5-1-1975

THE DIFFERENTIABILITY OF a^x

J. A. Eidswick
University of Nebraska - Lincoln

THE DIFFERENTIABILITY OF a^x

J. A. Eidswick

A "from scratch" proof of the differentiability of a^x, $a > 0$, is avoided by essentially all modern-day authors. A slick and popular way of handling the problem is to define a^x as $e^{x \log a}$ its differentiability and other properties following from that of the functions e^x and $\log x$. Unfortunately, the usual definitions of e^x and $\log x$ involve relatively sophisticated ideas (e.g., integration or power series). Furthermore, the student, having heard of e, the natural logarithm base, at an early stage of his development, is hardly enlightened when he is told that e is e^1. He would have a much better feeling for the "naturalness" of e if it were defined as that number a for which $(a^x)' = a^x$.

The purpose of this note is to provide a direct and relatively simple way of getting at the differentiability of a^x. We define $a^x = \lim a^r$ as $r \to x$ through rational values of r from which continuity and other basic properties follow (see e.g., [**1**, p. 63]). The differentiability question obviously reduces to showing that the function $F(x) = (a^x - 1)/x$ has a limit at 0. Since $F(-x) = a^{-x}F(x)$, it suffices to show only that the right-hand limit exists. By a similar observation, we may assume that $a > 1$. As a final reduction, we note that, for $a > 1$, F is bounded below on $(0, \infty)$ and, hence, it is sufficient to show that F is increasing on $(0, \infty)$.

Define $S(x, n) = 1 + x + \cdots + x^{n-1}$ so that $S(x, n)(x - 1) = x^n - 1$. Since

$$\begin{aligned} n(a^{1/n} - a^{1/(n+1)}) &= na^{1/(n+1)}(a^{1/n(n+1)} - 1) \\ &> S(a^{1/n(n+1)}, n)(a^{1/n(n+1)} - 1) \\ &= a^{1/(n+1)} - 1, \end{aligned}$$

the sequence $\{F(1/n)\}$ is decreasing. Therefore, for positive rational numbers $m/n < p/q$, we have

$$\begin{aligned} F(m/n) &= F(1/pn)S(a^{1/pn}, pm)/pm \\ &< F(1/qm)S(a^{1/qm}, pm)/pm = F(p/q). \end{aligned}$$

In other words, F is increasing on the positive rationals. By continuity, F is increasing on $(0, \infty)$.

The American Mathematical Monthly, Vol. 82, No. 5 (May, 1975), pp. 505-506

We conclude by noting that $(a^x)'$ is proportional to a^x and that the constant of proportionality can be taken to be 1 if a is chosen suitably, leading to an appealing definition of e (cf. [**2**, p. 41–44]).

References

1. Casper Goffman, Introduction to Real Analysis, Harper & Row, New York, 1966.

2. Edmund Landau, Differential and Integral Calculus, 2nd ed., Chelsea, New York, 1960.

DEPARTMENT OF MATHEMATICS AND STATISTICS, UNIVERSITY OF NEBRASKA, LINCOLN, NE 68508.