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MODULES OVER COMMUTATIVE RINGS

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CLASSROOM NOTES

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The following is another short proof of the fact that for a commutative ring with unit R, any finitely based R-module is "dimensional" in the sense that all of its bases have the same number of elements.

THEOREM. *Let R be a commutative ring with unit. If M is a unitary R-module with a basis of n elements, then all bases of M contain exactly n elements.*

Proof. (The method is that of [1], p. 115.) Let $\{\alpha_i\}$ $(i=1, \cdots, n)$ be a basis for M. It is easy to see that M cannot have an infinite basis. (See [2], p. 241–2. Applied to modules, the method shows that for a module with an infinite basis all bases have the same cardinality.) Thus let $\{\beta_j\}$ $(j=1, \cdots, m)$ be another

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basis of M. Write $\alpha_i = \sum_{j=1}^{m} a_{ij}\beta_j$ $(i = 1, \cdots, n)$ and $\beta_j = \sum_{k=1}^{n} b_{jk}\alpha_k$ $(j=1, \cdots, m)$. If $A = [a_{ij}]$ and $B = [b_{ij}]$, it follows from the independence of the α_i's and the β_j's that

$$(1)\ AB = I_n \quad \text{and} \quad (2)\ BA = I_m,$$

where I_n and I_m are unit matrices. Conversely, the existence of relations (1) and (2) in a ring R implies the existence of an R-module with bases of lengths m and n, namely the module of all m-tuples. This module has, of course, the rows of I_m as a basis, but also has as an alternative basis the rows of A. This is clear, since from (2) each row of I_m is a linear combination of the rows of A, while from (1), $XA = 0$ implies $XI_n = X = 0$, so the rows of A are independent.

Now any homomorphism of R preserves the relations (1) and (2), and so any nonzero homomorphic image of R also admits a module with bases of lengths m and n. But if we apply Zorn's lemma in the usual way (relative to ideals not containing the unit, partially ordered by set inclusion) we obtain a maximal ideal I of R. Since R/I is a field, its modules are vector spaces all of whose bases are of the same length. Thus since R/I is a homomorphic image of R, we must conclude that $m = n$.

References

1. W. G. Leavitt, The module type of a ring, Trans. Am. Math. Soc., 103 (1962) 113–130.
2. N. Jacobson, Lectures in abstract algebra, vol. II, Van Nostrand, New York, 1953.