Knörrer Periodicity and Bott Periodicity

Michael K. Brown
University of Nebraska-Lincoln, mbrown15@math.unl.edu

Follow this and additional works at: http://digitalcommons.unl.edu/mathstudent
Part of the Algebra Commons, and the Geometry and Topology Commons

http://digitalcommons.unl.edu/mathstudent/59

This Article is brought to you for free and open access by the Mathematics, Department of at DigitalCommons@University of Nebraska - Lincoln. It has been accepted for inclusion in Dissertations, Theses, and Student Research Papers in Mathematics by an authorized administrator of DigitalCommons@University of Nebraska - Lincoln.
KNÖRRER PERIODICITY AND BOTT PERIODICITY

by

Michael K. Brown

A DISSERTATION

Presented to the Faculty of

The Graduate College at the University of Nebraska

In Partial Fulfilment of Requirements

For the Degree of Doctor of Philosophy

Major: Mathematics

Under the Supervision of Mark E. Walker

Lincoln, Nebraska

May, 2015
The main goal of this dissertation is to explain a precise sense in which Kn" orrer periodicity in commutative algebra is a manifestation of Bott periodicity in topological K-theory. In Chapter 2, we motivate this project with a proof of the existence of an 8-periodic version of Kn" orrer periodicity for hypersurfaces defined over the real numbers. The 2- and 8-periodic versions of Kn" orrer periodicity for complex and real hypersurfaces, respectively, mirror the 2- and 8-periodic versions of Bott periodicity in $KU$- and $KO$-theory. In Chapter 3, we introduce the main tool we need to demonstrate the compatibility between Kn" orrer periodicity and Bott periodicity: a homomorphism from the Grothendieck group of the homotopy category of matrix factorizations associated to a complex (real) polynomial $f$ into the topological K-theory of its Milnor fiber (positive or negative Milnor fiber). A version of this map first appeared in the setting of complex isolated hypersurface singularities in the paper “An Index Theorem for Modules on a Hypersurface Singularity”, by Buchweitz and van Straten. We show that, when $f$ is non-degenerate quadratic (over the real or complex numbers), this map recovers the Atiyah-Bott-Shapiro construction in topology. In Chapter 4, we prove that when $f$ is a complex simple plane curve singularity, this homomorphism is injective.
ACKNOWLEDGMENTS

I must first of all thank my advisor, Mark Walker, for his support during the past four years; I could not have written this dissertation without his guidance. A student could not ask for a better advisor than Mark.

I thank Luchezar Avramov and Brian Harbourne for their reading of my thesis and comments on it. I also thank Ragnar Buchweitz, Jesse Burke, Michael Hopkins, and Claudia Miller for valuable conversations with regard to this work.
# Contents

1 Introduction  

2 Knörrer Periodicity over $\mathbb{R}$  
   2.1 Differential $\mathbb{Z}/2\mathbb{Z}$-graded categories  
   2.1.1 The derived category of a differential $\mathbb{Z}/2\mathbb{Z}$-graded category  
   2.1.2 Triangulated differential $\mathbb{Z}/2\mathbb{Z}$-graded categories  
   2.1.3 Hochschild homology and the Chern character  
   2.2 Matrix factorization categories  
   2.2.1 Definitions and some properties  
   2.2.2 Triangulated structure  
   2.2.3 The Hochster theta pairing  
   2.2.4 Stabilization  
   2.3 The tensor product of matrix factorizations  
   2.4 Clifford algebras  
   2.5 An example: $f = y^2 - x^2(x+1) \in \mathbb{C}[x,y]_{(x,y)}$  
   2.6 Periodicity  

3 Matrix Factorizations and the K-theory of the Milnor Fiber
3.1 The real and complex Milnor fibers ........................................ 59

3.1.1 Construction of the Milnor fibration and some properties of the Milnor fiber ....................................................... 60

3.1.2 The Sebastiani-Thom homotopy equivalence ...................... 62

3.1.3 An analogue of the Milnor fibration for polynomials over \( \mathbb{R} \) ... 67

3.2 Relative topological K-theory and the Euler characteristic .......... 68

3.3 A generalized Atiyah-Bott-Shapiro construction .................... 79

3.4 Knörrer periodicity and Bott periodicity ............................... 88

4 Examples: the ADE singularities ........................................... 97

4.1 Maximal Cohen-Macaulay modules over the ADE singularities .... 97

4.2 Hochschild homology of matrix factorization categories .......... 100

4.3 An application of the Hirzebruch-Riemann-Roch formula for differential \( \mathbb{Z}/2\mathbb{Z} \)-graded categories ............................... 103

Bibliography ................................................................. 108
Chapter 1

Introduction

Matrix factorizations were introduced by Eisenbud in [Eis80] as a tool for studying the homological behavior of modules over a hypersurface ring; that is, a quotient of a regular ring by a principal ideal generated by a non-unit, non-zero-divisor. Recently, matrix factorizations have begun appearing in a wide variety of contexts, for instance:

- Homological mirror symmetry (e.g. [KKP08], by Katzarkov-Kontsevich-Pantev)
- Knot theory (e.g. [KR04], by Khovanov-Rozansky)
- Singularity theory (e.g. [BVS12], by Buchweitz-van Straten)

The overall goal of this work is to continue the study of an interplay between matrix factorizations and topological K-theory that was begun in the inspiring paper [BVS12].

Let $f \in \mathbb{C}\{x_1,\ldots,x_n\}$ be a convergent power series such that

$$R = \mathbb{C}\{x_1,\ldots,x_n\}/(f)$$
defines an isolated hypersurface singularity. One of the key insights in [BVS12] is that, by passing to topological information about the hypersurface, the vanishing of the Hochster theta pairing associated to the hypersurface ring $R$ when $n$ is odd can be viewed as a consequence of Bott periodicity in topological K-theory. The main goal of this thesis is to express precisely the manner in which Bott periodicity manifests itself in commutative algebra: it turns out that the answer is Knörrer periodicity, a behavior of maximal Cohen-Macaulay modules over certain hypersurface rings discovered by Knörrer ([Knö87] Theorem 3.1).

In Chapter 2, we establish various results concerning differential $\mathbb{Z}/2\mathbb{Z}$-graded categories of matrix factorizations. Most of the results we discuss are well-known; among the new results in this chapter is an 8-periodic version of Knörrer periodicity for isolated hypersurface singularities over the real numbers.

**Theorem 1.0.1.** Let $Q := \mathbb{R}[x_1, \ldots, x_n]$ and $f \in Q$. Suppose $Q/(f)$ has an isolated singularity at the origin (i.e. $\dim_{\mathbb{R}} \mathbb{R}[[x_1, \ldots, x_n]](\partial f / \partial x_1, \ldots, \partial f / \partial x_n) < \infty$). Set $Q' := \mathbb{R}[u_1, \ldots, u_8]$, $q := u_1^2 + \cdots + u_8^2 \in Q'$, and $Q'' := Q \otimes_{\mathbb{R}} Q'$. Then there exists an equivalence of triangulated categories

$$[\text{MF}(\hat{Q}, f)] \xrightarrow{\sim} [\text{MF}(\hat{Q}'', f + q)],$$

where $\hat{(\quad)}$ denotes completion at the homogeneous maximal ideal.

We point out that the “period” here is exactly 8; that is, for $1 \leq l < 8$, it can happen that

$$[\text{MF}(\mathbb{R}[[x_1, \ldots, x_n]], f)] \not\cong [\text{MF}(\mathbb{R}[[x_1, \ldots, x_n, u_1, \ldots, u_l]], f + u_1^2 + \cdots + u_l^2)].$$

Our proof relies heavily on machinery developed by Dyckerhoff and Toën in [Dyc11] and [Toe07]. This result draws a distinction between the maximal Cohen-
Macaulay representation theory of hypersurface rings with ground field $\mathbb{R}$ and those whose ground field is algebraically closed and has characteristic not equal to 2, since the latter exhibit 2-periodic Knörrer periodicity. The maximal Cohen-Macaulay representation theory of hypersurface rings with ground field $\mathbb{R}$ does not seem to be well-studied, and we hope this work motivates further investigation in this direction.

The presence of 2- and 8-periodic versions of Knörrer periodicity over $\mathbb{C}$ and $\mathbb{R}$, respectively, suggests the possibility of a compatibility between Knörrer periodicity and Bott periodicity. Such a compatibility statement is formulated and proved in Chapter 3 (see Theorem 3.4.4):

**Theorem 1.0.2.** Let $Q := \mathbb{C}[x_1, \ldots, x_n]$, and suppose $f$ is an element of $Q$ such that $Q/(f)$ has an isolated singularity at the origin (i.e. $\dim_{\mathbb{C}} \frac{\mathbb{C}[x_1, \ldots, x_n]}{(\partial f/\partial x_1, \ldots, \partial f/\partial x_n)} < \infty$). Then there exists a commutative diagram

$$
\begin{array}{ccc}
K_0[\text{MF}(Q, f)] & \xrightarrow{\phi_f} & KU^0(B_\epsilon, F_f) \\
\downarrow & & \downarrow \beta \\
K & & KU^0(B_\epsilon, F_f) \otimes KU^0(B_{\epsilon'}, F_{u^2+v^2}) \\
\downarrow & & \downarrow \text{ST}_{KU} \\
K_0[\text{MF}(Q[u, v], f + u^2 + v^2)] & \xrightarrow{\phi_{f+u^2+v^2}} & KU^0(B_{\epsilon'}, F_{f+u^2+v^2})
\end{array}
$$

where $F_f$ denotes the Milnor fiber of $f$, $B_\epsilon$ is a closed ball of radius $\epsilon$ in $\mathbb{C}^n$, $K$ is induced by the Knörrer functor, $\beta$ is the Bott periodicity isomorphism, and $\text{ST}_{KU}$ is given by the product in relative $K$-theory followed by the inverse of the map induced by pullback along the Sebastiani-Thom homotopy equivalence.
The Sebastiani-Thom homotopy equivalence to which we refer in Theorem 1.0.2 is discussed in Section 3.1.2.

The key construction in this chapter gives a way of building the horizontal maps above; specifically, given a polynomial $f$ over the complex (real) numbers, we construct a map $\Phi_f$ that assigns to a matrix factorization of a complex (real) polynomial $f$ a class in the topological K-theory of the Milnor fiber (positive or negative Milnor fiber) of $f$; this map first appeared in [BVS12] in the setting of complex isolated hypersurface singularities. We prove that this construction induces a map $\phi_f$ on the Grothendieck group of the (triangulated) homotopy category of matrix factorizations of $f$, and we show that it recovers the Atiyah-Bott-Shapiro construction when $f$ is a non-degenerate quadratic (over $\mathbb{R}$ or $\mathbb{C}$). The Atiyah-Bott-Shapiro construction, introduced in Part III of [ABS64], provides the classical link between $\mathbb{Z}/2\mathbb{Z}$-graded modules over Clifford algebras and vector bundles over spheres; the map $\phi_f$ we discuss in Chapter 3 can be thought of as providing a more general link between algebra and topology.

In Chapter 4, we apply the Atiyah-Bott-Shapiro-type construction $\phi_f$ from Chapter 3 to matrix factorizations of the ADE singularities, or simple plane curve singularities. The main result of this chapter is:

**Theorem 1.0.3.** If $f \in \mathbb{C}[x_1, x_2]$ is an ADE singularity, $\phi_f$ is injective.

The proof makes heavy use of key results in [BVS12] and [PV12].
Chapter 2

Knörrer Periodicity over $\mathbb{R}$

In this chapter, we recall some foundational material concerning matrix factorizations in commutative algebra, and we exhibit an 8-periodic version of Knörrer periodicity for matrix factorization categories associated to isolated hypersurface singularities over the real numbers.

2.1 Differential $\mathbb{Z}/2\mathbb{Z}$-graded categories

Let $k$ be a field. We review some facts concerning $k$-linear differential $\mathbb{Z}/2\mathbb{Z}$-graded categories. In this section, all categories and functors are $k$-linear.

All of the results in Sections 2.1.1 and 2.1.2 are $\mathbb{Z}/2\mathbb{Z}$-graded variants of results in the setting of differential $\mathbb{Z}$-graded categories appearing in [Toë11]. We refer the reader to Section 5.1 of [Dyc11] for a discussion of how one may reformulate Töen’s homotopy theory of dg-categories so that it applies to the $\mathbb{Z}/2\mathbb{Z}$-graded setting.

Henceforth, when we use the term “dg category”, we mean “differential $\mathbb{Z}/2\mathbb{Z}$-graded category”.

2.1.1 The derived category of a differential $\mathbb{Z}/2\mathbb{Z}$-graded category

Define $C_{\mathbb{Z}/2\mathbb{Z}}(k)$ to be the dg category of $\mathbb{Z}/2\mathbb{Z}$-graded complexes of $k$-modules. Fix a dg category $\mathcal{T}$.

**Definition 2.1.1.** The *homotopy category* of $\mathcal{T}$, denoted by $[\mathcal{T}]$, is the category given by the following:

- Objects in $[\mathcal{T}]$ are the same as the objects in $\mathcal{T}$.
- Given two objects $X, Y$ of $[\mathcal{T}]$, the morphisms from $X$ to $Y$ are given by the $0^{th}$ cohomology vector space $H^0\text{Hom}_\mathcal{T}(X, Y)$ of the $\mathbb{Z}/2\mathbb{Z}$-graded complex $\text{Hom}_\mathcal{T}(X, Y)$.

**Remark 2.1.2.** A dg functor $F : \mathcal{S} \to \mathcal{T}$ determines an additive functor $[F] : [\mathcal{S}] \to [\mathcal{T}]$.

We introduce the *derived category* of $\mathcal{T}$:

**Definition 2.1.3.** A module $M$ over $\mathcal{T}$ is a dg functor

$$M : \mathcal{T} \to C_{\mathbb{Z}/2\mathbb{Z}}(k).$$

One may form the dg category $\text{Mod}(\mathcal{T})$ of modules over $\mathcal{T}$ in the evident way.

The dg category $\text{Mod}(\mathcal{T})$ may be equipped with a $C_{\mathbb{Z}/2\mathbb{Z}}(k)$-enriched model structure such that the weak equivalences are given by morphisms

$$F \to F'$$
having the property that the induced maps $F(x) \to F'(x)$ are quasi-isomorphisms of $\mathbb{Z}/2\mathbb{Z}$-graded complexes for all objects $x \in \mathcal{T}$. We refer the reader to Section 3.2 of [Toë11] for details.

**Definition 2.1.4.** The *derived category* of $\mathcal{T}$, denoted $D(\mathcal{T})$, is the homotopy category $Ho(\text{Mod}(\mathcal{T}))$. That is, there is a functor

$$L_{\mathcal{T}} : \text{Mod}(\mathcal{T}) \to Ho(\text{Mod}(\mathcal{T}))$$

sending weak equivalences to isomorphisms, and the pair $(L_{\mathcal{T}}, Ho(\text{Mod}(\mathcal{T})))$ is universal with respect to this property.

**Remark 2.1.5.** Note the distinction between the homotopy category $[\text{Mod}(\mathcal{T})]$ of $\text{Mod}(\mathcal{T})$ in the dg sense and the homotopy category $Ho(\text{Mod}(\mathcal{T}))$ of $\text{Mod}(\mathcal{T})$ in the model-theoretic sense. This collision of terminology should cause no confusion in what follows.

### 2.1.2 Triangulated differential $\mathbb{Z}/2\mathbb{Z}$-graded categories

Denote by $\mathcal{T}^{op}$ the opposite category of $\mathcal{T}$; that is, the category with the same objects, but with composition $f \circ g$ replaced with $(-1)^{|f||g|}g \circ f$. The Yoneda functor

$$h_{\mathcal{T}} : [\mathcal{T}] \to D(\mathcal{T}^{op})$$

is given, on objects, by

$$T \mapsto L_{\mathcal{T}}(S \mapsto (\text{Hom}_{\mathcal{T}}(S, T)))$$

and by the evident map on morphisms.
We say an object in $D(\mathcal{T}^{\text{op}})$ is \textit{quasi-representable} if it is in the essential image of $h_{\mathcal{T}}$. An object $M \in D(\mathcal{T}^{\text{op}})$ is \textit{compact} if $\text{Hom}_{D(\mathcal{T}^{\text{op}})}(M, -)$ commutes with coproducts. Quasi-representable objects are compact, but the converse is not true.

**Definition 2.1.6.** We say $\mathcal{T}$ is \textit{dg-triangulated} if every compact object in $D(\mathcal{T}^{\text{op}})$ is quasi-representable.

**Remark 2.1.7.** We give an example of a dg category that is not dg-triangulated in Section 2.5.

**Remark 2.1.8.** Denote by $D(\mathcal{T}^{\text{op}})_c$ the full subcategory of compact objects in $D(\mathcal{T}^{\text{op}})$. If $\mathcal{T}$ is dg-triangulated, $h_{\mathcal{T}} : [\mathcal{T}] \to D(\mathcal{T}^{\text{op}})_c$ is an equivalence. Since $D(\mathcal{T}^{\text{op}})_c$ is triangulated, it follows that, when $\mathcal{T}$ is dg-triangulated, $[\mathcal{T}]$ may be equipped with a canonical triangulated structure.

Every dg category may be embedded in a dg-triangulated category, its \textit{triangulated hull}. To define the triangulated hull, we must introduce the \textit{homotopy category of dg categories}.

**Definition 2.1.9.** A dg functor $F : \mathcal{S} \to \mathcal{T}$ is a \textit{quasi-equivalence} if $[F] : [\mathcal{S}] \to [\mathcal{T}]$ is an equivalence of categories.

Consider the category $\text{dg}_{\mathbb{Z}/2\mathbb{Z}}k\text{-cat}$ of $k$-linear dg categories. There exists a $C_{\mathbb{Z}/2\mathbb{Z}}(k)$-enriched model structure on $\text{dg}_{\mathbb{Z}/2\mathbb{Z}}k\text{-cat}$ with weak equivalences given by quasi-equivalences. For details, we refer the reader to Section 3.2 of [Toë11].

In particular, there is a category $\text{Ho}(\text{dg}_{\mathbb{Z}/2\mathbb{Z}}k\text{-cat})$ and a functor

$$L : \text{dg}_{\mathbb{Z}/2\mathbb{Z}}k\text{-cat} \to \text{Ho}(\text{dg}_{\mathbb{Z}/2\mathbb{Z}}k\text{-cat})$$

that maps quasi-equivalences to isomorphisms such that the pair $(L, \text{Ho}(\text{dg}_{\mathbb{Z}/2\mathbb{Z}}k\text{-cat}))$ is universal with respect to this property.
Denote by $\text{dg}_{Z/2Z}k$-cat the full subcategory of $\text{dg}_{Z/2Z}k$-cat given by dg-triangulated categories.

**Proposition 2.1.10 ([Toë11] Prop 4.4.2).** The inclusion functor

$$i : \text{Ho}(\text{dg}_{Z/2Z}k\text{-cat}^{\text{tr}}) \to \text{Ho}(\text{dg}_{Z/2Z}k\text{-cat})$$

admits a left adjoint

$$\text{Perf}(\cdot) : \text{Ho}(\text{dg}_{Z/2Z}k\text{-cat}) \to \text{Ho}(\text{dg}_{Z/2Z}k\text{-cat}^{\text{tr}}).$$

**Definition 2.1.11.** Given a dg category $\mathcal{T}$, we shall call $\text{Perf}(\mathcal{T})$ the triangulated hull of $\mathcal{T}$.

It will be useful for us to have an explicit model for the triangulated hull of a dg category. To construct it, we must introduce the notion of a perfect module over a dg category:

**Definition 2.1.12.** A module $M \in \text{Mod}(\mathcal{T})$ is perfect if $L_{\mathcal{T}}(M) \in D(\mathcal{T})$ is a compact object.

**Remark 2.1.13.** $\text{Perf}(\mathcal{T})$ coincides with the dg subcategory of $\text{Mod}(\mathcal{T})$ consisting of perfect modules, thought of as an object in $\text{Ho}(\text{dg}_{Z/2Z}k\text{-cat})$.

We may now introduce the notion of a Morita equivalence of dg categories:

**Definition 2.1.14 ([Toë11] Definition 4.4.4).** A morphism $F : \mathcal{S} \to \mathcal{T}$ in $\text{Ho}(\text{dg}_{Z/2Z}k\text{-cat})$ is called a Morita equivalence if $\text{Perf}(F)$ is an isomorphism.

If $F$ is a dg functor such that $L(F)$ is a Morita equivalence, we shall call $F$ a Morita equivalence as well.
Proposition 2.1.15. A dg functor $F : \mathcal{S} \to \mathcal{T}$ between two dg-triangulated categories is a Morita equivalence if and only if it is a quasi-equivalence.

Proof. It is immediate that $F$ is a Morita equivalence when $F$ is a quasi-equivalence. By Proposition 2.1.10, if $F$ is a Morita equivalence, $L(F)$ is an isomorphism. By Theorem 1.2.10 in [Hov07], $L(F)$ is an isomorphism if and only if $F$ is a quasi-equivalence. \qed

2.1.3 Hochschild homology and the Chern character

In this section, we assume char($k$) = 0. A $k$-linear dg category is a generalization of a dg $k$-algebra; in fact, a dg category with only one object is precisely a dg $k$-algebra. There exists a notion of Hochschild homology for $k$-linear dg categories that recovers the definition for dg $k$-algebras; we introduce this notion here, following Section 1.1 of [PV12].

Let $\mathcal{S}$ and $\mathcal{T}$ be dg categories.

Definition 2.1.16. The tensor product, $\mathcal{S} \otimes \mathcal{T}$, of $\mathcal{S}$ and $\mathcal{T}$ is the dg category given by the following:

- Objects are pairs $(S, T)$, where $S \in \text{Ob}(\mathcal{S})$ and $T \in \text{Ob}(\mathcal{T})$.
- $\text{Hom}_{\mathcal{S} \otimes \mathcal{T}}((S, T), (S', T')) := \text{Hom}_\mathcal{S}(S, S') \otimes_k \text{Hom}_\mathcal{T}(T, T')$.

Definition 2.1.17. An $\mathcal{S}$-$\mathcal{T}$ bimodule is a module over $(\mathcal{S} \otimes \mathcal{T}^{\text{op}})^{\text{op}} \cong \mathcal{S}^{\text{op}} \otimes \mathcal{T}$.

By Section 6.1 of [Kel94], an $\mathcal{S}$-$\mathcal{T}$ bimodule $X$ determines a functor

$$T_X : \text{Mod}(\mathcal{T}^{\text{op}}) \to \text{Mod}(\mathcal{S}^{\text{op}})$$
in the following way: given an object $M$ of $\text{Mod}(\mathcal{T}^{\text{op}})$, define a functor

$$\mathcal{S}^{\text{op}} \to C_{\mathbb{Z}/2\mathbb{Z}}(k)$$

given by

$$S \mapsto \text{coker}( \bigoplus_{T, T' \in \mathcal{T}} M(T') \otimes_k \text{Hom}_\mathcal{T}(T, T') \otimes_k X(S, T) \xrightarrow{\nu} \bigoplus_{T \in \mathcal{T}} M(T) \otimes_k X(S, T)),$$

where $\nu(y, f, x) = (M(f))(y) \otimes x - y \otimes X(\text{id}_S \otimes f)(x)$.

Remark 2.1.18. Suppose $S$ and $T$ have exactly one object; denote the unique object of $\mathcal{T}$ by $Y_\mathcal{T}$. In this case, $X$ is a right module over the dg algebra $\text{End}_\mathcal{T}(Y_\mathcal{T})$, and the functor $T_X$ amounts to the tensor product $- \otimes_{\text{End}_\mathcal{T}(Y_\mathcal{T})} X$.

There exists a left derived functor

$$\mathbb{L} T_X : D(\mathcal{T}^{\text{op}}) \to D(\mathcal{S}^{\text{op}})$$

of $T_X$; we refer the reader to [Kel94] for details.

