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Fractals and the Collage Theorem

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Fractals and the Collage Theorem
Expository Paper

Sandra S. Snyder

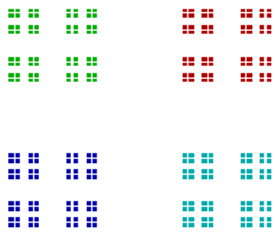
In partial fulfillment of the requirements for the Master of Arts in Teaching with a
Specialization in the Teaching of Middle Level Mathematics
in the Department of Mathematics.
David Fowler, Advisor.

July 2006

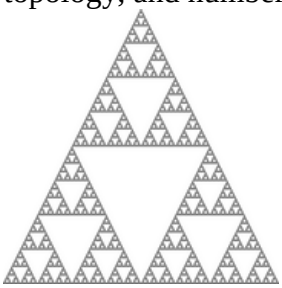
Fractals and the Collage Theorem[1]

A. Fractal History

The idea of fractals is relatively new, but their roots date back to 19th century mathematics. A fractal is a mathematically generated pattern that is reproducible at any magnification or reduction and the reproduction looks just like the original, or at least has a similar structure. Georg Cantor (1845-1918) founded set theory and introduced the concept of infinite numbers with his discovery of cardinal numbers. He gave examples of subsets of the real line with unusual properties. These Cantor sets are now recognized as fractals, with the most famous being the Cantor Square.

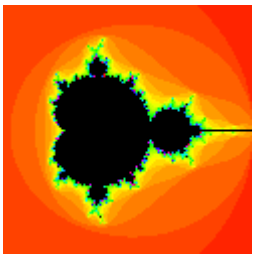


Waclaw Sierpinski (1882-1969), a Polish mathematician, worked in set theory, point set topology, and number theory. He is known for the Sierpinski Triangle.



However, there are many other Sierpinski fractals, such as the Sierpinski Carpet.

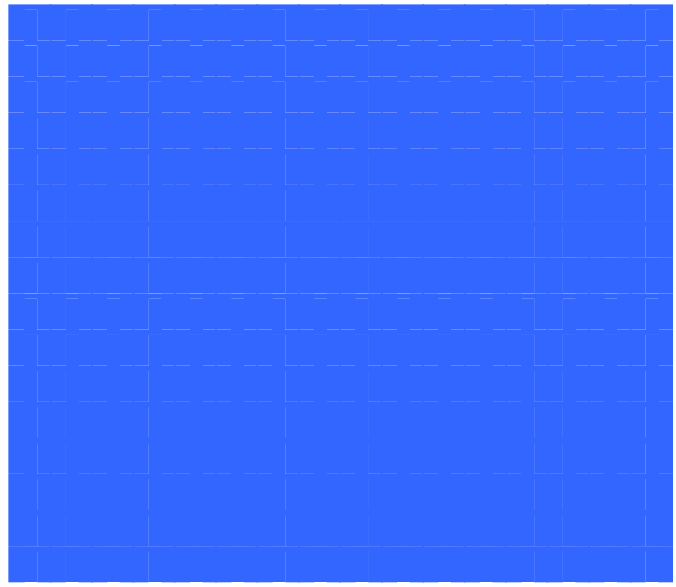
The term 'fractal' was coined in 1975 by Benoit Mandelbrot (1924 -) from the Latin *fractus*, meaning "broken" or "irregular." This term was used to describe shapes that have the characteristic of self-similarity, i.e. that when you magnify any part it looks just like (or has the same structure) as the original. He is widely known for the Mandelbrot set.



B. Basic Fractals

Now, let's try to create some basic fractals using functions on the plane. We can start with a

[1] Sections A, B, and E created in collaboration with Stacie Lefler.



square with corners at (0, 0), (1, 0), (0, 1), and (1, 1).

We will call our initial image of a square S_0.

We are interested in what happens to our square when we consider the functions $f_1(x, y) = \left(\frac{x}{2}, \frac{y}{2}\right)$, $f_2(x, y) = \left(\frac{x}{2} + \frac{1}{2}, \frac{y}{2}\right)$, and $f_3(x, y) = \left(\frac{x}{2} + \frac{1}{4}, \frac{y}{2} + \frac{1}{2}\right)$, and evaluate them at the vertices of our square.

When we evaluate $f_1(x, y) = \left(\frac{x}{2}, \frac{y}{2}\right)$ at the vertices of S_0 we get the following:

$f_1(0, 0) = (0, 0)$
$f_1(1, 0) = (½, 0)$
$f_1(0, 1) = (0, ½)$
$f_1(1, 1) = (½, ½)$

Notice that this takes S_0 and shrinks it to half of its original x length and half of its original y height.

When we evaluate $f_2(x, y) = \left(\frac{x}{2} + \frac{1}{2}, \frac{y}{2}\right)$ at the vertices of S_0 we get the following:

$f_2(0, 0) = (½, 0)$
$f_2(1, 0) = (1, 0)$
$f_2(0, 1) = (½, ½)$
$f_2(1, 1) = (1, ½)$

Note that this is the same image as we get from f_1, but it is shifted ½ unit to the right.

