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A PROOF OF NEWTON’S POWER SUM FORMULAS

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For a polynomial \( P(z) = \alpha_0 + \alpha_1 z + \cdots + \alpha_n z^n = \alpha_n (z - z_1) (z - z_2) \cdots (z - z_n) \),
the power sums \( S_m = \sum_{k=1}^{n} x_k^m \), \( m = 1, 2, \cdots \), can be calculated from the formulas

\[
\sum_{m=1}^{n} \alpha_{n-m+b} S_k = 0 \quad \text{if} \quad m \leq n,
\]

\[
\sum_{k=m-n}^{m} \alpha_{n-m+b} S_k = 0 \quad \text{if} \quad m > n.
\]

For example, if \( n = 3 \),
\[
S_1 = - \alpha_2 \alpha_3^{-1}, \quad S_2 = \alpha_2 \alpha_3^{-2} - 2 \alpha_1 \alpha_3^{-1}, \quad S_3 = - \alpha_3^{-3} + 3 \alpha_1 \alpha_2 \alpha_3^{-2} - 3 \alpha_0 \alpha_3^{-1},
\]
\[
S_4 = \alpha_2 \alpha_3^{-4} - 4 \alpha_1 \alpha_2 \alpha_3^{-3} + 4 \alpha_0 \alpha_2 \alpha_3^{-2} + 2 \alpha_1 \alpha_3^{-2}.
\]
One quick, but rather vague, method of proving (1) is to differentiate $P(z)$ two different ways, equating like powers of $z$ (see [1]). Another method is through the use of the theory of symmetric functions (see [2]). The student might find the following proof more satisfying: The logarithmic derivative of the polynomial \[ Q(z) = \alpha_n + \alpha_{n-1}z + \cdots + \alpha_0 z^n = \alpha_0 (z - z_1^{-1})(z - z_2^{-1}) \cdots (z - z_n^{-1}) \] (assuming, without loss of generality, that $P(0) \neq 0$) is
\[ F(z) = \frac{Q'(z)}{Q(z)} = \sum_{k=1}^{n} (z - z_k^{-1})^{-1} \]
which, when differentiated $k$ times, gives $F^{(k)}(0) = -k! S_{k+1}$. Since
\[ Q^{(m)}(z) = [F(z)Q(z)]^{(m-1)} = \sum_{k=0}^{m-1} \binom{m - 1}{k} F^{(k)}(z)Q^{(m-1-k)}(z), \]
we have
\[ \frac{Q^{(m)}(0)}{m!} = - \frac{1}{m} \sum_{k=0}^{m-1} \frac{Q^{(m-1-k)}(0)}{(m-1-k)!} S_{k+1} \]
or
\[ -m \alpha_{n-m} = \sum_{k=0}^{m-1} \alpha_{n-m+k+1} S_{k+1} \quad \text{if } m \leq n, \]
\[ 0 = \sum_{k=m-n-1}^{m-1} \alpha_{n-m+k+1} S_{k+1} \quad \text{if } m > n, \]
which is (1).

References
2. B. L. van der Waerden, Modern Algebra, Frederick Ungar, New York, 1953.