Now, fix a dg category $\mathcal{U}$. Let $\Delta$ denote the $\mathcal{U} \otimes \mathcal{U}^{\text{op}}$-module given, on objects, by

$$(U, V) \mapsto \text{Hom}_\mathcal{U}(V, U).$$

Considering $k$ as a dg category with one object whose endomorphism complex consists of the $k$-module $k$ concentrated in degree 0, clearly $\Delta$ is a $k-(\mathcal{U} \otimes \mathcal{U}^{\text{op}})$ bimodule. Thus, noting that $(\mathcal{U} \otimes \mathcal{U}^{\text{op}})^{\text{op}} \cong \mathcal{U}^{\text{op}} \otimes \mathcal{U}$, we have that $\Delta$ determines a functor

$$\mathbb{L} T_\Delta : D(\mathcal{U}^{\text{op}} \otimes \mathcal{U}) \to D(k).$$
\[ \Delta \] is also a \( \mathcal{U}^{\text{op}} \otimes \mathcal{U} \)-module in an evident way; we define the \textit{Hochschild complex} of \( \mathcal{U} \) to be the object

\[ \mathbb{L}T_{\Delta}(\Delta) \in D(k). \]

The \textit{Hochschild homology} of \( \mathcal{U} \), denoted \( HH_*(\mathcal{U}) \), is the homology of \( \mathbb{L}T_{\Delta}(\Delta) \).

As a reality check, let’s suppose \( \mathcal{U} \) has one object whose endomorphism ring \( A \) is concentrated in degree 0. Then \( \Delta \) is the left \( A \otimes_k A^{\text{op}} \)-module \( A \). Thus, by Remark 2.1.18, the Hochschild complex of \( \mathcal{U} \) is \( A \otimes_{A \otimes_k A^{\text{op}}} A \in D(k) \); this agrees with the \( \mathbb{Z}/2\mathbb{Z} \)-folding of the usual Hochschild complex.

We list two properties of Hochschild homology of dg categories that we will make use of:

- Hochschild homology is \textit{Morita invariant}; that is, there is a natural isomorphism

\[ HH_*(\mathcal{U}) \cong HH_*(\text{Perf}(\mathcal{U})) \]

(\cite{Toe11} Section 5.2).

- There is a Künneth isomorphism

\[ HH_*(S) \otimes_k HH_*(T) \cong HH_*(S \otimes T) \]

(\cite{PV12} Proposition 1.1.4).

Now, suppose \( \mathcal{U} \) has the following properties:

1. \( \Delta \in \text{Mod}(\mathcal{U} \otimes \mathcal{U}^{\text{op}}) \) is perfect

2. For every pair of objects \( U, V \) in \( \mathcal{U} \), \( \text{Hom}_\mathcal{U}(U, V) \) has finite-dimensional cohomology
(3) \( D(\mathcal{U}) \) admits a compact generator

By Section 1.2 of [PV12], \( \mathcal{U} \) is Morita equivalent to a *homologically smooth and proper* dg algebra. Also, when \( \mathcal{S} \) and \( \mathcal{T} \) are dg categories with the above properties, a dg functor

\[
F : \text{Perf}(\mathcal{S}) \to \text{Perf}(\mathcal{T})
\]

yields a map

\[
F_* : \text{HH}_*(\mathcal{S}) \to \text{HH}_*(\mathcal{T}).
\]

In particular, if \( U \) is an object in \( \mathcal{U} \), the functor

\[
1_U : k \to \mathcal{U}
\]

that sends the unique object of \( k \) to \( U \) yields a map

\[
(1_U)_* : k = \text{HH}_*(k) \to \text{HH}_*(\mathcal{U}).
\]

**Definition 2.1.19.** We define \( \text{ch}(U) := (1_U)_*(1) \in \text{HH}_*(\mathcal{U}) \) to be the *Chern character* of \( U \).

The functor

\[
T_\Delta : \text{Mod}(\mathcal{U}^{\text{op}} \otimes \mathcal{U}) \to \text{Mod}(k)
\]

restricts to a functor

\[
\text{Perf}(\mathcal{U}^{\text{op}} \otimes \mathcal{U}) \to \text{Perf}(k).
\]

Combining the map on Hochschild homology induced by this functor with the
K"unneth isomorphism, we have a canonical pairing

\[ \langle - , - \rangle_U : \text{HH}^\ast (U^{\text{op}}) \otimes_k \text{HH}^\ast (U) \to k. \]

On the other hand, one has the Euler pairing

\[ \chi : \text{Ob}(U) \times \text{Ob}(U) \to k \]

given by

\[ (U, V) \mapsto \dim_k H^0 \text{Hom}_U(U, V) - \dim_k H^1 \text{Hom}_U(U, V). \]

The following is an analogue of the Hirzebruch-Riemann-Roch formula:

**Theorem 2.1.20 ([PV12] Section 1.2).** Let \( \mathcal{U} \) be a \( k \)-linear dg category, where \( k \) is a field, and assume \( \mathcal{U} \) has properties (1) - (3) above. If \( U \) and \( V \) are objects in \( \mathcal{U} \),

\[ \chi(U, V) = \langle \text{ch}(U), \text{ch}(V) \rangle_U. \]

**Remark 2.1.21.** When \( \mathcal{U} \) is a dg category with properties (1) - (3) above, the pairing \( \langle - , - \rangle_{\mathcal{U}} \) is non-degenerate. In fact, the map

\[ \text{HH}^\ast (\mathcal{U}) \otimes_k \text{HH}^\ast (\mathcal{U}^{\text{op}}) \otimes_k \text{HH}^\ast (\mathcal{U}) \to \text{HH}^\ast (\mathcal{U}) \]

given by

\[ h \otimes h' \otimes h'' \mapsto \langle h, h' \rangle_{U^{\text{op}}} \cdot h'' \]

sends \( h \otimes \text{ch}(\Delta) \) to \( h \) for all \( h \in \text{HH}^\ast (\mathcal{U}) \), where \( \text{ch}(\Delta) \in \text{HH}^\ast (\text{Perf}(\mathcal{U}^{\text{op}} \otimes \mathcal{U})) \) is
identified with its image under the canonical isomorphisms

\[ HH_\ast(\text{Perf}({\mathcal{U}}^\text{op} \otimes {\mathcal{U}})) \cong HH_\ast({\mathcal{U}}^\text{op} \otimes {\mathcal{U}}) \cong HH_\ast({\mathcal{U}}^\text{op}) \otimes_k HH_\ast({\mathcal{U}}). \]

\section{2.2 Matrix factorization categories}

We provide some background on matrix factorization categories. Fix a commutative algebra \( Q \) over a field \( k \) and an element \( f \) of \( Q \). All categories and functors in this section are assumed to be \( k \)-linear.

\subsection{2.2.1 Definitions and some properties}

\textbf{Definition 2.2.1.} The dg category \( \text{MF}(Q,f) \) of matrix factorizations of \( f \) over \( Q \) is given by the following:

Objects in \( \text{MF}(Q,f) \) are pairs \((P,d)\), where \( P \) is a finitely-generated projective \( \mathbb{Z}/2\mathbb{Z} \)-graded \( Q \)-module, and \( d \) is an odd-degree endomorphism of \( P \) such that \( d^2 = f \cdot \text{id}_P \). Henceforth, we will often denote an object \((P,d)\) in \( \text{MF}(Q,f) \) by just \( P \).

The morphism complex of a pair of matrix factorizations \( P, P' \), which we will denote by \( \text{Hom}_{\text{MF}}(P, P') \), is the \( \mathbb{Z}/2\mathbb{Z} \)-graded module of \( Q \)-linear maps from \( P \) to \( P' \) equipped with the differential \( \partial \) given by

\[ \partial(\alpha) = d' \circ \alpha - (-1)^{|\alpha|} \alpha \circ d \]

for homogeneous maps \( \alpha : P \to P' \).

It will often be useful to express an object \( P \) in \( \text{MF}(Q,f) \) in the following way:

\[ P_1 \overset{d_1}{\underset{d_0}{\rightleftarrows}} P_0 \]
where $P_1, P_0$ are the odd and even degree summands of $P$, and $d_1, d_0$ are the restrictions of $d$ to $P_1$ and $P_0$, respectively.

We now establish several technical results concerning matrix factorization categories that we will need later on.

Define $\text{EMF}(Q, f)$ to be the category with the same objects as $\text{MF}(Q, f)$ and with morphisms given by the degree 0 cycles in $\text{MF}(Q, f)$. When $Q$ is regular with finite Krull dimension and $f$ is a regular element of $Q$ (i.e. $f$ is a non-unit, non-zero-divisor), $\text{EMF}(Q, f)$ is an exact category with the evident family of exact sequences ([Orl03] Section 3.1); the “E” stands for exact.

A degree 0 morphism $\alpha$ in $\text{MF}(Q, f)$ can be represented by a diagram of the following form:

\[
\begin{array}{ccc}
P_1 & \xrightarrow{d_1} & P_0 & \xrightarrow{d_0} & P_1 \\
\downarrow{\alpha_1} & & \downarrow{\alpha_0} & & \downarrow{\alpha_1} \\
P_1' & \xrightarrow{d_1'} & P_0' & \xrightarrow{d_0'} & P_1'
\end{array}
\]

It is straightforward to check that $\alpha$ is a cycle if and only if this diagram commutes. In fact, if $f \in Q$ is a non-zero-divisor, it is easy to see that the left square commutes if and only if the right square commutes.

Remark 2.2.2. If $P_1$ and $P_0$ are free and $f$ is non-zero-divisor, $P_1$ and $P_0$ must have the same rank.

It will be useful for us to have an alternative characterization for when a morphism in $\text{EMF}(Q, f)$ is a boundary in $\text{MF}(Q, f)$.

**Definition 2.2.3.** We call a matrix factorization *trivial* if it is a direct sum of matrix factorizations that are isomorphic in $\text{EMF}(Q, f)$ to either

\[
\begin{array}{ccc}
E & \xrightarrow{id_E \cdot f} & E \\
& \xleftarrow{id_E} & 
\end{array}
\]
or

\[ E \xrightarrow{id_E} E \xrightarrow{f \cdot id_E} E. \]

for some finitely generated projective \( Q \)-module \( E \).

**Proposition 2.2.4.** A morphism \( \alpha : P \to P' \) in \( \text{EMF}(Q, f) \) is a boundary in \( \text{MF}(Q, f) \) if and only if it factors through a trivial matrix factorization in \( \text{EMF}(Q, f) \).

**Proof.** Suppose \( \alpha \) factors through a trivial matrix factorization \( E \). It is easy to see that \( \text{id}_E \) is a boundary in \( \text{MF}(Q, f) \); it follows immediately that \( \alpha \) is as well.

Conversely, suppose \( \alpha \) is a boundary. Write

\[
P = (P_1 \xrightarrow{d_1} P_0), \quad P' = (P'_1 \xleftarrow{d'_1} P'_0).
\]

Since \( \alpha \) is a degree 0 cycle, there exist \( Q \)-linear maps

\[
\alpha_1 : P_1 \to P'_1, \quad \alpha_0 : P_0 \to P'_0
\]

such that \( \alpha = \alpha_1 + \alpha_0 \) and the following diagram commutes:

\[
\begin{array}{ccc}
P_1 & \xrightarrow{d_1} & P_0 \xrightarrow{d_0} P_1 \\
\alpha_1 \downarrow & & \alpha_0 \downarrow \downarrow \alpha_1 \\
P'_1 & \xrightarrow{d'_1} & P'_0 \xrightarrow{d'_0} P'_1
\end{array}
\]

Choose a \( Q \)-linear map

\[
h : P \to P'
\]

such that \( \partial(h) = \alpha \). Since \( \alpha \) has degree 0, \( \partial \) evaluated at the degree 0 component of \( h \) is 0. Thus, we may as well assume \( h \) is homogeneous of degree 1; that is, there are
\(Q\)-linear maps

\[
h_0 : P_0 \to P'_1
\]

\[
h_1 : P_1 \to P'_0
\]

such that \(h = h_0 + h_1\).

Define

\[
A : P_1 \oplus P'_1 \to P_1 \oplus P'_1
\]

\[
B : P_1 \oplus P'_1 \to P_1 \oplus P'_1
\]

to be given by

\[
A = \begin{pmatrix} f \cdot \text{id}_{P_1} & 0 \\ 0 & \text{id}_{P'_1} \end{pmatrix}
\]

\[
B = \begin{pmatrix} \text{id}_{P_1} & 0 \\ 0 & f \cdot \text{id}_{P'_1} \end{pmatrix}
\]

Notice that

\[
P_1 \oplus P'_1 \xrightarrow{A} P_1 \oplus P'_1 \xrightarrow{B} P_1 \oplus P'_1
\]

is a trivial matrix factorization, and we have the following commutative diagram:
Thus, $\alpha$ factors through a trivial matrix factorization. \qed

Here is another technical result that will be useful later on:

**Proposition 2.2.5.** Let $P = (P_1 \xleftarrow{d_1} d_0 P_0)$ be a matrix factorization of $f$ over $Q$. Assume $f$ is a non-zero-divisor. Then the following are equivalent:

1. $\text{coker}(d_1)$ is isomorphic to $L/fL$ for some projective $Q$-module $L$.

2. There exists a trivial matrix factorization $E$ and a matrix factorization $E'$ that is isomorphic in $\text{EMF}(Q, f)$ to one of the form

$$F \xleftarrow{id_{E'}} F$$

such that $P \oplus E'$ is isomorphic to $E$ in $\text{EMF}(Q, f)$.

Before proving the proposition, we establish a general fact about idempotent complete categories.
**Definition 2.2.6.** We say an additive category $\mathcal{C}$ is *idempotent complete*, or that $\mathcal{C}$ has *split idempotents*, if every idempotent endomorphism $\phi = \phi^2$ of an object $X$ splits; that is, there exists a factorization

$$X \xrightarrow{\pi} Y \xrightarrow{\iota} X$$

of $\phi$ with $\pi \circ \iota = \text{id}_Y$.

**Lemma 2.2.7.** Let $\mathcal{C}$ be an idempotent complete additive category, and let $\mathcal{E}$ be a collection of objects in $\mathcal{C}$ that is

- closed under isomorphisms,
- closed under finite coproducts, and
- closed under taking summands; that is, whenever $X$ is an object in $\mathcal{C}$ such that $\text{id}_X$ factors through an object of $\mathcal{E}$, $X$ is an object in $\mathcal{E}$.

Denote by $\mathcal{L}$ the quotient of $\mathcal{C}$ by those morphisms that factor through an object in $\mathcal{E}$. If $X$ and $Y$ are objects in $\mathcal{C}$, their images in $\mathcal{L}$ are isomorphic if and only if there exist objects $E_X$, $E_Y$ in $\mathcal{E}$ such that

$$X \oplus E_X \cong Y \oplus E_Y.$$

**Proof.** Let $X$ and $Y$ be objects in $\mathcal{C}$. Suppose there exist objects $E_X$, $E_Y$ in $\mathcal{E}$ such that

$$X \oplus E_X \cong Y \oplus E_Y.$$

The quotient functor from $\mathcal{C}$ to $\mathcal{L}$ is additive, and hence preserves finite coproducts. Thus, it suffices to show that objects in $\mathcal{E}$ are mapped to 0 under the quotient functor. This is clear, since, if $E$ is an object in $\mathcal{E}$, $\text{id}_E$ factors through an object in $\mathcal{E}$. 
Conversely, suppose the images of $X$ and $Y$ in $\mathcal{L}$ are isomorphic. Choose morphisms in $\mathcal{C}$

$$\alpha : X \to Y, \beta : Y \to X$$

whose images in $\mathcal{L}$ are mutually inverse. Choose an object $E_Y$ in $\mathcal{E}$ and morphisms

$$\delta : E_Y \to X, \epsilon : X \to E_Y$$

in $\mathcal{C}$ such that

$$\delta \circ \epsilon = \beta \circ \alpha - \text{id}_X.$$

We have morphisms

$$\phi := \begin{pmatrix} \alpha \\ \epsilon \end{pmatrix} : X \to Y \oplus E_Y, \psi := \begin{pmatrix} \beta & -\delta \end{pmatrix} : Y \oplus E_Y \to X$$

in $\mathcal{C}$. Notice that $\psi \circ \phi = \text{id}_X$. Also, an easy computation shows that

$$\sigma := \text{id}_{Y \oplus E_Y} - \phi \circ \psi$$

is idempotent. Choose an object $Z$ in $\mathcal{C}$ and morphisms

$$\tau : Z \to Y \oplus E_Y, \rho : Y \oplus E_Y \to Z$$

in $\mathcal{C}$ such that

$$\tau \circ \rho = \sigma \text{ and } \rho \circ \tau = \text{id}_Z.$$
Given two objects $A_1, A_2$ in $\mathcal{C}$, we denote by

$$\iota_{A_i} : A_i \to A_1 \oplus A_2, \quad \pi_{A_i} : A_1 \oplus A_2 \to A_i$$

the canonical maps associated to the coproduct of $A_1$ and $A_2$ in $\mathcal{C}$ (which is also the product of $A_1$ and $A_2$ in $\mathcal{C}$). We have mutually inverse morphisms

$$\begin{pmatrix} \alpha & \pi_Y \circ \tau \\ \epsilon & \pi_{E_Y} \circ \tau \end{pmatrix} : X \oplus Z \to Y \oplus E_Y$$

$$\begin{pmatrix} \beta & -\delta \\ \rho \circ \iota_Y & \rho \circ \iota_{E_Y} \end{pmatrix} : Y \oplus E_Y \to X \oplus Z,$$

so it suffices to show that $Z$ is in $\mathcal{E}$. We first show that $\phi$ descends to an isomorphism in $\mathcal{L}$. Recall that $\psi \circ \phi = \text{id}_X$ in $\mathcal{C}$. Write $\phi \circ \psi - \text{id}_{Y \oplus E_Y}$ as the $2 \times 2$ matrix

$$\begin{pmatrix} \alpha \circ \beta - \text{id}_Y & -\alpha \circ \delta \\ \epsilon \circ \beta & -\epsilon \circ \delta - \text{id}_{E_Y} \end{pmatrix}$$

Since each entry of this matrix factors through an object in $\mathcal{E}$, $\phi \circ \psi - \text{id}_{Y \oplus E_Y}$ descends to the zero map in $\mathcal{L}$. This shows that the images of $\phi$ and $\psi$ in $\mathcal{L}$ are mutually inverse.

Notice that

$$\phi = \begin{pmatrix} \alpha & \pi_Y \circ \tau \\ \epsilon & \pi_{E_Y} \circ \tau \end{pmatrix} \circ \iota_X$$

in $\mathcal{C}$. Thus, $\iota_X$ descends to an isomorphism in $\mathcal{L}$. Since $\pi_X \circ \iota_X = \text{id}_X$ in $\mathcal{C}$, it follows
that the images of \( \pi_X \) and \( \iota_X \) in \( \mathcal{L} \) are mutually inverse. Choose an object \( E \) in \( \mathcal{E} \) and morphisms

\[
f : E \to X \oplus Z, \quad g : X \oplus Z \to E
\]

such that

\[
\text{id}_{X \oplus Z} - \iota_X \circ \pi_X = f \circ g
\]

in \( \mathcal{C} \). Observe that

\[
\pi_Z \circ f \circ g \circ \iota_Z = \text{id}_Z - \pi_Z \circ \iota_X \circ \pi_X \circ \iota_Z = \text{id}_Z.
\]

Thus, \( \text{id}_Z \) factors through an object in \( \mathcal{E} \).

We are now ready to prove Proposition 2.2.5.

**Proof.** (2) \( \Rightarrow \) (1): Since the cokernel of \( d_1 \) is isomorphic to the cokernel of

\[
d_1 \oplus \text{id}_{E'} : P_1 \oplus E' \to P_0 \oplus E',
\]

we may assume \( P \) is trivial. In this case, the result is obvious.

(1) \( \Rightarrow \) (2): We have projective resolutions

\[
0 \to P_1 \xrightarrow{d_1} P_0 \to \ker(d_1) \to 0
\]

\[
0 \to L \xrightarrow{f} \to L/\ker f \to 0
\]

Thus, there exist maps

\[
\beta_i : P_i \to L, \quad \gamma_i : L \to P_i
\]
for $i = 0, 1$ making the following diagrams commute:

\[
\begin{array}{c}
0 \rightarrow P_1 \xrightarrow{d_1} P_0 \rightarrow \text{coker } d_1 \rightarrow 0 \\
\beta_1 \downarrow \quad \beta_0 \downarrow \quad \downarrow \cong \\
0 \rightarrow L \xrightarrow{f} L \rightarrow L/fL \rightarrow 0
\end{array}
\]

\[
\begin{array}{c}
0 \rightarrow L \xrightarrow{f} L \rightarrow L/fL \rightarrow 0 \\
\gamma_1 \downarrow \quad \gamma_0 \downarrow \quad \downarrow \cong \\
0 \rightarrow P_1 \xrightarrow{d_1} P_0 \rightarrow \text{coker } d_1 \rightarrow 0
\end{array}
\]

Hence, we have maps

\[
h_P : P_0 \rightarrow P_1, \; h_L : L \rightarrow L
\]

such that

\[
\gamma_1 \circ \beta_1 - \text{id}_{P_1} = h_P \circ d_1, \; \gamma_0 \circ \beta_0 - \text{id}_{P_0} = d_1 \circ h_P.
\]

\[
\beta_1 \circ \gamma_1 - \text{id}_L = f h_L, \; \beta_0 \circ \gamma_0 - \text{id}_L = f h_L.
\]

We have commutative diagrams

\[
\begin{array}{ccc}
P_1 & \xrightarrow{d_1} & P_0 \\
\downarrow h_P \circ d_1 & & \downarrow h_P \\
P_1 & \xrightarrow{\text{id}_{P_1}} & P_1
\end{array}
\quad
\begin{array}{ccc}
L & \xrightarrow{f \cdot \text{id}_L} & L \\
\downarrow f \cdot h_L & & \downarrow f \cdot h_L \\
L & \xrightarrow{\text{id}_L} & L
\end{array}
\quad
\begin{array}{ccc}
P_1 & \xrightarrow{d_1} & P_0 \\
\downarrow \text{id}_{P_1} & & \downarrow \text{id}_{P_1} \\
P_1 & \xrightarrow{d_1} & P_0
\end{array}
\quad
\begin{array}{ccc}
L & \xrightarrow{f \cdot \text{id}_L} & L \\
\downarrow f \cdot \text{id}_L & & \downarrow f \cdot \text{id}_L \\
L & \xrightarrow{\text{id}_L} & L
\end{array}
\]

Denote by $\mathcal{E}$ the collection of matrix factorizations of $f$ over $Q$ isomorphic in
EMF($Q, f$) to a matrix factorization of the form

$$E \xleftarrow{\text{id}_E} \xrightarrow{f} E.$$  

Notice that EMF($Q, f$) is an idempotent complete additive category, and $\mathcal{E}$ is closed under direct sums and direct summands in EMF($Q, f$). Letting $\mathcal{L}$ denote the quotient of EMF($Q, f$) by those morphisms that factor through an object in $\mathcal{E}$, we have that

$$(P_1 \xleftarrow{d_1} \xrightarrow{d_0} P_0) \cong (L \xleftarrow{f} \xrightarrow{\text{id}_L} L)$$

in $\mathcal{L}$. The result now follows from Lemma 2.2.7.