When we evaluate $f_3(x, y) = \left(\frac{x}{2} + \frac{1}{4}, \frac{y}{2} + \frac{1}{2}\right)$ at the vertices of S_0 we get the following:

$f_3(0, 0) = (¼, ½)$
$f_3(1, 0) = (¾, ½)$
$f_3(0, 1) = (¼, 1)$
$f_3(1, 1) = (¾, 1)$

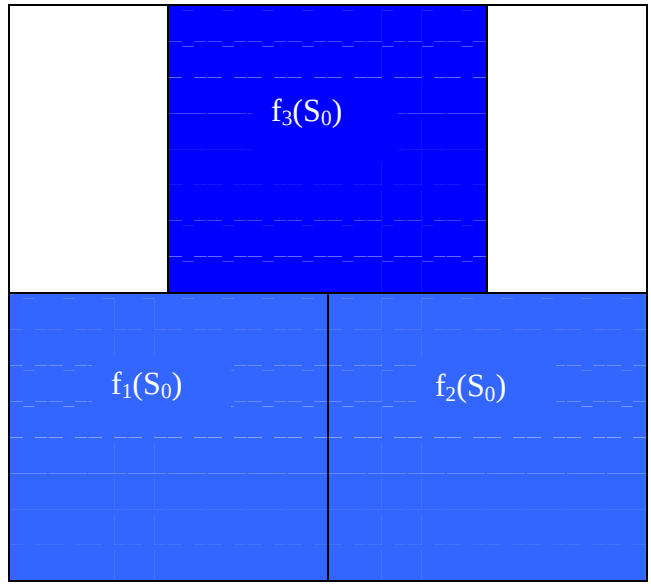
Note that this, too, is the same image as we get from f_1, but it is shifted ¼ unit to the right and ½ unit up.

We define the new function formed as:

$F(S) = f_1(S) \cup f_2(S) \cup f_3(S)$

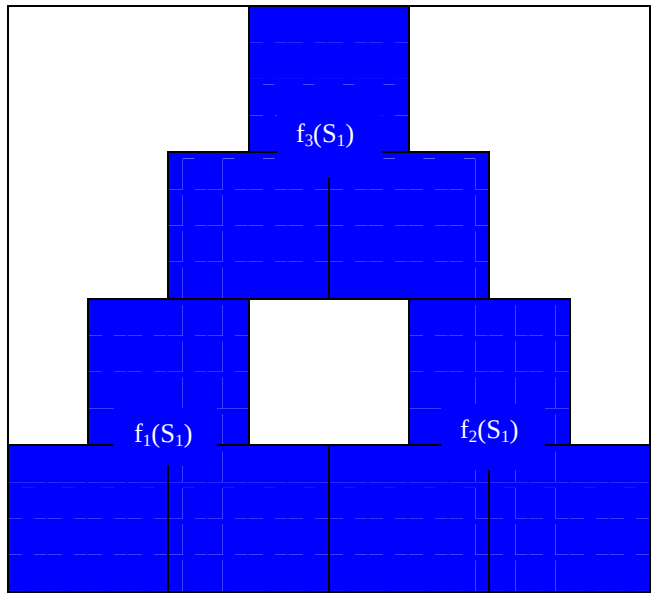
Here is $F(S_0)$:

We will call this new image S_1.



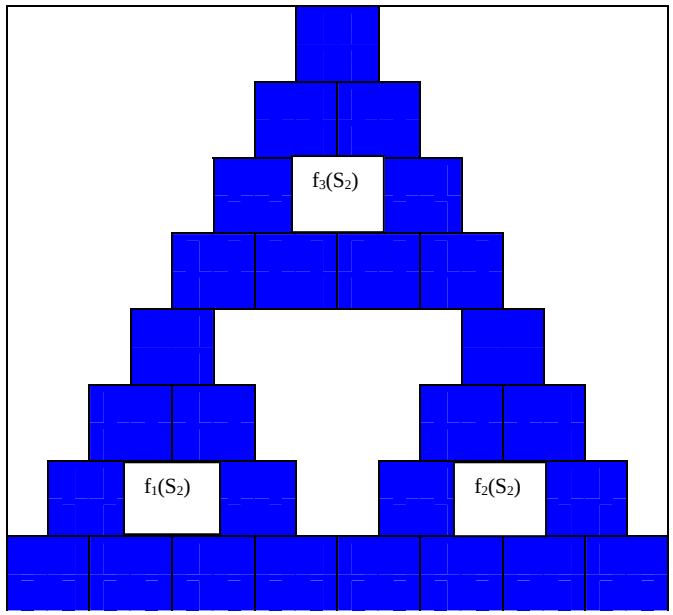
We can now iterate, or repeat, our image again by following the same pattern. To get S_2 we can start by evaluating f_1 at the vertices of S_1 which would shrink S_1 to half of its original x length and half of its original y height. Next, we would evaluate f_2 at the vertices of S_1. This simply gives a ½ to the right translation of $f_1(S_1)$. Last, we would evaluate f_3 at the vertices of S_1. This gives us a translation of $f_1(S_1)$, too. This one is shifted ¼ unit left, and ½ unit up.

We'll call this image S_2.



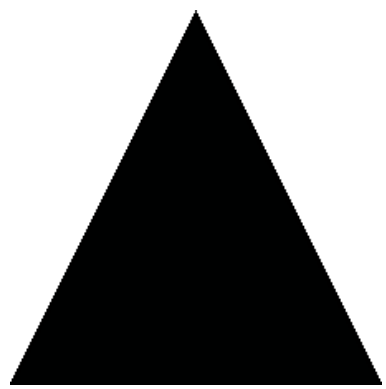
Iterate the image again by evaluating the same three functions at the vertices of S_2. The first function, f_1, will shrink the image, f_2 will translate the shrunken figure to the right ½, and f_3 will translate the shrunken figure to the right ¼, and up ½.

