8-2013

Results on Containments and Resurgences, with a Focus on Ideals of Points in the Plane

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A DISSERTATION

Presented to the Faculty of

The Graduate College at the University of Nebraska

In Partial Fulfilment of Requirements

For the Degree of Doctor of Philosophy

Major: Mathematics

Under the Supervision of Professor Brian Harbourne

Lincoln, Nebraska

August, 2013

RESULTS ON CONTAINMENTS AND RESURGENCES, WITH A FOCUS ON IDEALS OF POINTS IN THE PLANE

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University of Nebraska, 2013

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Let K be a field of arbitrary characteristic and let I be a nontrivial homogeneous ideal in $R = K[\mathbb{P}^N]$. Then we can take two different kinds of powers of I - ordinary powers of the form I^r and symbolic powers of the form $I^{(m)} = R \cap \left(\bigcap_{P \in Ass(I)} I^m R_P\right)$. A question that has been of particular interest to the mathematical community over the last two decades is that of the relationship between I^r and $I^{(m)}$. How do these two notions compare?

It can be shown that $I^r \subseteq I^{(m)}$ if and only if $r \geq m$, so the question that remains is what we can say about the containment of $I^{(m)}$ in I^r.

Results by Ein/Lazarsfeld/Smith and Hochster/Huneke showed that $I^{(m)} \subseteq I^r$ whenever $m \geq Nr$. Moreover, it can be shown that $I^{(m)} \nsubseteq I^r$ whenever $r > m$, and, in addition, improvements on the bound $m \geq Nr$ have been made in many specific situations. However, the question of exactly when $I^{(m)} \subseteq I^r$ holds for $r \leq m < Nr$ remains open in general.

An asymptotic variant of this Containment Question is the Resurgence Problem posed by Bocci and Harbourne: Define the resurgence of a homogeneous ideal I by $\rho(I) = \sup_{m,r} \left\{ \frac{m}{r} \;\middle|\; I^{(m)} \nsubseteq I^r \right\}$. What can we say about its value in general or for certain classes of ideals?

Most work in this direction has been done in the geometric setting of ideals of points in $\mathbb{P}^N$.

In this thesis we will address both the Containment Question and the Resurgence Problem for a family of ideals of points in $\mathbb{P}^2$. We start in chapter 2 with a point configuration for which we can give a complete answer to both questions. The key to comparing the symbolic and ordinary powers of I in these cases is to find a vector space basis for the homogeneous coordinate ring of the plane such that subsets of this basis give bases for all powers and symbolic powers of I. In chapter 3, we describe a way to obtain a lower bound for $\rho(I)$ for point configurations for which the vector space basis approach does not apply. We connect the configuration of points under consideration in this chapter to the one we examine in the preceding chapter to demonstrate how to obtain partial results through the vector space approach. In chapter 4, we develop computational methods for estimating resurgences arbitrarily accurately for nontrivial homogeneous ideals in $K[\mathbb{P}^N]$ whenever there is an $m \in \mathbb{N}$ such that powers of $I^{(m)}$ are symbolic, i.e. $I^{(mt)} = \left(I^{(m)}\right)^t$ for all $t \in \mathbb{N}$. Our main result here is Theorem 4.1.5, which gives for such ideals I a computational method for determining $\rho(I)$ to any desired accuracy. We also demonstrate the application in several examples.

ACKNOWLEDGMENTS

I owe a great deal to my adviser, Brian Harbourne, for his incredible patience and indispensable guidance. Thank you, Brian.

To my math brother Mike Janssen, thank you for your friendship and all your hard work.

I also thank my committee, Tom Marley, Srikanth Iyengar, Mikil Foss, and Matthew Cushing, for their helpful comments and encouragement, and the rest of the math faculty, staff, and fellow graduate students for their help, wisdom, and companionship. I learned a lot from you - in math, teaching, and life.

Finally, I couldn't have done it without my family and friends. You know who you are. Thanks and Danke.

Contents

Chapter 1

Introduction

Let R be a polynomial ring over a field K and let R contain a homogeneous ideal I. Then we can define the m^{th} symbolic power $I^{(m)}$ of I as

$$I^{(m)} = R \cap \left(\bigcap_P I^m R_P \right),$$

where all intersections take place in the fraction field of R, and the second intersection is taken over all associated primes P of I. This ideal has garnered quite a bit of interest in recent years. It is immediate from the definition that $I^m \subseteq I^{(m)}$ for all $m \in \mathbb{N}$ and any such ideal I. With his Unmixedness Theorem in [12], Macaulay showed that $I^m = I^{(m)}$ if I is a complete intersection, but what more can we say? One basic question in particular stands at the heart of this:

The Containment Question Given a homogeneous ideal $(0) \neq I \subsetneq R$, for which positive integers m and r is $I^{(m)} \subseteq I^r$ or $I^r \subseteq I^{(m)}$?

The second part of this question has an easy answer: $I^r \subseteq I^{(m)}$ if and only if $r \geq m$. (See, for example, [13].)

This fact also shows that for any investigation into the first part of the Containment Question, we need only consider m and r such that $m \geq r$, because $r > m$ implies that $I^r \subsetneq I^m \subseteq I^{(m)}$ and hence $I^{(m)} \subseteq I^r$ is impossible.

Moreover, a consequence of results obtained in [7] and [11] is that for homogeneous ideals I in $K[\mathbb{P}^N]$, we have $I^{(m)} \subseteq I^r$ if $m \geq Nr$. Attempts have been made to improve that bound, for example in [13], where Harbourne conjectured that for such I, the containment $I^{(m)} \subseteq I^r$ holds if $m \geq Nr - (N-1)$. In [10], Harbourne and Huneke conjectured that similar conditions apply to containment of a symbolic power in an ordinary power times a power of the unique homogeneous maximal ideal, also called the irrelevant ideal, of $K[\mathbb{P}^N]$. Specifically, they suggested that $I^{(m)} \subseteq \mathcal{M}^{r(N-1)}I^r$ when $m \geq Nr$ and $I^{(m)} \subseteq \mathcal{M}^{(r-1)(N-1)}I^r$ when $m \geq Nr - (N-1)$, where $I \subseteq K[\mathbb{P}^N]$ is a fat point ideal and $\mathcal{M}$ is the irrelevant ideal in $K[\mathbb{P}^N]$. The authors verified their conjectures for some special point configurations, among them general points and star configurations.

Harbourne's conjecture $I^{(Nr-(N-1))} \subseteq I^r$ is a generalization of a question posed by Huneke in 2006. Huneke asked if $I^{(3)} \subseteq I^2$ for any ideal I of finitely many points in projective 2-space. This conjecture has also been verified in many cases, among them star configurations and general point configurations, but earlier this year, Dumnicki, Szemberg and Tutaj-Gasinska came across a deceptively simple configuration involving only 12 points for which this conjecture does not hold. (See [5] for the counterexample.) Hence the answer to Huneke's question as it stands is negative, but given the rarity of counterexamples, it is of interest to understand better for which ideals the conjecture holds. We will see that it holds in the cases studied in this thesis.

Instead of calculating exactly for which combinations of m and r we have $I^{(m)} \subseteq I^r$, one can also approach the problem asymptotically. In [2], Bocci and Harbourne defined an asymptotic invariant for a homogeneous ideal I in $K[\mathbb{P}^N]$, known as the resurgence

$$\rho(I) = \sup \left\{ m/r \mid I^{(m)} \nsubseteq I^r \right\}.$$

This gives rise to the asymptotic version of the Containment Question.

The Resurgence Problem Given a homogeneous ideal $(0) \neq I \subsetneq R$, what is $\rho(I)$?

The results stated above prove that $1 \leq \rho(I) \leq N$, and additional results in [10] sharpen those bounds, for some choices of I considerably so. But in general, computing $\rho(I)$ remains an open problem. Given an arbitrary $\varepsilon > 0$, there is, for example, no general algorithm for computing $\rho(I)$ to within ε. We address this problem and give a partial solution in chapter 4.

While the Containment Question is of interest for homogeneous ideals in general, most previous work has been done in a more geometric setting, namely in the context of ideals of points in projective space.

Let K be a field of arbitrary characteristic and let R be the homogeneous coordinate ring $R = K[\mathbb{P}^N] = K[x_0, x_1, \ldots, x_N]$. If $p_1, \ldots, p_s \in \mathbb{P}^N$ are distinct points, then $Z = p_1 + \cdots + p_s \subseteq \mathbb{P}^2$ is the zero-dimensional subscheme defined by the ideal $I = \mathcal{I}(Z)$ generated by all forms vanishing on the points p_i. We can write I as

$$I = \bigcap_i \mathcal{I}(p_i),$$

where $\mathcal{I}(p_i)$ is the ideal generated by all forms vanishing at p_i. Then the symbolic powers of I take the form

$$I^{(m)} = \bigcap_i \mathcal{I}(p_i)^m$$

and the ordinary powers of I the form

$$I^r = \left(\bigcap_i \mathcal{I}(p_i)\right)^r.$$

Therefore, comparing symbolic and ordinary powers amounts in essence to comparing intersections of powers and powers of intersections.

One of the reasons many mathematicians start with such a relatively specific ideal is that ideals of points are somewhat easier to understand than general ideals, which is helpful given how little is known about the Containment Question for general ideals. Also, ideals of points have provided a plethora of examples and partial results so far; For example, Bocci and Harbourne found in [2] that for such an ideal, the resurgence is bounded by

$$\frac{\alpha(I)}{\gamma(I)} \leq \rho(I) \leq \frac{\mathrm{reg}(I)}{\gamma(I)},$$

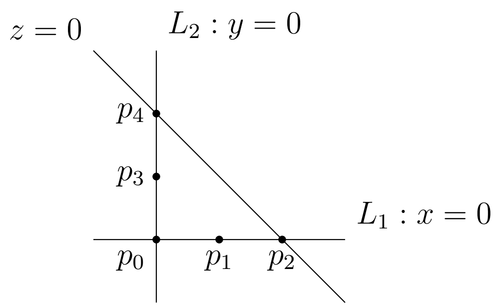
where $\alpha(I)$ denotes the least degree $t > 0$ such that $I_t \neq 0$ and $\gamma(I)$ is the Waldschmidt constant $\gamma(I) = \lim_{m\to\infty} \frac{\alpha(I^{(m)})}{m}$.

Given two integers m and r and an ideal I, two straightforward methods for showing $I^{(m)} \not\subseteq I^r$ and thus obtaining lower bounds on $\rho(I)$ involve comparing fixed components or Hilbert functions. However, as the following example will show, the lower bounds obtained by these methods can fail to be sharp.

Example 1.0.1. Let I be the ideal defined by the vanishing on the points $p_0, \ldots, p_4 \in \mathbb{P}^2$, where $p_0 = [0,0,1]$, $p_1 = [0,1,1]$, $p_2 = [0,1,0]$, $p_3 = [1,0,1]$, and $p_4 = [1,0,0]$.

Then we can write I and its m^{th} symbolic power as $I = (xy, xF, yF)$ and $I^{(m)} = (x,y)^m \cap (xy, F)^m$, where $F = xz + yz - z^2$ (see Proposition 2.1.1). Here, the reducible conic is defined by the lines $L_1 : x = 0$ and $L_2 : y = 0$.

Let $m = 6$ and $r = 5$. Say we want to know whether $I^{(6)} \subseteq I^5$ or not. Using a program (see Appendix 2 for the code) or Lemma 2.2.8, we find that $I^{(6)} \not\subseteq I^5$ because $x^3y^3F^3$ is in $I^{(6)}$ but not



in I^5. But what information can we obtain from comparing fixed components or Hilbert functions?

Consider the fixed components of $\left(I^{(m)}\right)_t$ and $(I^r)_t$, or equivalently, the $\gcd$'s (i.e. a common factor of greatest degree) of the homogeneous components of $I^{(m)}$ and I^r in some degree $t \geq \max\left(\alpha\left(I^{(m)}\right), \alpha\left(I^r\right)\right)$. Note that if I^r contains $I^{(m)}$, then for every t, the $\gcd$ of $(I^r)_t$ divides the $\gcd$ of $\left(I^{(m)}\right)_t$, and thus if for some t the $\gcd$ of $(I^r)_t$ does not divide the $\gcd$ of $\left(I^{(m)}\right)_t$, then I^r cannot contain $I^{(m)}$.

We note that $\alpha\left(I^{(m)}\right) = 2m$ and $\alpha\left(I^r\right) = 2r$ (see Lemma 2.7.1, for example), and that by symmetry L_1 and L_2 occur with the same multiplicities. In the case of $m = 6$ and $r = 5$, we have $\max\left(\alpha\left(I^{(6)}\right), \alpha\left(I^5\right)\right) = 12$ and we obtain the following data.

degree	12	13	14	15	16	17	18
multiplicity of L_1, L_2 in $I^{(6)}$	3	3	2	2	1	1	0
multiplicity of L_1, L_2 in I^5	3	2	1	0	0	0	0

Here, the term multiplicity has the traditional meaning, namely it denotes the exponent of the factor defining L_1 and L_2 in the $\gcd$.

We see in every degree that the $\gcd$'s of I^5 divide those of $I^{(6)}$, and hence that this data is not enough for us to verify that $I^{(6)} \nsubseteq I^5$.

However, we can determine a lower bound for the resurgence of I using fixed components. It can be shown that each fixed component of $I^{(m)}$ in degree $2m \leq k \leq 3m$ has multiplicity $a = \left\lceil \frac{3m-k}{2} \right\rceil$, and each fixed component of I^r in degree $2r \leq k \leq 3r$ has multiplicity $b = 3r - k$. Now, taking $m = 6t + 8$, $r = 5t + 7$, and $k = 2m = 12t + 16$ gives $a = \left\lceil \frac{3m-2m}{2} \right\rceil = 3t + 4$ and $b = 3(5t+7) - (12t+16) = 3t+5$. But then $a < b$, which means that in degree $2m$, the multiplicity of the fixed components of $I^{(m)}$ is strictly less than the multiplicity of the fixed components of I^r. Hence $I^{(m)}$ cannot be contained in I^r for these choices of m and r. Thus $\rho(I) \geq \frac{6t+8}{5t+7}$ for all $t \in \mathbb{N}$, which gives $\rho(I) \geq \frac{6}{5}$.

This is the best lower bound one can obtain from comparing fixed components. Using the methods we present in this thesis, we show in Theorem 2.3.1 that in fact $\rho(I) = \frac{4}{3}$.

Alternatively, Hilbert functions also give a criterion for failure of containment, since if the dimension $H(I^r, t)$ of $(I^r)_t$ is less than the dimension $H(I^{(m)}, t)$ of $\left(I^{(m)}\right)_t$ for some t, then I^r cannot contain $I^{(m)}$.

Note that we only need to consider the Hilbert functions up to degree $\operatorname{reg}(I^r)-1$, as $\dim\,(I^r)_t = \dim\left(I^{(r)}\right)_t$ for $t \geq \operatorname{reg}(I^r)$. Since $I^{(r)} \supseteq I^{(m)}$ for $m \geq r$, we have $\dim\,(I^r)_t = \dim\left(I^{(r)}\right)_t \geq \dim\left(I^{(m)}\right)_t$ for $t \geq \operatorname{reg}(I^r)$.

We show in Theorem 2.9.2 and Corollary 2.9.3 that in the case $m = 6$ and $r = 5$, we get $\operatorname{reg}(I^5) = 15$, so we only need to consider the Hilbert functions up to degree 14. We get the following table. (See Appendix 1 for the code.)

degree t	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
$H(I^{(6)}, t)$	0	0	0	0	0	0	0	0	0	0	0	0	4	12	23
$H(I^5, t)$	0	0	0	0	0	0	0	0	0	0	1	5	13	25	41

Then $H(I^{(6)}, t) \leq H(I^5, t)$ for all $t \in \mathbb{N}$, so again, it is possible to have $I^{(6)}$ contained in I^5. Moreover, it can be very difficult, if not impossible, to calculate the Hilbert functions of large powers or symbolic powers of an ideal of points.

The above two analyses show that approaching the Containment Questions geometrically from the directions of fixed components or Hilbert functions can fail to yield definitive answers even for simple point configurations. In this paper, we exhibit an algebraic approach to the Containment Question and the Resurgence Problem that proves to be more effective for some configurations of points.

We will examine a few special configurations of points in projective 2-space in light of the Containment Question and its asymptotic variant, the Resurgence Problem. We also take a closer look at the conjectures made in [1] and [10] pertaining to symbolic powers, and verify them for one of the point configurations in this paper. Where possible, we calculate the exact resurgence of the ideals defining these configurations, and give upper or lower bounds where an exact answer is more difficult to compute. In the last chapter, we examine a general algorithm for computing the resurgence of homogeneous ideals in $K[\mathbb{P}^N]$.

Perhaps one of the most basic, and therefore interesting, methods we employ in this paper follows the model of monomial ideals. We give a description of a rather simple K-vector space basis for $K[\mathbb{P}^2]$ that allows us to write all symbolic and ordinary powers of our ideal I as spans of subsets of this basis. Many of these results appear in [4], where this vector space approach is applied to obtain results for two different point configurations, one of which we analyze in depth in this paper. We expand the vector space descriptions to verify the conjectures mentioned above, and several more. Later on, we highlight some of the shortcomings of the vector space approach and present alternatives for computing upper and lower bounds for the resurgence.

Comment 1.0.2. Throughout this paper, we assume that K is a field of arbitrary characteristic. Most of our ideas don't necessarily need K to be algebraically closed, and oftentimes it may be enough to demand that K be sufficiently large. However, for simplicity we will just assume that K is algebraically closed.

In addition, we fix the following notation for the entire paper: Let R be the polynomial ring in the indeterminates x, y, and z, so $R = K[x, y, z] = K[\mathbb{P}^2]$, and let $n \in \mathbb{N}$.

We will not make changes to K or R, and if we make additional assumptions about n, we will explicitly state them.

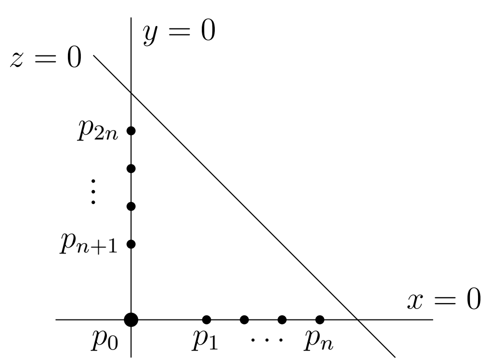
Chapter 2

Ideals of Nearly-Complete Intersections

2.1 Preliminaries

Throughout this chapter, I will always denote the ideal of the following configuration of points.

Suppose we have $2n+1$ points $p_i \in \mathbb{P}^2$ with n points on $L_1 : x = 0$, say $p_1, \ldots, p_n$, and n points on $L_2 : y = 0$, say $p_{n+1}, \ldots, p_{2n}$, all of multiplicity 1. We assume that there is one additional point p_0 of multiplicity 1 at the intersection of L_1 and L_2.



These $2n+1$ points define a 0-dimensional subscheme $Z = p_0 + \ldots + p_{2n}$ of $\mathbb{P}^2$ and an ideal of points $I = \mathcal{I}(Z)$. The ideal vanishing on Z is

$$I = (x,y) \cap \left(\bigcap_{i=1}^{n} (x, z - \alpha_i y) \cap \bigcap_{i=1}^{n} (y, z - \beta_i x) \right),$$

where $\alpha_i, \beta_i \in K$, $\mathcal{I}(p_i) = (x, z - \alpha_i y)$, and $\mathcal{I}(p_{n+i}) = (y, z - \beta_i x)$ for $i = 1, \ldots, n$.

Then for any $m \in \mathbb{N}$, the m^{th} symbolic power of I is

$$I^{(m)} = (x,y)^m \cap \bigcap_{i=1}^{2n} \mathcal{I}(p_i)^m.$$

The m^{th} symbolic power of I also describes the ideal $\mathcal{I}(mZ)$, i.e. the ideal of the fat point subscheme obtained by assigning each of the points p_i the multiplicity m.

We can define a polynomial $F \in R$ by

$$F = z^n - \prod_{i=1}^{n} (z - \beta_i x) - \prod_{i=1}^{n} (z - \alpha_i y)$$

and express I in terms of this polynomial.

Proposition 2.1.1. *Let I and F be as defined above. Then*

$$I = (x,y) \cap (xy, F) = (xy, xF, yF)$$

and for any $m \in \mathbb{N}$,

$$I^{(m)} = (x,y)^m \cap (xy, F)^m.$$

Proof. Let $I' := \bigcap_{i=1}^{n} (x, z - \alpha_i y) \cap \bigcap_{i=1}^{n} (y, z - \beta_i x)$ and consider the two curves $C_1 : xy = 0$ and $C_2 : F = 0$. Then C_1 and C_2 intersect exactly at the $2n$ points $p_1, \ldots, p_{2n}$, and transversely at that. Therefore I' is generated by the forms defining C_1 and C_2, hence $I' = (xy, F)$ and therefore $I = (x,y) \cap (xy, F)$.

It is easy to see that $(x,y) \cap (xy, F) \supseteq (xy, xF, yF)$.

To see the reverse containment, suppose that $g \in (x, y) \cap (xy, F)$, say $g = k_1(xy) + k_2F$ with $k_1, k_2 \in R$. Since $g \in (x, y)$, we have

$$\begin{aligned} 0 &= g([0,0,1]) \\ &= (k_1(xy) + k_2F)([0,0,1]) \\ &= 0 + k_2([0,0,1]) \cdot F([0,0,1]) \\ &= k_2([0,0,1]) \cdot (-1) \\ &= -k_2([0,0,1]) \end{aligned}$$

by definition of F. Thus $k_2([0,0,1]) = 0$ and hence $k_2 \in (x, y)$.

But then $g = k_1(xy) + k_2F \in (xy, (x, y)F) = (xy, xF, yF)$ and hence $(x, y) \cap (xy, F) \subseteq (xy, xF, yF)$. Therefore $I = (x, y) \cap (xy, F) = (xy, xF, yF)$ as desired.

Since the ideals (x, y) and (xy, F) are each generated by a regular sequence, they are complete intersections. Therefore, by [15], symbolic and ordinary powers of the ideals coincide. It follows that $I^{(m)} = (x, y)^{(m)} \cap (xy, F)^{(m)} = (x, y)^m \cap (xy, F)^m$. □

Example 2.1.2. The simplest, but for us also least interesting example is when $n = 1$. Then the ideal can be written as

$$I = (x, y) \cap (x, z) \cap (y, z) = (xy, xz, yz),$$

a monomial ideal which is also an ideal of points. This ideal has been thoroughly studied and we will only mention it here for the sake of completeness. We shall see, however, that the results in the case $n = 1$ often coincide with the results in the case $n \geq 2$. Most ideas and proofs will only assume $n \geq 1$, and hence we will include previously known results throughout this chapter. When a result for $n \geq 2$ differs significantly from the corresponding result for $n = 1$, we will usually include both cases.

The standard example is the case $n = 2$. Here, possibly after a change of coordinates, the ideal can be taken to be

$$I = (x, y) \cap (x, z) \cap (x, z - y) \cap (y, z) \cap (y, z - x).$$

With $F = z^2 - z(z-x) - z(z-y) = xz + yz - z^2$, we can write

$$I = (x, y) \cap (xy, F) = (xy, xF, yF) \text{ and } I^{(m)} = (x, y)^m \cap (xy, F)^m.$$

For example,

$$I^2 = (xy, xF, yF)^2 = (x^2y^2, x^2F^2, y^2F^2, x^2yF, xy^2F, xyF^2)$$

whereas

$$I^{(2)} = (x, y)^2 \cap (xy, F)^2 = (x^2y^2, x^2F^2, y^2F^2, xyF)$$

(see Appendix 3 for the code).

2.2 Descriptions of I, I^r, and $I^{(m)}$

We will use vector space theory to describe I and its powers, symbolic as well as ordinary. Throughout, we will use angle brackets $\langle , \rangle$ to denote a K-vector space span, and parentheses $(,)$ to denote ideal generation in R.

In this section we show that barring a few restrictions, we can treat polynomials of the form $x^a y^b z^c F^d$ as monomials and use them as a basis for our polynomial ring. This will allow us to obtain a lot of results with basic vector space theory.

A similar approach is used for the almost collinear point configuration in [4], where almost collinear point configurations on $n+1$ points were defined to have n points on one line and one point off the line. Though the method is quite similar, the basis for the vector space as well as the general results differ. One main difference is that most of our results here are independent of n. For almost collinear points, however, most results are highly dependent on the number of points.