\square

2.2.2 Triangulated structure

Suppose $Q$ is regular with finite Krull dimension and $f$ is a regular element of $Q$. A feature of the homotopy category $[\text{MF}(Q, f)]$ is that it may be equipped with a triangulated structure in the following way ([Orl03] Section 3.1):

The shift functor maps the object

$$P = (P_1 \leftrightarrow_{d_1} \xrightarrow{d_0} P_0)$$

to the object

$$P[1] = (P_0 \leftrightarrow_{-d_0} \xrightarrow{-d_1} P_1).$$

That is, shifting a matrix factorization flips the grading on the module and negates
the odd-degree endomorphism. On morphisms, the shift functor maps the cycle

\[
P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} P_1 \\
\downarrow \alpha_1 \downarrow \alpha_0 \downarrow \alpha_1 \\
P'_1 \xrightarrow{d'_1} P'_0 \xrightarrow{d'_0} P'_1
\]

to the cycle

\[
P_0 \xrightarrow{-d_0} P_1 \xrightarrow{-d_1} P_0 \\
\downarrow \alpha_0 \downarrow \alpha_1 \downarrow \alpha_0 \\
P'_0 \xrightarrow{-d'_0} P'_1 \xrightarrow{-d'_1} P'_0
\]

Notice that the shift functor applied twice is the identity functor.

Given a morphism \( \alpha : (P_1 \xleftarrow{d_1} P_0) \to (P'_1 \xleftarrow{d'_1} P'_0) \) in \( \text{EMF}(Q, f) \), we define the mapping cone of \( \alpha \) as follows:

\[
\text{cone}(\alpha) = (P'_0 \oplus P_1 \xrightarrow{\begin{pmatrix} \alpha_0 & 0 \\ d'_0 & -d_1 \end{pmatrix}} P'_1 \oplus P_0)
\]

There are canonical morphisms \( P' \to \text{cone}(\alpha) \) and \( \text{cone}(\alpha) \to P[1] \) in \( \text{EMF}(Q, f) \). Taking the distinguished triangles in \([\text{MF}(Q, f)]\) to be the triangles isomorphic to those of the form

\[
P \xrightarrow{\alpha} P' \to \text{cone}(\alpha) \to P[1],
\]

[\(\text{MF}(Q, f)\)] may be equipped with the structure of a triangulated category.

We define the Grothendieck group

\[
K_0[\text{MF}(Q, f)]
\]
to be the free abelian group generated by isomorphism classes of \([\text{MF}(Q,f)]\) modulo elements of the form \([P_1] - [P_2] + [P_3]\), where \(P_1, P_2,\) and \(P_3\) fit into a distinguished triangle in the following way:

\[
P_1 \to P_2 \to P_3 \to P_1[1].
\]

Remark 2.2.8. The category \(\text{MF}(Q,f)\) is not always dg-triangulated in this setting; a counterexample is given in Section 2.5. When \(\text{MF}(Q,f)\) is dg-triangulated, the induced triangulated structure on \([\text{MF}(Q,f)]\) agrees with the triangulated structure just described.

Remark 2.2.9. When \(Q\) is a regular local ring and \(f\) is a regular element of \(Q\), one has an equivalence of triangulated categories

\[
[\text{MF}(Q,f)] \xrightarrow{\sim} \text{MCM}(Q/(f)),
\]

where \(\text{MCM}(Q/(f))\) denotes the stable category of maximal Cohen-Macaulay (MCM) modules over the ring \(Q/(f)\). The stable category of MCM modules is obtained by taking the quotient of the category of MCM modules over \(Q/(f)\) by those morphisms that factor through a projective \(Q/(f)\)-module. The above equivalence is given, on objects, by

\[
(P_1 \xrightarrow{d_1} P_0) \mapsto \text{coker}(d_1).
\]

Matrix factorizations were first defined by Eisenbud in [Eis80]; this interplay between matrix factorizations and MCM modules over hypersurface rings provided the original motivation for the study of matrix factorization categories.
2.2.3 The Hochster theta pairing

We begin this section with a technical definition:

**Definition 2.2.10.** If the pair \((Q, f)\) satisfies

- \(Q\) is essentially of finite type over \(k\)
- \(Q\) is equidimensional of dimension \(n\)
- The module \(\Omega^1_{Q/k}\) of Kähler differentials is locally free of rank \(n\)
- The zero locus of \(df \in \Omega^1_{Q/k}\) is a 0-dimensional scheme supported on a unique closed point \(m\) of \(\text{Spec}(Q)\) with residue field \(k\) and \(f \in m\)

we shall call \(Q/(f)\) an *isolated hypersurface singularity*, or IHS. We will sometimes just say \(f\) is IHS, if the ambient ring \(Q\) is clear.

**Remark 2.2.11.** Our IHS condition above is precisely condition (B) in Section 3.2 of [Dyc11]. As noted in loc. cit., if \(Q/(f)\) and \(Q/(f')\) are IHS, \(Q \otimes_k Q'/(f \otimes 1 + 1 \otimes f')\) is as well.

Now, assume \(\text{char}(k) = 0\) and that \(Q\) is a regular local ring such that \(Q/(f)\) is IHS. Set \(R := Q/(f)\). One may define a symmetric pairing

\[
\theta : K_0[\text{MF}(Q, f)] \times K_0[\text{MF}(Q, f)] \to \mathbb{Z},
\]

called the *Hochster theta pairing*, that maps a pair

\[
([P_1 \xrightarrow{d_1} d_0 \xrightarrow{P_0}], [P'_1 \xleftarrow{d'_1} d'_0 \xrightarrow{P'_0}])
\]

to

\[
l(\text{Tor}_2^R(\text{coker}(d_1), \text{coker}(d'_1))) - l(\text{Tor}_1^R(\text{coker}(d_1), \text{coker}(d'_1))),
\]
where \( l \) denotes length as an \( R \)-module; our assumption that the singular locus of \( R \) is dimension 0 guarantees that these lengths are finite. The pairing \( \theta \) was introduced in \cite{Hoc81}; for more detailed discussions related to this pairing, we refer the reader to \cite{BVS12}, \cite{MPSW11}, \cite{PV12}, and \cite{Wal14b}.

The Euler pairing \( \chi \) from Section 2.1.3 applied to the dg category \( \text{MF}(Q, f) \) can be thought of as a pairing on the homotopy category, since, for matrix factorizations \( P, P' \in \text{MF}(Q, f) \),

\[
\chi(P, P') = \dim_k H^0 \text{Hom}_{\text{MF}}(P, P') - \dim_k H^1 \text{Hom}_{\text{MF}}(P, P')
\]

\[
= \dim_k H^0 \text{Hom}_{\text{MF}}(P, P') - \dim_k H^0 \text{Hom}_{\text{MF}}(P, P'[1]).
\]

It is straightforward to check that \( \chi \) induces a pairing on \( K_0[\text{MF}(Q, f)] \).

Write

\[
P = (P_1 \xleftarrow{d_1} P_0), \quad P' = (P'_1 \xleftarrow{d'_1} P'_0).
\]

By Section 3 of \cite{Wall14a}, the Euler pairing \( \chi \) applied to \( (P, P') \) corresponds, via the equivalence

\[
[\text{MF}(Q, f)] \xrightarrow{\text{coker}} \text{MCM}(Q/(f)),
\]

to the pairing

\[
(\text{coker}(d_1), \text{coker}(d'_1)) \mapsto l(\text{Ext}_R^2(\text{coker}(d_1), \text{coker}(d'_1))) - l(\text{Ext}_R^1(\text{coker}(d_1), \text{coker}(d'_1))).
\]

As above, these lengths must be finite because of our assumption on the singular locus of \( R \). By Remark 3.2 of \cite{BVS12}, it follows that

\[
\chi(P, P') = \theta(\text{coker}(d_1)^*, \text{coker}(d'_1)),
\]
where $(-)^*$ denotes the $R$-linear dual $\text{Hom}_R(-, R)$. In particular, since MCM modules over $R$ are reflexive,
\[
\chi(-, [P]) : K_0[\text{MF}(Q, f)] \to \mathbb{Z}
\]
is the zero map if and only if
\[
\theta(-, \text{coker}(d_1)) : K_0[\text{MF}(Q, f)] \to \mathbb{Z}
\]
is the zero map.

### 2.2.4 Stabilization

Assume now that $Q$ is a regular local ring of Krull dimension $n$, and suppose $f$ is a regular element of $Q$. Denote by $\text{D}^b(Q/(f))$ the bounded derived category of $Q/(f)$. We will say an object $C$ in $\text{D}^b(Q/(f))$ is perfect if it is isomorphic, in $\text{D}^b(Q/(f))$, to a complex of finitely generated projective $Q/(f)$-modules; set $\text{D}^b_{\text{perf}}(Q/(f))$ to be the full subcategory of $\text{D}^b(Q/(f))$ given by perfect complexes. It turns out that $\text{D}^b_{\text{perf}}(Q/(f))$ is a thick subcategory of $\text{D}^b(Q/(f))$; define $\text{D}^b(Q/(f))$ to be the Verdier quotient of $\text{D}^b(Q/(f))$ by $\text{D}^b_{\text{perf}}(Q/(f))$. In [Buc86], Buchweitz defines this quotient to be the stabilized derived category of $Q/(f)$.

By [Buc86], the functor
\[
\text{MCM}(Q/(f)) \to \text{D}^b(Q/(f))
\]
that sends an MCM module $M$ to the complex with $M$ concentrated in degree 0 is a triangulated equivalence. Hence, composing with the equivalence in Remark 2.2.9, one has an equivalence
\[
[\text{MF}(Q, f)] \to \text{D}^b(Q/(f))
\]
Following [Dyc11], given an object $C$ in $\mathbf{D}^b(Q/(f))$, we denote by $C^{\text{stab}}$ the isomorphism class in $[\text{MF}(Q,f)]$ corresponding to $C$ under the above equivalence ("stab" stands for "stabilization").

In particular, thinking of the residue field $k$ of $Q/(f)$ as a complex concentrated in degree 0, we may associate to $k$ an isomorphism class $k^{\text{stab}}$ in $[\text{MF}(Q,f)]$. We now construct an object $E_f$ in $\text{MF}(Q,f)$ that represents $k^{\text{stab}}$; this construction appears in [Dyc11]. Choose a regular system of parameters $x_1, \ldots, x_n$ for $Q$, and consider the Koszul complex

$$\bigoplus_{i=0}^n \wedge^i Q^n, s_0$$

as a $\mathbb{Z}/2\mathbb{Z}$-graded complex of free $Q$-modules with even (odd) degree piece given by the direct sum of the even (odd) exterior powers of $Q^n$. Here, $s_0$ denotes the $\mathbb{Z}/2\mathbb{Z}$-folding of the Koszul differential associated to $x_1, \ldots, x_n$. Choose an expression of $f \in Q$ of the form

$$f = g_1 x_1 + \cdots + g_n x_n.$$

Fix a basis $e_1, \ldots, e_n$ of $Q^n$, and set $s_1$ to be the odd-degree endomorphism of

$$\bigoplus_{i=0}^n \wedge^i Q^n$$

given by exterior multiplication on the left by $g_1 e_1 + \cdots + g_n e_n$. Set

$$E_f := \left( \bigoplus_{i=0}^n \wedge^i Q^n, s_0 + s_1 \right).$$

It is easy to check that $E_f$ is a matrix factorization of $f$. By Corollary 2.7 in [Dyc11], $E_f$ represents $k^{\text{stab}}$ in $[\text{MF}(Q,f)]$. In particular, $E_f$ does not depend on the choice of regular system of parameters $x_1, \ldots, x_n$ or coefficients $g_1, \ldots, g_n$ up to homotopy equivalence.

We will be interested in the dga $\text{End}_{\text{MF}}(E_f)$. $\text{End}_{\text{MF}}(E_f)$ may be expressed, in
terms of generators and relations, as the $\mathbb{Z}/2\mathbb{Z}$-graded $Q$-algebra

$$Q\langle \lambda_1, \ldots, \lambda_n, c_1, \ldots, c_n \rangle/(\{[\lambda_i, \lambda_j], [c_i, c_j], [\lambda_i, c_j] - \delta_{ij}\})$$

equipped with the differential $\partial$ determined by $\partial(\lambda_i) = x_i$ and $\partial(c_i) = g_i$. Here, the $\lambda_i$ and $c_i$ are non-commuting variables of odd degree, $[-,-]$ denotes the $\mathbb{Z}/2\mathbb{Z}$-graded commutator, and $\delta_{ij}$ is the Kronecker delta. An isomorphism from this algebra to $\text{End}_{\text{MF}}(E_f)$ is given by

$$\lambda_i \mapsto \text{left multiplication by } e_i$$

$$c_i \mapsto \text{contraction by } e_i$$

where, by contraction by $e_i$, we mean the map that sends a basis element

$$e_{i_1} \cdots e_{i_r}$$

to 0 if $i \notin \{i_1, \ldots, i_r\}$ and to $(-1)^{r-1}e_{i_1} \cdots \widehat{e}_i \cdots e_{i_r}$ otherwise.

We set

$$A_{(Q,f)} := \text{End}_{\text{MF}}(E_f),$$

and we emphasize that $A_{(Q,f)}$ does not depend on the choice of regular system of parameters $x_1, \ldots, x_n$ or coefficients $g_1, \ldots, g_n$ up to quasi-isomorphism.
2.3 The tensor product of matrix factorizations

Suppose $Q$ and $Q'$ are commutative algebras over a field $k$. Given objects $P$ and $P'$ in $\text{MF}(Q,f)$, $\text{MF}(Q',f')$, one can form their tensor product over $k$:

$$P \otimes_{\text{MF}} P' := ((P_1 \otimes_k P'_0) \oplus (P_0 \otimes_k P'_1)) \to (P_0 \otimes_k P'_0) \oplus (P_1 \otimes_k P'_1).$$

This construction first appeared in [Yos98]; it can be thought of as a $\mathbb{Z}/2\mathbb{Z}$-graded analogue of the tensor product of complexes. It is straightforward to check that $P \otimes_{\text{MF}} P'$ is an object in $\text{MF}(Q \otimes_k Q', f \otimes 1 + 1 \otimes f')$.

In fact, setting $f \oplus f' := f \otimes 1 + 1 \otimes f' \in Q \otimes_k Q'$, and noting that there is a canonical map

$$\text{Hom}_{\text{MF}}(P, L) \otimes_k \text{Hom}_{\text{MF}}(P', L') \to \text{Hom}_{\text{MF}}(P \otimes_{\text{MF}} P', L \otimes_{\text{MF}} L'),$$

we have the following:

**Proposition 2.3.1.** There is a dg functor

$$\text{ST}_{\text{MF}} : \text{MF}(Q, f) \otimes_k \text{MF}(Q', f') \to \text{MF}(Q \otimes_k Q', f \oplus f')$$

that sends an object $(P, P')$ to $P \otimes_{\text{MF}} P'$.

Further,
Proposition 2.3.2. \( ST_{MF} \) induces a pairing

\[
K_0[MF(Q, f)] \otimes K_0[MF(Q', f')] \to K_0[MF(Q \otimes_k Q', f \oplus f')].
\]

Proof. Suppose \( P \) is a contractible matrix factorization in \( MF(Q, f) \). Choose a contracting homotopy \((h_0, h_1)\). Then, if \( P' \) is any matrix factorization in \( MF(Q', f') \), the maps

\[
\begin{pmatrix}
  h_0 \otimes \text{id}_{P_0'} & 0 \\
  0 & h_1 \otimes \text{id}_{P_1'}
\end{pmatrix}
\]

\[
(P_0 \otimes P_0') \oplus (P_1 \otimes P_1') \longrightarrow
\]

\[
\begin{pmatrix}
  h_1 \otimes \text{id}_{P_0'} & 0 \\
  0 & h_0 \otimes \text{id}_{P_1'}
\end{pmatrix}
\]

\[
(P_1 \otimes P_0') \oplus (P_0 \otimes P_1') \longrightarrow
\]

yield a contracting homotopy of \( P \otimes_{MF} P' \).

Suppose \( \alpha : L \to L' \) is a morphism in \( EMF(Q, f) \). One easily checks that, if \( B \) is a matrix factorization in \( MF(Q', f') \),

\[
\text{cone}(\alpha) \otimes_{MF} B = \text{cone}(\alpha \otimes \text{id}_B).
\]

Thus,

\[
[(L \oplus \text{cone}(\alpha)) \otimes_{MF} B] - [L' \otimes_{MF} B] = [L \otimes_{MF} B] + [\text{cone}(\alpha \otimes \text{id}_B)] - [L' \otimes_{MF} B] = 0.
\]

If \( \alpha : P \to P' \) is an isomorphism in \([MF(Q, f)]\), then \( \text{cone}(\alpha) \) is contractible.
Thus, for all matrix factorizations $B$ of $f'$ over $Q'$,

$$[P \otimes_{\text{MF}} B] = [P' \otimes_{\text{MF}} B].$$

It follows that the pairing respects isomorphism in the homotopy category. Since every distinguished triangle is isomorphic to one of the form

$$P \xrightarrow{\alpha} P' \rightarrow \text{cone}(\alpha) \rightarrow P[1],$$

and we have shown that the pairing preserves triangles of this form, this finishes the proof. \hfill \Box

**Remark 2.3.3.** The “ST” in the name $\text{ST}_{\text{MF}}$ stands for “Sebastiani-Thom”, since this tensor product operation is related to the Sebastiani-Thom homotopy equivalence discussed in Section 3.1.2. A precise sense in which the tensor product of matrix factorizations is related to the Sebastiani-Thom homotopy equivalence is illustrated by the proof of Proposition 3.4.1 below; see Remark 3.4.3 for further details.

Now, suppose $Q/(f)$ and $Q'/(f')$ are IHS (see Definition 2.2.10 above). Henceforth, we will denote by $\hat{Q}$ the $m$-adic completion of $Q_m$, where $m$ is as in the definition of IHS.

Set $Q'' := Q \otimes_k Q'$, and define

$$\phi: \hat{Q} \otimes_k \hat{Q}' \rightarrow \hat{Q}''$$

to be the canonical ring homomorphism. $\phi$ induces a dg functor

$$\text{MF}(\phi): \text{MF}(\hat{Q} \otimes_k \hat{Q}', f \oplus f') \rightarrow \text{MF}(\hat{Q}'', f \oplus f').$$
Set $\widehat{ST}_{MF} := MF(\phi) \circ ST_{MF}$.

**Proposition 2.3.4.** If $Q/(f)$ and $Q'//(f')$ are IHS,

$$\widehat{ST}_{MF} : MF(\widehat{Q}, f) \otimes_k MF(\widehat{Q'}, f') \rightarrow MF(\widehat{Q''}, f \oplus f')$$

is a Morita equivalence.

**Remark 2.3.5.** This proposition is really just a straightforward application of several results in [Dyc11].

**Proof.** Suppose $Q_m$ and $Q_{m'}'$ have Krull dimensions $n$ and $m$, respectively. $Q_m$ and $Q_{m'}'$ are regular local rings; choose regular systems of parameters $x_1, \ldots, x_n$ and $y_1, \ldots, y_m$ in $Q_m$ and $Q_{m'}'$, and choose expressions

$$f = g_1x_1 + \cdots + g_nx_n$$

$$f' = h_1y_1 + \cdots + h_my_m$$

of $f$ and $f'$. Use these expressions to construct the dga’s $A(Q_m,f)$ and $A(Q_{m'}',f')$.

Note that $x_1, \ldots, x_n$ and $y_1, \ldots, y_m$ form regular systems of parameters in $\widehat{Q}$ and $\widehat{Q'}$ as well, so we may use these expressions to construct $A(\widehat{Q}_m,f)$ and $A(\widehat{Q}_{m'}',f')$. Also, $x_1 \otimes 1, \ldots, x_n \otimes 1, 1 \otimes y_1, \ldots, 1 \otimes y_m$ is a regular system of parameters in $Q''_{m'}$, where $m'' := m \otimes 1 + 1 \otimes m'$, so we may use the expression

$$f \oplus f' = g_1x_1 \otimes 1 + \cdots + g_nx_n \otimes 1 + 1 \otimes h_1y_1 + \cdots + 1 \otimes h_my_m$$

to construct $A(Q_{m''},f \oplus f')$ and $A(\widehat{Q''},f \oplus f')$. 
By Section 6.1 of \cite{Dyc11}, we have a quasi-isomorphism

\[ F : A(Q_m,f) \otimes_k A(Q'_m,f') \xrightarrow{\sim} A(Q''_m,f \oplus f'). \]

We also have a canonical map

\[ G : A(\hat{Q},f) \otimes_k A(\hat{Q}',f') \rightarrow A(\hat{Q}'',f \oplus f'). \]

By the proof of Theorem 5.7 in \cite{Dyc11}, the inclusions

\[ A(Q_m,f) \hookrightarrow A(\hat{Q},f) \]

\[ A(Q'_m,f') \hookrightarrow A(\hat{Q}',f') \]

\[ A(Q''_m,f \oplus f') \hookrightarrow A(\hat{Q}'',f \oplus f') \]

are all quasi-isomorphisms.

By Exercise 4.4.11 in \cite{Toö11}, it follows that the induced map

\[ \text{Perf}(A(Q_m,f) \otimes_k A(Q'_m,f')) \rightarrow \text{Perf}(A(\hat{Q},f) \otimes_k A(\hat{Q}',f')) \]

is an isomorphism in \( Ho(dg_{\mathbb{Z}/2\mathbb{Z}} k\text{-cat}) \).

It is clear that we have the following commutative square in \( Ho(dg_{\mathbb{Z}/2\mathbb{Z}} k\text{-cat}) \):

\[
\begin{array}{ccc}
\text{Perf}(A(Q_m,f) \otimes_k A(Q'_m,f')) & \xrightarrow{\sim} & \text{Perf}(A(\hat{Q},f) \otimes_k A(\hat{Q}',f')) \\
\downarrow{\cong} & & \downarrow{\cong} \\
\text{Perf}(A(Q''_m,f \oplus f')) & \xrightarrow{\sim} & \text{Perf}(A(\hat{Q}'',f \oplus f'))
\end{array}
\]
It follows that $\text{Perf}(L(G))$ is an isomorphism in $Ho(\text{dg}_{\mathbb{Z}/2\mathbb{Z}} k\text{-cat})$.

One may think of a dga as a dg category with a single object. Adopting this point of view, we have inclusion functors

$$i : A_{(\hat{Q}, f)} \hookrightarrow MF(\hat{Q}, f)$$

$$j : A_{(\hat{Q}', f')} \hookrightarrow MF(\hat{Q}', f')$$

$$l : A_{(\hat{Q}'', f \oplus f')} \hookrightarrow MF(\hat{Q}'', f \oplus f')$$

Combining Theorem 5.2 and Lemma 5.6 in [Dyc11], we conclude that $i$, $j$, and $l$ are Morita equivalences. In particular, applying Exercise 4.4.11 in [Toë11] again, we have that

$$\text{Perf}(L(i) \otimes L(j)) : \text{Perf}(A_{(\hat{Q}, f)} \otimes_k A_{(\hat{Q}', f')}) \to \text{Perf}(MF(\hat{Q}, f) \otimes_k MF(\hat{Q}', f'))$$

is an isomorphism in $Ho(\text{dg}_{\mathbb{Z}/2\mathbb{Z}} k\text{-cat})$.

Finally, observe the following commutative diagram in $Ho(\text{dg}_{\mathbb{Z}/2\mathbb{Z}} k\text{-cat})$:

$$\begin{array}{ccc}
\text{Perf}(A_{(\hat{Q}, f)} \otimes_k A_{(\hat{Q}', f')}) & \xrightarrow{\text{Perf}(L(i) \otimes L(j))} & \text{Perf}(MF(\hat{Q}, f) \otimes_k MF(\hat{Q}', f')) \\
\text{Perf}(L(G)) & \xleftarrow{} & \text{Perf}(L(\hat{TS}_{MF})) \\
\text{Perf}(A_{(\hat{Q}'', f \oplus f')}) & \xrightarrow{\text{Perf}(L(l))} & \text{Perf}(MF(\hat{Q}'', f \oplus f')) \\
\end{array}$$

Since the left-most vertical map and both horizontal maps are isomorphisms in $Ho(\text{dg}_{\mathbb{Z}/2\mathbb{Z}} k\text{-cat})$, $\text{Perf}(L(\hat{TS}_{MF}))$ is as well.
Remark 2.3.6. Under the assumptions of Proposition \[2.3.4\], the functor

\[
\text{MF}(Q, f) \otimes_k \text{MF}(Q', f') \to \text{MF}(Q'', f \oplus f')
\]
given by tensor product of matrix factorizations is also a Morita equivalence.