We'll call this image S_3.



This is starting to look like the Sierpinski Triangle. We might wonder if our three functions gave us the Sierpinski Triangle because we started with four corners of a square. Let's see what happens if we start with an isosceles triangle instead of a square. We start with an isosceles triangle with corners at (0, 0), (1, 0), and (½, 1).

The initial image is the equilateral triangle.
We will call this image S_0.



When we evaluate $f_1(x, y) = \left(\frac{x}{2}, \frac{y}{2}\right)$ at the vertices of S_0 we get the following:

$f_1(0, 0) = (0, 0)$
$f_1(1, 0) = (½, 0)$
$f_1(½, 1) = (¼, ½)$

Notice that evaluating f_1 with the vertices of S_0 shrinks the image to half of the original length of x and half the original height of y.

When we evaluate f_2 (x, y) = $\left(\frac{x}{2}+\frac{1}{2},\frac{y}{2}\right)$ at the vertices of S_0 we get the following:

$f_2(0, 0) = (½ , 0)$
$f_2(1, 0) = (1, 0)$
$f_2(½ , 1) = (¾ , ½)$

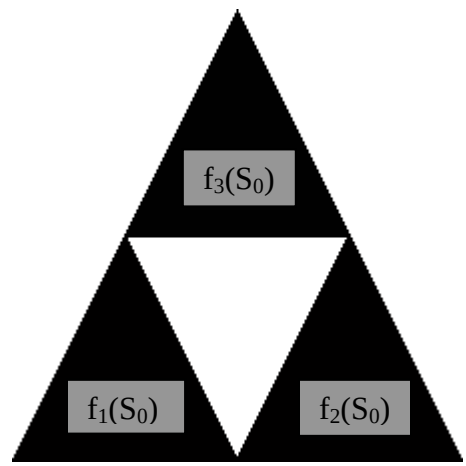
Notice that this gives the same image that we achieved with f_1, but it has been shifted ½ to the right.

When we evaluate $f_3(x, y) = \left(\frac{x}{2}+\frac{1}{4},\frac{y}{2}+\frac{1}{2}\right)$ at the vertices of S_0 we get the following:

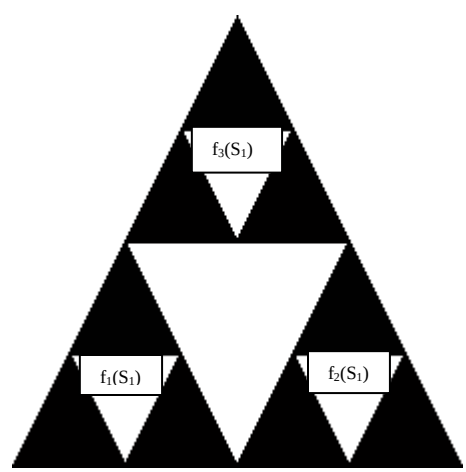
$f_3(0, 0) = (¼ , ½)$
$f_3(1, 0) = (¾ , ½)$
$f_3(½ , 1) = (½ , 1)$

This, too, gives the same image as in f_1, but it has been shifted ¼ to the right, and ½ up.

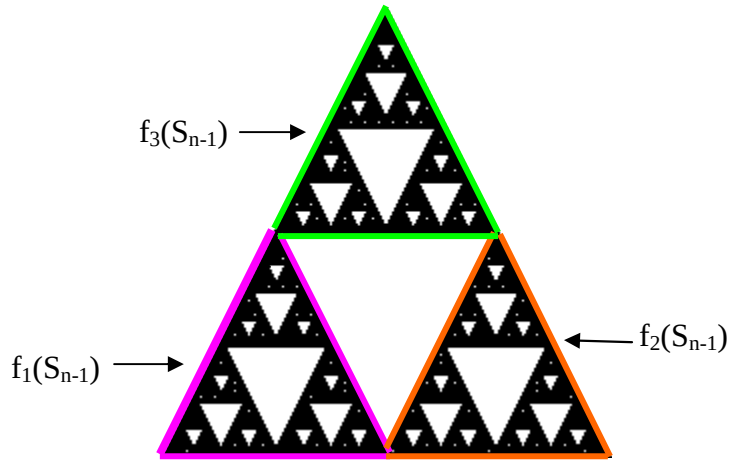
We will call this new image S_1.



After the second iteration we have the new image, S_2.



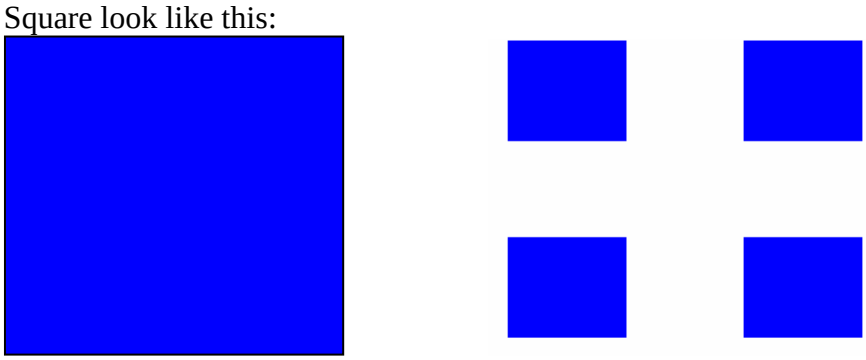
After the *n* iterations we have the new image, S_n.



