Steady-Periodic Heating of a Cylinder

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1 Introduction

Steady-periodic heat transfer in cylinders is important in several engineering applications including annular fins [1–3], rotating machinery such as electromagnetic bearings [4], and in devices experiencing periodic thermal contact such as engine exhaust valves [5].

In this paper, steady-periodic heat conduction in cylinders is presented with the method of Green’s functions (GFs). This approach provides a comprehensive set of solutions and specific strategies for improving the numerical evaluation of these solutions.

There are several recent books on GF applied to heat conduction [6–8]. The book by Mandelis [9] is devoted exclusively to steady-periodic heat conduction with the method of GFs. Of the solutions given for cylindrical geometries, however, only three kinds of boundary conditions are treated, and only one form of the GF is given for each geometry.

The contributions of this paper are threefold. First, a great many steady-periodic solutions are presented systematically with the method of GFs for several cylindrical geometries. Second, five kinds of boundary conditions are treated for one-, two-, and three-dimensional geometries. For some geometries an alternate form of the Green’s function is given, which can be used for improvement of series convergence and for checking purposes to produce highly accurate numerical values. Numerical examples are given.

[DOI: 10.1115/1.3139107]

Keywords: frequency response, thermal transient, oscillating heat source, convective boundary, lumped boundary, surface-film boundary, extended surface

2 Steady-Periodic Relations

Steady-periodic heating is an important experimental technique for measurement of thermal properties. In these methods the thermal properties are deduced from a systematic comparison between the data (such as temperature) and a detailed thermal model. This paper addresses steady-periodic heat transfer on cylindrical geometries with application to thermal-property measurements. The method of Green’s functions is used to provide a comprehensive collection of exact analytical expressions for temperature in cylinders. Five kinds of boundary conditions are treated for one-, two-, and three-dimensional geometries. For some geometries an alternate form of the Green’s function is given, which can be used for improvement of series convergence and for checking purposes to produce highly accurate numerical values. Numerical examples are given.

\[
T(r, \omega) = \frac{\alpha}{k} \int g(r', \omega)G(r, r', \omega)dr' + \alpha \sum \int f_i(r', \omega) \times \left[ -\frac{\partial G_i}{\partial n_i} \text{ (first kind only)} \right] \frac{1}{k} G(r, r', \omega) \left( \text{second to fifth kinds} \right) ds_i' \quad (3)
\]

The first integral is the effect of internal heat generation \( g \), and the second integral is the effect of each nonhomogeneous boundary term \( f_i \). Note that the same GF appears in each integral but it is evaluated at locations appropriate for each integral.

The GF associated with Eqs. (1) and (2) is the response at \( r \) to a steady-periodic heat source located at \( r' \), and the GF satisfies

\[
\nabla^2 G - \alpha^2 G = - \frac{1}{\alpha} \delta(r - r') \quad (4)
\]
Here $\iota = \iota_1 + j\omega (\rho c_e)$, and $\delta(r-r')$ is the Dirac delta function. The coefficient $1/\alpha$ preceding the delta function in Eq. (4) provides units for the steady-periodic GF that are consistent with earlier work [10]. The boundary conditions for the GF are homogeneous but of the same kind (see Ref. [6], Chap. 2) as for the temperature problem of interest. Thus, a different GF is needed for each geometry and for each combination of boundary conditions.

In Secs. 4–6, the method of GFs is applied to one-, two-, and three-dimensional cylinders. First, the GFs will be identified, and then temperature examples will be given.

4 Long Cylinder

In this section, we consider steady-periodic heating of an infinitely long cylinder. The heat conduction is along the radial direction only. The GF in this case satisfies the following equation:

$$\frac{\partial^2 G}{\partial r^2} + \frac{1}{r} \frac{\partial G}{\partial r} - \alpha^2 G = -\frac{1}{\alpha} \delta(r-r') \quad a < r < b \quad (6)$$

Using variation of parameters, the solution may be found in the form

$$G(r,r') = \frac{1}{2\pi\alpha(1-A_1A_2)} \times \left[ A_1 \left(I_0(\alpha r) + K_0(\alpha r')\right) + A_2 \left(I_0(\alpha r') + K_0(\alpha r)\right) \right]$$

where

$$A_1 = \begin{cases} 0 & \text{if } a = 0 \text{ (solid cylinder)} \\ -I_0(\alpha a) / K_0(\alpha a) & \text{if kind 1 at } r = a \\ I_0(\alpha a) / K_0(\alpha a) & \text{if kind 2 at } r = a \\ k\alpha I_0(\alpha a) - \lambda_1 I_0(\alpha a) / \lambda_1 K_0(\alpha a) & \text{if kind 3, 4, or 5 at } r = a \\ k\alpha K_0(\alpha a) + \lambda_1 K_0(\alpha a) / \lambda_1 K_0(\alpha a) & \text{if kind 3, 4, or 5 at } r = a \end{cases}$$