Proposition 2.2.1. *Let $t \in \mathbb{N}_0$. Then every monomial of the form $x^i y^j z^k$ with $k \leq t$ and $i, j, k \in \mathbb{N}_0$ is contained in the K-span $\mathcal{S}_t$, where*

$$\mathcal{S}_t = \langle x^a y^b z^c F^d | a, b, c, d \in \mathbb{N}_0, c < n, c + dn \leq t \rangle.$$

Proof. Note that $\mathcal{S}_0 \subseteq \mathcal{S}_1 \subseteq \ldots$ and use induction on t. For $t < n$, the condition $c + dn \leq t$ implies that $d = 0$, so $\mathcal{S}_t = \langle \{ x^a y^b z^c \mid a, b, c \in \mathbb{N}_0, c < n \} \rangle$. In particular, $x^a y^b z^c$ with $c \leq t < n$ is in $\mathcal{S}_t$.

For $t \geq n$, assume that $x^a y^b z^c \in \mathcal{S}_t$ for every $a, b \in \mathbb{N}_0$ and $c < t$. Take $a, b \in \mathbb{N}_0$ and consider the polynomial $G = x^a y^b z^t - x^a y^b z^r (-F)^q$ with $t = qn + r$ and $0 \leq r < n$. Then by definition of F, G is a polynomial of degree less than t in z with coefficients in $K[x, y]$, so $G \in \mathcal{S}_{t-1}$ by induction. But $r < n$, so $x^a y^b z^r (-F)^q = (-1)^q x^a y^b z^r F^q \in \mathcal{S}_t$ and hence $x^a y^b z^t = G + x^a y^b z^r (-F)^q \in \mathcal{S}_t$. $\square$

Lemma 2.2.2. *The set*

$$\mathcal{A} = \left\{ x^a y^b z^c F^d \;\middle|\; a, b, c, d \in \mathbb{N}_0, c < n \right\}$$

is linearly independent over K and spans R as a K-vector space.

Proof. By Proposition 2.2.1, each monomial $x^a y^b z^c$ is in $\langle \mathcal{A} \rangle$, and since $R = \langle \left\{ x^a y^b z^c \;\middle|\; a, b, c \in \mathbb{N}_0 \right\} \rangle$, we see that $R = \langle \mathcal{A} \rangle$.

For each $s \in \mathbb{N}_0$, define a subset $\mathcal{A}_s$ of $\mathcal{A}$ by

$$\mathcal{A}_s := \left\{ x^a y^b z^c F^d \;\middle|\; a, b, c, d \in \mathbb{N}_0, c < n, a + b + c + dn = s \right\}.$$

Then $\mathcal{A}$ is the disjoint union $\mathcal{A} = \bigsqcup_{s=0}^{\infty} \mathcal{A}_s$ and the elements of $\mathcal{A}_s$ are homogeneous of degree s, and therefore $\langle \mathcal{A}_s \rangle = R_s$. By homogeneity of $\mathcal{A}_s$, all elements in $\mathcal{A}_s$ are linearly independent from elements in $\mathcal{A} \setminus \mathcal{A}_s$, and $|\mathcal{A}_s| = \binom{s+(3-1)}{3-1} = \dim R_s$ (the number of partitions of s into three parts in the non-negative integers), which means that $\mathcal{A}_s$ not only spans R_s but also has the same size as a (linearly independent) monomial basis for R_s. Therefore, $\mathcal{A}_s$ is linearly independent as well, and so is $\mathcal{A}$. $\square$

The following lemma will be needed to describe $I^{(m)}$ and I^r.

Lemma 2.2.3. *Let V be a K-vector space with basis $\mathcal{B}$. If $U, W \leq V$ are subspaces with bases $\mathcal{B}_U := \mathcal{B} \cap U$ and $\mathcal{B}_W := \mathcal{B} \cap W$, respectively, then $\mathcal{B}_{U \cap W} := \mathcal{B}_U \cap \mathcal{B}_W$ is a basis for $U \cap W$.*

Proof. Since $\mathcal{B}_{U\cap W} \subseteq \mathcal{B}$, it is linearly independent over K and we only have to show $\langle \mathcal{B}_{U\cap W} \rangle = U \cap W$.

Let $u \in (U \cap W) \setminus \{0\}$. Then there exist unique basis vectors $\{v_1, \ldots, v_s\} \subseteq \mathcal{B}_U$ and $\{v'_1, \ldots, v'_t\} \subseteq \mathcal{B}_W$ such that

$$u = \sum_{i=1}^{s} \alpha_i v_i = \sum_{j=1}^{t} \beta_j v'_j \qquad (*)$$

for some $\alpha_i, \beta_j \in K^*$.

If $\{v_1, \ldots, v_s\} = \{v'_1, \ldots, v'_t\}$, then we're done. So assume that $\{v_1, \ldots, v_s\} \neq \{v'_1, \ldots, v'_t\}$, say $v_1 \notin \{v'_1, \dots, v'_t\}$. By $(*)$, we in particular have

$$0 = \alpha_1 v_1 + \sum_{i=2}^{s} \alpha_i v_i + \sum_{j=1}^{t} (-\beta_j) v'_j.$$

But $v_1 \notin \{v'_1, \dots, v'_t\} \cup \{v_2, \dots, v_s\}$ and the v_i and v'_j are all elements in the linearly independent set $\mathcal{B}$, which means that necessarily $\alpha_1 = 0$. This contradicts our choice of the α_i.

Therefore $\{v_1, \dots, v_s\} = \{v'_1, \dots, v'_t\}$ and since u was chosen arbitrarily, we have $\langle \mathcal{B}_{U \cap W} \rangle = U \cap W$ as desired. □

The fact above allows us to describe the symbolic powers of our ideal I as vector spaces.

Proposition 2.2.4. *Let $m \in \mathbb{N}$. Then*

1. *The set $\mathcal{B} = \left\{ x^a y^b z^c F^d \;\middle|\; a, b, c, d \in \mathbb{N}_0, c < n, a + b \geq m \right\}$ is linearly independent over K and spans $(x, y)^m$ as a K-vector space.*
2. *The set $\mathcal{C} = \left\{ x^a y^b z^c F^d \;\middle|\; a, b, c, d \in \mathbb{N}_0, c < n, \min(a, b) + d \geq m \right\}$ is linearly independent over K and spans $(xy, F)^m$ as a K-vector space.*

Therefore,

$$I^{(m)} = \langle x^a y^b z^c F^d | a, b, c, d \in \mathbb{N}_0, c < n, a + b \geq m, \min(a, b) + d \geq m \rangle.$$

Proof. Let $\mathcal{A}$ be as in Lemma 2.2.2. Since $\mathcal{B}, \mathcal{C} \subseteq \mathcal{A}$, linear independence of $\mathcal{B}$ and $\mathcal{C}$ over K is immediate.

By definition and Lemma 2.2.2,

$$\begin{aligned}
(x, y)^m &= (x^a y^b | a, b \in \mathbb{N}_0, a + b = m) \\
&= \sum_{a+b=m} x^a y^b R \\
&\subseteq \langle x^a y^b z^c F^d | a, b, c, d \in \mathbb{N}_0, c < n, a + b \geq m \rangle \\
&\subseteq (x, y)^m
\end{aligned}$$

and likewise

$$
\begin{aligned}
(xy, F)^m &= (x^a y^a F^d | a, d \in \mathbb{N}_0, a + d = m) \\
&= \langle x^a y^b z^c F^d | a, b, c, d \in \mathbb{N}_0, c < n, \min(a, b) + d \geq m \rangle.
\end{aligned}
$$

Hence $I^{(m)} = (x, y)^m \cap (xy, F)^m = \langle \mathcal{B} \rangle \cap \langle \mathcal{C} \rangle$, and Lemma 2.2.3 then gives $I^{(m)} = \langle \mathcal{B} \cap \mathcal{C} \rangle = \langle x^a y^b z^c F^d | a, b, c, d \in \mathbb{N}_0, c < n, a + b \geq m, \min(a, b) + d \geq m \rangle$. $\square$

We will describe I^r in a similar fashion.

Lemma 2.2.5. *Let $r \in \mathbb{N}$. Define one set, $\mathcal{S}_r$, by*

$$\mathcal{S}_r = \left\{ (a, b, c, d) \in \mathbb{N}_0^4 \;\middle|\; c < n, \min(a, b) + d \geq r, a + b + d \geq 2r, \textit{ and } a + b \geq r \right\},$$

and another set, $\mathcal{T}_r$, by

$$\mathcal{T}_r = \{(a, b, c, d) \in \mathbb{N}_0^4 | c < n, (\min(a, b) + d \geq r \textit{ if } d \leq \max(a, b) - \min(a, b)),$$

$$(a + b + d \geq 2r \textit{ if } \max(a, b) - \min(a, b) < d < b + a), \textit{ and } (a + b \geq r \textit{ if } a + b \leq d)\}.$$

Then $\mathcal{S}_r = \mathcal{T}_r$.

Proof. If $(a, b, c, d) \in \mathcal{S}_r$, then it is also always in $\mathcal{T}_r$. Now suppose $(a, b, c, d) \in \mathcal{T}_r$. We will consider three different cases, as given in the definition of $\mathcal{T}_r$, to show that then also $(a, b, c, d) \in \mathcal{S}_r$. We may assume, without loss of generality, that $a \leq b$, so $\min(a, b) = a$ and $\max(a, b) = b$.

(a) If $d \leq b - a$ and consequently $a + d = \min(a, b) + d \geq r$, then $b \geq d + a \geq r$ and hence $a + b \geq r$ and $(a + d) + b \geq 2r$. Therefore, $(a, b, c, d) \in \mathcal{S}_r$.

(b) If $b - a < d < b + a$ and consequently $a + b + d \geq 2r$, then $2(a + b) > (a + b) + d \geq 2r$, so $a + b \geq r$. If $b > r$, then $\min(a, b) + d = a + d > (b - a) + a = b > r$, and if $b \leq r$, then $r - b \geq 0$ and so $\min(a, b) + d = a + d \geq 2r - b = r + (r - b) \geq r$. Therefore, $(a, b, c, d) \in \mathcal{S}_r$.

(c) Finally, if $a + b \leq d$ and consequently $a + b \geq r$, then $d \geq a + b \geq r$ and hence $\min(a, b) + d \geq d \geq r$ and $(a + b) + d \geq 2r$. Therefore, $(a, b, c, d) \in \mathcal{S}_r$. $\square$

Lemma 2.2.6. *Let $r \in \mathbb{N}$ and $\mathcal{S}_r$ and $\mathcal{T}_r$ be as above. Then*

$$J := \langle x^a y^b z^c F^d | (a,b,c,d) \in \mathcal{S}_r \rangle = \langle x^a y^b z^c F^d | (a,b,c,d) \in \mathcal{T}_r \rangle$$

is an ideal.

Proof. By Lemma 2.2.5, the second equality is immediate.

Note that it is enough to check that $(x^a y^b z^c F^d)(x^{a'} y^{b'} z^{c'} F^{d'}) \in J$ whenever $(a,b,c,d) \in \mathcal{S}_r$ and $a', b', c', d' \geq 0$. The definition of $\mathcal{S}_r$ immediately gives that $(a,b,c,d) \in \mathcal{S}_r$ implies $(a+a', b+b', c, d+d') \in \mathcal{S}_r$ for all $a', b', d' \in \mathbb{N}_0$, because $(a+a') + (b+b') \geq a+b \geq r$, $\min(a+a', b+b') + (d+d') \geq \min(a,b) + d \geq r$, and $(a+a') + (b+b') + (d+d') \geq a+b+d \geq 2r$. So now we only need to show that $x^a y^b z^{c+c'} F^d \in J$ for all $(a,b,c,d) \in \mathcal{S}_r$ and all $c' \in \mathbb{N}_0$.

To see this, induct on c'. Take $(a,b,c,d) \in \mathcal{S}_r$ and let $h = x^a y^b z^{c+c'} F^d$. If $c' < n - c$, then $(a,b,c+c',d) \in \mathcal{S}_r$ and hence $h \in J$. If $c' \geq n - c$, then $c' + c \geq n$, say $c' + c = qn + p$ with $q \geq 1$ and $p < n$. Assume that $x^a y^b z^{c+t} F^d \in J$ for all $(a,b,c,d) \in \mathcal{S}_r$ and $t \leq c' - 1$. By definition of F, $L := F + z^n$, considered as a polynomial in z with coefficients in $K[x,y]$, has degree at most $n-1$. Thus $h = x^a y^b z^p F^d (z^n)^q = x^a y^b z^p F^d (L - F)^q = x^a y^b z^p F^d \sum_i \eta_i L^{q-i} F^i$ where L^{q-i} is a sum of terms $x^{a_{i,j}} y^{b_{i,j}} z^{c_{i,j}}$ with each $c_{i,j} \leq (q-i)(n-1)$. Therefore, we can write $h = \sum_{i,j} \lambda_{i,j} x^{a+a_{i,j}} y^{b+b_{i,j}} z^{p+c_{i,j}} F^{d+i}$ for some $\lambda_{i,j} \in K$. But $p + c_{i,j} \leq p + (q-i)(n-1) < p + qn = c + c'$ for all i,j by choice of L, and by our induction assumption, we have $x^{a+a_{i,j}} y^{b+b_{i,j}} z^{p+c_{i,j}} F^{d+i} \in J$ for all i,j.

Hence $h \in J$ and J is an ideal. □

Proposition 2.2.7. *Let $r \in \mathbb{N}$ and $\mathcal{S}_r$ and $\mathcal{T}_r$ be as above. Then*

$$I^r = \langle x^a y^b z^c F^d | (a,b,c,d) \in \mathcal{S}_r \rangle = \langle x^a y^b z^c F^d | (a,b,c,d) \in \mathcal{T}_r \rangle.$$

Proof. By Lemma 2.2.5, the second equality is immediate.

Define $J := \langle x^a y^b z^c F^d | (a,b,c,d) \in \mathcal{S}_r \rangle = \langle x^a y^b z^c F^d | (a,b,c,d) \in \mathcal{T}_r \rangle$ as above.

Since $I = (xy, xF, yF)$, we know I^r is generated by

$$G = \left\{ (xy)^s (xF)^t (yF)^u \;\middle|\; s,t,u \in \mathbb{N}_0, s+t+u = r \right\}.$$

Thus, as J is an ideal, $G \subseteq J$ implies $I^r \subseteq J$.

So take a generator $g = (xy)^s(xF)^t(yF)^u \in I^r$, i.e. pick $s, t, u \in \mathbb{N}_0$ such that $s + t + u = r$, and show that $g \in J$. Then $u = r - s - t$ and $g = (xy)^s(xF)^t(yF)^{r-s-t} = x^{s+t}y^{r-t}F^{r-s}$. But $(s + t) + (r - t) = r + s \geq r$, and $(s + t) + (r - s) = r + t \geq r$ and $(r - t) + (r - s) = r+(r-s-t) = r+u \geq r$, so $\min(s+t, r-t)+(r-s) \geq r$, and $(s+t)+(r-t)+(r-s) = 2r$, so $(a = s + t, b = r - t, c = 0, d = r - s) \in \mathcal{S}_r$ and hence $g \in J$ as desired.

For the reverse containment, we take a basis element $g = x^a y^b z^c F^d \in J$, where $(a, b, c, d) \in \mathcal{T}_r$, and show that $g \in I^r$. Since $g \in J$ if and only if $x^a y^b F^d \in J$, and $x^a y^b F^d \in I^r$ implies $g \in I^r$, we may take c to be 0. Without loss of generality, we may also assume that $a \leq b$, so $a = \min(a, b)$ and $b = \max(a, b)$. We will consider three different cases, as given in the definition of $\mathcal{T}_r$.

(a) If $d \leq b - a$ and consequently $a + d \geq r$, then $b - d - a \geq 0$ and we can write $g = (xy)^a(yF)^d \cdot y^{b-(d+a)} \in I^r$ since $a + d \geq r$ and $b - a - d \geq 0$.

(b) Now assume $b - a < d < b + a$. Then either (b.i) $b > r$ or (b.ii) $b \leq r$.

(b.i) If $b > r$, then we can write $g = (xy)^a(yF)^{b-a} \cdot F^{d+a-b} \in I^r$ since $a + (b - a) = b > r$.

(b.ii) If $b \leq r$, then we can write $g = (xy)^{r-d}(xF)^{r-b}(yF)^{(b+d)-r} \cdot x^{a+b+d-2r} \in I^r$ for $d \leq r$ and $g = (xF)^a(yF)^{d-a} \cdot y^{b+a-d} \in I^r$ for $d > r$ since $a \leq b \leq r < d < a + b$.

(c) Finally, assume $a + b \leq d$ and consequently $a + b \geq r$. Then $d - a - b \geq 0$ and we can write $g = (xF)^a(yF)^b \cdot F^{d-a-b} \in I^r$.

Therefore, we always have $g \in I^r$, and hence $J \subseteq I^r$. $\square$

Combining Propositions 2.2.4 and 2.2.7 gives the following criterion for containment of $I^{(m)}$ in I^r.

Lemma 2.2.8. *For $m, r \in \mathbb{N}$, $I^{(m)} \nsubseteq I^r$ if and only if*

- *either $r > m$ or*
- *$r \leq m$ and there exists an element $x^a y^b F^d \in I^{(m)}$ such that $a + b + d < 2r$.*

Proof. There are elements in $I^{(m)}$ such that $\min(a, b) + d = m$, where $a, b, d \in \mathbb{N}_0$ are as in the description of $I^{(m)}$ in Proposition 2.2.4. One such example is $g = x^{\lceil \frac{m}{2} \rceil} y^{\lfloor \frac{m}{2} \rfloor} F^{\lceil \frac{m}{2} \rceil} \in I^{(m)}$.

Therefore, $I^{(m)} \nsubseteq I^r$ if either $r > m$ (since $g \notin I^r$ because $\lfloor \frac{m}{2} \rfloor + \lceil \frac{m}{2} \rceil = m < r$ and hence the conditions that $a + b \geq r$ and $\min(a, b) + d \geq r$ are not satisfied) or (by Propositions 2.2.4 and 2.2.7) $r \leq m$ and there exists an element $x^a y^b F^d \in I^{(m)}$ such that $a + b + d < 2r$.

Conversely, if $I^{(m)} \nsubseteq I^r$, then Propositions 2.2.4 and 2.2.7 say that there is a basis element h of $I^{(m)}$ that violates at least one of the conditions for I^r. If either the condition $a + b \geq r$ or the condition $\min(a,b)+d \geq r$ is violated, then $h \in I^{(m)}$ means $a+b \geq m$ and $\min(a,b)+d \geq m$ and hence $r > m$. If $r \leq m$ and hence h satisfies $a+b \geq r$ and $\min(a,b)+d \geq r$, then it has to violate the third condition, and thus $a+b+d < 2r$. □

The preceding lemma is useful for finding failures of containment and also for calculating the resurgence of I, as we will see in the next section.

Example 2.2.9. We know that $I^{(26)} \nsubseteq I^{20}$ because the element $g = x^{13}y^{13}F^{13}$ is in $I^{(26)}$ (by Proposition 2.2.4, since any two exponents add to 26) but $g \notin I^{20}$ because $13+13+13 = 39 < 40 = 2 \cdot 20$.

2.3 The Resurgence

In this section, we determine the resurgence

$$\rho(I) = \sup_{m,r} \left\{ \frac{m}{r} \;\middle|\; I^{(m)} \nsubseteq I^r \right\}.$$

In fact, we will exhibit a condition on m and r that is necessary and sufficient for $I^{(m)} \subseteq I^r$. In other words, given m we will find the largest r such that we have containment of $I^{(m)}$ in I^r.

Theorem 2.3.1. *We have $I^{(m)} \subseteq I^r$ if and only if $4r \leq 3m+1$.*
In particular, the resurgence is $\rho(I) = \frac{4}{3}$.

Proof. Suppose that $m, r \in \mathbb{N}$ are such that $I^{(m)} \subseteq I^r$. By Lemma 2.2.8, this means that $m \geq r$. If $4r > 3m+1$, we show that there exists an element $x^a y^b F^d$ in $I^{(m)}$ such that $a+b+d < 2r$, which is a contradiction and hence shows that $4r \leq 3m+1$ is required. Consider two cases, m even and m odd.

(a) If m is even, then $a = b = d = \frac{m}{2}$ satisfies the requirements for $I^{(m)}$ as any two of the exponents add to m, but $a+b+d = \frac{3}{2}m < 2r - \frac{1}{2} < 2r$, so $x^{\frac{m}{2}} y^{\frac{m}{2}} F^{\frac{m}{2}} \in I^{(m)} \setminus I^r$.

(b) If m is odd, then $a = d = \frac{m+1}{2}$ and $b = \frac{m-1}{2}$ satisfy the requirements for $I^{(m)}$ as any two of the exponents add to at least m, but $a + b + d = \frac{3}{2}m + \frac{1}{2} < 2r$, so $x^{\frac{m+1}{2}} y^{\frac{m-1}{2}} F^{\frac{m+1}{2}}$ is in $I^{(m)}$ but not in I^r.

Conversely, suppose that $m, r \in \mathbb{N}$ are such that $4r \leq 3m + 1$, and therefore $r \leq m$. We will now show that $I^{(m)} \subseteq I^r$.

Let $x^a y^b F^d \in I^{(m)}$, so $a + b \geq m$ and $\min(a, b) + d \geq m$. By symmetry, we may assume that $a \leq b$, so $a + d \geq m \geq r$. Then also $b + d \geq m$, and therefore $(a + b) + (a + d) + (b + d) \geq 3m$, i.e. $2(a + b + d) \geq 3m \geq 4r - 1$, which means $a + b + d \geq 2r - \frac{1}{2}$. Since $a, b, d, r \in \mathbb{N}_0$, we get $a + b + d \geq 2r$ and therefore $I^{(m)} \subseteq I^r$ by Lemma 2.2.8.

For the resurgence, notice first that $\rho \leq \frac{4}{3}$ because $\frac{m}{r} \geq \frac{4}{3}$ implies $I^{(m)} \subseteq I^r$.
Now, since $I^{(m)} \subseteq I^r$ if and only if $4r \leq 3m+1$, we get $I^{(m)} \nsubseteq I^r$ if and only if $4r > 3m+1$, i.e. $I^{(m)} \nsubseteq I^r$ if and only if $\frac{4}{3} - \frac{1}{3r} > \frac{m}{r}$. Given $r \in \{3k+1\}_{k \in \mathbb{N}}$, $m = \frac{4r-1}{3} - 1 = \frac{4(r-1)}{3} = 4k \in \mathbb{N}$ satisfies $4r = 12k+4 > 12k+1 = 3m+1$ and hence $I^{(m)} \nsubseteq I^r$. But then $\frac{m}{r} = \frac{4}{3}\left(\frac{r-1}{r}\right) = \frac{4k}{3k+1} \leq \rho$ for all $k \in \mathbb{N}$, so $\rho \geq \frac{4}{3}$. Therefore $\rho = \frac{4}{3}$. $\square$

Comment 2.3.2. The result above also allows us to give an explicit formula for all m and r such that $I^{(m)} \subseteq I^r$:
Given $m \in \mathbb{N}$, we have $I^{(m)} \subseteq I^r$ exactly when $r \in \{1, \ldots, m - \lceil \frac{m-1}{4} \rceil\}$ as $m - \lceil \frac{m-1}{4} \rceil = \lfloor \frac{3m+1}{4} \rfloor$.

Example 2.3.3. Instead of exhibiting a specific element that is contained in $I^{(26)}$ but not in I^{20}, we can also calculate that $26 - \lceil \frac{26-1}{4} \rceil = 19$, so $I^{(26)} \subseteq I^r$ for $r \in \{1, \ldots, 19\}$ but not for $r = 20$.

2.4 Alternative Descriptions for $I^{(m)}$ and I^r

We can also obtain interesting descriptions for I^r and $I^{(m)}$ using sums of ideals rather than vector spaces.

Recall that $I = (xy, (x,y)F) = (x,y) \cap (xy, F)$. Then

$$I^r = (xy, (x,y)F)^r = \sum_{j=0}^{r} (xy)^j (x,y)^{r-j} F^{r-j}.$$

For each $0 \le j \le r$, define the ideals

$$K_j := (xy)^j (x,y)^{r-j} F^{r-j},$$

hence

$$I^r = \sum_{j=0}^{r} K_j.$$

For the symbolic powers of I we get a similar result.

Lemma 2.4.1. *For each* $0 \le i \le m$*, define the ideals*

$$J_i := (xy)^i (x,y)^{(m-2i)_+} F^{m-i},$$

where $(m-2i)_+ = \max(m-2i, 0)$*. Then*

$$I^{(m)} = \sum_{i=0}^{m} J_i.$$

Proof. By Proposition 2.2.4,

$$\begin{aligned} I^{(m)} &= \langle x^a y^b z^c F^d | a,b,c,d \in \mathbb{N}_0, c < n, a+b \ge m, \min(a,b) + d \ge m \rangle \\ &= (x^a y^b z^c F^d | a,b,c,d \in \mathbb{N}_0, c < n, a+b \ge m, \min(a,b) + d \ge m) \\ &= (x^a y^b F^d | a,b,d \in \mathbb{N}_0, a+b \ge m \text{ and } \min(a,b) + d \ge m) \end{aligned}$$

and it follows that

$$\sum_{i=0}^{m} (xy)^i (x,y)^{(m-2i)_+} F^{m-i} \subseteq I^{(m)},$$

so it's enough to show $I^{(m)} \subseteq \sum_{i=0}^{m} J_i$.