After the nth iteration, we can see that when we start with a triangle the functions are affecting the image in the same way as when we started with a square. In fact, we could start with any initial image, even a silhouette of Jim Lewis, and after enough iterations using our three functions we would begin to see the Sierpinski Triangle. The final image is actually independent of the initial image. If we do enough iterations, the initial image gets smaller and smaller, becoming a dot, and so the final image is in a sense made up of an infinite number of dots. It does not matter what shape we start with, if we apply the same three functions,

$f_1(x, y) = \left(\frac{x}{2}, \frac{y}{2}\right)$, $f_2(x, y) = \left(\frac{x}{2} + \frac{1}{2}, \frac{y}{2}\right)$, and $f_3(x, y) = \left(\frac{x}{2} + \frac{1}{4}, \frac{y}{2} + \frac{1}{2}\right)$, we will get the Sierpinski Triangle. (for more information, visit http://www.maths.anu.edu.au/~barnsley/pdfs/V-var_super_fractals.pdf)

Another famous iteration is known as the Cantor Square. The Cantor Square, in contrast, is an iteration of four functions. We learned that the first two iterations of the Cantor Square look like this:



From that, we were able to determine the functions that generate the fractal. To create the Cantor Square, we begin with a 1 x 1 square. To this image, we apply the following functions:

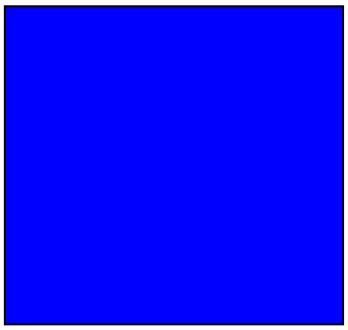
$$f_1(x, y) = \left(\frac{x}{3}, \frac{y}{3}\right) \qquad f_3(x, y) = \left(\frac{x}{3}, \frac{y}{3} + \frac{2}{3}\right)$$

$$f_2(x, y) = \left(\frac{x}{3} + \frac{2}{3}, \frac{y}{3}\right) \qquad f_4(x, y) = \left(\frac{x}{3} + \frac{2}{3}, \frac{y}{3} + \frac{2}{3}\right)$$

We did not know these four functions before creating the fractal. We determined these functions by examining the S_0 and S_1 images. The functions came from discovering the shrinks and translations applied to the initial image, S_0.

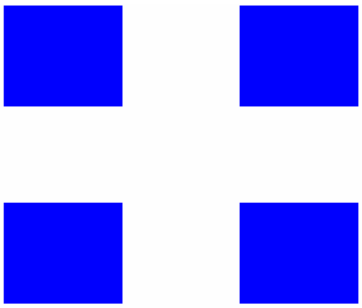
The initial image of the Cantor Square is to the right.

We will call this image S_0.



$F(S_0)$ is an image that looks like this:

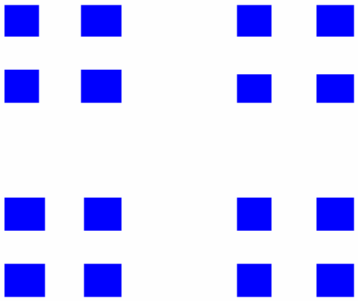
We will call this image S_1.



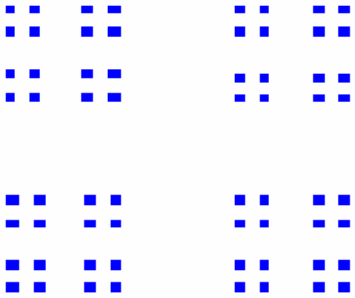
We get these four squares by applying the four functions. The function f_1 simply shrinks the image. The function f_2 translates the shrunken image to the right. The function f_3 translates the shrunken image up, and the function f_4 translates the shrunken image to the right and up.

If we repeat the iteration on the previous image, we get:

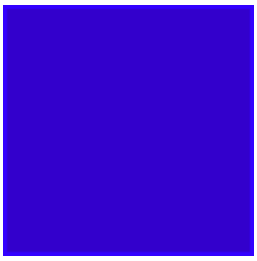
We will call this image S_2.

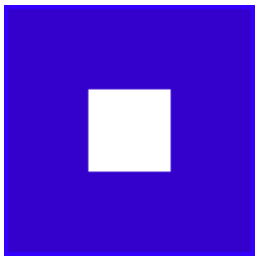


And, after one more iteration, we produce this image:
We will call this image S_3.



The Sierpinski Carpet is another unique fractal. We learned that the first two images of the Sierpinski Carpet look like the following:





By looking at these images, we determined the eight different functions necessary to generate the Sierpinski Carpet.

$$f_1(x,y)=\left(\frac{x}{3},\frac{y}{3}\right)$$

$$f_2(x,y)=\left(\frac{x}{3}+\frac{1}{3},\frac{y}{3}\right)$$

$$f_3(x,y)=\left(\frac{x}{3}+\frac{2}{3},\frac{y}{3}\right)$$

$$f_4(x,y)=\left(\frac{x}{3},\frac{y}{3}+\frac{1}{3}\right)$$

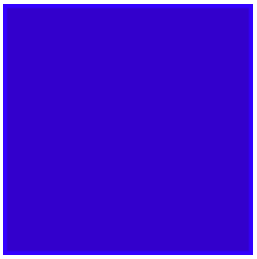
$$f_5(x,y)=\left(\frac{x}{3}+\frac{2}{3},\frac{y}{3}+\frac{1}{3}\right)$$