and

$$A_2 = \begin{cases} 0 & b \to \infty \\ -K_0(\alpha b) / I_0(\alpha b) & \text{if kind 1 at } r = b \\ K_0(\alpha b) / I_0(\alpha b) & \text{if kind 2 at } r = b \\ k\alpha K_0(\alpha b) - \lambda_2 K_0(\alpha b) / \lambda_2 K_0(\alpha b) & \text{if kind 3, 4, or 5 at } r = b \\ k\alpha I_0(\alpha b) + \lambda_2 I_0(\alpha b) / \lambda_2 K_0(\alpha b) & \text{if kind 3, 4, or 5 at } r = b \end{cases}$$

The set of GFs given in Eq. (7) applies to 36 combinations of boundary conditions—six kinds at $r_{\text{min}}$ and six at $r_{\text{max}}$. This includes the zeroth kind to represent a nonphysical boundary at $r = 0$ or $r \to \infty$. We are using a numbering system to identify these GFs in the form RJI, where R represents the radial coordinate, and I, J = 0, 1, ..., 5 represent the kinds of boundary condition present. For example, R11 denotes a hollow cylinder with type 1 boundaries. For more information on the numbering system, see Ref. [6] (Chap. 2).

5 Finite Cylinder With $T=T(r,z)$

Consider a finite-length hollow cylinder with outer radius $b$, inner radius $a$, and length $L$. This geometry can also describe a solid cylinder if $a=0$. Suppose the steady-periodic temperature in the finite cylinder does not depend on angle, then the temperature satisfies

$$\frac{1}{r} \frac{\partial}{\partial r} \left[ r \frac{\partial T}{\partial r} \right] + \frac{\partial^2 T}{\partial z^2} - \alpha^2 T = -\frac{g(r,z,\omega)}{k} \quad (10)$$

and at the boundaries

$$k \frac{\partial T}{\partial n_i} + [h_i + j\omega (\rho c_e)] T = f_i(r_i,z) \quad (11)$$

where $i = 1, 2, 3,$ and 4 represents boundaries at $r = a, r = b, z = 0,$ and $z = L$, respectively. Quantity $e_i$ is the thickness of a surface layer with high conductivity that may be present. The associated GF for the finite cylinder satisfies

$$\frac{1}{r} \frac{\partial}{\partial r} \left[ r \frac{\partial G}{\partial r} \right] + \frac{\partial^2 G}{\partial z^2} - \alpha^2 G = -\frac{1}{\alpha} \delta(r-r') \delta(z-z') \quad (12)$$

and at the boundaries

$$k \frac{\partial G}{\partial n_i} + \lambda_i G = 0 \quad (13)$$

where $\iota_i = \iota_1 + j\omega (\rho c_e)$.

The set of GFs given by Eqs. (12) and (13) represents a large number of geometries, with 36 combinations of boundary conditions along $r$ and 36 combinations along $z$ for a total of 1296 combinations. The number system for these GFs have the form R11DKL, where $R$ and $Z$ represent the coordinate directions, and $I$, $J$, $K$, and $L$ represent the types of boundary conditions present. For example, number R023Z3 represents a solid cylinder with convection (third kind) over the boundaries at $r = b, z = 0,$ and $z = L$.

There are two forms of the single-sum GF, one with eigenfunctions along the $z$-direction and the other with eigenfunction along the $r$-direction. Both are important, as one can be used to check the other, and where one converges slowly the other generally converges rapidly [11].

5.1 2D GF With Eigenfunctions Along $z$. The single-sum steady-periodic GF with eigenfunctions along the $z$-direction has the form

$$G(r,z|z',\omega) = \sum_{p=0}^{\infty} \frac{Z_p(z)Z_p(z')}{N_p(\nu_p)} Q_p(r,r') \quad (14)$$

where eigenfunctions $Z_p$ satisfy

$$Z_p'' + \nu_p^2 Z_p = 0$$

along with boundary conditions at $z = 0$ and $z = L$ drawn from those for $G$. Eigenfunctions $Z_p$, norm $N_p$, and eigenvalues $\nu_p$ are well-known values given in Table 1 (see also Refs. [11,12] (p. 35)).

Kernel function $Q_p(r,r')$ is identical with the 1D GF discussed earlier. That is,

$$Q_p(r,r') = G(r,r')|_{\alpha=\beta_p} \quad (15)$$

where $\beta_p^2 = \nu_p^2 + \alpha^2$.

