12-1-1964

MODULES OVER COMMUTATIVE RINGS

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CLASSROOM NOTES

EDITED BY GERTRUDE EHRLICH, University of Maryland

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MODULES OVER COMMUTATIVE RINGS

W. G. LEAVITT, University of Nebraska

The following is another short proof of the fact that for a commutative ring with unit $R$, any finitely based $R$-module is "dimensional" in the sense that all of its bases have the same number of elements.

**THEOREM.** Let $R$ be a commutative ring with unit. If $M$ is a unitary $R$-module with a basis of $n$ elements, then all bases of $M$ contain exactly $n$ elements.

**Proof.** (The method is that of [1], p. 115.) Let $\{\alpha_i\} (i = 1, \ldots, n)$ be a basis for $M$. It is easy to see that $M$ cannot have an infinite basis. (See [2], p. 241–2. Applied to modules, the method shows that for a module with an infinite basis all bases have the same cardinality.) Thus let $\{\beta_j\} (j = 1, \ldots, m)$ be another
basis of $M$. Write $\alpha_i = \sum_{j=1}^{n} a_{ij} \beta_j$ ($i = 1, \ldots, n$) and $\beta_j = \sum_{k=1}^{n} b_{jk} \alpha_k$ ($j = 1, \ldots, m$). If $A = [a_{ij}]$ and $B = [b_{ij}]$, it follows from the independence of the $\alpha_i$'s and the $\beta_j$'s that

$$AB = I_n \quad \text{and} \quad BA = I_m,$$

where $I_n$ and $I_m$ are unit matrices. Conversely, the existence of relations (1) and (2) in a ring $R$ implies the existence of an $R$-module with bases of lengths $m$ and $n$, namely the module of all $m$-tuples. This module has, of course, the rows of $I_m$ as a basis, but also has as an alternative basis the rows of $A$. This is clear, since from (2) each row of $I_m$ is a linear combination of the rows of $A$. This is clear, since from (2) each row of $I_m$ is a linear combination of the rows of $A$, while from (1), $XA = 0$ implies $XI_n = X = 0$, so the rows of $A$ are independent.

Now any homomorphism of $R$ preserves the relations (1) and (2), and so any nonzero homomorphic image of $R$ also admits a module with bases of lengths $m$ and $n$. But if we apply Zorn's lemma in the usual way (relative to ideals not containing the unit, partially ordered by set inclusion) we obtain a maximal ideal $I$ of $R$. Since $R/I$ is a field, its modules are vector spaces all of whose bases are of the same length. Thus since $R/I$ is a homomorphic image of $R$, we must conclude that $m = n$.

References