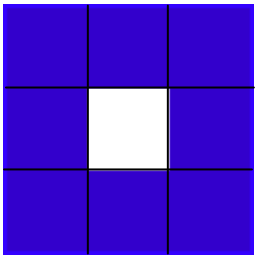
$$f_6(x,y)=\left(\frac{x}{3},\frac{y}{3}+\frac{2}{3}\right)$$

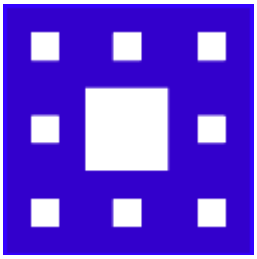
$$f_7(x,y)=\left(\frac{x}{3}+\frac{1}{3},\frac{y}{3}+\frac{2}{3}\right)$$

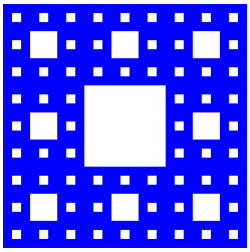
$$f_8(x,y)=\left(\frac{x}{3}+\frac{2}{3},\frac{y}{3}+\frac{2}{3}\right)$$

We found these eight functions by observing the given images S_0 and S_1 and how S_0 was transformed to achieve S_1. Each part of S_1 was formed by either shrinking or shrinking and translating S_0. The first four images are shown below. Notice that the second image is actually eight shrunken copies of the previous image, seven of which are also translated. This is why we have eight functions necessary to create the Sierpinski Carpet.

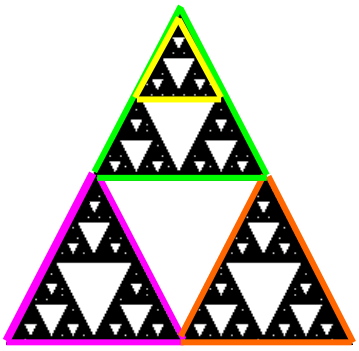








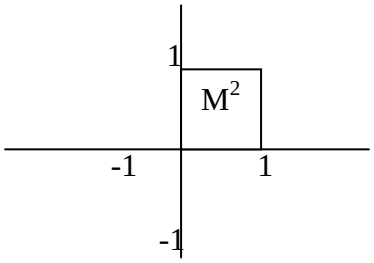
All of the fractals we have looked at so far share the common characteristic in that they are self-similar. An object is said to be self-similar if it looks "roughly" the same on any scale of magnification. We can choose a small part of the image and it will look very similar to the whole image. For example, consider the Sierpinski Triangle. Zoom in on a section, say the pink region. It is a miniature duplicate or a copy of the whole triangle, as is the yellow region. It is self-similar. In fact, all fractals share this characteristic of self-similarity.

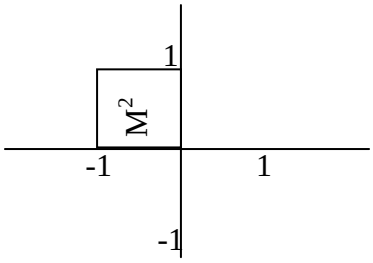


When we evaluate the previous image with the function f_1, it gives us the lower left corner. When we evaluate the previous image with the function f_2,it gives us the lower right corner, and f_3 gives the upper corner. Many objects in nature also have the self-similarity property as we will see later. (for more information, visit http://math.bu.edu/DYSYS/chaos-game/node5.html)

C. More about Functions on the Plane

In previous examples, our fractals were created by shrinking and translating shapes. To get even more interesting fractals we can incorporate functions that will rotate and flip shapes. It is important to find the basic functions to achieve each of these. For example, let's determine a function that will rotate a shape 90°. Let's start with a basic shape, a square positioned at the origin, with other vertices at (1,0), (1,1), and (0,1). We will find a function that gives the rotated image:



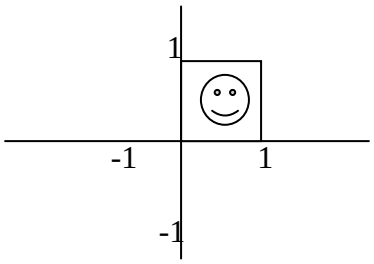


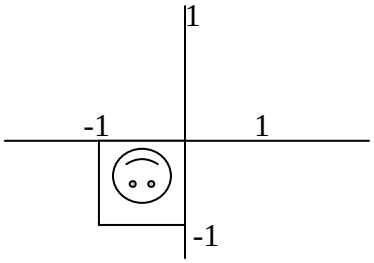
First, I mapped where the original vertices fell in the rotated image.