5.2 2D GF With Eigenfunctions Along $r$. An alternate GF that satisfies Eq. (12) may also be constructed using eigenfunctions along the $r$-direction. If the $r$-direction eigenfunctions are denoted $R_m(r)$, then the alternate single-sum GF may be written

$$G(r,z|z',\omega) = \sum_{m=0}^{\infty} \frac{R_m(r)R_m(r')}{N_m(\gamma_m)} P_m(z,z') \quad (16)$$

Eigenfunctions $R_m$ satisfy

$\n$
Table 1: Eigenfunctions along the z-direction. Note B = hL/k; B = hL/k. (a) Eigenfunctions and (b) inverse norm and eigenvalues or conditions.

<table>
<thead>
<tr>
<th>Case</th>
<th>Zp(z)</th>
<th>Np−1</th>
<th>Eigencondition</th>
</tr>
</thead>
<tbody>
<tr>
<td>Z11, Z12, and Z13</td>
<td>sin(νpL)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Z21, Z22, and Z23</td>
<td>cos(νpL)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Z31, Z32, and Z33</td>
<td>νpL cos(νpL) + B1 sin(νpL)</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

(b) Conditions for computing eigenvalues

<table>
<thead>
<tr>
<th>Case</th>
<th>Np−1</th>
<th>νp or eigencondition</th>
</tr>
</thead>
<tbody>
<tr>
<td>Z11</td>
<td>2/L</td>
<td>pπ/L</td>
</tr>
<tr>
<td>Z12</td>
<td>2/L</td>
<td>(2p−1)/2</td>
</tr>
<tr>
<td>Z13</td>
<td>2Dp/L</td>
<td>νpL cot(νnL) = −Bz</td>
</tr>
<tr>
<td>Z21</td>
<td>2/L</td>
<td>(2p−1)/2</td>
</tr>
<tr>
<td>Z22</td>
<td>2/L</td>
<td>νpL cot(νnL) = 0</td>
</tr>
<tr>
<td>Z23</td>
<td>2Dp/L</td>
<td>νpL tan(νnL) = Bz</td>
</tr>
<tr>
<td>Z31</td>
<td>2</td>
<td>νpL cot(νnL) = −Bz</td>
</tr>
<tr>
<td>Z32</td>
<td>2</td>
<td>νpL tan(νnL) = Bz</td>
</tr>
</tbody>
</table>
| Z33 | 2Dp/L | tan(νnL) = BpL(B1 + B2) /
| &nbsp; | &nbsp; | (νnL)2 − B2 |


Table 2: Eigenfunctions and norm for solid cylinders. Note B = kL/h.

<table>
<thead>
<tr>
<th>Case</th>
<th>Rmn</th>
<th>Condition at r = b</th>
<th>(Nn)−1 for n ≠ 0</th>
<th>(Nn)−1 for n = 0</th>
</tr>
</thead>
<tbody>
<tr>
<td>R01</td>
<td>Rmn = 0</td>
<td>Jn(γmnb)</td>
<td>2</td>
<td>b2Jn(γmnb)</td>
</tr>
<tr>
<td>R02</td>
<td>dRmn / dr = 0</td>
<td>Jn(γmnb)</td>
<td>2</td>
<td>b2Jn(γmnb)</td>
</tr>
<tr>
<td>R03</td>
<td>k dRmn / dr + hRmn = 0</td>
<td>Jn(γmnb)</td>
<td>2</td>
<td>b2Jn(γmnb)</td>
</tr>
</tbody>
</table>

Table 3: Eigenconditions for solid cylinders

<table>
<thead>
<tr>
<th>Case</th>
<th>Eigencondition</th>
</tr>
</thead>
<tbody>
<tr>
<td>R01</td>
<td>Jn(γmnb) = 0</td>
</tr>
<tr>
<td>R02</td>
<td>Jn(γmnb) = 0</td>
</tr>
<tr>
<td>R03</td>
<td>γmnbL(Jn(γmnb) + B2Jn(γmnb)) = 0</td>
</tr>
</tbody>
</table>

Note: L = hL/k = B = hL/k. The eigenfunctions and the inverse norm are given in Table 2 for solid cylinders (0 < r < b). For hollow cylinders see Ref. [12] (pp. 108–113) or Ref. [13].