Take a generator $g = x^a y^b F^d$ of $I^{(m)}$. Then $a + b \ge m$ and we may assume $b \ge a$. If

$b > m$, then g is divisible by $x^{\min(a,m)}y^mF^d \in I^{(m)}$ and showing $x^{\min(a,m)}y^mF^d \in \sum_i J_i$ also shows that $g \in \sum_i J_i$, so we have reduced to the case that $b \leq m$. Let $a + b = m + k$, so $m \geq b = m + k - a \geq a$. Then $d = m - a + t \geq 0$ for some $t \in \mathbb{N}_0$, so $g = x^a y^{m+k-a} F^{m-a+t} = (xy)^a (y^{(m-2a)_+}) F^{m-a} \cdot y^{m+k-2a-(m-2a)_+} F^t$ which is contained in $(xy)^a (x,y)^{(m-2a)_+} F^{m-a} = J_a$.

Thus $g \in \sum_{i=0}^m J_i$ and therefore $I^{(m)} = \sum_{i=0}^m J_i$. □

Comment 2.4.2. For $0 \leq j \leq r$ and $0 \leq i \leq m$, we can also write K_j and J_i in vector space form. We have

$$\begin{aligned} K_j &= (xy)^j (x,y)^{r-j} F^{r-j} \\ &= (x^a y^b F^{r-j} | a + b = r + j, \min(a,b) \geq j) \\ &= \langle x^a y^b z^c F^d | a,b,c,d \in \mathbb{N}_0, c < n, d \geq r - j, a + b \geq r + j, \min(a,b) \geq j \rangle. \end{aligned}$$

Similarly, if $i < \frac{m}{2}$, we get

$$\begin{aligned} J_i &= (xy)^i (x,y)^{m-2i} F^{m-i} \\ &= (x^a y^b F^{m-i} | a + b = m, \min(a,b) \geq i) \\ &= \langle x^a y^b z^c F^d | a,b,c,d \in \mathbb{N}_0, c < n, d \geq m - i, a + b \geq m, \min(a,b) \geq i \rangle \end{aligned}$$

and for $i \geq \frac{m}{2}$ we get

$$\begin{aligned} J_i &= (xy)^i F^{m-i} \\ &= (x^i y^i F^{m-i}) \\ &= \langle x^a y^b z^c F^d | a,b,c,d \in \mathbb{N}_0, c < n, d \geq m - i, \min(a,b) \geq i \rangle. \end{aligned}$$

Lemma 2.4.3. *Using the above notation for I^r and $I^{(m)}$, we have $I^{(m)} \subseteq I^r$ if and only if for each $i \leq m$ there exists $j \leq r$ such that $J_i \subseteq K_j$.*

Proof. Clearly, $I^{(m)} \subseteq I^r$ if for each $i \leq m$ there exists a $j \leq r$ such that $J_i \subseteq K_j$.

Suppose conversely that $I^{(m)} \subseteq I^r$. Since the definition of J_i changes depending on whether $i \geq \frac{m}{2}$ or $i < \frac{m}{2}$, we will look at two separate cases.

(a) For $i \geq \frac{m}{2}$, J_i has a single homogeneous generator, which is contained in $I^r = \sum K_j$. Thus it has to be contained in some K_j, hence $J_i \subseteq K_j$ for some j.

(b) For $i < \frac{m}{2}$, recall that $K_j = (x^a y^b F^{r-j} | a + b = r + j, \min(a, b) \geq j)$ and $J_i = (x^a y^b F^{m-i} | a + b = m, \min(a, b) \geq i)\,(*)$. Note that $\min(a, b) = i$ is attained for some elements of J_i, so if we want $J_i \subseteq K_j$, then we need $i \geq j$ to satisfy $\min(a, b) \geq i, j$. Since $i < \frac{m}{2}$ implies $i < m-i$, we actually have that $J_i = (x^i y^{m-i} F^{m-i}, x^{i+1} y^{m-(i+1)} F^{m-i}, \ldots, x^{m-i} y^i F^{m-i})$. So for J_i to be contained in K_j, we need that $x^a y^{m-a} F^{m-i} \in K_j$ for $a = i, \ldots, m - i$.

Let $i \leq a \leq m - i$. Then $x^a y^{m-a} F^{m-i} \in J_i \subseteq I^{(m)} \subseteq I^r$, so there exists a $j \leq r$ such that $x^a y^{m-a} F^{m-i} \in K_j$. By the conditions given in the descriptions of J_i and K_j in $(*)$ and comparing exponents on x, y, and F, this means that $\min(a, m - a) \geq i \geq j$, $m \geq r + j$, and $m - i \geq r - j$.

By choice of a, we also get $i < a + 1 \leq m - i$ and therefore $x^{a+1} y^{m-(a+1)} F^{m-i} \in J_i$. But that means that $\min(a + 1, m - (a + 1)) \geq i \geq j$, and still $m \geq r + j$ and $m - i \geq r - j$. Hence $x^{a+1} y^{m-(a+1)} F^{m-i} \in K_j$ as well. Repeating this shows that $x^{a+t} y^{m-(a+t)} F^{m-i} \in K_j$ also as long as $a + t \leq m - i$. Hence there exists one j such that K_j contains all of J_i.

Thus for each i there is a j such that $J_i \subseteq K_j$, as desired. $\square$

Comment 2.4.4. It is straightforward to recover Theorem 2.3.1 from the previous result.

2.5 Splitting up Symbolic Powers: Consequences

It will come in handy later if we can express a symbolic power of I as either the product of two other, smaller symbolic powers, or possibly even as the ordinary power of a small symbolic power of I. It turns out that we can express all symbolic powers as an ordinary power of $I^{(2)}$.

Theorem 2.5.1. *Let $\alpha, \beta \in \mathbb{N}$. Then we have $I^{(\alpha+\beta)} \supseteq I^{(\alpha)}I^{(\beta)}$ with equality if at least one of α and β is even.*

Proof. We will first show that $I^{(\alpha+\beta)} \supseteq I^{(\alpha)}I^{(\beta)}$. Note that it suffices to find a set G of generators for $I^{(\alpha)}$ and a set H of generators for $I^{(\beta)}$ and show that $gh \in I^{(\alpha+\beta)}$ for each $g \in G$ and $h \in H$.

Using Proposition 2.2.4 again, we may assume that $g = x^{a_1}y^{b_1}F^{d_1}$ and $h = x^{a_2}y^{b_2}F^{d_2}$, where for $i = 1, 2$, $a_i, b_i, d_i, s_i, t_i \in \mathbb{N}_0$ are such that $a_1+b_1 = \alpha+s_1$, $a_2+b_2 = \beta+s_2$, $\min(a_1, b_1)+d_1 = \alpha+t_1$, and $\min(a_2, b_2)+d_2 = \beta+t_2$. We immediately obtain $(a_1+a_2)+(b_1+b_2) = \alpha+\beta+(s_1+s_2)$ and $\min(a_1+a_2, b_1+b_2)+(d_1+d_2) \geq \min(a_1, b_1)+\min(a_2, b_2)+(d_1+d_2) = \alpha+\beta+(t_1+t_2)$, so $gh = x^{a_1+a_2}y^{b_1+b_2}F^{d_1+d_2} \in I^{(\alpha+\beta)}$ as desired.

To show that $I^{(\alpha+\beta)} = I^{(\alpha)}I^{(\beta)}$ if at least one of α and β is even, assume without loss of generality that α is even. In order to show $I^{(\alpha+\beta)} \subseteq I^{(\alpha)}I^{(\beta)}$, it is enough to find a set of ideal generators g of $I^{(\alpha+\beta)}$ in $I^{(\alpha)}I^{(\beta)}$, so let $g \in I^{(\alpha+\beta)}$. By our description for $I^{(\alpha+\beta)}$ from Proposition 2.2.4, we may assume that $g = x^a y^b F^d$, where $a, b, d, s, t \in \mathbb{N}_0$ are such that $a + b = \alpha + \beta + s$ and $\min(a, b) + d = a + d = \alpha + \beta + t$, where by symmetry we may also assume $a \leq b$. There are three possibilities: $a \geq \frac{\alpha}{2}$ and $d \geq \frac{\alpha}{2}$; $a \geq \frac{\alpha}{2}$ and $d < \frac{\alpha}{2}$; or $a < \frac{\alpha}{2}$. We treat each one in turn.

(a) If $b \geq a \geq \frac{\alpha}{2}$ and $d \geq \frac{\alpha}{2}$, g can be written as $g = \underbrace{x^{\frac{\alpha}{2}}y^{\frac{\alpha}{2}}F^{\frac{\alpha}{2}}}_{\in I^{(\alpha)}} \cdot \underbrace{x^{a-\frac{\alpha}{2}}y^{b-\frac{\alpha}{2}}F^{d-\frac{\alpha}{2}}}_{\in I^{(\beta)}}$ where the first factor is in $I^{(\alpha)}$ because any two of the exponents add to α, and the second factor is in $I^{(\beta)}$ because $(a - \frac{\alpha}{2}) + (b - \frac{\alpha}{2}) = a + b - \alpha = \beta + s$ and $(\min(a, b) - \frac{\alpha}{2}) + (d - \frac{\alpha}{2}) = \beta + t$.

(b) If $b \geq a \geq \frac{\alpha}{2}$ but $d < \frac{\alpha}{2}$, then $b \geq a \geq \alpha - d$ as $b + d \geq a + d = \alpha + \beta + t \geq \alpha$. Also, $a - (\alpha - d) = \beta + t$ and $b - (\alpha - d) = \beta + t + v$ for some $v \in \mathbb{N}_0$. Then we can write $g = \underbrace{x^{\alpha-d}y^{\alpha-d}F^d}_{\in I^{(\alpha)}} \cdot \underbrace{x^{\beta+t}y^{\beta+t+v}}_{\in I^{(\beta)}}$ where the first factor is in $I^{(\alpha)}$ because $(\alpha - d) + d = \alpha$ and $2\alpha - 2d > \alpha$, and the second factor is in $I^{(\beta)}$ because $s, t, v \in \mathbb{N}_0$.

(c) If $a < \frac{\alpha}{2}$, then $a + b \geq \alpha > 2a$ and $a + d \geq \alpha > 2a$, so $b \geq \alpha - a > a$ and $d \geq \alpha - a > a$. Also, $b + a - \alpha = \beta + s$ and $d + a - \alpha = \beta + t$. Therefore we can write $g = \underbrace{x^a y^{\alpha-a}F^{\alpha-a}}_{\in I^{(\alpha)}} \cdot \underbrace{y^{\beta+s}F^{\beta+t}}_{\in I^{(\beta)}}$

where the first factor is in $I^{(\alpha)}$ because $a + (\alpha - a) = \alpha$ and $2(\alpha - a) > \alpha$, and the second factor is in $I^{(\beta)}$ because $s, t \in \mathbb{N}_0$.

Therefore, if α is even, we get $I^{(\alpha+\beta)} \subseteq I^{(\alpha)} I^{(\beta)}$ and consequently $I^{(\alpha+\beta)} = I^{(\alpha)} I^{(\beta)}$. $\square$

Corollary 2.5.2. *Let* $r, s, t \in \mathbb{N}$. *Then* $I^{(2st)} = \left(I^{(2s)}\right)^t$ *and* $I^{(2st+rt)} = I^{(2st)} I^{(rt)}$.

Proof. Both equations follow immediately from Theorem 2.5.1. $\square$

To discuss the odd equivalent of Theorem 2.5.1, we need to return to explicit vector space notation for a moment.

Lemma 2.5.3. *Let* $m \in \mathbb{N}$ *be odd. Then*

$$I^{(m)} I = \langle x^a y^b z^c F^d | a, b, c, d \in \mathbb{N}_0, c < n, a + b \geq m + 1, \min(a, b) + d \geq m + 1,$$

$$\textit{and } a + b + d \geq \frac{3m + 1}{2} + 2 \rangle.$$

Proof. Let $\mathcal{S} := \langle x^a y^b z^c F^d | a, b, c, d \in \mathbb{N}_0, c < n, a + b \geq m + 1, \min(a, b) + d \geq m + 1, a + b + d \geq \frac{3m+1}{2} + 2 \rangle$. By Proposition 2.2.4, $I^{(m)} I$ is generated by elements of the form $g = \underbrace{x^{a_1} y^{b_1} z^{c_1} F^{d_1}}_{\in I^{(m)}} \cdot \underbrace{x^{a_2} y^{b_2} z^{c_2} F^{d_2}}_{\in I}$, where for $i = 1, 2$, $a_i, b_i, c_i, d_i \in \mathbb{N}_0$ are such that $c_1 < n$, $c_2 < n$, $a_1 + b_1 \geq m$, $\min(a_1, b_1) + d_1 \geq m$, $a_2 + b_2 \geq 1$, and $\min(a_2, b_2) + d_2 \geq 1$. Note that also $2(a_1 + b_1 + d_1) = (a_1 + b_1) + (a_1 + d_1) + (b_1 + d_1) \geq 3m$, so $a_1 + b_1 + d_1 \geq \frac{3m}{2}$. But m is odd, so we actually have $a_1 + b_1 + d_1 \geq \frac{3m+1}{2}$, and likewise $a_2 + b_2 + d_2 \geq 2$.

Thus any such generator g for $I^{(m)} I$ is of the form $x^{a_1+a_2} y^{b_1+b_2} z^{c_1+c_2} F^{d_1+d_2}$ where $(a_1 + a_2) + (b_1 + b_2) \geq m + 1$, $\min(a_1 + a_2, b_1 + b_2) + (d_1 + d_2) \geq \min(a_1, b_1) + \min(a_2, b_2) + (d_1 + d_2) \geq m + 1$, $(a_1 + a_2) + (b_1 + b_2) + (d_1 + d_2) \geq \frac{3m+1}{2} + 2$, and, by Lemma 2.2.2, we may replace $z^{c_1+c_2}$ by a linear combination of elements of the form $x^a y^b z^d F^d$ with $a, b, d \geq 0$ and $0 \leq c < n$. Therefore, $g \in \mathcal{S}$ and hence $I^{(m)} I \subseteq \mathcal{S}$.

For the other containment, let $g = x^a y^b z^c F^d \in \mathcal{S}$, where it is enough to assume that $c = 0$. Let $s, t, u \in \mathbb{N}_0$ be such that $a + b = m + 1 + s$, $a + d = m + 1 + t$, and $b + d = m + 1 + u$. Then $a + b + d = \frac{3m+3}{2} + \frac{s+t+u}{2} \geq \frac{3m+1}{2} + 2$ implies that $\frac{s+t+u}{2} \geq 1$, and $a + b = m + 1 + s$ implies that $a \geq 1$ or $b \geq 1$. By symmetry, we may assume that $a \geq 1$. Then either $d = 0$ or $d \geq 1$.

(a) If $d = 0$, then $a - 1 = m + t$ and $b - 1 = m + u$, so $g = \underbrace{x^{a-1}y^{b-1}}_{\in I^{(m)}} \cdot \underbrace{xy}_{\in I} \in I^{(m)}I$.

(b) So suppose that $d \geq 1$. If $s \geq 1$, we get $(a-1) + d = m + t$, $(b-1) + d = m + u$, and $(a-1)+(b-1) = m+(s-1) \geq m$, so $g = \underbrace{x^{a-1}y^{b-1}F^d}_{\in I^{(m)}} \underbrace{xy}_{\in I} \in I^{(m)}I$. If $s = 0$ and $u, t \geq 1$, then $(a-1)+b = m$, $(a-1)+(d-1) = m+(t-1) \geq m$, and $(b-1)+(d-1) = m+(u-1) \geq m$, so $g = \underbrace{x^{a-1}y^{b}F^{d-1}}_{\in I^{(m)}} \underbrace{xF}_{\in I} \in I^{(m)}I$. Finally, suppose $s = 0$ and either of u or t is also zero, say $t = 0$. Then $u \geq 2$ as $s + t + u \geq 2$, and $b = d \geq 1$. Thus we get $a + (b-1) = m$, $a + (d-1) = m$, and $(b-1) + (d-1) = m + (u-1) \geq m$, so $g = \underbrace{x^{a}y^{b-1}F^{d-1}}_{\in I^{(m)}} \underbrace{yF}_{\in I} \in I^{(m)}I$.

Therefore $g \in I^{(m)}I$ and hence $\mathcal{S} \subseteq I^{(m)}I$. $\square$

We can now prove that the condition that at least one of α or β be even in Theorem 2.5.1 is necessary.

Theorem 2.5.4. *Let $\alpha, \beta \in \mathbb{N}$ both be odd. Then $I^{(\alpha+\beta)} \supsetneq I^{(\alpha)}I^{(\beta)} = I^{(\alpha+\beta-1)}I$.*

Proof. By Theorem 2.5.1, for any odd number $m \in \mathbb{N}$, we have $I^{(m)} = I^{(m-1+1)} = I^{(m-1)}I^{(1)} = I^{(m-1)}I$ since $m - 1$ is even.

By Theorem 2.5.1, we have $I^{(\alpha+\beta)} \supseteq I^{(\alpha)}I^{(\beta)}$.

Since $\alpha = 2k+1$ and $\beta = 2l+1$ for some $k, l \in \mathbb{N}_0$, then $I^{(\alpha)} = \left(I^{(2)}\right)^k I$ and $I^{(\beta)} = \left(I^{(2)}\right)^l I$ by Corollary 2.5.2. Then $\alpha + \beta - 2$ is even and thus $I^{(\alpha)}I^{(\beta)} = \left(I^{(2)}\right)^{k+l} I^2 = I^{(2k+2l)}I^2 = \left(I^{(\alpha+\beta-2)}I\right) I = I^{(\alpha+\beta-1)}I$. By Lemma 2.5.3,
$I^{(\alpha+\beta-1)}I = \langle x^a y^b z^c F^d | a, b, c, d \in \mathbb{N}_0, c < n, a+b \geq \alpha+\beta, \min(a,b)+d \geq \alpha+\beta,$ and $a+b+d \geq \frac{3(\alpha+\beta)}{2} + 1\rangle$.

But $g = x^{\frac{\alpha+\beta}{2}} y^{\frac{\alpha+\beta}{2}} F^{\frac{\alpha+\beta}{2}} \in I^{(\alpha+\beta)}$ and $3\left(\frac{\alpha+\beta}{2}\right) < \frac{3(\alpha+\beta)}{2} + 1$, so g does not satisfy the condition that $a + b + d \geq \frac{3(\alpha+\beta)}{2} + 1$ and therefore g is contained in $I^{(\alpha+\beta)}$ but not in $I^{(\alpha)}I^{(\beta)}$. $\square$

Corollary 2.5.5. *For $m \in \mathbb{N}$, we have $I^{(m)} = \left(I^{(2)}\right)^{\frac{m}{2}}$ if m is even, and $I^{(m)} = \left(I^{(2)}\right)^{\frac{m-1}{2}} I$ if m is odd.*

Proof. The even case follows by Corollary 2.5.2 and the odd case by Corollary 2.5.2 and Theorem 2.5.4. $\square$

Comment 2.5.6. A consequence of Corollary 2.5.5 is that the symbolic power algebra $\bigoplus_m I^{(m)}$ is Noetherian. (See [14], for example.)

Comment 2.5.7. In [10], Harbourne and Huneke conjecture that $J^{(Nr)} \subseteq \mathcal{M}^{r(N-1)}J^r$ and $J^{(Nr-N+1)} \subseteq \mathcal{M}^{(r-1)(N-1)}J^r$ for an ideal J of finitely many points in $\mathbb{P}^N$, $\mathcal{M}$ the irrelevant ideal in $K[\mathbb{P}^N]$, and any $r > 0$. They verify their conjecture for some ideals, for example for those arising from taking general points in $\mathbb{P}^2$. We verify those two conjectures for our ideal I at this point. We will take a closer look at other conjectures in a later section.

Proposition 2.5.8. *Let $r \in \mathbb{N}$ and let $\mathcal{M} = (x, y, z)$. Then $I^{(2r)} \subseteq \mathcal{M}^r I^r$ and $I^{(2r-1)} \subseteq \mathcal{M}^{r-1}I^r$.*

Proof. Notice that $I^{(2)} = (x^2y^2, x^2F^2, y^2F^2, xyF)$ (see Appendix 3, for example) and

$$\mathcal{M}I = (x, y, z)(xy, xF, yF) = (x^2y, x^2F, xyF, xy^2, y^2F, xyz, xFz, yFz),$$

and thus $I^{(2)} \subseteq \mathcal{M}I$. Then Corollary 2.5.5 gives $I^{(2r)} = \left(I^{(2)}\right)^r \subseteq (\mathcal{M}I)^r = \mathcal{M}^r I^r$ and $I^{(2r-1)} = \left(I^{(2)}\right)^{r-1} I \subseteq (\mathcal{M}I)^{r-1} I = \mathcal{M}^{r-1}I^r$. $\square$

2.6 Complete Description of $I^{(m)} \subseteq \mathcal{M}^t I^r$

In fact, we can improve on the exponents on $\mathcal{M}$ in Proposition 2.5.8. Let F, $I = (xy, xF, yF)$, and $\mathcal{M} = (x, y, z)$ be as above. Then what can we say about r and t such that $I^{(m)} \subseteq \mathcal{M}^t I^r$?

In this section, we give an exact criterion for how much we can shrink an ordinary power of I by multiplication with a power of the irrelevant ideal of R such that the product still contains a given symbolic power of I.

Clearly, $I^{(m)} \not\subseteq I^r$ implies $I^{(m)} \not\subseteq \mathcal{M}^t I^r$ for all $t \geq 0$. Hence, by Comment 2.3.2, we will assume $r \leq m - \lceil \frac{m-1}{4} \rceil$.

Recall that for $r \in \mathbb{N}$, we defined a set $\mathcal{S}_r$ by

$$\mathcal{S}_r = \left\{ (a,b,c,d) \in \mathbb{N}_0^4 \;\middle|\; c < n, \min(a,b) + d \geq r, a+b+d \geq 2r, \text{ and } a+b \geq r \right\}.$$

We first describe $\mathcal{M}^t I^r$ as a vector space, so we can use the vector space description of the symbolic powers later.

Lemma 2.6.1. *Let $r, t \in \mathbb{N}$ and $\mathcal{M}$, I be as above. Then*

$$\mathcal{M}^t I^r = \langle x^a y^b z^c F^d | \;\exists\, 0 \leq \alpha \leq a, 0 \leq \beta \leq b, 0 \leq \delta \leq d$$

$$\textit{such that } (a-\alpha, b-\beta, c, d-\delta) \in \mathcal{S}_r \textit{ and } \alpha + \beta + n\delta \geq (t-c)_+\rangle.$$

Proof. Let $J := \langle x^a y^b z^c F^d | \;\exists\, 0 \leq \alpha \leq a, 0 \leq \beta \leq b, 0 \leq \delta \leq d$ such that $(a-\alpha, b-\beta, c, d-\delta) \in \mathcal{S}_r$ and $\alpha + \beta + n\delta \geq (t-c)_+\rangle$.

We will show that J is an ideal and that $\mathcal{M}^t I^r \supseteq J \supseteq \mathcal{M}^t I^r$.

To see that J is an ideal in R, it suffices to show that for any basis element $g = x^a y^b z^c F^d$ of J, the polynomials xg, yg, and zg are also in J. Since $g \in J$, there exist α, β, and δ such that $(a-\alpha, b-\beta, c, d-\delta) \in \mathcal{S}_r$ such that $\alpha+\beta+n\delta \geq (t-c)_+$. But then $(a+1-\alpha, b-\beta, c, d-\delta)$ and $(a-\alpha, b+1-\beta, c, d-\delta)$ are also in $\mathcal{S}_r$, so xg and yg are in J.

Also, if $c < n-1$, then $c+1 \leq n-1$ and $\alpha+\beta+n\delta \geq (t-c)_+ \geq (t-(c+1))_+$, so $(a-\alpha, b-\beta, c+1, d-\delta) \in \mathcal{S}_r$ and $zg \in J$.