Here is a proof: by Theorem 5.2 in \[Dyc11\], the inclusion functors

\[
A(Q_m, f) \hookrightarrow \text{MF}(Q_m, f)
\]

\[
A(Q_{m'}, f') \hookrightarrow \text{MF}(Q_{m'}, f')
\]

are Morita equivalences.

By arguments similar to those in the proof Proposition \[2.3.4\] one has a commutative square in \(Ho(\text{dg}_{\mathbb{Z}/2\mathbb{Z}}k\text{-cat})\):

\[
\begin{array}{ccc}
\text{Perf}(A(Q_m, f) \otimes_k A(Q_{m'}, f')) & \cong & \text{Perf}(\text{MF}(Q_m, f) \otimes_k \text{MF}(Q_{m'}, f')) \\
\downarrow & & \downarrow \\
\text{Perf}(A(Q_{m''}, f \oplus f')) & \cong & \text{Perf}(\text{MF}(Q_{m''}, f \oplus f'))
\end{array}
\]

It follows that the dg functor

\[
\text{MF}(Q_m, f) \otimes_k \text{MF}(Q_{m'}, f') \to \text{MF}(Q_{m''}, f \oplus f')
\]
given by tensor product of matrix factorizations is a Morita equivalence. Finally, consider the square
where the vertical maps are induced by localization. By Theorems 4.11 and 5.2 in [Dyc11] and an application of Exercise 4.4.11 in [Toé11], the vertical maps are Morita equivalences; hence, the top map is a Morita equivalence.

2.4 Clifford algebras

Fix a field $k$ such that char($k$) $\neq 2$ and a finite-dimensional vector space $V$ over $k$. Let $q : V \to k$ be a quadratic form.

The Clifford algebra, Cliff$_k(q)$, of $q$ over $k$ is defined to be the quotient

$$\mathcal{T}(V)/(v \otimes v - q(v)),$$

where $\mathcal{T}(V)$ denotes the tensor algebra of $V$ over $k$.

Cliff$_k(q)$ is a $\mathbb{Z}/2\mathbb{Z}$-graded $k$-algebra; let mod$_{\mathbb{Z}/2\mathbb{Z}}$(Cliff$_k(q)$) denote the category of finitely generated $\mathbb{Z}/2\mathbb{Z}$-graded left modules over Cliff$_k(q)$. Henceforth, when we refer to a module over a Clifford algebra, we will always mean it to be a left module.

Assume $q$ is non-degenerate, and choose a basis $\{e_1, \ldots, e_n\}$ of $V$ with respect to which $q$ is diagonal; that is,

$$q = a_1x_1^2 + \cdots + a_nx_n^2 \in S^2(V^*)$$

where the $x_i$ comprise the dual basis corresponding to the $e_i$, and the $a_i$ are nonzero.
elements of $k$. Denote by $Q$ the localization of $S(V^*)$ at the ideal $(x_1, \ldots, x_n)$.

We quote the following theorem from Section 14 of [Yos90]; it is due to Buchweitz-Eisenbud-Herzog ([BEH87]).

**Theorem 2.4.1.** $[\text{MF}(\hat{Q}, q)]$ and $\text{mod}_{\mathbb{Z}/2\mathbb{Z}}(\text{Cliff}_k(q))$ are equivalent $k$-linear categories.

It will be useful for us to exhibit a bijection between the isomorphism classes of objects of these two categories; this bijection is described in Section 14 of [Yos90]:

Given an isomorphism class $[P]$, where $P$ is an object in $[\text{MF}(\hat{Q}, q)]$, we may choose an object

$$\tilde{P} = (\tilde{P}_1 \xleftarrow{d_1} \tilde{P}_0)$$

in $[\text{MF}(\hat{Q}, q)]$ such that

(a) $\tilde{P} \cong P$ in $[\text{MF}(\hat{Q}, q)]$, and

(b) there exist choices of bases of $\tilde{P}_0$ and $\tilde{P}_1$ as free modules over $\hat{Q}$ so that $d_1$ and $d_0$ may be expressed by matrices $A$ and $B$ with entries in $S^1(V^*) = V^*$.

That such an object $\tilde{P}$ exists is a theorem due to Buchweitz-Eisenbud-Herzog in [BEH87].

Recall that $\tilde{P}_0$ and $\tilde{P}_1$ have the same rank as free $\hat{Q}$-modules. Set $m$ to be this rank. Let $W$ be a $k$-vector space of dimension $m$ equipped with a basis. Set $U_0$ and $U_1$ to be copies of $W$.

Given $x \in V$ and a matrix $D$ with entries in $V^*$, define

$$\text{ev}_x(D)$$

to be the matrix over $k$ given by evaluating the entries of $D$ at $x$. 
Define an action of $\mathcal{T}(V)$ on $U_1 \oplus U_0$ by

$$x \cdot u_1 = ev_x(A) \cdot u_1 \in U_0$$

for $x \in V$ and $u_1 \in U_1$, and

$$x \cdot u_0 = ev_x(B) \cdot u_0 \in U_1$$

for $x \in V$ and $u_0 \in U_0$. Notice that $(v \otimes v)u = q(v)u$ for all $u \in U_1 \oplus U_0$ and $v \in V$. It follows that $U_1 \oplus U_0$ is a finitely generated $\text{Cliff}_k(V)$-module. It turns out that $U_1 \oplus U_0$ does not depend on the choice of $\tilde{P}$ up to isomorphism of $\text{Cliff}_k(V)$-modules, so that we may set $\Delta_q([P])$ to be the isomorphism class of the $\text{Cliff}_k(V)$-module $U_1 \oplus U_0$.

Going the other direction, let $[M]$ denote the isomorphism class of a finitely generated $\mathbb{Z}/2\mathbb{Z}$-graded $\text{Cliff}_k(q)$-module $M = M_1 \oplus M_0$. Let $v \in V$. Multiplication by $v$ determines $k$-linear maps

$$\phi(v) : M_1 \rightarrow M_0$$

$$\psi(v) : M_0 \rightarrow M_1$$

Since $q$ is non-degenerate, we may choose $w \in V$ such that $q(w) \neq 0$. It follows that $\phi(w)$ and $\psi(w)$ are isomorphisms; in particular, $M_1$ and $M_0$ have the same rank $m$ as $k$-vector spaces.

Choosing bases of $M_0$ and $M_1$, we may think of the maps $\phi(v)$ and $\psi(v)$ as $m \times m$ matrices with entries in $k$.

Define

$$\phi_{ij} : V \rightarrow k$$

to be the $k$-linear map assigning an element $v$ of $V$ to the $(i,j)$ entry of $\phi(v)$. Define
\(\psi_{ij}\) similarly.

The maps \(\phi_{ij}, \psi_{ij}\) are elements of \(V^*\), so they may be written as linear combinations of \(x_1, \ldots, x_n \in S^1(V^*)\).

Define \(\Theta_q(M)\) to be the isomorphism class of the matrix factorization

\[
\tilde{Q}^m \leftrightarrow_{\phi} \tilde{Q}^m
\]

where \(\phi\) is the square matrix with entries \(\phi_{ij}\), and \(\psi\) is defined similarly. It is elementary to check that the assignments \(\Delta_q\) and \(\Theta_q\) are inverses on isomorphism classes.

Remark 2.4.2. Note that the inclusion

\[k[x_1, \ldots, x_n] \hookrightarrow \tilde{Q}\]

induces an equivalence

\[[MF(k[x_1, \ldots, x_n], q_n)] \xrightarrow{\cong} [MF(\tilde{Q}, q_n)].\]

To see this, we first recall that, as noted above, every matrix factorization of \(q_n\) over \(\tilde{Q}\) is isomorphic in \([MF(\tilde{Q}, q_n)]\) to one with (linear) polynomial entries (Proposition 14.3, [Yos90]); hence, the functor is essentially surjective.

Also, one has a commutative diagram

\[
\begin{array}{ccc}
[MF(Q, q_n)] & \xrightarrow{\cong} & \text{MCM}(Q/(q_n)) \\
\downarrow & & \downarrow \\
[MF(\tilde{Q}, q_n)] & \xrightarrow{\cong} & \text{MCM}(\tilde{Q}/(q_n))
\end{array}
\]
The morphism sets in \( \text{MCM}(Q/(q_n)) \) are Artinian modules, and hence complete. Thus, the functor on the right is fully faithful, and so the functor on the left is as well.

It now follows easily from Corollary 4.11, Theorem 5.2, and Theorem 5.7 in [Dyc11] that the functor

\[
[\text{MF}(k[x_1, \ldots, x_n], q_n)] \rightarrow [\text{MF}(\hat{Q}, q_n)]
\]

is fully faithful.

**Remark 2.4.3.** Suppose \( q' : V' \rightarrow k \) is another non-degenerate quadratic form. Choose a basis of \( V' \) with respect to which \( q' \) is diagonal, and let \( x_1, \ldots, x_m \) denote the basis of \( (V')^* \) corresponding to this choice of basis. As above, we may think of \( q' \) as an element of \( S^2((V')^*) \). Set \( Q' \) to be the localization of \( S((V')^*) \) at the ideal \( (x_1, \ldots, x_m) \).

It is well-known that the \( \mathbb{Z}/2\mathbb{Z} \)-graded tensor product of \( \text{Cliff}_k(q) \) and \( \text{Cliff}_k(q') \) is canonically isomorphic to \( \text{Cliff}_k(q + q') \). Further, by Remark 1.3 in [Yos98], the \( \mathbb{Z}/2\mathbb{Z} \)-graded tensor product of Clifford modules is compatible, via this canonical isomorphism and the equivalence in Theorem 2.4.1, with the tensor product \( \text{ST}_{\text{MF}} \) in Proposition 2.3.1. That is, one has a commutative diagram

\[
\begin{array}{ccc}
[\text{Ob}(\text{mod}_{\mathbb{Z}/2\mathbb{Z}}(\text{Cliff}_k(q)))] & \times & [\text{Ob}(\text{mod}_{\mathbb{Z}/2\mathbb{Z}}(\text{Cliff}_k(q')))] \\
\downarrow \Theta_q \times \Theta_{q'} & & \downarrow \Theta_{q + q'} \\
[\text{Ob}([\text{MF}(Q, q)])] & \times & [\text{Ob}([\text{MF}(Q', q')])] \\
\text{ST}_{\text{MF}} & - & \text{ST}_{\text{MF}} \\
[\text{Ob}([\text{MF}(Q \otimes_k Q', q + q')])]
\end{array}
\]

where \( [\text{Ob}(\mathcal{C})] \) denotes the collection of isomorphism classes of objects in a category \( \mathcal{C} \).

Let \( C \) be a rank 1 free \( \mathbb{Z}/2\mathbb{Z} \)-graded \( \text{Cliff}_k(q) \)-module. If \( \dim(V) = 1 \) and \( q = x^2 \), it is easy to see that \( \Theta_q([C]) = k^{\text{stab}} \), where \( k^{\text{stab}} \) is as defined in Section 2.2.4.
By the discussion in Section 6.1 of [Dyc11], the tensor product of a stabilization of the residue field of $k[x]/(x^2)$ with itself is a stabilization of the residue field of $k[x_1, x_2](x_1, x_2)/(x_1^2 + x_2^3)$. Thus, by Remark 2.4.3 we have:

**Proposition 2.4.4.** If $a_i = 1$ for $1 \leq i \leq n$, $\Theta_q([C]) = k^{stab}$.

**Corollary 2.4.5.** If $k$ is algebraically closed, $\Theta_q(C) = k^{stab}$.

### 2.5 An example: $f = y^2 - x^2(x + 1) \in \mathbb{C}[x, y]_{(x, y)}$.

We now consider the category of matrix factorizations of $f = y^2 - x^2(x + 1)$ over the ring $Q = \mathbb{C}[x, y]_{(x, y)}$. Our main goal in this section is to show that $[\text{MF}(Q, f)]$ is not idempotent complete.

Let $\widehat{Q}$ denote the $(x, y)$-adic completion of $Q$. We first prove:

**Proposition 2.5.1.** $K_0[\text{MF}(Q, f)] \neq K_0[\text{MF}(\widehat{Q}, f)]$.

*Proof. We will show that $K_0[\text{MF}(Q, f)]$ is a torsion group, while $K_0[\text{MF}(\widehat{Q}, f)]$ is not."

Set $R := Q/(f)$, let $\text{MCM}(R)$ denote the category of maximal Cohen-Macaulay modules over $R$, and let $\text{mod}(R)$ denote the category of finitely generated modules over $R$.

The inclusion of exact categories

$$\text{MCM}(R) \hookrightarrow \text{mod}(R).$$

induces an isomorphism on Grothendieck groups. By the bottom of page 7 of [Dyc11], $\text{MCM}(R)$ is the stable category of the Frobenius exact category $\text{MCM}(R)$; hence, by
Section 7.4 of \cite{Kra07}, one has a well-defined map

\[ K_0(\operatorname{MCM}(R)) \to K_0(\operatorname{MCM}(R)) \]

given by \([M] \mapsto [M]\).

Thus, one has a surjection

\[ \Phi : G_0(R) \to K_0(\operatorname{MCM}(R)). \]

Since \(f = y^2 - x^2(x + 1)\) is irreducible over \(\mathbb{C}[x, y]\), \(R\) has exactly two prime ideals: (0) and \((x, y)\). It follows that \(G_0(R)\) is generated by \([R]\) and \([R/(x, y)]\).

Choose a nonzero element \(r \in R\). One has an exact sequence

\[ 0 \to R \xrightarrow{\cdot r} R \to R/(r) \to 0, \]

and \(R/(r)\) is a finite length \(R\)-module. Thus, the class \([R/(x, y)] \in G_0(R)\) is torsion.

Since \(\Phi([R]) = 0\), it follows that \(K_0(\operatorname{MCM}(R))\) is torsion. Hence, by Remark 2.2.9, \(K_0[\operatorname{MF}(\hat{Q}, y^2 - x^2)]\) is a torsion group.

\(x + 1 \in \hat{Q}\) has a square root \(z \in \hat{Q}\); this follows, for instance, from Hensel’s Lemma. Thus, there is an isomorphism

\[ \hat{Q}/(y^2 - x^2) \to \hat{Q}/(f) \]

given by \(x \mapsto zx\) and \(y \mapsto y\).

Finally, note that \(K_0[\operatorname{MF}(\hat{Q}, y^2 - x^2)]\) is not a torsion group. One way to see this
is that the Hochster theta pairing (see Section 2.2.3)

\[ \theta : K_0[\text{MF}(\hat{Q}, y^2 - x^2)] \times K_0[\text{MF}(\hat{Q}, y^2 - x^2)] \to \mathbb{Z} \]

is non-zero (Examples 1.1 in [BVS12]). \(\square\)

Remark 2.5.2. In fact, \(K_0[\text{MF}(\hat{Q}, y^2 - x^2)] \cong \mathbb{Z}\). To see this, we need only show that \(K_0[\text{MF}(\hat{Q}, y^2 - x^2)]\) is cyclic, since we demonstrated above that it is not torsion.

The only primes of \(\hat{Q}/(y^2 - x^2)\) are \((x,y), (x),\) and \((y)\). Hence, \(G_0(\hat{Q}/(y^2 - x^2))\) is generated by \([\hat{Q}/(x,y)], [\hat{Q}/(x)],\) and \([\hat{Q}/(y)]\). By the reasoning in the proof above, one has a surjection

\[ G_0(\hat{Q}/(y^2 - x^2)) \to K_0[\text{MF}(\hat{Q}, y^2 - x^2)]. \]

Since there is an exact sequence

\[ 0 \to \hat{Q}/(y) \to \hat{Q}/(y) \to \hat{Q}/(x,y) \to 0, \]

\([\hat{Q}/(x,y)] = 0\) in \(G_0(\hat{Q}/(y^2 - x^2))\). It is also easy to see that the image of \([\hat{Q}/(x)]\) in \(K_0[\text{MF}(\hat{Q}, y^2 - x^2)]\) is the negative of the image of \([\hat{Q}/(y)]\). Thus, \(K_0[\text{MF}(\hat{Q}, y^2 - x^2)]\) is cyclic.

The following is now a straightforward consequence of Proposition 2.5 and several results in [Dyc11]:

**Proposition 2.5.3.** The triangulated category \([\text{MF}(Q,f)]\) is not idempotent complete.

**Proof.** \(Q/(f)\) is an isolated hypersurface singularity in the sense of Definition 2.2.10, thus, by Theorem 5.2 and Theorem 5.7 in [Dyc11], \([\text{MF}(\hat{Q}, f)]\) is the idempotent
completion of $[\text{MF}(Q, f)]$. By Prop 2.5 this means $[\text{MF}(Q, f)]$ cannot be idempotent complete.

Since homotopy categories of dg-triangulated categories are idempotent complete, we immediately obtain:

**Corollary 2.5.4.** $\text{MF}(Q, f)$ is not dg-triangulated.

It is an illuminating exercise to produce an idempotent morphism in $[\text{MF}(Q, f)]$ that does not split; we conclude this section by doing so.

As discussed in the proof of Proposition 2.5, there is a ring isomorphism

$$\hat{Q}/(x^2 - y^2) \cong \hat{Q}/(f).$$

We construct a model for the stabilization of the residue field of $\hat{Q}/(x^2 - y^2)$, as in Section 2.2.4.

Let $F$ be a rank 2 free $\hat{Q}$-module. Choose a basis $e_1, e_2$ of $F$, so that one has a basis $1, e_1, e_2, e_1e_2$ of $\bigoplus_{i=0}^{2} \Lambda^i F$.

As in Section 2.2.4, we may use the expression

$$x^2 - y^2 = x \cdot x + (-y) \cdot y$$

to build the matrix factorization

$$E_{x^2-y^2} = (\Lambda^0 F \oplus \Lambda^2 F) \xrightarrow{(x \quad -y \quad -y \quad x)} \Lambda^1 F).$$
Again following Section 2.2.4, express $\text{End}_{\text{MF}}(E_{x^2-y^2})$ in terms of generators and relations as the dg $\hat{Q}$-algebra

$$\hat{Q}\langle \lambda_1, \lambda_2, c_1, c_2 \rangle/([\lambda_i, \lambda_j], [c_i, c_j], [\lambda_i, c_j] - \delta_{ij})$$

with differential $\partial$ determined by $\partial(\lambda_1) = x = \partial(c_1)$, $\partial(\lambda_2) = y$, and $\partial(c_2) = -y$.

As observed in Section 5.5 of [Dyc11], the cycles $z_1 = c_1 - \lambda_1$ and $z_2 = c_2 + \lambda_2$ generate $H^0\text{End}_{\text{MF}}(E_{x^2-y^2})$ as a $\mathbb{C}$-algebra (in fact, $H^0\text{End}_{\text{MF}}(E_{x^2-y^2})$ is isomorphic to the Clifford algebra $\text{Cliff}_\mathbb{C}(y^2 - x^2)$).

Notice that the element $1 + z_1 z_2$ of the algebra $H^0\text{End}_{\text{MF}}(E_{x^2-y^2})$ is idempotent.

Since $Q/(f)$ is IHS, the functor

$$[\text{MF}(Q, f)] \to [\text{MF}(\hat{Q}, f)]$$

induced by the inclusion $Q \hookrightarrow \hat{Q}$ is fully faithful. Letting $E_f$ and $E_{\hat{f}}$ denote the stabilizations of the residue fields of $Q/(f)$ and $\hat{Q}/(f)$, we may trace through the isomorphisms

$$H^0\text{End}_{\text{MF}}(E_f) \cong H^0\text{End}_{\text{MF}}(E_{\hat{f}}) \cong H^0\text{End}_{\text{MF}}(E_{x^2-y^2})$$

to obtain an idempotent $z$ of $[\text{MF}(Q, f)]$. We now demonstrate that $z$ does not split.

The morphism $1 + z_1 z_2$, thought of as an element of $\text{End}_{\text{MF}}(E_{x^2-y^2})$, may be expressed by the diagram
where $P$ is the matrix \[
\begin{pmatrix}
\frac{1}{2} & -\frac{1}{2} \\
-\frac{1}{2} & \frac{1}{2}
\end{pmatrix}
\].

Since idempotents split in $\text{MF}(\hat{Q}, x^2 - y^2)$, the kernel and cokernel of $\frac{1+z_1z_2}{2}$ determine objects in $\text{MF}(\hat{Q}, x^2 - y^2)$. It is easy to see that $\text{ker}(\frac{1+z_1z_2}{2})$ is isomorphic to $\hat{Q} \xrightarrow{x-y} \hat{Q}$.

As established in Remark 2.5.2, $[E_f] = 0$ and $[\hat{Q} \xleftarrow{\frac{x-y}{x+y}} \hat{Q}] \neq 0$ in $K_0[\text{MF}(\hat{Q}, x^2 - y^2)]$; this implies that $[\text{coker}(\frac{1+z_1z_2}{2})] = -[\hat{Q} \xleftarrow{\frac{x-y}{x+y}} \hat{Q}] \neq 0$.

Now, suppose $z$ splits. Choose an object $Y$ of $[\text{MF}(Q, f)]$ such that there is a factorization

$E_f \xrightarrow{\pi} Y \xrightarrow{\iota} E_f$

of $z$, where $\pi \circ \iota = \text{id}_Y$. Applying the composition

$\Theta : [\text{MF}(Q, f)] \to [\text{MF}(\hat{Q}, f)] \xrightarrow{\sim} [\text{MF}(\hat{Q}, x^2 - y^2)]$

yields a splitting of the idempotent $\frac{1+z_1z_2}{2}$; this means $\Theta(Y) \cong \text{coker}(\frac{1+z_1z_2}{2})$ in $[\text{MF}(\hat{Q}, x^2 - y^2)]$. 
Since the class \([\text{coker}(\frac{1+z_1+z_2}{2})]\) is nonzero in \(K_0[\text{MF}(\widehat{Q}, x^2 - y^2)] \cong \mathbb{Z}\), this implies that the map
\[
K_0[\text{MF}(Q, f)] \to K_0[\text{MF}(\widehat{Q}, x^2 - y^2)]
\]
induced by \(\Theta\) is nonzero. But this is impossible, since \(K_0[\text{MF}(Q, f)]\) is torsion. Thus, \(z\) does not split.

## 2.6 Periodicity

The following phenomenon, discovered by Kn"orrer in [Kn"orrer 1987], is known as Kn"orrer periodicity:

**Theorem 2.6.1.** Suppose \(k\) is an algebraically closed field and \(\text{char}(k) \neq 2\). Let \(q = u^2 + v^2 \in k[[u, v]]\). If \(f \in (x_1, \ldots, x_n) \setminus \{0\} \subseteq k[[x_1, \ldots, x_n]]\), there is a triangulated equivalence
\[
K : [\text{MF}(k[[x_1, \ldots, x_n]], f)] \to [\text{MF}(k[[x_1, \ldots, x_n, u, v]], f + q)].
\]

**Remark 2.6.2.** Set \(X\) to be the matrix factorization
\[
k[[u, v]] \xrightarrow{u+iv} k[[u, v]] \xleftarrow{u-iv} k[[u, v]]
\]
of \(u^2 + v^2\) over \(k[[u, v]]\). \(K\) may be given by
\[
P \mapsto P \otimes_{\text{MF}} X
\]
on objects and
\[
\alpha \mapsto \alpha \otimes \text{id}_X
\]
on morphisms.