Kernel function Pm in is given by Ref. [10]:

\[ P_m(\xi,\gamma) = \frac{S_m(S_1 e^{-\beta_m(2L-\xi)} + S_1 e^{\beta_m(2z-\xi)})}{2\alpha \beta_m(S_1 - \gamma \beta_m e^{-2\beta_m \xi})} + \frac{S_m(S_1 e^{-\beta_m(\xi-\gamma)} + S_1 e^{\beta_m(\xi-\gamma)})}{2\alpha \beta_m(S_1 - \gamma \beta_m e^{-2\beta_m \xi})} \]

where \( \beta_m^2 = \gamma^2 + \sigma^2 \) and where subscripts 1 and 2 indicate sides z and \( \xi \), respectively.

6 Finite Cylinder With \( T = T(r, \phi, z) \)

In this section, the finite-length cylinder with three-dimensional heat conduction is treated. That is, temperature depends on spatial coordinates \( (r, \phi, z) \). The steady-periodic temperature satisfies

\[ \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial T}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 T}{\partial \phi^2} + \frac{\partial^2 T}{\partial z^2} = \frac{g(r, \phi, z)}{k} \]

and at the boundaries

\[ k \frac{\partial T}{\partial n_i} + \lambda T = f(r, \phi, z) \]

where \( i = 1, 2, 3, \) and 4 represents boundaries at \( r = a, r = b, z = 0, \) and \( z = L \), respectively.

The associated GF for 3D steady-periodic heat conduction in the finite cylinder satisfies

\[ \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial G}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 G}{\partial \phi^2} + \frac{\partial^2 G}{\partial z^2} = \frac{1}{\alpha} \delta(r - r') \delta(z - z') \delta(\phi - \phi') \]

and at the boundaries

\[ k \frac{\partial G}{\partial n_i} + \lambda G = 0 \]

The set of GFs represented by Eqs. (22) and (23) represent 1296 combinations of boundary conditions (36 along r and 36 along z), denoted by GF number RIJKZKL000. Here 000 denotes the angular dependence for the full cylinder, and \( I, J, K, L = 0, 1, \ldots, 5 \) denote the types of boundary conditions present. Two forms of the double-sum GF are discussed in Secs. 6.1 and 6.2.
6.1 3D GF With Eigenfunctions Along z. The double-sum steady-periodic GF with eigenfunctions along the z-direction has the form

\[ G(r, \phi, z')r', \phi', z', \omega) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{Z_p(n, m)(\omega)}{N_p(n, m)} Q_{np}(r, r') \]

The boundary conditions at \( z=0 \) and \( z=L \) are satisfied by the eigenfunctions in the z-direction, and the conditions at \( \phi=0 \) and \( \phi=2\pi \) are satisfied by the eigenfunctions in the \( \phi \)-direction. Here \( N_p \) is equal to \( \pi \) for \( n=0 \) and \( 2\pi \) for \( n=1 \).

Kernel function \( Q_{np} \), can be shown to have the form

\[ Q_{np}(r, r') = \frac{1}{2 \pi \alpha(1 - A_1 A_2)} \]

\[ \left\{ \begin{array}{ll} A_1 I_0(\beta_1 r) + A_2 K_0(\beta_2 r), & r < r' \\ A_1 I_0(\beta_1 r) + A_2 K_0(\beta_2 r), & r > r' \end{array} \right. \]

where

\[ A_1 = \frac{\beta_1 I_0(\beta_1 a) + n I_1(\beta_1 a)}{\beta_1 I_0(\beta_1 b) - n K_0(\beta_1 b)} \]

\[ A_2 = \frac{\beta_2 I_0(\beta_2 a) - n K_0(\beta_2 a)}{\beta_2 I_0(\beta_2 b) + n K_0(\beta_2 b)} \]

The quantities \( B_1 = \lambda_1 a/k_1 \) and \( B_2 = \lambda_2 b/k_2 \) are modified Biot numbers at the inner and outer radii, respectively. The above values for \( A_1 \) and \( A_2 \) are for the most general boundary condition (fifth kind). Values for other kinds of boundaries can be found by analogy with Eqs. (27)-(29). Some care is required when combining eigenfunctions \( Z_p \) and kernel functions \( Q_{np} \) as they depend on different Biot numbers. In the finite-length cylinder, eigenfunctions \( Z_p \) depend on Biot numbers \( B_M = \lambda_M k/l \), where \( \lambda_M \) is associated with the boundary conditions at \( z=0 \) and \( z=L \). As many as four distinct Biot numbers may be present in the finite cylinder.