Finally, if $c = n-1$, then $zg = x^a y^b z^n F^d$. By definition of F, there exists a polynomial L in z of degree at most $n-1$ with coefficients in $K[x,y]$ such that $F = L - z^n$, and L has the form $L = \sum_{i=0}^{n-1} (\eta_i x^{n-i} + \eta_i' y^{n-i}) z^i$ for some $\eta_i, \eta_i' \in K$. So $z^n = L - F$ and consequently $zg = x^a y^b F^d (L-F) = \sum_{i=0}^{n-1} \eta_i x^{a+n-i} y^b z^i F^d + \sum_{i=0}^{n-1} \eta_i' x^a y^{b+n-i} z^i F^d - x^a y^b F^{d+1}$. Now, for all i, we have $((a+n-i)-(\alpha+n-i), b-\beta, i, d) \in \mathcal{S}_r$ and $(\alpha+n-i)+\beta+n\delta \geq (t-(n-1))_+ + (n-i) = \max(n-i, t+1-i) \geq (t-i)_+$, so $\sum_{i=0}^{n-1} \eta_i x^{a+n-i} y^b z^i F^d \in J$. Likewise $(a-\alpha, (b+n-i)-(\beta+n-i), i, d) \in \mathcal{S}_r$ and $\alpha + (\beta+n-i) + n\delta \geq (t-i)_+$, so $\sum_{i=0}^{n-1} \eta_i' x^a y^{b+n-i} z^i F^d \in J$. Finally, $(a-\alpha, b-\beta, c, d-\delta) \in \mathcal{S}_r$ with $\alpha+\beta+n\delta \geq (t-(n-1))_+$ also means that $(a-\alpha, b-\beta, 0, (d+1)-(\delta+1)) \in \mathcal{S}_r$ with $\alpha+\beta+n(\delta+1) \geq (t-(n-1))_+ + n = \max(n, t+1) \geq (t-0)_+$, so $x^a y^b F^{d+1} \in J$.

Therefore, $zg \in J$ and J is an ideal.

Let $j = x^a y^b z^c F^d \in J$ be a basis element and α, β, and δ be accordingly. Then by Proposition 2.2.7 $x^{a-\alpha} y^{b-\beta} F^{d-\delta} \in I^r$ and $x^\alpha y^\beta z^c F^\delta \in \mathcal{M}^t$, so $j = x^\alpha y^\beta z^c F^\delta \cdot x^{a-\alpha} y^{b-\beta} F^{d-\delta} \in \mathcal{M}^t I^r$. Therefore, $\mathcal{M}^t I^r \supseteq J$.

Let $h \in \mathcal{M}^t I^r$ be of the form $h = x^a y^b z^c F^d \cdot h'$, where $(a,b,c,d) \in \mathcal{S}_r$ and $h' \in \mathcal{M}^t$, $h' = x^{t_1} y^{t_2} z^{t_3}$ for some $t_1, t_2, t_3 \in \mathbb{N}_0$ with $t_1 + t_2 + t_3 = t$. Thus $h = x^{a+t_1} y^{b+t_2} z^{c+t_3} F^d$. The set of all such h generates $\mathcal{M}^t I^r$. Argue by induction on $c + t_3$. If $c + t_3 < n$, then letting $\alpha = t_1, \beta = t_2$, and $\delta = 0$ gives $\alpha + \beta + n\delta = t_1 + t_2 = t - t_3 \geq (t - (c + t_3))_+$ and $(a,b,c,d) \in \mathcal{S}_r$, so $h \in J$. Now assume that $h \in J$ as long as $c + t_3 < s$ for some $s \geq n$. Say $c + t_3 = s = qn + p$, where $0 \leq p < n$. Then, like above, $h = x^{a+t_1} y^{b+t_2} z^s F^d = x^{a+t_1} y^{b+t_2} z^p F^d (L - F)^q$. By definition of L, all summands of $(L - F)^q = \sum_{i=0}^q \binom{q}{i} L^i F^{q-i}$ other than F^q are going to be of degree less than qn in z, and so by induction, all those respective summands of $h = x^{a+t_1} y^{b+t_2} z^p F^d \left(\sum_{i=0}^q \binom{q}{i} L^i F^{q-i}\right)$ are going to be in J. Hence it suffices to show that the remaining summand $x^{a+t_1} y^{b+t_2} z^p F^{d+q}$ is in J. But $(a,b,p,d) \in \mathcal{S}_r$ and choosing $\alpha = t_1, \beta = t_2$, and $\delta = q$ gives $\alpha + \beta + n\delta = t_1 + t_2 + qn = t_1 + t_2 + (c + t_3 - p) = t + c - p \geq (t - p)_+$, so $h \in J$.

Therefore, $J \supseteq \mathcal{M}^t I^r$ and the proof is complete. $\square$

To answer the containment question $I^{(m)} \subseteq \mathcal{M}^t I^r$ completely, we will consider three cases, which jointly include all possible combinations of n, m, and r.

For the first case, Proposition 2.6.2, we will assume that r is maximal such that $I^{(m)} \subseteq I^r$. For the second and third case, Proposition 2.6.3 and Theorem 2.6.4, respectively, we will assume a smaller r and instead differentiate by the magnitude of n, either $n = 1$ or $n \geq 2$.

The $n = 1$ case is interesting because unlike most of our results, it behaves quite differently from the $n \geq 2$ case.

Proposition 2.6.2. *Let $n \geq 1$, $m \in \mathbb{N}$, and set $r := m - \lceil \frac{m-1}{4} \rceil$, so r is maximal such that $I^{(m)} \subseteq I^r$. Then*

1. $I^{(m)} \not\subseteq \mathcal{M} I^r$ *if* $m \equiv 0$ *or* $1 \mod 4$ *and*

2. $I^{(m)} \subseteq \mathcal{M}^t I^r$ *but* $I^{(m)} \not\subseteq \mathcal{M}^{t+1} I^r$ *for* $t = \min(n, 2m - 2r)$ *if* $m \equiv 2$ *or* $3 \mod 4$.

Proof. 1. Suppose $m \equiv 0$ or $1 \mod 4$.

We have $r = \frac{3m}{4}$ or $\frac{3m+1}{4}$, so $2r = \lceil \frac{3m}{2} \rceil$. But $g = x^{\lceil \frac{m}{2} \rceil} y^{\lfloor \frac{m}{2} \rfloor} F^{\lceil \frac{m}{2} \rceil} \in I^{(m)}$ and $a + b + d = \lceil \frac{m}{2} \rceil + \lfloor \frac{m}{2} \rfloor + \lceil \frac{m}{2} \rceil = \lceil \frac{3m}{2} \rceil = 2r$, so the only nonnegative α, β, and δ such that $(\lceil \frac{m}{2} \rceil - \alpha, \lfloor \frac{m}{2} \rfloor - \beta, 0, \lceil \frac{m}{2} \rceil - \delta) \in \mathcal{S}_r$ are $\alpha = \beta = \delta = 0$. However, by Lemma 2.6.1, for $g \in \mathcal{M}I^r$ we would need at least one of α, β, or δ to be positive, as $\alpha + \beta + n\delta$ needs to be at least $(1-c)_+ = 1$. Thus $g \notin \mathcal{M}I^r$ and hence $I^{(m)} \not\subseteq \mathcal{M}I^r$.

2. Suppose $m \equiv 2$ or $3 \mod 4$. Then $r = \frac{3m-2}{4}$ or $\frac{3m-1}{4}$ and thus $2r = \lceil \frac{3m}{2} \rceil - 1$.

   Suppose $g = x^a y^b z^c F^d \in I^{(m)}$. Then (by Proposition 2.2.4) $a + b, a + d, b + d \geq m > r$, so $2(a + b + d) \geq 3m$ and therefore $a + b + d \geq \lceil \frac{3m}{2} \rceil = 2r + 1$. Thus we may choose at least one of α, β, δ to be positive and still have $(a - \alpha, b - \beta, c, d - \delta) \in \mathcal{S}_r$.

   (a) If $d > 0$, take $\alpha = \beta = 0$ and $\delta = 1$. Then, if $m \geq 2$ and thus $r \leq m - 1$, we have $a+(d-1), b+(d-1) \geq m-1 \geq r$, so $(a, b, c, d-1) \in \mathcal{S}_r$ and $\alpha+\beta+n\delta = n \geq (n-c)_+$ and therefore $g \in \mathcal{M}^n I^r$. Note that $x^{\lceil \frac{m}{2} \rceil} y^{\lfloor \frac{m}{2} \rfloor} F^{\lceil \frac{m}{2} \rceil} \in I^{(m)}$, so here $a+b+d = \lceil \frac{3m}{2} \rceil = 2r+1$ and $\alpha = \beta = 0$, $\delta = 1$ is the best we can do. Hence we indeed have $I^{(m)} \not\subseteq \mathcal{M}^{n+1} I^r$.

   (b) If $d = 0$, then $a, b \geq m$ and we may take $\alpha = m - r = \beta$ and $\delta = 0$. Then $\alpha + \beta + n\delta = 2m - 2r \geq (2m - 2r - c)_+$ and $g \in \mathcal{M}^{2m-2r} I^r$. Note that $x^m y^m \in I^{(m)}$, so here $a + b + d = 2m = 2r + (2m - 2r)$ and $\alpha = \beta = m - r$, $\delta = 0$ is the best we can do. Hence we indeed have $I^{(m)} \not\subseteq \mathcal{M}^{2m-2r+1} I^r$.

   Therefore, for $t = \min(n, 2m - 2r)$, we have $I^{(m)} \subseteq \mathcal{M}^t I^r$ but $I^{(m)} \not\subseteq \mathcal{M}^{t+1} I^r$.

$\square$

Proposition 2.6.3. *Suppose $n = 1$. Let $m, r \in \mathbb{N}$ be such that $r < m - \lceil \frac{m-1}{4} \rceil$. Then*

$$I^{(m)} \subseteq \mathcal{M}^{\lceil \frac{3m}{2} \rceil - 2r} I^r \textit{ but } I^{(m)} \not\subseteq \mathcal{M}^{\lceil \frac{3m}{2} \rceil - 2r + 1} I^r.$$

Proof. Note that $n = 1$ implies $F = -z$, which means that $I = (xy, xz, yz)$, $I^{(m)} = \langle x^a y^b z^d | a + b \geq m, \min(a, b) + d \geq m \rangle$, and $I^r = \langle x^a y^b z^d | (a, b, 0, d) \in \mathcal{S}_r \rangle$.

We have $g = x^{\lceil \frac{m}{2} \rceil} y^{\lfloor \frac{m}{2} \rfloor} z^{\lceil \frac{m}{2} \rceil} \in I^{(m)}$, so $a + b + d = \lceil \frac{3m}{2} \rceil = 2r + (\lceil \frac{3m}{2} \rceil - 2r)$. But this means that we can't find α, β, δ such that $\alpha + \beta + n\delta = \alpha + \beta + \delta \geq (\lceil \frac{3m}{2} \rceil - 2r) + 1$ and still have $(a-\alpha)+(b-\beta)+(d-\delta) \geq 2r$. Therefore, $g \notin \mathcal{M}^{\lceil \frac{3m}{2} \rceil - 2r + 1} I^r$ and hence $I^{(m)} \not\subseteq \mathcal{M}^{\lceil \frac{3m}{2} \rceil - 2r + 1} I^r$.

We still need to show $I^{(m)} \subseteq \mathcal{M}^{\lceil \frac{3m}{2} \rceil - 2r} I^r$. For that, let $g = x^a y^b z^d \in I^{(m)}$ and note that $r < m - \lceil \frac{m-1}{4} \rceil$ implies $\lceil \frac{3m}{2} \rceil - 2r > 0$. We proceed by looking at two main cases, $r \geq \lceil \frac{m}{2} \rceil$

and $r < \left\lceil \frac{m}{2} \right\rceil$. Each case will be further divided into two subcases, (i) and (ii), depending on the magnitude of a.

(a) First consider the case that $r \geq \left\lceil \frac{m}{2} \right\rceil$. Then $m - r \geq \left\lceil \frac{3m}{2} \right\rceil - 2r > 0$. Let $\delta = 0$, $\alpha = \min(a, \left\lceil \frac{3m}{2} \right\rceil - 2r)$, and $\beta = \left\lceil \frac{3m}{2} \right\rceil - 2r - \alpha$, so $\beta = \left\lceil \frac{3m}{2} \right\rceil - 2r - a > 0$ if $a < \left\lceil \frac{3m}{2} \right\rceil - 2r$ and $\beta = 0$ if $a \geq \left\lceil \frac{3m}{2} \right\rceil - 2r$. Also, $b \geq \beta$ as $b \geq 0$ and $b \geq m - a \geq \left\lceil \frac{3m}{2} \right\rceil - 2r - a$.

(a.i) If $a = \alpha$, then $a \leq \left\lceil \frac{3m}{2} \right\rceil - 2r \leq m - r$, so $b, d \geq r$ as $g \in I^{(m)}$. Consequently, $(a - \alpha) + (b - \beta) = 0 + (a + b + 2r - \left\lceil \frac{3m}{2} \right\rceil) \geq m + 2r - \left\lceil \frac{3m}{2} \right\rceil = r + (m + r - \left\lceil \frac{3m}{2} \right\rceil) \geq r$, $(a - \alpha) + (d - \delta) = d \geq r$, $(b - \beta) + (d - \delta) \geq d \geq r$, and $(a - \alpha) + (b - \beta) + (d - \delta) = b + d - (\left\lceil \frac{3m}{2} \right\rceil - 2r - a) = 2r + (a + b + d - \left\lceil \frac{3m}{2} \right\rceil) \geq 2r$. Therefore, $(a - \alpha, b - \beta, 0, d - \delta) \in \mathcal{S}_r$. But $\alpha + \beta + \delta = \left\lceil \frac{3m}{2} \right\rceil - 2r$, so $g \in \mathcal{M}^{\left\lceil \frac{3m}{2} \right\rceil - 2r} I^r$.

(a.ii) If $a \geq \alpha = \left\lceil \frac{3m}{2} \right\rceil - 2r$, then $(a - \alpha) + (b - \beta) = a + b - \alpha = r + (a + b - r) - (\left\lceil \frac{3m}{2} \right\rceil - 2r) \geq r + \big((m - r) - (\left\lceil \frac{3m}{2} \right\rceil - 2r)\big) \geq r$, $(a - \alpha) + (d - \delta) = a + d - \alpha \geq r + (m - r) - (\left\lceil \frac{3m}{2} \right\rceil - 2r) \geq r$, $(b - \beta) + (d - \delta) = b + d \geq m \geq r$, and $(a - \alpha) + (b - \beta) + (d - \delta) = a + b + d - \alpha \geq \left\lceil \frac{3m}{2} \right\rceil - (\left\lceil \frac{3m}{2} \right\rceil - 2r) = 2r$. Therefore, $(a - \alpha, b - \beta, 0, d - \delta) \in \mathcal{S}_r$. But $\alpha + \beta + \delta = \left\lceil \frac{3m}{2} \right\rceil - 2r$, so $g \in \mathcal{M}^{\left\lceil \frac{3m}{2} \right\rceil - 2r} I^r$.

(b) Now consider the case that $r < \left\lceil \frac{m}{2} \right\rceil$, which implies $0 < r < m - r$. By symmetry, we may assume that $a = \min(a, b, d)$.

(b.i) If $a \geq r$, then $b, d \geq r$ and we may take $\alpha = a - r$, $\beta = b - r$, and $\delta = d$. Then $(a - \alpha) + (b - \beta) = 2r$, $(a - \alpha) + (d - \delta) = r$, $(b - \beta) + (d - \delta) = r$, and $(a - \alpha) + (b - \beta) + (d - \delta) = 2r$, so $(a - \alpha, b - \beta, 0, d - \delta) \in \mathcal{S}_r$. Also, $\alpha + \beta + \delta = a + b + d - 2r \geq \left\lceil \frac{3m}{2} \right\rceil - 2r$, so $g \in \mathcal{M}^{\left\lceil \frac{3m}{2} \right\rceil - 2r} I^r$.

(b.ii) If $a < r$, then $b, d \geq m - a > m - r > r$ and we may take $\alpha = 0$, $\beta = b - r$, and $\delta = d - (r - a) > 0$ as $a + d \geq m > r$. Then $(a - \alpha) + (b - \beta) = a + b - r \geq m - r > r$, $(a - \alpha) + (d - \delta) = a + (r - a) = r$, $(b - \beta) + (d - \delta) = r + (r - a) > r$, and $(a - \alpha) + (b - \beta) + (d - \delta) = a + r + (r - a) = 2r$, so $(a - \alpha, b - \beta, 0, d - \delta) \in \mathcal{S}_r$. Also, $\alpha + \beta + \delta = 0 + (b - r) + (d + a - r) = a + b + d - 2r \geq \left\lceil \frac{3m}{2} \right\rceil - 2r$, so $g \in \mathcal{M}^{\left\lceil \frac{3m}{2} \right\rceil - 2r} I^r$.

Therefore, we always get $g \in \mathcal{M}^{\left\lceil \frac{3m}{2} \right\rceil - 2r} I^r$, and hence $I^{(m)} \subseteq \mathcal{M}^{\left\lceil \frac{3m}{2} \right\rceil - 2r} I^r$. □

Theorem 2.6.4. *Suppose $n \geq 2$. Let $m, r \in \mathbb{N}$ be such that $r < m - \left\lceil \frac{m-1}{4} \right\rceil$. Then*

$$I^{(m)} \subseteq \mathcal{M}^t I^r \text{ but } I^{(m)} \not\subseteq \mathcal{M}^{t+1} I^r$$

where

- $t = 2m - 2r$ *if* $r \leq \left\lfloor \frac{m}{2} \right\rfloor$ *and*

- $t = \min(2m - 2r, n\left(\left\lceil \frac{3m}{2} \right\rceil - 2r\right))$ *if* $r > \left\lfloor \frac{m}{2} \right\rfloor$.

Proof. Let $g = x^a y^b z^c F^d \in I^{(m)}$. Recall that then $a + b \geq m$ and $\min(a, b) + d \geq m$. Similarly to before, we proceed by looking at the two cases $r \leq \left\lfloor \frac{m}{2} \right\rfloor$ and $r > \left\lfloor \frac{m}{2} \right\rfloor$. Each case will be divided into subcases depending on the magnitude of d.

(a) First, assume that $r \leq \left\lfloor \frac{m}{2} \right\rfloor$.

(a.i) If $d = 0$, then $a, b \geq m$ and we may choose $\alpha = \beta = m - r$ and $\delta = 0$. Then $(a - \alpha) + (b - \beta) = a + b - 2(m - r) = 2r + (a - m) + (b - m) \geq 2r \geq r$ and $\min(a - \alpha, b - \beta) + (d - \delta) = \min(a, b) - (m - r) = r + (\min(a, b) - m) \geq r$, so $(a - \alpha, b - \beta, c, d - \delta) \in \mathcal{S}_r$. Since $\alpha + \beta + n\delta = 2(m - r) \geq (2m - 2r - c)_+$, we get that $g \in \mathcal{M}^{2m-2r} I^r$.

(a.ii) If $0 < d < m - r$, then $a, b > r$. We may choose $\delta = d$ and $\alpha = m - r - d = \beta$. Then $a - \alpha = a + d - m + r \geq r$ and $b - \beta \geq r$, so $(a - \alpha) + (b - \beta) \geq 2r \geq r$, $\min(a - \alpha, b - \beta) + (d - \delta) = \min(a - \alpha, b - \beta) \geq r$, and $(a - \alpha) + (b - \beta) + (d - \delta) = (a - \alpha) + (b - \beta) \geq 2r$, so $(a - \alpha, b - \beta, c, d - \delta) \in \mathcal{S}_r$. Also, $\alpha + \beta + n\delta = 2m - 2r + (n - 2)d \geq 2m - 2r \geq (2m - 2r - c)_+$, so $g \in \mathcal{M}^{2m-2r} I^r$.

(a.iii) Finally, suppose $d \geq m - r$. Since $r \leq \left\lfloor \frac{m}{2} \right\rfloor$, we have $d \geq m - r \geq r, \left\lceil \frac{m}{2} \right\rceil$. Since $a + b \geq m \geq 2r$, one of a and b has to be at least r. Say $b \geq r$. Now choose $\alpha = (a - r)_+$, $\beta = b - r$ and $\delta = d - (r - a)_+ \geq m - r \geq \left\lceil \frac{m}{2} \right\rceil$. Then $a - \alpha = \min(a, r)$, $b - \beta = r$, and $d - \delta = (r - a)_+$ and therefore $(a - \alpha) + (b - \beta) \geq b - \beta = r$, $(a - \alpha) + (d - \delta) = r$, $(b - \beta) + (d - \delta) \geq b - \beta = r$, and $(a - \alpha) + (b - \beta) + (d - \delta) = 2r$, so $(a - \alpha, b - \beta, c, d - \delta) \in \mathcal{S}_r$. Also, if $a < r$, then $\alpha + \beta + n\delta = 0 + (b - r) + n(d + a - r) = (a + b + d - 2r) + (n - 1)\delta \geq (\left\lceil \frac{3m}{2} \right\rceil - 2r) + (n - 1)\left\lceil \frac{m}{2} \right\rceil \geq 2m - 2r \geq (2m - 2r - c)_+$, so $g \in \mathcal{M}^{2m-2r} I^r$. If $a \geq r$, then $\alpha + \beta + n\delta = (a - r) + (b - r) + nd = (a + b - 2r) + nd \geq m - 2r + n\left\lceil \frac{m}{2} \right\rceil \geq 2m - 2r \geq (2m - 2r - c)_+$, so $g \in \mathcal{M}^{2m-2r} I^r$.

Therefore $r \leq \left\lfloor \frac{m}{2} \right\rfloor$ gives $I^{(m)} \subseteq \mathcal{M}^{2m-2r} I^r$.

(b) Now suppose that $r > \left\lfloor \frac{m}{2} \right\rfloor$.

(b.i) and (b.ii) The above proofs for $d = 0$ and $0 < d < m - r$ didn't use the restriction on r, so they still work here. As $2m - 2r \geq \min(2m - 2r, n\left(\left\lceil \frac{3m}{2} \right\rceil - 2r\right))$, we get $g \in \mathcal{M}^{\min(2m-2r, n\left(\left\lceil \frac{3m}{2} \right\rceil - 2r\right))} I^r$.

(b.iii) Now suppose $d \geq m - r$. Since $r > \left\lfloor \frac{m}{2} \right\rfloor$, we get $d \geq m - r \geq \left\lceil \frac{3m}{2} \right\rceil - 2r > 0$. Choose $\alpha = \beta = 0$ and $\delta = \left\lceil \frac{3m}{2} \right\rceil - 2r$. Then $(a - \alpha) + (b - \beta) = a + b \geq r$, $\min(a - \alpha, b - \beta) + (d - \delta) = \min(a, b) + d - (\left\lceil \frac{3m}{2} \right\rceil - 2r) \geq m - (m - r) = r$, and $(a - \alpha) + (b - \beta) + (d - \delta) = a + b + d - (\left\lceil \frac{3m}{2} \right\rceil - 2r) \geq \left\lceil \frac{3m}{2} \right\rceil - (\left\lceil \frac{3m}{2} \right\rceil - 2r) = 2r$, so $(a - \alpha, b - \beta, c, d - \delta) \in \mathcal{S}_r$. As $\alpha + \beta + n\delta = n\left(\left\lceil \frac{3m}{2} \right\rceil - 2r\right) \geq \min(2m - 2r, n\left(\left\lceil \frac{3m}{2} \right\rceil - 2r\right))$, we have $g \in \mathcal{M}^{\min(2m-2r, n\left(\left\lceil \frac{3m}{2} \right\rceil - 2r\right))} I^r$.

Therefore $r \leq \lfloor \frac{m}{2} \rfloor$ gives $I^{(m)} \subseteq \mathcal{M}^{\min(2m-2r, n(\lceil \frac{3m}{2} \rceil - 2r))} I^r$.

This proves that $I^{(m)} \subseteq \mathcal{M}^t I^r$ where $t = 2m - 2r$ if $r \leq \lfloor \frac{m}{2} \rfloor$ and $t = \min(2m - 2r, n\left(\lceil \frac{3m}{2} \rceil - 2r\right))$ if $r > \lfloor \frac{m}{2} \rfloor$.

To see that $I^{(m)} \nsubseteq \mathcal{M}^{t+1} I^r$ in both cases, it suffices to exhibit one element in $I^{(m)}$ that is not contained in $\mathcal{M}^{t+1} I^r$.

First note that for an arbitrary $r < m - \lceil \frac{m-1}{4} \rceil$, we have $g = x^m y^m \in I^{(m)}$. For g to be in $\mathcal{M}^{2m-2r+1} I^r$, we need $\alpha, \beta \leq m$ and $\delta = 0$ such that $\alpha + \beta + n\delta = \alpha + \beta \geq (2m - 2r + 1)_+ = 2m - 2r + 1 > 0$ since $m \geq r$. Say, $\alpha + \beta = 2m - 2r + 1$. Then the conditions that $r \leq (a - \alpha) + (d - \delta) = m - \alpha$ and likewise $r \leq m - \beta$ imply that $m - r \geq \alpha, \beta$ and hence $2(m - r) \geq \alpha + \beta = 2m - 2r + 1$, a contradiction. Therefore $g \notin \mathcal{M}^{2m-2r+1} I^r$ and hence $I^{(m)} \nsubseteq \mathcal{M}^{2m-2r+1} I^r$ for all $r < m - \lceil \frac{m-1}{4} \rceil$.