A version of Knörrer periodicity for isolated hypersurface singularities may be deduced from the following proposition:

**Proposition 2.6.3.** Suppose $Q$ and $Q'$ are algebras over a field $k$. Let $f \in Q$ and $f' \in Q'$, and suppose $Q/(f)$ and $Q'/(f')$ are IHS. Set $Q'' := Q \otimes_k Q'$. If there exists an object $X$ in $\text{MF}(Q', f')$ such that

(a) $X$ is a compact generator of $[\text{MF}(\widehat{Q}', f')]$, and

(b) the inclusion $k \hookrightarrow \text{End}_{\text{MF}(\widehat{Q}', f')}(X)$ is a quasi-isomorphism

then the dg functor

$$K_X : \text{MF}(\widehat{Q}, f) \to \text{MF}(\widehat{Q}'', f \oplus f')$$

given by

$$P \mapsto P \otimes_{\text{MF}} X$$

on objects and

$$\alpha \mapsto \alpha \otimes \text{id}_X$$

on morphisms is a quasi-equivalence.

**Proof.** By Sections 4 and 5 of [Dac11], the inclusion

$$\text{End}_{\text{MF}}(X) \hookrightarrow \text{MF}(\widehat{Q}', f').$$

is a Morita equivalence. Applying Exercise 4.4.11 in [Toë11], we have a chain of Morita equivalences

$$\text{MF}(\widehat{Q}, f) \otimes_k k \hookrightarrow \text{MF}(\widehat{Q}, f) \otimes_k \text{End}_{\text{MF}}(X) \hookrightarrow \text{MF}(\widehat{Q}, f) \otimes_k \text{MF}(\widehat{Q}', f').$$
Composing with $\widehat{ST}_{MF}$, Proposition 2.3.4 yields a Morita equivalence

$$MF(\widehat{Q}, f) \to MF(\widehat{Q}'', f \oplus f').$$

This composition is clearly the functor $K_X$. Since both $MF(\widehat{Q}, f)$ and $MF(\widehat{Q}'', f \oplus f')$ are dg-triangulated by Lemma 5.6 in [Dyc11], an application of Proposition 2.1.15 finishes the proof.

To deduce a version of Knörrer periodicity for isolated hypersurface singularities, assume $k$ to be an algebraically closed field such that $\text{char}(k) \neq 2$, set $Q' = k[u, v]$ and $f' = u^2 + v^2$, and take $X$ to be the matrix factorization

$$k[u, v] \xrightarrow{u+iv} k[u, v].$$

This is the approach taken in Section 5.3 of [Dyc11].

We point out that $k$ is not assumed to be algebraically closed in Proposition 2.6.3 and no assumptions on the characteristic of $k$ are made, either. In particular, we may use Proposition 2.6.3 to prove an 8-periodic version of Knörrer periodicity over $\mathbb{R}$ (this result implies Theorem 1.0.1 from the introduction):

**Theorem 2.6.4.** Suppose $Q$ is an $\mathbb{R}$-algebra. Let $f \in Q$, and suppose $Q/(f)$ is IHS. Set $Q' := \mathbb{R}[u_1, \ldots, u_8]$, $q := u_1^2 + \cdots + u_8^2 \in Q'$, and $Q'' := Q \otimes_\mathbb{R} Q'$. Then there exists a matrix factorization $X$ of $q$ over $Q'$ such that the dg functor

$$MF(\widehat{Q}, f) \to MF(\widehat{Q}'', f + q)$$

given by

$$P \mapsto P \otimes_{MF} X$$
on objects and
\[ \alpha \mapsto \alpha \otimes \text{id}_X \]
on morphisms is a quasi-equivalence.

Proof. We equip the matrix algebra \( \text{Mat}_{16}(\mathbb{R}) \) of \( 16 \times 16 \) of matrices over \( \mathbb{R} \) with a \( \mathbb{Z}/2\mathbb{Z} \)-grading in the following way: \( A = (a_{ij}) \) is homogeneous of even degree if \( a_{ij} = 0 \) whenever \( i + j \) is odd, and \( A \) is homogeneous of odd degree if \( a_{ij} = 0 \) whenever \( i + j \) is even. Then, by Proposition V.4.2 in [Lam05],

\[ \text{Cliff}_k(q) \cong \text{Mat}_{16}(\mathbb{R}) \]
as \( \mathbb{Z}/2\mathbb{Z} \)-graded algebras. In particular,

\[ [\text{MF}(\hat{Q}', q)] \cong \text{mod}_{\mathbb{Z}/2\mathbb{Z}}(\text{Mat}_{16}(\mathbb{R})) \]
by Theorem 2.4.1

Let \( M \in \text{mod}_{\mathbb{Z}/2\mathbb{Z}}(\text{Mat}_{16}(\mathbb{R})) \) be the module of matrices with nonzero entries only in the first column, and let \( X \) be a matrix factorization corresponding to \( M \) under the equivalence of categories in Theorem 2.4.1. Since

\[ \text{End}_{\text{Mat}_{16}(\mathbb{R})}(M) \cong \mathbb{R} \]
as \( \mathbb{Z}/2\mathbb{Z} \)-graded algebras, where \( \mathbb{R} \) is concentrated in even degree, we have

\[ H^0(\text{End}_{\text{MF}}(X)) \cong \mathbb{R} \]
Thus, the inclusion
\[ \mathbb{R} \hookrightarrow \text{End}_{\text{MF}}(X) \]
is a quasi-isomorphism. Also, by Proposition 2.4.4

\[ (X \oplus X[1])^{\oplus 8} \cong \mathbb{R}^{\text{stab}} \]
in \text{MF}(\hat{Q}, q); it follows from Sections 4 and 5 of [Dyc11] that \( X \) is a compact generator of \text{MF}(\hat{Q}, q). Now apply Proposition 2.6.3 \( \square \)

Remark 2.6.5. Explicitly, one may take \( X \) to be the matrix factorization

\[ \hat{Q}'^{\oplus 8} \xrightarrow{A} \hat{Q}'^{\oplus 8}, \]

where

\[
A = \begin{pmatrix}
  u_1 & u_2 & -u_3 & u_4 & u_5 & -u_6 & -u_7 & -u_8 \\
-u_2 & u_1 & -u_4 & -u_3 & u_6 & u_5 & -u_8 & u_7 \\
u_3 & u_4 & u_1 & -u_2 & u_7 & u_8 & u_5 & -u_6 \\
-u_4 & u_3 & u_2 & u_1 & u_8 & -u_7 & u_6 & u_5 \\
u_5 & u_6 & u_7 & u_8 & u_1 & -u_2 & u_3 & -u_4 \\
-u_6 & u_5 & u_8 & -u_7 & u_2 & u_1 & u_4 & u_3 \\
-u_7 & -u_8 & u_5 & u_6 & -u_3 & -u_4 & u_1 & u_2 \\
-u_8 & u_7 & -u_6 & u_5 & u_4 & -u_3 & -u_2 & u_1 
\end{pmatrix}
\]
and

\[
B = \begin{pmatrix}
  u_1 & -u_2 & u_3 & -u_4 & u_5 & u_6 & u_7 & u_8 \\
  u_2 & u_1 & u_4 & u_3 & -u_6 & u_5 & u_8 & -u_7 \\
  -u_3 & -u_4 & u_1 & u_2 & -u_7 & -u_8 & u_5 & u_6 \\
  u_4 & -u_3 & -u_2 & u_1 & -u_8 & u_7 & -u_6 & u_5 \\
  -u_5 & u_6 & u_7 & u_8 & u_1 & u_2 & -u_3 & u_4 \\
  -u_6 & -u_5 & u_8 & -u_7 & -u_2 & u_1 & -u_4 & -u_3 \\
  -u_7 & -u_8 & -u_5 & u_6 & u_3 & u_4 & u_1 & -u_2 \\
  -u_8 & u_7 & -u_6 & -u_5 & -u_4 & u_3 & u_2 & u_1
\end{pmatrix}
\]

This can be verified by computing $\Theta_{u_1^2 + \cdots + u_8^2}(M)$ using the formula provided in Section 2.4.

Remark 2.6.6. Theorem 2.6.4 implies the existence of a Knörrer-type periodicity for matrix factorizations over $\mathbb{R}$ of period at most 8. We point out that the period is exactly 8, since the Brauer-Wall group of $\mathbb{R}$ is the cyclic group $\mathbb{Z}/8\mathbb{Z}$ generated by the class of Cliff$_{\mathbb{R}}(x^2)$ (see [Yos90] Remark 14.9).

It is natural to ask whether one may use Proposition 2.6.3 to exhibit additional periodic behaviors of matrix factorization categories. We conclude this section with some remarks in this direction.

The existence of an object $X$ as in the setup of 2.6.3 implies that the dga $A_{(Q',f')}$ is formal, since, by Theorem 5.1 in [Dyc11], $A_{(Q',f')}$ is quasi-isomorphic, in this setting, to the endomorphism dga of a $\mathbb{Z}/2\mathbb{Z}$-graded complex of $k$-vector spaces. On the other hand, by Theorem 5.9 in [Dyc11], $A_{(Q',f')}$ can only be formal when $f'$ has no terms of degree higher than 2. It follows that, when char($k$) $\neq 2$, we may use the Buchweitz-Eisenbud-Herzog equivalence (Theorem 2.4.1) to reduce the problem of determining
whether $\text{MF}(Q', f')$ is Morita equivalent to $k$ to studying the image of the group homomorphism

$$\text{WG}(k) \to \text{BW}(k)$$

where $\text{WG}(k)$ is the Witt-Grothendieck ring of $k$, thought of as an additive group, and $\text{BW}(k)$ is the Brauer-Wall group of $k$. When $\text{char}(k) = 2$, less is known about the structure of matrix factorization categories over non-degenerate quadratics. We leave for future work the problem of finding sufficient conditions for such a matrix factorization category to be Morita equivalent to its ground field.
Chapter 3

Matrix Factorizations and the K-theory of the Milnor Fiber

We have demonstrated that matrix factorization categories associated to isolated hypersurface singularities over $\mathbb{C}$ and $\mathbb{R}$ exhibit 2- and 8-periodic versions of Knörrer periodicity, respectively. This pattern resembles Bott periodicity in topological K-theory; the goal of this chapter is to explain this resemblance.

We give a rough sketch of our approach. The classical link between the periodicity of Clifford algebras up to $\mathbb{Z}/2\mathbb{Z}$-graded Morita equivalence and Bott periodicity in topological K-theory is the Atiyah-Bott-Shapiro construction, which first appeared in Part III of [ABS64]. Loosely speaking, the Atiyah-Bott-Shapiro construction is a way of mapping a finitely generated $\mathbb{Z}/2\mathbb{Z}$-graded module over a real or complex Clifford algebra to a class in the K-theory of a sphere.

Composing the Buchweitz-Eisenbud-Herzog equivalence (Theorem 2.4.1) with the Atiyah-Bott-Shapiro construction, we have a way of assigning a class in the topological K-theory of a sphere to a matrix factorization of a non-degenerate quadratic form over
The idea is to lift this composition; that is, we wish to associate a space $X_f$ to a real or complex polynomial $f$ and construct a map from matrix factorizations of $f$ to the topological K-theory of $X_f$ so that the diagram

\[
\begin{array}{c}
\text{mf’s of real/complex quadratics} \\ \downarrow \\
\text{K-theory of spheres}
\end{array}
\xrightarrow{\text{ABS} \circ \text{BEH}}
\begin{array}{c}
\text{mf’s of real/complex polynomials} \\ \uparrow \\
\text{K-theory of spaces of the form } X_f
\end{array}
\]

commutes.

It turns out that the right choice of $X_f$ is the Milnor fiber (resp. positive or negative Milnor fiber) associated to the complex (resp. real) polynomial.

We begin this chapter with discussions of known results concerning the Milnor fiber and relative topological K-theory. Then, using the work of Atiyah-Bott-Shapiro in [ABS64] as a guide, we will complete the above diagram, and we will use the bottom arrow to explain a precise sense in which Knörrer periodicity and Bott periodicity are compatible phenomena.

### 3.1 The real and complex Milnor fibers

Let $f \in \mathbb{C}[x_0, \ldots, x_n]$, and suppose $f(0) = 0$. We begin this section by describing the construction of the Milnor fiber associated to $f$, following the exposition in Section 1 of [BVS12]. We then discuss various properties of the Milnor fiber that we will make
3.1.1 Construction of the Milnor fibration and some properties of the Milnor fiber

For $\epsilon > 0$, define $B_\epsilon$ to be the closed ball centered at the origin of radius $\epsilon$ in $\mathbb{C}^{n+1}$, and for $\delta > 0$, set $D_\delta^*$ to be the punctured disk centered at the origin in $\mathbb{C}$ of radius $\delta$.

Choose $\epsilon > 0$ so that, for $0 < \epsilon' \leq \epsilon$, $\partial B_{\epsilon'}$ intersects $f^{-1}(0)$ transversely. Upon choosing such an $\epsilon$, choose $\delta \in (0, \epsilon)$ such that $f^{-1}(t)$ intersects $\partial B_\epsilon$ transversely for all $t \in D_\delta^*$. Then the map

$$\psi : B_\epsilon \cap f^{-1}(D_\delta^*) \to D_\delta^*$$

given by $\psi(x) = f(x)$ is a locally trivial fibration.

The map $\psi$ depends, of course, on our choices of $\epsilon$ and $\delta$. However, if $\epsilon', \delta'$ is another pair of positive numbers satisfying the above conditions, the fibration associated to these choices is equivalent to the one above (see Definition 1.5 in Chapter 2 of [Dim92] for a description of what it means for two fibrations to be equivalent). We are thus justified in calling $\psi$ the Milnor fibration associated to $f$.

Remark 3.1.1. The Milnor fibration was originally introduced in [Mil68]. The above construction is not the same as the construction of the Milnor fibration in [Mil68], but the two constructions yield equivalent fibrations; this is a result due to Lê in [Lê76].

We will call the fiber of this fibration the Milnor fiber of $f$ and denote it by $F_f$. $F_f$ is independent of our choices of $\epsilon$ and $\delta$ up to homeomorphism, so we suppress these choices in the notation. However, these choices will be significant at various points later on.
Definition 3.1.2. Let $k$ be a field. A polynomial $f \in k[x_0, \ldots, x_n]$ is called quasi-homogeneous of degree $d$ if there exist positive integers $w_0, \ldots, w_n$ such that $f$ is a homogeneous element of degree $d$ in the $\mathbb{Z}$-graded ring $k[x_0, \ldots, x_n]$, where each variable $x_i$ has degree $w_i$.

Remark 3.1.3. Suppose $f$ is quasi-homogeneous, and let $t > 0$. Define

$$h : \mathbb{C}[x_0, \ldots, x_n] \to \mathbb{C}[x_0, \ldots, x_n]$$

to be the ring automorphism given by

$$x_i \mapsto \frac{x_i}{t^{w_i/d}}.$$ 

Then, if the Milnor fiber of $f$ may be taken to be a fiber of $f$ over $t$, the Milnor fiber of $h(f)$ may be taken to be a fiber over 1; hence one may often assume without loss that the Milnor fiber associated to a quasi-homogeneous polynomial is a fiber over 1.

If $\mathbb{C}[x_0, \ldots, x_n](x_0, \ldots, x_n)/(f)$ is IHS (see Definition 2.2.10), set

$$\mu := \dim_{\mathbb{C}} \frac{\mathbb{C}[x_0, \ldots, x_n](x_0, \ldots, x_n)}{(\frac{\partial f}{\partial x_0}, \ldots, \frac{\partial f}{\partial x_n})} < \infty,$$

the Milnor number of $f$.

Theorem 3.1.4 (Milnor, 1968). If $\mathbb{C}[x_0, \ldots, x_n](x_0, \ldots, x_n)/(f)$ is IHS, $F_f$ is homotopy equivalent to a wedge sum of $\mu$ copies of $S^n$.

Remark 3.1.5. Since $\psi$ restricts to a fibration over a circle, $F_f$ comes equipped with a monodromy homeomorphism

$$h_f : F_f \xrightarrow{\sim} F_f.$$
3.1.2 The Sebastiani-Thom homotopy equivalence

We recall the definition of the join of two topological spaces:

**Definition 3.1.6.** Let $X$ and $Y$ be compact Hausdorff spaces. The join of $X$ and $Y$, denoted $X \ast Y$, is the quotient of $X \times Y \times I$ by the relations

$$(x_1, y, 0) \sim (x_2, y, 0)$$

$$(x, y_1, 1) \sim (x, y_2, 1)$$

equipped with the quotient topology.

**Remark 3.1.7.** The cone $CX$ over a compact Hausdorff space $X$ can be expressed explicitly as the quotient of

$$X \times [0, 1]$$

by the relation

$$(x_1, 0) \sim (x_2, 0)$$

for all $x_1, x_2 \in X$.

When $X$ and $Y$ are compact Hausdorff,

$$X \ast Y \cong (CX \times Y) \cup (X \times CY) \subseteq CX \times CY;$$

here, we identify $X$ and $Y$ with the subsets $X \times \{1\}$ and $Y \times \{1\}$ of $CX$ and $CY$, respectively.

By [Bro06] 5.7.4, one has an explicit homeomorphism

$$CX \times CY \to C(X \ast Y)$$
given by

\[(x, t, y, t') \mapsto ((x, y, \frac{t}{2t'}), t'), \text{ if } t' \geq t, t' \neq 0\]

\[(x, t, y, t') \mapsto ((x, y, 1 - \frac{t'}{2t}), t), \text{ if } t \geq t', t \neq 0\]

\[(x, 0, y, 0) \mapsto ((x, y, 0), 0),\]

and this map restricts to a homeomorphism

\[(CX \times Y) \cup (X \times CY) \rightarrow X \ast Y.\]

**Example 3.1.8.** Let \(X\) be a compact Hausdorff space.

- \(X \ast \text{point} \cong CX\).
- \(X \ast S^0 \cong SX\), the suspension of \(X\).

**Remark 3.1.9.** When forming the join of spaces \(X\) and \(Y\) that are not necessarily compact Hausdorff, the set \(X \ast Y\) is typically equipped with the weakest topology such that the coordinate projections from \(X \times Y \times I\) to \(X \ast Y\) are continuous. When \(X\) and \(Y\) are compact Hausdorff, this topology coincides with the quotient topology ([Eh92] Section 3.2).

Now, suppose \(f \in \mathbb{C}[x_0, \ldots, x_n], f' \in \mathbb{C}[y_0, \ldots, y_m]\), and \(f(0) = 0 = f'(0)\). Let \(f \oplus f'\) denote the sum of \(f\) and \(f'\) thought of as an element of \(\mathbb{C}[x_0, \ldots, x_n, y_0, \ldots, y_m]\).

We have the following classical result relating the Milnor fibers of \(f, f',\) and \(f \oplus f'\), due to Sebastiani-Thom:

**Theorem 3.1.10 ([ST71]).** There is a homotopy equivalence

\[ST : F_f \ast F_{f'} \rightarrow F_{f \oplus f'}.\]
that is compatible with monodromy; that is, the square

\[
\begin{array}{ccc}
F_f \ast F_{f'} & \xrightarrow{\text{ST}} & F_{f \oplus f'} \\
\downarrow h_f \ast h_{f'} & & \downarrow h_{f \oplus f'} \\
F_f \ast F_{f'} & \xrightarrow{\text{ST}} & F_{f \oplus f'}
\end{array}
\]

commutes up to homotopy.

We refer the reader to Section 2.7 of [AGZV12], §3 of Chapter 3 in [Dim92], and [Oka73] for discussions related to Theorem 3.1.10.

Suppose \( \mathbb{C}[x_0, \ldots, x_n]/(x_0, \ldots, x_n)/(f) \), \( \mathbb{C}[y_0, \ldots, y_m]/(y_0, \ldots, y_m)/(f') \) are IHS (see Definition 2.2.10). We now exhibit an explicit map realizing the homotopy equivalence in Theorem 3.1.10 in this setting, following Section 2.7 of [AGZV12].

Choose real numbers \( \epsilon'', \delta'' \), such that the map

\[
B_{\epsilon''} \cap (f \oplus f')^{-1}(D_{\delta''}^*) \to D_{\delta''}^*
\]

given by \( x \mapsto (f \oplus f')(x) \) is a locally trivial fibration, as above.

Similarly, choose \( \epsilon, \delta \) and \( \epsilon', \delta' \), as well as \( t'' \in D_{\delta''}^* \), so that the analogous maps

\[
B_{\epsilon} \cap f^{-1}(D_{\delta}^*) \to D_{\delta}^*
\]

\[
B_{\epsilon'} \cap (f')^{-1}(D_{\delta'}^*) \to D_{\delta'}^*
\]

are locally trivial fibrations, and also so that

(a) \( \epsilon, \epsilon' \) are sufficiently small so that \( B_{\epsilon} \times B_{\epsilon'} \subseteq B_{\epsilon''} \).

(b) \( |t''| < \min\{\delta, \delta'\} \).

Set \( F_f \), \( F_{f'} \), and \( F_{f \oplus f'} \) to be the Milnor fibers of \( f \), \( f' \), and \( f \oplus f' \) over \( t'' \).
The goal is to construct an injective homotopy equivalence

\[ CF_f \times F_{f'} \cup F_f \times CF_{f'} \rightarrow F_{f \oplus f'}. \]

Applying Lemma 2.10 in [AGZV12], choose an injection

\[ H : CF_f \rightarrow B_{\epsilon} \]

such that

- \( H(x, 1) = x \in F_f \subseteq B_{\epsilon}, \)
- \( H(-, s) : F_f \rightarrow B_{\epsilon} \) maps into the Milnor fiber \( B_{\epsilon} \cap f^{-1}(st'') \) for \( s \in (0, 1) \), and
- \( H(x, 0) = 0 \) for all \( x \in F_f \)

**Example 3.1.11.** If \( f \) is quasi-homogeneous of degree \( d \) with weights \( w_0, \ldots, w_d \), such a map \( H \) may be given by

\[ (x, s) \mapsto (s^{\frac{w_0}{d}} x_0, \ldots, s^{\frac{w_d}{d}} x_n). \]

Notice that our isolated singularity assumption is not necessary in this example.

Choose \( H' \) similarly for the Milnor fiber \( F_{f'} \).

By the discussion on pages 54-55 of [AGZV12] and Remark 3.1.7, there is an injective homotopy equivalence

\[ (\text{im}(H) \times F_{f'} \cup F_f \times \text{im}(H')) \rightarrow F_{f \oplus f'}. \]
given by

\[(H(x, s), H'(y, s')) \mapsto (H(x, \frac{1 + s - s'}{2}), H'(y, \frac{1 - s + s'}{2})).\]

Composing, one has an injective homotopy equivalence

\[g : CF_f \times F_f' \cup F_f \times CF_f' \rightarrow F_f \oplus F_f',\]

as desired. The map obtained by composing \(g\) with the inverse of the homeomorphism from \(CF_f \times F_f' \cup F_f \times CF_f'\) to \(F_f \ast F_f'\) in Remark 3.1.7 enjoys the same properties as the map \(ST\) in Theorem 2.3.4.

**Remark 3.1.12.** The homotopy equivalence

\[(\text{im}(H) \times F_f' \cup F_f \times \text{im}(H')) \rightarrow F_f \oplus F_f',\]

above extends to an injection of pairs

\[G : (\text{im}(H) \times \text{im}(H'), \text{im}(H) \times F_f' \cup F_f \times \text{im}(H')) \rightarrow (B_{\epsilon''}, F_f \oplus F_f')\]

that maps a point \((H(x, s), H'(y, s'))\) to

\[(H(x, \frac{s}{2}), H'(y, \frac{2s' - s}{2}), \text{if } s \leq s', s' \neq 0)\]

\[(H(x, \frac{2s - s'}{2}), H'(y, \frac{s'}{2}), \text{if } s' \leq s, s \neq 0)\]

\[0, \text{if } s = 0 = s'.\]

The image of \(\text{im}(H) \times \text{im}(H')\) under this injection is homeomorphic to \(CF_{f \oplus f'}\) in an evident way.
3.1.3 An analogue of the Milnor fibration for polynomials over \( \mathbb{R} \)

Now, suppose \( f \in \mathbb{R}[x_0, \ldots, x_n] \) and \( f(0) = 0 \). One may construct a topological locally trivial fibration

\[
\psi : B_\epsilon \cap f^{-1}\left((-\delta, 0) \cup (0, \delta)\right) \to (-\delta, 0) \cup (0, \delta)
\]

for some \( \epsilon > 0 \) and \( \delta \) such that \( 0 < \delta << \epsilon \) in the same way as above, where \( B_\epsilon \) is now the closed ball of radius \( \epsilon \) centered at the origin in \( \mathbb{R}^{n+1} \).

