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THE DIFFERENTIABILITY OF $a^x$

J. A. Erdswick

A "from scratch" proof of the differentiability of $a^x$, $a > 0$, is avoided by essentially all modern-day authors. A slick and popular way of handling the problem is to define $a^x$ as $e^{x \log a}$ its differentiability and other properties following from that of the functions $e^x$ and $\log x$. Unfortunately, the usual definitions of $e^x$ and $\log x$ involve relatively sophisticated ideas (e.g., integration or power series). Furthermore, the student, having heard of $e$, the natural logarithm base, at an early stage of his development, is hardly enlightened when he is told that $e$ is $e^1$. He would have a much better feeling for the "naturalness" of $e$ if it were defined as that number $a$ for which $(a^x)' = a^x$.

The purpose of this note is to provide a direct and relatively simple way of getting at the differentiability of $a^x$. We define $a^x = \lim_{r \to x} a^r$ through rational values of $r$ from which continuity and other basic properties follow (see e.g., [1, p. 63]). The differentiability question obviously reduces to showing that the function $F(x) = (a^x - 1)/x$ has a limit at 0. Since $F(-x) = a^{-x}F(x)$, it suffices to show only that the right-hand limit exists. By a similar observation, we may assume that $a > 1$. As a final reduction, we note that, for $a > 1$, $F$ is bounded below on $(0, \infty)$ and, hence, it is sufficient to show that $F$ is increasing on $(0, \infty)$.

Define $S(x, n) = 1 + x + \cdots + x^{n-1}$ so that $S(x, n)(x-1) = x^n - 1$. Since
\[ n(a^{1/n} - a^{1/(n+1)}) = na^{1/(n+1)}(a^{1/(n+1)} - 1) > S(a^{1/(n+1)}, n)(a^{1/(n+1)} - 1) = a^{1/(n+1)} - 1, \]
the sequence $\{F(1/n)\}$ is decreasing. Therefore, for positive rational numbers $m/n < p/q$, we have
\[ F(m/n) = F(1/pn)S(a^{1/pn}, pn)/pm < F(1/qm)S(a^{1/qm}, pm)/pm = F(p/q). \]

In other words, $F$ is increasing on the positive rationals. By continuity, $F$ is increasing on $(0, \infty)$. 

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We conclude by noting that \((a^x)'\) is proportional to \(a^x\) and that the constant of proportionality can be taken to be 1 if \(a\) is chosen suitably, leading to an appealing definition of \(e\) (cf. [2, p. 41–44]).

References


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