Now, to see that $I^{(m)} \nsubseteq \mathcal{M}^{\min(2m-2r, n(\lceil \frac{3m}{2} \rceil - 2r))+1} I^r$ for $r > \lfloor \frac{m}{2} \rfloor$, note that we're done by the preceding paragraph if $\min(2m - 2r, n\left(\lceil \frac{3m}{2} \rceil - 2r\right)) = 2m - 2r$. If $2m - 2r > n\left(\lceil \frac{3m}{2} \rceil - 2r\right)$, consider $g = x^{\lceil \frac{m}{2} \rceil} y^{\lfloor \frac{m}{2} \rfloor} F^{\lceil \frac{m}{2} \rceil}$. Then $a + b + d = 2r + (\lceil \frac{3m}{2} \rceil - 2r)$, so we cannot choose α, β, δ such that $\alpha + \beta + n\delta \geq n\left(\lceil \frac{3m}{2} \rceil - 2r\right) + 1$ and still have $(a - \alpha, b - \beta, c, d - \delta) \in \mathcal{S}_r$. Therefore $g \notin \mathcal{M}^{n(\lceil \frac{3m}{2} \rceil - 2r)+1} I^r$ if $r > \lfloor \frac{m}{2} \rfloor$ and $2m - 2r > n\left(\lceil \frac{3m}{2} \rceil - 2r\right)$.

This concludes the proof. $\square$

2.7 Conjectures

In chapter 3 of [1], Bocci, Cooper and Harbourne pose some conjectures that have only been verified in a few sporadic cases so far. In the paper, the authors discuss those conjectures for star configurations and for points on a smooth plane conic, amongst other cases.

Here we will verify them for our ideal I defining a nearly-complete intersection.

Recall that for a nontrivial homogeneous ideal J, $\alpha(J)$ is defined to be the least degree t such that $J_t \neq 0$. In particular, $\alpha(J)$ is the least degree of a nonzero form in J.

Lemma 2.7.1. *For $m, r \in \mathbb{N}$, we have*

- $\alpha(I) = 2$,
- $\alpha(I^r) = 2r$, *and*
- $\alpha(I^{(m)}) = \lceil \frac{3m}{2} \rceil$ *if* $n = 1$ *and* $2m$ *if* $n \geq 2$.

Proof. Since $I = (xy, xF, yF)$ and F has degree $n \geq 1$, we immediately get $\alpha(I) = 2$. Moreover, I^r has generators of degree $r\alpha(I)$ and no generator of lesser degree, hence $\alpha(I^r) = r\alpha(I)$ so $\alpha(I^r) = 2r$.

To find $\alpha(I^{(m)})$, recall that $x^m y^m \in I^{(m)}$ for any $n \geq 1$, so $\alpha(I^{(m)}) \leq 2m$. Also, for $n = 1$, we have $x^{\lceil \frac{m}{2} \rceil} y^{\lceil \frac{m}{2} \rceil} F^{\lfloor \frac{m}{2} \rfloor} = x^{\lceil \frac{m}{2} \rceil} y^{\lceil \frac{m}{2} \rceil} z^{\lfloor \frac{m}{2} \rfloor} \in I^{(m)}$, which has degree $\lceil \frac{3m}{2} \rceil \leq 2m$. Therefore, $\alpha(I^{(m)}) \leq 2m$ if $n \geq 2$ and $\alpha(I^{(m)}) \leq \lceil \frac{3m}{2} \rceil$ if $n = 1$.

We will show that we also have the other inequalities, i.e. $\alpha(I^{(m)}) \geq 2m$ if $n \geq 2$ and $\alpha(I^{(m)}) \geq \lceil \frac{3m}{2} \rceil$ if $n = 1$.
Consider an element $g = x^a y^b F^d \in I^{(m)}$. If $n = 1$, then $a + b \geq m$ and $\min(a, b) + d \geq m$ imply that $2(a + b + d) \geq 3m$, so $\deg(g) = a + b + nd = a + b + d \geq \lceil \frac{3m}{2} \rceil$, hence $\alpha(I^{(m)}) \geq \lceil \frac{3m}{2} \rceil$ as g was arbitrary. If $n \geq 2$, then $\deg g = a + b + nd \geq 2m$ if $d \geq m$, so we may assume that $d \leq m$. Then $a, b \geq m - d \geq 0$ as $\min(a, b) + d \geq m$. Thus $\deg(g) = a + b + nd \geq (m - d) + (m - d) + nd = 2m + (n - 2)d \geq 2m$ and hence $\alpha(I^{(m)}) \geq 2m$ as g was arbitrary. $\square$

Chapter 1.4 and the above complete description for conditions on m, r, and t such that $I^{(m)} \subseteq \mathcal{M}^t I^r$ show that the following conjectures (3.1 to 3.6 in [1]) hold for our ideal I.

1. (Conjecture 3.1) $I^{(2r)} \subseteq \mathcal{M}^r I^r$ for all $r \in \mathbb{N}$.

   This was shown in Proposition 2.5.8.

2. (Conjecture 3.2) $I^{(2r-1)} \subseteq I^r$ for all $r \in \mathbb{N}$.

   Since $\mathcal{M}^{r-1} I^r \subseteq I^r$, this was also shown in Proposition 2.5.8.

3. (Conjecture 3.3) $I^{(m)} \subseteq I^r$ holds whenever $\frac{m}{r} > \frac{2\alpha(I)}{\alpha(I)+1}$.

By Lemma 2.7.1, $\frac{2\alpha(I)}{\alpha(I)+1} = \frac{4}{3}$, which is $\rho(I)$. By definition of the resurgence, $I^{(m)} \subseteq I^r$ holds whenever $\frac{m}{r} > \rho(I)$.

4. (Conjecture 3.4) $I^{(2r-1)} \subseteq \mathcal{M}^{r-1}I^r$ for all $r \in \mathbb{N}$.

   This is part of Proposition 2.5.8.

5. (Conjecture 3.5) $\alpha(I^{(2r-1)}) \geq r\alpha(I) + r - 1$ for all $r \in \mathbb{N}$.

   Also by Lemma 2.7.1, $\alpha(I^{(2r-1)}) = \left\lceil \frac{3(2r-1)}{2} \right\rceil = 3r - 1 \geq 3r - 1 = r\alpha(I) + r - 1$ if $n = 1$ and $\alpha(I^{(2r-1)}) = 2(2r-1) \geq 3r - 1 = r\alpha(I) + r - 1$ if $n \geq 2$.

6. (Conjecture 3.6) $\frac{\alpha(I^{(m)})+1}{m+1} \leq \frac{\alpha(I^{(r)})}{r}$ for all $r, m \in \mathbb{N}$.

   By Lemma 2.7.1, $\frac{\alpha(I^{(m)})+1}{m+1} = \frac{\left\lceil \frac{3m}{2} \right\rceil + 1}{m+1} \leq 2 = \frac{2r}{r} = \frac{\alpha(I^{(r)})}{r}$ if $n = 1$ and $\frac{\alpha(I^{(m)})+1}{m+1} = \frac{2m+1}{m+1} \leq 2 = \frac{2r}{r} = \frac{\alpha(I^{(r)})}{r}$ if $n \geq 2$.

To show that conjectures 3.7 to 3.9 in [1] also hold for our ideal, we need the next two lemmas.

Lemma 2.7.2. *As in Lemma 2.6.1, we can describe $\mathcal{M}^t I^{(m)}$ using our vector space results:*

$$\mathcal{M}^t I^{(m)} = \langle x^a y^b z^c F^d | c < n,\ \exists\, 0 \leq \alpha \leq a, 0 \leq \beta \leq b, 0 \leq \delta \leq d$$

$$\textit{such that } (a-\alpha) + (b-\beta) \geq m, \min(a-\alpha, b-\beta) + (d-\delta) \geq m, \textit{ and}$$

$$\alpha + \beta + n\delta \geq (t-c)_+ \rangle$$

for all $m, t \in \mathbb{N}$.

Proof. We employ the same method we used to prove Lemma 2.6.1. Let

$$\mathcal{P} = \left\{ (a,b,c,d) \in \mathbb{N}_0^4 \;\middle|\; c < n, a + b \geq m, \min(a,b) + d \geq m \right\}$$

and define

$$J := \langle x^a y^b z^c F^d | \; \exists\, \alpha \leq a, \beta \leq b, \delta \leq d \ni (a-\alpha, b-\beta, c, d-\delta) \in \mathcal{P} \text{ and } \alpha+\beta+n\delta \geq (t-c)_+ \rangle.$$

We will show that J is an ideal and that $\mathcal{M}^t I^{(m)} \supseteq J \supseteq \mathcal{M}^t I^{(m)}$.

To see that J is an ideal in R, it suffices to show that for any basis element $g = x^a y^b z^c F^d$ of J, the polynomials xg, yg, and zg are also in J. Since $g \in J$, there exist α, β, and δ such that $(a-\alpha, b-\beta, c, d-\delta) \in \mathcal{P}$ and $\alpha + \beta + n\delta \geq (t-c)_+$. But then $(a+1-\alpha, b-\beta, c, d-\delta)$ and $(a-\alpha, b+1-\beta, c, d-\delta)$ are also in $\mathcal{P}$, so xg and yg are in J. If $c < n-1$, then $c+1 \leq n-1$ and $\alpha+\beta+n\delta \geq (t-c)_+ \geq (t-(c+1))_+$, so $(a-\alpha, b-\beta, c+1, d-\delta) \in \mathcal{P}$ and $zg \in J$. Finally, if $c = n-1$, then $zg = x^a y^b z^n F^d$. By definition of F, there exists a polynomial L in z of degree at most $n-1$ with coefficients in $K[x,y]$ such that $F = L - z^n$, and L has the form $L = \sum_{i=0}^{n-1} (\eta_i x^{n-i} + \eta_i' y^{n-i}) z^i$ for some $\eta_i, \eta_i' \in K$. So $z^n = L - F$ and consequently $zg = x^a y^b F^d (L-F) = \sum_{i=0}^{n-1} \eta_i x^{a+n-i} y^b z^i F^d + \sum_{i=0}^{n-1} \eta_i' x^a y^{b+n-i} z^i F^d - x^a y^b F^{d+1}$. Now, for all i, we have $((a+n-i)-(\alpha+n-i), b-\beta, i, d) \in \mathcal{P}$ and $(\alpha+n-i)+\beta+n\delta \geq (t-(n-1))_+ + (n-i) = \max(n-i, t+1-i) \geq (t-i)_+$, so $\sum_{i=0}^{n-1} \eta_i x^{a+n-i} y^b z^i F^d \in J$. Likewise $(a-\alpha, (b+n-i)-(\beta+n-i), i, d) \in \mathcal{P}$ and $\alpha + (\beta+n-i) + n\delta \geq (t-i)_+$, so $\sum_{i=0}^{n-1} \eta_i' x^a y^{b+n-i} z^i F^d \in J$. Finally, $(a-\alpha, b-\beta, c, d-\delta) \in \mathcal{P}$ with $\alpha+\beta+n\delta \geq (t-(n-1))_+$ also means that $(a-\alpha, b-\beta, 0, (d+1)-(\delta+1)) \in \mathcal{P}$ with $\alpha+\beta+n(\delta+1) \geq (t-(n-1))_+ + n = \max(n, t+1) \geq (t-0)_+$, so $x^a y^b F^{d+1} \in J$.

Therefore, $zg \in J$ and J is an ideal.

Let $j = x^a y^b z^c F^d \in J$ be a basis element and choose α, β, and δ accordingly. Then by Proposition 2.2.4 $x^{a-\alpha} y^{b-\beta} F^{d-\delta} \in I^{(m)}$ and $x^\alpha y^\beta z^c F^\delta \in \mathcal{M}^t$, so $j = x^\alpha y^\beta z^c F^\delta \cdot x^{a-\alpha} y^{b-\beta} F^{d-\delta} \in \mathcal{M}^t I^{(m)}$. Therefore, $\mathcal{M}^t I^{(m)} \supseteq J$.

Let $h \in \mathcal{M}^t I^{(m)}$ be of the form $h = x^a y^b z^c F^d \cdot h'$, where $(a,b,c,d) \in \mathcal{P}$ and $h' \in \mathcal{M}^t$, $h' = x^{t_1} y^{t_2} z^{t_3}$ for some $t_1, t_2, t_3 \in \mathbb{N}_0$ with $t_1 + t_2 + t_3 = t$. Thus $h = x^{a+t_1} y^{b+t_2} z^{c+t_3} F^d$. The set of all such h generates $\mathcal{M}^t I^{(m)}$. We will argue by induction on $c + t_3$. If $c + t_3 < n$, then letting $\alpha = t_1, \beta = t_2$, and $\delta = 0$ gives $\alpha+\beta+n\delta = t_1+t_2 = t-t_3 \geq (t-(c+t_3))_+$ and $(a,b,c,d) \in \mathcal{P}$, so $h \in J$. Now assume that $h \in J$ as long as $c+t_3 < s$ for some $s \geq n$. Say $c+t_3 = s = qn+p$, where $0 \leq p < n$. Then, like above, $h = x^{a+t_1} y^{b+t_2} z^s F^d = x^{a+t_1} y^{b+t_2} z^p F^d (L-F)^q$. By definition of L, all summands of $(L-F)^q = \sum_{i=0}^{q} \binom{q}{i} L^i F^{q-i}$ other than F^q are going to be of degree less than qn in z, and so by induction, all those respective summands of $h = x^{a+t_1} y^{b+t_2} z^p F^d \left(\sum_{i=0}^{q} \binom{q}{i} L^i F^{q-i} \right)$ are going to be in J. Hence it suffices to show that the remaining summand $x^{a+t_1} y^{b+t_2} z^p F^{d+q}$ is in J. But $(a,b,p,d) \in \mathcal{P}$ and choosing $\alpha = t_1, \beta = t_2$, and $\delta = q$ gives $\alpha + \beta + n\delta = t_1 + t_2 + qn =$

$t_1 + t_2 + (c + t_3 - p) = t + c - p \geq (t-p)_+$, so $h \in J$.

Therefore, $J \supseteq \mathcal{M}^t I^{(m)}$ and the proof is complete. $\square$

In addition to our complete description of $I^{(m)} \subseteq \mathcal{M}^t I^r$, we can also give the following containment criterion.

Lemma 2.7.3. *We have $I^{(m+t)} \subseteq \mathcal{M}^t I^{(m)}$ for all $m, t \in \mathbb{N}$.*

Proof. Let $x^a y^b z^c F^d$ be a (vector space) generator of $I^{(m+t)}$, so $c < n$, $a + b \geq m + t$, and $\min(a, b) + d \geq m + t$. We may assume that $c = 0$ and $a \leq b$. We will show that there exist $0 \leq \alpha \leq a$, $0 \leq \beta \leq b$, and $0 \leq \delta \leq d$ such that $x^\alpha y^\beta F^\delta \in \mathcal{M}^t$ and $x^{a-\alpha} y^{b-\beta} F^{d-\delta} \in I^{(m)}$. Consider three cases, depending on the magnitude of d.

(a) $d = 0$. Then $a, b \geq m + t$. Take $\alpha = \beta = t$ and $\delta = 0$. Then $\alpha + \beta + n\delta = 2t > (t - c)_+$, so $x^\alpha y^\beta F^\delta \in \mathcal{M}^t$. Also, $(a - \alpha) + (b - \beta) \geq 2m$ and $\min(a - \alpha, b - \beta) + (d - \delta) \geq m$, hence $x^{a-\alpha} y^{b-\beta} F^{d-\delta} \in I^{(m)}$. Hence $x^a y^b F^d \in \mathcal{M}^t I^{(m)}$.

(b) $0 < d < m$. Let $\alpha = (a - m)_+$, $\beta = 0$, and $\delta = d - (m - a)_+$. Then $\alpha + \beta + n\delta = n(d+a-m) \geq nt \geq (t-c)_+$ if $a < m$, and $\alpha+\beta+n\delta = (a-m)+nd = (a+d-m)+(n-1)d \geq a + d - m \geq t = (t - c)_+$ if $a \geq m$, and therefore $x^\alpha y^\beta F^\delta \in \mathcal{M}^t$. Moreover, $(a - \alpha) + (b - \beta) = a + b - (a - m)_+ \geq m$, $(a - \alpha) + (d - \delta) = m$, and $(b - \beta) + (d - \delta) = b + (m - a)_+ \geq m$ as $a \leq b$. Therefore, $x^{a-\alpha} y^{b-\beta} F^{d-\delta} \in I^{(m)}$ and hence $x^a y^b F^d \in \mathcal{M}^t I^{(m)}$.

(c) $d \geq m$. Let $\alpha = a - \min(a, m)$, $\beta = b - (m - a)_+$, and $\delta = d - m$. Then $\alpha + \beta + n\delta = a + b - m + n(d - m) \geq a + b - m \geq t = (t - c)_+$, thus $x^\alpha y^\beta F^\delta \in \mathcal{M}^t$. Moreover, $(a - \alpha) + (b - \beta) = \min(a, m) + (m - a)_+ = m$ and $\min(a - \alpha, b - \beta) + (d - \delta) \geq \delta = m$, so $x^{a-\alpha} y^{b-\beta} F^{d-\delta} \in I^{(m)}$. Therefore $x^a y^b F^d \in \mathcal{M}^t I^{(m)}$.

This means that $I^{(m+t)} \subseteq \mathcal{M}^t I^{(m)}$, which completes the proof. $\square$

Conjectures 3.7, 3.8, and 3.9 as well as Proposition 2.3 in [1] assert and prove, respectively, that certain containments should or do hold. Specifically, since $\mathbb{P}^N = \mathbb{P}^2$ in our paper, we have $I^{(tm+t)} \subseteq \mathcal{M}^t \left(I^{(m)}\right)^t$ and $I^{(tm+t-1)} \subseteq \mathcal{M}^{t-1} \left(I^{(m)}\right)^t$, and thus $I^{(tm+t-1)} \subseteq \left(I^{(m)}\right)^t$, for all m and t in $\mathbb{N}$.

We verify these containments for our ideal I, using Lemma 2.7.3 and a similar approach to the one we took to prove Lemma 2.7.2.

Proposition 2.7.4. *([1] Conjectures 3.7, 3.8) Given $m \in \mathbb{N}$, we have $I^{(tm+t)} \subseteq \mathcal{M}^t \left(I^{(m)}\right)^t$ for all $t \in \mathbb{N}$.*

Proof. If m is odd, then $m+1$ is even, so Corollary 2.5.2 gives that $I^{(tm+t)} = \left(I^{(m+1)}\right)^t$. By Lemma 2.7.3, we have $I^{(m+1)} \subseteq \mathcal{M}I^{(m)}$, so $I^{(tm+t)} = \left(I^{(m+1)}\right)^t \subseteq \left(\mathcal{M}I^{(m)}\right)^t = \mathcal{M}^t \left(I^{(m)}\right)^t$ as desired.

If m is even, we can use Corollary 2.5.2 to write $\left(I^{(m)}\right)^t$ as $I^{(tm)}$. By Lemma 2.7.3, we then have $I^{(tm+t)} \subseteq \mathcal{M}^t I^{(tm)} = \mathcal{M}^t \left(I^{(m)}\right)^t$ as desired. $\square$

Rounding off this section on conjectures in [1] is the following lemma.

Lemma 2.7.5. *([1] Conjecture 3.9 or Proposition 2.3) For all $t, m \in \mathbb{N}$, we have $I^{(tm+t-1)} \subseteq \mathcal{M}^{t-1} \left(I^{(m)}\right)^t \subseteq \left(I^{(m)}\right)^t$.*

Proof. The containment $\mathcal{M}^{t-1} \left(I^{(m)}\right)^t \subseteq \left(I^{(m)}\right)^t$ is clear for all $t, m \in \mathbb{N}$. Given $t, m \in \mathbb{N}$, we will show $(*) : I^{(tm+t-1)} \subseteq \mathcal{M}^{t-1} \left(I^{(m)}\right)^t$.

First, notice that if either of t or m is 1, we get previously obtained results. If $t = 1$, then $(*)$ becomes $I^{(m)} \subseteq I^{(m)}$, which is trivially true. If $m = 1$, then $(*)$ becomes $I^{(2t-1)} \subseteq \mathcal{M}^{t-1} I^t$, which we proved to be true in Proposition 2.5.8. Therefore, we may assume that $m, t \geq 2$. We differentiate two cases, m being even and m being odd. In the latter case, we will consider four subcases, divided by the corresponding size of d.

(a) Assume that m is even. Then $\left(I^{(m)}\right)^t = I^{(tm)}$ by Corollary 2.5.2. Thus by Lemma 2.7.2, $I^{(tm+t-1)} \subseteq \mathcal{M}^{t-1} I^{(tm)} = \mathcal{M}^{t-1} \left(I^{(m)}\right)^t$ as desired.

(b) Now suppose m is odd. If m is odd, then $m-1$ is even and by Corollary 2.5.2 $\left(I^{(m)}\right)^t = \left(I^{(m-1)} I\right)^t = I^{(tm-t)} I^t$, so we want to show that $I^{(tm+t-1)} \subseteq \mathcal{M}^{t-1} I^t I^{(tm-t)}$.

Let $x^a y^b z^c F^d$ be a (vector space) generator of $I^{(tm+t-1)}$, so $c < n$, $a + b \geq tm + t - 1$, and $\min(a, b) + d \geq tm + t - 1$. We may assume that $c = 0$ and $a \leq b$. We will show that there exist $0 \leq \alpha_1, \alpha_2, \alpha_3 \leq a$, $0 \leq \beta_1, \beta_2, \beta_3 \leq b$, and $0 \leq \delta_1, \delta_2, \delta_3 \leq d$ such that $\sum_{i=1}^3 \alpha_i = a$, $\sum_{i=1}^3 \beta_i = b$, $\sum_{i=1}^3 \delta_i = d$, as well as $x^{\alpha_1} y^{\beta_1} F^{\delta_1} \in \mathcal{M}^{t-1}$, $x^{\alpha_2} y^{\beta_2} F^{\delta_2} \in I^t$, and $x^{\alpha_3} y^{\beta_3} F^{\delta_3} \in I^{(tm-t)}$. The latter three conditions can be expressed as

$(\star)\ :\ \alpha_1+\beta_1+n\delta_1 \geq t-1=(t-1-c)_+$,
$(\star\star)\ :\ \alpha_2+\beta_2 \geq t$, $\min(\alpha_2,\beta_2)+\delta_2 \geq t$, $\alpha_2+\beta_2+\delta_2 \geq 2t$, and
$(\star\star\star)\ :\ \alpha_3+\beta_3 \geq tm-t$, $\min(\alpha_3,\beta_3)+\delta_3 \geq tm-t$.

Now look at the four subcases: (b.i) Say $d=0=\delta_i$ for $i=1,2,3$. Then $a,b \geq tm+t-1$. Let $\alpha_1=a-tm$, $\beta_1=b-tm$, $\alpha_2=\beta_2=t$, and $\alpha_3=\beta_3=tm-t$. Then $(\star\star)$ and $(\star\star\star)$ are clearly satisfied. For $(\star)$, note that $\alpha_1+\beta_1+n\delta_1=a+b-2tm \geq 2(t-1)>t-1$. Hence $x^ay^bF^d \in \mathcal{M}^{t-1}I^tI^{(tm-t)}$.

(b.ii) Suppose $0<d\leq t$. Then $a,b\geq tm-1$. Let $\alpha_1=\left\lfloor\frac{t-1}{2}\right\rfloor$, $\alpha_2=a-\alpha_1-\alpha_3$, $\alpha_3=\beta_3=tm-t-1$, $\beta_1=\left\lceil\frac{t-1}{2}\right\rceil$, $\beta_2=b-\beta_1-\beta_3$, $\delta_1=0$, $\delta_2=d-1$, and $\delta_3=1$. Then it is easy to see that $(\star)$ and $(\star\star\star)$ hold. To see $(\star\star)$, notice that $\alpha_2+\beta_2=a+b+2t+2-2tm-(t-1)=(a+1-tm)+(b+1-tm)+t+1\geq t$, $\alpha_2+\delta_2=(a+d-tm)+t-\left\lfloor\frac{t-1}{2}\right\rfloor \geq 2t-1-\left\lfloor\frac{t-1}{2}\right\rfloor \geq \frac{3}{2}t-\frac{1}{2}\geq t$ and similarly $\beta_2+\delta_2\geq t$. Finally, since $\alpha_2+\beta_2+\delta_2=a+b+d+2t+2-2tm=(a+1-tm)+(b+d-tm+1)+2t\geq 2t$, $(\star\star)$ holds and $x^ay^bF^d\in\mathcal{M}^{t-1}I^tI^{(tm-t)}$.