6.2 3D GF With Eigenfunctions Along r. An alternate GF that satisfies Eq. (22) may be constructed using eigenfunctions along the z-direction. If the r-direction eigenfunctions are denoted \( R_{mn}(r) \), then the alternate double-sum GF may be written as

\[ G(r, \phi, z')r', \phi', z', \omega) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{R_{mn}(r)R_{nm}(r') \cos[n(\phi - \phi')]}{N_p(n, m)} \]

\[ \times \times P_m(z, z') \]

7 Temperature Examples

In the two next sections, numerical examples are given of the temperature in cylindrical geometries caused by steady-periodic heating. The first example is a cylinder heated at one end and experiencing axisymmetric convective heat loss from the other surfaces. The second example is a cylinder heated over a small region on its surface with convective heat loss.

8 Pin Fin With Heat Flux at Base

Steady-periodic heat transfer in fins has been studied several times [1–3]. Generally, a fin is long and thin and the temperature varies only along the axis of the fin; however, Kraus et al. [14] (Chap. 17) described the two-dimensional temperature in a rectangular fin with an oscillating base temperature. This example is concerned with a short cylindrical fin in which two-dimensional heat transfer is present. The base of the fin is uniformly heated by a steady-periodic heat flux, and the other surfaces are cooled by convection. This geometry is R03Z23. The temperature satisfies the following equations:

\[ \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial T}{\partial r} \right) + \frac{\partial^2 T}{\partial z^2} - \sigma^2 T = 0 \]

Mathematically, the temperature has a unique solution. However, there are two series forms of the GF that can provide two distinct series expressions for the temperature.

8.1 Eigenfunctions Along z. With eigenfunctions along the z-direction, the GF is given by

\[ G(r, z')z', \omega) = \sum_{p=1}^{\infty} \frac{Z_p(n, m)(\omega)}{N_p(n, m)} Q_{np}(r, r') \]

where the eigenfunction and norm are given by (Table 1)

\[ Z_p(z) = \cos(nz) \]

\[ N_p(n, m) = L(n, m) + \frac{B_2}{L(n, m)^2 + B_2} \]

where \( B_2 = hL/k \). Eigenvalues \( v_p \) satisfy \( v_p L \tan(v_p L) = hL/k \).

The kernel function is given by

\[ Q_{np}(r, r') = \frac{1}{2 \pi \alpha} \left\{ \begin{array}{ll} A_1 I_0(\beta_1 r) + A_2 K_0(\beta_2 r), & r < r' \\ A_1 I_0(\beta_1 r) + A_2 K_0(\beta_2 r), & r > r' \end{array} \right. \]

where \( A_2 \) is given by

\[ A_2 = \frac{\beta_2 L K_0(\beta_2 b) - B_2 K_0(\beta_2 b)}{\beta_2 L I_0(\beta_2 b) + B_2 I_0(\beta_2 b)} \]

and where \( \beta_2^2 = \gamma_2^2 + \sigma^2 \). Replace this GF into the temperature integral, Eq. (33), and evaluate the integral over \( r' \) to find
The cylinder is heated at \( z=0 \) and cooled by convection at \( r=b \) and \( z=L \). The heating frequency is fixed at \( \omega b^2/\alpha = 1.0 \) and the boundary convection is given by \( hb/k=0.2, 1.0, \) and \( 5.0 \) for the top, middle, and bottom of the figure, respectively.

The amplitude of the temperature in a cylinder of aspect ratio \( b/L=0.5 \) is given by

\[
G(r,z') = \sum_{n=1}^{\infty} R_n(r) I_n(z') N_n(\gamma_n) P_n(z, z') = 0
\]

where the eigenfunction and norm are given by (Table 2)

\[
R_n(r) = J_n(\gamma_n r)
\]

\[
\frac{1}{N_n} = \frac{2}{J_n^2(\gamma_n b)} \left[ (h/b) b^2 + b^2 \gamma_n^2 \right]
\]

where eigenvalue \( \gamma_n \) satisfies (Table 3)

\[
\gamma_n J_n(\gamma_n b) + B J_0(\gamma_n b) = 0
\]

Kernel function \( P \) is given by Eq. (17) for a type 2 boundary at \( z=0 \) and a type 3 boundary at \( z=L \) (case 223):

\[
P(z, z') = S_2 e^{-\beta_m(2L-z)} + S_2 e^{-\beta_m(z-L)}
\]

where \( \beta_m^2 = \gamma_m^2 + \sigma^2 \), \( S_1 = \beta_m L - hL/k \), and \( S_2 = \beta_m L + hL/k \). This form of the GF may be substituted into the temperature integral to find an alternate series expression for the temperature:

\[
T(r,z', \omega) = \sum_{n=1}^{\infty} \frac{1}{\gamma_n b} J_n(\gamma_n r) I_n(\gamma_n b) / J_0(\gamma_n b) \left[ (h/b) b^2 + b^2 \gamma_n^2 \right]
\]

\[
\times \frac{2 \gamma_n^2 b^2}{(\beta_m L - hL/k)e^{-\beta_m(2L-z)} + (\beta_m L + hL/k)e^{-\beta_m(z-L)}}
\]

