. Since $e \geq 3$, the growth of the Betti numbers of k is exponential; see [2, Thm 4.1]. Thus we may focus on the Koszul modules.

Let M be a nonfree indecomposable Koszul R-module. The ring R is Koszul by [2, Thm 4.1] since $\operatorname{rank}_k(0:\mathfrak{m}) \leq e-1$. We note that $\mathfrak{m}^2 M = 0$.

Indeed, if $\mathfrak{m}^2 M \neq 0$, choose $x \in M$ such that $\mathfrak{m}^2 x \neq 0$. Let $\varphi : R \to M$ be the homomorphism with $\varphi(1) = x$. Since R is Gorenstein, $(0:\mathfrak{m}) \subseteq I$ for every nonzero ideal I of R. Then, since φ is injective on $(0:\mathfrak{m})$, it is injective on R. Since R is self-injective, φ splits. Thus R is a direct summand of M, which is a contradiction.

Then [2, Prop 3.1] yields the following equality:

$$P_M^R(t) = \frac{H_M(-t)}{H_R(-t)}, \tag{5.2}$$

where $H_M(t)$ is the *Hilbert series* of M, that is,

$$H_M(t) = \sum_{i=0}^{\infty} \operatorname{rank}_k \left(\mathfrak{m}^i M / \mathfrak{m}^{i+1} M\right) t^i.$$

Next, we show that the Betti numbers of M are eventually strictly increasing. Since $\mathfrak{m}^2 M = 0$, we have $H_M(-t) = a - bt$ for some $a, b \in \mathbb{Z}_{\geq 0}$. Then Equation (5.2) yields the following equality:

$$a - bt = (1 - et + t^2) P_M^R(t).$$

By comparing coefficients, we find the following recursion relation on the Betti numbers of M.

$$\beta_0(M) = a.$$
$$\beta_1(M) = e\beta_0(M) - b.$$
$$\beta_{n+1}(M) = e\beta_n(M) - \beta_{n-1}(M) \quad \text{for } n \geq 1.$$

As M is not free, M has infinite projective dimension by the Auslander-Buchsbaum formula [4, Thm 1.3.3]. Then since Betti numbers are nonnegative integers, there exists an $N \in \mathbb{N}$ such that $\beta_{N+1}(M) \geq \beta_N(M)$. Using this inequality and the recursion relation, we have

$$\begin{aligned} \beta_{N+2}(M) &= e\beta_{N+1}(M) - \beta_N(M) \\ &\geq (e-1)\beta_{N+1}(M). \end{aligned}$$

Since $e \geq 3$, this inequality implies that $\beta_{N+2}(M) > \beta_{N+1}(M)$. Iterating this process, we find that $\beta_{n+2}(M) > \beta_{n+1}(M)$ for all $n \geq N$. Thus the Betti numbers of M eventually strictly increase, and therefore M is not eventually periodic. $\square$

Example 5.2.18. The ring $R = k[x,y,z]/(x^2-y^2, x^2-z^2, xy, xz, yz)$ is a Gorenstein local ring with $\mathfrak{m}^3 = 0$ and $e = 3$. By Proposition 5.2.17, R has no nonzero nonfree eventually periodic modules.

Bibliography

[1] L.L. Avramov, "Problems on infinite free resolutions" in *Free resolutions in commutative algebra and algebraic geometry* (Sundance, UT, 1990), Res. Notes Math. **2**, Jones and Bartlett, Boston (1992), 3-23.

[2] L.L. Avramov, S.B. Iyengar, L.M. Şega, *Free resolutions over short local rings*, J. London Math. Soc. **78** (2008), 459-476.

[3] H. Bass, M.P. Murthy, *Grothendieck groups and Picard groups of abelian group rings*, Ann. Math. **86** (1967), 16-73.

[4] W. Bruns, J. Herzog, *Cohen-Macaulay Rings*, revised ed., Studies in Advanced Mathematics **39**, Cambridge University Press, Cambridge, 1998.

[5] D. Eisenbud, *Homological algebra on a complete intersection, with an application to group representations*, Trans. Amer. Math. Soc., **260** (1980), no. 1, 35-64.

[6] E. Enochs, O.M.G. Jenda, *Relative Homological Algebra*, de Gruyter Exp. Math. **30**, de Gruyter, Berlin, 2000.

[7] V. Gasharov, I. Peeva, *Boundedness versus periodicity over commutative local rings*, Trans. Amer. Math. Soc. **320** (1990), 569-580.

[8] T.H. Gulliksen, *A proof of the existence of minimal R-algebra resolutions*, Acta Math. **120** (1968), 53-58.

[9] D. Hilbert, *Uber die Theorie der algebraischen Formen*, Math. Ann. **36** (1890), 473-534.

[10] J. Herzog, *Ringe mit nur endlich vielen Isomorphieklassen von maximalen, unzerlegbaren Cohen-Macaulay-Moduln*, Math. Ann. **233** (1978), 21-34.

[11] C. Huneke, D. Jorgensen, R. Wiegand, *Vanishing Theorems for Complete Intersections*, J. Algebra **238** (2001), 684-702.

[12] F. Ischebeck, *Eine Dualität zwischen den Funktoren Ext und Tor*, J. Algebra **11** (1969), 510-531.

[13] D.R. Jordan, *A Grothendieck module with applications to Poincaré rationality*, J. Pure Appl. Algebra **214** (2010), 616-621.

[14] I. Kaplansky, *Commutative Rings*, revised ed., University of Chicago Press, Chicago, 1974.

[15] S. Lang, *Algebra*, revised 3rd ed., Graduate Texts in Mathematics **211**, Springer, New York, 2002.

[16] G. Leuschke, R. Wiegand, *Cohen-Macaulay Representations*, Math. Surveys Monogr. **181**, Amer. Math. Soc., Providence, 2012.

[17] W. Vasconcelos, *Reflexive modules over Gorenstein rings*, Proc. Amer. Math. Soc. **19** (1968), 1349-1355.

[18] C. Weibel, *An introduction to homological algebra*, Studies in Advanced Mathematics **38**, Cambridge University Press, Cambridge, 2003.

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