(b.iii) Now assume $t<d<tm$. Then either $a<b$ or $a=b$, and our choice of the $\alpha_i,\beta_i,\delta_i$ changes accordingly.

(b.iii.1) If $a<b$, choose $\alpha_1=\beta_1=0$, $\alpha_2=\delta_1=t-1$, $\alpha_3=a+1-t$, $\beta_2=t$, $\beta_3=b-t$, $\delta_2=1$, and $\delta_3=d-t$. Then $(\star)$ and $(\star\star)$ follow immediately. For $(\star\star\star)$, note that $\alpha_3+\beta_3=a+b+1-2t\geq tm-t$ and $\beta_3+\delta_3\geq\alpha_3+\delta_3=a+d+1-2t\geq tm-t$ as $b\geq a+1$. Therefore, $(\star\star\star)$ holds and $x^ay^bF^d\in\mathcal{M}^{t-1}I^tI^{(tm-t)}$.

(b.iii.2) If, however, $a=b$, then $a+b\geq tm+t-1$ and m being odd means that $a=b\geq\frac{t(m+1)}{2}$ and thus actually $2a\geq tm+t$. Choose $\alpha_1=0$, $\alpha_2=t$, $\alpha_3=\beta_3=a-t$, $\beta_1=\delta_2=1$, $\delta_1=t-2$, and $\delta_3=d+1-t$. Again, $(\star)$ and $(\star\star)$ are immediate, whereas $(\star\star\star)$ requires a closer look. But $\alpha_3+\beta_3=2a-2t\geq tm-t$ and $\alpha_3+\delta_3=\beta_3+\delta_3=a+d+1-2t\geq tm-t$, hence $(\star\star\star)$ holds and $x^ay^bF^d\in\mathcal{M}^{t-1}I^tI^{(tm-t)}$.

(b.iv) Finally, let $d\geq tm$. Then we can differentiate further by the relationship between b and tm.

(b.iv.1) Suppose first that $b\geq tm$. Let $\alpha_1=a$, $\alpha_2=\alpha_3=0$, $\beta_1=b-tm$, $\beta_2=\delta_2=t$, $\beta_3=\delta_3=tm-t$, and $\delta_1=d-tm$. Then $(\star\star)$ and $(\star\star\star)$ are clearly satisfied, and $(\star)$ holds because $\alpha_1+\beta_1+n\delta_1=(a+b-tm)+n(d-tm)\geq(t-1)+n(d-tm)\geq t-1$. Therefore $x^ay^bF^d\in\mathcal{M}^{t-1}I^tI^{(tm-t)}$.

(b.iv.2) Now suppose that $tm-t\leq b<tm$. Then choosing $\alpha_1=a+b-tm$, $\alpha_2=tm-b$, $\alpha_3=\beta_1=\delta_1=0$, $\beta_2=b+t-tm$, $\beta_3=\delta_3=tm-t$, and $\delta_2=d+t-tm$ gives $(\star\star\star)$ immediately. Also, $(\star)$ holds because $\alpha_1+\beta_1+n\delta_1=a+b-tm\geq t-1$, and to see $(\star\star)$, note that $\delta_2\geq t$ and $\alpha_2+\beta_2=tm-b+b+t-tm=t$. Again, $x^ay^bF^d\in\mathcal{M}^{t-1}I^tI^{(tm-t)}$.

(b.iv.3) Finally, suppose that $b < tm - t$. Let $\alpha_1 = \beta_1 = 0$, $\alpha_2 = (t - (d + 1 - tm))_+$, $\alpha_3 = a - \alpha_2$, $\beta_2 = b - \beta_3$, $\beta_3 = (tm - t - \alpha_3)_+$, $\delta_1 = t - 1$, $\delta_2 = d + 1 - tm$, and $\delta_3 = tm - t$. Then it is clear that $(\star)$ and $(\star\star\star)$ hold. To see $(\star\star)$, note that $\alpha_2 + \delta_2 \geq t$ by definition and $\alpha_2 + \beta_2 = a + b + t - tm \geq 2t - 1 \geq t$. If $\delta_2 = d + 1 - tm \geq t$, then $\beta_2 + \delta_2 \geq t$. If $\delta_2 < t$, then $\beta_2 + \delta_2 = (a + b - tm + \delta_2) + \delta_2 \geq t - 1 + 2\delta_2 > t$ as $a \leq b < tm - t$. Finally, if $\delta_2 < t$, then $\alpha_2 + (\beta_2 + \delta_2) \geq \alpha_2 + (t - 1 + 2\delta_2) = t + (\alpha_2 + \delta_2) + (\delta_2 - 1) \geq 2t$ by the preceding results, and if $\delta_2 \geq t$, then $(\alpha_2 + \beta_2) + \delta_2 \geq 2t - 1 + t \geq 2t$. Hence $(\star\star)$ holds and $x^a y^b F^d \in \mathcal{M}^{t-1} I^t I^{(tm-t)}$.

Consequently, $I^{(tm+t-1)} \subseteq \mathcal{M}^{t-1} I^t I^{(tm-t)} = \mathcal{M}^{t-1} \left(I^{(m)}\right)^t$ as desired. $\square$

2.8 The Saturation Degree

The saturation degree of a homogeneous ideal J is defined to be the least degree t such that $(J)_s = \left(J^{sat}\right)_s$ for all $s \geq t$. Here, J^{sat} denotes the saturation of the ideal J, i.e. J^{sat} is the smallest ideal such that $J \subseteq J^{sat}$ and $(J^{sat} : \mathcal{M}) = J^{sat}$. If J is a power of a (saturated) ideal L defining a zero-dimensional subscheme, say $J = L^r$, then $J^{sat} = L^{(r)}$. We will use this fact to find the saturation degree of powers of our ideal I.

We start by finding the saturation degree for I^m in the case that $m > 1$, which turns out to be $\operatorname{satdeg}(I^m) = (n+1)m$. We then exhibit two other cases for which $\left(I^{(m)}\right)_t = (I^m)_t$. Afterwards, we will prove that these cases are indeed the only ones for which we have equality.

Theorem 2.8.1. *For $m > 1$ and $n \geq 2$, we have $\left(I^{(m)}\right)_t = (I^m)_t$ when $t \geq (n+1)m$ and $\left(I^{(m)}\right)_t \neq (I^m)_t$ when $t = (n+1)m - 1$.*
In particular, $\operatorname{satdeg}(I^m) = (n+1)m$.

Proof. Note that for all $t \in \mathbb{N}_0$, $\left(I^{(m)}\right)_t \supseteq (I^m)_t$, so we will either show that $\left(I^{(m)}\right)_t \subseteq (I^m)_t$, which thus implies $\left(I^{(m)}\right)_t = (I^m)_t$, or $\left(I^{(m)}\right)_t \nsubseteq (I^m)_t$, which thus implies $\left(I^{(m)}\right)_t \neq (I^m)_t$.

Let $t = (n+1)m + k$ for some $k \geq 0$. It suffices to show that each $g \in \left(I^{(m)}\right)_t$ of the form $g = x^a y^b z^c F^d$ with $c < n$ is in $(I^m)_t$. So take $g = x^a y^b z^c F^d \in \left(I^{(m)}\right)_t$, i.e. $a + b \geq m$, $\min(a, b) + d \geq m$, and $\deg g = a + b + c + nd = t$. We need to show that $a + b + d \geq 2m$. Assume that not, so $2m > a+b+d$. Then $2m > a+b+d = t-(c+(n-1)d) = (n+1)m+k-(c+(n-1)d)$, so $c+(n-1)d > (n-1)m+k$. But $n-1 \geq c$, so $(n-1)(d+1) = n-1+(n-1)d > (n-1)m+k$ and hence $d+1 > m+\frac{k}{n-1} \geq m$. But then $d \geq m$ and we get $2m > (a+b)+d \geq m+m = 2m$, a contradiction. Therefore, $g \in (I^m)_t$ and hence $\left(I^{(m)}\right)_t \subseteq (I^m)_t$ as desired.

For $t = (n+1)m-1$, it is easy to see that $g = x^{\left\lfloor \frac{m}{2} \right\rfloor} y^{\left\lceil \frac{m}{2} \right\rceil} z^{n-1} F^{m-1}$ is an element of $I^{(m)}$. Also, $g \notin I^m$ by Lemma 2.2.8, as $a+b+d = \left\lfloor \frac{m}{2} \right\rfloor + \left\lceil \frac{m}{2} \right\rceil + m - 1 = 2m-1 < 2m$. But $\deg g = m+n-1+n(m-1) = (n+1)m-1 = t$, so $\left(I^{(m)}\right)_t \neq (I^m)_t$. Since the saturation degree of I is defined to be the smallest degree t such that $(I^m)_s = \left(I^{(m)}\right)_s$ for all $s \geq t$, it follows that $\operatorname{satdeg}(I) = (n+1)m$. □

Moreover, we can give additional values of t for which $\left(I^{(m)}\right)_t = (I^m)_t$.

Proposition 2.8.2. *For $n \geq 2$, we have $\left(I^{(m)}\right)_t = (I^m)_t$ also when one of the following holds.*

1. $m = 1$ *and* $t \in \mathbb{N}_0$

2. $m > 1$ *and* $t < \max(2m+1, 2m+n-2)$

Proof. 1. If $m = 1$, then $I^{(1)} = I$, so $\left(I^{(1)}\right)_t = (I)_t$ for all $t \in \mathbb{N}$.

Suppose now that $m > 1$.

2. First, notice that $\alpha(I^{(m)}) = \alpha(I^m) = 2m$, so $\left(I^{(m)}\right)_t = 0 = (I^m)_t$ for all $t < 2m$. Also, both $(I^m)_{2m}$ and $\left(I^{(m)}\right)_{2m}$ are generated by $x^m y^m$ (as a K-vector space), hence $(I^m)_{2m} = \left(I^{(m)}\right)_{2m}$. So, for $n = 2$ or 3, the case $t < \max(2m+1, 2m+n-2) = 2m+1$ is done. Assume that $n > 3$. Then $\max(2m+1, 2m+n-2) = 2m+n-2$. Let $t = 2m+k$ where $1 \leq k \leq n-3$ and take $g = x^a y^b z^c F^d \in \left(I^{(m)}\right)_t$, so $a+b \geq m$, $\min(a,b)+d \geq m$, and $\deg g = a+b+c+nd = t$. If $g \notin (I^m)_t$, then $2m > a+b+d \geq (m-d)+(m-d)+d = 2m-d$ since $a, b \geq m-d$, and hence $d \geq 1$. But then $2m-d \leq a+b+d = t-(c+(n-1)d) = 2m+k-(c+(n-1)d)$, so $c+(n-2)d \leq k \leq n-3$, which is a contradiction as $c \geq 0$, $d \geq 1$, and $n-3 > 0$. Therefore, $g \in (I^m)_t$ and $\left(I^{(m)}\right)_t \subseteq (I^m)_t$ as desired.

We will show in Lemma 2.8.3 that $\left(I^{(m)}\right)_t \nsubseteq (I^m)_t$ for $\max(2m+1, 2m+n-2) \leq t \leq (n+1)m-1$, which shows that $(I^m)_t = \left(I^{(m)}\right)_t$ exactly for the m and t listed in the statement of this proposition and Theorem 2.8.1. □

We now finish completely describing the containment of $\left(I^{(m)}\right)_t$ in $(I^m)_t$.

Lemma 2.8.3. *Let* $m > 1$. *For* $n \geq 2$, *we have* $\left(I^{(m)}\right)_t \neq (I^m)_t$ *when* $\max(2m+1, 2m+n-2) \leq t \leq (n+1)m - 1$.

Proof. We will show that $\left(I^{(m)}\right)_t \nsubseteq (I^m)_t$ for $\max(2m+1, 2m+n-2) \leq t \leq (n+1)m-1$ by finding an element $g \in \left(I^{(m)}\right)_t \setminus (I^m)_t$. We proceed by considering several different cases, depending on the magnitudes of n and m. Together, these cases cover all possible combinations of n, m, and t under consideration. We will use Lemma 2.2.8 extensively.

(a) First suppose that $n = 2$. Then $\max(2m+1, 2m+n-2) = 2m+1$, so the range for t is $2m+1 \leq t \leq 3m-1$. Write $t = 2m+k$ for some $1 \leq k \leq m-1$. Set $i = (k+m) \mod 2$ and consider the element $g = x^{\lceil \frac{m}{2} \rceil} y^{\lfloor \frac{m}{2} \rfloor} z^i F^{\lceil \frac{m}{2} \rceil + j}$, where $0 \leq j \leq \lfloor \frac{m}{2} \rfloor - 1$. Since $i \in \{0,1\}$, $\lceil \frac{m}{2} \rceil + \lfloor \frac{m}{2} \rfloor = m$, and $\lfloor \frac{m}{2} \rfloor + \lceil \frac{m}{2} \rceil + j = m + j \geq m$, we have that $g \in I^{(m)}$. Moreover, $\deg g = \lfloor \frac{m}{2} \rfloor + \lceil \frac{m}{2} \rceil + i + 2\left(\lceil \frac{m}{2} \rceil + j\right)$, which is $2m+2j+i$ if m is even and $2m+2j+i+1$ if m is odd. Note that we can write $k = 2\lfloor \frac{k}{2} \rfloor + (k \mod 2)$.

(a.i) If m is even, then choosing $j = \lfloor \frac{k}{2} \rfloor \leq \frac{m}{2} - 1$ gives $2j + i = 2\lfloor \frac{k}{2} \rfloor + (k \mod 2) = k$, since $i = (m+k) \mod 2 = k \mod 2$. Hence $\deg g = 2m+k$ as desired.

(a.ii) If m is odd and k even, then $i = (m+k) \mod 2 = 1$, and choosing $j = \frac{k}{2} - 1 \leq \frac{m}{2} - 1$ gives $2j+i+1 = 2\left(\frac{k}{2} - 1\right) + 1 + 1 = k$, and hence $\deg g = 2m+k$.

(a.iii) Finally, if m and k are both odd, then $i = (m+k) \mod 2 = 0$, and choosing $j = \frac{k-1}{2} \leq \frac{m}{2} - 1$ gives $2j+i+1 = 2\left(\frac{k-1}{2}\right) + 1 = k$, and hence $\deg g = 2m+k$ as desired.

Therefore $g \in \left(I^{(m)}\right)_t$. However, $\lfloor \frac{m}{2} \rfloor + \lceil \frac{m}{2} \rceil + \lceil \frac{m}{2} \rceil + j = m + (\lceil \frac{m}{2} \rceil + j) < 2m$ as $j \leq \lfloor \frac{m}{2} \rfloor - 1$. Hence $g \notin I^m$..

(b) Now assume that $n > 2$ but $m = 2$. Then $\max(2m+1, 2m+n-2) = 2m+n-2 = n+2$, so the range for t is $n+2 \leq t \leq 2n+1$. Write $t = n+2+k$ for some $0 \leq k \leq n-1$. Consider the element $g = xyz^kF$. Then $1+1 = 2$, $k < n$, and $\deg g = 1+1+k+n(1) = n+2+k = t$, so $g \in \left(I^{(2)}\right)_t$. However, $1+1+1 = 3 < 4 = 2m$, so $g \notin I^2$.

(c) Now assume that $n > 2$ but $m = 4$. Then $\max(2m+1, 2m+n-2) = 2m+n-2 = n+6$, so the range for t is $n+6 \leq t \leq 4n+3$. We split this interval up into three parts.

(c.i) If $n+6 \leq t \leq 2n+3$, write $t = n+6+k$ for some $0 \leq k \leq n-3$. Consider the element $g = x^3y^3z^kF$. Then $3+3 = 6 > 4$, $3+1 = 4$, and $k < n$, so $g \in I^{(4)}$. Also, $\deg g = 3+3+k+n(1) = n+6+k = t$, so $g \in \left(I^{(4)}\right)_t$. But $3+3+1 = 7 < 8 = 2m$, so $g \notin I^4$.

(c.ii) If $2n+4 \leq t \leq 3n+3$, write $t = 2n+4+k$ for some $0 \leq k \leq n-1$. Consider the element $g = x^2y^2z^kF^2$. Then $2+2=4$ and $k<n$, so $g \in I^{(4)}$. Also, $\deg g = 2+2+k+n(2) = 2n+4+k = t$, so $g \in \left(I^{(4)}\right)_t$. But $2+2+2 = 6 < 8 = 2m$, so $g \notin I^4$.

(c.iii) If $3n+4 \leq t \leq 4n+3$, write $t = 3n+4+k$ for some $0 \leq k \leq n-1$. Consider the element $g = x^2y^2z^kF^3$. Then $2+2=4$, $2+3=5>4$, and $k<n$, so $g \in I^{(4)}$. Also, $\deg g = 2+2+k+n(3) = 3n+4+k = t$, so $g \in \left(I^{(4)}\right)_t$. But $2+2+3 = 7 < 8 = 2m$, so $g \notin I^4$.

(d) Finally, assume that $n > 2$ and $m \neq 2, 4$. Then $\max(2m+1, 2m+n-2) = 2m+n-2$, so the range for t is $2m+n-2 \leq t \leq (n+1)m-1$. We split this interval into the two parts (d.i) $(n+1)m - \left(n\left\lfloor \frac{m}{2}\right\rfloor\right) \leq t \leq (n+1)m-1$ and (d.ii) $2m+n-2 \leq t \leq (2m+n-2) + \left((n-2)\left(\left\lfloor \frac{m}{2}\right\rfloor\right) + n - 3\right)$. Note that $[(n+1)m-1] - [2m+n-2] = nm-m-n+1$, and $\left(n\left\lfloor \frac{m}{2}\right\rfloor\right) + \left((n-2)\left\lfloor \frac{m}{2}\right\rfloor + n-3\right)$ is $nm-m+n-3 > nm-m-n+1$ if m is even, and $nm-m-2 \geq nm-m-n+1$ if m is odd.

Hence we have that $(n+1)m - \left(n\left\lfloor \frac{m}{2}\right\rfloor\right) \leq (2m+n-2) + \left((n-2)\left\lfloor \frac{m}{2}\right\rfloor + n-3\right)$ and that these two parts do indeed cover the entirety of the original interval $2m+n-2 \leq t \leq (n+1)m-1$.

(d.i) Write $t = (n+1)m - k$ for some $k \in \{1, \ldots, n\left\lfloor \frac{m}{2}\right\rfloor\}$. Write $k = np+q$, where $0 \leq q < n$. Then, by choice of k, we have $1 \leq p \leq \left\lfloor \frac{m}{2}\right\rfloor$ if $q=0$, and $0 \leq p \leq \left\lfloor \frac{m}{2}\right\rfloor - 1$ if $q \neq 0$. Define i as 0 if $q = 0$ and $n-q$ if $q \neq 0$. In particular, $i < n$. Also, define j as p if $q=0$ and $p+1$ if $q \neq 0$. In particular, $1 \leq j \leq \left\lfloor \frac{m}{2}\right\rfloor$. Now consider $g = x^{\left\lceil \frac{m}{2}\right\rceil}y^{\left\lfloor \frac{m}{2}\right\rfloor}z^iF^{m-j}$. Then $g \in I^{(m)}$, as $\left\lceil \frac{m}{2}\right\rceil + \left\lfloor \frac{m}{2}\right\rfloor = m$ and $\left\lfloor \frac{m}{2}\right\rfloor + (m-j) = m + (\left\lfloor \frac{m}{2}\right\rfloor - j) \geq m$ by definition of j. However, $g \notin I^m$ as $a+b+d = m+(m-j) = 2m-j < 2m$ by definition of j. Finally, $\deg g = m+i+n(m-j) = (n+1)m-(nj-i) = (n+1)m-k = t$ by definition of i and j. Hence $\left(I^{(m)}\right)_t \nsubseteq (I^m)_t$ as desired.

(d.ii) And finally, let $t = 2m+n-2+k$ for some $k \in \{0, \ldots, (n-2)\left(\left\lfloor \frac{m}{2}\right\rfloor\right) + n-3\}$. Write $k = (n-2)p+q$, where $0 \leq q < n-2$ and $0 \leq p \leq \left\lfloor \frac{m}{2}\right\rfloor$. We distinguish two cases by magnitude of p, either $p \leq \left\lfloor \frac{m}{2}\right\rfloor - 1$ or $p = \left\lfloor \frac{m}{2}\right\rfloor$.

(d.ii.1) Let $p \leq \left\lfloor \frac{m}{2}\right\rfloor - 1$. Consider $g = x^{m-(p+1)}y^{m-(p+1)}z^qF^{p+1}$. Then $g \in I^{(m)}$ as $m-(p+1)+(p+1) = m$ and $2(m-(p+1)) = m+(m-2(p+1)) \geq m$ by choice of p. However, $g \notin I^m$ as $a+b+d = 2(m-(p+1))+(p+1) = 2m-(p+1) < 2m$, again by choice of p. Finally, $\deg g = 2(m-(p+1))+q+n(p+1) = 2m+n-2+((n-2)p+q) = 2m+n-2+k = t$. Therefore, $\left(I^{(m)}\right)_t \nsubseteq (I^m)_t$ as desired.

(d.ii.2) Now take $p = \left\lfloor \frac{m}{2}\right\rfloor$. Then $k = (n-2)\left\lfloor \frac{m}{2}\right\rfloor + q$ for some $0 \leq q \leq n-3$. Then either (d.ii.2.I) $n-2 < \left\lfloor \frac{m}{2}\right\rfloor$ or (d.ii.2.II) $n-2 \geq \left\lfloor \frac{m}{2}\right\rfloor$.

(d.ii.2.I) Assume $n-2 < \left\lfloor \frac{m}{2}\right\rfloor$. Consider $g = x^{\left\lfloor \frac{m}{2}\right\rfloor + n-2+i}y^{\left\lceil \frac{m}{2}\right\rceil}z^qF^{\left\lfloor \frac{m}{2}\right\rfloor}$, where $i = m \bmod 2$. Then $g \in I^{(m)}$ as $\left\lfloor \frac{m}{2}\right\rfloor + n-2+i \geq \left\lceil \frac{m}{2}\right\rceil$ and $\left\lfloor \frac{m}{2}\right\rfloor + n-2+i+\left\lceil \frac{m}{2}\right\rceil > m$. However,

$g \notin I^m$ as $a+b+d = \lfloor \frac{m}{2} \rfloor + n - 2 + i + \lceil \frac{m}{2} \rceil + \lfloor \frac{m}{2} \rfloor = m + (\lfloor \frac{m}{2} \rfloor + n - 2 + i)$, which is strictly less than $2m$ as $n-2+i < \lceil \frac{m}{2} \rceil$. Also, $\deg g = \lfloor \frac{m}{2} \rfloor n - 2 + i + \lceil \frac{m}{2} \rceil + q + n(\lfloor \frac{m}{2} \rfloor) = m + n - 2 + i + q + n\left(\lfloor \frac{m}{2} \rfloor\right)$, which is $2m + n - 2 + (n-2)\left(\frac{m}{2}\right) + q = 2m + n - 2 + k = t$ if m is even and $2m + n - 2 + (n-2)\left(\lfloor \frac{m}{2} \rfloor\right) + q + (i-1) = 2m + n - 2 + k = t$ if m is odd. Therefore, $\left(I^{(m)}\right)_t \nsubseteq (I^m)_t$ as desired.

(d.ii.2.II) Now assume $n - 2 \geq \lfloor \frac{m}{2} \rfloor$. Consider two subcases according to the parity of m.

(d.ii.2.II.1) Suppose m is even. Then $m \geq 6$ by assumption. If $q = 0$, then it is easy to see that $g = x^{\frac{m}{2}} y^{\frac{m}{2}} z^{n-2} F^{\frac{m}{2}} \in I^{(m)}$ but $g \notin I^m$. Also, $\deg g = m + n - 2 + n\left(\frac{m}{2}\right) = 2m + n - 2 + (n-2)\left(\frac{m}{2}\right) = 2m + n - 2 + k = t$ as desired. Likewise, if $q = 1$, then it is easy to see that $g = x^{\frac{m}{2}} y^{\frac{m}{2}} z^{n-1} F^{\frac{m}{2}} \in \left(I^{(m)}\right)_t \setminus (I^m)_t$. For $q \geq 2$, let $g = x^{\frac{m}{2}} y^{\frac{m}{2}} z^{q-2} F^{\frac{m}{2}+1}$. Then is is again easy to see that $g \in I^{(m)}$ but $g \notin I^m$ as $m \geq 6$. Also, $\deg g = m + q - 2 + n\left(\frac{m}{2} + 1\right) = 2m + n - 2 + (n-2)\left(\frac{m}{2}\right) + q = t$. Therefore, $\left(I^{(m)}\right)_t \nsubseteq (I^m)_t$ as desired.