(0,0) → (0,0)
(1,0) → (0,1)
(1,1) → (-1,1)
(0,1) → (-1,0)

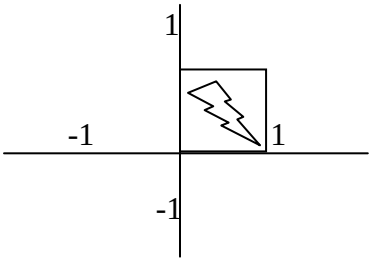
I can see that the original *x* values are now the y values, and the opposites of the original *y* values are now the *x* values. I can write this in function form: $f(x,y) = (-y,x)$

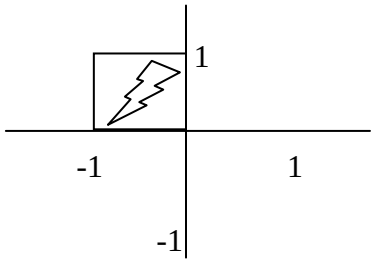
Next, I found a function that would rotate an image 180°. I started with the same basic situation:



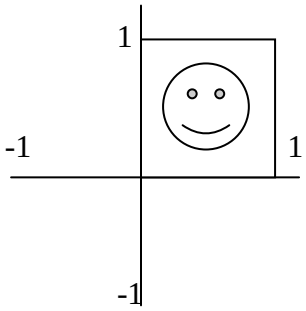


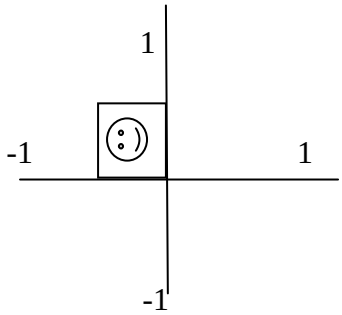
My mapping was:
(0,0) → (0,0)
(1,0) → (-1,0)
(1,1) → (-1,-1)
(0,1) → (0,-1)
I can see that the second ordered pairs can be written as the opposites of the first ordered pairs. I can write this in function form: $f(x,y) = (-x,-y)$
Last, I found a function that would flip an image. I again started with the same basic situation:



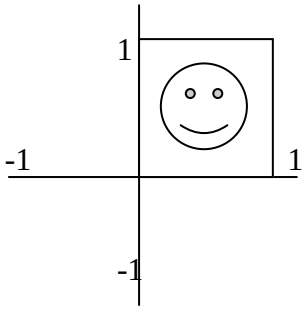


My mapping was:
(0,0) → (0,0)
(1,0) → (-1,0)
(1,1) → (-1,1)
(0,1) → (0,1)
The new x values are the opposite of the originals, and the y values remain the same.
This can be written as: $f(x,y) = (-x,y)$
We can put rotations, shrinks, translations, flips, etc. altogether to do interesting things.
For example, I found functions that turn the first box into the second.

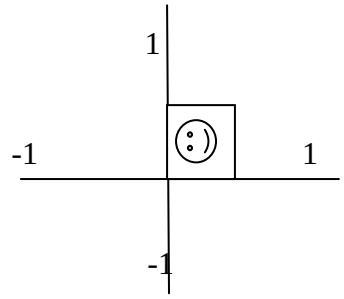




$$f(x,y) = \left(\frac{-y}{2}, \frac{x}{2}\right)$$



$$f(x,y) = \left(\frac{-y}{2} + \frac{1}{2}, \frac{x}{2}\right)$$



D. The Collage Theorem

In the previous examples we have been taking an image and finding the functions that generated that image. Michael Barnsley, a professor at Georgia Tech, proved a theorem that actually supports this idea and gives an idea how to do it. It is called the Collage Theorem. It states:

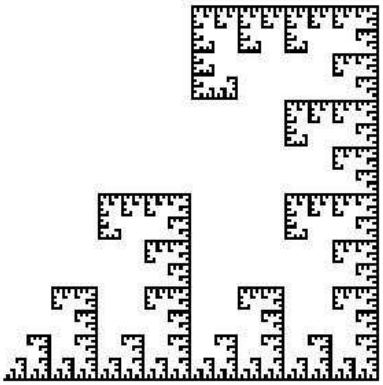
> *If the image you want to get is called L, then you need to find functions f such that F(L) = L. Then no matter what initial image you start with, if you iterate F, you'll "eventually" get L, where "eventually" means you'll get closer and closer to it and after awhile your image will be indistinguishable form L.* (direct correspondence, W. Hines)

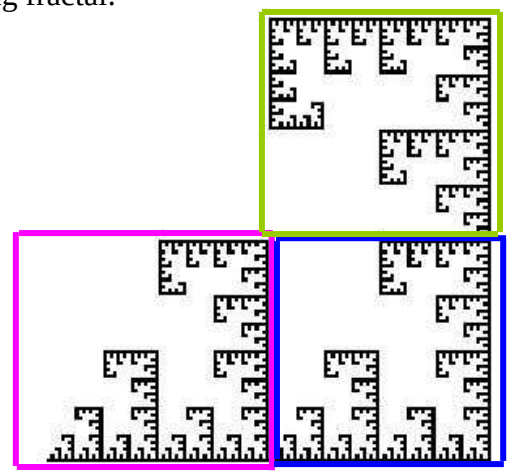
To make this more precise, we should add:

> *If F(S) = S and S contains L, then S = L. That is, no larger set is 'fixed' by F. Without this, many F's will not work, the simplest F being the function*
> *F(x,y) = (x,y), for all (x,y). This function satisfies F(L) = L, but it will not generate L.* (direct correspondence, G. Woodward)

We can look at an iterated image, and try to find the functions that we applied to any initial shape to generate the final image. Take for example the Cantor Square. In looking at S_3, I see four identical "parts". This is a clue to me that there are four functions that will be applied to any basic shape. Each shape has one-third the dimensions of S_2. One of the functions will then shrink the previous image. We can then slide that shrunken image to three different locations. The remaining three functions must translate the image to each of the three positions. We actually did the inverse of this theorem when we were given the functions for the Sierpinski Triangle, and we applied them to different initial images. "Eventually" we got the same picture.