Numerical values for the temperature in the pin fin were computed using both temperature series, Eqs. (39) and (45), and the results agree to five significant figures, providing a very strong check on the correctness of the results. As both series are exact, closer agreement could have been secured by including more terms in the truncated sum. In Fig. 1, the contour plots of the amplitude and phase of the temperature are given for a fin of aspect ratio \( b/L=0.5 \). The frequency is fixed at \( \omega b^2/\alpha = 1.0 \) and the results for Biot number \( hb/k=0.2, 1.0, \) and \( 5.0 \) are shown at the top, middle, and bottom of the figure, respectively. The amplitude of the temperature is largest where the heat is added \( (z=0) \) and decreases along the length of the fin. Heat leaves the fin along the \( r/b=1 \)

Fig. 1 Effect of varying convection on the amplitude and phase of the temperature in a cylinder of aspect ratio \( b/L=0.5 \). The cylinder is heated at \( z=0 \) and cooled by convection at \( r=b \) and \( z=L \). The heating frequency is fixed at \( \omega b^2/\alpha = 1.0 \) and the boundary convection is given by \( hb/k=0.2, 1.0, \) and \( 5.0 \) for the top, middle, and bottom of the figure, respectively.

8.2 Eigenfunctions Along \( r \). An alternate form of the GF has eigenfunctions along the \( r \)-direction, and is given by

\[
T(r,z,\omega) = \sum_{\beta_m} \cos(\beta_m r) \frac{2b}{b} \left( \frac{rL}{bL} \right)^2 + b^2 + B^2
\]

\[
\times \left( \frac{1}{\beta_m^2} \left[ A \left[ (\beta_m b) - K_i(\beta_m b) \right] I_0(\beta_m r) + \frac{1}{\beta_m^2 b^2} \right] \right)
\]

(39)

8.2 Eigenfunctions Along \( r \). An alternate form of the GF has eigenfunctions along the \( r \)-direction, and is given by

\[
G(r,z') = \sum_{m=1}^{\infty} R_m(r) I_m(z') N_m(\gamma_m) P_m(z, z') = 0
\]

where the eigenfunction and norm are given by (Table 2)

\[
R_m(r) = J_m(\gamma_m r)
\]

\[
\frac{1}{N_m} = \frac{2}{J_m^2(\gamma_m b)} \left[ (h/b) b^2 + b^2 \gamma_m^2 \right]
\]

where eigenvalue \( \gamma_m \) satisfies (Table 3)

\[
\gamma_m J_m(\gamma_m b) + B J_0(\gamma_m b) = 0
\]

Kernel function \( P \) is given by Eq. (17) for a type 2 boundary at \( z=0 \) and a type 3 boundary at \( z=L \) (case 223):

\[
P(z, z') = S_2 e^{-\beta_m(2L-z)} + S_2 e^{-\beta_m(z-L)}
\]

where \( \beta_m^2 = \gamma_m^2 + \sigma^2 \), \( S_1 = \beta_m L - hL/k \), and \( S_2 = \beta_m L + hL/k \). This form of the GF may be substituted into the temperature integral to find an alternate series expression for the temperature:

\[
T(r,z,\omega) = \sum_{m=1}^{\infty} \frac{1}{\gamma_m b} J_m(\gamma_m r) I_m(\gamma_m b) / J_0(\gamma_m b) \left[ (h/b) b^2 + b^2 \gamma_m^2 \right]
\]

\[
\times \left( \frac{2 \gamma_m^2 b^2}{(\beta_m L - hL/k)e^{-\beta_m(2L-z)} + (\beta_m L + hL/k)e^{-\beta_m(z-L)}} \right)
\]

(45)

Numerical values for the temperature in the pin fin were computed using both temperature series, Eqs. (39) and (45), and the results agree to five significant figures, providing a very strong check on the correctness of the results. As both series are exact, closer agreement could have been secured by including more terms in the truncated sum. In Fig. 1, the contour plots of the amplitude and phase of the temperature are given for a fin of aspect ratio \( b/L=0.5 \). The frequency is fixed at \( \omega b^2/\alpha = 1.0 \) and the results for Biot number \( hb/k=0.2, 1.0, \) and \( 5.0 \) are shown at the top, middle, and bottom of the figure, respectively. The amplitude of the temperature is largest where the heat is added \( (z=0) \) and decreases along the length of the fin. Heat leaves the fin along the \( r/b=1 \)

Fig. 2 Fin effectiveness in the pin fin heated at the base \( (z=0) \) as a function of Biot number and dimensionless frequency \( \omega b^2/\alpha \) for aspect ratios \( b/L=0.1, 0.5, \) and \( 1.0 \)
boundary, demonstrated by the slope of the temperature at the boundary, which is proportional to heat flux. Thus, as the Biot number increases, the slope at the boundary increases along with the boundary heat flux.