(d.ii.2.II.2) Now let m be odd. If $q = 0$, then it is easy to see that $g = x^{\lceil \frac{m}{2} \rceil} y^{\lceil \frac{m}{2} \rceil} z^{n-2} F^{\lfloor \frac{m}{2} \rfloor} \in I^{(m)}$ but $g \notin I^m$. Also, $\deg g = 2\left(\lceil \frac{m}{2} \rceil\right) + n - 2 + n\left(\lfloor \frac{m}{2} \rfloor\right) = 2m + n - 2 + (n-2)\left(\lfloor \frac{m}{2} \rfloor\right) = t$. If $q \geq 1$, let $g = x^{\lfloor \frac{m}{2} \rfloor} y^{\lceil \frac{m}{2} \rceil} z^{q-1} F^{\lfloor \frac{m}{2} \rfloor + 1}$. Again, it is easy to see that $g \in I^{(m)}$ but $g \notin I^m$. Moreover, $\deg g = m + q - 1 + n\left(\lfloor \frac{m}{2} \rfloor + 1\right) = 2m + n - 2 + (n-2)\left(\lfloor \frac{m}{2} \rfloor\right) + q$. Therefore, $\left(I^{(m)}\right)_t \nsubseteq (I^m)_t$ as desired. $\square$

Knowing the saturation degree of an ideal is interesting in its own right, but it is also a useful tool for computing other values, for example the Castlenuovo-Mumford regularity. The regularity of an ideal plays an important role in computing the ideal's Hilbert series, among other things. We now turn to the regularity for our ideal.

2.9 The Regularity

There are many ways to define the Castelnuovo-Mumford regularity of an ideal.

Here, we make use of the following two facts to compute the regularities of $I^{(m)}$ and I^m.

For a homogeneous ideal J defining a 0-dimensional subscheme of $\mathbb{P}^N$, the regularity of J is known to satisfy $\operatorname{reg}(J) = \max(\operatorname{satdeg}(J), \operatorname{reg}(J^{sat}))$. (See, for example, [8].)

If J is saturated, the regularity of J can be defined to be the least $t > 0$ such that $\dim{(R/J)}_t = \dim{(R/J)}_{t-1}$.

One can apply the results of Harbourne's paper [9] to show that the regularity for our ideal is $\operatorname{reg}(I^{(m)}) = (n+1)m$. We will show how to use our methods to obtain this result and expand their application to computing the regularity for ordinary powers of I.

Lemma 2.9.1. *We have* $\operatorname{reg}(I) = n+1$ *for any* $n \in \mathbb{N}$.

Proof. As I has necessary generators of degree $n+1$, namely xF and yF, we know from the definition of the Castlenuovo-Mumford regularity in [6] that $\operatorname{reg}(I) \geq n+1$. To show the other inequality, we will use the fact that I, as an ideal of fat points, is saturated. If $\dim{(R/I)}_n = \dim{(R/I)}_{n+1}$, then $\operatorname{reg}(I) \leq n+1$ and we are done.

So show that $\dim{(R/I)}_n = \dim{(R/I)}_{n+1}$. Since $\dim R_t = \binom{2+t}{2} = \frac{1}{2}(t+2)(t+1)$, and showing $\dim{(R/I)}_n = \dim{(R/I)}_{n+1}$ amounts to showing that $\dim R_n - \dim I_n = \dim R_{n+1} - \dim I_{n+1}$, we want to prove that $\dim I_{n+1} = \dim I_n + n + 2$.

If $n = 1$, then $I = (xy, xz, yz)$. Hence $\dim I_1 = 0$ and $\dim I_2 = 3 = 0 + (1+2)$ as desired.

If $n \geq 2$, then $I = (xy, xF, yF)$ has a generator of degree 2 and two generators of degree $n+1$. Therefore, $I_n = {(xy \cdot \mathcal{M}^{n-2})}_n$, where $\mathcal{M} = (x, y, z)$ is the irrelevant ideal, and $\dim I_n = \dim{(xy \cdot \mathcal{M}^{n-2})}_n = \dim{(\mathcal{M}^{n-2})}_n = \binom{3+(n-2)-1}{n-2} = \binom{n}{n-2}$, the number of possible n-combinations on 3 elements with repetitions. Likewise, $I_{n+1} = (xy \cdot \mathcal{M}^{n-1}, xF, yF)_{n+1}$, so $\dim I_{n+1} = \dim{(xy \cdot \mathcal{M}^{n-1})}_{n+1} + 2 = \dim{(\mathcal{M}^{n-1})}_{n+1} + 2$, where $\dim{(\mathcal{M}^{n-1})}_{n+1} = \binom{3+(n-1)-1}{n-1} = \binom{n+1}{n-1}$.

Hence $\dim I_{n+1} = \binom{n+1}{n-1} + 2 = \binom{n}{n-2} + n + 2 = \dim I_n + n + 2$ as desired. $\square$

We can generalize this result for all $m \geq 2$.

Theorem 2.9.2. *For* $n \geq 2$ *and* $m \geq 2$*, we have* $\operatorname{reg}(I^{(m)}) = (n+1)m$.

Proof. By definition of the Castlenuovo-Mumford regularity in [6], we immediately obtain $\operatorname{reg}(I^{(m)}) \geq (n+1)m$, as for $0 \leq s \leq m$, each $x^s y^{m-s} F^m$ is a necessary generator of $I^{(m)}$ of degree $(n+1)m$, which cannot be replaced by combinations of generators of lower degree.

Hence it remains to prove that $\operatorname{reg}(I^{(m)}) \leq (n+1)m$. But $\operatorname{reg}(I^{(m)}) \leq \operatorname{reg}(I^m)$ and $\operatorname{reg}(I^m) \leq m\operatorname{reg}(I)$ by [8], so $\operatorname{reg}(I^{(m)}) \leq m(\operatorname{reg}(I)) = (n+1)m$ by Lemma 2.9.1. $\square$

Using this, we immediately obtain the regularity of I^m.

Corollary 2.9.3. *For* $n \geq 2$ *and* $m \geq 2$*, we have* $\operatorname{reg}(I^m) = (n+1)m$.

Proof. Using that $(I^m)^{sat} = I^{(m)}$, the fact from [8], and Theorems 2.8.1 and 2.9.2, we get $\operatorname{reg}(I^m) = \max(\operatorname{satdeg}(I^m), \operatorname{reg}((I^m)^{sat})) = \max(\operatorname{satdeg}(I^m), \operatorname{reg}(I^{(m)})) = \max((n+1)m, (n+1)m) = (n+1)m$. □

Comment 2.9.4. We could also have used our vector space description to prove that $\operatorname{reg}(I^{(m)})$, and hence also $\operatorname{reg}(I^m)$, is at most $(n+1)m$.

To see this, we show that $\dim \left(R/I^{(m)}\right)_{(n+1)m} = \dim \left(R/I^{(m)}\right)_{(n+1)m-1}$. As the regularity is defined to be the least positive degree for which we have this equality, we then have $\operatorname{reg}(I^{(m)}) \leq (n+1)m$.

As in Lemma 2.9.1, showing $\dim \left(R/I^{(m)}\right)_{(n+1)m} = \dim \left(R/I^{(m)}\right)_{(n+1)m-1}$ means showing $\dim R_{(n+1)m} - \dim \left(I^{(m)}\right)_{(n+1)m} = \dim R_{(n+1)m-1} - \dim \left(I^{(m)}\right)_{(n+1)m-1}$, i.e. $(*): \dim \left(I^{(m)}\right)_{(n+1)m} = \dim \left(I^{(m)}\right)_{(n+1)m-1} + (n+1)m + 1$.

Since a (vector space) generator $g = x^a y^b z^c F^d \in \left(I^{(m)}\right)_{(n+1)m}$ can either be divisible by another (vector space) generator of $I^{(m)}$ or not, a straightforward counting argument proves $(*)$: Using the conditions on a, b, c, d we found in Proposition 2.2.4, we can show that g has to be either divisible by another generator of $I^{(m)}$, in which case $g \in A = (x, y, z)\left(I^{(m)}\right)_{(n+1)m-1}$ or $g \in B = \langle x^s y^{m-s} F^m | 0 \leq s \leq m \rangle$, or g is not divisible by another generator, in which case g is also in B. Using basic combinatorics, we can calculate the vector space dimensions of A and B to get the desired equality $(*)$.

Comment 2.9.5. As mentioned previously, it can be quite difficult, if not impossible, to calculate the resurgence of an ideal with the currently existing methods. This holds even for ideals of fat points in $\mathbb{P}^2$, which, as mentioned in the introduction of this paper, have proven to be more accessible in general than arbitrary homogeneous ideals.

Note that [2] gives bounds on the resurgence $\rho(I)$ when I is the ideal of a 0-dimensional subscheme, such as in our situation. They show that $\frac{\alpha(I)}{\gamma(I)} \leq \rho(I) \leq \frac{\operatorname{reg}(I)}{\gamma(I)}$, where $\gamma(I) = \lim_{m\to\infty} \frac{\alpha(I^{(m)})}{m}$ is the Waldschmidt constant. For our ideal of points, we know that $\alpha(I) = 2$ and $\alpha(I^{(m)}) = 2m$, so $\gamma(I) = 2$. Hence we obtain a lower bound of $\frac{2}{2} = 1$, which does not provide any new information. We already know that we always have $1 \leq \rho(I)$, as $m \geq r$ or, alternatively, $m\alpha(I) \geq \alpha\left(I^{(m)}\right)$ i.e. $\alpha(I) \geq \gamma(I)$, which then implies that $1 \leq \frac{\alpha(I)}{\gamma(I)} \leq \rho(I)$. Also, we have $\operatorname{reg}(I) = n + 1$, so we obtain an upper bound for the resurgence of $\frac{n+1}{2}$, which is at least 1.5 as $n \geq 2$. As mentioned in the introduction, it is known that the resurgence of a homogeneous ideal in $K[\mathbb{P}^N]$ is bounded above by N, which in our case is 2. Therefore, we get the - quite imprecise - estimate $1 \leq \rho(I) \leq \min(2, \frac{n+1}{2})$.

With our method, however, we successfully calculated the exact resurgence, $\frac{4}{3}$, of our ideal of points in Theorem 2.3.1.

Chapter 3

Ideals of Nearly-Complete Intersections With an Additional Point on $x = 0$

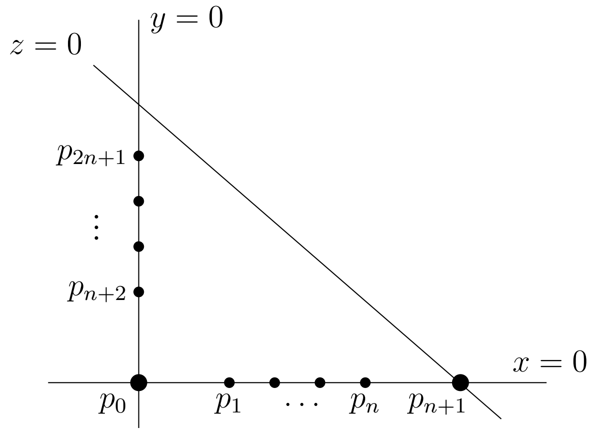
3.1 Preliminaries

After having thoroughly analyzed the point configuration with equally many points on $x = 0$ and $y = 0$, plus a point at the origin, we now move on to the next closest point configuration that still maintains the additional point at the origin.

Throughout this chapter, suppose that $n \geq 2$ and that we have $n + 1$ points on $L_1 : x = 0$, say $p_1, \ldots, p_{n+1}$, and n points on $L_2 : y = 0$, say $p_{n+2}, \ldots, p_{2n+1}$, all of multiplicity m. We are still considering the case where there is an additional point p_0 of multiplicity m at the intersection of L_1 and L_2. Without loss of generality, we may assume that p_0 and p_{n+1} are the only points at the intersections of the axes, so $\mathcal{I}(p_0) = (x, y)$ and $\mathcal{I}(p_{n+1}) = (x, z)$.

These $2n+2$ points define a 0-dimensional subscheme $Z = p_0 + \ldots + p_{2n+1}$ of $\mathbb{P}^2$ and an ideal of points $I = \mathcal{I}(Z)$. The ideal vanishing on Z is

$$I = (x, y) \cap (x, z) \cap \left(\bigcap_{i=1}^{n} (x, z - \alpha_i y) \cap \bigcap_{i=1}^{n} (y, z - \beta_i x) \right),$$



where $\alpha_i, \beta_i \in K^*$, and for $i = 1, \ldots, n$, we have $\mathcal{I}(p_i) = (x, z - \alpha_i y)$ and $\mathcal{I}(p_{n+1+i}) = (y, z - \beta_i x)$.

Define a degree n polynomials $F \in R$ by

$$F = z^n - \prod_{i=1}^{n}(z - \beta_i x) - \prod_{i=1}^{n}(z - \alpha_i y),$$

hence F is defined in the same way it was in the previous chapter.

Comment 3.1.1. Note that the case $n = 1$ gives an almost collinear point configuration, which was mentioned in the previous chapter and analyzed in [4]. Therefore we will concentrate on $n \geq 2$, even though most of the ideas and proofs in this chapter don't require that.

3.2 Descriptions of I, I^r, and $I^{(m)}$

Proposition 3.2.1. *With I and F as above, we have*

$$I = (x, y) \cap (xy, F) \cap (x, z) = (x, yz) \cap (xy, F) = (xy, (x, yz)F).$$

For $m, r \in \mathbb{N}$, we get the following descriptions of the ordinary r^{th} power of I and the symbolic m^{th} power of I:

$$I^r = (xy, (x, yz)F)^r = \sum_{j=1}^{r} (xy)^j (x, yz)^{r-j} F^{r-j}$$

and

$$I^{(m)} = (x, yz)^m \cap (xy, F)^m.$$

Proof. Since F is defined as it was in the previous chapter and (x, y) and (x, z) are monomial ideal, the equalities $I = (x, y) \cap (xy, F) \cap (x, z) = (x, yz) \cap (xy, F)$ are immediate. Therefore, we can write

$$\begin{aligned} I &= (x, y) \cap (x, z) \cap (xy, F) \\ &= (xy, xF, yF) \cap (x, z) \\ &\overset{(*)}{=} (xy, xF, yzF) \\ &= (xy, (x, yz)F) \\ &= (x, yz) \cap (xy, F) \end{aligned}$$

where $(*)$ holds for the following reasons: Certainly,

$$(xy, xF, yF) \cap (x, z) \supseteq (xy, xF, yzF).$$

The reverse containment holds because if $g \in (xy, xF, yF) \cap (x, z)$, then we can write $g = (xy)g_1 + (xF)g_2 + (yF)g_3$ for some $g_1, g_2, g_3 \in R$. Assume without loss of generality that x does not divide any term of g_3, so we cannot group any part of $(yF)g_3$ together with either of the first two summands. We want to show that $g_3 = 0$ or $z \mid g_3$. If $g_3 = 0$, then $g \in (xy, xF, yzF)$ as desired, so suppose $g_3 \neq 0$. Since $g \in (x, z)$, we have

$$\begin{aligned} 0 &= g([0, 1, 0]) \\ &= ((xy)g_1)([0, 1, 0]) + ((xF)g_2)([0, 1, 0]) + ((yF)g_3)([0, 1, 0]) \\ &= 0 + 0 + F([0, 1, 0]) \cdot g_3([0, 1, 0]) \end{aligned}$$

Now, $F([0, 1, 0]) = -\prod_{i=1}^{n}(-\alpha_i) \neq 0$ by definition of F, so we have $g_3([0, 1, 0]) = 0$. But x does not divide any term of g_3, so we have in fact that $g_3([x, 1, 0]) = 0$ for any $x \in K$. Hence z has to

divide g_3. Thus
$$(xy, xF, yF) \cap (x, z) \subseteq (xy, xF, yzF)$$
and we have the desired equality $(*)$.

Therefore
$$I^r = (xy, (x, yz)F)^r = \sum_{j=0}^{r} (xy)^j (x, yz)^{r-j} F^{r-j}$$
and
$$I^{(m)} = (x, yz)^{(m)} \cap (xy, F)^{(m)} = (x, yz)^m \cap (xy, F)^m$$
as (x, yz) and (xy, F) are complete intersections. □

Example 3.2.2. For $n = 2$, if we set

$$I = (x, y) \cap (x, z) \cap (x, z - y) \cap (x, z - 2y) \cap (y, z - x) \cap (y, z - 2x),$$

which we may after possibly a change of coordinates, we get $F = z^2 - (z - x)(z - 2x) - (z - y)(z - 2y)$, so
$$F = -2x^2 - 2y^2 + 3xz + 3yz - z^2.$$
Also,
$$I^2 = (xy, xF, yzF)^2 = (x^2y^2, x^2F^2, y^2z^2F^2, x^2yF, xy^2zF, xyzF^2)$$
and
$$I^{(2)} = (x, yz)^2 \cap (xy, F)^2 = (x^2y^2, x^2F^2, y^2z^2F^2, x^2yF, xyzF)$$
(see Appendix 4 for the code).

3.3 A Lower Bound for the Resurgence

Unfortunately, the vector space approach we used in the previous chapter doesn't translate nicely to this new point configuration. Finding more accessible descriptions for $I^{(m)}$ and I^r than the ones given above is difficult. Instead, we replace I^r by a slightly larger ideal J^r and work with that.

In particular, we can use this to give a lower bound for the resurgence $\rho(I)$.

Comment 3.3.1. [2] gives one lower bound for the resurgence as the ratio $\frac{\alpha(I)}{\gamma(I)}$, where $\gamma(I) = \lim_{m\to\infty} \frac{\alpha(I^{(m)})}{m}$ is the Waldschmidt constant for I. For this configuration of points, it is easy to see that $\alpha(I)$, the least degree $t > 0$ such that $I_t \neq 0$, is equal to 2, as $xy \in I$ and $\deg F = n \geq 2$. Likewise, it is easy to check that $x^m y^m \in I^{(m)}$ and that this is a generator of least degree. Therefore, $\alpha(I^{(m)}) = 2m$ and hence $\gamma(I) = 2$. It follows that the bound in [2] gives us $\rho(I) \geq \frac{2}{2} = 1$, which doesn't provide any extra information.

So even if we cannot provide an exact value for the resurgence, finding decent bounds for it in this situation is interesting and non-trivial.

Lemma 3.3.2. *Let $I = (xy, xF, yzF)$ be defined as above and let $J = (xy, xF, yF)$ be the ideal we examined in chapter 2. Then $J \supseteq I$.*

Proof. Let $Z_1 \subseteq \mathbb{P}^2$ be the 0-dimensional subscheme defining I, and let $Z_2 \subseteq \mathbb{P}^2$ be the 0-dimensional subscheme defining J. Since, up to change of coordinates, $Z_1 \supseteq Z_2$, we have $I = \mathcal{I}(Z_1) \subseteq \mathcal{I}(Z_2) = J$. □

This containment means that for any m and r in $\mathbb{N}$ such that $I^{(m)} \nsubseteq J^r$, we also have $I^{(m)} \nsubseteq I^r$. We can compute a lower bound for $\rho(I)$ by finding some ordered pairs $(m, r) \in \mathbb{N}^2$ such that $I^{(m)} \nsubseteq J^r$.

Theorem 3.3.3. *Let $n \geq 2$, $k \in \mathbb{N}$, and set $g = x^{2kn}y^{2kn}z^{2kn}F^{2kn}$. Then $g \in I^{(4kn)}$ but $g \notin J^r$ for any $r \geq 3kn + k + 1$. In particular, $\rho(I) \geq \frac{4n}{3n+1}$.*

Proof. It is easy to see that $g \in I^{(4kn)} = (x, yz)^{4kn} \cap (xy, F)^{4kn}$.

Recall that by Proposition 2.2.7, for an element $x^a y^b z^c F^d$ to be in J^r, we require $a + b \geq r$, $\min(a, b) + d \geq r$, and $a + b + d \geq 2r$. Moreover, we showed in Lemma 2.2.6 that we have a basis for J^r where $c < n$, so we may replace each copy of z^n with a copy of F. Our element g has $2k$ copies of z^n, hence we may consider the element $g' = x^{2kn}y^{2kn}F^{2kn+2k}$ and show instead that $g' \notin J^r$ for any $r \geq 3kn + k + 1$.

As $n \geq 2$ and $k \geq 1$, we have $a + b = 4kn \geq 3kn + k + 1$ and $\min(a, b) + d = 4kn + 2k > 3kn + k + 1$ but $a + b + d = 6kn + 2k < 6kn + 2k + 2 = 2(3kn + k + 1)$. Hence $g' \notin J^{3kn+k+1}$ and therefore also $g \notin J^{3kn+k+1} \supseteq J^r$ for $r \geq 3kn + k + 1$.

Since g is then also not contained in $I^{3kn+k+1}$, we have $I^{(4kn)} \nsubseteq I^{3kn+k+1}$. Hence $\rho(I) \geq \frac{4kn}{3kn+k+1}$ for each $k \in \mathbb{N}$. Therefore, we have in fact that $\rho(I) \geq \lim_{k\to\infty} \frac{4nk}{3nk+k+1} = \frac{4n}{3n+1}$. $\square$

Comment 3.3.4. Note that the limit of this lower bound, as n becomes larger, is the resurgence we found in the previous chapter, namely $\frac{4}{3}$.

The reason we found this particular limit is that we chose an element which is also contained in $J^{(4n)} = (x, y)^{4kn} \cap (xy, F)^{4kn}$, and we know that $\rho(J) = \frac{4}{3}$. However, computational evidence suggests that the resurgence of I is also close to $\frac{4}{3}$, so there is reason to believe that the limit may be the better approximation to $\rho(I)$ than the lower bound we found.

Comment 3.3.5. We mentioned earlier in this chapter that the case $n = 1$ is an almost collinear point configuration, which was analyzed in [4]. In this paper, it is shown that for the ideal defining three collinear points and one point off the line, the resurgence is $\frac{3^2}{3^2-3+1} = \frac{9}{7}$. Our proof for Theorem 3.3.3 requires $n \geq 2$, so it doesn't hold for $n = 1$, but even if it did, we would get a lower bound of $\frac{4(1)}{3(1)+1} = \frac{4}{4} = 1$ in this situation, which is significantly lower than the actual resurgence calculated in [4]. In fact, since $m \geq r$, we always have $\rho(I) \geq 1$, so the lower bound of $\frac{4n}{3n+1}$ wouldn't have provided new information in the case $n = 1$ anyway.

However, for $n \geq 2$, we get $\frac{4n}{3n+1} > 1$, so the usefulness of the lower bound improves and we obtain better results.

Example 3.3.6. For $n = 2$, we get a lower bound of $\frac{4(2)}{3(2)+1} = \frac{8}{7}$ for $\rho(I)$. We return to this example in the next chapter, where we develop a method for approximating the resurgence from above, in some cases to any desired accuracy.

Chapter 4

A Computational Approach for the Resurgence

In the previous chapter, we found a lower bound for the resurgence by exhibiting a small ideal J^r containing I^r and computing when $I^{(m)}$ was not contained in J^r. A natural approach to finding an upper bound for $\rho(I)$ would be to find an ideal smaller than I^r containing $I^{(m)}$ and calculate when that is contained in I^r. However, finding a suitable ideal proved difficult, at least for the ideal of points we were examining. Hence we need to find a different method to bound the resurgence from above.

In this chapter, we develop computational methods for estimating resurgences for nontrivial homogeneous ideals $I \subseteq K[\mathbb{P}^N]$ whenever there is an $m \in \mathbb{N}$ such that powers of $I^{(m)}$ are symbolic, i.e. $I^{(mt)} = \left(I^{(m)}\right)^t$ for all $t \in \mathbb{N}$. Our main result here is Theorem 4.1.5, which gives a computational method for such ideals for determining $\rho(I)$ to any desired accuracy. We demonstrate the method on some examples.

Until further notice, let $(0) \neq I \subsetneq K[\mathbb{P}^N]$ be an arbitrary nontrivial homogeneous ideal.

4.1 Calculating an Upper Bound for ρ and Examples

We start by describing the combinations of m and r whose ratio has a given lower bound.

Lemma 4.1.1. *Suppose that $I^{(\alpha)} \subseteq I^{\beta}$ for some $\alpha, \beta \in \mathbb{N}$. Also assume that $I^{(\alpha t)} = \left(I^{(\alpha)}\right)^t$ for all $t \in \mathbb{N}$. Then $I^{(m)} \subseteq I^r$ for all*

$$\frac{m}{r} \geq \frac{\alpha}{\beta}\left(1 + \frac{j}{r}\right),$$

where $j = 0$ if β divides r, and otherwise $j = \beta - i$ where $i = r \mod \beta$.