But now, fibers over \((-\delta, 0)\) and \((0, \delta)\) need not be homeomorphic. For instance, if \( f = x_0^2 + \cdots + x_n^2 \), the positive fibers of \( \psi \) are homeomorphic to \( S^n \), while the negative fibers are empty.

We denote by \( F^+_f \) and \( F^-_f \) the positive and negative Milnor fibers of \( f \). The topology of the real Milnor fibers is more complicated than that of the complex Milnor fiber. However, there is a version of Theorem \[3.1.10\] for real Milnor fibers:

**Theorem 3.1.13** ([DP92] Remark 11). Suppose

\[
f \in \mathbb{R}[x_0, \ldots, x_n], \ g \in \mathbb{R}[y_0, \ldots, y_m]
\]

are quasi-homogeneous, and \( f(0) = 0 = g(0) \).

If \( F^+_f \) and \( F^+_g \) are nonempty, there is a homotopy equivalence

\[
ST : F^+_f \ast F^+_g \to F^+_{f \circ g}.
\]

**Remark 3.1.14.** Since \( F^-_f = F^-_f \), we have a similar result for negative Milnor fibers.
3.2 Relative topological K-theory and the Euler characteristic

We introduce some facts concerning relative topological K-theory that we will need along the way. All results in this section are essentially due to Atiyah-Bott-Shapiro in [ABS64], though we modify their exposition at various points to fit our purposes.

Let $X$ be a compact topological space, and let $Y$ be a closed subspace of $X$ such that there exists a homotopy equivalence of pairs between $(X, Y)$ and a finite CW pair; that is, a pair $(X', Y')$ where $X'$ is a finite CW complex and $Y'$ is a subcomplex of $X'$. We construct a category $\mathcal{C}_1(X, Y)$ from $(X, Y)$ in the following way:

- Objects of $\mathcal{C}_1(X, Y)$ are pairs of real vector bundles $V_1, V_0$ over $X$ equipped with isomorphisms
  \[ V_1|_Y \xrightarrow{\sigma} V_0|_Y. \]

Denote objects of $\mathcal{C}_1(X, Y)$ by $(V_1, V_0; \sigma)$.

- Morphisms in $\mathcal{C}_1(X, Y)$ are pairs of morphisms of vector bundles over $X$
  \[ \alpha_1 : V_1 \to V'_1, \alpha_0 : V_0 \to V'_0 \]
  such that the following diagram of maps of vector bundles over $Y$ commutes:
  \[
  \begin{array}{ccc}
  V_1|_Y & \xrightarrow{\sigma} & V_0|_Y \\
  \alpha_1|_Y & & \alpha_0|_Y \\
  V'_1|_Y & \xrightarrow{\sigma'} & V'_0|_Y 
  \end{array}
  \]

We write morphisms in $\mathcal{C}_1(X, Y)$ as ordered pairs $(\alpha_1, \alpha_0)$.

Remark 3.2.1. The reason for the subscript in the notation $\mathcal{C}_1(X, Y)$ is that, for any $n \geq 1$, one may build a category $\mathcal{C}_n(X, Y)$ with objects given by ordered $n + 1$-tuples.
of vector spaces on $X$ whose restrictions to $Y$ fit into an exact sequence (cf. \cite{ABS64} §8).

**Remark 3.2.2.** We will work with real vector bundles throughout this section; however, there is an analogous version of every result in this section for complex vector bundles.

**Proposition 3.2.3.** A map $g : (X_1, Y_1) \rightarrow (X_2, Y_2)$ of pairs of spaces as above induces a functor

$$g^* : C_1(X_2, Y_2) \rightarrow C_1(X_1, Y_1).$$

**Proof.** On objects,

$$g^*((V_1, V_0; \sigma)) = (g^*(V_1), g^*(V_0); (g|_{Y_1})^*(\sigma)).$$

If $\alpha = (\alpha_1, \alpha_0) : (V_1, V_0; \sigma) \rightarrow (V'_1, V'_0; \sigma')$ is a morphism,

$$g^*(\alpha) = (g^*(\alpha_1), g^*(\alpha_0)).$$

The diagram

$$\begin{array}{ccc}
g^*(V_1)|_{Y_1} & \xrightarrow{(g|_{Y_1})^*(\sigma)} & g^*(V_0)|_{Y_1} \\
g^*(\alpha_1)|_{Y_1} \downarrow & & \downarrow g^*(\alpha_0)|_{Y_1} \\
g^*(V'_1)|_{Y_1} & \xrightarrow{(g|_{Y_1})^*(\sigma')} & g^*(V'_0)|_{Y_1}
\end{array}$$

commutes, since pullback of vector bundles respects composition.

Given objects $V = (V_1, V_0; \sigma)$ and $V' = (V'_1, V'_0; \sigma')$ in $C_1(X, Y)$, define an object

$$V \oplus V' := (V_1 \oplus V'_1, V_0 \oplus V'_0; \sigma \oplus \sigma').$$
We have evident canonical morphisms

\[ \iota_V := V \to V \oplus V' \]

\[ \iota_{V'} := V' \to V \oplus V' \]

**Proposition 3.2.4.** Let \( V = (V_1, V_0; \sigma) \) and \( V' = (V'_1, V'_0; \sigma') \) be objects in \( \mathcal{C}_1(X,Y) \).

Then \( (V \oplus V', \iota_V, \iota_{V'}) \)

is the coproduct of \( V \) and \( V' \) in \( \mathcal{C}_1(X,Y) \).

**Proof.** This follows easily from the fact that

\[ (V_1 \oplus V'_1, (\iota_V)_1, (\iota_{V'})_1) \]

and

\[ (V_0 \oplus V'_0, (\iota_V)_0, (\iota_{V'})_0) \]

are the coproducts in the category of vector bundles over \( X \) of \( V_1, V'_1 \) and \( V_0, V'_0 \).

**Proposition 3.2.5.** \( \mathcal{C}_1(X,Y) \) is an additive category.

**Proof.** It is well-known that the category of vector bundles over any topological space is additive ([Kar08] Theorem I.6.1).

Given morphisms

\[ (\alpha_1, \alpha_0), (\beta_1, \beta_0) : (V_1, V_0; \sigma) \to (V'_1, V'_0; \sigma'), \]

define

\[ \alpha + \beta := (\alpha_1 + \beta_1, \alpha_0 + \beta_0). \]
It is easy to check that $\text{Hom}_{C(X,Y)}((V_1, V_0; \sigma), (V'_1, V'_0; \sigma'))$, equipped with this operation, is an abelian group and that composition in $C_1(X,Y)$ is $\mathbb{Z}$-bilinear.

Finally, apply Proposition 3.2.4 to conclude that $C_1(X,Y)$ admits finite coproducts. \hfill $\square$

**Remark 3.2.6.** A morphism in $(\alpha_1, \alpha_0)$ in $C_1(X,Y)$ is an isomorphism (resp. monomorphism, epimorphism) if and only if $\alpha_1$ and $\alpha_0$ are isomorphisms (resp. monomorphisms, epimorphisms) of vector bundles over $X$.

We shall call an object of $C_1(X,Y)$ *elementary* if it is isomorphic to an object of the form $(V, V; \text{id}_V|_Y)$. Notice that the direct sum of two elementary objects in $C_1(X,Y)$ is again elementary.

There is a useful alternative definition of an elementary object:

**Lemma 3.2.7.** Let $(V_1, V_0; \sigma)$ be an object in $C_1(X,Y)$. The following are equivalent:

1. $\sigma$ can be extended to an isomorphism $\tilde{\sigma} : V_1 \to V_0$.

2. $(V_1, V_0; \sigma)$ is elementary.

**Proof.** Suppose $(V_1, V_0; \sigma)$ is elementary. Then we have a commutative square on $Y$:

\[
\begin{array}{ccc}
V_1|_Y & \xrightarrow{\sigma} & V_0|_Y \\
\downarrow{\alpha_1|_Y} & & \downarrow{\alpha_0|_Y} \\
V|_Y & \xrightarrow{\text{id}_V|_Y} & V|_Y
\end{array}
\]

where $V$ is a vector bundle over $X$, and

$$\alpha_1 : V_1 \to V, \: \alpha_0 : V \to V_0$$

are isomorphisms of vector bundles over $X$. Observe that $\sigma$ lifts to $\alpha_0^{-1} \circ \alpha_1$. 
Conversely, suppose $\sigma$ can be extended to an isomorphism $\tilde{\sigma} : V_1 \to V_0$. Then we have a commutative square of maps of vector bundles on $X$:

\[
\begin{array}{ccc}
V_1 & \xrightarrow{\tilde{\sigma}} & V_0 \\
\downarrow{\text{id}_{V_1}} & & \downarrow{\tilde{\sigma}^{-1}} \\
V_1 & \xrightarrow{\text{id}_{V_1}} & V_1
\end{array}
\]

If $V$ and $V'$ are objects in $C_1(X, Y)$, we will say $V \sim V'$ if and only if there exist elementary objects $E, E'$ such that

\[V \oplus E \cong V' \oplus E'.\]

The relation $\sim$ is an equivalence relation. Define $L_1(X, Y)$ to be the commutative monoid of equivalence classes under $\sim$ with operation given by $\oplus$.

**Remark 3.2.8.** Let $(X_1, Y_1), (X_2, Y_2)$ be pairs as above, and let $g : (X_1, Y_1) \to (X_2, Y_2)$ be a map of pairs. Then the functor

\[g^* : C_1(X_2, Y_2) \to C_1(X_1, Y_1)\]

applied to an elementary object is again elementary. Hence, $g^*$ induces a map of monoids

\[\overline{g}^* : L_1(X_2, Y_2) \to L_1(X_1, Y_1).\]

One may similarly define monoids $L_n(X, Y)$ involving longer sequences of bundles; see [ABS64] Definition 7.1 for details. Denote elements of $L_n(X, Y)$ by

\[[V_n, \ldots, V_0; \sigma_n, \ldots, \sigma_1].\]
We point out that there is an inclusion map

\[ j_n : L_1(X, Y) \to L_n(X, Y) \]

given by

\[ [V_1, V_0; \sigma] \mapsto [0, \ldots, 0, V_1, V_0; 0, \ldots, 0, \sigma], \]

and, by Proposition 7.4 in [ABS64], \( j_n \) is an isomorphism for all \( n \).

The main reason we are interested in the monoid \( L_1(X, Y) \) is the following result due to Atiyah-Bott-Shapiro:

**Proposition 3.2.9** ([ABS64] 9.1). There exists a unique natural homomorphism

\[ \chi : L_1(X, Y) \to KO^0(X, Y) \]

which, when \( Y = \emptyset \), is given by

\[ \chi(E) = [V_0] - [V_1]. \]

Moreover, \( \chi \) is an isomorphism.

In particular, \( L_1(X, Y) \) is an abelian group. Atiyah-Bott-Shapiro call the map \( \chi \) an *Euler characteristic*.

Let \( (X, Y), (X', Y') \) be pairs as above. We conclude this section by exhibiting a product map

\[ L_1(X, Y) \otimes L_1(X', Y') \to L_1(X \times X', X \times Y' \cup Y \times X') \]

that agrees, via \( \chi \), with the usual product on relative K-theory.
Let $V = (V_1, V_0; \sigma) \in \text{Ob}(\mathcal{C}_1(X, Y))$ and $V' = (V'_1, V'_0; \sigma') \in \text{Ob}(\mathcal{C}_1(X', Y'))$. By Proposition 10.1 in [ABS64], we may lift $\sigma, \sigma'$ to maps $\tilde{\sigma}, \tilde{\sigma}'$ of bundles over $X$ and $X'$, respectively.

Thinking of

$$0 \to V_1 \tilde{\sigma} \to V_0 \to 0$$

$$0 \to V'_1 \tilde{\sigma}' \to V'_0 \to 0$$

as complexes of bundles with $V_1, V'_1$ in degree 1 and $V_0, V'_0$ in degree 0, we may take their tensor product

$$V \otimes V' = 0 \to V_1 \otimes V'_1 \tau_2 \to (V_1 \otimes V'_0) \oplus (V_0 \otimes V'_1) \tau_1 \to V_0 \otimes V'_0 \to 0,$$

where

$$\tau_1 = \begin{pmatrix} \tilde{\sigma} \otimes \text{id}_{V'_0} & \text{id}_{V_0} \otimes \tilde{\sigma}' \end{pmatrix}$$

$$\tau_2 = \begin{pmatrix} -\text{id}_{V_1} \otimes \tilde{\sigma}' \\ \tilde{\sigma} \otimes \text{id}_{V'_1} \end{pmatrix}$$

The result is a complex of vector bundles over $X \times X'$ that is exact upon restriction to $X \times Y' \cup Y \times X'$.

Choose a splitting $\pi$ of $\tau_2|_{X \times Y' \cup Y \times X'}$. Then,

$$[(V_1 \otimes V'_0) \oplus (V_0 \otimes V'_1), (V_0 \otimes V'_0) \oplus (V_1 \otimes V'_1); \left(\tau_1|_{X \times Y' \cup Y \times X'}\right) \pi]$$

is an element of $L_1(X \times X', X \times Y' \cup Y \times X')$. 
Now, the pairing

\[ L_1(X, Y) \otimes L_1(X', Y') \to L_1(X \times X', X \times Y' \cup Y \times X') \]

described in Proposition 10.4 of \textcite{ABS64} is given by sending a simple tensor

\[ [V_1, V_0; \sigma] \otimes [V'_1, V'_0; \sigma'] \]

to

\[ j_2^{-1}([V_1 \otimes V'_1, (V_1 \otimes V'_0) \oplus (V_0 \otimes V'_1), V_0 \otimes V'_0; \tau_2|_{X \times Y' \cup Y \times X'}, \tau_1|_{X \times Y' \cup Y \times X'})]; \]

this follows from the proof of Proposition 10.4.

Thus, in order to show that the assignment

\[ \text{Ob}(\mathcal{C}_1(X, Y)) \times \text{Ob}(\mathcal{C}_1(X', Y')) \to L_1(X \times X', X \times Y' \cup Y \times X') \]

given by

\[ (V, V') \mapsto [(V_1 \otimes V'_1) \oplus (V_0 \otimes V'_0), (V_0 \otimes V'_0) \oplus (V_1 \otimes V'_1); \begin{pmatrix} \tau_1|_{X \times Y' \cup Y \times X'} \\ \pi \end{pmatrix}] \]

determines

(a) a well-defined pairing on \text{Ob}(\mathcal{C}_1(X, Y)) \times \text{Ob}(\mathcal{C}_1(X', Y')) up to our choices of liftings \( \tilde{\sigma}, \tilde{\sigma}' \) and splitting \( \pi \), and
(b) a pairing

\[ L_1(X, Y) \otimes L_1(X', Y') \rightarrow L_1(X \times X', X \times Y' \cup Y \times X') \]

that coincides with the pairing in Proposition 10.4 of [ABS64], we need only prove:

**Lemma 3.2.10.** Let \((X, Y)\) be a pair as above, and let \([V_2, V_1, V_0; \sigma_2, \sigma_1] \in L_2(X, Y)\).
If \(\pi\) is a splitting of \(\sigma_2\),

\[ j_2([V_1, V_0 \oplus V_2; \begin{pmatrix} \sigma_1 \\ \pi \end{pmatrix}]) = [V_2, V_1, V_0; \sigma_2, \sigma_1]. \]

**Proof.** First, suppose \(\dim(V_1) > \dim(V_2) + \dim(X)\). Apply Lemma 7.2 in [ABS64] to construct a monomorphism

\[ h : V_2 \rightarrow V_1 \]

that extends \(\sigma_2\). By the proof of Lemma 7.3 in [ABS64],

\[ j_2([\text{coker}(h), V_0; \sigma_1]) = [V_2, V_1, V_0; \sigma_2, \sigma_1], \]

and so

\[ j_2([\text{coker}(h) \oplus V_2, V_0 \oplus V_2; A]) = [V_2, V_1, V_0; \sigma_2, \sigma_1], \]

where

\[ A = \begin{pmatrix} \sigma_1 & 0 \\ 0 & \text{id}_{V_2}|_Y \end{pmatrix}. \]
Hence, it suffices to show

\[ [\text{coker}(h) \oplus V_2; V_0 \oplus V_2; A] = [V_1, V_0 \oplus V_2; \begin{pmatrix} \sigma_1 \\ \pi \end{pmatrix}] \]

Choose a splitting \( s \) of \( h \), and let

\[ p : V_1 \to \text{coker}(h) \]

denote the canonical surjection. Then we have an isomorphism

\[
\begin{pmatrix} p \\ s \end{pmatrix} : V_1 \to \text{coker}(h) \oplus V_2.
\]

Since \( s|_Y \) is a splitting of \( \sigma_2 \), we also have an isomorphism

\[
\begin{pmatrix} \sigma_1 \\ s|_Y \end{pmatrix} : V_1|_Y \to V_0|_Y \oplus V_2|_Y.
\]

We have a commutative square

\[
\begin{array}{ccc}
V_1|_Y & \xrightarrow{\begin{pmatrix} \sigma_1 \\ s|_Y \end{pmatrix}} & V_0|_Y \oplus V_2|_Y \\
\downarrow{\begin{pmatrix} p|_Y \\ s|_Y \end{pmatrix}} & & \downarrow{\text{id}_{V_0|_Y \oplus V_2|_Y}} \\
\text{coker}(h)|_Y \oplus V_2|_Y & \xrightarrow{A} & V_0|_Y \oplus V_2|_Y
\end{array}
\]
Thus,

\[ [\text{coker}(h) \oplus V_2, V_0 \oplus V_2; A] = [V_1, V_0 \oplus V_2; \begin{pmatrix} \sigma_1 \\ s|_Y \end{pmatrix}] \]

Notice that we have an object

\[ [V_1 \times I, (V_0 \oplus V_2) \times I; t \begin{pmatrix} \sigma_1 \\ s|_Y \end{pmatrix} + (1 - t) \begin{pmatrix} \sigma_1 \\ \pi \end{pmatrix}] \]

in \( C_1(X \times I, Y \times I) \) whose restrictions to \( X \times \{0\} \) and \( X \times \{1\} \) are \( [V_1, V_0 \oplus V_2; \begin{pmatrix} \sigma_1 \\ \pi \end{pmatrix}] \)

and \( [V_1, V_0 \oplus V_2; \begin{pmatrix} \sigma_1 \\ s|_Y \end{pmatrix}] \), respectively. It now follows from Proposition 9.2 in \( \text{ABS64} \) that

\[ [V_1, V_0 \oplus V_2; \begin{pmatrix} \sigma_1 \\ s|_Y \end{pmatrix}] = [V_1, V_0 \oplus V_2; \begin{pmatrix} \sigma_1 \\ \pi \end{pmatrix}] \]

This finishes the case where \( \dim(V_1) > \dim(V_2) + \dim(X) \).

For the general case, choose a bundle \( E \) such that

\[ \dim(E) + \dim(V_1) > \dim(V_2) + \dim(X) \]

Define

\[ U := [V_2, V_1 \oplus E, V_0 \oplus E; \begin{pmatrix} \sigma_2 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \sigma_1 & 0 \\ 0 \& \text{id}_{V_1}|_Y \end{pmatrix}] \]
\[ U' := [V_1 \oplus E, V_0 \oplus E \oplus V_2; \begin{pmatrix} \sigma_1 & 0 \\ 0 & \text{id}_{Y_1} \\ \pi & 0 \end{pmatrix}] \]

Notice that

\[ [V_2, V_1, V_0; \sigma_2, \sigma_1] = U, \]

and

\[ [V_1, V_0 \oplus V_2; \begin{pmatrix} \sigma_1 \\ \pi \end{pmatrix}] = U', \]

so that it suffices to show that \( j(U') = U \). Since \( \begin{pmatrix} \pi & 0 \end{pmatrix} \) is a splitting of \( \begin{pmatrix} \sigma_2 \\ 0 \end{pmatrix} \), this follows from the case we have already considered.

Let \([V], [V']\) denote the classes of \(V, V'\) in \(L_1(X, Y), L_1(X', Y')\), and define

\[ [V] \otimes_{L_1} [V'] := [(V_1 \otimes V_0') \oplus (V_0 \otimes V_1'), (V_0 \otimes V_0') \oplus (V_1 \otimes V_1'); \begin{pmatrix} \tau_1 |_{X \times X', X \times Y \cup Y \times X'} \\ \pi \end{pmatrix}] \]

Remark 3.2.11. By Proposition 10.4 in [ABS64] and the above remarks,

\[ \chi([V]) \otimes \chi([V']) = \chi([V] \otimes_{L_1} [V']). \]

3.3 A generalized Atiyah-Bott-Shapiro construction

We now recall the classical Atiyah-Bott-Shapiro construction ([ABS64] Part III). Following Atiyah-Bott-Shapiro, we work with real Clifford algebras and \(KO\)-theory, and
we point out that one may perform a similar construction involving complex Clifford algebras and $KU$-theory.

Define

$$q_n := -x_1^2 - \cdots - x_n^2 \in \mathbb{R}[x_1, \ldots, x_n]$$

for all $n \geq 1$, and set $C_n := \text{Cliff}_\mathbb{R}(q_n)$. We also set $C_0 := \mathbb{R}$.

Let $M(C_n)$ denote the free abelian group generated by isomorphism classes of finitely-generated, indecomposable $\mathbb{Z}/2\mathbb{Z}$-graded modules over $C_n$. When we say a $\mathbb{Z}/2\mathbb{Z}$-graded module is indecomposable, we mean that if the module is written as a direct sum of two $\mathbb{Z}/2\mathbb{Z}$-graded modules, then one of the two summands must be 0.

There are evident injective maps

$$i_n : C_n \to C_{n+1}$$

for all $n \geq 0$; these injections induce homomorphisms

$$i_n^* : M(C_{n+1}) \to M(C_n)$$

via restriction of scalars. Set

$$A_n := M(C_n)/i_n^*(M(C_{n+1}))$$

Define $D^n$ to be the closed disk of radius 1 in $\mathbb{R}^n$. An important special case of the classical Atiyah-Bott-Shapiro construction is the group isomorphism

$$\alpha_n : A_n \xrightarrow{\cong} L_1(D^n, \partial D^n)$$

that appears in [ABS64] Theorem 11.5.
\( \alpha_n \) is defined as follows: let \( M = M_1 \oplus M_0 \) be a finitely generated \( \mathbb{Z}/2\mathbb{Z} \)-graded \( C_n \)-module. We use the \( \mathbb{R} \)-vector spaces \( M_1 \) and \( M_0 \) to construct real vector bundles over \( D^n \):

\[
V_1 := D^n \times M_1 \\
V_0 := D^n \times M_0
\]

and we define a map

\[
\sigma : V_1 \to V_0
\]

given by \((x, m) \mapsto (x, x \cdot m)\), where \( \cdot \) denotes the action of \( C_n \) on \( M \). Here, we are thinking of \( D^n \subseteq \mathbb{R}^n \) as a subset of \( C_n \). Notice that \( \sigma \) restricts to an isomorphism of bundles over \( \partial D^n \). Thus, we have constructed an element \([V_1, V_0; \sigma] \in L_1(D^n, \partial D^n)\).

Define

\[
\alpha_n([M]) = [V_1, V_0; \sigma].
\]

We refer the reader to [ABS64] for verification that the mapping

\[
[M] \mapsto [V_1, V_0; \sigma]
\]

is well-defined on the quotient \( A_n \) and determines an isomorphism.