The phase of the temperature shown in Fig. 1 is negative and closest to zero at the heating location \( z=0 \). As \( z \) increases, the phase becomes more negative (moving further from zero). For the smallest Biot number (at the top of the figure), the change in phase along the fin is most pronounced, and as the Biot number increases there is less change in phase along the fin.

In Fig. 2, the effectiveness of fins with three different aspect ratios are graphed as a function of the Biot number. As frequency increases, the fin effectiveness is an approximate thermal model of a hot-film sensor used to determine the temperature fluctuations for the heat to transfer down the fin. It is also interesting to note that thinner fins are more effective at lower frequencies but this is not necessarily true at higher frequencies. For example, a fin with a Biot number of \( 10^{-3} \), an aspect ratio of 1, and an effectiveness of 2 would have an effectiveness somewhat less than 2 if the aspect ratio was changed to 0.1 while holding all of the other parameters constant.

### 9 Solid Cylinder Heated Over a Sector of Its Surface and Cooled by Convection

Consider a solid cylinder with steady-periodic heating over an angular sector of the curved surface, parallel to the cylinder axis, and cooled by convection over the entire surface. The flat ends of the cylinder are fixed at the fluid temperature. This geometry is an approximate thermal model of a hot-film sensor used to measure fluid flow. The temperature satisfies the following equations:

\[
\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial T}{\partial r} \right) + \frac{\partial^2 T}{\partial z^2} - \sigma^2 T + \frac{g(r, \phi, z)}{k} = 0 \tag{46}
\]

where:

- \( T(z=0) = T_0 \)
- \( T(z=L) = T_\infty \)
- \( T(r=b, z) = \frac{\partial T}{\partial r} + hT = hT_\infty \)

The heating function is given by

\[
g(r, \phi, z) = \begin{cases} 
q_0 \delta(r-b) & 0 < \phi < \phi_0 \\
q_0 & \phi_0 < \phi < 2\pi 
\end{cases} \tag{50}
\]

Note that the heat is introduced at surface \( r=b \). This is geometry R03Z11F00 in the heat conduction numbering system. The temperature may be stated in the form of an integral with the GF, as follows:

\[
T(r, \phi, z, \omega) - T_\infty = \frac{1}{k} \phi_0 \int_{\phi_0}^{\phi} \int_{z=0}^{L} q_0 G(r, \phi, z, \omega') \, dz' \, d\phi' \, b \tag{51}
\]

There are two forms of the GF that allow for two distinct series expressions for the temperature.

#### 9.1 Eigenfunctions Along \( z \)

With eigenfunctions along the \( z \)-direction, the GF is given by Eq. (24). The eigenfunction and norm are given by Table 2 (case Z11) and the kernel function is given by Eq. (25) (for case R03). Replace the GF into the temperature integral, Eq. (51), and evaluate the integrals on \( \phi' \) and \( z' \):

\[
T(r, \phi, z) - T_\infty = \frac{1}{k} \phi_0 \int_{\phi_0}^{\phi} \int_{z=0}^{L} q_0 G(r, \phi, z, \omega') \, dz' \, d\phi' \, b \tag{52}
\]

where

\[
C_\phi = \begin{cases} 
\phi_0 / \pi, & n = 0 \\
(n\pi - \sin(n\phi - \phi_0))/(2\pi n), & n \neq 0
\end{cases}
\]

and where

\[
A_n = \frac{[\beta_h h L_n(n) - nK_0(\beta_h)] - B_n K_0(\beta_h)}{[\beta_h h L_{n+1}(n) + nL_0(\beta_h)] + B_n L_0(\beta_h)} \tag{53}
\]

with \( B_n = \pi b / k \). Note that the integral over \( \phi' \) must be treated separately when \( n=0 \).

#### 9.2 Eigenfunctions Along \( r \)

An alternate form of the GF, with eigenfunctions along the \( r \)-direction, is given by Eq. (28). The eigenfunction and norm are given by case R03 in Table 3, and the kernel function \( P_n \) is given by Eq. (17), with \( \beta^2 = \gamma^2_m + \sigma^2 \). Replace the alternate GF into the integral expression for the GF and evaluate the integrals over \( \phi' \) and \( z' \) to find the alternate series expression for the temperature.