Proof. Let $m, r \in \mathbb{N}$. Then either (a) $\beta \mid r$, or (b) $\beta \nmid r$.

(a) Suppose $\beta \mid r$, say $\beta s = r$. Assume $\frac{m}{r} \geq \frac{\alpha}{\beta}$. Then we want to show that $I^{(m)} \subseteq I^r$.

We have $\frac{m}{r} \geq \frac{\alpha}{\beta}$, so $m \geq \alpha \frac{r}{\beta} = \alpha s$, which implies that $I^{(m)} \subseteq I^{(\alpha s)} = \left(I^{(\alpha)}\right)^s \subseteq \left(I^{\beta}\right)^s = I^{\beta s} = I^r$ as desired.

(b) Now assume that $\beta \nmid r$. Let $s \in \mathbb{N}$ be such that $r = \beta s + i$ for some $0 < i < \beta$. Assume also that $\frac{m}{r} \geq \frac{\alpha}{\beta}\left(1 + \frac{\beta - i}{r}\right)$. To see that then $I^{(m)} \subseteq I^r$, notice that $\frac{m}{r} \geq \frac{\alpha}{\beta}\left(1 + \frac{\beta - i}{r}\right)$ implies $m \geq \frac{\alpha}{\beta}(r + \beta - i) = \frac{\alpha}{\beta}(\beta(s+1)) = \alpha(s+1)$. Then $I^{(m)} \subseteq I^{(\alpha(s+1))} = \left(I^{(\alpha)}\right)^{s+1} \subseteq \left(I^{\beta}\right)^{s+1} = I^{\beta(s+1)} = I^{r+(\beta-i)} \subseteq I^r$as desired. $\square$

We can loosen the conditions on $\frac{m}{r}$ a bit, provided we choose r, and thus m, large enough.

Lemma 4.1.2. *Suppose that $I^{(\alpha)} \subseteq I^{\beta}$ for some $\alpha, \beta \in \mathbb{N}$, and assume that $I^{(\alpha t)} = \left(I^{(\alpha)}\right)^t$ for all $t \in \mathbb{N}$. Let $\varepsilon > 0$. Then $I^{(m)} \subseteq I^r$ for all*

$$\frac{m}{r} \geq \frac{\alpha}{\beta} + \varepsilon \text{ with } r \geq \frac{\beta - 1}{\varepsilon}.$$

Proof. Let $m, r \in \mathbb{N}$. Suppose we have $(*) : \frac{m}{r} \geq \frac{\alpha}{\beta} + \varepsilon$ and $r \geq \frac{\beta-1}{\varepsilon}$, i.e. $\varepsilon \geq \frac{\beta-1}{r}$. It suffices to show that this implies $(\star) : \frac{m}{r} \geq \frac{\alpha}{\beta}\left(1 + \frac{j}{r}\right)$, where j is as in the previous proposition. If $\frac{m}{r}$ has the required lower bound, Lemma 4.1.1 gives $I^{(m)} \subseteq I^r$.

(a) If $\beta \mid r$, then $(*)$ implies $(\star)$ and we are done.

(b) If $\beta \nmid r$, then $\frac{j}{r} = \frac{\beta - i}{r} \leq \frac{\beta - 1}{r} \leq \varepsilon$, so $\frac{m}{r} \geq \frac{\alpha}{\beta} + \varepsilon \geq \frac{\alpha}{\beta} + \frac{\beta-1}{r} \geq \frac{\alpha}{\beta}\left(1 + \frac{j}{r}\right)$ as desired. $\square$

We can now either find $\rho(I)$ exactly or give an upper bound for it.

Theorem 4.1.3. *Suppose that $I^{(\alpha)} \subseteq I^{\beta}$ for some $\alpha, \beta \in \mathbb{N}$, and assume that $I^{(\alpha t)} = \left(I^{(\alpha)}\right)^t$ for all $t \in \mathbb{N}$. Let $\varepsilon > 0$. Then all ordered pairs $(m, r) \in \mathbb{N}^2$ with $I^{(m)} \not\subseteq I^r$ but with $\frac{m}{r} > \frac{\alpha}{\beta} + \varepsilon$ satisfy*

$$(**): \qquad Nr > m > r\left(\frac{\alpha}{\beta} + \varepsilon\right) \text{ and } r < \frac{\beta - 1}{\varepsilon}.$$

In particular, either

- $\rho(I) \leq \frac{\alpha}{\beta} + \varepsilon$ *or*
- $\rho(I) = \frac{m_0}{r_0}$, *where* $\frac{m_0}{r_0}$ *is the largest among the ratios* $\frac{m}{r}$ *such that* m *and* r *satisfy* $(**)$ *with* $I^{(m)} \not\subseteq I^r$.

Proof. Let $(m, r) \in \mathbb{N}^2$ be such that $I^{(m)} \not\subseteq I^r$ and $(*): \frac{m}{r} > \frac{\alpha}{\beta} + \varepsilon$. We know that $m \geq Nr$ implies $I^{(m)} \subseteq I^r$, so we have to have $Nr > m$. Combining this with $(*)$ gives $Nr > m > r\left(\frac{\alpha}{\beta} + \varepsilon\right)$. Lemma 4.1.2 gives $r < \frac{\beta - 1}{\varepsilon}$, for otherwise $I^{(m)} \subseteq I^r$.

The claims follow immediately.

- If no such $(m, r) \in \mathbb{N}^2$ exists, i.e. $\frac{m}{r} > \frac{\alpha}{\beta} + \varepsilon$ always implies $I^{(m)} \subseteq I^r$, then $\rho(I) \leq \frac{\alpha}{\beta} + \varepsilon$ by definition of $\rho(I)$.
- If there are some $(m_i, r_i) \in \mathbb{N}^2$ such that $I^{(m_i)} \not\subseteq I^{r_i}$ and $\frac{m_i}{r_i} > \frac{\alpha}{\beta} + \varepsilon$, then the condition that $r_i < \frac{\beta - 1}{\varepsilon}$ implies that there are only finitely many such r_i. The condition that $Nr_i > m_i > r_i\left(\frac{\alpha}{\beta} + \varepsilon\right)$ implies that for each r_i, there are only finitely many choices for m as well. Hence there are only finitely many pairs (m_i, r_i), and we can order them by magnitude of their ratios, say $N > \frac{m_0}{r_0} \geq \frac{m_1}{r_1} \geq \ldots > \frac{\alpha}{\beta} + \varepsilon$. Since $\frac{m_0}{r_0}$ is then the largest such ratio, we have $\rho(I) = \frac{m_0}{r_0}$.

□

Example 4.1.4. Results based on computation were obtained using CoCoA, working over $K = \mathbb{Q}$.

Let $n = 2$, $F = 3xz + 3yz - 2x^2 - 2y^2 - z^2$, and $I = (x, yz) \cap (xy, F)$ as in chapter 3. Then $I^{(5)} \subseteq I^r$ for $r \leq 3$ (see Appendix 5 for the code). Computational evidence suggests that $I^{(5t)} = \left(I^{(5)}\right)^t$ for all $t \in \mathbb{N}$. Assuming that this is indeed true, Theorem 4.1.3 gives that for any $\varepsilon > 0$, either $\rho(I) \leq \frac{5}{3} + \varepsilon$ or $\rho(I) = \frac{m_0}{r_0}$, where $\frac{m_0}{r_0}$ is largest such that $2r > m > r\left(\frac{5}{3} + \varepsilon\right)$ and $r < \frac{2}{\varepsilon}$.

We showed in Example 3.3.6 that $\frac{8}{7} \leq \rho(I)$. Therefore, we either have $\rho(I) = \frac{m_0}{r_0}$ or $\frac{8}{7} \leq \rho(I) \leq \frac{5}{3} + \varepsilon$. Computational evidence shows that $\rho(I)$ is approximately $\frac{4}{3}$, so the latter seems to be the case.

But suppose we want to know definitively whether $\rho(I) \leq 1.75$. Can we confirm that, and if so, at what cost?

By Lemma 4.1.2, we know that $I^{(m)} \subseteq I^r$ whenever $\frac{m}{r} \geq \frac{5}{3} + \varepsilon$ and $r \geq \frac{2}{\varepsilon}$. So let $\varepsilon = \frac{1}{12}$. Then $\frac{5}{3} + \varepsilon = \frac{7}{4} = 1.75$ and $\frac{2}{\varepsilon} = 24$. Therefore, $I^{(m)} \subseteq I^r$ whenever $\frac{m}{r} \geq 1.75$ and $r \geq 24$. So the only ratios $\frac{m}{r} \geq 1.75$ with possibly $I^{(m)} \not\subseteq I^r$ have $r < 24$. We can list and test those ratios easily.

We obtain the following data (where the code is as in Appendix 5), where $1 \leq r \leq 23$ is given and m is the smallest integer such that $\frac{m}{r} \geq 1.75$.

r	1	2	3	4	5	6	7	8	9	10	11
m	2	4	6	7	9	11	13	14	16	18	20
$\frac{m}{r}$	2	2	2	1.75	1.8	1.83	1.86	1.75	1.78	1.8	1.82
$I^{(m)} \subseteq I^r$?	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y

and

12	13	14	15	16	17	18	19	20	21	22	23
21	23	25	27	28	30	32	34	35	37	39	41
1.75	1.77	1.79	1.8	1.75	1.76	1.78	1.79	1.75	1.76	1.77	1.78
Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y

Since we're assuming that $I^{(5t)} = \left(I^{(5)}\right)^t$ for all t, we have in fact that $I^{(m)} \subseteq I^r$ whenever $\frac{m}{r} \geq 1.75$ and $r \geq 1$, which means that $\rho(I) \leq 1.75$.

Using this method, we can test $\rho(I)$ to any desired accuracy. However, it may be difficult or impractical to require too great an accuracy. For example, even if we only want to see whether we can improve the upper bound to 1.7, we have to set $\varepsilon = \frac{1}{30}$ and hence test $1 \leq r \leq 59$.

We now show how to compute $\rho(I)$, in principle, to any desired accuracy.

Theorem 4.1.5. *Assume there is a positive integer α such that $I^{(\alpha t)} = \left(I^{(\alpha)}\right)^t$ for all $t \in \mathbb{N}$. Let β be any positive integer with $\alpha < \beta$. Then there exists $\sigma \in \mathbb{N}$ such that $I^{(\alpha\sigma)} \nsubseteq I^\beta$ but $I^{(\alpha(\sigma+1))} \subseteq I^\beta$. Moreover, one of the following applies.*

- *Either $\frac{\alpha\sigma}{\beta} \leq \rho(I) \leq \frac{\alpha\sigma}{\beta} + \frac{\alpha+1}{\beta}$ or*
- *$\rho(I) = \frac{m_0}{r_0}$, where $\frac{m_0}{r_0}$ is the largest ratio among the ordered pairs (m, r) such that $I^{(m)} \nsubseteq I^r$ with $Nr > m > r\left(\frac{\alpha(\sigma+1)}{\beta} + \frac{1}{\beta}\right)$ and $r < \beta(\beta - 1)$.*

Proof. Since $I^{(Nr)} \subseteq I^r$, there exists some $\sigma < \frac{Nr}{\alpha}$ such that $I^{(\alpha\sigma)} \nsubseteq I^\beta$ but $I^{(\alpha(\sigma+1))} \subseteq I^\beta$. Since $I^{(\alpha\sigma)} \nsubseteq I^\beta$, we have a lower bound of $\frac{\alpha\sigma}{\beta}$ for $\rho(I)$. Let $\varepsilon = \frac{1}{\beta}$. Then all pairs $(m, r) \in \mathbb{N}^2$ with $I^{(m)} \nsubseteq I^r$ but with $\frac{m}{r} > \frac{\alpha(\sigma+1)}{\beta} + \frac{1}{\beta} = \frac{\alpha\sigma}{\beta} + \frac{\alpha+1}{\beta}$ satisfy $Nr > m > r\left(\frac{\alpha(\sigma+1)}{\beta} + \frac{1}{\beta}\right)$ and $r < \beta(\beta - 1)$. By Theorem 4.1.3 we get the desired results. $\square$

Example 4.1.6. For the same setup and assumptions as in Example 4.1.4, we can verify that $I^{(5\cdot 21)} \nsubseteq I^{80}$ but $I^{(5\cdot 22)} \subseteq I^{80}$ (see Appendix 6 for the code). By Theorem 4.1.5, for each $\varepsilon > 0$ we either have $\rho(I) = \frac{m_0}{r_0}$, where $\frac{m_0}{r_0}$ is largest such that $I^{(m)} \nsubseteq I^r$ with $2r > m > r\left(\frac{110}{80} + \frac{1}{80}\right) = 1.3875r$ and $r < 80 \cdot 79 = 6320$, or we have $1.3125 = \frac{105}{80} \leq \rho(I) \leq \frac{105}{80} + \frac{6}{80} = 1.3875$.

Example 4.1.7. Let $I = (xy, xF, yF)$ be as in chapter 2. We showed in Corollary 2.5.5 that $I^{(2t)} = \left(I^{(2)}\right)^t$ for all $t \in \mathbb{N}$. Suppose we didn't know that $\rho(I) = \frac{4}{3}$. Then by Theorem 4.1.5, either $\frac{2\sigma}{\beta} \leq \rho(I) \leq \frac{2\sigma}{\beta} + \frac{3}{\beta}$ or $\rho(I) = \frac{m_0}{r_0}$.

Note that our choice of β not only affects our accuracy in the first case but also determines how many ratios $\frac{m_i}{r_i}$ we have to check in the second case. If we want to calculate the resurgence to within a certain margin of error, say 0.1, we need to pick a $\beta > 2$ such that $\frac{3}{\beta} \leq 0.1$. Moreover, the corresponding σ needs to have $I^{(2\sigma)} \nsubseteq I^\beta$ but $I^{(2\sigma+2)} \subseteq I^\beta$.

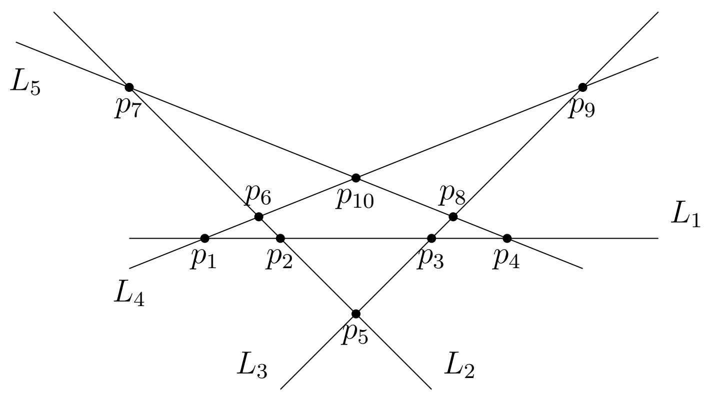
If we choose the least permissible β, namely $\beta = 30$, we can either use our containment criterion Theorem 2.3.1 or guess a number for σ and verify it computationally. We get that $\sigma = 19$ and therefore either $1.267 = \frac{2\cdot 19}{30} \leq \rho(I) \leq \frac{2\cdot 19}{30} + \frac{3}{30} = 1.367$ or $\rho(I) = \frac{m_0}{r_0}$, where (m_0, r_0) has the largest ratio among the pairs (m, r) that satisfy $I^{(m)} \nsubseteq I^r$, $2r > m > 1.367r$, and $r < 15 \cdot 14 = 210$.

If we want to decrease our margin of error or simply like $\beta = 100$ better, we get $\sigma = 66$ and hence either $1.32 = \frac{2\cdot 66}{100} \leq \rho(I) \leq \frac{2\cdot 66}{100} + \frac{3}{100} = 1.35$ or $\rho(I) = \frac{m_0}{r_0}$, where (m_0, r_0) has the largest ratio among the pairs (m, r) that satisfy $I^{(m)} \nsubseteq I^r$, $2r > m > 1.35r$ and $r < 100 \cdot 99 = 9900$.

Comment 4.1.8. We now mention a method which can sometimes be applied in the case of an ideal of points in $\mathbb{P}^2$ to verify for some $m \in \mathbb{N}$ that $I^{(mt)} = \left(I^{(m)}\right)^t$ for all $t \in \mathbb{N}$.

In [10], Harbourne and Huneke show that if I is an ideal of n points in $\mathbb{P}^2$ and $m \in \mathbb{N}$, then $I^{(mt)} = \left(I^{(m)}\right)^t$ for all $t \in \mathbb{N}$ if $\alpha(I^{(m)}) \cdot \beta_m = m^2 n$, where β_m is the least integer k for which $\left(I^{(m)}\right)_k$ contains a regular sequence of length two.

Example 4.1.9. Consider a new example. Suppose ten points $p_1, \ldots, p_{10} \in \mathbb{P}^2$ and five lines $L_1, \ldots, L_5$ form a star configuration. Let I be the ideal generated by the vanishing on these points.



Star configurations have been studied extensively in the context of the resurgence and the conjectures discussed in section 2.6 (see, for example, [1] or [10]). Here, we apply our methods to this particular configuration of points.

First, we want to show that $I^{(2t)} = \left(I^{(2)}\right)^t$ for all $t \in \mathbb{N}$. It is easy to see that $\alpha\left(I^{(2)}\right) = 5$, since these five lines together give each of the points multiplicity 2, hence $\alpha\left(I^{(2)}\right) \leq 5$, and $\alpha\left(I^{(2)}\right) \geq 5$ by Bézout or [10]. Also, note that β_2 is the least degree t such that the base locus of I_t is at most 0-dimensional. By Bézout, each of the lines is in the base locus of $\left(I^{(2)}\right)_7$, since each of the L_i contains four of the points, each with multiplicity 2. Hence $\beta_2 \geq 2 \cdot 4 = 8$. By [2], $\operatorname{reg}(I) = 4$, and it is known that $\beta_2 \leq \operatorname{reg}\left(I^{(2)}\right) \leq \operatorname{reg}\left(I^2\right) \leq 2 \cdot \operatorname{reg}(I) = 8$, so we have indeed $\beta_2 = 8$. Therefore, $\alpha\left(I^{(2)}\right) \cdot \beta_2 = 5 \cdot 8 = 40 = 2^2 \cdot 10 = m^2 \cdot n$, and by Comment 4.1.8, we have $I^{(2t)} = \left(I^{(2)}\right)^t$ for all $t \in \mathbb{N}$.

Suppose we wish to calculate $\rho(I)$ to within 0.2. Then, as in Example 4.1.7, we have $\frac{3}{\beta} \leq 0.2$, so $\beta \geq 15$. Say $\beta = 15$.

Bocci and Harbourne showed in [2] that $\alpha(I) = \operatorname{reg}(I)$, so by a result in [3], we have that $I^{(m)} \subseteq I^r$ if and only if $\alpha\left(I^{(m)}\right) \geq r\alpha(I)$. Hence $I^{(24)} \subseteq I^{15}$ since $\alpha(I) = 4$ and we have the equation $\alpha\left(I^{(24)}\right) = \alpha\left(\left(I^{(2)}\right)^{12}\right) = 12\alpha\left(I^{(2)}\right) = 12 \cdot 5 = 60 = 15 \cdot 4 = r\alpha(I)$. However, this equality also means that $I^{(m)} \nsubseteq I^{15}$ for any $m \leq 23$. Thus $I^{(22)} \nsubseteq I^{15}$ and consequently $\sigma = 11$. We therefore obtain the bounds $1.467 = \frac{22}{15} \leq \rho(I) \leq \frac{22}{15} + \frac{3}{15} = 1.67$ for the resurgence. This is consistent with the known value of $\rho(I) = 1.6$ (see [2]), so our approach of bounding the resurgence works nicely for this situation as well.

Comment 4.1.10. For both of the preceding examples, the exact resurgence was known. In general, it can be very difficult to prove that the assumption $I^{(\alpha t)} = \left(I^{(\alpha)}\right)^t$ holds for the ideal under consideration, and many of the point configurations in $\mathbb{P}^2$ for which we can use the method described in Comment 4.1.8 have known resurgences. However, using the procedure we presented above, we can get a good bound on the resurgence of ideals for which we have computational evidence of the hypothesis $I^{(\alpha t)} = \left(I^{(\alpha)}\right)^t$ for small enough t.

Appendix A

CoCoA Codes

The computer program we mainly used is CoCoA (Computations in Commutative Algebra, see http://cocoa.dima.unige.it).

CoCoA is very similar to M2 (Macaulay 2, see http://www.math.uiuc.edu/Macaulay2), so the codes we reference here can be adapted to M2 with few changes.

1. Calculating the Hilbert functions for Example 1.0.1.

   INPUT:

   ```
   Use R ::= QQ[x,y,z];
   F := xz + yz - z^2;
   I := Ideal(xy,xF,yF)^5;
   J := Intersection(Ideal(x,y)^6, Ideal(xy,F)^6);
   Hilbert(I);
   Hilbert(J);
   ```

   OUTPUT:

H(0) = 0	H(0) = 0

H(1) = 0

H(2) = 0

H(3) = 0

H(4) = 0

H(5) = 0

H(6) = 0

H(7) = 0

H(8) = 0

H(9) = 0

H(10) = 1

H(11) = 5

H(12) = 13

H(13) = 25

H(14) = 41

H(t) = 1/2tˆ2 + 3/2t - 74 for t ¿= 15

H(1) = 0

H(2) = 0

H(3) = 0

H(4) = 0

H(5) = 0

H(6) = 0

H(7) = 0

H(8) = 0

H(9) = 0

H(10) = 0

H(11) = 0

H(12) = 4

H(13) = 12

H(14) = 23

H(15) = 35

H(16) = 50

H(t) = 1/2tˆ2 + 3/2t - 104 for t ¿= 17

2. Showing that $I^{(6)} \not\subseteq I^5$ in Example 1.0.1.

INPUT:

```
Use R ::= QQ[x,y,z];

F := xz + yz - zˆ2;

I := Ideal(xy,xF,yF)ˆ5;

J := Intersection(Ideal(x,y)ˆ6, Ideal(xy,F)ˆ6);

G := xˆ3*yˆ3*Fˆ3;
```

```
G IsIn J;
G IsIn I;
```

OUTPUT:

```
True
- - - - - - - - - - - - - - -
False
```

3. Showing that $I^{(2)} = (x^2y^2, x^2F^2, y^2F^2, xyF)$ in Example 2.1.2.

INPUT:

```
Use R ::= QQ[x,y,z];
F := xz + yz - z^2;
I := Intersection(Ideal(x,y)^2, Ideal(xy,F)^2);
J := Ideal(x^2*y^2,x^2*F^2,y^2*F^2,x*y*F);
I=J;
```

OUTPUT:

```
True
```

4. Showing that $I^{(2)} = (x^2y^2, x^2F^2, y^2z^2F^2, x^2yF, xyzF)$ in Example 3.2.2.

INPUT:

```
Use R ::= QQ[x,y,z];
F := -2*x^2 - 2*y^2 + 3*x*z + 3*y*z - z^2;
```

```
I := Intersection(Ideal(x,y)^2, Ideal(xy,F)^2);
J := Ideal(x^2*y^2,x^2*F^2,y^2*z^2*F^2,x^2*y*F, x*y*z*F);
I=J;
```

OUTPUT:

```
True
```

5. Showing that $I^{(5)} \subseteq I^3$ in 4.1.4.

INPUT:

```
Use R::= QQ[x,y,z];
F := -2*x^2 - 2*y^2 + 3*x*z + 3*y*z - z^2;
I := Ideal(xy,xF,yzF);
N := [True];


For S := 3 To 4 Do
J := Intersection(Ideal(x,yz)^5, Ideal(xy,F)^5);
G := Gens(J);
ForEach T In G Do
If T IsIn I^S Then
Append(N,True);
Else Append(N,False)
EndIf;
EndForEach;
Print S;
```

```
NewLine();
EqSet(N,[True]);
NewLine();
EndFor;
```

OUTPUT:

```
3
True
4
False
```

6. Showing that $I^{(5\cdot 21)} \not\subseteq I^{80}$ but $I^{(5\cdot 22)} \subseteq I^{80}$ in 4.1.6.

INPUT:

```
Use R::= QQ[x,y,z];
F := -2*x^2 - 2*y^2 + 3*x*z + 3*y*z - z^2;
I := Ideal(xy,xF,yzF);
N := [True];


For S := 0 To 1 Do
J := Intersection(Ideal(x,yz)^(5*(21+S)), Ideal(xy,F)^(5*(21+S)));
G := Gens(J);
ForEach T In G Do
If T IsIn I^(80) Then
Append(N,True);
```

```
Else Append(N,False)
EndIf;
EndForEach;
Print 5*(21+S);
NewLine();
EqSet(N,[True]);
NewLine();
EndFor;
```

OUTPUT:

```
105
False
110
True
```

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