Now, let \( f \in (x_1, \ldots, x_n) \subseteq Q := \mathbb{R}[x_1, \ldots, x_n] \). Choose real numbers \( \epsilon, \delta, \) and \( t \) such that \( \epsilon > 0, 0 < \delta << \epsilon, \) and \( t \in (-\delta, 0) \) in such a way that we may construct a negative Milnor fiber \( F_f^- \) associated to \( f \) as in Section 3.1.3.

Denote by \( B_\epsilon \) the closed ball of radius \( \epsilon \) in \( \mathbb{R}^n \) centered at the origin. We now construct a map

\[
\text{Ob}(\text{MF}(Q, f)) \to L_1(B_\epsilon, F_f^-)
\]

that
(a) recovers the Atiyah-Bott-Shapiro construction via the Buchweitz-Eisenbud-Herzog
equivalence (Theorem 2.4.1) when \( f = q_n \), and

(b) descends to a group homomorphism

\[
K_0[\text{MF}(Q,f)] \to L_1(B_\epsilon, F_f^-).
\]

We emphasize that a similar construction involving complex polynomials and their
Milnor fibers may be performed \textit{mutatis mutandis}. One may also perform the follow-
ing construction using the positive Milnor fiber \( F_f^+ \) of \( f \).

Let

\[
P = (P_1 \xleftarrow{d_1} P_0)
\]

be a matrix factorization of \( f \) over \( Q \). Denote by \( C(B_\epsilon) \) the ring of \( \mathbb{R} \)-valued contin-
uous functions on \( B_\epsilon \).

Applying extension of scalars along the inclusion

\[
Q \hookrightarrow C(B_\epsilon),
\]

we obtain a map

\[
P_1 \otimes_Q C(B_\epsilon) \xrightarrow{d_1 \otimes \text{id}} P_0 \otimes_Q C(B_\epsilon)
\]

of finitely generated projective \( C(B_\epsilon) \)-modules.

The category of real vector bundles over \( B_\epsilon \) is equivalent to the category of finitely
generated projective \( C(B_\epsilon) \)-modules; on objects, the equivalence sends a bundle to
its space of sections. Let

\[
V_1 \xrightarrow{d_1} V_0
\]

be a map of real vector bundles over \( B_\epsilon \) corresponding to the above map \( d_1 \otimes \text{id} \) under
this equivalence.

Notice that \( d_1|_{F^{-}_f} \) is an isomorphism; its inverse is the restriction to \( F^{-}_f \) of the map \( d_0 : V_0 \to V_1 \) determined by

\[
P_0 \otimes_Q C(B_\epsilon) \xrightarrow{\frac{1}{d_0 \otimes \text{id}}} P_1 \otimes_Q C(B_\epsilon).
\]

Define \( \Phi_f(P_1 \xrightarrow{d_1}{d_0} P_0) = (V_1, V_0; d_1|_{F^{-}_f}) \in \text{Ob}(C_1(B_\epsilon, F^{-}_f)). \)

**Remark 3.3.1.** The map analogous to \( \Phi_f \) in the setting of polynomials over \( \mathbb{C} \) and \( KU \)-theory appears in [BVS12]; we discuss this in detail in the proof of Proposition 4.3.2.

A morphism in \( \text{EMF}(Q, f) \) determines a morphism in \( C_1(B_\epsilon, F^{-}_f) \) in an obvious way (see Section 2.2.1 for the definition of the category \( \text{EMF}(Q, f) \)). Hence, we have shown:

**Proposition 3.3.2.** There is an additive functor

\[
\Phi_f : \text{EMF}(Q, f) \to C_1(B_\epsilon, F^{-}_f)
\]

given, on objects, by

\[
(P_1 \xrightarrow{d_1}{d_0} P_0) \mapsto [V_1, V_0; d_1|_{F^{-}_f}].
\]

In particular, we have a map

\[
\text{Ob}(\text{MF}(Q, f)) \to L_1(B_\epsilon, F^{-}_f).
\]

Suppose \( f = q_n \). Then \( \epsilon \) can be chosen to be 1 in the construction of the negative Milnor fiber \( F^{-}_f \), and the fiber can be chosen to be exactly \( S^{n-1} \subseteq \mathbb{R}^n \).

Let \( [\text{Ob}([\text{MF}(Q, f)])] \) denote the set of isomorphism classes of objects in \([\text{MF}(Q, f)]\). It is easy to check that one has a commutative triangle
where BEH denotes the bijection induced by the Buchweitz-Eisenbud-Herzog equivalence (discussed in detail in Section 2.4), and ABS denotes the Atiyah-Bott-Shapiro construction. Hence, our construction recovers the Atiyah-Bott-Shapiro construction via the Buchweitz-Eisenbud-Herzog equivalence when \( f = q_n \).

Our next goal is to show that \( \Phi_f \) induces a map on K-theory:

\[ \text{Proposition 3.3.3. } \Phi_f \text{ induces a group homomorphism } \]

\[ \phi_f : K_0[\text{MF}(Q, f)] \to L_1(B_\epsilon, F^-_t). \]

We will adopt the following notational conventions for the purposes of this proof:

1. A pair \((\epsilon, t)\) is a good pair if \( \epsilon > 0, t < 0 \), and the map

\[ \psi : B_\epsilon \cap f^{-1}((-\delta, 0) \cup (0, \delta)) \to (-\delta, 0) \cup (0, \delta) \]

from Section 3.1 is a locally trivial fibration for some \( \delta > 0 \) such that

\[ 0 < |t| < \delta << \epsilon. \]

2. If \((\epsilon, t)\) is a good pair, we denote the negative Milnor fiber \( B_\epsilon \cap f^{-1}(t) \) by \( F^-_t \).

We will need the following technical lemma:
Lemma 3.3.4. Let \((\epsilon_1, t_1), (\epsilon_2, t_2)\) be good pairs. Then there is an isomorphism

\[
g : L_1(B_{\epsilon_1}, F_{t_1}^-) \xrightarrow{\cong} L_1(B_{\epsilon_2}, F_{t_2}^-)
\]

yielding a commutative triangle

\[
\begin{array}{ccc}
L_1(B_{\epsilon_1}, F_{t_1}^-) & \xrightarrow{g} & L_1(B_{\epsilon_2}, F_{t_2}^-) \\
\Phi_f & & \Phi_f \\
\text{Ob}(\text{MF}(Q, f)) & & \\
\end{array}
\]

Proof. The case where \(t_1 = t_2\) is immediate, so we may assume \(t_1 \neq t_2\). First, suppose \(\epsilon_1 = \epsilon_2\). Without loss, assume \(t_2 < t_1\).

Set \(F^-_{[t_2, t_1]} := f^{-1}([t_2, t_1])\). Since the inclusions

\[
F^-_{t_1} \hookrightarrow F^-_{[t_2, t_1]}
\]

\[
F^-_{t_2} \hookrightarrow F^-_{[t_2, t_1]}
\]

are homotopy equivalences, the pullback maps

\[
L_1(B_{\epsilon_1}, F_{[t_2, t_1]}) \to L_1(B_{\epsilon_1}, F_{t_1})
\]

\[
L_1(B_{\epsilon_1}, F_{[t_2, t_1]}) \to L_1(B_{\epsilon_1}, F_{t_2})
\]

are isomorphisms.

We have commuting triangles
for $i = 1, 2$. It follows that the result holds when $\epsilon_1 = \epsilon_2$.

For the general case, assume, without loss, that $|t_2| < |t_1|$. Then $(\epsilon_1, t_2)$ is also a good pair. By the cases we’ve already considered, the result holds for the pairs $(\epsilon_1, t_1)$ and $(\epsilon_1, t_2)$, and also for the pairs $(\epsilon_1, t_2)$ and $(\epsilon_2, t_2)$. Hence, the result holds for the pairs $(\epsilon_1, t_1), (\epsilon_2, t_2)$.

We now prove Proposition 3.3.3.

**Proof.** It is not hard to see that $\Phi_f(P \oplus P') = \Phi_f(P) \oplus \Phi_f(P')$; we need only show that $\phi_f$ is well-defined. First, suppose $P \cong 0$ in $[\text{MF}(Q, f)]$. Then $\text{id}_P$ is a boundary in $\text{MF}(Q, f)$, and so $\text{id}_P$ factors through a trivial matrix factorization, by Proposition 2.2.4.

Write

$$P = (P_1 \xrightarrow{d_1} P_0).$$

Since $P$ is a summand of a trivial matrix factorization, $\text{coker}(d_1)$ is a projective $Q/(f)$ module. Choose $g \in Q$ such that $g(0) \neq 0$ and $\text{coker}(d_1)_g$ is free over $Q_g/(f)$, and choose $\epsilon' \in (0, \epsilon)$ such that $B_{\epsilon'} \cap g^{-1}(0) = \emptyset$.

The inclusion

$$Q \hookrightarrow Q_g$$

induces a functor

$$\text{MF}(Q, f) \to \text{MF}(Q_g, f).$$
Choose \( t' \) such that \((\epsilon', t')\) is a good pair. Applying Lemma 3.3.4, we have a commutative diagram

\[
\begin{array}{ccc}
L_1(B_{\epsilon}, F_{t}^-) & \cong & L_1(B_{\epsilon'}, F_{t'}^-) \\
\Phi_f & & \Phi_f \\
\Phi_f & & \Phi_f \\
\text{Ob}(\text{MF}(Q, f)) & \longrightarrow & \text{Ob}(\text{MF}(Q_g, f))
\end{array}
\]

By Proposition 2.2.5, the image of \( P \) in \( \text{Ob}(\text{MF}(Q_g, f)) \) maps to 0 via \( \Phi_f \). Hence, the map \( \Phi_f : \text{Ob}(\text{MF}(Q, f)) \to L_1(B_{\epsilon}, F_{t}^-) \) sends \( P \) to 0, as required.

We now show that, if \( \alpha : P \to P' \) is a morphism in \( \text{EMF}(Q, f) \), \( \Phi_f(P) \oplus \Phi_f(\text{cone}(\alpha)) \) and \( \Phi_f(P') \) represent the same class in \( L_1(B_{\epsilon}, F_{t}^-) \).

We start by showing \( \Phi_f(P[1]) = -\Phi_f(P) \) in \( L_1(B_{\epsilon}, F_{t}^-) \). Write \( \Phi_f(P) = (V_1, V_0; d_1|_{F_{t}^-}) \), so that \( \Phi_f(P[1]) = (V_0, V_1; -d_0|_{F_{t}^-}) \). Since \( \text{cone}(\text{id}_P) \) is contractible, the class represented by

\[
\Phi_f(\text{cone}(\text{id}_P)) = (V_0 \oplus V_1, V_1 \oplus V_0; \begin{pmatrix} d_0|_{F_{t}^-} & \text{id} \\ 0 & -d_1|_{F_{t}^-} \end{pmatrix})
\]

in \( L_1(B_{\epsilon}, F_{t}^-) \) is 0.

The object

\[
(V_0 \oplus V_1, V_1 \oplus V_0; \begin{pmatrix} d_0|_{F_{t}^-} & t \cdot \text{id} \\ 0 & -d_1|_{F_{t}^-} \end{pmatrix})
\]

of \( \mathcal{C}_1(B_{\epsilon} \times I, F_{t}^- \times I) \) restricts to \( \Phi_f(\text{cone}(\text{id}_P)) \) at \( t = 1 \) and \( \Phi_f((P \oplus P[1])[1]) \) at \( t = 0 \). Since \( (P \oplus P[1])[1] \cong P \oplus P[1] \), we may use Proposition 9.2 in [ABS64] to conclude that \( \Phi_f(P[1]) = -\Phi_f(P) \) in \( L_1(B_{\epsilon}, F_{t}^-) \).
Now, we have

\[ \Phi_f(\text{cone}(\alpha)) = (V_0 \oplus V_1, V_1 \oplus V_0; \begin{pmatrix} d_0|_{F_f^{-}} & \alpha_1 \\ 0 & -d_1'|_{F_f^{-}} \end{pmatrix}). \]

Using Proposition 9.2 in [ABS64] in the same manner as above, we may conclude that \( \Phi_f(\text{cone}(\alpha)) \) and \( \Phi_f(P') \oplus \Phi_f(P[1]) \) represent the same class in \( L_1(B_\epsilon, F^-) \).

Finally, suppose \( \alpha : P \cong P' \) is an isomorphism in \([\text{MF}(Q, f)]\). Then \( \text{cone}(\alpha) \) is contractible, and so the results we just established imply that \( \Phi_f(P) = \Phi_f(P') \). Thus, \( \Phi_f \) preserves isomorphisms in \([\text{MF}(Q, f)]\). Since every distinguished triangle in \([\text{MF}(Q, f)]\) is isomorphic to one of the form

\[ P \xrightarrow{\alpha} P' \to \text{cone}(\alpha) \to P[1], \]

and we have shown that \( \Phi_f \) preserves such triangles, we are done.

We now use our construction \( \phi_f \) to exhibit a compatibility between Knörrer periodicity (Theorem 2.6.1) and Bott periodicity; we study the map \( \phi_f \) more closely in the case where \( f \) is an ADE singularity in Chapter 4.

### 3.4 Knörrer periodicity and Bott periodicity

In this section, we work with polynomials and vector bundles over \( \mathbb{C} \). The author fully expects results analogous to those in this section to hold for polynomials and vector bundles over \( \mathbb{R} \); we leave the details for future work.

Set

\[ Q := \mathbb{C}[x_1, \ldots, x_n], \quad Q' := \mathbb{C}[y_1, \ldots, y_m] \]
and let
\[ f \in (x_0, \ldots, x_n) \subseteq Q, \ f' \in (y_0, \ldots, y_m) \subseteq Q' \]
be such that \( Q_{(x_1, \ldots, x_n)}/(f)\), \( Q'_{(y_1, \ldots, y_m)}/(f')\) are IHS.

We now construct the Milnor fibers of \( f \) and \( f' \). Choose real numbers \( \epsilon'', \delta'' \), such that the map
\[ B_{\epsilon''} \cap (f \oplus f')^{-1}(D_{\delta''}^*) \to D_{\delta''}^* \]
given by \( x \mapsto (f \oplus f')(x) \) is a locally trivial fibration.

Similarly, choose \( \epsilon, \delta \) and \( \epsilon', \delta' \), as well as \( t'' \in D_{\delta''}^* \), so that the analogous maps
\[ B_{\epsilon} \cap f^{-1}(D_{\delta}^*) \to D_{\delta}^* \]
\[ B_{\epsilon'} \cap (f')^{-1}(D_{\delta'}^*) \to D_{\delta'}^* \]
are locally trivial fibrations, and also so that

(a) \( \epsilon, \epsilon' \) are sufficiently small so that \( B_{\epsilon} \times B_{\epsilon'} \subseteq B_{\epsilon''} \).

(b) \( |t''| < \min\{\delta, \delta'\} \).

Set \( F_f, F_{f'}, \) and \( F_{f \oplus f'} \) to be the Milnor fibers of \( f \), \( f' \), and \( f \oplus f' \) over \( t'' \).

Recall from Proposition 2.3.2 that we have a map
\[ K_0[\text{MF}(Q, f)] \otimes K_0[\text{MF}(Q', f')] \to K_0[\text{MF}(Q \otimes C Q', f \oplus f')] \]
given by
\[ [P] \otimes [P'] \mapsto [P \otimes \text{MF} P']. \]

The following proposition is the key technical result in this section.
Proposition 3.4.1. There exists a map

\[ \text{ST}_{L_1} : L_1(B_{\epsilon}, F_f) \otimes L_1(B_{\epsilon'}, F_{f'}) \to L_1(B_{\epsilon''}, F_{f \oplus f'}) \]

such that, given matrix factorizations \( P \) and \( P' \) of \( f \) and \( f' \), respectively,

\[ \text{ST}_{L_1}(\phi_f([P]) \otimes \phi_f'([P'])) = \phi_{f \oplus f'}([P \otimes_{MF} P']). \]

Proof. Write

\[ P = (P_1 \xleftrightarrow{d_1} P_0), \quad P' = (P'_1 \xleftrightarrow{d'_1} P'_0) \]

and

\[ \Phi_f(P) = [V_1, V_0; d_1|_{F_f}], \quad \Phi_{f'}(P') = [V'_1, V'_0; d'_1|_{F_{f'}}]. \]

We note that

\[ \phi_{f \oplus f'}([P \otimes_{MF} P']) = [(V_1 \otimes V'_0) \oplus (V_0 \otimes V'_1), (V_0 \otimes V'_0) \oplus (V_1 \otimes V'_1); A], \]

where \( A \) is the restriction of the matrix

\[
\begin{pmatrix}
    d_1 \otimes \text{id} & \text{id} \otimes d'_1 \\
    -\text{id} \otimes d'_0 & d_0 \otimes \text{id}
\end{pmatrix}
\]

to \( F_{f \oplus f'} \).

As in Section 3.1.2, choose an injection

\[ H : CF_f \to B_\epsilon \]

such that
• \( H(x, 1) = x \in F_f \subseteq B_e, \)

• \( H(-, s) : F_f \rightarrow B_e \) maps into the Milnor fiber \( B_e \cap f^{-1}(st'') \) for \( s \in (0, 1) \), and

• \( H(x, 0) = 0 \) for all \( x \in F_f \)

Choose \( H' : CF_{f'} \rightarrow B_{e'} \) similarly.

Clearly \( \text{im}(H) \) is contractible, since it is homeomorphic to \( CF_f \) (and of course the same is true for \( \text{im}(H') \)). It follows that the inclusions of pairs

\[
g : (\text{im}(H), F_f) \hookrightarrow (B_e, F_f)
\]

\[
g' : (\text{im}(H'), F_{f'}) \hookrightarrow (B_{e'}, F_{f'})
\]

induce isomorphisms on \( L_1 \) upon pullback; this is immediate from the long exact sequence in topological K-theory and the naturality of the Euler characteristic from Section 3.2 with respect to maps of pairs.

Recall from Section 3.2 that we have a map

\[
L_1(\text{im}(H), F_f) \otimes L_1(\text{im}(H'), F_{f'}) \rightarrow L_1(\text{im}(H) \times \text{im}(H'), \text{im}(H) \times F_{f'} \cup F_f \times \text{im}(H'))
\]

denoted by

\[
[V] \otimes [V'] \mapsto [V] \otimes_{L_1} [V'].
\]

Define

\[
\text{ST}_{L_1} : L_1(B_e, F_f) \otimes L_1(B_{e'}, F_{f'}) \rightarrow L_1(B_{e''}, F_{f''})
\]

to be given by

\[
[V] \otimes [V'] \mapsto (G')^{-1}(g^*([V]) \otimes_{L_1} (g')^*([V'])),
\]
where \( G \) is the homotopy equivalence of pairs in Remark 3.1.12. We now compute

\[
g^*(\phi_f(P)) \otimes_{L_1} (g')^* (\phi_f(P')) = g^*([V_1, V_0; d|_{F_f}]) \otimes_{L_1} (g')^*([V'_1, V'_0; d'|_{F_{f'}}])
\]

explicitly.

There are obvious liftings of \( d_1|_{F_f} \) and \( d'_1|_{F_{f'}} \) to maps of bundles over \( \text{im}(H) \) and \( \text{im}(H') \), namely \( d_1|_{\text{im}(H)} \) and \( d'_1|_{\text{im}(H')} \). A splitting of the restriction of

\[
\begin{pmatrix}
-\text{id} \otimes d'_1|_{\text{im}(H')}
\end{pmatrix}
\]

to \( \text{im}(H) \times F_{f'} \cup F_f \times \text{im}(H') \) is given, on the fiber over \((H(x, s), H'(y, s'))\), by

\[
\frac{1}{f(H(x, s)) + f'(H'(y, s'))} \begin{pmatrix}
-\text{id} \otimes d'_0|_{\text{im}(H')} & d_0|_{\text{im}(H)} \otimes \text{id}
\end{pmatrix}
\]

(notice that \( f(H(x, s)) + f'(H'(y, s')) = (s + s')t'' \neq 0 \) when \((H(x, s), H'(y, s')) \in \text{im}(H) \times F_{f'} \cup F_f \times \text{im}(H')\), since either \( s \) or \( s' \) is equal to 1).

Thus, by the discussion at the end of Section 3.2, the product

\[
g^*([V_1, V_0; d|_{F_f}]) \otimes_{L_1} (g')^*([V'_1, V'_0; d'|_{F_{f'}}])
\]

is equal to

\[
[(V_1|_{\text{im}(H)} \otimes V'_0|_{\text{im}(H')}) \oplus (V_0|_{\text{im}(H)} \otimes V'_1|_{\text{im}(H')}), (V_0|_{\text{im}(H)} \otimes V'_0|_{\text{im}(H')} \oplus (V_1|_{\text{im}(H)} \otimes V'_1|_{\text{im}(H')}); B],
\]

where \( B \) is given, on the fiber over \((H(x, s), H'(y, s')) \in \text{im}(H) \times F_{f'} \cup F_f \times \text{im}(H')\),
by the matrix
\[
\begin{pmatrix}
d_1|_{\text{im}(H)} \otimes \text{id} & \text{id} \otimes d'_1|_{\text{im}(H')} \\
\frac{1}{f(H(x,s)) + f'(H'(y,s'))} \left( -\text{id} \otimes d'_0|_{\text{im}(H')} \right) & \frac{1}{f(H(x,s)) + f'(H'(y,s'))} \left( d_0|_{\text{im}(H)} \otimes \text{id} \right)
\end{pmatrix}
\]
restricted to $\text{im}(H) \times F_f \cup F_f \times \text{im}(H')$.

We wish to show that, upon applying $(G^*)^{-1}$ to this class, one obtains
\[
[(V_1 \otimes V'_0) \oplus (V_0 \otimes V'_1), (V_0 \otimes V'_0) \oplus (V_1 \otimes V'_1); C],
\]
where $C$ is the restriction of the matrix
\[
\begin{pmatrix}
d_1 \otimes \text{id} & \text{id} \otimes d'_1 \\
\frac{1}{\rho} \left( -\text{id} \otimes d'_0 \right) & \frac{1}{\rho} \left( d_0 \otimes \text{id} \right)
\end{pmatrix}
\]
to $F_{f \oplus f'}$. This will finish the proof, since the class
\[
[(V_1 \otimes V'_0) \oplus (V_0 \otimes V'_1), (V_0 \otimes V'_0) \oplus (V_1 \otimes V'_1); C]
\]
is clearly equal to
\[
[(V_1 \otimes V'_0) \oplus (V_0 \otimes V'_1), (V_0 \otimes V'_0) \oplus (V_1 \otimes V'_1); A],
\]
where $A$ is as above.

Observe that we have an object
\[
[((V_1 \otimes V'_0) \oplus (V_0 \otimes V'_1)) \times I, ((V_0 \otimes V'_0) \oplus (V_1 \otimes V'_1)) \times I; D]
\]
in $C_1(\text{im}(H) \times \text{im}(H') \times I, (\text{im}(H) \times F_{f'} \cup F_f \times \text{im}(H')) \times I)$, where $D$ is given, on
the fiber over

\[(H(x, s), H'(y, s'), T) \in (\text{im}(H) \times F_{f'} \cup F_f \times \text{im}(H')) \times I,\]

by the matrix

\[
\begin{pmatrix}
  d_1 \otimes \text{id} & \text{id} \otimes d'_1 \\
  \frac{1}{f(a(T)) + f'(b(T))} (-\text{id} \otimes d'_0) & \frac{1}{f(a(T)) + f'(b(T))} (d_0 \otimes \text{id})
\end{pmatrix}.
\]

Here, \(f, f', \) and the entries of \(d_1, d'_1, d_0, d'_0\) are evaluated at the point

\[(a(T), b(T)) := (H(x, \frac{T(1 - s' - s) + 2s}{2}), H'(y, \frac{T(1 - s' - s) + 2s'}{2})).\]

Notice that \(f(a(T)) + f'(b(T)) \neq 0\) for all

\[(H(x, s), H'(y, s'), T) \in (\text{im}(H) \times F_{f'} \cup F_f \times \text{im}(H')) \times I,\]

so this matrix is indeed an isomorphism on every fiber over \((\text{im}(H) \times F_{f'} \cup F_f \times \text{im}(H')) \times I\).