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**Fig. 3** Amplitude and phase of the temperature around the circumference of a cylinder \( r=b, z=L/2 \) for several values of the (dimensionless) heating frequency. The cylinder surface is heated steady periodically over a small strip \( 0<\phi<0.2 \) and the convection on the curved surface is characterized by \( B_n=1 \).
\[ T(r, \phi, z) - T_m = \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} J_n(\gamma_m r) 2\gamma_m^2 J_n(\gamma_m b) (b/2)^n \left( b^2 \gamma_m^2 - n^2 \right) \times C_n \left[ e^{-\beta(r-L_z)} - e^{-\beta(r-L_z)} - e^{-2\beta L_z} - e^{-\beta L_z} \right] \]

(54)

where \( C_n \) is given above. Note that additive term \( 1/\beta^2 \), from integration on \( z' \) of the kernel function, may cause slow series convergence because this portion of the series does not contain a convergence-promoting exponential function. The series containing this additive term can be shown to correspond to a two-dimensional temperature distribution that does not depend on coordinate \( z \) and it can be replaced by a faster-converging single-sum form (see Ref. [11]).

Numerical values were computed for the amplitude and phase of the dimensionless temperature on the cylinder surface \( r=b \) and at the midpoint \( z=L/2 \). The heated strip is located on \( 0<\phi <0.2 \) and the aspect ratio of the cylinder is \( b/L=0.2 \).

Figure 3 shows the temperature amplitude and phase at the spatial location \( (r=b, \ z=L/2) \), with several different heating frequencies and the Biot number is fixed at \( B_i=1 \). The temperature amplitude decreases as the frequency increases, also the temperature becomes more localized to the heater at higher frequencies. As an explanation, there is less time for angular heat diffusion at higher frequencies. For all frequencies, the phase contains a flat region far from the heater; however, the level of this flat region increases with increasing frequency.

10 Summary

In this paper, a family of solutions to steady-periodic heating in cylinders has been presented with the method of Green’s functions. Five types of boundary conditions have been treated. Solutions are given for cylinder geometries described by one, two, and three spatial coordinates, along with numerical examples. For some geometries alternate forms of the GF are given, which can be used for checking purposes and for improving the convergence behavior of the resulting temperature solutions. One application of these steady-periodic heat conduction solutions is for use with inverse methods for determining thermal properties from experimental temperature data.

Nomenclature

- \( a \) = inner radius (m)
- \( b \) = outer radius (m)
- \( B_i \) = Biot number at boundary \( i \)
- \( f_i \) = known effect at boundary \( i \)
- \( g \) = internal heating at frequency \( \omega \)
- \( G \) = steady-periodic Green’s function
- \( h_i \) = heat transfer coefficient (W m\(^{-2}\) K\(^{-1}\))
- \( i \) = index
- \( I_n \) = modified Bessel function, order \( n \)
- \( j \) = imaginary number, \( \sqrt{-1} \)
- \( k \) = thermal conductivity (W m\(^{-1}\) K\(^{-1}\))
- \( K_n \) = modified Bessel function, order \( n \)
- \( L \) = length of domain in \( z \)-direction (m)
- \( m \) = index
- \( n \) = index
- \( n_i \) = outward-facing unit normal vector on boundary \( i \)
- \( N \) = norm
- \( p \) = index
- \( P \) = kernel function along \( z \) direction
- \( q \) = steady-periodic heat flux (W m\(^{-2}\))
- \( Q \) = kernel function along \( r \) direction
- \( r \) = radial coordinate
- \( R \) = eigenfunction along \( r \)-direction
- \( S_M \) = coefficient for kernel function \( P \)
- \( t \) = time (s)
- \( T \) = steady-periodic temperature (K)
- \( z \) = axial coordinate
- \( Z_p \) = eigenfunction along \( z \)-direction

Greek Symbols

- \( \alpha \) = thermal diffusivity (m\(^2\) s\(^{-1}\))
- \( \beta \) = parameter for kernel function
- \( \delta \) = Dirac delta function
- \( \epsilon \) = thickness of surface layer on boundary
- \( \gamma \) = eigenvalue associated with \( R \)
- \( \lambda \) = boundary parameter, type 3, 4, or 5
- \( \nu_p \) = eigenvalue associated with \( Z_p \)
- \( \rho \) = density (kg m\(^{-3}\))
- \( \sigma \) = \([jo/\alpha]^{1/2}\)
- \( \phi \) = angular coordinate (rad)
- \( \omega \) = frequency (rad s\(^{-1}\))

References