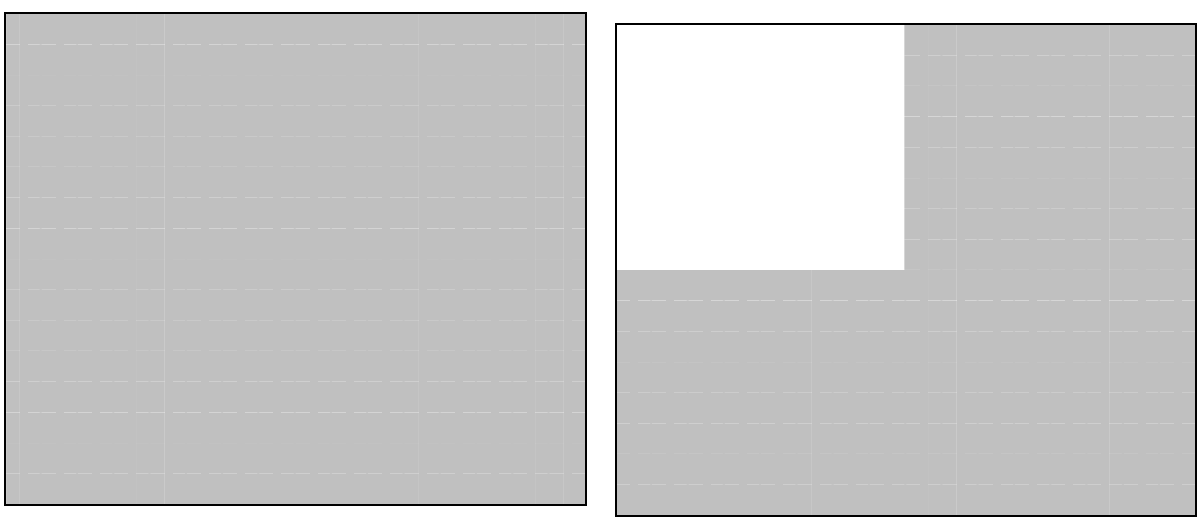
My next challenge is to use The Collage Theorem to determine the functions that give me the following fractal:

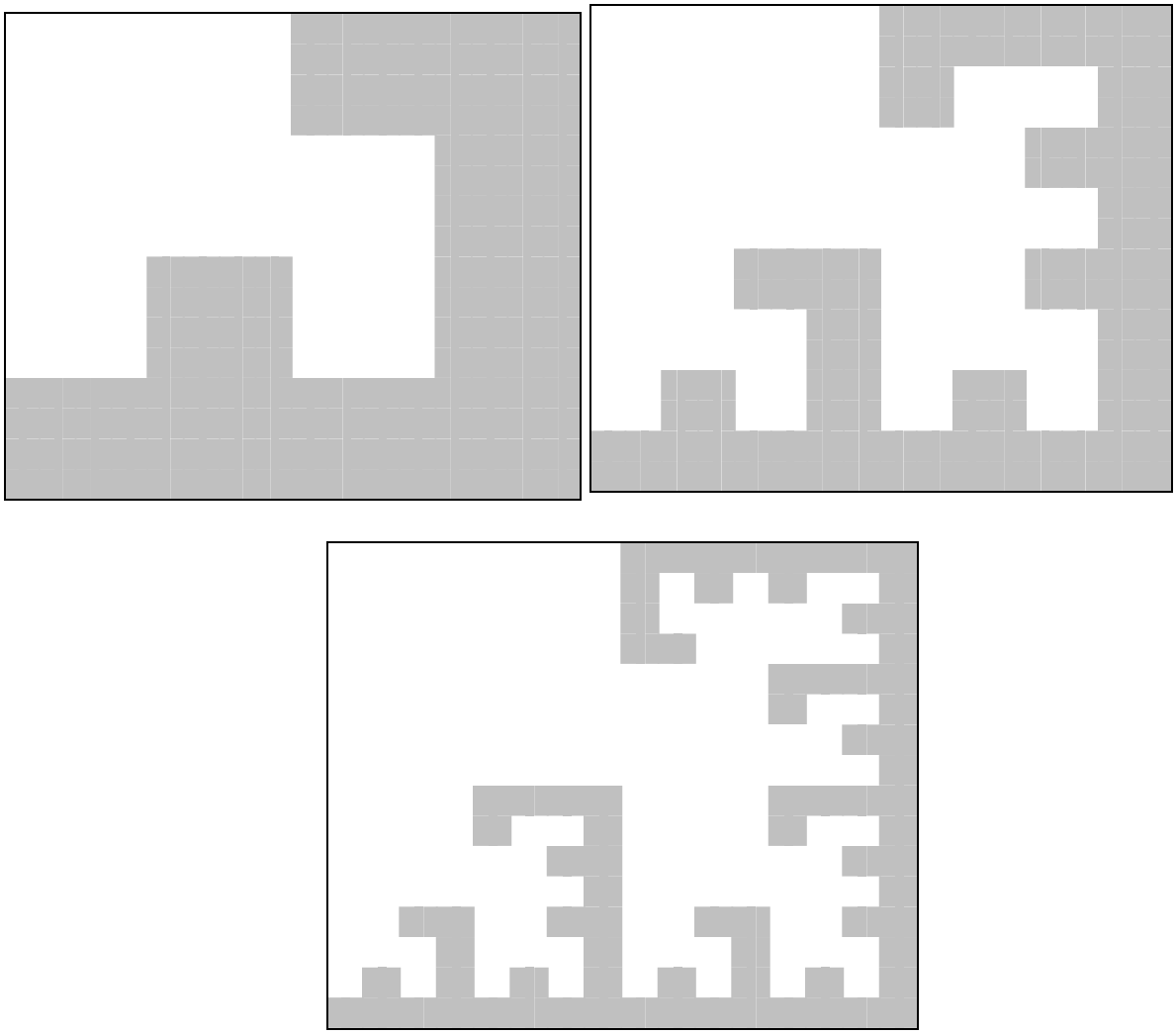




I started by staring at the picture. I viewed this as part of a square with vertices (0,0), (0,1), (1,0), and (1,1). With a little prodding, I did see three, smaller replicas of the big picture. One of the pictures was simply a shrink of the big picture. Both dimensions were half of the original, *x* and *y* by the vertices used for the original image. (Marked in pink on the above diagram.) This gave me one of my functions: $f(x,y)=\left(\frac{x}{2},\frac{y}{2}\right)$.

Another was a translation of the shrunken image. I determined that it was translated ½ to the right. (Marked in blue on the above diagram.) This allowed me to find the second function: $f(x,y) = \left(\frac{x}{2}+\frac{1}{2},\frac{y}{2}\right)$. It took me quite a while to find the third formula because I was not finding the third image! Finally, and with some assistance, I found the shrunken image translated to the right and up, and then rotated. (Marked in green on the above diagram.) Because of the rotational relationships found previously, this allowed me to write the third function: $f(x,y) = \left(\frac{-y}{2}+1,\frac{x}{2}+\frac{1}{2}\right)$. Now, I needed to test my formulas. I decided to use a 1x1 square as my initial (S_0) shape. I then applied four iterations, shown in progression below. If I were to continue, I would indeed have my starting image. My functions are appropriate for the fractal.



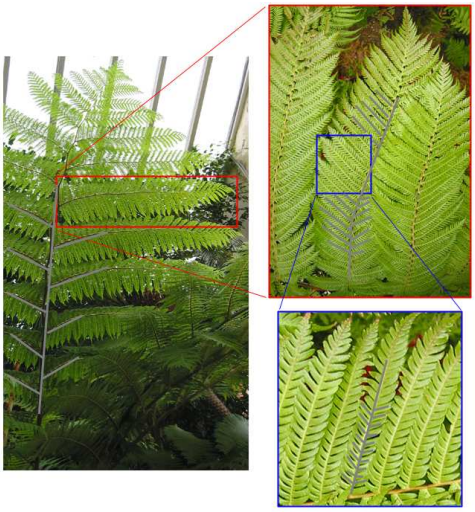


E. Fractals in Nature

This was not Barnsley's only contribution in the area of fractals. Let's say you want to store a picture of the hook picture from before. That would take much storage space if we saved the information by the pixel. Barnsley realized there is no need to store the whole picture, just store the three functions needed to create the picture. When you want the picture, run a program that iterates the functions as many times as you choose, and you will get a picture like the hooks. He also used four functions to generate a very natural-looking image of a fern, called the Barnsley fern. This prompted others to try their hand at generating all sorts of natural images. Believe it or not, this image is actually computer-generated with the use of fractals, and not taken with a powerful camera.

It has also become increasingly popular to use fractal technology to create computer-generated special effects.

Nature holds an array of items that have fractal properties. Among them are a fern and romanesco. As you see in the pictures, both have the self-similarity property, and both repeat.



Trees, clouds, snowflakes, mountains, rivers, and coastlines are also naturally occurring fractals.

Hopefully you have seen, as did I, some wonderful aspects of fractals. Not only do they hold the beauty of the mathematics used to create them, they have the beauty of occurring naturally. It is amazing that this exciting mathematical application is around us at every turn and has been forever, but that it took so much time to give it a name.

F. Summary

I have thoroughly enjoyed this opportunity to study fractals. This was a topic that I had heard of, but had not had a chance to learn about it. This study strengthened my knowledge and confidence when working with functions with two variables. By the time I was done with the Sierpinski Triangle, I was pretty confident that I knew what there was to know. Then, I moved to the Cantor Square and one more function to find. I was still in my comfort zone. But, then we got to the Sierpinski Carpet with eight functions. Overwhelming was my first thought, and all of a sudden, the functions were there! That was a big sense of accomplishment for me. I then went on to research the Collage Theorem, which backed up the previous work. The inclusion of rotations as an additional transformation technique was challenging, but not too extreme. I definitely felt a sense of achievement when Jamie Radcliffe told me my work on the hook fractal was correct.

In researching this topic, I primarily used the Internet. One challenge was to read the functions given on a web page. They are not always written in the most appropriate form. Another challenge was finding appropriate material. Wendy Hines was an invaluable resource for this topic. She was definitely better than any book or web page! She was able to give the important facts and let me experiment and discover on my own. This is an area I need to work on as a teacher. So often, I want to give all the pertinent information to my students instead of allowing them the time to research and test ideas. I would like to do some work with my students pertaining to the fractals I learned about. I am sure my 8th graders would enjoy this. For my own knowledge, I plan to find some more information about the Mandelbrot set. I think this may be a little too in depth with the complex plane for middle school students, but the images are beautiful and can be enjoyed by students of all ages.

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