Restricting to \(T = 0\), one obtains the object

\[((V_1|_{\text{im}(H)} \otimes V'_0|_{\text{im}(H')}) \oplus (V_0|_{\text{im}(H)} \otimes V'_1|_{\text{im}(H')}), (V_0|_{\text{im}(H)} \otimes V'_0|_{\text{im}(H')}) \oplus (V_1|_{\text{im}(H)} \otimes V'_1|_{\text{im}(H')}); B).\]

Restricting to \(T = 1\) and applying \(G^{*-1}\), one obtains

\[((V_1 \otimes V'_0) \oplus (V_0 \otimes V'_1), (V_0 \otimes V'_0) \oplus (V_1 \otimes V'_1); C).\]

Now apply Proposition 9.2 in [ABS64].
Remark 3.4.2. It follows easily from the naturality of the Euler characteristic $\chi$ from Section 3.2 and Remark 3.2.11 that $ST_{L_1}$ induces a map

$$ST_{KU} : KU^0(B_\epsilon, F_f) \otimes KU^0(B_\epsilon', F_{f'}) \to KU^0(B_{\epsilon''}, F_{f \oplus f'}).$$

Remark 3.4.3. We point out that the group homomorphism $ST_{L_1}$ in Proposition 3.4.1 is given by the composition of the tensor product in topological K-theory with a specific formulation of the Sebastiani-Thom homotopy equivalence. Hence, Proposition 3.4.1 yields a precise sense in which the tensor product of matrix factorizations is related to the Sebastiani-Thom homotopy equivalence (cf. Remark 2.3.3).

Let us now consider the case where $Q' = \mathbb{C}[u, v]$ and $f = u^2 + v^2$. Note that $K_0[MF(\mathbb{C}[u, v], u^2 + v^2)] \cong \mathbb{Z}$ (Remark 2.5.2 and Remark 2.4.2); it is generated by the class $X = [\mathbb{C}[u, v] \xrightarrow{u+iv} \mathbb{C}[u, v]].$

Also, by Theorem 3.1.4 $F_{u^2+v^2}$ is homotopy equivalent to $S^1$, and so $L_1(B_\epsilon, F_{u^2+v^2})$ is isomorphic to $\mathbb{Z}$. This group is generated by $\phi_{u^2+v^2}(X)$; a way to see this is to apply Theorem 11.5 in [ABS64] and observe the compatibility between the Atiyah-Bott-Shapiro construction and the map $\phi_f$ via the Buchweitz-Eisenbud-Herzog functor, as discussed in the previous section. Thus, $\phi_{u^2+v^2}(X)$ is a Bott element in the group $L_1(B_\epsilon', F_{u^2+v^2}) \cong \widetilde{KU}^0(S^2)$; we shall denote by $\beta$ the map

$$KU^0(B_\epsilon, F_f) \to KU^0(B_\epsilon, F_f) \otimes KU^0(B_\epsilon, F_{u^2+v^2})$$

given by $(\chi \otimes \chi) \circ (- \otimes \phi_{u^2+v^2}(X)) \circ \chi^{-1}$. $\beta$ is the Bott periodicity isomorphism.

Since Knörrer periodicity is induced by tensoring with the matrix factorization
$\mathbb{C}[u,v] \xleftarrow{u+iv} \xrightarrow{u-iv} \mathbb{C}[u,v]$, we will denote by $K$ the map

$$K_0[M\text{F}(Q,f)] \to K_0[M\text{F}(Q[u,v], f \oplus u^2 + v^2)]$$

given by $- \otimes_{\text{MF}} X$.

The following result gives a precise sense in which Bott periodicity and Knörrer periodicity are compatible; it follows immediately from Proposition 3.4.1:

**Theorem 3.4.4.** Let $f \in (x_1, \ldots, x_n) \subseteq \mathbb{C}[x_1, \ldots, x_n]$, and suppose the hypersurface $\mathbb{C}[x_1, \ldots, x_n]_{(x_1, \ldots, x_n)}/(f)$ is IHS (see Definition 2.2.10). Then the diagram

\[
\begin{array}{ccc}
K_0[M\text{F}(Q,f)] & \xrightarrow{\chi \circ \phi_f} & KU^0(B_{\epsilon}, F_{f}) \\
\downarrow K & & \downarrow \beta \\
K_0[M\text{F}(Q[u,v], f \oplus u^2 + v^2)] & \xrightarrow{\chi \circ \phi_{f \oplus u^2 + v^2}} & KU^0(B_{\epsilon'}, F_{f \oplus u^2 + v^2}) \\
\end{array}
\]

commutes.
Chapter 4

Examples: the ADE singularities

In Section 3.3 we constructed a map $\phi_f$ from the Grothendieck group of the homotopy category of matrix factorizations associated to a complex (real) polynomial $f$ into the topological K-theory of its Milnor fiber (positive or negative Milnor fiber). We established that, when $f$ is a non-degenerate quadratic, this map recovers the Atiyah-Bott-Shapiro construction.

In this chapter, we examine some properties of the map $\phi_f$ when $f \in \mathbb{C}[x_1, \ldots, x_n]$ is an ADE singularity.

4.1 Maximal Cohen-Macaulay modules over the ADE singularities

Let $f \in (x_1, \ldots, x_n) \subseteq \mathbb{C}[x_1, \ldots, x_n]$, and assume the hypersurface

$$\mathbb{C}[x_1, \ldots, x_n]_{(x_1, \ldots, x_n)}/(f)$$

is IHS (Definition 2.2.10).
Choose $\epsilon, \delta > 0$ so that the map

$$B_\epsilon \cap f^{-1}(D_\delta^*) \to D_\delta^*$$

given by $x \mapsto f(x)$ is a locally trivial fibration, as in Section 3.1. Let $F_f$ denote the fiber of this fibration, the Milnor fiber of $f$.

Recall that $F_f$ is homotopy equivalent to a wedge sum of $\mu$ copies of $S^{n-1}$, where $\mu$ is the Milnor number of $f$ (Theorem 3.1.4).

Suppose $n$ is odd. Then

$$KU^0(B_\epsilon, F_f) \cong \tilde{KU}^0(\Sigma F_f) \cong \bigoplus_\mu \tilde{KU}^0(S^n) = 0$$

Thus, in this case, the map

$$\phi_f : K_0[MF(\mathbb{C}[x_1, \ldots, x_n], f)] \to L_1(B_\epsilon, F_f)$$

is the zero map.

For more interesting examples, we look to the ADE singularities, or simple plane curve singularities:

- $A_k = x_1^{k+1} + x_2^2, \quad k \geq 1$
- $D_k = x_1^{k-1} + x_1x_2^2, \quad k \geq 4$
- $E_6 = x_1^3 + x_2^4$
- $E_7 = x_1^3 + x_1x_2^3$
- $E_8 = x_1^3 + x_2^5$
It turns out that, if $f$ is a simple plane curve singularity and $n \geq 2$, the ring

$$R = \mathbb{C}[[x_1, \ldots, x_n]]/(f + x_3^2 + \cdots + x_n^2)$$

has finite MCM type; that is, $R$ has only finitely many indecomposable MCM modules up to isomorphism (this follows from results in [Yos90] Chapters 9 and 11 along with Knörrer periodicity). By a theorem of Buchweitz-Greuel-Schreyer in 1987, the converse is also true:

**Theorem 4.1.1** ([BGS87]). If $R = \mathbb{C}[[x_1, \ldots, x_n]]/(f)$ has finite MCM type and $n \geq 2$, $R \cong \mathbb{C}[[x_1, \ldots, x_n]]/(g + x_3^2 + \cdots + x_n^2)$, where $g \in \mathbb{C}[[x_1, x_n]]$ is an ADE singularity.

**Remark 4.1.2.** A more general result is stated in Theorem 9.8 of [LW12].

In particular, when $f \in \mathbb{C}[[x, y]]$ is an ADE singularity, $K_0[M\text{F}(\mathbb{C}[[x, y]], f)]$ is a finitely generated abelian group; this makes the ADE singularities a convenient source of examples for studying the properties of the map

$$\phi_f : K_0[M\text{F}(\mathbb{C}[x, y], f)] \to L_1(B_c, F_f).$$

The results we mentioned above involve ADE singularities thought of as elements of power series rings, not polynomial rings. But, for the purposes of studying homotopy categories of matrix factorizations, this makes no difference:

**Proposition 4.1.3.** If $f \in \mathbb{C}[x, y]$ is an ADE singularity, the functor

$$i : [M\text{F}(\mathbb{C}[x, y], f)] \to [M\text{F}(\mathbb{C}[[x, y]], f)]$$

induced by inclusion is an equivalence.
Proof. Every matrix factorization of $f$ over $\mathbb{C}[[x, y]]$ can be expressed, up to isomorphism in $[\text{MF}(\mathbb{C}[[x, y]], f)]$, as one involving a pair of matrices with polynomial entries ([Yos90] Chapter 9). Thus, $i$ is essentially surjective. One can argue that $i$ is fully faithful in the same manner as in Remark 2.4.2.

Before going further, we return to a discussion of formalities involving Hochschild homology of dg categories, this time applied to matrix factorization categories.

### 4.2 Hochschild homology of matrix factorization categories

Set $Q := \mathbb{C}[x_1, \ldots, x_n]$, and let $f \in (x_1, \ldots, x_n) \subseteq Q$ so that $Q_{(x_1, \ldots, x_n)}/(f)$ is IHS. Assume $n$ is even.

Let $\Omega_{Q/\mathbb{C}}^1$ denote the module of Kähler differentials of $Q$ over $\mathbb{C}$. We consider the exterior algebra

$$\bigwedge \Omega_{Q/\mathbb{C}}^1,$$

as a $\mathbb{Z}/2\mathbb{Z}$-graded complex of $Q$-modules with odd (even) degree piece given by the direct sum of the odd (even) exterior powers, equipped with differential given by left exterior multiplication by $df$.

A computation due to Dyckerhoff in Section 6 of [Dyc11] yields a canonical isomorphism of $\mathbb{Z}/2\mathbb{Z}$-graded complexes of $\mathbb{C}$-vector spaces between the above complex and Hochschild complex of the dg category $\text{MF}(Q, f)$, and hence a canonical isomorphism

$$HH_*(\text{MF}(Q, f)) \xrightarrow{\cong} \Omega_{Q/\mathbb{C}}^n/(df \wedge \Omega_{Q/\mathbb{C}}^n) \cong J_f \otimes_Q \Omega_{Q/\mathbb{C}}^n.$$
where \( J_f \) is the algebra
\[
\mathbb{C}[x_1, \ldots, x_n] \left/ \left( \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n} \right) \right.
\]
thought of as a \( Q \)-module. In particular, \( \text{HH}^*(\text{MF}(Q, f)) \) is concentrated in even degree.

Let \( P = (P_1 \xleftarrow{A} \ xrightarrow{B} P_0) \) be a matrix factorization of \( f \) over \( Q \). Choose bases of \( P_1, P_0 \), so that we may view \( A \) and \( B \) as matrices with entries in \( Q \).

By Example 2.30 in [Wal14a], upon applying the above isomorphism
\[
\text{HH}^*(\text{MF}(Q, f)) \xrightarrow{\sim} \Omega^n_{Q/\mathbb{C}}/(df \wedge \Omega^n_{Q/\mathbb{C}}),
\]
the Chern character
\[
\text{ch} : \text{Ob} (\text{MF}(Q, f)) \to \Omega^n_{Q/\mathbb{C}}/(df \wedge \Omega^n_{Q/\mathbb{C}})
\]
is given by
\[
(P_1 \xleftarrow{A} \ xrightarrow{B} P_0) \mapsto \frac{2}{n!}(-1)^{\binom{n}{2}} \text{tr}(dA dB \cdots dA dB) dx_1 \wedge \cdots \wedge dx_n
\]
where \( dA \) and \( dB \) denote the matrices resulting from applying \( d : Q \to \Omega^1_{Q/\mathbb{C}} \) to the entries of \( A \) and \( B \).

By Corollary 5.12 in [Yu15], the Chern character map descends, in this setting, to a map
\[
\text{ch} : K_0[\text{MF}(Q, f)] \to \Omega^n_{Q/\mathbb{C}}/(df \wedge \Omega^n_{Q/\mathbb{C}}).
\]

**Example 4.2.1.** It will be useful for us to have a formula for the Chern character of
\[
\Delta \in \text{Perf}(\text{MF}(Q, f)^{\text{op}} \otimes \text{MF}(Q, f)),
\]
where \( \Delta \) is as in Section 2.1.3.

Applying [PV12] (2.14) and Remark 2.3.6, we have natural isomorphisms

\[
HH_*(\text{Perf}(\text{MF}(Q, f)^{\text{op}} \otimes \text{MF}(Q, f))) \cong HH_*(\text{MF}(Q, -f) \otimes \text{MF}(Q, f)).
\]

\[
\cong HH_*(\text{MF}(Q \otimes_{\mathbb{C}} Q, -f \oplus f)).
\]

For \( 1 \leq j \leq n \) and \( g \in Q \), set \( \Delta_j(g) \in \mathbb{C}[x_1, \ldots, x_n, y_1, \ldots, y_n] \) to be the polynomial

\[
g(x_1, \ldots, x_{j-1}, y_j, y_{j+1}, \ldots, y_n) - g(x_1, \ldots, x_{j-1}, x_j, y_{j+1}, \ldots, y_n) \frac{y_j - x_j}{y_j - x_j}.
\]

Let \( C \) denote the \( n \times n \) matrix over \( \mathbb{C}[x_1, \ldots, x_n, y_1, \ldots, y_n] \) with \( C_{ij} = \Delta_j(\frac{\partial f}{\partial x_i}) \).

By Proposition 4.1.1 in [PV12], \( \text{ch}(\Delta) \) corresponds, via the above isomorphisms, to the class

\[
(-1)^{\binom{n}{2}} \cdot \det(C) \cdot dx_1 \wedge \cdots \wedge dx_n \wedge dy_1 \wedge \cdots \wedge dy_n \in \Omega^n_{Q/\mathbb{C}}/(df \wedge \Omega^n_{Q/\mathbb{C}})^{\otimes 2}.
\]

We now wish to use our formula for the Chern character map, along with Theorem 2.1.20 and several results of [BVS12], to examine the map

\[
\phi_f : K_0[\text{MF}(Q, f)] \to KU^0(B_\epsilon, F_f)
\]

when \( f \) is an ADE singularity.
4.3 An application of the Hirzebruch-Riemann-Roch formula for differential $\mathbb{Z}/2\mathbb{Z}$-graded categories

Let $f = x_1^3 + x_1x_2^2 \in Q = \mathbb{C}[x_1, x_2]$, the $D_4$ singularity. By results of Chapters 9 and 13 in [Yos90], $K_0[MF(Q, f)]$ is generated as an abelian group by the classes

\[
\left[ Q \xrightarrow{x_1} \frac{x_1}{x_1^2 + x_2^2} \right], \left[ Q \xrightarrow{x_1(x_1 + ix_2)} \frac{x_1}{x_1 - ix_2} \right].
\]

The Chern characters of these classes are

\[-2x_2 dx_1 dx_2, (3ix_1 - x_2)dx_1 dx_2 \in \Omega^2_{Q/\mathbb{C}}/(df \wedge \Omega^2_{Q/\mathbb{C}})\]

These classes are linearly independent over $\mathbb{C}$. This implies that

(a) $K_0[MF(Q, f)]$ is a rank 2 free abelian group generated by the two classes above, and

(b) the Chern character map $\text{ch} : K_0[MF(Q, f)] \rightarrow \Omega^2_{Q/\mathbb{C}}/(df \wedge \Omega^2_{Q/\mathbb{C}})$ is injective.

Remark 4.3.1. Using the same argument, it is straightforward to check that, if $f$ is any ADE singularity, $K_0[MF(Q, f)]$ is free abelian and the Chern character map

\[\text{ch} : K_0[MF(Q, f)] \rightarrow \Omega^2_{Q/\mathbb{C}}/(df \wedge \Omega^2_{Q/\mathbb{C}}).\]

is injective.

We are now prepared to prove:
Proposition 4.3.2. If \( f = x_1^3 + x_1 x_2^2 \in Q = \mathbb{C}[x_1, x_2] \), the homomorphism

\[
\phi_f : K_0[\text{MF}(Q,f)] \to L_1(B_\epsilon, F_f) \cong \mathbb{Z}^\oplus 4
\]

is injective.

Proof. Suppose \( \phi_f([P]) = 0 \). As alluded to in Remark \[3.3.1\] the map

\[
\Phi_f : \text{Ob}(\text{MF}(Q,f)) \to L_1(B_\epsilon, F_f)
\]

in this setting agrees with a map discussed in \[BVS12\]. More specifically,

\[
\Phi_f(E = (E_1 \xleftarrow{d_1} \xrightarrow{d_0} E_0)) = \alpha(\text{coker}(d_1))|_{F_f}
\]

for all matrix factorizations \( E \) of \( f \), where \( \alpha \) is as described on page 252 of \[BVS12\].

It follows from Proposition 4.1 and Theorem 4.2 in \[BVS12\], as well as the discussion in Section \[2.2.3\] that

\[
\chi(-, [P]) : K_0[\text{MF}(Q,f)] \to \mathbb{Z}
\]

is the zero map. Thus, by the Hirzebruch-Riemann-Roch formula in Theorem \[2.1.20\]

\[
\langle \text{ch}((-)^\vee), \text{ch}([P]) \rangle_{\text{MF}(Q,f)} : K_0[\text{MF}(Q,f)] \to \mathbb{C}
\]

is the zero map, where \( E^\vee \) is the matrix factorization of \(-f\) corresponding to the object \( E \in \text{MF}(Q,f)^{\text{op}} \) under the equivalence in \[PV12\] (2.14). By the discussion on
page 11 of [PV12], it follows that

$$\langle \text{ch}([P]), \text{ch}([(-)^\vee]) \rangle_{\text{MF}(Q, -f)} : K_0[\text{MF}(Q, f)] \to \mathbb{C}$$

is the zero map.

Clearly $\Omega^2_{Q/\mathbb{C}}/(d(-f) \wedge \Omega^2_{Q/\mathbb{C}}) = \Omega^2_{Q/\mathbb{C}}/(df \wedge \Omega^2_{Q/\mathbb{C}})$. Also, the images of the Chern character maps

$$\text{ch} : K_0[\text{MF}(Q, -f)] \to \Omega^2_{Q/\mathbb{C}}/(df \wedge \Omega^2_{Q/\mathbb{C}})$$

$$\text{ch} : K_0[\text{MF}(Q, f)] \to \Omega^2_{Q/\mathbb{C}}/(df \wedge \Omega^2_{Q/\mathbb{C}})$$

are identical, since one has an isomorphism

$$K_0[\text{MF}(Q, f)] \xrightarrow{\cong} K_0[\text{MF}(Q, -f)]$$

given by

$$[P_1 \xleftarrow{d_1} P_0] \mapsto [P_1 \xleftarrow{-d_1} P_0].$$

By the computations above, the $\mathbb{C}$-span of the image of the Chern character map in $\Omega^2_{Q/\mathbb{C}}/(df \wedge \Omega^2_{Q/\mathbb{C}})$ is precisely the elements of the form $ldx_1dx_2$, where $l \in Q$ is a homogeneous linear form. Thus, $\langle \text{ch}([P]), ldx_1dx_2 \rangle_{\text{MF}(Q, -f)} = 0$ for all homogeneous linear forms $l \in Q$.

Let $\Delta \in \text{Ob}(\text{MF}(\mathbb{C}[x_1, \ldots, x_n, y_1, \ldots, y_n], -f \oplus f))$ denote the matrix factorization described in Example 4.2.1. An easy computation yields

$$\text{ch}(\Delta) = (-6x_1^2 - 6x_1y_1 + 2x_2y_2 + 2y_2^2) \cdot dx_1 \wedge dx_2 \wedge dy_1 \wedge dy_2.$$
By Remark 2.1.21 the map

\[ \Omega^2_{Q/C}/(df \wedge \Omega^2_{Q/C})^{\otimes 3} \to \Omega^2_{Q/C}/(df \wedge \Omega^2_{Q/C}) \]

given by

\[ h \otimes h' \otimes h'' \mapsto \langle h, h' \rangle_{\text{MF}(Q,-f)} \cdot h'' \]

maps \( \text{ch}([P]) \otimes \text{ch}(\Delta) \) to \( \text{ch}([P]) \). That is, denoting \( \langle -,- \rangle_{\text{MF}(Q,-f)} \) by just \( \langle -,- \rangle \), we have

\[ -6\langle \text{ch}([P]), 1 \rangle \cdot x_1^2 - 6\langle \text{ch}([P]), y_1 \rangle \cdot x_1 + 2\langle \text{ch}([P]), y_2 \rangle \cdot x_2 + 2\langle \text{ch}([P]), y_2^2 \rangle \cdot 1 \]

\[ = -6\langle \text{ch}([P]), 1 \rangle \cdot x_1^2 + 2\langle \text{ch}([P]), y_2^2 \rangle \cdot 1 = \text{ch}([P]). \]

Since \( x_1^2 \) and 1 are not homogeneous linear forms, the only way this equality can hold is if \( \text{ch}([P]) = 0 \). Since

\[ \text{ch} : K_0[\text{MF}(Q,f)] \to \Omega^2_{Q/C}/(df \wedge \Omega^2_{Q/C}) \]

is injective, it follows that \([P] = 0\). \( \square \)

Remark 4.3.3. The only properties of the polynomial \( D_4 \) that we used in the proof were

(1) \( \mathbb{C}[x_1, x_2]_{(x_1, x_2)}/D_4 \) is IHS.

(2) The map

\[ K_0[\text{MF}(\mathbb{C}[x_1, x_2], D_4)] \to K_0[\text{MF}(\mathbb{C}[x_1, x_2], D_4)] \]
induced by inclusion is an isomorphism, and

(3) If
\[ (\text{ch}([P]), \text{ch}([(-)^\vee]))_{\text{MF}(\mathbb{C}[x_1, x_2], -D_4)} : K_0[\text{MF}(\mathbb{C}[x_1, x_2], D_4)] \to \mathbb{C} \]

is the zero map, then \([P] = 0\).

The ADE singularities clearly have property (1), and we showed in Proposition 4.1.3 that they have property (2). An easy (but tedious) series of computations shows that the ADE singularities satisfy property (3) as well; one can show this for each ADE singularity using exactly the same argument that we used in the \(D_4\) example above. Hence, if \(f \in \mathbb{C}[x_1, x_2]\) is an ADE singularity, \(\phi_f\) is injective (this is Theorem 1.0.3 in the introduction). In fact, more is true:

**Theorem 4.3.4.** If \(f \in \mathbb{C}[x_1, x_2]\) is an ADE singularity and \(n \geq 0\) is even, \(\phi_f(x_1^2 + \cdots + x_n^2)\) is injective.

**Proof.** Since \(\phi_f\) is injective, this follows immediately from Theorem 3.4.4. \(\square\)
Bibliography


Dim92 Alexandru Dimca, Singularities and topology of hypersurfaces, Springer, 1992. 60 64

DP92 Alexandra Dimca and Laurentiu Pienescu, Real singularities and dihedral representations, School of Mathematics and Statistics, University of Sydney, 1992. 67


Eis80 David Eisenbud, Homological algebra on a complete intersection, with an application to group representations, Transactions of the American Mathematical Society 260 (1980), no. 1, 35–64. 1 27


Hov07 Mark Hovey, Model categories, Mathematical Surveys and Monographs, no. 63, American Mathematical Soc., 2007. 10


